Anagram-free Graph Colouring and Colour Schemes

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Abstract

The primary topic of this thesis is graph colouring, which is the study of pattern avoidance in assignments of colours to vertices or edges of graphs. In this thesis, I introduce the study of anagram-free colouring and formulate an axiomatic generalisation of colouring in which results about many variants of colouring can be proven.

The study of anagram-free colouring was suggested by Alon et al. [Random Structures & Algorithms, 2002], and was first studied by the author and Wood [The Electronic Journal of Combinatorics, 2018] and independently by Kamčev, Łuczak and Sudakov [Combinatorics, Probability and Computing, 2017]. An anagram is a word of the form WP, where P is a permutation of W. An anagram-free colouring is a graph colouring in which the sequence of symbols along each path in the graph is not an anagram. Anagram-free colouring is an extension of square-free colouring, which is a well studied variant of graph colouring. I review square-free colouring, as it is a rich source of questions to ask about anagram-free colouring.

Alon et al. asked whether the anagram-free chromatic number is bounded on graphs of bounded maximum degree. I answer this question in the negative by constructing graphs with maximum degree 3 and unbounded anagram-free chromatic number. Furthermore, I investigate the behaviour of ϕ, and its generalisations, on various classes of graphs and trees, with upper and lower bounds for trees of bounded radius. Most results about the anagram-free chromatic number show that it is unbounded on many classes of graphs. This motivates the search for a setting in which the anagram-free chromatic number is bounded. To this end, I introduce the study of anagram-free colourings of graph subdivisions. Graph subdivisions are studied in square-free colouring, with one result being that every graph has a square-free 3-colourable subdivision [Andrzej and Michał, Electron. J. Combin, 2009]. Analogously, I show that every graph has an anagram-free 8-colourable subdivision. For trees, I construct anagram-free 10-colourable subdivisions with many fewer division vertices per edge than in the construction for general graphs.

Colour schemes are an axiomatic generalisation of graph colouring which is introduced and developed in this thesis. Most variations of graph colouring found in the literature can be formulated as colour schemes. The underlying axioms of colour schemes are used to derive properties and prove general results that have applications across many areas of graph colouring. These results include conditions for a variant of graph colouring to be bounded on sufficiently subdivided graphs, and on graphs of bounded maximum degree. Colour schemes are also used to construct new variants of graph colouring with novel properties to demonstrate the diversity available within this set of natural axioms. One example is a variant of colouring where every tree has a 4-colourable subdivision, but for which no similar result holds for graphs.

Declaration

This thesis contains no material which has been accepted for the award of any other degree or diploma at any university or equivalent institution and that, to the best of my knowledge and belief, this thesis contains no material previously published or written by another person, except where due reference is made in the text of the thesis.

[redacted]

Tim Wilson
2 October 2018

Publications During Enrolment

This thesis includes the following published work:

- Chapter 2 contains a result from the publication *Transversals in Latin arrays with many distinct symbols* in *Journal of Combinatorial Designs* in 2017 [19]. The entire paper was joint work with Darcy Best, Kevin Hendrey, Ian Wanless and David Wood.
- Chapter 3 is joint work with David Wood and is based on the publication *Anagram-Free Graph Colouring* in *Electronic Journal of Combinatorics* in 2018 [131].
- Chapter 4 is joint work with David Wood and is based on the publication *Anagram-free colourings of graph subdivisions* in *SIAM Journal on Discrete Mathematics* in 2018 [132].

For everyone who enjoys
self-referential dedications,
the world needs more of you.

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Chapter 1

Introduction

This thesis studies anagram-free colouring and proposes an axiomatic generalisation of graph colouring, called a colour scheme. Anagram-free colouring is a newly studied area of graph colouring that was proposed by Alon et al. [10] as an extension of square-free colouring. This chapter begins with a review of square-free colouring and many of its extensions, since square-free colouring provides context for anagram-free colouring. I then introduce original results in anagram-free colouring, along with an overview of results from the literature. Lastly, I introduce colour schemes as an axiomatic approach to graph colouring, with the goal of generalising many types of graph colouring present in the literature.

1.1 Graph colouring

Graph colouring is a well-known area within combinatorics that studies pattern avoidance in assignments of colours to the vertices or edges of graphs. My focus is on finite, simple graphs; see Diestel [48]. Chapters 2, 3 and 4 are concerned entirely with finite graphs. The study of colour schemes in Chapters 5 and 6 applies to infinite graphs.

A *graph* is a pair, $G = (V(G), E(G))$, where $V(G)$ is the set of *vertices* of G and $E(G)$ is the set of *edges* of G. The vertex set is any set. The edge set contains elements of the form $\{u, v\}$ where $u, v \in V(G)$ and $u \neq v$. The edge, $\{u, v\} \in E(G)$, is often written as uv for brevity. An edge is *incident* to a vertex if it contains the vertex. Two vertices, $u, v \in V(G)$, are *adjacent* if $uv \in E(G)$. Two edges are *adjacent* if they are incident to a common vertex. The *neighbourhood* of a vertex v, denoted $N(v)$, is the set of all vertices adjacent to v. The *closed neighbourhood*, $N[v]$, of a vertex v is $N(v) \cup \{v\}$. The *degree* of v, denoted $\deg(v)$, is the size of its neighbourhood. The *maximum degree*, $\Delta(G)$, of a graph G is $\max_{v \in V(G)} \deg(v)$. The *order* of a graph is $|V(G)|$, where $|S|$ denotes the number of elements in a set S.

A *subgraph*, H, of a graph G is a graph such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$; we say that G *contains* H. The *induced subgraph* of a set of vertices, $X \subseteq V(G)$, is the subgraph of G with vertex set X and edge set $\{xy : x, y \in X, xy \in E(G)\}$. The *disjoint union* of two graphs, G and H, with $V(G) \cap V(H) = \emptyset$, is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$.

A *vertex colouring* of a graph, G, is a function that assigns one colour to each vertex of G. Similarly, an *edge colouring* is a function that assigns one colour to each edge of G. These colourings are often denoted by functions $f : V(G) \to C$ or $f : E(G) \to C$, where C is a set of colours. For brevity, a *colouring* of a graph is either a vertex or an edge colouring, depending on context. A colouring of a graph is *proper* if all adjacent coloured elements, either vertices or edges, have distinct colours. A *monochromatic edge* in a vertex colouring of a graph, G, is an edge, $uv \in E(G)$, such that u and v have the same colour.

The *chromatic number*, $\chi(G)$, of a graph G, is the minimum number of colours in a proper vertex colouring of G. The *chromatic index*, $\chi'(G)$, is the minimum number of colours in a proper edge colouring of a graph G. The chromatic number is the central parameter studied within graph colouring. This thesis studies particular variations and extensions of the chromatic number.

Square-free colouring is defined in terms of paths. A *path* in a graph, G, is a finite sequence, $P := v_1 v_2 \dots v_n$, of vertices, $v_i \in V(G)$, such that consecutive vertices in P are adjacent in G and each vertex occurs at most once in P. A path, P, in a graph, G, is said to *contain* $uv \in E(G)$ if u and v are consecutive vertices in P. Paths can be written as a finite sequence of consecutively adjacent edges, provided that care is taken to ensure that no vertex is visited more than once. The *order* and *length* of a path is its number of vertices and edges respectively. The *path of order* n, also known as the *path of length* $n-1$, is the graph, denoted P_n, on n vertices and $n-1$ edges which contains a path of order n. The vertices at either end of a path are known as its *endpoints*. A *path between two vertices*, u and v, is a path with u and v as endpoints. The *distance* between two vertices, u and v, denoted $\operatorname{dist}(u, v)$, is the minimum length of a path between u and v, and it is undefined if no such path exists. Define the *distance* between an edge, $ab \in E(G)$, and a vertex, $v \in V(G)$, of a graph, G, to be the minimum of $\operatorname{dist}(a, v)$ and $\operatorname{dist}(b, v)$. A graph, G, is *connected* if there is a path between every pair of vertices in $V(G)$. A *connected component* of a graph, G, is a maximal connected subgraph of G. A *subpath* of a graph, G, is a subgraph of G which is isomorphic to P_n for some $n \geqslant 1$.

1.2 Square-free graph colouring

The study of square-free graph colouring was introduced in 2002 by Alon et al. [10] and has since received much interest [7, 13–15, 27, 29–31, 49, 50, 64, 69, 70, 72, 73, 75, 79, 91, 94, 97, 110, 114, 135]. The area is a generalisation of the study of square-free words. Let $[k] := \{1, 2, \dots, k\}$. A *word* is a sequence of symbols, $W := w_1 w_2 \dots w_n$, and a *subword* of W is a contiguous subsequence, $w_i w_{i+1} \dots w_j$, of W, with $i, j \in [n]$ and $i \leqslant j$. The *concatenation*, AB, of two words, A and B, is the symbols of A followed by the symbols of B. A *square* is a word of the form WW, where W is any non-empty word. A word is *square-free* if it does not contain any square as a subword. A graph colouring is *square-free* if the sequence of colours read along each of its paths is square-free. Note that this definition can apply to both vertex colouring and edge colouring, and in fact, both forms of colouring have been studied. See Figure 1.1 for an example of a graph with a square-free vertex colouring.

The terms 'square-free' and 'nonrepetitive' are used interchangeably throughout the literature. For consistency, I rephrase all results to use the terminology 'square-free' as it is an easier term to generalise. The *square-free chromatic number* of a graph G, denoted $\pi(G)$, is the minimum number of colours in a square-free vertex colouring of G. Similarly, the *square-free chromatic index* of a graph G, denoted $\pi'(G)$, is the minimum number of colours in a square-free edge colouring of G.

1.2.1 Background

Square-free colourings were originally studied as square-free words within the area of combinatorics on words. A foundational result in this area is the following theorem of Thue

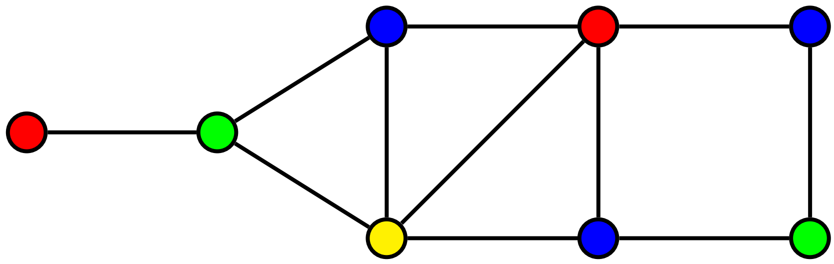


Figure 1.1: A graph with a square-free vertex 4-colouring. The validity of the colouring can be verified by checking that every path in the graph is not a square.

from 1906.

Theorem 1.1 (Thue [126])**.** *There exist arbitrarily long square-free words on three symbols.*

Thue's proof of Theorem 1.1 explicitly constructs a square-free word of arbitrary length with the following iterative process. Let $\{1, 2, 3\}$ be the symbols of our word and let f be the function $f(1) := 12312$, $f(2) := 131232$, $f(3) := 1323132$. It can be shown that if the word $w_1 w_2 \dots w_n$ is square-free, then $f(w_1)f(w_2)\dots f(w_n)$ is square-free. The theorem follows by induction. In the context of square-free graph colouring, Theorem 1.1 states that $\pi(P_n) \leqslant 3$ and $\pi'(P_n) \leqslant 3$, for all $n \in \mathbb{Z}^+$. It is easily observed that the longest square-free words on 1 and 2 symbols have length 1 and 3 respectively. Therefore π is fully characterized on paths.

Alon et al. [10] generalised the study of square-free words to the field of graph colouring and proved that there is a constant c' such that $\pi'(G) \leqslant c'\Delta(G)^2$ for all graphs G. Most subsequent work on square-free colouring focuses on vertex colouring rather than edge colouring. The proof method used to show $\pi'(G) \leqslant c'\Delta(G)^2$ is easily translated to show $\pi(G) \leqslant c\Delta(G)^2$, so in a sense there is little distinction between π' and π when bounding them on graphs of bounded maximum degree. The focus switched to vertex colouring because every bound on π' as a function of maximum degree is implied by a similar bound on $\pi(G)$ by applying a line-graph construction.

The *line-graph*, $L(G)$, of a graph G is the graph with $V(L(G)) := E(G)$, where $\{e_1, e_2\} \in E(L(G))$ if and only if e_1 and e_2 are adjacent edges in G. It is well known that $\chi'(G) = \chi(L(G))$ because two vertices, $u, v \in V(L(G))$, are adjacent in $L(G)$ if and only if the edges corresponding to u and v in $E(G)$ are adjacent in G.

Line-graphs are used in square-free colouring to translate bounds from π to π'. To do this, note that every path $e_1 e_2 \dots e_i$ in G, written as a sequence of edges of G, has a corresponding path in $L(G)$ where $e_1 e_2 \dots e_i$ is now a sequence of vertices of $L(G)$. Therefore every square-free vertex colouring of $L(G)$ corresponds to a square-free edge colouring of G. Let f be a function such that π is bounded by $f(\Delta)$ on graphs on maximum degree Δ. Let G be a graph and note that $\Delta(L(G)) \leqslant 2(\Delta(G) - 1)$, with equality attained by edges of G that are incident to two vertices of degree $\Delta(G)$. Since $\pi(L(G)) \leqslant f(\Delta(L(G)))$, $\pi'(G) \leqslant f(2(\Delta(G) - 1))$.

1.2.2 Bounding by maximum degree

Optimizing the constant and lower order terms in $\pi(G) \leqslant c\Delta(G)^2$ has received much attention. Most of the following results use variants of the Lovász Local Lemma. The Lovász Local Lemma is a probabilistic tool used to prove the existence of combinatorial objects with features that depend on local structures, in this case square-free colourings.

Recall that Alon et al. [10] initially proved $\pi'(G) \leqslant c'\Delta(G)^2$ for some constant c'. In fact, Alon et al. proved that $c' \leqslant 2e^{16}$ and did not attempt to optimize c'. Grytczuk [70] optimised this proof method, giving the bound $\pi(G) \leqslant 16\Delta(G)^2$. This bound was improved by Harant and Jendrol' [75] to $\pi(G) \leqslant \lceil 12.92(\Delta(G)-1)^2 \rceil$, for $\Delta(G) \geqslant 3$. Kolipaka et al. [89] created a stronger version of the Lovász Local Lemma and, as an application, used it to prove $\pi(G) \leqslant 10.4\Delta(G)^2$. Most recently, Dujmović et al. [50] used entropy compression to improve the bound to $\pi(G) \leqslant (1+o(1))\Delta(G)^2$. Entropy compression is a method which can often be used to prove stronger results than those proven with the Lovász Local Lemma. See Chapter 2 for a more detailed review of these optimizations, as well as a review of the Lovász Local Lemma and entropy compression. The approach of Dujmović et al. was later extended by Esperet and Parreau [60] to prove bounds for many other types of graph colouring.

The upper bound $\pi(G) \leqslant c\Delta(G)^2$ is known to be tight up to a logarithmic factor. Alon et al. [10] prove that, for all $\Delta \in \mathbb{Z}^+$, there exists a graph, G, of maximum degree Δ, with $\frac{c\Delta^2}{\log \Delta} \leqslant \pi(G)$. This bound is proven using a random graph construction.

1.2.3 k-power-free colouring

Squares have a natural extension to cubes and higher powers. A *k-power* is a word of the form W^k, meaning k repetitions of a non-empty word W. A word is *k-power-free* if it has no k-powers as subwords. k-power-free words are also known as 'k-nonrepetitive' words and are well studied in combinatorics of words [6, 40, 43, 127]. Similarly to squares, a graph colouring is *k-power-free* if the sequence of symbols along each path in the graph is not a k-power. The *k-power-free chromatic number* or *k-power-free chromatic index* of a graph, G, denoted $\pi_k(G)$ or $\pi'_k(G)$, is the minimum number of colours in a k-power-free vertex or edge colouring of G, respectively. Thue [127] proved that $\pi_3(P_n) \leqslant 2$, for all $n \geqslant 3$, with a method of proof similar to their result that $\pi(P_n) \leqslant 3$.

Two observations relate k-power-free colouring to square-free colouring. Firstly, every k-power-free colouring is also $(k+1)$-power-free, hence $\pi_k(G) \geqslant \pi_{k+1}(G)$, and $\pi'_k(G) \geqslant \pi'_{k+1}(G)$, for every graph G. In particular, upper bounds on square-free chromatic numbers apply more generally to k-power-free chromatic numbers. Secondly, for $k \geqslant 3$, a k-power-free colouring is not necessarily proper.

Alon and Grytczuk [7] generalise the bounds on square-free colouring by showing $\pi_k(G) \leqslant \left\lceil (6\Delta(G))^{k/(k-1)} \right\rceil$, for all $k \geqslant 2$ and graphs G. This proof uses the Lovász Local Lemma. Moreover, they use random graphs to show that there is a constant c such that for all d there are infinitely many graphs, G, with $\Delta(G) \leqslant d$ and $\pi_k(G) \geqslant \frac{c\Delta(G)^{k/(k-1)}}{k(\log \Delta(G))^{1/(k-1)}}$.

1.2.4 Cycles

Cycles have been thoroughly studied, both in square-free graph colouring and the study of circular words. A *cycle* is a connected graph where all vertices have degree 2. There is a unique cycle, denoted C_n, for each n for $n \geqslant 3$. Currie [41] show that $\pi(C_n) = 3$ for all $n \geqslant 3$, with the exception of $n = 5, 7, 9, 10, 14, 17$, in which case $\pi(C_n) = 4$. Subsequently Currie and Fitzpatrick [42] showed that $\pi_3(C_n) = 2$ for all $n \geqslant 3$. This completes the picture for cycles, since $L(C_n) = C_n$, there is no distinction between π and π' in this area. Combined with Thue's results, everything is known about k-power-free colouring on graphs of maximum degree 2.

1.2.5 Choosability

Choosability, also known as list colouring, is a well known generalisation of graph colouring, which has received interest within the study of square-free colouring [59]. A *list assignment* of a graph, G, is a function, L, such that $L(v)$ is a set of colours, called the *admissible colours*, for each vertex $v \in V(G)$. An L*-colouring* of a graph G is a colouring of G where every vertex v receives a colour from $L(v)$. A k*-list-assignment* is a list assignment where each set of admissible colours has size at least k. A graph is k*-choosable* if it has a proper L-colouring for every k-list-assignment L. Edge colouring can be extended similarly by making L a function from edges to sets of admissible colours. The *choosability*, $\text{ch}(G)$, of a graph G is the minimum k such that G is k-choosable. Note that $\chi(G) \leqslant \text{ch}(G)$, for every graph G, because graph colouring is a special case of L-colouring in which the set of admissible colours is the same for each vertex.

All variants of square-free colouring can be further generalised to choosability in the obvious way. A graph is *square-free* k*-choosable* if it has a square-free L-colouring for every k-list-assignment L. The *square-free choice number* and *square-free choice index* are denoted π_{ch} and π'_{ch}, respectively. The k-power-free extensions of π_{ch} and π'_{ch} are denoted $\pi_{k\,\text{ch}}$ and $\pi'_{k\,\text{ch}}$ respectively. Grytczuk et al. [72] prove that $\pi_{\text{ch}}(P_n) \leqslant 4$ using the left-handed local lemma; Grytczuk et al. [73] later proved the same result using a simple entropy compression argument. Furthermore, Dujmović et al. [50] observe that the known results bounding π as a function of maximum degree trivially extend to π_{ch}, with near-identical proofs. This widespread generalisation is due to the way in which the Lovász Local Lemma was applied to prove the known results. In particular, the proofs using the Lovász Local Lemma prove results about choosability because they only depend on the size of the sets of admissible colours, instead of requiring that the sets be equal, as is the case in standard graph colouring.

It is open whether $\pi_{\text{ch}}(P) \leqslant 3$ for all paths P. Zhao and Zhu [135] show that $\pi_{(2+\varepsilon)\,\text{ch}}(P_n) \leqslant 3$, for all $\varepsilon > 0$, using the following extension of k-power-free colouring to non-integer values of k. A word $w_1 \dots w_n$ is a k-power for $k > 1$ if $n = \lceil kq \rceil$, for some integer q, and $w_i = w_{i+q}$ for all $i \leqslant n - q$. The intuition behind this definition is we are still looking at k blocks of repetition, as in the case of integer k, but that the last block is truncated when k is not an integer. Other aspects of square-free choosability are discussed in later sections.

1.2.6 Subdivisions

A *subdivision* of a graph, G, is a graph obtained from G by replacing each edge, $uv \in E(G)$, by a path with endpoints u and v. If an edge, $uv \in E(G)$, is replaced by a path, $uw_1w_2 \dots w_iv$, of length $i + 1$, then we say that uv was *subdivided* i *times* and call the vertices $w_1, \dots, w_i$ its *division vertices*. The *original vertices* of a subdivision are the vertices which were present in the non-subdivided graph. The *length* of an edge of G is the length of its replacement path in S, which is one more than its number of division vertices. The k*-subdivision* of a graph, G, is the subdivision in which every edge of G is subdivided exactly k times. Similarly, a $(\leqslant k)$*-subdivision* of a graph G is a subdivision in which every edge of G is subdivided at most k times.

Highly subdivided graphs locally look like long paths or subdivisions of stars, which have bounded π, so one would intuitively expect π to be low on highly subdivided graphs. Let $\pi_{\text{sub}}(G)$ denote the minimum of $\pi(H)$ over all subdivisions, H, of a graph G. Grytczuk [70] note that Theorem 1.1 implies $\pi_{\text{sub}}(G) \leqslant 5$ for every graph G. To see this, let S be a subdivision of G where every edge is subdived an odd and unique number of times. Colour

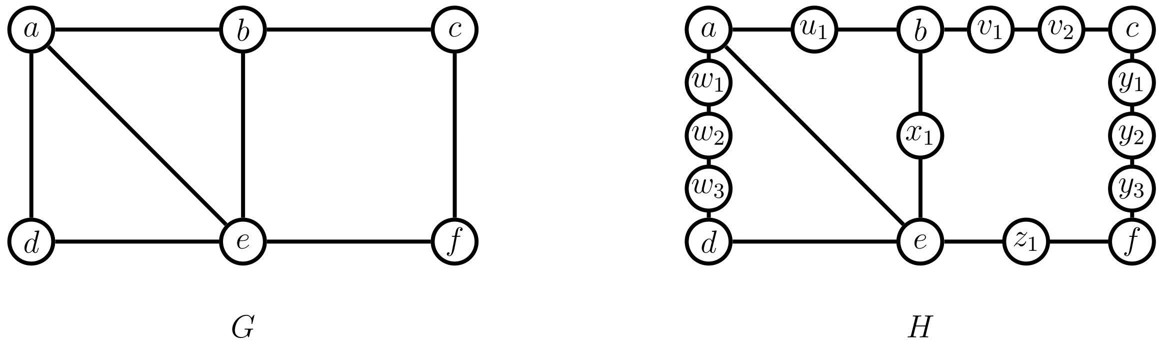


Figure 1.2: A graph, G, and a ($\leqslant 3$)-subdivision, H, of G. The vertices $a, b, c, d, e, f \in V(H)$ are original vertices, and the other vertices of H are division vertices. $v_1, v_2 \in V(H)$ are the division vertices of $bc \in E(G)$.

every original vertex of S red and colour the centre vertex of each subdivided edge blue. The remaining vertices induce a graph of disjoint paths, so we can square-free 3-colour each disjoint path with the three remaining colours. Now we consider a path, P, in S and show that it is not a square. If P contains no red or blue vertices then it is not a square as it is within a square-free 3-coloured subpath of division vertices. Now note that every path between two red or two blue vertices must contain a blue or red vertex respectively. Therefore, if P contains at least one red or blue vertex then, to be a square, it must contain at least two red vertices and two blue vertices. However, P cannot be a square because the unique edge lengths of the subdivision result in unique spacings between consecutive red and blue vertices in P, prohibiting a square.

Barát and Wood [15] improve the bound by proving $\pi_{\text{sub}}(G) \leqslant 4$, for every graph G, via the palindrome lemma of Kündgen and Pelsmajer [94]; see Lemma 1.2. Marx and Schaefer [97] give a different proof of $\pi_{\text{sub}}(S) \leqslant 4$ using a method similar to the above method used by Grytczuk [70]. The concept of levels, as seen in the palindrome lemma, is a reusable insight, so here is the outline of the proof by Barát and Wood [15]. The *levelling* of a connected graph, G, rooted at a vertex r is the function, $\lambda : V(G) \to \mathbb{Z}_{\geqslant 0}$, where $\lambda(v) = \text{dist}(v, r)$. A *level* of a graph with a levelling, λ, is a set of vertices $\lambda^{-1}(x)$ for, $x \geqslant 0$, such that $\lambda^{-1}(x)$ is non-empty.

Lemma 1.2 (Palindrome lemma [94]). *For every levelling λ of a graph G, there is a vertex 4-colouring of G such that every square path $v_1, \ldots, v_{2n}$ in G satisfies $\lambda(v_i) = \lambda(v_{n+i})$ for all $i \in [n]$.*

The palindrome lemma is so named because its 4-colouring derives from a word which is both square-free and palindrome-free. A *palindrome* is a word that reads the same forwards and backwards. Barát and Wood [15] show that every graph has a subdivision, S, with a levelling that satisfies the following two properties. Firstly, every level of S contains exactly one original vertex. Secondly, the two neighbours, $u_1, u_2 \in V(S)$, of each division vertex $v \in V(S)$ satisfies $\lambda(u_1) < \lambda(v) < \lambda(u_2)$. Note that the set of vertices of each level of S is an independent set, that is, a set of mutually non-adjacent vertices. It follows that S has a square-free 4-colouring because, by Lemma 1.2, each square path, $v_1, \ldots, v_{2n}$, in S must turn around twice in the same level, since $\lambda(v_1) = \lambda(v_{n+1})$. More precisely, there exists a $j \in [n]$ such that $\lambda(v_{j-1}) = \lambda(v_{j+1}) = \lambda(v_{n+j-1}) = \lambda(v_{n+j+1})$. However, paths are only able to turn around at an original vertex and each level has only one original vertex. Therefore S has a square-free 4-colouring.

Pezarski and Zmarz [114] improve the bound by proving $\pi_{\text{sub}}(G) \leqslant 3$ for every graph G, which is best possible since $\pi(P_4) = 3$. Their approach is similar to the proof of $\pi_{\text{sub}}(G) \leqslant 5$ but, instead of using additional colours, they use unique patterns in the colouring around the original vertices and central division vertices. The maximum number of division vertices per edge in their construction grows linearly in $|E(G)|$ which motivates the search for efficient subdivisions. Similarly Barát and Wood [15] show $\pi_{\text{sub}}(G) \leqslant 4$ using subdivisions where the number of division vertices per edge grows linearly with $|V(G)|$. Marx and Schaefer [97] also show a linear bound, with the distinction that their subdivision can be further subdivided without affecting their upper bound on π, whereas the results of Pezarski and Zmarz [114] and Barát and Wood [15] rely on using a particular number of divisions per edge.

The bounds on π_{sub} motivate the study of tradeoffs between the number of colours and the maximum number of division vertices per edge. Nešetřil et al. [110] prove that there is a constant, c, such that every graph G has a subdivision, H, with $\pi(H) \leqslant 17$ and less than $c \log |V(G)|$ subdivisions per edge. They show that, for constant π, this bound is best possible, up to the value of c, by studying subdivisions of the complete graph. The *complete graph of order* n, denoted K_n, is the graph on n vertices which are all pairwise adjacent.

Theorem 1.3 (Nešetřil et al. [110])**.** *For $k \geqslant 2$, the k-subdivision of K_n, denoted H, satisfies*

$$\left(\frac{n}{2}\right)^{1/(k+1)} \leqslant \pi(H) \leqslant 9\lceil n^{1/(k+1)} \rceil.$$

For a constant $c := \pi(H)$, Theorem 1.3 implies $\log_c \frac{n}{2} \leqslant k+1$. Nešetřil et al. [110] also show there is a function f such that $\pi(G) \leqslant f(\pi(H), k)$ for every $(\leqslant k)$-subdivision H of G. The lower bound in Theorem 1.3 shows π is unbounded on 1-subdivisions of K_n, so π is not bounded on graphs of bounded average degree. Dujmović et al. [50] study subdivisions in the context of square-free choosability. They prove the following using entropy compression.

Theorem 1.4 (Dujmović et al. [50])**.** *Let H be a subdivision of a graph G, such that every edge $vw \in E(G)$ is subdivided at least $\lceil 10^5 \log(\deg(v)+1) \rceil + \lceil 10^5 \log(\deg(w)+1) \rceil + 2$ times. Then $\pi_{\text{ch}}(H) \leqslant 5$.*

1.2.7 Trees

A *tree* is a connected graph which contains no cycles. A *rooted tree* is a tree with a vertex labelled as the root. For a rooted tree, T, with root r, the *depth* of a vertex, $v \in V(T)$, is the distance between v and r. A vertex, u, in a rooted tree, T, is a child of a vertex, $v \in V(T)$, if u and v are adjacent and u is deeper than v. A *leaf* of a tree, T, is a vertex, $v \in V(T)$, with $\deg(v) = 1$, unless T is a rooted tree with root v. The *height* of a rooted tree is the depth of its deepest vertex. A *d-ary tree* is a rooted tree with at most d children per vertex. The *complete d-ary tree of height h* is the rooted tree such that every non-leaf vertex has d children and every leaf has depth h. The complete 2-ary tree is called the *complete binary tree*. Brešar et al. [30] studied square-free colourings of trees, showing that, for every tree T, $\pi(T) \leqslant 4$, and that T has a subdivision, S, with $\pi(S) \leqslant 3$.

The behaviour of π_{ch} on trees has received a lot of interest, sparked by Fiorenzi et al. [63] proving that π_{ch} is unbounded on trees, which is in stark contrast to the bound of 4 for π on trees. Kozik and Micek [91] prove that, for all $\varepsilon > 0$, there is a constant, c, such that

$\pi_{\text{ch}}(T) \leqslant c\Delta(T)^{1+\varepsilon}$ for all trees T, which is stronger than the bound of $c\Delta^2$ on graphs of maximum degree Δ. Gągol et al. [64] show that there is a function, f, such that every tree, T, is square-free $f(\text{pw}(T))$-choosable, where $\text{pw}(T)$ is the pathwidth of T, as defined in Section 1.2.8. Moreover, they show that graphs of pathwidth 2 have unbounded square-free choosability.

1.2.8 Pathwidth and treewidth

Let G be a graph. Pathwidth and treewidth are graph parameters denoted $\text{pw}(G)$ and $\text{tw}(G)$ respectively. A *tree-decomposition* of G is a tree, T, with a function $B : V(T) \to \mathscr{P}(V(G))$, where $\mathscr{P}$ denotes the powerset, which satisfies the following two properties. First, for every edge, $e \in E(G)$, there is a $u \in V(T)$ such that e is an edge of the subgraph of G induced by $B(u)$. Second, for all $v \in V(G)$, the set of vertices $\{u \in V(T) : v \in B(u)\}$ induces a non-empty connected subgraph of T. The *width* of a tree-decomposition T is the maximum of $|B(w)| - 1$ over all $w \in V(T)$. The *treewidth* of G is the minimum width over all of its tree-decompositions. A *path-decomposition* is a tree-decomposition where the underlying tree is a path. The *pathwidth* of G is the minimum width over all of its path-decompositions.

Treewidth and Pathwidth were first defined by Robertson and Seymour [119] and have been applied to many areas. The appeal of pathwidth and treewidth is that, in a sense, they measure how similar a graph is to a path or a tree respectively. The parameters are of interest in square-free colouring because π is bounded on paths and trees. Kündgen and Pelsmajer [94] and Barát and Varjú [13] independently proved that π is bounded by a function of treewidth. Kündgen and Pelsmajer [94] give the best known bound of $\pi(G) \leqslant 4^{\text{tw}(G)}$. Albertson et al. [5] proved that there exists a graph G with $\pi(G) \geqslant \binom{\text{tw}(G)+2}{2}$. It is open as to whether there is a polynomial upper bound. For pathwidth, Dujmović et al. [50] prove the upper bound $\pi(G) \leqslant 2\,\text{pw}(G)^2 + 6\,\text{pw}(G) + 1$. It is open whether there is a linear upper bound.

1.2.9 Walks and trails

Walks and trails are sequences of consecutively adjacent vertices of a graph, like paths, but with fewer restrictions. A *walk* is a sequence of vertices, S, of a graph G such that each pair of consecutive vertices in S are adjacent in G. A *trail* is a walk, S, of a graph G such that each edge of G occurs at most once in S. Similarly to paths, walks and trails can each be treated either as a sequence of vertices or edges, depending on convenience. The only caveat is to ensure that, regardless of its representation, every edge occurs at most once in a trail.

It is not immediately clear how to define a square walk-avoiding version of square-free colouring, due to trivially square walks. For example, every vertex colouring of the walk $uvuv$, where u and v are adjacent vertices, is a square. Some set of square walks must be ignored if we are to have anything interesting to study. There have been two distinct definitions which disagree on which walks, known as the *admissible* walks, are required to be square-free in their respective colourings. An *open walk* is a walk with two distinct vertices as endpoints. Brešar and Klavzar [29] define square-free walk colouring with open walks as the admissible walks. To disambiguate the two definitions, I denote the corresponding *square-free open walk chromatic index* by σ'_{open}. They prove that $\sigma'_{\text{open}}(G) = \pi'(G)$ for all trees and cycles G. Unfortunately, this definition is only useful for edge colouring, since the walk $uvuv$, on adjacent vertices u and v, is square under every vertex colouring, so

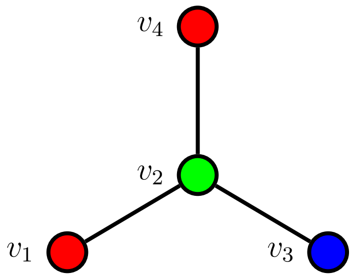


Figure 1.3: A square-free vertex colouring of the star of order 4 which is not a square-free walk vertex colouring. The admissible walk $v_1v_2v_3v_2v_4v_2v_3v_2$ is a square. This example can be extended to show that σ is unbounded on stars.

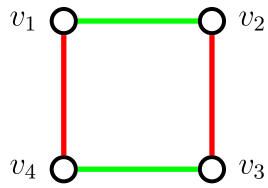


Figure 1.4: A square-free open walk edge colouring which is not a square-free walk edge colouring. The walk $v_1v_2v_3v_4v_1$ is a square. Since each colours in a square occurs an even number of times, a parity argument shows that all the square walks in this colouring are not open.

σ_{open} is not defined on graphs with at least one edge.

Barát and Varjú [13] define square-free walk colouring where a walk, $v_1, \ldots, v_{2n}$, is admissible if there exists $i \in \{1, \ldots, n\}$ such that $v_i \neq v_{i+n}$. This definition of admissible is equivalent to requiring that the sequence of vertices of a walk not be a square. They define σ to be the corresponding *square-free walk chromatic number*. Further work uses σ, so I define a *square-free walk colouring* of a graph to be a colouring with no square admissible walks of the form defined by Barát and Varjú [13]. For edge colouring, the admissible walks are those where the sequence of edges is not a square. Let σ' denote the corresponding *square-free walk chromatic index*. See Figure 1.3 for an example of a square-free colouring which is not a square-free walk colouring. See Figure 1.4 for a square-free open walk colouring which is not a square-free walk colouring.

Since all paths are admissible walks, a square-free walk colouring of a graph is also a square-free colouring. It follows that $\pi(G) \leqslant \sigma(G)$ and $\pi'(G) \leqslant \sigma'(G)$ for all graphs G. This provides the study of σ with some lower bounds and motivates upper bounds, as upper bounds on σ are upper bounds on π. Barát and Wood [15] bound σ by treewidth and maximun degree, proving that $\sigma(G) \leqslant 10(\text{tw}(G)+1)(\frac{7}{2}\Delta(G)-1)(\Delta(G)^2+1)$ for all graphs G.

1.2.10 Planar graphs

Planar graphs are a popular class of graphs in graph colouring, and determining whether π is bounded on planar graphs is widely considered to be the most important problem in square-free colouring. A graph is *planar* if it can be drawn in the plane with no edge crossings. A *face* of an embedding of a planar graph is a contiguous region of the plane enclosed by vertices and edges. Note that the faces of a planar graph depend on the way it is drawn. An *outerplanar* graph is a planar graph with a face that contains all the vertices in its boundary. A *facial walk* in a planar graph is a walk that traces the boundary of a face.

Barát and Varjú [13] and Kündgen and Pelsmajer [94] independently prove that $\pi(G) \leqslant 12$ for outerplanar graphs and conjecture that π is bounded on planar graphs. Barát and Varjú [13] also show that there is an outerplanar graph, G, with $\pi(G) \geqslant 7$, and construct planar graphs with $\pi(G) \geqslant 10$. Ochem (see [49]) improved the lower bound for outerplanar graphs from 7 to 11. Dujmović et al. [49] prove that there is a constant

c such that $\pi(G) \leqslant c \log |V(G)|$ for all planar graphs G. Dujmović et al. [51] generalise this bound to surfaces of higher genus, proving that each graph G with Euler genus g has $\pi(G) \leqslant 8g + 12\left(1 + \log_{3/2} |V(G)|\right)$. The *Euler genus* of a graph is the minimum integer g such that it can be embedded in an orientable surface of genus $g/2$ or a non-orientable surface of genus g.

A graph, H, is a *minor* of a graph, G, if H is isomorphic to a graph obtained from G by a sequence of edge contractions, edge deletions, and vertex deletions. An *edge contraction* deletes both vertices of an edge uv and adds a new vertex adjacent to all previous neighbours of u and v. Minors are well studied and, most famously, appear in the graph minor theorem which states that graph classes closed under taking minors can be characterised by avoiding a finite set of forbidden minors [120]. The study of square-free colourings of planar graphs has lead to the study of square-free colouring on graphs that exclude variants of minors. A graph, H, is a *topological minor* of a graph G if a subdivision of H is a subgraph of G. Dujmović et al. [51] show that for every fixed graph, H, every graph, G, without H as a topological minor has a square-free $O(\log |V(G)|)$-colouring. Wollan and Wood [133] study another variant of minors, called immersions, and show that for every graph, H, there exists a $k \in \mathbb{Z}^+$ such that every graph, G, that does not contain H as an immersion is square-free k-colourable.

Havet et al. [79] introduce a type of square-free colouring in which square *facial walks* must be avoided. This type has received some interest, with results for both the vertex colouring and edge colouring variants [12, 27]. The body of this thesis does not concern planar graphs so I avoid going into further detail.

1.3 Anagram-free graph colouring

Anagram-free colouring was first suggested by Alon et al. [10] and introduced by Kamčev, Łuczak, and Sudakov [83] as well as by myself and my supervisor, David Wood, in 2016 [131]. In fact, our arXiv papers appeared within days of each other [82, 130]. In this section I present an overview of our results as well as the other results in the area. My initial results, which are independent of the work of Kamčev et al. [83], are introduced in Section 1.3.2 and presented in detail in Chapter 3. My later work on subdivisions, which postdates the work of Kamčev et al., is introduced in Section 1.3.4 and presented in detail in Chapter 4. Results in Chapter 3 and 4 are original contributions, unless indicated otherwise.

1.3.1 Background

An *anagram* is a word of the form WP, where W is a non-empty word and P is a permutation of W. A word is *anagram-free* if it contains no anagram as a subword. In the literature, anagrams are also known as 'abelian squares' and anagram-free words are also known as 'abelian square-free words' or 'strongly non-repetitive sequences'.

Anagrams are studied within combinatorics on words, and the first thing to note is their similarity to square-free words [38]. A square is also an anagram, which follows from the observation that a square is a word $W\sigma(W)$ where σ is the identity permutation. A natural question to ask is whether there are arbitrarily long anagram-free words on a constant number of symbols. Recall that the longest square-free words on 2 symbols has length 3, and that Thue [126] proved that there are arbitrarily long square-free words on three symbols. More than three symbols are required for arbitrarily long angram-free words, as the longest anagram-free words on three symbols have length at most 7, for example *abcbabc* [39].

Pleasants [115] prove that there are anagram-free words of arbitrary length of five symbols using a similar method of proof as Thue [126] (see Section 1.2.1). Pleasants defines the function

$$\begin{aligned} f(1) &:= 213151314151412 \\ f(2) &:= \sigma(f(1)) \\ f(3) &:= \sigma^2(f(1)) \\ f(4) &:= \sigma^3(f(1)) \\ f(5) &:= \sigma^4(f(1)) \end{aligned}$$

where σ is the cyclic shift $\sigma(12345) := 23451$. The proof that $f(w_1)f(w_2)\dots f(w_i)$ is anagram-free if the word $w_1w_2\dots w_i$ is anagram-free requires a lot of case analysis, which I do not reproduce here. Instead, I note some structural properties of f and outline the central idea of the proof. For brevity, $f(W)$ means f applied to each symbol of a word W. Note that, for $k \in \{1,2,3,4,5\}$, $f(k)$ contains seven occurrences of k, and two occurrences of each other symbol. Therefore, for all words W, $f(W)$ contains $5x + 2|W|$ occurrences of symbol k, where x is the number of occurrences of k in W. Since the two halves of an anagram have the same number of occurrences of each symbol, $f(W_1W_2)$ is an anagram if and only if W_1W_2 is an anagram. It follows that, for all subwords, U, of, $f(W)$, the parts of U that align with the blocks created by f do not contribute to U being an anagram. The remainder of the proof is case analysis concerning the relatively small parts of U which do not align with W.

Keränen [86, 87] proved that there are anagram-free words of arbitrary length on four symbols, closing the gap with the lower bound. He used the same method of proof with

$$\begin{aligned} f(1) :=&1231343234314342412131214212324232132343132 \\ &124121314323431342321323431343243414243231 \end{aligned}$$

and $f(2)$, $f(3)$ and $f(4)$ defined by cyclic shifts of $f(1)$. Notably, this replacement word, which has length 85, is much longer than the one used by Pleasants and each symbol in the word occurs a unique number of times.

Alon, Grytczuk, Hałuszczak, and Riordan [10] proposed anagram-free graph colouring as a subject of study as a generalisation of square-free colouring and it is defined similarly to square-free colouring. A graph colouring is *anagram-free* if the sequence of colours read along each of its paths is not an anagram. See Figure 1.5 for an example of an anagram-free colouring and Figure 1.6 for a square-free colouring which is not an anagram-free colouring. This thesis studies both the vertex colouring and edge colouring variants of anagram-free colouring. The *anagram-free chromatic number* of a graph, denoted $\phi(G)$, is the minimum number of colours in a anagram-free vertex colouring of G. Similarly, the *anagram-free chromatic index* of a graph, denoted $\phi'(G)$, is the minimum number of colours in a anagram-free edge colouring of G. In the context of anagram-free graph colouring, Keränen's result is $\phi(P) \leqslant 4$ for every path P. Throughout this thesis there are many examples of significant differences in behaviour between ϕ and π. This is somewhat surprising considering the similarity of Keränen's and Thue's results.

1.3.2 Bounds on graphs and trees

Anagram-free colouring is a generalisation of square-free colouring in the sense that $\pi(G) \leqslant \phi(G)$, for all graphs G, because every square is an anagram. It follows that all lower bounds

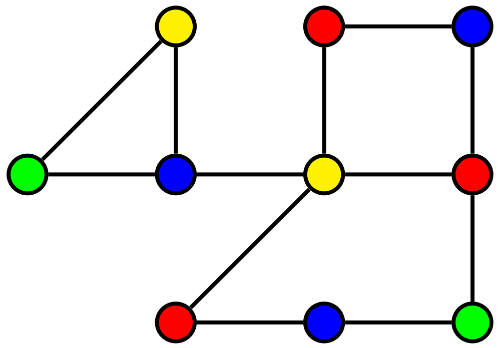


Figure 1.5: An anagram-free graph colouring.

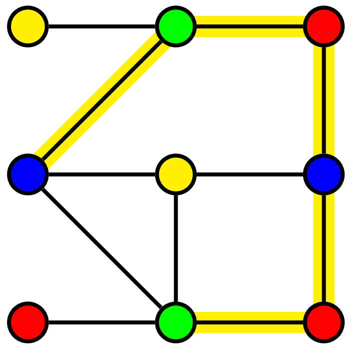


Figure 1.6: A square-free colouring which is not anagram-free. The highlighted path is an anagram because blue, green, red is a permutation of blue, red, green.

on π generalise to lower bounds on ϕ. This motivates the investigation of anagram-free colourings of classes of graphs with bounded π, to determine which, if any, upper bounds on π generalise to ϕ. As such, we begin by studying anagram-free colourings of trees and graphs of bounded degree.

Bounding by maximum degree

Alon et al. [10] states as an open problem whether ϕ is bounded on graphs of bounded degree. I answer this question, demonstrating the first significant difference in behaviour between π and ϕ. More specifically, ϕ is unbounded on outerplanar graphs of maximum degree 3, so many of the open problems about the behaviour of π on planar graphs do not correspond to interesting questions in the case of ϕ.

Theorem 3.2 (§3.1.2)**.** *Outerplanar graphs of maximum degree* 3 *have unbounded anagram-free chromatic number.*

Theorem 3.2 complements the result of Richmond and Shallit [118] regarding enumeration of anagrams. They counted anagrams with the aim of using the Lovász Local Lemma to prove that ϕ is bounded on paths, and concluded that there are too many anagrams for this method of proof to be feasible. As paths are a special case of graphs of bounded maximum degree, this also suggests that the Lovász Local Lemma cannot be used to bound ϕ on graphs of bounded degree, which makes sense in light of Theorem 3.2.

I investigate variations of anagram-free colouring that are analogous to variations of square-free colouring, starting with edge colouring. Recall that an edge colouring of a graph, G, is *anagram-free* if every path of G has an anagram-free colour sequence along its edges, and that $\phi'(G)$ is the minimum number of colours in an anagram-free edge colouring of G. The obvious bound, $\phi'(G) \geqslant \Delta(G)$, follows from the observation that edges incident to a common vertex receive distinct colours in an anagram-free edge colouring. I prove a significant improvement on this bound with the following result.

Theorem 3.1 (§3.1.1)**.** *Trees of maximum degree* 3 *have unbounded anagram-free chromatic index.*

Note that Theorems 3.2 and 3.1 already answer the question as to whether ϕ and ϕ' are bounded on outerplanar graphs. Furthermore, the constant of 3 in Theorems 3.1 and 3.2 is optimal because ϕ is bounded on graphs of maximum degree 2. Recall that $\phi(P) \leqslant 4$

for all paths P. It follows that $\phi(C) \leqslant 5$ for all cycles C, because we can colour a single vertex of C with a unique colour and anagram-free 4-colour the remaining path. Every graph, G, of maximum degree 2 is a disjoint union of paths and cycles, so $\phi(G) \leqslant 5$. The same result and proof shows that $\phi'(G) \leqslant 5$ for all graphs of maximum degree 2. It is open as to whether the constant of 5 can be reduced to 4.

Trees of bounded radius or pathwidth

The anagram-free chromatic number is unbounded on graphs of maximum degree 3, which motivates the study of ϕ on trees. Recall π is bounded on trees, as Brešar et al. [30] proved $\pi(T) \leqslant 4$ for every tree T. We answer the question as to whether ϕ is bounded on trees in the negative.

Theorem 3.4 (§3.1.2)**.** *Trees have unbounded anagram-free chromatic number.*

The proof of Theorem 3.4 is constructive, and uses a set of trees with unbounded maximum degree. This raises the question of whether ϕ is bounded on trees of maximum degree 3, answered by Kamčev et al. [83]. We also investigate bounding ϕ by parameters other than maximum degree, and show that ϕ is bounded on trees of bounded pathwidth and trees of bounded radius. The *radius* of a tree, T, is the minimum, taken over all vertices u in T, of the maximum of $\text{dist}(u, v)$, for all $v \in V(T)$. We obtain the following tight bound on ϕ on trees of bounded radius.

Theorem 3.7 (§3.2)**.** *Every tree T of radius h has $\phi(T) \leqslant h + 1$. Moreover, for every $h \geqslant 0$ there is a tree T of radius h such that $\phi(T) \geqslant h$.*

The radius of a path, P, is roughly half its length, so the bound in Theorem 3.7 is poor for paths. To address this, we bound ϕ on trees of bounded pathwidth. Pathwidth is a well studied parameter within square-free colouring [50, 64] as well as more generally. Recall that $\text{pw}(T)$, defined in Section 1.2.8, denotes the pathwidth of a tree T.

Theorem 3.10 (§3.2)**.** *For every tree T, $\phi(T) \leqslant 4\,\text{pw}(T) + 1$. Moreover, for every $p \geqslant 0$ there is a tree T such that $\phi(T) \geqslant p \geqslant \text{pw}(T)$.*

Note that since every tree, T, on n vertices, has pathwidth $O(\log n)$ [121], Theorem 3.10 implies that $\phi(T) \leqslant O(\log n)$. The graphs of pathwidth 1 are the *caterpillars*, which are trees consisting of a path, P, with additional leaf vertices adjacent to P. It is open whether $\phi(G) \leqslant 4$ for graphs of pathwidth 1.

k-anagram-free colouring

Recall that square-free colouring can be generalised to k-power-free colouring. Similarly, we generalise anagram-free colouring to k-anagram-free colouring. For $k \geqslant 2$, a *k-anagram* is a word $W_1 W_2 \dots W_k$ where each W_i is a permutation of a non-empty word W. k-anagrams are an established object of study in combinatorics on words and also known as 'abelian k-powers' or 'strong k-repetitions' [47]. A colouring of a graph, G, is *k-anagram-free* if the sequence of colours on each of its paths is not a k-anagram. This definition applies to both vertex and edge colourings. The corresponding *k-anagram-free chromatic number* is denoted by $\phi_k(G)$ and the *k-anagram-free chromatic index* is denoted by $\phi'_k(G)$.

Every $(k+1)$-anagram contains a k-anagram, so a k-anagram-free graph colouring is also $(k+1)$-anagram-free. Thus, for every graph G,

$$\phi(G) = \phi_2(G) \geqslant \phi_3(G) \geqslant \phi_4(G) \geqslant \dots$$

with an analogous expression for ϕ'_k. Therefore, for all $k \geqslant 2$, upper bounds on ϕ and ϕ' apply to ϕ_k and ϕ'_k respectively. However, recall that our only upper bounds on ϕ are for trees of bounded radius and trees of bounded pathwidth. This motivates the study of lower bounds for ϕ_k and ϕ'_k to determine whether the unbounded behaviour of ϕ and ϕ' extends to ϕ_k and ϕ'_k. We first show that Theorem 3.2, which says that ϕ is unbounded on planar graphs of maximum degree 3, generalises to ϕ_k.

Theorem 3.12 (§3.3.1)**.** *For $k \geqslant 2$, the k-anagram-free chromatic number is unbounded on planar graphs of maximum degree $k+1$.*

Note that Theorem 3.12 is not a full generalisation of Theorem 3.2 as the class of graphs are not outerplanar for $k \geqslant 3$. Also, the bound in Theorem 3.12 depends on k, so it is open whether there exists a $d \in \mathbb{Z}^+$ such that, for all $k \geqslant 2$, ϕ_k is unbounded on graphs of maximum degree d.

Results about ϕ and ϕ' do not always generalise to ϕ_k and ϕ'_k, as shown by the following contrasting result for trees.

Theorem 3.13 (§3.3.2)**.** *$\phi_k(T) \leqslant 4$ and $\phi'_k(T) \leqslant 4$ for every tree T and $k \geqslant 4$.*

This result is somewhat surprising given Theorems 3.2 and 3.4, which show that ϕ_2 and ϕ'_2 are unbounded on trees. Note that Theorem 3.13 leaves a gap at $k = 3$. Whether ϕ_3 and ϕ'_3 are bounded on trees is an open problem. Since upper bounds on ϕ apply to ϕ_3, ϕ_3 is bounded on trees of bounded radius and trees of bounded pathwidth, by Theorems 3.7 and 3.10. We show similar bounds for ϕ'_3.

Theorem 3.14 (§3.3.2)**.** *For every tree T, $\phi'_3(T) \leqslant 4\,\mathrm{pw}(T)$.*

The bounds on ϕ_k and ϕ'_k in Theorem 3.13 can be improved for larger values of k. The constant can be lowered to 3 for $k \geqslant 6$ and to 2 for $k \geqslant 8$. These improvements follow from bounds, proven by Dekking [47], on the minimum number of symbols in arbitrarily long 3-anagram-free and 4-anagram-free words, and the following theorem.

Theorem 3.15 (§3.3.2)**.** *For all $z \geqslant 1$ and $k \geqslant 2z$, if $\phi_z(P) \leqslant y$ for all paths P, then $\phi_k(T) \leqslant y$ and $\phi'_k(T) \leqslant y$ for all trees T.*

1.3.3 Results of Kamčev, Łuczak and Sudakov

Recall that Kamčev et al. [83] released an important paper on anagram-free colouring, independent of the results in Section 1.3.2. Among other results, they answer my question on the status of the binary tree. Here is a brief summary of their results.

Binary trees

Kamčev et al. [83] answer the question of whether ϕ is bounded on trees of maximum degree 3 with the following result.

Theorem 1.5 (Kamčev et al. [83])**.** *If T_h is the complete binary tree of height h, then*

$$\sqrt{\frac{h}{\log_2 h}} \leqslant \phi(T_h) \leqslant h+1.$$

The upper bound, $\phi(T) \leqslant h+1$, holds for every tree, T, of height h, and is obtained by colouring vertices by their depth. This upper bound is almost best possible, for general trees, as Theorem 3.7 states that there is a tree, T, for every height h with $\phi(T) \geqslant h$. I generalise Theorem 1.5 in Section 1.3.4 for use in my study of subdivisions.

Excluded minors

Recall from Section 1.2.10 that a graph, H, is a *minor* of a graph, G, if H is isomorphic to a graph obtained from G by a sequence of edge contractions, edge deletions and vertex deletions. Kamčev et al. [83] prove the following.

Theorem 1.6 (Kamčev et al. [83]). *For every graph G without H as a minor*

$$\phi(G) \leqslant 10|V(H)|^{3/2}|V(G)|^{1/2}.$$

This is a notable improvement over the trivial bound $\phi(G) \leqslant |V(G)|$ obtained by colouring each vertex with a unique colour, provided that the graph excludes a relatively small minor.

Random graphs

A graph is *d-regular* if all of its vertices have degree d. Let $G_{n,d}$ be the *random regular graph* which is chosen uniformly at random from all d-regular graphs of order n. Note that $G_{n,d}$ only exists when nd is even because, by the handshaking lemma, the sum of degrees in every graph is even.

Theorem 1.7 (Kamčev et al. [83]). *There exists a constant C such that for sufficiently large d the random regular graph $G_{n,d}$ satisfies*

$$\left(1 - \frac{C\log d}{d}\right)n \leqslant \phi(G_{n,d}) \leqslant \left(1 - \frac{\log d}{d}\right)n$$

with probability tending to 1 *as n tends to infinity, when $G_{n,d}$ exists.*

Note that $\left(1 - \frac{C\log d}{d}\right)$ tends to 1 as d tends to infinity, implying that nearly every vertex requires a unique colour in most regular graphs of sufficiently high degree.

1.3.4 Results on subdivisions

The results so far distinguish ϕ from π by showing that ϕ is unbounded on many classes of graphs for which π is bounded. This trend matches the intuition that, since there are many more squares than anagrams, it is much harder to avoid anagrams than it is to avoid squares. These results motivate the search for classes of graphs with bounded ϕ. Since ϕ is unbounded on graphs of maximum degree 3, we look to a class of graphs with low average degree, hence the study of highly subdivided graphs. Subdivisions are also of interest because bounding π on sufficiently subdivided graphs is a well studied topic within square-free colouring, see Section 1.2.6.

Subdivisions of trees

The results about π on highly subdivided trees generalise to similar results for ϕ. We first construct anagram-free 8-colourable subdivisions of binary trees.

Theorem 4.2 (§4.1.1). *Every binary tree, T, of height h, has a $(\leqslant 3^{h-1} - 1)$-subdivision, S, with $\phi(S) \leqslant 8$.*

We then construct anagram-free 10-colourable subdivisions of complete d-ary trees. Theorem 4.4 implies that all trees have a subdivision with ϕ at most 10 because every tree is a subtree of a d-ary tree, for some d.

Theorem 4.4 (§4.1.1)**.** *Every d-ary tree, T, of height h, has a* $\left(\leqslant 2d(d+1)^{h-1}\right)$*-subdivision, S, with $\phi(S) \leqslant 10$.*

The number of division vertices per edge in Theorems 4.2 and 4.4 grows exponentially with the height of the original tree. This raises the question of whether constructions with fewer division vertices exist. In particular, it is natural to ask whether every tree of bounded degree has an anagram-free c-colourable subdivision with the number of division vertices per edge growing slower than exponentially with height. This question is answered in the negative by the following theorem.

Theorem 4.9 (§4.1.3)**.** *The k-subdivision, S, of the complete d-ary tree of height h satisfies*

$$\sqrt{\frac{h}{\log_{\min\{d,(h(k+1))^2\}}(h(k+1))}} \leqslant \phi(S) \leqslant \frac{10h}{\log_{d+1}(k/2d)} + 14.$$

Theorem 4.9 implies that, for sufficiently large height h, the number of division vertices per edge, k, in an anagram-free c-colourable subdivision of the complete d-ary tree is at least

$$k \geqslant \frac{d^{h/c^2}}{h} - 1,$$

which is exponential in h for fixed c. The upper bound in Theorem 4.9 is obtained by applying Theorem 4.4 to independent subtrees of the complete d-ary tree. The lower bound is a generalisation of Theorem 1.5 of Kamčev et al. [83], and the proof is an extension of their method.

Subdivisions of General Graphs

We also show that ϕ is bounded on sufficiently subdivided graphs.

Theorem 4.11 (§4.2)**.** *Every graph G has a* $\left(\leqslant 6(2)^{2|E(G)|-1} - 1\right)$*-subdivision, S, with $\phi(S) \leqslant 14$.*

The bound on ϕ is improved in the following theorem, at the cost of a larger base in the exponent for the number of division vertices per edge.

Theorem 4.13 (§4.2)**.** *Every graph G has a* $\left(\leqslant 90\left(\frac{75}{9}+1\right)^{2|E(G)|-1}\right)$*-subdivision, S, with $\phi(S) \leqslant 8$.*

The bound $\phi(S) \leqslant 8$ in Theorem 4.13 is the best known bound for sufficiently subdivided graphs, and, notably, it is better than the bound for subdivisions of trees in Theorem 4.4. The advantage of Theorem 4.4 is that it uses many fewer division vertices per edge. Indeed, for the complete d-ary tree, T, the number of division vertices per edge in Theorem 4.4 grows polynomially as a function of $|E(T)|$.

To investigate the optimality, in terms of division vertices per edge, of Theorems 4.11 and 4.13, we study lower bounds for ϕ for subdivisions of K_n, the complete graph on n vertices. Recall that Nešetřil et al. [110] proved $\pi(S) \geqslant \left(\frac{n}{2}\right)^{1/(k+1)}$ with Theorem 1.3. Therefore, since $\phi(G) \geqslant \pi(G)$, $k \geqslant \log_c(n/2) - 1$ for every anagram-free c-colourable k-subdivision of K_n. We prove the following improvement.

Theorem 4.14 (§4.2.1)**.** *Let S be a $(\leqslant k)$-subdivision of K_n. If S is anagram-free c-colourable then*

$$k \geqslant \left(c!\left(\frac{n}{c}-1\right)\right)^{1/c} - c.$$

For fixed c, the bound in Theorem 4.14 is $k \geqslant xn^{1/c}$, for some x, which is larger than the logarithmic bound implied by Theorem 1.3. Still, this lower bound is much less than the exponential upper bound implied by Theorem 4.11. We expect that both our upper and lower bounds on k can be significantly improved.

1.3.5 Subsequent results from the literature

At the time of writing, two papers on anagram-free colouring have been published subsequent to the work in Sections 1.3.2 and 1.3.4.

Czap et al. [44] study facial anagram-free edge colourings of planar graphs in which trails restricted to the boundary of a face must not be anagrams. This is similar to the square-free colouring variant. They show that every planar graph has a facial anagram-free edge colouring with at most 11 colours.

The results in Section 1.3.2 motivate the question of whether ϕ is bounded on graphs of bounded pathwidth. This question was answered in the negative by Carmi et al. [35], by showing that ϕ is unbounded on the family of graphs, $\mathcal{G}$, known as the *cross-ladders*, where $\mathcal{G} := \{P_2 \boxtimes P_i : i \in \mathbb{Z}^+\}$. The graph, $G := H \boxtimes H'$, is the *strong product* of H and H', with vertex set, $V(G) := V(H) \times V(H')$, and an edge $(u, u')(v, v') \in E(G)$, between distinct vertices of $V(G)$, if $u \in N[v]$ and $u' \in N[v']$. See Figure 1.7 for an example of a cross-ladder in $\mathcal{G}$, and note that cross-ladders are planar. Carmi et al. [35] show that, for all $G \in \mathcal{G}$, $\log_2 |V(G)| \leqslant \phi(G)$.

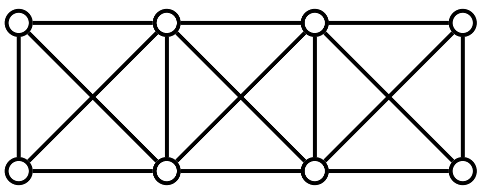


Figure 1.7: The cross-ladder of order 8.

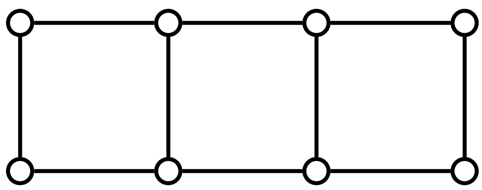


Figure 1.8: The ladder of order 8.

Carmi et al. [35] generalise their result to graphs with larger pathwidth with the following theorem.

Theorem 1.8 (Carmi et al. [35])**.** *For all integers $n \geqslant 1$ and $k \geqslant 3$, there exists a graph, G, of order kn, pathwidth $2k-1$, and maximum degree $3k-1$ with $\phi(G) \geqslant (k-2)\log_2(n/2)$.*

A remaining open problem is whether graphs of pathwidth 2 have bounded anagram-free chromatic number. A key example is the family of graphs, $\mathcal{H}$, known as the *ladders*, where $\mathcal{H} := \{P_2 \square P_i : i \in \mathbb{Z}^+\}$. The graph, $G := H \square H'$, is the *Cartesian product* of H and H', with vertex set, $V(G) := V(H) \times V(H')$, and an edge $(u, u')(v, v') \in E(G)$, if either $u = v$ and $u' \in N(v')$, or $u' = v'$ and $u \in N(v)$. See Figure 1.8 for an example of a ladder.

1.4 Colour schemes

Many variants of graph colouring are studied within the area of graph colouring. Some examples include acyclic colouring [9, 22–24, 60, 65, 66, 68, 84, 90, 134], star colouring

[5, 15, 84, 134], parity colouring [26, 34], centred colouring [109], conflict-free colouring [37], and pattern-free colouring [71]. Many types of colourings have common features or properties, yet they are mostly studied independently of each other. Transferring relevant insights between types of colouring tends to involve re-proving similar results, often with an extension or reformulation of an existing proof from one type of colouring to another. This requirement slows down exploration of the space of potential types of colouring, and makes it difficult to prove relationships between existing types of colouring. Currently, investigating a new variant of an established type of colouring involves rewriting many proofs or constructing arguments as to why existing proofs apply to the new variant.

In response, Chapter 5 proposes an axiomatic approach which unifies many types of graph colouring. In particular, I review many types of colouring, identify some shared properties, and propose a set of axioms to encompass these properties. A *colour scheme* is defined to be a set of coloured graphs that obey these axioms. For example, the colour scheme that corresponds to proper colouring is the set of coloured graphs with no monochromatic edges. Care needs to be taken to define axioms which are neither too restrictive nor too general. A set of axioms which is too restrictive has limited use, as it could only be used to study a limited number of variants of graph colouring. Conversely, a set of axioms which is too permissive would not have enough structure to guarantee standard graph colouring operations, such as the ability to take subgraphs of graphs with a valid colouring. In Chapter 5, the axioms are justified and used to establish many of the standard tools and operations used in graph colouring.

Chapter 6 is concerned with the identification of general properties that determine the behaviour of a colour scheme, with a focus on colour schemes which have their behaviour determined by the colours along the paths in a graph. This is an interesting class of colour schemes as it includes many types of colouring studied in the literature, including anagram-free colouring. Furthermore, many of the colour schemes in this class can be placed in a hierarchy, which motivates the generation and study of new types of colouring which fill gaps in the hierarchy. The gaps around anagram-free colouring are of particular interest, as ϕ is bounded on paths but not on many other classes of graphs.

1.4.1 Examples from the literature

The literature contains many types of graph colouring, some of which are reviewed in this section. The reviews here focus on the definition and results for each type of colouring. See Section 5.3 to see how each type relates to the study of colour schemes and a more detailed comparison of their properties.

Distance colouring

For $k \in \mathbb{Z}^+$, a *distance-k colouring* of a graph, G, is a colouring such that vertices at distance at most k receive distinct colours [92]. This is an extension of proper colouring, as proper colouring is distance-1 colouring. The *distance-k chromatic number*, $\chi_k(G)$, of a graph G is the minimum number of colours in a distance-k colouring of G. A distance-k colouring of a graph G is equivalent to a proper colouring of G^k, called the k^{th} *power* of G, where G^k is the graph with vertices $V(G)$ and an edge, $uv \in E(G^k)$, if $\text{dist}(u, v) \leqslant k$ in G. It follows that $\chi_k(G) \leqslant \Delta(G)\,(\Delta(G) - 1)^{k-1} + 1$ for all graphs G. Distance-k colourings are studied on many classes of graphs, for example, the distance-2 chromatic number of planar graphs has received much interest [11, 104, 128]. Agnarsson and Halldórsson [2] show that, for all $k \in \mathbb{Z}^+$, there exists c such that every planar graph, G, is distance-k $\left(c\Delta(G)^{\lfloor k/2 \rfloor}\right)$-choosable and that this bound is tight.

An *exact distance-k colouring* of a graph, G, is a colouring such that vertices at distance exactly k receive distinct colours [109]. Exact distance-k colouring has distinctive behaviour compared to distance-k colouring. For example, the (non-exact) distance-k chromatic number is unbounded on stars, for $k \geqslant 2$, whereas, for odd ℓ, the exact distance-ℓ chromatic number is bounded on planar graphs [110, 129].

Pattern-free colouring

Grytczuk [71] introduced p-free colouring, where p is a pattern, as a generalisation of square-free colouring. A *pattern* is a finite word which encodes disallowed sequences of blocks. A finite word W is said to *match* a pattern $p = x_1x_2 \dots x_n$ if W can be divided into n non-empty blocks, denoted $W = B_1B_2 \dots B_n$, such that $B_i = B_j$ if $x_i = x_j$, for all $i, j \in [n]$. A coloured graph is *p-free* if the sequence of colours read along each of its subpaths does not match p. The *p-chromatic number* of G is the minimum number of colours in a p-free colouring of G.

Pattern-free colouring is an extension of square-free colouring and, more generally, k-power-free colouring. Formulated as pattern-free colouring, k-power-free colouring is a^k-free colouring, where a^k denotes the symbol a repeated k times. Pattern-free colouring and square-free colouring also have similar inspirations, as they both originate in combinatorics of words. A pattern p is said to be *avoidable on graphs* if its chromatic number is bounded by a function of maximum degree. The study of avoidable patterns on paths was introduced and characterized by Zimin [136] and Bean et al. [16] in the context of combinatorics on words.

Grytczuk [71] proved that there is a constant, c, such that the p-free chromatic number of a graph G is at most $c\Delta(G)^{m/(m-1)}$, where m is the number of occurrences of the least occurring symbol in p. His proof uses the Lovász Local Lemma and has the same exponent as the upper bound on k-power-free colouring [7].

Parity colouring

A *parity path* in a coloured graph, G, is a path, P in G, in which every colour in P occurs an even number of times. A *parity colouring* of a graph, G, is a colouring of G such that G has no parity paths. Parity colouring was introduced by Bunde et al. [34] in the context of edge colouring. They also studied *strong parity edge colouring*, in which a graph colouring of G is admissible if no open walk in G has every colour occur an even number of times. Borowiecki et al. [26] study parity vertex colouring. I focus on the vertex version of parity colouring because my study of colour schemes is focused on vertex colouring. I denote the parity vertex chromatic number of a graph G by $\chi_{\mathbf{I}}(G)$.

Parity colouring is stronger than anagram-free colouring, in the sense that every anagram-free colouring is a parity colouring, because every symbol in an anagram occurs an even number of times. Bunde et al. [34] prove that $\chi_{\mathbf{I}}$ is unbounded on paths by showing $\chi_{\mathbf{I}}(P_n) = \lceil \log_2(n+1) \rceil$ for all n. Borowiecki et al. [26] prove the upper bound $\chi(G) \leqslant \chi_{\mathbf{I}}(G) \leqslant |V(G)| - \alpha(G) + 1$ where $\alpha(G)$ is the *independence number* of G; the size of the largest set of pairwise non-adjacent vertices in G.

Centred and conflict-free colouring

A *centred colouring* of a graph, G, is a colouring in which every connected subgraph of G has a colour which occurs exactly once [109]. I denote the centred chromatic number of a graph G by $\chi_{\mathbf{X}}$ [109]. Centred colouring can be used to define the tree-depth, $\mathrm{td}(G)$ of a graph G, as $\mathrm{td}(G) = \chi_{\mathbf{X}}(G)$. Centred colouring is also related to parity colouring, since $\chi_{\mathbf{I}}(P_n) = \chi_{\mathbf{X}}(P_n) = \lceil \log_2(n+1) \rceil$ for all n [34, 109].

Conflict-free colouring was introduced by Even et al. [61] to study conflicts in mobile phone networks. A colouring of a graph, G, is *conflict-free* if every path in G contains a colour which occurs exactly once. I denote the *conflict-free chromatic number* of G by $\chi_{\mathbf{C}}(G)$. Conflict-free colouring can be seen as a weakening of centred colouring, as centred colouring avoids strictly more 'bad' subgraphs than conflict-free colouring. As such $\chi_{\mathbf{C}}(G) \leqslant \chi_{\mathbf{X}}(G)$, with equality on, but not necessarily limited to, paths. In this sense, conflict-free colouring is more closely related to parity colouring, since $\chi_{\mathbf{I}}(G) \leqslant \chi_{\mathbf{C}}(G) \leqslant \chi_{\mathbf{X}}(G)$, for all graphs G. This bound follows from the observation that parity paths contain no colours that occur once.

Acyclic colouring

An *acyclic vertex colouring* of a graph, G, is a proper colouring in which every pair of colour classes induces a graph with no cycles [90]. See Figure 1.9 for an example of an acyclic vertex colouring of a graph. An *acyclic edge colouring* of a graph, G, is a proper edge colouring in which every subcycle of G contains at least three colours. The acyclic chromatic number and index of a graph, G, are denoted $\chi_a(G)$ and $\chi'_a(G)$, respectively.

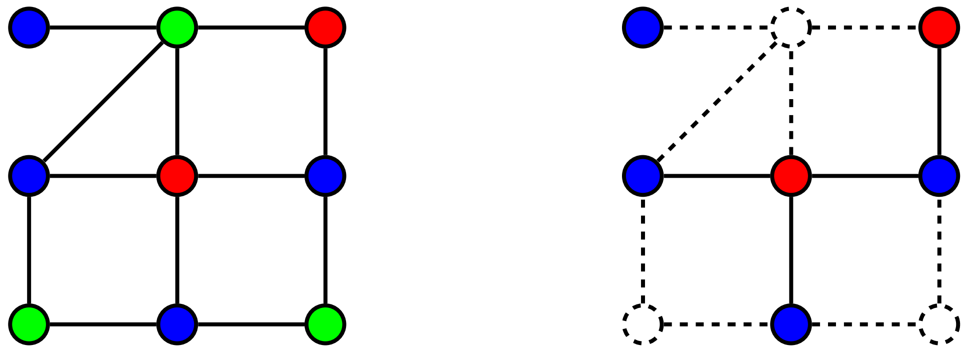


Figure 1.9: A graph, G, on the left, with the subgraph of G induced by the red and blue vertices on the right. This colouring of G is acyclic because every subgraph of G induced a pair of colours is a forest.

The study of acyclic colourings was introduced by Grünbaum [68], with a focus on planar graphs. Grünbaum [68] showed that $\chi_a(G) \leqslant 9$, for every planar graph G, and found a graph, G, with $\chi_a(G) = 5$. The upper bound was lowered to 8 by Mitchem [100], and, subsequently, to 7 by Albertson and Berman [4]. Finally, Borodin [22] showed that $\chi_a(G) \leqslant 5$ for every planar graph, G, proving a conjecture of Grünbaum [68]. Acyclic colouring has been studied for more general graph classes. A graph, G, is 1-*planar* if it can be drawn in the plane such that each edge of G crosses at most one other edge of G. Borodin et al. [23] show that $\chi_a(G) \leqslant 20$, for every 1-planar graph, G. The list-colouring variant of acyclic colouring has also been studied. Borodin et al. [24] show that $\chi_{a\,\mathrm{ch}}(G) \leqslant 7$, for every planar graph G, where $\chi_{a\,\mathrm{ch}}(G)$ denotes the acyclic choice-number of G.

Bounding χ_a and χ'_a on graphs of bounded degree has received much interest. Variants of the Lovász Local Lemma are often used to prove these bounds, with later results using entropy compression. Alon et al. [9] prove $\chi'_a(G) \leqslant 64\Delta(G)$ and that, for some c, $\chi_a(G) \leqslant c\Delta(G)^{4/3}$, for all graphs G. Molloy and Reed [101] improve the bound on χ'_a to $\chi'_a(G) \leqslant$

$16\Delta(G)$. This is further improved to $\chi'_a(G) \leqslant \lceil 9.62(\Delta(G)-1) \rceil$ by Ndreca et al. [108], who also show $\chi_a(G) \leqslant \lceil 6.59\Delta(G)^{4/3} + 3.3\Delta(G) \rceil$. Esperet and Parreau [60] improve the bound to $\chi'_a(G) \leqslant 4\Delta(G)-4$ using entropy compression. Also using entropy compression, Giotis et al. [66] improve the bound to $\chi'_a(G) \leqslant \lceil 3.75(\Delta(G)-1) \rceil + 1$. Gonçalves et al. [67] use entropy compression to show that there is a constant, c, such that $a(G) \leqslant \frac{3}{2}\Delta(G)^{4/3} + c\Delta(G)$. For lower bounds, $\chi'_a(G) \geqslant \Delta(G)$ because $\chi'(G) \geqslant \Delta(G)$, and Alon et al. [9] prove $\chi_a(G) \geqslant c\frac{\Delta(G)^{4/3}}{(\log \Delta(G))^{1/3}}$, for some c and graph G.

Star colouring

A *star colouring* of a graph, G, is a proper colouring in which every pair of colour classes induces a forest of stars [84]. A *star* is a tree with at most one vertex of degree exceeding 1 and a *forest of stars* is a disjoint union of stars.

Star colouring and acyclic colouring share a history as star colouring was introduced by Grünbaum [68] in a paper on acyclic colourings of planar graphs. Let $\chi_s(G)$ denote the star chromatic number of a graph G. Albertson et al. [5] show that every planar graph, G, has $\chi_s(G) \leqslant 20$ and find a planar graph, G, with $\chi_s(G) \geqslant 10$. Bu et al. [33] study star colourings of sparse graphs and prove $\chi_s(G) \leqslant 4$ for every graph, G, with maximum average degree less than $\frac{26}{11}$. The *maximum average degree* of a graph, G, is the maximum of the average degree over all subgraphs of G. Fertin et al. [62] study star colourings of many classes of planar graphs, including planar graphs, trees and graphs of high girth. The *girth* of a graph, G, is the length of the shortest subcycle in G. Fertin et al. [62] show that $\chi_s(G) \leqslant 12$ for every planar graph, G, with girth at least 7. Kündgen and Timmons [95] improve this bound to $\chi_s(G) \leqslant 7$ and generalise the result to list star colouring. Fertin et al. [62] also prove relationships on compositions of graphs, showing that $\chi_s(G \square H) \leqslant \chi_s(G)\chi_s(H)$, for all graphs G and H, where $G \square H$ is the Cartesian of G and H, see Section 1.3.5.

Fertin et al. [62] note that $\chi_a(G) \leqslant \chi_s(G)$, for all graphs G, because all 2-coloured cycles contain either a monochromatic edge or a 2-coloured P_4 as a subgraph. Furthermore, Fertin et al. [62] prove that there exists a constant, c, such that, for all graphs G, $\chi_s(G) \leqslant c\Delta(G)^{3/2}$, and that this bound is tight up to a factor of $(\log \Delta(G))^{1/2}$. This bound is improved by Esperet and Parreau [60], who use entropy compression to show that $\chi_s(G) \leqslant 2\sqrt{2}\Delta(G)^{3/2} + \Delta(G)$ for all graphs G.

Frugal colouring

A *k-frugal colouring* of a graph, G, is a proper vertex colouring where, for all $v \in V(G)$, each colour occurs at most k times in $N(v)$. Frugal colouring was introduced by Hind et al. [80], in part, for its application to total colouring. A *total colouring* of a graph, G, is a proper colouring of $E(G)$ and $V(G)$ with the additional requirement that every edge has a distinct colour from each of its endpoints [81]. Let $\chi_{\mathbf{G}_k}(G)$ denote the k-frugal chromatic number of a graph G.

First note that a colouring of a graph is 1-frugal if and only if it is a distance-2 colouring. Also, every k-frugal colouring is a $(k+1)$-frugal colouring. A central question in the field has been to determine, for which k, every graph of maximum degree Δ has a k-frugal $(\Delta+1)$-colouring. The question is asked for $(\Delta+1)$-colouring since $\chi(G) \leqslant \Delta(G)+1$, for all graphs G. Initially, Hind et al. [80] show that every graph with maximum degree $\Delta \geqslant e^{10^7}$ is $(\log^8 \Delta)$-frugal $(\Delta+1)$-colourable. This bound was improved by Pemmaraju and Srinivasan [113], who show that there exists a constant, c, such that every graph with maximum degree Δ is $(c\log^2\Delta/\log\log\Delta)$-frugal $(\Delta+1)$-colourable. This is further optimized by Molloy and Reed [103], who show that every graph with sufficiently large maximum degree, Δ, has a $(50\log\Delta/\log\log\Delta)$-frugal $(\Delta+1)$-colouring. This bound is tight up to a constant factor as Alon (see [80]) showed that there is a class of graphs which are not $(\log\Delta/\log\log\Delta)$-frugal $(\Delta+1)$-colourable. The proofs of lower bounds use variants of the Lovász Local Lemma. For upper bounds, Hind et al. [80] show that, for every graph, G, of sufficiently large maximum degree, $\chi_{\mathbf{G}_k}(G) \leqslant \max\{(k+1)\Delta(G), \lceil (e^3\Delta(G)^{(k+1)/k})/k\rceil\}$. Kang and Müller [84] improve upon this bound by showing that, for all k, there exists c such that $\chi_{\mathbf{G}_k}(G) \leqslant \frac{c}{k}\Delta(G)^{(k+1)/k}$, for all graphs G.

1.4.2 Results

The primary result of my investigation of colour schemes is that many types of graph colouring can be unified using five natural axioms. A colour scheme is a set of coloured graphs that satisfy these axioms. For example, the colour scheme corresponding to proper colouring, denoted $\mathbf{P}$, is the set of coloured graphs with no monochromatic edges. It is difficult to go into more detail without defining the axioms and delving into their implications, so instead, I outline two applications of colour schemes. The applications tend to be generalisations of results from Chapters 3 and 4, and include theorems about general properties, as well as the construction of new variants of graph colouring with novel properties. See Chapter 5 for the definition of colour schemes as well as an investigation of the general properties that follow from this definition. See Chapter 6 for theorems about general properties of colour schemes that apply to many types of graph colouring.

The first application regards subdivisions of graphs. The central result of Chapter 4 is that ϕ is bounded on sufficiently subdivided graphs. This raises the question of whether being bounded on subdivisions of trees implies that a colour scheme is bounded on sufficiently subdivided graphs, as there are no known counter-examples. I investigate this question in the context of colour schemes, and answer it in the negative. I construct a colour scheme which is bounded on sufficiently subdivided trees but not on sufficiently subdivided graphs. I also construct a colour scheme which is bounded on subdivisions of stars, but not on sufficiently subdivided trees. Conversely, I generalise the results in Chapter 4 by establishing some natural sufficient conditions for a colour scheme to be bounded on sufficiently subdivided graphs.

The second application is to explore variants of graph colouring which are, in a sense, 'close' to anagram-free colouring. This sense of closeness places square-free colouring closer to anagram-free colouring than anagram-free colouring is to proper colouring, since $\chi(G) \leqslant \pi(G) \leqslant \phi(G)$, for all graphs G. Anagram-free colouring is bounded by square-free colouring on one side and parity colouring on the other side. However, there are gaps between each variant of colouring, which I investigate by constructing new variants of graph colouring. These variants can have novel properties, for example, I construct a colour scheme which is bounded on graphs of bounded maximum degree, like π, but which is unbounded on trees, like ϕ. I also define ε-uniform-free colouring which is an extension of anagram-free colouring. A word AB is ε-uniform if the number of every symbol, c, in A is within a factor of ε of the number occurrences of c in B. I show that the ε-uniform-free chromatic number is unbounded on paths, for all $\varepsilon \in (0,1) \subseteq \mathbb{R}$.

1.4.3 Related concepts

The literature contains concepts and results with some similarity to colour schemes. Comparisons between types of graph colouring have been made within the study of graph colouring, and there are some theorems and techniques that apply to multiple types of graph colouring. Also, the formulation of colour schemes as sets of coloured graphs is related to the formulation of graph properties.

A *graph property* is a set of graphs closed under isomorphism. The most interesting sets of graphs correspond to established properties in graph theory, such as bipartite graphs or graphs of maximum degree at most k [109, Chapter 3]. Many graph properties in the literature share characteristics, such as being hereditary or additive [25]. A graph property, $\mathcal{G}$, is *hereditary* if, for all $G \in \mathcal{G}$, every subgraph of G is in $\mathcal{G}$. A graph property, $\mathcal{G}$, is *additive* if, for all $\mathcal{H} \subseteq \mathcal{G}$, the disjoint union of the graphs in $\mathcal{H}$ is in $\mathcal{G}$. To take a common example, let $\mathcal{S}_k$ be the graphs of maximum degree at most k. For each $G \in \mathcal{S}_k$, every subgraph of G has maximum degree at most k, and every disjoint union of graphs of maximum degree at most k has maximum degree at most k, so $\mathcal{S}_k$ is additive and hereditary. See [25] for a survey of graph properties. Chromatic numbers can be defined as a property. For example, let $\mathcal{P}_k$ be the set of properly k-colourable graphs, and note that $\mathcal{P}_k$ is additive and hereditary. Broere et al. [32] study a generalisation of chromatic numbers based on arbitrary hereditary properties. There are two main differences between colour schemes and graph properties. The first is that colour schemes are sets of coloured graphs, whereas graph properties are sets of graphs. This is an important distinction, since many useful operations can be explicitly applied to properly k-coloured graphs, with no analogous operation for graphs of chromatic number k. The second distinction is that the colour scheme axioms are designed to restrict colour schemes to the sets of coloured graphs which correspond to commonly held notions of graph colouring.

Part of the motivation behind colour schemes is to make it easier to compare and combine variants of graph colouring. Some comparisons and combinations of types of graph colouring are present in the literature. We have already seen an example, that $\chi_a(G) \leqslant \chi_s(G)$, for all graphs G [62]. Another example is the generalisation of star, acyclic, and distance-2 colouring, by Pór and Wood [116], to types of graph colouring defined by avoiding subgraphs in the induced subgraphs of pairs of colour classes.

Kang and Müller [84] combine frugal, acyclic and star colouring by defining *k-frugal star colouring* and *k-frugal acyclic colouring* to be k-frugal graph colourings which are also star colourings or acyclic colourings, respectively. They denote the k-frugal, k-frugal star, and k-frugal acyclic chromatic numbers χ^k, χ_s^k, and χ_a^k respectively. They note that, for all graphs G,

$$\frac{\Delta(G)}{k} \leqslant \chi^k(G) \leqslant \chi_a^k(G) \leqslant \chi_s^k F(G).$$

Cheilaris et al. [37] study conflict free colouring and note that its chromatic number, $\chi_{\mathbf{F}}$, is bounded above and below by other chromatic numbers. They form the hierarchy

$$\chi(G) \leqslant \pi(G) \leqslant \chi_{\mathbf{F}}(G) \leqslant \chi_{\mathrm{ord}}(G)$$

which holds for all graphs G. The chromatic number, χ_{ord}, corresponds to ordered colouring. An *ordered colouring* of G is a colouring, by integers, such that, for each path in G, the largest colour in the path occurs exactly once in the path. Chapter 6 presents an extended hierarchy of colour schemes, which includes newly constructed colour schemes placed in many of the gaps between colour schemes found in the literature.

The Lovász Local Lemma and entropy compression can be used to prove upper bounds in many areas of graph colouring, which motivates the creation of tools which can be applied to many types of graph colouring. Esperet and Parreau [60] use entropy compression to prove $\chi'_a(G) \leqslant 4\Delta(G) - 4$ in a way which is relatively easy to translate to other variants of graph colouring. Colour schemes provides a framework for results of this nature and make it easier to apply general results to multiple types of graph colouring. Theorem 6.21 generalises the bound on ϕ, for graphs of bounded degree, to types of graph colouring which avoid sufficiently few patterns along paths.

Chapter 2

Lovász Local Lemma

The probabilistic method is a powerful tool used within combinatorics which is often used to prove the existence of combinatorial objects [8, 102]. A typical application of the probabilistic method involves setting up a probability space and using probabilistic tools to show that a randomly selected object has a non-zero probability of satisfying a desired property. It can then be concluded that some positive number of the objects have the property. For example, consider a probabilistic proof that a graph G has a proper k-colouring. We start with the space of all k-colourings of G. Then, using properties of G, we show that a random k-colouring of G is proper with some positive probability. We then conclude that G has at least one proper k-colouring, since the probability space is discrete. Many of the tools in the probabilistic method give conditions on probability spaces that ensure that an object with the desired properties can be found.

The Lovász Local Lemma, henceforth called the Local Lemma, is a well studied tool in the probabilistic method with many extensions and variants. It was developed by Erdős and Lovász [57] in 1975 for the purpose of hypergraph colouring. More generally, the Local Lemma can be applied to find combinatorial objects with properties that depend on avoiding a set of local 'bad' patterns. This makes the Local Lemma particularly well suited to studying graph colouring, hence it is the focus of this chapter. To take our earlier example, a k-colouring of a graph is proper if it contains no monochromatic edge. Here we have expressed a global property, proper colouring, in terms of a set of local patterns which we are required to avoid.

In general, all variants of the Local Lemma require a probability space with a set of 'bad' events. The bad events are designed to encompass all ways in which an object may fail to satisfy the desired property. Given a set of bad events, the Local Lemma provides sufficient conditions for there to be a positive probability that none of the bad events occur. This condition depends on the probability of each bad event as well as the interdependence structure of the events. There are many levels of generality of the Local Lemma, each suited to particular situations. For example the Symmetric Local Lemma is most applicable to problems with symmetries in the probability and interdependence structure of their bad events. There are stronger versions, such as the General Local Lemma, which can be used to prove a wider range of results, but are harder to apply. Stronger variants of the Local Lemma tend to either use more details about the interdependency structure of the bad events, such as the Clique and Improved Local Lemmas, or weaken the dependency requirements of the events, such as the Lopsided Local Lemma.

The results presented in this chapter are non-original, with the exception of Theorem 2.11, which is an original contribution published in [19].

2.1 Formulations of the Local Lemma

All variants of the Local Lemma allow some dependence between the underlying bad events. The level of dependence is encoded by a dependency graph, with the bad events as vertices and an edge between two bad events if they are 'dependent'. Let $\mathcal{A}$ be a set of events in a probability space. A graph, G, is a *dependency graph* of $\mathcal{A}$ if $V(G) = \mathcal{A}$ and each event $A \in \mathcal{A}$ is mutually independent of $V(G) \setminus N[A]$. Recall that $N[A]$ denotes the closed neighbourhood of A, which is the set including A and the vertices adjacent to A. Two events, A and B, are *mutually independent* if conditioning on one event does not affect the other, that is, $\mathbb{P}(A|B) = \mathbb{P}(A)$ and $\mathbb{P}(B|A) = \mathbb{P}(B)$. An event, A, is *mutually independent* of a set of events $\mathcal{B}$, if A is mutually independent of each $B \in \mathcal{B}$. The mutual independence between bad events required by the Local Lemma is expressed in terms of restrictions on the structure of their dependency graph. Lower mutual dependence is better, so applications of the Local Lemma involve the construction of a sufficiently sparse dependency graph.

2.1.1 Symmetric Local Lemma

The simplest variant of the Local Lemma is the Symmetric Local Lemma. This variant expresses the independence requirement as a bound on the maximum degree of the dependency graph.

Lemma 2.1 (Symmetric Local Lemma [123])**.** *Let $\mathcal{A}$ be a set of events with dependency graph D and p be a number such that $\mathbb{P}(A) \leqslant p$ for all $A \in \mathcal{A}$. If $ep(\Delta(D) + 1) < 1$ then*

$$\mathbb{P}\left(\bigcap_{A \in \mathcal{A}} \overline{A}\right) > 0.$$

Shearer [122] shows that the constant 'e' is tight. The Symmetric Local Lemma is suited to applications where the bad events have uniform dependence and probability. As an example, consider bounding the distance-k chromatic number on graphs of bounded maximum degree. Recall that a distance-k colouring is a colouring in which vertices at distance at most k receive distinct colours. Let $\chi_k(G)$ denote the distance-k chromatic number of a graph G. Theorem 2.2 is a example of a bound that may be proven with the Symmetric Local Lemma. The theorem is purely demonstrative, as there are better known bounds for distance-k colouring.

Theorem 2.2. $\chi_k(G) \leqslant \lceil 2e\Delta(G)^k \rceil$ *for all graphs G and $k \geqslant 1$.*

Proof. Let $c := \lceil 2e\Delta(G)^k \rceil$ be a number of colours. Let $f : V(G) \to [c]$ be a random c-colouring of G with each $f(u) \in f(V(G))$ selected uniformly and independently. Define the set of bad events, $\mathcal{A}$, such that $A_{u,v} \in \mathcal{A}$ is the event $f(u) = f(v)$, for all $u, v \in V(G)$ with $\text{dist}(u, v) \leqslant k$.

Now construct a sufficiently sparse dependency graph, D, for the set of events $\mathcal{A}$. First set $V(D) = \mathcal{A}$. To determine adjacency, note that $A_{u,v}$ is mutually independent of $f(V(G) \setminus \{u, v\})$. Therefore, for D to be a dependency graph, it is sufficient for D to have an edge between each pair of events which reference a common vertex of G. In particular, $\{A_{x,y}, A_{u,v}\} \in E(D)$ if $x = u$, $x = v$, $y = u$ or $y = v$. We now calculate $\Delta(D)$. First observe that, for each vertex $u \in V(G)$, there are at most $\Delta(G)^k$ vertices, $v \in V(G)$, such that $\text{dist}(u, v) \leqslant k$. Therefore each vertex of G is referenced by at most $\Delta(G)^k$ events in $\mathcal{A}$. Each event references two vertices so $\Delta(D) \leqslant 2\Delta(G)^k - 2$.

Finally, we require a bound, p, on $\mathbb{P}(A_{u,v})$. For every $u, v \in V(G)$, the probability that $f(v) = f(u)$ is $1/c$ since all colours are assigned independently. Therefore $\mathbb{P}(A_{u,v}) = 1/c$. Now

$$ep(\Delta(D)+1) < \frac{e}{c}(2\Delta(G)^k - 1 + 1) = \frac{2e\Delta(G)^k}{\lceil 2e\Delta(G)^k \rceil} \leqslant 1$$

so the theorem follows by Lemma 2.1. □

2.1.2 General Local Lemma

Many extensions of the Local Lemma use variation between the probabilities and interdependence of the bad events to obtain better bounds. One such extension is the General Local Lemma which, loosely speaking, allows for an event to have higher than average dependence as long as it has a lower than average probability.

Lemma 2.3 (General Local Lemma [8]). *Let $A_1, \dots, A_n$ be events with dependency graph D and suppose there are real numbers $x_1, \dots, x_n$ such that $x_i \in [0, 1) \subseteq \mathbb{R}$ and*

$$\mathbb{P}(A_i) \leqslant x_i \prod_{A_j \in N(A_i)} (1 - x_j)$$

for all $i \in [n]$. Then

$$\mathbb{P}\left(\bigcap_{i=1}^{n} \overline{A_i}\right) \geqslant \prod_{i=1}^{n} (1 - x_i) > 0.$$

The General Local Lemma allows for the events $A_1, \dots, A_n$ to have varying probabilities and level of mutual independence. As such, the General Local Lemma can be used to prove bounds on chromatic numbers that depend on avoiding arbitrarily large patterns of colours. Two examples are square-free colouring and acyclic colouring. The former avoids squares on arbitrarily long paths in a graph while the latter avoids all 2-coloured subcycles of a graph. The dependency graph in Lemma 2.3 can be replaced with a dependency digraph [8], however, as directed graphs are not used in this thesis, we do not go into any further detail.

Recall that Grytczuk [70] prove $\pi(G) \leqslant 16\Delta(G)^2$. As an example, we use the General Local Lemma to reproduce a similar result using their technique. Similarly to distance-k colouring, the probability space is the k-colourings of a graph G and each bad event corresponds to a path being a square. The central component of the proof is to count the intersections between paths, since the intersecting paths correspond to adjacent bad events in the dependency graph. The exponentially increasing number of intersections is counterbalanced by the exponentially decreasing probability of a long path being a square. An analogous insight is at the core of all applications of the Local Lemma in similar situations.

Theorem 2.4. *$\pi(G) \leqslant 64\Delta(G)^2$ for all graphs G.*

Proof. Let $k := (8\Delta)^2$, $n := |V(G)|$, and $\mathcal{Q}$ be the set of paths in G. Colour each vertex of G independently and randomly with k colours. For each $P \in \mathcal{Q}$, let A_P be the event that P is a square. Define the dependency graph, D, with vertex set $V(D) := \{A_P : P \in \mathcal{Q}\}$. An edge $\{A_P, A_Q\}$ is in $E(D)$ if the paths P and Q share a vertex in G. Observe that two events, A_P and A_Q, are mutually independent if P and Q have no common vertex, so D

is a dependency graph. Partition $V(D)$ by the sets $S_i := \{A_P : P \text{ is a path of order } i\}$ for $i \in [n]$.

Let $\Delta := \Delta(G)$. We now show that, for all $i, j \in [n]$ and $P \in S_i$, $d(i,j) := ij\Delta^j$ is an upper bound on $|N(A_P) \cap S_j|$. To do so we consider a vertex, $v \in V(G)$, and bound the number of paths of order j that contain v. Note that there are at most Δ^{a-1} paths of order a in G which start at v. Therefore there are at most $\Delta^{b-1}\Delta^{j-b}$ paths in S_j with v as the b^{th} vertex. Since v can occur at any of j positions in a path of order j, there at most $j\Delta^j$ paths of order j going through v. It follows that a path on i vertices intersects at most $ij\Delta^j$ paths of order j, so $|N(A_P) \cap S_j| \leqslant ij\Delta^j$. Let $x_i = (3\Delta)^{-i}$, for all $i \in [n]$, and let $x_P := x_{|V(P)|}$ be the parameter, required by Lemma 2.3, corresponding to A_P, for all paths $P \in \mathcal{Q}$. By definition,

$$\prod_{A_Q \in N(A_P)} (1 - x_Q) \geqslant \prod_{j=1}^{n} (1 - x_j)^{d(|V(P)|,j)},$$

for all $P \in \mathcal{Q}$. For all $i \in [n]$, $(1 - x_i) \geqslant e^{-\frac{5x_i}{4}}$ because $x_i \leqslant 3^{-1}$, so

$$\begin{aligned} x_i \prod_{j=1}^{n} (1 - x_j)^{d(i,j)} &\geqslant (3\Delta)^{-i} \prod_{j=1}^{n} e^{\frac{-5}{4} x_j d(i,j)} \\ &= (3\Delta)^{-i} \prod_{j=1}^{n} e^{\frac{-5}{4}(3\Delta)^{-j} ij\Delta^j} \\ &> (3\Delta)^{-i} \exp\left(\frac{-5i}{4} \sum_{j=1}^{\infty} \left(j/3^j\right) \right). \end{aligned}$$

Let $P \in \mathcal{Q}$ and $i = |V(P)|$. Note that $\mathbb{P}(A_P) = k^{-i/2}$ because, if P is a square, the second half of P is determined by the first half. The series $\sum_{j=1}^{\infty} j/3^j$ converges to $3/4$ so

$$x_P \prod_{A_Q \in N(A_P)} (1 - x_Q) \geqslant x_i \prod_{j=1}^{n} (1 - x_j)^{d(i,j)} > (3\Delta)^{-i} e^{-15i/16} > (8\Delta)^{-i} = k^{-i/2} = \mathbb{P}(A_P).$$

Therefore $\mathbb{P}\left(\bigcap_{\mathcal{Q}} \overline{A_P}\right) > 0$, by Lemma 2.3. It follows that G has a square-free k-colouring. □

Recall that the factor of 64 in $\pi(G) \leqslant 64\Delta(G)^2$, has been the subject of much optimization, see Section 1.2.2. The bound in Theorem 2.4 is unoptimized, both for simplicity and to demonstrate some optimizations from the literature. The first optimization follows from the observation that paths of odd order cannot be squares. Including paths of odd order doubles $|V(D)|$ and quadruples $|E(D)|$. Removing the events that corresponds to paths of odd order results in the bound $\pi(G) \leqslant 16\Delta(G)^2$, which was obtained by Grytczuk [70].

To further optimize Theorem 2.4 consider the parameter $x_i = (3\Delta)^{-i}$. Ideally, x_i would be as small as possible, as a tighter upper bound on x_i allows us to reduce the exponent in $(1 - x_i) \geqslant e^{-\frac{5x_i}{4}}$. Theorem 2.4 uses $x_i \leqslant 3^{-1}$, which can be improved to $x_i \leqslant 9^{-1}$ by considering graphs of maximum degree 3. This is among the optimizations employed by Harant and Jendrol' [75] for their bound $\pi(G) \leqslant \lceil 12.92(\Delta(G) - 1)^2 \rceil$, where $\Delta(G) \geqslant 3$. Further improvements to this bound were obtained by extending the Local Lemma to take more details of the dependency graph into account.

2.1.3 Weighted Local Lemma

The Weighted Local Lemma strikes a balance between the generality of the General Local Lemma and the approachability of the Symmetric Local Lemma. It applies naturally to dependency graphs of bad events with exponentially increasing degree and exponentially decreasing probability.

Lemma 2.5 (Weighted Local Lemma [102]). *Let $A_1, \ldots, A_n$ be events with dependency graph D and suppose there are real numbers $x_1, \ldots, x_n$ such that $x_i \geqslant 1$. If there is a $p \in [0, \frac{1}{4}]$ such that*

$$\mathbb{P}(A_i) \leqslant p^{x_i} \text{ and}$$
$$\sum_{A_j \in N(A_i)} (2p)^{x_j} \leqslant \frac{x_i}{2}$$

for all $i \in [n]$, then $\mathbb{P}\left(\bigcap_{i=1}^{n} \overline{A_i}\right) > 0$.

The Weighted Local Lemma allows for a shorter proof of Theorem 2.4. It uses the same dependency graph, with the events corresponding to paths of order 1 removed. The proof follows from Lemma 2.5, with $p = 1/k$ and $x_P = |V(P)|/2$, for $|V(P)| \geqslant 2$.

Proof outline of Theorem 2.4. Use the same setup as the previous proof of Theorem 2.4, with the exception that $\mathcal{Q}$ excludes paths of order 1. Let $P \in \mathcal{Q}$ and $i := |V(P)|$. Recall that $\mathbb{P}(A_P) = k^{-i/2}$ so let $x_P = i/2$ and $p = 1/k$. The theorem follows by Lemma 2.5, because

$$\begin{aligned}
\sum_{A_Q \in N(A_P)} (2p)^{x_Q} &\leqslant \sum_{j=2}^{n} \Delta_{ij} (2p)^{x_j} \\
&= i \sum_{j=2}^{n} j \Delta^j \left(\frac{64\Delta^2}{2}\right)^{-j/2} \\
&= i \sum_{j=2}^{n} j \left(\sqrt{32}\right)^{-j} \\
&< \frac{i\left(2\sqrt{32}-1\right)}{\left(\sqrt{32}-1\right)^2 \sqrt{32}} \\
&< \frac{i}{4} \\
&= \frac{x_P}{2}
\end{aligned}$$

□

The Weighted Local Lemma is a special case of the General Local Lemma, reformulated into a more readily applicable form.

2.1.4 Detailed features of dependency graphs

The General Local Lemma has been extended in many ways, often with the goal of obtaining stronger bounds or to widen the applicability of the Local Lemma. One approach is to use an increasingly detailed view of the dependency graph. The effectiveness of this approach can be seen in the extension from the Symmetric Local Lemma to the General Local Lemma, as the former cannot be used to obtain bounds on π.

Bissacot et al. [20] extend the General Local Lemma by replacing the restriction on the degree of each event, A_i, with a restriction on the independent sets of the neighbourhood of A_i. An *independent set*, S, of a graph, G, is a set of vertices such that $uv \notin E(G)$ for all $u, v \in S$. The set of independent sets of a neighbourhood of a vertex, v, denoted $\mathrm{IN}(v)$, is the set of independent sets in G which are subsets of $N(v)$. The independent set extension of the Local Lemma is formulated as follows.

Lemma 2.6 (Improved Local Lemma [20]). *Let $A_1, \ldots, A_n$ be events with dependency graph G and suppose there are real numbers $x_1, \ldots, x_n$ such that $x_i > 0$ and*

$$x_i \geqslant f(i)\, \mathbb{P}(A_i)$$

where

$$f(i) := \sum_{S \in \mathrm{IN}(A_i)} \prod_{A_j \in S}^{k} x_j$$

for all $i \in [n]$. Then

$$\mathbb{P}\left(\bigcap_{i=1}^{n} \overline{A_i}\right) \geqslant \prod_{i=1}^{n} (1 - \mathbb{P}(A_i))^{f(i)} > 0.$$

The intuition behind this improvement is that the General Local Lemma contains a product which expands to a sum of all products of the weights of the events in the neighbourhood of an event, A. The function, f, in the Improved Local Lemma is analogous to this product, but it omits the terms with weights corresponding to adjacent events. Other differences between the formulations are primarily due to the x_i terms taking on a slightly different role, for example they can exceed 1.

Other details of the dependency graph can be used, such as the approach taken by Kolipaka et al. [89], who extend the Local Lemma to a variant which imposes conditions on a decomposition of the dependency graph. A *decomposition* of a graph, G, is a set of induced subgraphs of G, $\{G_1, G_2, \ldots, G_k\}$, such that $e \in E(G_i)$, for some i, for every $e \in E(G)$. They call their variant of the Local Lemma the Decomposition Theorem, which I avoid stating here as it requires too much additional background. Instead, in Section 2.1.6, I state a symmetric formulation of the Clique Local Lemma, which is a specialisation of the Decomposition Theorem, and use it to prove an original result.

The Decomposition Theorem is an extension of the General Local Lemma in the sense that the General Local Lemma is recovered by decomposing a dependency graph into paths of order 2. Kolipaka et al. [89] use the Clique Local Lemma, a specialisation of the Decomposition Theorem, to improve upon two previous applications of the Local Lemma. In the first application they show that the acyclic chromatic index of a graph G is at most $8.6\Delta(G)$. In the second application, Kolipaka et al. improve the bound on π obtained by Harant and Jendrol' [75] from $\pi(G) \leqslant \lceil 12.92(\Delta(G) - 1)^2 \rceil$, for $\Delta(G) \geqslant 3$, to $\pi(G) \leqslant 10.4\Delta(G)^2$.

2.1.5 Lopsidependency

Another way to extend the General Local Lemma is to consider extensions of dependency graphs with less restrictive adjacency requirements. In terms of the underlying probability space, a weakened adjacency requirement is a narrowing of what it means for two events to be 'local'. This type of extension allows sparser dependency graphs for the same set of bad events, which makes the Local Lemma easier to apply. Some applications of the Local Lemma, such as to spaces of random permutations, are dependent on a narrower notion of dependency because every event may be dependent on every other, in the standard sense. The prime example of this form of extension is lopsidependency graphs.

Erdős and Spencer [58] introduced the Lopsided Local Lemma as an extension of the Symmetric Local Lemma. This extension essentially involved reproving the Symmetric Local Lemma with a lopsidependency graph instead of a dependency graph. A graph, G, with vertex set $\mathcal{A}$, of bad events, is a *lopsidependency graph* if for all $A_i \in \mathcal{A}$,

$$\mathbb{P}\left(A_i \middle| \bigcap_{j\in S} \overline{A_j}\right) \leqslant \mathbb{P}(A_i),$$

for every subset, S, of the complement of the closed neighbourhood of A_i in G. Intuitively, a lopsidependency graph requires that the probability of an event does not increase when conditioned on an arbitrary set of non-adjacent events not occurring. Since a dependency graph is a graph that requires the probability of an event to remain constant when conditioned on an arbitrary set of non-adjacent events not occurring, all dependency graphs are lopsidependency graphs. Erdős and Spencer [58] prove the lopsidependence version of the Symmetric Local Lemma.

Lemma 2.7 (Lopsided Lovász Local Lemma [58])**.** *Let $\mathcal{A}$ be a set of events with lopsidependency graph G and p be such that $\mathbb{P}(A) \leqslant p$ for all $A \in \mathcal{A}$. If $4p\Delta(G) < 1$ then*

$$\mathbb{P}\left(\bigcap_{A\in\mathcal{A}} \overline{A}\right) > 0.$$

There is no important distinction between the two conditions $4p\Delta(G) < 1$ and $ep(\Delta(G)+1) < 1$, as both formulations of the Symmetric Local Lemma exist for dependency and lopsidependency graphs. Most variants of the Local Lemma have an analogous extension which uses a lopsidependency graph. As noted earlier, every dependency graph is a lopsidependency graph so such extensions are stronger than the original. Both extensions in Section 2.1.4 have variants that use a lopsidependency graph. In fact, lopsidependency was introduced long before the extensions of the previous section.

To illustrate lopsidependence, we briefly leave graph theory and look at Latin arrays. A *Latin array* of size n is an $n \times n$ array of cells filled with $m \geqslant n$ distinct symbols, such that no row or column contains more than one copy of a symbol. For a Latin array L, the symbol in cell (i, j) is denoted $L(i, j)$. Given a Latin array, L, of size, n, a permutation $\sigma \in S_n$ is a *transversal* if all the symbols $L(i, \sigma(i))$, $i \in \{1, \ldots, n\}$ are distinct. See Figure 2.1 for an example of a Latin array with a transversal. Erdős and Spencer [58] introduced lopsidependence to prove the following result.

Theorem 2.8 (Erdős and Spencer [58])**.** *Let $n > 1$ and $k \leqslant \frac{n-1}{16}$. Every $n \times n$ Latin array with no more than k occurrences of each symbol has a transversal.*

The bound of $k \leqslant \frac{n-1}{16}$ was improved to $k < \frac{27}{256}n$ by Bissacot et al. [20] with the application of the lopsidependency extension of the Improved Local Lemma.

The proof of Theorem 2.8 uses the Lopsided Local Lemma on the space of permutations, σ, on a fixed $n \times n$ Latin array, L. Each permutation corresponds to the cells $L(i, \sigma(i))$, for all $i \in [n]$, which is a potential transversal because each row and column has one cell. The bad events correspond to pairs of cells which share a symbol, with a bad event occurring if

1	**2**	3	4	5	6
3	7	6	1	2	5
2	3	7	**8**	4	1
4	1	5	2	**7**	8
6	4	**1**	7	3	2
7	6	2	5	1	**4**

Figure 2.1: A Latin array of order 6 on 8 symbols, with a transversal highlighted.

both cells are selected by σ. Two events are adjacent in the lopsidependency graph, D, if any of their corresponding cells share a row or column. Note that D is not a dependency graph because if we know that a set of cells are in σ, then it becomes more likely that other valid cells are in σ. Erdős and Spencer [58] show that D is a lopsidependency graph as follows.

Lemma 2.9 (Erdős and Spencer [58]). *D is a lopsidependency graph.*

Proof. Let $A \in \mathcal{A}$. Without loss of generality, let cells $(1,1)$ and $(2,2)$ contain the same symbol and A be the event that $\sigma(1) = 1$ and $\sigma(2) = 2$. We are required to show

$$\mathbb{P}(A| \bigcap_{B \in S} \overline{B}) \leqslant \mathbb{P}(A), \tag{2.1}$$

for all sets of events, S, which are not adjacent to A.

Fix a set, $S \subseteq (V(D) \setminus N[A])$, and, for brevity, define the clause $C := \bigcap_{B \in S} \overline{B}$. Define s_{ij} to be the set of permutations with $\sigma(1) = i$ and $\sigma(2) = j$ that satisfy C. Also note that $\mathbb{P}(A) = \frac{1}{n(n-1)}$ by simple counting. It follows that (2.1) can be extended to

$$\frac{|s_{12}|}{\sum_{i \neq j} |s_{ij}|} = \mathbb{P}(A|C) \leqslant \mathbb{P}(A) = \frac{1}{n(n-1)}.$$

Therefore, we just have to prove

$$\frac{|s_{12}|}{\sum_{i \neq j} |s_{ij}|} \leqslant \frac{1}{n(n-1)}. \tag{2.2}$$

(2.2) is satisfied if $|s_{12}| \leqslant |s_{ij}|$ for all i, j because

$$|s_{12}| \leqslant |s_{ij}| \implies \frac{|s_{12}|}{\sum_{i \neq j} |s_{ij}|} \leqslant \frac{|s_{12}|}{\sum_{i \neq j} |s_{12}|} = \frac{|s_{12}|}{|s_{12}| \sum_{i \neq j} 1} = \frac{1}{n(n-1)}.$$

Therefore we just have to prove $|s_{12}| \leqslant |s_{ij}|$. This will be achieved with an injective function from s_{12} to s_{ij}.

Fix values for i and j. Let $\alpha \in s_{12}$ and let x and y be such that $\alpha(x) = i$ and $\alpha(y) = j$. Define $f(\alpha) := \alpha^*$ where α^* is a copy of α with the modifications $\alpha^*(1) = i$, $\alpha^*(x) = 1$, $\alpha^*(2) = j$ and $\alpha^*(y) = 2$. Note that $\alpha^* \in s_{ij}$ because $\alpha^*(1) = i$, $\alpha^*(2) = j$ and α^* satisfies C. It satisfies C because the new cells, $(1,i)$, $(x,1)$, $(2,j)$ and $(y,2)$, contain 1 or 2 as a row or column, so cannot cause α^* to disagree with C. The other cells are from α, which satisfies C by definition. So $\alpha^* \in s_{ij}$, which means that $f : s_{12} \to s_{ij}$.

To show that f is injective, consider $\alpha^* \in s_{ij}$ and run the definition of f backwards to construct α such that $f(\alpha) = \alpha^*$. It is then clear that α is unique. It is irrelevant that α, constructed by running f backwards, does not necessarily satisfy C. Since f is injective, $|s_{12}| \leqslant |s_{ij}|$. $\square$

2.1.6 Lopsided Clique Local Lemma

I now present an original result which answers a similar question to that of Theorem 2.8. Akbari and Alipour [3] ask for a characterisation of the pairs of integers, m and n, for which all $n \times n$ Latin arrays with exactly m symbols have a transversal. This is a particularly interesting question because increasing the number of symbols does not necessarily make it easier to find a Latin array with a transversal. For example, every 5×5 Latin array with 5 symbols has a transversal, but there is a 5×5 Latin array on 6 symbols with no transversal. Akbari and Alipour [3] conjecture that every Latin array with $n \geqslant 3$ and at least $\frac{n^2}{2}$ symbols has a transversal. In fact, whether every $n \times n$ Latin array with $(1-\varepsilon)n^2$ symbols has a transversal, for fixed $\varepsilon > 0$, was an open problem prior to Theorem 2.11. The problem can be solved with an application of the lopsidependency extension of the Clique Local Lemma, specialised to equal weights.

Lemma 2.10 (Lopsided Symmetric Clique Local Lemma [89]). *Let $\mathcal{A} = \{A_1, \ldots, A_n\}$ be a set of events with lopsidependency graph D. Let $\{K_1, \ldots, K_m\}$ be a set of cliques that cover all edges in D and let $\kappa = \max_i |K_i|$. Suppose that no event A_i is in more than μ of the cliques $K_1, \ldots, K_m$. If there exist $x \in (0, 1/\kappa)$ such that*

$$\mathbb{P}(A_i) \leqslant x\,(1-\kappa x)^{\mu-1} \tag{2.3}$$

for all i, then $\mathbb{P}\left(\bigcap_{i=1}^{n} \overline{A_i}\right) > 0$.

Theorem 2.11 uses the same lopsidependency graph as Theorem 2.8. The main distinction between Theorem 2.8 and Theorem 2.11 is that a small number of symbols may occur n times in L, while in Theorem 2.8 there is a limit to the occurrence of each symbol. This difference allows for regions of high density in the corresponding lopsidependency graph, hence the Symmetric Lopsided Local Lemma is unsuitable. To apply the Clique Local Lemma we consider a set of $2n$ cliques that cover the lopsidependency graph, one for each row or column.

Theorem 2.11. *Every $n \times n$ Latin array with at least $(229n^2 + 27n)/256$ distinct symbols has a transversal.*

Proof. Let L be a fixed $n \times n$ Latin array with $n^2 - cn^2 - dn$ distinct symbols, with $c := 27/256$ and $d := -27/256$. Let σ be a permutation picked uniformly at random from $\mathcal{S}_n$, the symmetric group on $\{1, 2, \ldots, n\}$. We think of σ as choosing the set of cells $(i, \sigma(i))$, for $i \in [n]$, which might correspond to a transversal, as one cell is picked for each row and column. Note that σ fails to be a transversal if at least two of its corresponding cells contain the same symbol. Accordingly, the bad events are defined as

$$\mathcal{A} := \{(i, j, i', j') : i, i' \in [n], i < i', \sigma(i) = j, \sigma(i') = j', L_{ij} = L_{i'j'}\}.$$

Keep in mind that L is fixed and it is σ that is randomly sampled. To prove that a transversal exists we need to prove that, with positive probability, none of the bad events occur.

The next task is to define the lopsidependency graph which will be used in applying Lemma 2.10. Let D be a graph with vertex set $\mathcal{A}$. An edge $\{(a, b, x, y), (a', b', x', y')\}$ is in D if and only if some of the cells (a, b), (x, y), (a', b') and (x', y') share a row or column. This occurs only if at least one of $x = x'$, $x = a'$, $a = x'$, $a = a'$, $y = y'$, $y = b'$, $b = y'$, or $b = b'$. D is a lopsidependency graph by Lemma 2.9. By this point we have a set of bad events $\mathcal{A}$ with a lopsidependency graph D. The remaining ingredient is the set of cliques which cover the edges of D.

Let $\mathcal{K} := \{K_1, K_2, \ldots, K_{2n}\}$ be a set of cliques of D defined as follows. Each clique corresponds to a row or column of L. An event (a, b, x, y) is in a clique K_i if (a, b) or (x, y) is in the row or column corresponding to K_i. We know that $K_i \in \mathcal{K}$ is a clique because the events in K_i share a row or column (the one corresponding to K_i) and so they are adjacent in D. These cliques cover every edge of D because two events are adjacent only if they share a row or column. It follows that both events are both in the clique corresponding to their shared row or column.

Each event in $\mathcal{A}$ corresponds to two cells in distinct rows and columns, so each event is within exactly four cliques. Thus we take $\mu = 4$. To find the bound κ, we consider a clique $K \in \mathcal{K}$ which, without loss of generality, corresponds to the first row. Each event in K corresponds to two cells of L, one in the first row and one not in that row. Let C be the set of cells outside the first row that are included in some event in K. Each cell in C shares a symbol with exactly one cell in the first row. Hence $|K| = |C|$ and the cells not in C contain as many distinct symbols as L does. Hence $n^2 - |K| \geqslant n^2 - cn^2 - dn$, which means that we may take $\kappa = cn^2 + dn$.

Taking $x = 1/(4\kappa)$, we find that to apply Lemma 2.10 we need

$$\frac{1}{n(n-1)} = \mathbb{P}(A_i) \leqslant x(1 - \kappa x)^3 = \frac{27}{256\kappa},$$

which is satisfied, because $c = 27/256$ and $d = -27/256$. □

Subsequently, in the same paper as Theorem 2.11, a non-probabilistic approach is used to show that every $n \times n$ Latin array with at least $(2 - \sqrt{2})n^2$ distinct symbols has a transversal. Montgomery et al. [105] improve upon this result by showing that, for all $\varepsilon > 0$ and sufficiently large n, every $n \times n$ Latin array with at least εn^2 symbols has a transversal. Keevash and Yepremyan [85] improve the exponent, by showing that, for sufficiently large n, every Latin array with $n^{399/200}$ symbols has a transversal.

2.2 Entropy compression

Until recently, a major limitation of the Local Lemma is that it only proved the existence of an object, without providing any method by which to construct the object. Some attempt has been made to de-randomise the Local Lemma to enable it to prove bounds on the complexity of algorithms for generating certain objects [17, 101]. These attempts are restricted to particular problems and rely on non-constructive proofs of the Local Lemma. In 2009, Moser [106] overcame these limitations by introducing a technique known as entropy compression, which he used to prove a constructive version of the Symmetric Local Lemma. This breakthrough has initiated a lot of work on algorithmic versions of the Local Lemma [36, 74, 76, 98, 112]. Moser and Tardos [107] apply entropy compression to prove a constructive versions of the General Local Lemma, as well as variants of the Lopsided Local Lemma. Since then, entropy compression has been used to prove results as strong as those proved with the Local Lemma, but constructively and in a way which yields randomised algorithms with known time complexity. Applications include square-free sequences [73], square-free colouring [50], pattern avoidance [111], permutations [77] and acyclic edge colourings [65].

The central idea behind entropy compression is to define a randomised, usually naive, algorithm designed to explore the object space and find an instance with the desired properties. It is useful to think of the algorithm as a deterministic algorithm initialised with a large random string. Each time the algorithm would take a random sample, it instead consumes a part of the random string. Additionally, the algorithm outputs a history that, along with a constant amount of information, is sufficient to reconstruct the consumed portion of the random string. The algorithm is designed so that it finds an object with the desired property, if it halts, but it is usually a greedy recursive algorithm which does not obviously halt. The core insight of entropy compression is the method by which the algorithm is shown to halt. Entropy compression uses the structure of the space of objects to encode the history such that, eventually, the algorithm losslessly compresses the random string. This is done by using the same sort of locality required by variants of the Local Lemma. In many cases, the algorithm searches the object space by recursively solving local problems. Solving a local problem can only introduce a limited set of new problems, because distant problems are unaffected by the solution, so there are restrictions on the walk that the algorithm can take through the object space. The trick to entropy compression is to use the structure of the problem to show that there is less information in a feasible walk than in the random string used to construct the walk. It follows that, if the algorithm were to run long enough, it would losslessly compress a random string, which is impossible, so the algorithm must halt before the constant amount of information required to reconstruct the random string is outweighed by the history. Since the algorithm halts, it must find an object, so an instance of the object exists. Furthermore, the algorithm can be analysed to determine its running time and it provides a constructive proof that the object exists.

Extensions and uses of entropy compression have become increasingly diverse. Here are some examples. The Local Action Lemma [18] formulates entropy compression as a semigroup which acts on an object to move around the object space and find one with the desired properties. Harvey and Vondrák provide a general algorithm which uses a re-sampling oracle [78] and allows for lopsidependency graphs. Achlioptas and Iliopoulos model the entropy compression algorithm as a random walk on a graph with arbitrary state transitions [1]. As seen in Chapter 1, many of the more recent results in square-free colouring, and related variants of graph colouring, use entropy compression to attain better bounds than previous results with the Local Lemma.

2.3 The Local Lemma and anagram-free colouring

Given the success of entropy compression and the Local Lemma in bounding π, it is natural to consider their application to similar problems. Richmond and Shallit [118] investigate whether the Local Lemma could be used to bound the number of symbols in an arbitrarily long anagram-free word. They conclude that no variant of the Local Lemma could be applied, and did so by counting anagrams.

In general, to apply the Local Lemma we require a set of bad events with sufficiently low probability. Let $\mathbb{P}(\text{square})$ and $\mathbb{P}(\text{anagram})$ be the probability that a word of order $2n$ on k symbols is a square or anagram, respectively. The proof of Theorem 2.4 uses $\mathbb{P}(\text{square}) = k^{-n}$, which follows from the observation that there are k^n squares of order $2n$ on k symbols. The exponential decline of $\mathbb{P}(\text{square})$, with fixed k, is required to counteract the exponentially increasing interdependence of bad events that correspond to long paths.

Let $f_k(n)$ be the number of anagrams of order $2n$ on k symbols. To count anagrams, we count pairs of words, W_1 and W_2, where W_1 is a permutation of W_2. For all sequences $n_1, \dots, n_k$, such that $n_1 + n_2 + \cdots + n_k = n$, there are $n!/(n_1!n_2!\dots n_k!)$ words of order n

on k symbols in which symbol i occurs n_i times. It follows that

$$f_k(n) = \sum_{n_1+n_2+\cdots+n_k=n} \left(\frac{n!}{n_1!n_2!\ldots n_k!} \right)^2 . \tag{2.4}$$

Richmond and Shallit use (2.4) to determine the asymptotics of $f_k(n)$.

Theorem 2.12 (Richmond and Shallit [118])**.** *Let k be an integer and $n \geqslant 2$. Then, as $n \to \infty$,*

$$f_k(n) \sim k^{2n+2} \left(4\pi n\right)^{(1-k)/2} .$$

It follows from Theorem 2.12 that, the probability that a random word of order $2n$ on k symbols is an anagram is

$$\mathbb{P}(\text{anagram}) = f_k(n)/k^{2n} \sim k^2 \left(4\pi n\right)^{(1-k)/2} .$$

The qualitative difference between $\mathbb{P}(\text{square})$ and $\mathbb{P}(\text{anagram})$ is that, for fixed k, $\mathbb{P}(\text{square})$ decreases exponentially with n while $\mathbb{P}(\text{anagram})$ only decreases polynomially with n. The polynomial decrease of $\mathbb{P}(\text{anagram})$ is insufficient to counteract the exponentially increasing interdependence of bad events. Richmond and Shallit remark that the Local Lemma is unlikely to yield bounds on the number of symbols in an arbitrarily long anagram-free word. More generally, the same can be said of using the Local Lemma to prove bounds on the anagram-free chromatic number on graphs of bounded degree.

Chapter 3

Anagram-free colourings of graphs and trees

This chapter contains original results regarding bounds on the anagram-free chromatic number of graphs and trees. Recall that an *anagram* is a word of the form WP, where W is a non-empty word and P is a permutation of W. A graph colouring is *anagram-free* if the sequence of colours read along each of its paths is anagram-free. The *anagram-free chromatic number*, $\phi(G)$, of a graph, G, is the minimum number of colours in an anagram-free vertex colouring of G. Similarly, the *anagram-free chromatic index*, $\phi'(G)$, of a graph, G, is the minimum number of colours in an anagram-free edge colouring of G.

The results in this chapter are based on the paper *Anagram-Free Graph Colouring* [131]. At the time, anagrams had only been studied in the context of combinatorics of words. Our foray into anagram-free graph colourings is in the context of two contrasting results from the combinatorics of words. The first result is the bound $\phi(P) \leqslant 4$, for all paths P, by Keränen [86] which shows a similarity between anagram-free colouring and square-free colouring (see Section 1.3.1). The second is the enumeration of anagrams by Richmond and Shallit [118], which suggests that Lovász Local Lemma cannot be used to obtain bounds on ϕ (see Section 2.3).

Alon et al. [10] ask whether ϕ is bounded on graphs of bounded maximum degree. Theorem 3.2 answers this question in the negative. Furthermore, ϕ and ϕ' are unbounded on trees. The behaviour of ϕ on trees is studied in more detail, with upper and lower bounds for ϕ given on trees of bounded radius or pathwidth. We also prove a non-trivial lower bound on $\phi(T)$, as a function of height, for the complete binary tree T. In Section 3.3, we study the extension of k-anagram-free colouring to determine whether results on ϕ and ϕ' generalised to ϕ_k and ϕ'_k.

3.1 Lower bounds

We begin by proving lower bounds for ϕ and ϕ' on various classes of graphs. In these proofs, we often fix an arbitrary k-colouring of a sufficiently large graph, G, from the class of graphs in question, and then proceed to find an anagram. We often find anagrams in a graph, G, by constructing a large set of paths, $\mathcal{S}$, in G, such that G contains an anagram if two paths in $\mathcal{S}$ have the same number of occurrences of each colour. To facilitate this, we define a *colour multiset* of size n on c colours to be a multiset of size n with entries from $[c]$. For a coloured graph or path, G, let $M(G)$ be the multiset of colours that occur in G and call $M(G)$ the *colour multiset* of G. Equivalent formulations of the colour multiset of a word, W, have previously been called the *signature* of W or, more commonly in computer science, the *Parikh vector* of W [118].

Let $\mathcal{M}_{n,c}$ be the set of colour multisets of size n on c colours. Note that $|\mathcal{M}_{n,c}|$ is the number of ways to partition a set of n unlabelled objects into c labelled partitions, which is well known to equal $\binom{n+c-1}{c-1}$; see [124, Section 1.9]. For convenience, we use the bound

$$|\mathcal{M}_{n,c}| \leqslant (n+1)^c, \tag{3.1}$$

which is obtained by noting that the number of occurrences of each colour is at least zero and at most n. This bound is sufficient for our purposes, as we only require that, for fixed c, $|\mathcal{M}_{n,c}|$ is bounded above by a polynomial in n. Recall that a *c-colouring* of a graph, G, is a colouring, of either the vertices or edges of G, which uses at most c colours. Note that, if G has a vertex c-colouring, $M(G) \in \mathcal{M}_{|V(G)|,c}$, and, if G has an edge c-colouring, $M(G) \in \mathcal{M}_{|E(G)|,c}$. The study of edge colouring and vertex colouring is kept sufficiently separate for $M(G)$ to be clear from context. For a set of colours C let $M_C(G)$ be $M(G)$ restricted to C. For a graph, G, and set of colours, C, let $V_C(G)$ and $E_C(G)$ be the vertices or edges of G that are assigned a colour from C, respectively.

Anagram-free colouring can be defined in terms of colour multisets. For $i \geqslant 1$, a vertex coloured path $v_1, \dots, v_{2i}$ is an anagram if and only if

$$M(v_1, \dots, v_i) = M(v_{i+1}, \dots, v_{2i}).$$

Similarly, for $i \geqslant 1$, an edge coloured path $v_1, \dots, v_{2i+1}$ is an anagram if and only if

$$M(v_1v_2, \dots, v_iv_{i+1}) = M(v_{i+1}v_{i+2}, \dots, v_{2i}v_{2i+1}).$$

The indices in these expressions highlight a distinction between ϕ and ϕ'. In a vertex colouring only the paths of even order can be anagrams. However, in an edge colouring only the paths of even length can be anagrams.

3.1.1 Edge colouring

A good introductory result is that ϕ' is unbounded on trees of maximum degree 3. The proof uses the fact that the number of leaves in a complete binary tree grows exponentially with height while, for fixed c, $|\mathcal{M}_{n,c}|$ is bounded by a polynomial in n. The theorem follows by using two root-to-leaf paths, with a common colour multiset, to construct an anagram, where a *root-to-leaf path* is a path between the root and a leaf of a tree. Recall that a *binary tree* is a rooted tree such that every vertex has at most two children. A *complete* binary tree is a rooted tree such that every non-leaf vertex has two children and the leaves have equal distance to the root.

Theorem 3.1. *Trees of maximum degree* 3 *have unbounded anagram-free chromatic index.*

Proof. Fix $c \geqslant 1$ and choose $h \in \mathbb{Z}^+$ so that $2^h > (h+1)^c$. Let T be the rooted complete binary tree of height h with root vertex r. Fix an arbitrary edge c-colouring of T. By our choice of h and Equation (3.1)

$$|\mathcal{M}_{h,c}| \leqslant (h+1)^c < 2^h = \#\text{leaves of } T.$$

Since each root-to-leaf path in T has h edges, the number of leaves in T is greater than the number of distinct colour multisets on root-to-leaf paths in T. Therefore, there are two leaves, p and q, such that $M(P) = M(Q)$, where P is the rp-path and Q is the rq-path. The anagram that can be found with p and q is illustrated in Figure 3.1.

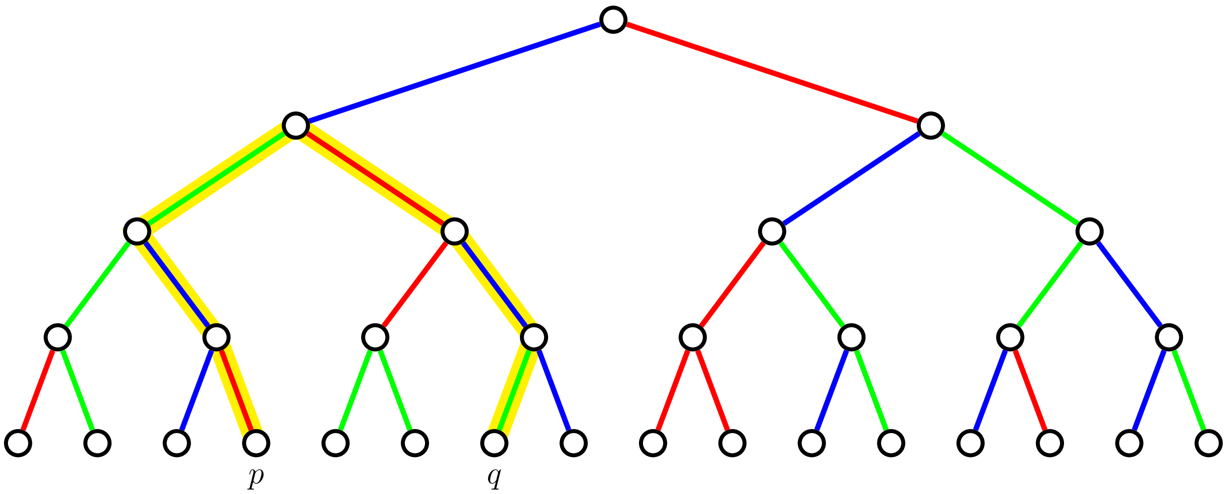


Figure 3.1: The complete binary tree used in Theorem 3.1 with $h = 4$, an edge 3-colouring and leaves p and q which lead to an anagram.

Let v be the least common ancestor of p and q. Split these paths into three disjoint parts by defining R as the rv-path, P' as the vp-path and Q' as the vq-path. Note that $E(R)$ is exactly the set of edges shared by P and Q. Therefore

$$M(P) = M(P') \cup M(R) \text{ and } M(Q) = M(Q') \cup M(R).$$

Thus $M(P') = M(Q')$. Finally, note that $P'Q'$ is a path in T, so T contains an anagram. □

3.1.2 Vertex colouring

As a corollary of Theorem 3.1, ϕ is unbounded on graphs of maximum degree 4, by a line graph construction. Recall, from Section 1.2.1, that line graphs are used in square-free colouring to prove upper bounds on π from upper bounds on π'. Similarly, line graphs can be used to translate bounds between ϕ and ϕ'. Upper bounds on ϕ' can be derived from upper bounds on ϕ and lower bounds on ϕ can be derived from lower bounds ϕ'.

For completeness, we use line graphs to show that ϕ is unbounded on graphs of maximum degree 4. Note that $\phi'(G) \leqslant \phi(L(G))$, since every path in G, written as a sequence of edges, $e_1, e_2, \ldots, e_n$, corresponds to a path in $L(G)$ with vertex sequence $e_1, e_2, \ldots, e_n$. By Theorem 3.1, for every $c \in \mathbb{Z}^+$, there is a binary tree such that $\phi'(T) > c$. Note that $L(T)$ has maximum degree at most 4 and recall $\phi'(T) \leqslant \phi(L(T))$. Therefore, ϕ is unbounded on graphs of maximum degree 4, which raises the question of whether ϕ is bounded on graphs of maximum degree 3.

Theorem 3.2 shows that ϕ is unbounded on graphs of maximum degree 3. Furthermore, ϕ is unbounded on outerplanar graphs of maximum degree 3. The proof proceeds similarly to the proof of Theorem 3.1, with the construction of a family of graphs which exhibit some exponential growth. The exponential growth is exploited to find two paths which share a colour multiset and can be concatenated to find an anagram.

Theorem 3.2. *Outerplanar graphs of maximum degree* 3 *have unbounded anagram-free chromatic number.*

Proof. Let $c \geqslant 1$ and let $h \in \mathbb{Z}^+$ be odd such that $2^{(h+1)/2} > (h+2)^c$. Let T be the rooted tree, with root r, such that:

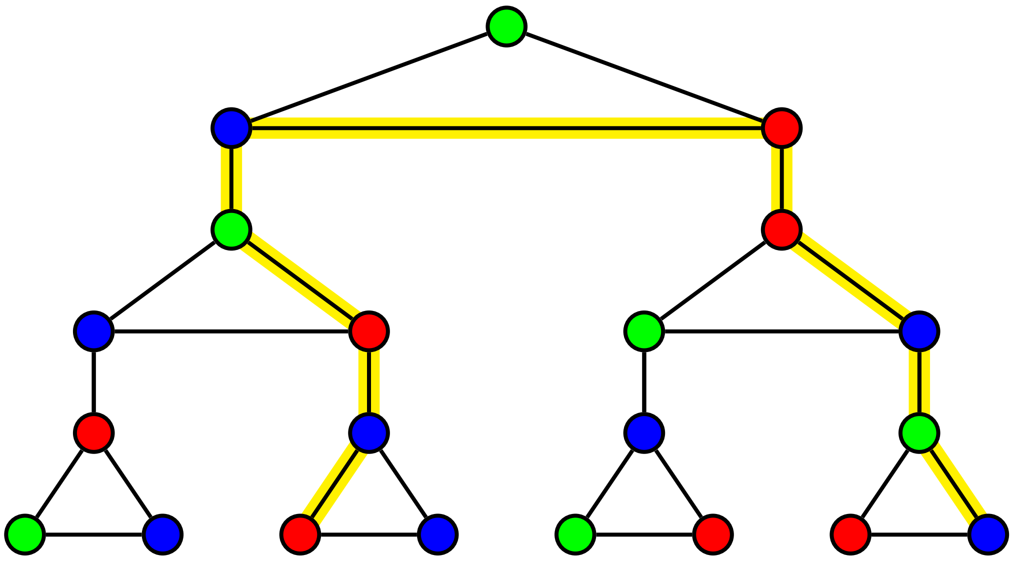


Figure 3.2: The graph used in Theorem 3.2 with $h = 5$, a vertex 3-colouring and vertices p and q which lead to an anagram.

- vertices of depth h are leaves,
- vertices of even depth have two children,
- non-leaf vertices of odd depth have one child.

Recall that the *depth* of a vertex is its distance from the root. Let G be the graph obtained from T by adding an edge between every pair of vertices in T that share a parent, as illustrated in Figure 3.2. Fix an arbitrary c-colouring of G. We now show that G contains an anagram. By Equation (3.1) and our choice of h,

$$|\mathcal{M}_{h+1,c}| \leqslant (h+2)^c < 2^{(h+1)/2} = \#\text{leaves of } T.$$

Since each root-to-leaf path in T has $h + 1$ vertices, the number of leaves in T is greater than the number of distinct colour multisets on root-to-leaf paths in T. Therefore there are two leaves of T, p and q, such that $M(P) = M(Q)$ where P and Q are the rp-path and rq-path in T, respectively. Split the two paths into three vertex-disjoint paths $R = P \cap Q$, $P' = P - V(R)$ and $Q' = Q - V(R)$. Note that

$$M(P) = M(P') \cup M(R) \text{ and } M(Q) = M(Q') \cup M(R).$$

Thus $M(P') = M(Q')$. By construction, the graph induced by $V(P') \cup V(Q')$ is a subpath of G. Therefore G is not anagram-free. □

Theorem 3.2 only proves that ϕ is unbounded on graphs. In subsequent sections we study ϕ on trees. A particularly interesting question is whether there is a result analogous to Theorem 3.1: is ϕ bounded on trees of maximum degree 3? This motivates further investigation of ϕ on trees.

3.1.3 Vertex colouring trees

In light of the results from the previous two sections, the most pressing question is whether ϕ is bounded on trees. We prove lower bounds on two classes of trees, both of which follow from Theorem 3.3. The *complete d-ary tree of height h* is the rooted tree such that every internal vertex has d children and all leaves are distance h from the root.

Theorem 3.3. *The complete d-ary tree of height h does not have an anagram-free c-colouring when $d^c \leqslant (d/c)^h$.*

Proof. Let T be the complete d-ary tree of height h with root r. Let L be the set of leaves of T, and fix an arbitrary c-colouring, $\psi : V(T) \to [c]$, of T. For each $v \in L$ let S_v be the sequence of colours on the rv-path.

There are at most c^h sequences of colours in root-to-leaf paths since each sequence has length $h+1$ and they all start with $\psi(r)$. Since $|L| = d^h$ there is a set, $C \subseteq L$, of size at least d^h/c^h such that $S_v = S_w$ for all $v, w \in C$. Note that C is a large set of leaves with the same colour sequence on their root-to-leaf paths, as illustrated in Figure 3.3. Let R be the subtree of T induced by the set of all ancestors of leaves in C. The remainder of the proof finds an anagram in R.

Define a *level of R* to be a maximal set of vertices of R that all have equal depth. R is coloured by level, as $\psi(u) = \psi(v)$ for every pair of vertices $u, v \in V(R)$ with the same depth. Let $\ell_0, \ell_1, \dots, \ell_h$ be the sets of vertices corresponding to levels of R, where $\ell_0 = \{r\}$ and $\ell_h = C$. A level, ℓ_i, is *bad* if every vertex $v \in \ell_i$ has exactly one child in R. A level is *good* if it is not bad. Note that only level ℓ_h contains vertices with no children. Let g be the number of good levels of R and b be the number of bad levels of R. By definition, $h+1 = g+b$. We now prove that there are at least $c+1$ good levels and so at least two good levels share a colour.

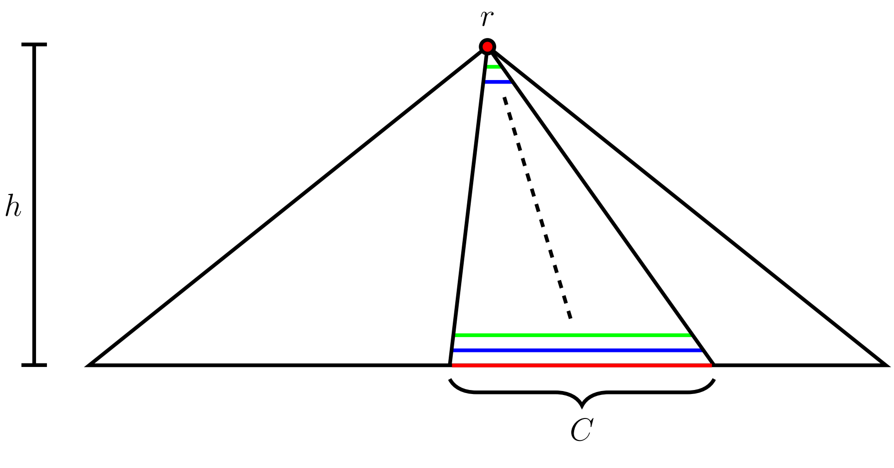


Figure 3.3: The complete d-ary tree of height h with a large set, C, of leaves which have the same root-to-leaf path colour sequence.

We bound the number of bad levels by considering the number of good levels required to attain $|\ell_h| \geqslant (d/c)^h$. If ℓ_i is bad, then $|\ell_i| = |\ell_{i+1}|$, and if ℓ_i is good, then $|\ell_i| < |\ell_{i+1}| \leqslant d|\ell_i|$. It follows that

$$|\ell_k| \leqslant d^{\#\text{preceding good levels}}.$$

Since ℓ_h is the final good level, it is preceded by $g-1$ good levels. Thus

$$(d/c)^h \leqslant |\ell_h| \leqslant d^{g-1}$$

so $d^{h+1}c^{-h} \leqslant d^g$. Recall that $d^{c+1} \leqslant d^{h+1}c^{-h}$, so $c+1 \leqslant g$. Therefore there are at least two good levels with the same colour.

Let A and B be two good levels that have the same colour denoted such that vertices in A are closer to r than vertices in B. Let $a \in A$ be a vertex with at least two children. All vertices in the levels between A and B have at least one child so there are two vertices $v, w \in B$ such that a is their least common ancestor.

Let $p_0, p_1, \dots, p_n$ be the va-path and $q_0, q_1, \dots, q_n$ be the wa-path. Since R is coloured by level, $\psi(q_i) = \psi(p_i)$, for all $i \in \{0, \dots, n\}$. Recall that $\psi(p_0) = \psi(p_n)$ and that $p_n = q_n$. It follows that $p_0 p_1 \dots p_, q_{n-1} \dots q_1$ is a path in T and

$$M(p_0, p_1, \dots, p_{n-1}) = M(q_n, q_{n-1} \dots q_1)$$

so $\psi(p_0, p_1, \dots, p_n, q_{n-1}, \dots, q_1)$ is an anagram. □

Interestingly, Theorem 3.3 proves more than is required. Instead of just proving that every c-colouring of the tree contains a general anagram, the proof finds a coloured path of the form $xWx\overleftarrow{W}$, where x is a symbol, W is a word and $\overleftarrow{W}$ is W written backwards. Let $xWx\overleftarrow{W}$-free colouring be a variant of graph colouring that avoids paths with words of the form $xWx\overleftarrow{W}$. The proof of Theorem 3.3 shows that $xWx\overleftarrow{W}$-free colouring is unbounded on trees. This observation is particularly interesting in light of the result that $\pi(T) \leqslant 4$ for all trees T [30]. Both square-free colouring and $xWx\overleftarrow{W}$-free colouring avoid paths with colour sequence $W\sigma_n(W)$, each for a single σ_n, for each $n \geqslant 1$, so, in a sense, they avoid the same number of paths. However, they have contrasting behaviour on trees. This motivates a more general study of permutation-avoiding colouring, which we leave until the study of colour schemes in Chapter 6. For now, note that Theorem 3.4 is a corollary of Theorem 3.3.

Theorem 3.4. *Trees have unbounded anagram-free chromatic number.*

Furthermore, we can vary the height and maximum degree in Theorem 3.3 to prove bounds on classes of graphs that represent each extreme. In particular, Theorem 3.3 implies the following two theorems.

Theorem 3.5. *For every integer $h \geqslant 2$ the complete $(h-1)^h$-ary tree of height h has $\phi(T) \geqslant h$.*

Proof. Let $c \geqslant 1$, $h := c+1$ and $d := c^{c+1}$. The conditions of Theorem 3.3 are satisfied because

$$d^c = c^{c(c+1)} = \left(c^{c(c+1)}c^{c+1}\right)c^{-(c+1)} = d^{c+1}c^{-(c+1)} = (d/c)^h.$$

Therefore the complete c^{c+1}-ary tree of height $c+1$ does not have an anagram-free c-colouring. □

Theorem 3.6. *For every integer $d' \geqslant 1$ there exists a tree T with maximum degree d' such that $\phi(T) \geqslant d' - 1$.*

Proof. Fix c and choose h so that

$$(c+1)^c \leqslant \left(1 + \frac{1}{c}\right)^h = ((c+1)/c)^h.$$

It follows that the conditions of Theorem 3.3 are satisfied with $d := c+1$. Therefore the complete $(c+1)$-ary tree of height h does not have an anagram-free c-colouring. This tree has maximum degree $d' = c+2$ and so $\phi(T) \geqslant c+1 = d'-1$. □

Theorems 3.5 and 3.6 demonstrate a trade-off between height and maximum degree, with the former result holding for trees of relatively large maximum degree and the latter holding for trees of relatively large height.

3.2 Upper bounds for ϕ on trees

In this section, we complement the lower bounds of the previous section with upper bounds for ϕ on trees. The first bound is derived from centred colouring. Recall that a vertex colouring of a graph, G, is *centred* if every subtree, T of G, contains a vertex whose colour appears exactly once in T. All centred colourings are anagram-free, since every anagram contains an even number of occurrences of each colour and every path in a centred colouring contains a colour that occurs exactly once. Therefore, for every graph G, the centred chromatic number of G is an upper bound on $\phi(G)$. It is easily seen that every tree, T, of radius h, has centred chromatic number at most $h+1$, see [109, Section 6.5], thus $\phi(T) \leqslant h+1$. This bound is achieved by colouring each vertex by its distance from a centre of T. This colouring shows that the lower bound in Theorem 3.7 is tight.

Theorem 3.7. *Every tree T of radius h has $\phi(T) \leqslant h+1$. Moreover, for every $h \geqslant 0$, there is a tree T of radius h such that $\phi(T) \geqslant h$.*

Proof. The upper bound follows from centred colouring. The lower bound follows from Theorem 3.5 for $h \geqslant 2$, and an inspection of the trees of radius 0 or 1, for $h \leqslant 1$. □

We now show that ϕ is bounded on trees of bounded pathwidth. See Section 1.2.8 for a definition of pathwidth. The only non-trivial property of pathwidth that we require is stated in the following lemma.

Lemma 3.8 (Suderman [125], Lemma 5)**.** *Every tree, T, with at least one edge contains a path P such* $\mathrm{pw}(T - V(P)) \leqslant \mathrm{pw}(T) - 1$.

We also require two trivial properties of pathwidth. The first is that edgeless graphs have pathwidth 0, and the second is that the pathwidth of a disconnected graph equals the maximum pathwidth of its components.

Theorem 3.9. *Every tree of pathwidth $m \in \mathbb{Z}_{\geqslant 0}$ has an anagram-free vertex $(4m+1)$-colouring.*

Proof. Note that every tree, T, of pathwidth zero is edgeless and thus anagram-free 1-colourable. We proceed by induction on m.

Let T be a tree of pathwidth $m+1$. By Lemma 3.8, there exists a path $P \subseteq T$ such that every component H of $T - V(P)$ has $\mathrm{pw}(H) \leqslant m$. By induction we may anagram-free colour each component of $T - V(P)$ with a common set of $4m+1$ colours. To complete the colouring of T, use four additional colours to anagram-free colour P, by [87].

We now show that this colouring is anagram-free. Let Q be a path in T. If Q is entirely contained within a component of $T - V(P)$, then by induction, Q is not an anagram. Otherwise, Q intersects P. The intersection of Q and P is an anagram-free subpath of P and the colours in $Q \cap V(P)$ occur nowhere else in Q. Therefore Q is not an anagram. □

Theorem 3.5 shows that the upper bound in Theorem 3.9 is tight, up to a constant factor.

Theorem 3.10. *For every tree T, $\phi(T) \leqslant 4\,\mathrm{pw}(T) + 1$. Moreover, for every $p \geqslant 0$ there is a tree T such that $\phi(T) \geqslant p \geqslant \mathrm{pw}(T)$.*

Proof. The first part follows directly from Theorem 3.9. For the second part it is well known, and easily proved, that the pathwidth of a tree is at most its radius. Therefore, by Theorem 3.5, there exists a tree T with $\phi(T) \geqslant p \geqslant \text{pw}(T)$ for all $p \geqslant 2$. For the remaining cases use the path of order 2 for $p = 1$ and the empty graph for $p = 0$. □

Theorem 3.10 raises the question of whether ϕ is tied to pathwidth on trees, that is, whether there exists a function f such that $\text{pw}(T) \leqslant f(\phi(T))$ for every tree T. We answer this question in the negative with a result from Chapter 4. Theorem 4.3 shows that every tree, T, has a subdivision, H, such that $\phi(H) \leqslant 10$, and it is well known that $\text{pw}(T) = \text{pw}(H)$. Therefore, there exist trees of arbitrarily large pathwitdh with ϕ at most 10, so ϕ is not tied to pathwidth on trees.

A remaining question is whether ϕ is bounded for trees of maximum degree 3. The complete binary tree of height h is the key example. Centred colourings provides a trivial upper bound of $h+1$ (Theorem 3.7). We give the following non-trivial colouring of complete binary trees, to demonstrate a case in which the bound attained by centred colouring is not optimal.

Theorem 3.11. *If T is the complete binary tree of height h, then*

$$\phi(T) \leqslant \frac{h}{2} + \frac{1}{2}\log_2(h+1) + 1. \tag{3.2}$$

Proof. We proceed by induction on h. The base case is satisfied as follows:

$$\phi(\text{singleton vertex}) = 1 \leqslant \frac{0}{2} + \frac{1}{2}\log_2(0+1) + 1 = 1.$$

Now assume the result holds up to $h-1$. Let T be the complete binary tree of height h with root r. Let $t := 2\lfloor \frac{h+2}{4} \rfloor$ and $T_t \subseteq T$ be the complete binary tree of height t with root r. T_t is the top half of T, which we colour directly. Let $b := h - t - 1$, this is the height of each subtree which will be coloured by induction. Colour T_t as follows:

- All vertices with even depth receive the same colour. Call this colour c.
- Each odd level is allocated distinct set of two colours. Vertices of odd depth are coloured with one of the two colours allocated to their level, so that each vertex receives a colour distinct from their sibling.

Note that the leaves of T_t have even depth so have colour c. This colouring is shown for $h = 8$ in Figure 3.4.

Colour each remaining subtree of T by induction, avoiding colours that occur on their ancestors in T_t.

Claim 1. *This colouring of T is anagram-free.*

Proof. Let P be a path in T with even order at least 2. Let u be the shallowest vertex in P. If $u \notin V(T_t)$ then, by induction, P is not an anagram. Now consider the case where $u \in V(T_t)$. If u has odd depth then its colour is unique in P. Indeed, in T_t the colour of u only occurs in the level of u and in $T - V(T_t)$ the descendants of u avoid its colour. Similarly, if u has even depth and u is an endpoint of P then the child of u in P is uniquely coloured in P.

The remainder of the proof is concerned with the case where $u \in V(T_t)$, u has even depth and neither endpoint of P is u. Let $x_1, x_2 \in V(P)$ be the endpoints of P and let $v_1, v_2 \in V(P)$ be the children of u such that v_i is an ancestor of x_i for $i \in [2]$. If $x_i \in V(T_t)$

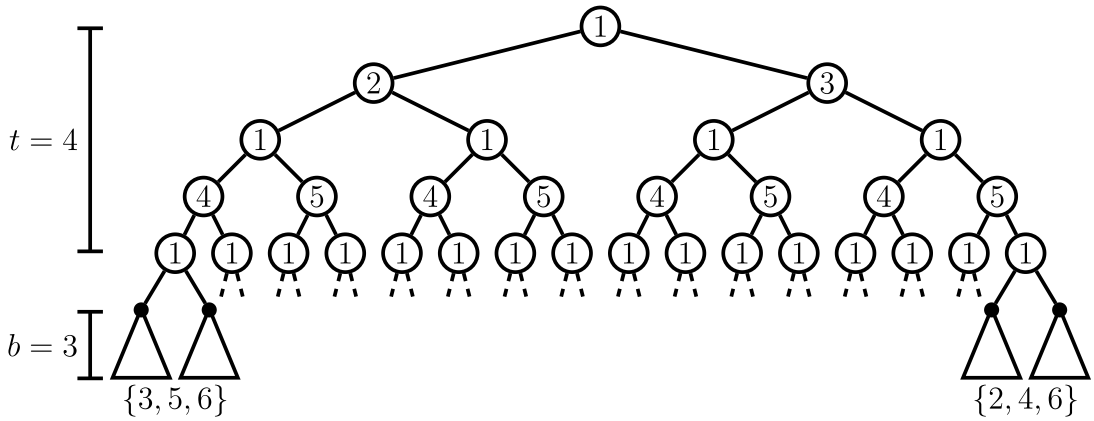


Figure 3.4: Schematic of colouring the complete binary tree with $h = 8$. The set of colours used in the subtrees of height b are shown below the trees.

then u_{2-i} has a unique colour in P because the $x_i v_i$-path does not contain the colour on u_{2-i}.

If both $x_1, x_2 \notin V(T_t)$ then the colour c occurs an odd number of times in P. Indeed, u has colour c and each subpath of P to either side of u contain the same number of occurrences of c. □

To complete the proof we show that this colouring satisfies Equation (3.2). Our colouring of T_t uses $t+1$ colours as the even levels share a colour and there are $\frac{t}{2}$ odd levels which each use 2 colours.

Let v be a child of a leaf of T_t and T_b be the subtree of T rooted at v. Recall that colours on the vr-path do not occur in T_b. The number of distinct colours on the vr-path is $\frac{t}{2}+1$ because the path contains $t+1$ vertices and $\frac{t}{2}+1$ of them have even depth so share colour c. Therefore the colouring of T_b can reuse $\frac{t}{2}$ of the colours used to colour T_t. So our colouring of T requires $\phi(T_b) - \frac{t}{2}$ colours in addition to those used to colour T_t. So

$$\phi(T) \leqslant t + 1 + \left(\phi(T_b) - \frac{t}{2}\right) = \frac{t}{2} + 1 + \phi\left(T_{h-t-1}\right).$$

By induction, since T_b is a complete binary tree of height $h-t-1$,

$$\begin{aligned}
\phi(T) &\leqslant \frac{t}{2} + 1 + \left(\frac{h-t-1}{2} + \frac{1}{2}\log_2(h-t) + 1\right) \\
&= \frac{h}{2} + \frac{1}{2} + \frac{1}{2}\log_2(h-t) + 1 \\
&= \frac{h}{2} + \frac{1}{2} + \frac{1}{2}\log_2\left(h - 2\left\lfloor\frac{h+2}{4}\right\rfloor\right) + 1 \\
&\leqslant \frac{h}{2} + \frac{1}{2} + \frac{1}{2}\log_2\left(h - \frac{h-1}{2}\right) + 1 \\
&= \frac{h}{2} + \frac{1}{2} + \frac{1}{2}\log_2\left(\frac{h+1}{2}\right) + 1 \\
&= \frac{h}{2} + \frac{1}{2}\log_2(h+1) + 1.
\end{aligned}$$

□

3.3 k-anagram-free colourings

A k-anagram is a word $W_1 W_2 \ldots W_k$ where W_i is a permutation of W_j, for all $i, j \in [k]$. Equivalently, a word, $W_1 W_2 \ldots W_k$, is a k-anagram if and only if, for all $i, j \in [k]$,

$$M(W_i) = M(W_j).$$

Recall that a graph colouring is k-anagram-free if the sequence of colours read along each of its paths are not k-anagrams. The k-anagram-free chromatic number and index of a graph G are denoted $\phi_k(G)$ and $\phi'_k(G)$, respectively. Note that k-anagram-free colouring is a generalisation of anagram-free colouring, as $\phi = \phi_2$ and $\phi' = \phi'_2$. Since every k-anagram-free word is $(k+1)$-anagram-free,

$$\phi_{k+1}(G) \leqslant \phi_k(G),$$
$$\phi'_{k+1}(G) \leqslant \phi'_k(G),$$

for all $k \geqslant 2$ and graphs G. It follows that upper bounds on ϕ and ϕ' apply to ϕ_k and ϕ'_k. However, we only have upper bounds for ϕ, and they are on trees of bounded pathwidth or radius. In this section we show that ϕ_k is unbounded on graphs of maximum degree $k+1$ and that ϕ_4 and ϕ'_4 are bounded on trees. The first result is a generalisation of Theorem 3.2, that ϕ is unbounded on graphs of maximum degree 3. The second result contrasts with Theorems 3.1 and 3.4 because ϕ_k and ϕ'_k are unbounded on trees for $k = 2$ and bounded on trees for $k \geqslant 4$.

3.3.1 Lower bounds

We first show that ϕ_k is unbounded on graphs of bounded degree. The method is similar to that used in Theorem 3.2 to prove ϕ is not bounded by maximum degree. For each k and c we construct a graph, G, such that every c-colouring of G contains a k-anagram. The proof generalises most of Theorem 3.2 as, with the exception of outerplanarity, it is implied by Theorem 3.12 for $k = 2$.

Theorem 3.12. *For $k \geqslant 2$, the k-anagram-free chromatic number is unbounded on planar graphs of maximum degree $k+1$.*

Proof. Let $S(t)$ be the proposition that there exists a planar graph, G, with $\Delta(G) \leqslant k+1$, special vertices, u and v, such that $\deg(u) = \deg(v) = 1$, and, for every vertex colouring of G, at least one of the following holds:

- G contains a k-anagram, or
- $|D| \geqslant \left(\frac{k}{k-1}\right)^t$, where D is the set of colour multisets on uv-paths of length $4t$.

Claim 2. *$S(t)$ is true for all $t \geqslant 1$.*

Proof. We first prove $S(1)$. Let G be the disjoint union of the path, P, of order k, and the four vertices a, b, u, v, with the addition of edges ua, vb, ap, and bp, for all $p \in V(P)$. Note that G satisfies the degree requirements of $S(1)$. Fix a colouring of G. If P is monochromatic then it is a k-anagram. If P is not monochromatic, then there are two paths u, a, p_1, b, v and u, a, p_2, b, v with distinct colour multisets. Therefore $|D| \geqslant 2 \geqslant \frac{k}{k-1}$ and so $S(1)$ is true. This graph is shown in Figure 3.5 for $k = 4$.

We proceed by induction on t by taking $S(t)$ to be true for some $t \geqslant 1$. Let $G_1, \ldots, G_k$ be copies of the graph guaranteed to exist by $S(t)$. Denote the two special vertices of G_i by u_i and v_i. Let H be the graph with $V(G) = \{u, v, a, b\}$ and $E(G) = \{ua, vb\}$. Let G be the disjoint union of H and $G_1, \ldots, G_k$, with the additional edges:

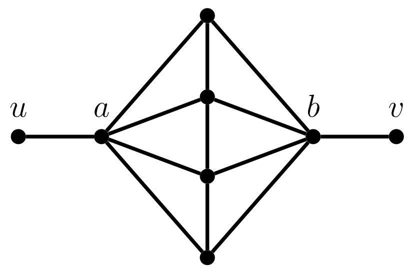


Figure 3.5: A graph satisfying $S(1)$ for $k = 4$.

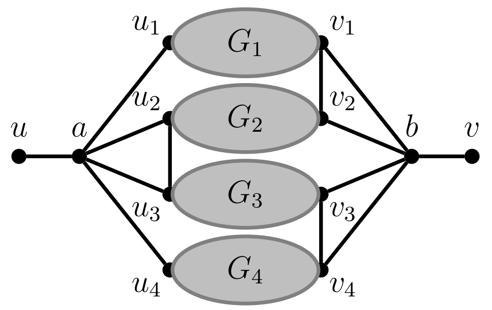


Figure 3.6: A graph satisfying $S(t)$ for $k = 4$. G_1, G_2, G_3 and G_4 are graphs satisfying $S(t-1)$.

(i) $u_i a$ for all $i \in [k]$.

(ii) $v_i b$ for all $i \in [k]$.

(iii) $u_i u_{i+1}$ for all even $i \in [k-1]$.

(iv) $v_i v_{i+1}$ for all odd $i \in [k-1]$.

This construction is shown in Figure 3.6 for $k = 4$.

First we show that G satisfies the degree requirements of $S(t+1)$. Clearly, $\deg(u) = \deg(v) = 1$ and $\deg(a) = \deg(b) = k+1$. For all $i \in [k]$, u_i has degree 1 in G_i, so $\deg(u_i) \leqslant 3 \leqslant k+1$. Similarly, $\deg(v_i) \leqslant 3 \leqslant k+1$. Every remaining vertex, $v \in V(G)$, is in G_i, for some $i \in [k]$, and v is only adjacent to vertices in G_i, so, by $S(t)$, $\deg(v) \leqslant k+1$.

Now fix a colouring of G. $S(t+1)$ is satisfied if some G_i contains a k-anagram, so take the case that G_i is k-anagram-free, for all $i \in [k]$. Let D_i be the set of colour multisets on paths of length $4t$ in G_i with endpoints u_i and v_i. By $S(t)$, we have $|D_i| \geqslant (k/(k-1))^t$ for all $i \in [k]$. We now split the proof into two cases.

In the first case, there exists a colour multiset T such that $T \in D_i$, for all $i \in [k]$. This means that, for all $i \in [k]$, G_i contains a $u_i v_i$-path, P_i, with $M(P_i) = T$. By the existence of type (iii) and (iv) edges between special vertices of G_i and G_{i+1}, $P_1 P_2 \dots P_k$ is a path in G. This path is a k-anagram and so the colouring of G satisfies $S(t+1)$.

In the second case, there is no colour multiset that occurs in every D_i. Define the union of colour multisets to be $\mathcal{U} := \bigcup_{i \in [k]} D_i$. For a colour multiset $T \in \mathcal{U}$, let $f(T)$ be the number of sets from $\{D_1, \dots, D_k\}$ that contain T. Since no colour multiset occurs in every D_i, $f(T) \leqslant k-1$. Therefore

$$|\mathcal{U}|(k-1) \geqslant \sum_{T \in \mathcal{U}} f(T) = \sum_{i \in [k]} |D_i| \geqslant k \left(\frac{k}{k-1}\right)^t,$$

so $|\mathcal{U}| \geqslant \left(\frac{k}{k-1}\right)^{t+1}$. There is a bijection from $\mathcal{U}$ to D because every uv-path of length $4t+4$ shares vertices a, b, u and v. Therefore $|D| = |\mathcal{U}|$ and so $S(t+1)$ is satisfied. □

Let $c \in \mathbb{Z}^+$ and let $t \in \mathbb{Z}^+$ be such that

$$\left(\frac{k}{k-1}\right)^t > (4t+2)^c.$$

Let G be the graph guaranteed to exist by $S(t)$ and fix an arbitrary c-colouring of G. Let D be the set of colour multisets as defined previously. By (3.1) there are at most $(4t+2)^c$

colour multisets of size $4t+1$. Therefore $|D| \leqslant (4t+2)^c < \left(\frac{k}{k-1}\right)^t$, so, by $S(t)$, G contains a k-anagram. □

Theorem 3.12 raises the question of whether there is a fixed integer d such that, for all $k \geqslant 2$, ϕ_k is unbounded on graphs of maximum degree d. Additionally, the analogous problem for edge colouring is open. For $k \geqslant 3$, we know of no family of graphs of bounded degree for which ϕ'_k is unbounded.

3.3.2 Upper bounds on trees

In this section we show that ϕ_k and ϕ'_k are bounded on trees, for all $k \geqslant 4$. The proof uses an anagram-free word on 4 symbols to colour trees by a levelling. This colouring is k-anagram-free for $k \geqslant 4$ because at least half of every path, P, has the level of its vertices monotonically increase or decrease in every levelling of a tree T.

Theorem 3.13. *$\phi_k(T) \leqslant 4$ and $\phi'_k(T) \leqslant 4$ for every tree T and $k \geqslant 4$.*

Proof. The proofs for ϕ_k and ϕ'_k are very similar. We carry them both out simultaneously by defining $X(G) := V(G)$ for the ϕ_k proof and $X(G) := E(G)$ for the ϕ'_k proof.

Root T at an arbitrary vertex $r \in V(T)$ and let h be the height of the resulting rooted tree. Let $C = c_o \dots c_h$ be an anagram-free word on four symbols. Colour each $x \in X(T)$ by c_i where i is the distance between x and r. Recall that the distance between an edge, uv, and a vertex, r, is the minimum of $\operatorname{dist}(u,r)$ and $\operatorname{dist}(v,r)$.

Let $P = P_1 \dots P_k$ be a path in T such that $|X(P_i)| = |X(P_j)|$ for all $i,j \in [k]$. Note that $|X(P)|$ is a multiple of k, and that this is the only type of path which can be a k-anagram. Recall that P is a k-anagram if and only if $M(P_i) = M(P_j)$ for all $i,j \in [k]$.

Let $v \in V(P)$ be the unique vertex of P closest to the root, r, of T. If $v \in V(P_i)$ for $i \geqslant 3$, then the colour sequence along P_1P_2 appears in C, so $M(P_1) \neq M(P_2)$. Otherwise, $v \in V(P_i)$ for $i \leqslant 2$, so the colour sequence along P_3P_4 appears in C, implying $M(P_3) \neq M(P_4)$. In each case, P is not a k-anagram. Hence $\phi_k(T) \leqslant 4$ and $\phi'_k(T) \leqslant 4$. □

Theorem 3.13 demonstrates a qualitative change in behaviour as k increases. The case of $k = 3$ is an open problem that sits between bounded and unbounded behaviour on trees. For ϕ_3 we have upper bounds on trees of bounded height and pathwidth due to Theorems 3.7 and 3.10. We prove a similar bound on pathwidth for ϕ'_3 using Dekking's [47] result, $\phi_3(P) \leqslant 3$.

Theorem 3.14. *For every tree T, $\phi'_3(T) \leqslant 4\,\mathrm{pw}(T)$.*

Proof. Trees of pathwidth zero satisfy $\phi'_3(T) \leqslant 4\,\mathrm{pw}(T)$ because they are edgeless. We proceed by induction on m.

Let T be a tree of pathwidth $m+1$. By Lemma 3.8, there exists a path, P, in T such that $\mathrm{pw}(T - V(P)) \leqslant m$. Each component of $T - V(P)$ has pathwidth at most m, so, by induction, each component of $T - V(P)$ can be 3-anagram-free edge-coloured with a shared set of $4m$ colours. We now use four additional colours to colour the remaining edges. Dekking [47] proves $\phi_3(P) = 3$ so we can 3-anagram-free edge-colour P with three colours. The fourth extra colour is used to colour the edges between P and $T - V(P)$.

We now show that this colouring is 3-anagram-free. Let Q be a path in T. If Q is entirely contained within a component of $T - V(P)$ then, by induction, Q is not a 3-anagram. Otherwise Q intersects P. The intersection of Q and P is a 3-anagram-free subpath of P and the colours in $Q \cap P$ occur nowhere else in Q. Therefore Q is not a 3-anagram. □

Note that no similar bound exists for ϕ'_2 because stars have pathwidth 1 and ϕ'_2 is unbounded on stars. A bound on $\phi'_3(T)$ as a function of radius follows from the relation between pathwidth and radius in trees. Also, since $\phi_3(P) \leqslant 3$, the bound in Theorem 3.9 can be improved to $\phi_3(T) \leqslant 3\,\mathrm{pw}(T) + 1$.

Dekking [47] also proves $\phi_4(P) = 2$, and we use both results to improve upon Theorem 3.13 for larger values of k.

Theorem 3.15. *For all $z \geqslant 1$ and $k \geqslant 2z$, if $\phi_z(P) \leqslant y$ for all paths P, then $\phi_k(T) \leqslant y$ and $\phi'_k(T) \leqslant y$ for all trees T.*

Proof. As before, the proofs for ϕ_k and ϕ'_k are very similar. We carry them both out simultaneously by defining $X(G) := V(G)$ for the ϕ_k proof and $X(G) := E(G)$ for the ϕ'_k proof.

Let T be a tree with root r and height h. Let $C = c_o \dots c_h$ be a z-anagram-free word on y symbols. Colour each vertex $x \in X(T)$ by c_i where i is the distance between x and r.

Let $P = P_1 \dots P_k$ be a path in T such that $|X(P_i)| = |X(P_j)|$ for all $i, j \in [k]$. Note that $|X(P)|$ is a multiple of k and that this is the only type of path which can be a k-anagram. Recall that P is a k-anagram if and only if $M(P_i) = M(P_j)$ for all $i, j \in [k]$.

P contains a unique vertex v closest to r. If $v \in V(P_i)$ with $i > z$ then the colour sequence along $P_1 P_2 \dots P_z$ appears in C. In the other case, $v \in V(P_i)$ with $i \leqslant z$, so the colour sequence along $P_{z+1} P_{z+2} \dots P_{2z}$ appears in C. In each case there exist a, b such that $M(P_a) \neq M(P_b)$ because C is z-anagram-free. Therefore P is not a k-anagram. Hence $\phi_k(T) \leqslant y$ and $\phi'_k(T) \leqslant y$. □

Dekking [47] proves $\phi_3(P) = 3$ and $\phi_4(P) = 2$, so Theorem 3.15 implies $\phi_6(T) \leqslant 3$ and $\phi_8(T) \leqslant 2$, for all trees T.

Chapter 4

Anagram-free colourings of graph subdivisions

There are very few classes of graphs which are known to have bounded anagram-free chromatic number, with the examples in Chapter 3 consisting entirely of trees. Many commonly studied classes, such as graphs of bounded degree and planar graphs, are known to have unbounded ϕ. This further motivates the search for a class of graphs with bounded ϕ. To this end, this chapter studies highly subdivided graphs. Subdivisions are of particular interest because $\pi_{\text{sub}}(G) \leqslant 3$, for every graph G [114], which raises the question of whether an analogous result holds for anagram-free colouring, as very few other results about π correspond to similar results for ϕ. Conversely, every highly subdivided graph is, locally, a subdivided star or long path, which suggests that ϕ may be bounded on sufficiently subdivided graphs. The results in this chapter are based on the paper *Anagram-free colourings of graph subdivisions* [132].

For a path P and set of colours C, define P *restricted to* C to be the word $\omega_C(P) := f(v_1)f(v_2)\dots f(v_x)$, where $v_1, v_2, \dots, v_x$ are the vertices in $V_C(P)$, in the order defined by P, and f is the colouring of P. Similarly, for a word $W = w_1w_2\dots w_n$ and set of symbols C, define W *restricted to* C, denoted $\omega_C(W) := f(w_1)f(w_2)\dots f(w_n)$, where $f(w) = w$ if $w \in C$ and $f(w)$ is the empty character otherwise. A path is *even* if it has even order. Define a pair of paths, L and R, to be the *split* of an even path, P, if $P = LR$ and $|V(L)| = |V(R)|$. Note that a coloured path, P, with split, L and R, is an anagram if and only if $M(L) = M(R)$. Equivalently, P is not an anagram if $M_C(L) \neq M_C(R)$ for some set of colours C. We encapsulate this observation in the following lemma.

Lemma 4.1. *A path, P, coloured by C, is an anagram if and only if, for all $C' \subseteq C$, P restricted to C' is an anagram or the empty word.*

Proof. We first prove the forward implication. Let $C' \subseteq C$ be such that $\omega_{C'}(P)$ is nonempty, since the empty case is trivial. Let L and R be the split of P. Note that $M_{C'}(L) = M(\omega_{C'}(L))$ and $M_{C'}(R) = M(\omega_{C'}(R))$. Since P is an anagram,

$$M(\omega_{C'}(L)) = M_{C'}(L) = M_{C'}(R) = M(\omega_{C'}(R)).$$

Therefore $\omega_{C'}(P)$ is an anagram.

To prove the reverse implication take $C' = C$. Then P restricted to C', which is all of P, is an anagram. □

4.1 Subdivisions of trees

In this section we show that ϕ is bounded by a constant on highly subdivided trees. For every vertex, v, in a rooted tree T, define $A_T(v)$ to be the set of ancestors and descendants of v in T. A *branch vertex* is a vertex of a rooted tree with at least two children. The *midedge* of an even path P with split L and R is the edge of P not contained in L or R.

4.1.1 Upper bounds

We start with a result for binary trees.

Theorem 4.2. *Every binary tree, T, of height h, has a ($\leqslant 3^{h-1} - 1$)-subdivision, S, with $\phi(S) \leqslant 8$.*

Proof. 2-colour the edges of T with $\{1,2\}$ such that for every branch vertex, $v \in V(T)$, with children c_1 and c_2, the edge vc_i receives colour i. Colour the remaining edges arbitrarily from $\{1,2\}$. Let S be the subdivision of T such that edges at distance x from the root are subdivided $3^{h-x-1} - 1$ times. Note that edges incident to leaves of depth h are not subdivided.

Let r be the root of S. Label the vertices of S according to the edge 2-colouring of T as follows:

- Label division vertices with the colour of the corresponding edge in T.
- Label r with 1.
- Label the original non-root vertices with the label of their parent edge in T.

Let $W = w_1 w_2 \ldots$ be an anagram-free word on $\{1,2,3,4\}$. Define $V_\ell(S)$ to be the set of vertices with label ℓ in S. Colour every vertex $v \in V(S)$ by (ℓ, w_x) where ℓ is the label of v and x is the number of vertices with label ℓ on the vr-path. We now show that this 8-colouring of S is anagram-free.

Let P be an even path in S. Consider the case where there is some $\ell \in \{1,2\}$ such that $V_\ell(P) \subseteq A_S(v)$ for all vertices $v \in V_\ell(P)$. If $V_\ell(P) = \emptyset$, then P is not an anagram because, by construction, S restricted to a label is anagram-free. So now consider $V_\ell(P) \neq \emptyset$ and let $C' = \{\ell\} \times \{1,2,3,4\}$. Then $\omega_{C'}(P) = (\ell, w_y)(\ell, w_{y+1}) \ldots (\ell, w_{y+|V_\ell(P)|})$, for some integer y, because the number of ℓ labelled vertices on the vr-path increments by 1 for all vertices $v \in V(P_\ell)$ along P. Therefore $\omega_{C'}(P)$ is a subword of W so, by Lemma 4.1, P is not an anagram.

Now consider the case where for every $\ell \in \{1,2\}$ there exists a $v \in V_\ell(P)$ such that $V_\ell(P) \nsubseteq A_S(v)$. Let u be the minimum depth vertex in $V(P)$. Both labels have vertices that are not mutual ancestors or descendants so u has two children in T, $x, y \in V(T)$, and, $x, y \in V(P)$.

Partition $V(P)$ into $X := (V(P) \cap A_S(x)) \setminus \{u\}$ and $Y := V(P) \cap A_S(y)$. Let L and R be the split of P such that, without loss of generality, $u \in V(R)$. Also without loss of generality, choose x and y such that $Y \subseteq V(R)$. Note that $Y \cap V(L) = \emptyset$ so

$$|X \cap V(L)| = |V(L)| = |V(R)| = |X \cap V(R)| + |Y \cap V(R)|.$$

Let z be the integer such that $3^z + 1$ is the order of the ux-path in S. We will prove an upper bound on $|X \cap V(R)|$ to show that the midedge of P is 'close' to u. Due to the case under consideration, $V(R) - u$ contains vertices of both labels and $|V(P)| > 3$. Therefore,

y is not an endpoint of P, because y has the label of the division vertices of the uy-path. It follows that

$$|Y \cap V(R)| \geqslant 3^z + 2$$

because the edge uy was subdivided $3^z - 1$ times. $|X|$ is at most the length of a path from u to a leaf so

$$|X \cap V(L)| \leqslant 3^z + 3^{z-1} + \cdots + 3^1 + 1 - |X \cap V(R)| = \frac{3}{2}3^z - \frac{1}{2} - |X \cap V(R)|.$$

Therefore

$$|X \cap V(R)| = |X \cap V(L)| - |Y \cap V(R)| \leqslant \frac{3}{2}3^z - \frac{1}{2} - |X \cap V(R)| - 3^z - 2.$$

Thus

$$|X \cap V(R)| \leqslant \frac{1}{4}3^z - \frac{5}{4}.$$

Without loss of generality, let x have label 1 and y have label 2. Indeed, the labels of x and y are distinct because edges ux and uy have different colours in T. All vertices on the ux-path (except possibly u) have label 1 so

$$|V_1(L)| \geqslant 3^z - |X \cap V(R)| \geqslant \frac{3}{4}3^z + \frac{5}{4}.$$

To put an upper bound on $|V_1(R)|$ assume the worse case, that u has label 1. Then

$$\begin{aligned}|V_1(R)| &\leqslant |X \cap V_1(R)| + 1 + 3^{z-1} + 3^{z-2} + \cdots + 3^1 + 1 \\ &\leqslant \frac{1}{4}3^z - \frac{5}{4} + 1 + \frac{3}{2}3^{z-1} - \frac{1}{2} \\ &= \frac{3}{4}3^z - \frac{3}{4}.\end{aligned}$$

It follows that $|V_1(R)| < |V_1(L)|$, so P is not an anagram. □

The construction in Theorem 4.2 does not extend to a good bound on ϕ for subdivisions of complete d-ary trees. The obvious extension, using d labels for the edge colouring, shows that the complete d-ary tree has a $4d$-colourable subdivision. This is improved upon by the following result for complete d-ary trees.

Theorem 4.3. *The complete d-ary tree, T, of height h, has a $\left(\leqslant 2d(d+1)^{h-1}\right)$-subdivision, S, with $\phi(S) \leqslant 10$.*

Proof. Let r be the root of T. For all $x \in [h]$ and $y \in [d]$, let $t_{x,y} := y(d+1)^{x-1}$. Define the labelling $\ell : E(T) \to [d]$ such that edges incident to the same parent vertex receive distinct labels. Let S be the subdivision of T such that every edge $e \in E(T)$ is subdivided $2t_{h-z,\ell(e)}$ times where z is the depth of e. Note that $z \in \{0, \ldots, h-1\}$.

Let $\ell_T : V(T) \to \{\text{black}, \text{white}\}$ be a proper vertex 2-colouring of T. Define the labelling $\ell_S : V(S) \to \{\text{black}, \text{white}, \text{red}, \text{green}\}$ as follows. If $v \in V(S)$ is an original vertex, then $\ell_S(v) := \ell_T(v)$. Otherwise, let $v' \in V(S)$ be the closest original vertex to v and $e \in E(T)$ be the edge such that v is a division vertex of e. If v' is the vertex of e closest to the root, then $\ell_S(v) :=$ red; otherwise $\ell_S(v) :=$ green. Note that v' is well defined because all

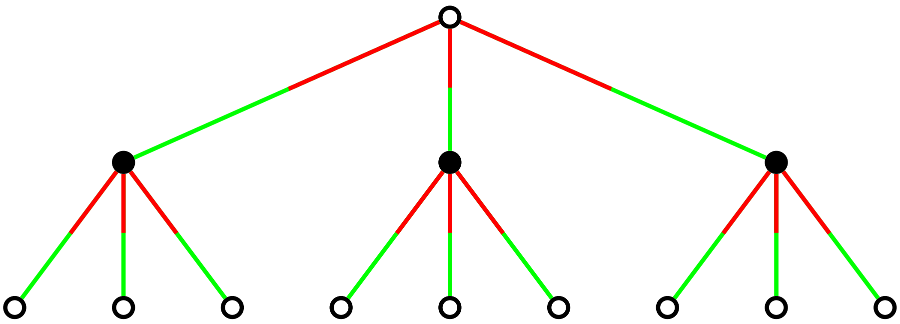


Figure 4.1: S for the complete 3-ary tree of height 2. The edges represent a number of division vertices denoted by their label.

edges of T have an even number of division vertices. See Figure 4.1 for an example of this construction.

Define the *red-depth* of a vertex $v \in V(S)$ to be the number of red vertices on the vr-path in S and define *green-depth* analogously. Let $W = w_1 w_2 \ldots$ be a long anagram-free word on $\{1, 2, 3, 4\}$. Define the vertex colouring f as follows. If $v \in V(S)$ is an original vertex then colour v by $\ell_S(v)$. Otherwise, let i be the $\ell_S(v)$-depth of v and define $f(v) := (w_i, \ell_S(v))$. A vertex has label black or white if and only if it is an original vertex, so f is a 10-colouring of S.

Let P be a path in S, and assume for the sake of contradiction that P is an anagram. P contains at least one division vertex because the original vertices have a proper colouring in T, and all edges not incident to leaves have at least one division vertex. Let u be the vertex with minimum depth in P.

First consider the case where u is an endpoint of P. In this case $V(P) \subseteq A_S(v)$ for all vertices $v \in V(P)$. For subscripts of the functions V and ω, let red $:= \{1, 2, 3, 4\} \times \{\text{red}\}$ and green $:= \{1, 2, 3, 4\} \times \{\text{green}\}$. Without loss of generality, P contains a red division vertex. Therefore the red-depth increments by one along the vertices in $V_{\text{red}}(P)$. It follows that the sequence of colours along the red vertices of P is a subword of W, so $\omega_{\text{red}}(P)$ is not an anagram. Therefore, by Lemma 4.1, P is not an anagram.

The remaining case is where u is not an endpoint of P. In this case, u is an original vertex. For all $e \in E(T)$, let D_e be the division vertices of e. Say that P *hits* an edge $e \in E(T)$ if $D_e \cap V(P) \neq \emptyset$, and that P *contains* e if $D_e \subseteq V(P)$. Let α be the largest edge in T (the edge with most division vertices in S) hit by P, and β be the second largest edge in T hit by P. Since $t_{x,y} > t_{x',y'}$, for all y, y', and $x > x'$ the edges of T are larger nearer the root, so both α and β are adjacent to u. Let v_α and v_β be the endpoints of P denoted such that the uv_α-path hits D_α.

Let $C' = \{\text{red}, \text{green}\} \times \{1, 2, 3, 4\}$ and define W_L, W_α, W_β and W_R so that the concatenation $W_L W_\alpha W_\beta W_R = \omega_{C'}(P)$ and

- W_L is the subword corresponding to the division vertices in $V(uv_\alpha\text{-path}) \setminus D_\alpha$,
- W_α is the subword corresponding to the vertices $V(P) \cap D_\alpha$,
- W_β is the subword corresponding to the vertices $V(P) \cap D_\beta$, and
- W_R is the subword corresponding to the remaining division vertices of P.

Note that each of W_L and W_R may be the empty word.

Let x_α, y_α and y_β be such that $|D_\alpha| = 2t_{x_\alpha,y_\alpha}$ and $|D_\beta| = 2t_{x_\alpha,y_\beta}$. Firstly,

$$|W_L| \leqslant b := 2\sum_{i=1}^{x_\alpha-1} t_{i,d}$$

because $|W_L|$ is at most the number of division vertices on the longest path from the child of α to a leaf of S. Similarly $|W_R| \leqslant b$. For all $x \in [h]$,

$$t_{x,1} = 1 + \sum_{i=1}^{x-1} t_{i,d}$$

because, by induction on x,

$$t_{x,1} = (d+1)t_{x-1,1} = (d+1)\left(1 + \sum_{i=1}^{x-2} t_{i,d}\right) = (d+1) + \sum_{i=2}^{x-1} t_{i,d} = 1 + \sum_{i=1}^{x-1} t_{i,d}.$$

Therefore

$$|D_\alpha| = 2t_{x_\alpha,y_\alpha} \geqslant 2t_{x_\alpha,1} = 2 + 2\sum_{i=1}^{x_\alpha-1} t_{i,d} > b.$$

Similarly $|D_\beta| > b$. Also,

$$|D_\alpha| = 2y_\alpha t_{x_\alpha,1} \geqslant 2y_\beta t_{x_\alpha,1} + 2t_{x_\alpha,1} = 2t_{x_\alpha,y_\beta} + 2 + 2\sum_{i=1}^{x_\alpha-1} t_{i,d} > |D_\beta| + b.$$

Recall that the vertex colouring of T is a proper 2-colouring and that $V(P)$ contains an original vertex. Let L and R be the split of P, with $v_\alpha \in V(L)$. The shortest anagram in a proper 2-colouring has four vertices. Therefore, by Lemma 4.1, both L and R contain at least two original vertices so P contains at least three edges of T. This implies that at least one of W_L and W_R is not the empty word. Thus, at least one of α and β is contained in P.

Consider the case where α is not contained in P. Then $|W_\beta| = |D_\beta| > b \geqslant |W_R|$ so W_R is a subword of $\omega_{C'}(R)$. This implies that L only contains one original vertex, which is a contradiction, so P is not an anagram.

Now consider the case where α is contained in P. Then $|W_\alpha| = |D_\alpha|$. Since exactly half the division vertices of each edge are labelled red,

$$\begin{aligned}
|\omega_{\text{red}}(W_\alpha)| = |\omega_{\text{green}}(W_\alpha)| &= \frac{|W_\alpha|}{2},\\
|\omega_{\text{green}}(W_\beta)| &\leqslant \frac{|W_\beta|}{2},\\
|\omega_{\text{red}}(W_L)| &\leqslant \frac{b}{2},\\
|\omega_{\text{green}}(W_R)| &\leqslant \frac{b}{2}.
\end{aligned}$$

If all vertices corresponding to $\omega_{\text{green}}(W_\alpha)$ are in L, then

$$|\omega_{\text{green}}(L)| \geqslant |\omega_{\text{green}}(W_\alpha)| > \frac{|D_\beta| + b}{2} \geqslant |\omega_{\text{green}}(W_\beta)| + |\omega_{\text{green}}(W_R)| \geqslant |\omega_{\text{green}}(R)|.$$

Thus P is not an anagram. If some vertices corresponding to $\omega_{\text{green}}(W_\alpha)$ are not in L, then all vertices corresponding to $\omega_{\text{red}}(W_\alpha)$ are in R, see Figure 4.1. Therefore,

$$|\omega_{\text{red}}(R)| \geqslant |\omega_{\text{red}}(W_\alpha)| > \frac{b}{2} \geqslant |\omega_{\text{red}}(W_L)| \geqslant |\omega_{\text{red}}(L)|.$$

Thus P is not an anagram. This covers all cases since $v_\alpha \in V(L)$. □

Theorem 4.4 follows as a corollary of Theorem 4.3.

Theorem 4.4. *Every d-ary tree, T, of height h, has a $\left(\leqslant 2d(d+1)^{h-1}\right)$-subdivision, S, with $\phi(S) \leqslant 10$.*

Proof. Apply Theorem 4.3 to the complete d-ary tree of height h, and take the appropriate subgraph of the resulting subdivision. □

In Section 4.1.3 we show that the exponential upper bound on the number of division vertices per edge in Theorem 4.4 is necessary. To achieve this, we first extend a result of Kamčev et al. [83].

4.1.2 Extension of the lower bound for complete binary trees

This subsection extends Theorem 1.5, for complete binary trees, by Kamčev et al. [83]. We generalise their method of proof to obtain a result about subdivisions of high degree trees. The following definitions are extensions of those found in their original paper.

Let T be a rooted tree with root r. The *effective vertices* of T are its leaves and branch vertices. The *effective root* of T is the closest effective vertex to r, including r. The *effective height* of T is the minimum, over the leaves of T, of the number of branch vertices on each root to leaf path. Call T *essentially i-monochromatic* if all of its effective vertices are coloured i. Call T *essentially monochromatic* if it is essentially i-monochromatic for some i. For $d \geqslant 2$, a *d-branch tree* is a rooted tree such that every branch vertex has at least d children.

Lemma 4.5. *For all integers $a_1, \dots, a_c \geqslant 0$ and $d \geqslant 2$, every d-branch tree with vertices coloured by $[c]$ and effective height at least $\sum_{i=1}^{c} a_i$, contains an essentially i-monochromatic d-branch subtree of effective height at least a_i for some $i \in [c]$.*

Proof. We proceed by induction on $\sum_{i=1}^{c} a_i$. The base case, $a_1 = \cdots = a_c = 0$, is satisfied by taking a single vertex as the required d-branch subtree.

Let T be a d-branch tree of effective height $a_1 + \cdots + a_c \geqslant 1$ with vertices coloured by $[c]$. Without loss of generality, its effective root, v, has colour 1. Let $v_1, \dots, v_d$ be children of v. Let T_j be the subtree rooted at v_j. Note that T_j has effective height at least $(a_1 - 1) + a_2 + \cdots + a_c$. If, for some $j \in [d]$ and $i \in \{2, \dots, c\}$, T_j contains an essentially i-monochromatic subtree of effective height a_i then we are done. Otherwise, by induction, each T_j contains an essentially 1-monochromatic d-branch subtree of effective height $a_1 - 1$. These subtrees, together with v, are an essentially 1-monochromatic d-branch subtree of T, as required. □

We now prove a lower bound on ϕ by using an essentially monochromatic subtree to find anagrams in sufficiently large trees.

Theorem 4.6. *Let T be a d-branch tree of effective height at least h' and height at most h. If $h \geqslant \max\{2, \sqrt{d}\}$, then*

$$\phi(T) \geqslant c := \sqrt{\frac{h'}{\log_d h}}\ .$$

Proof. If $c \leqslant 1$, the theorem follows trivially, so assume $c > 1$. Let T be coloured with x colours, where $1 \leqslant x \leqslant c-1$. Our goal is to show that T contains an anagram. For $i \in [x]$, define $a_i \in \{\lfloor h'/x \rfloor, \lceil h'/x \rceil\}$ such that $\sum_{i=1}^{x} a_i = h'$. By Lemma 4.5, and without loss of generality, T contains an essentially 1-monochromatic d-branch subtree, S, of effective height at least $\lfloor h'/x \rfloor$.

Let r be the root of S. There are at least $d^{\lfloor h'/x \rfloor}$ paths from r to the leaves of S, and the colouring of each path has a colour multiset of order at most $h+1$. Since each path shares the colour of r, there are at most h^x distinct multisets that can occur on the paths. Since $x \leqslant c-1$,

$$\#\text{multisets} \leqslant h^x < h^{(c^2/x)-2}.$$

Since $h \geqslant \sqrt{d}$

$$h^{(c^2/x)-2} \leqslant \frac{1}{d} h^{(c^2/x)}.$$

Therefore

$$\#\text{multisets} < \frac{1}{d} h^{(c^2/x)} = \frac{1}{d} \left(h^{\frac{1}{\log_d h}} \right)^{(h'/x)} = d^{(h'/x)-1} \leqslant d^{\lfloor h'/x \rfloor} \leqslant \#\text{paths}.$$

So there is a multiset that occurs on two different paths, P_1 and P_2, from r to the leaves of S. Let v be the lowest common vertex of P_1 and P_2, and let ℓ_i be the leaf endpoint of P_i. By definition, $M(P_1) = M(P_2)$ so $M(P_1 - P_2) = M(P_2 - P_1)$. Since S is essentially 1-monochromatic, the vertices v, ℓ_1, and ℓ_2 have colour 1 so $((P_1 - P_2) \setminus \{\ell_1\})((P_2 - P_1) \setminus \{v\})$ is an anagram. □

4.1.3 Optimality of the number of division vertices

To investigate the optimality, in terms of division vertices per edge, of the subdivision in Theorem 4.3, we consider the c-colourable k-subdivisions of the complete d-ary tree of height h. We start with an upper bound on ϕ for $(\leqslant k)$-subdivisions.

Corollary 4.7. *For every $k \geqslant 0$ and every complete d-ary tree of height h', T, there exists a $(\leqslant k)$-subdivision, S, such that*

$$\phi(S) \leqslant c := 10 \left\lceil \frac{h'}{\log_{d+1}(k/2d)} \right\rceil.$$

Proof. Let $x := c/10$ and let $B \subseteq E(T)$ be the set of edges with depths $i\lceil h'/x \rceil - 1$ for $i \in \{0, \dots, x-1\}$, recalling that the depth of an edge is the minimum depth of its endpoints. Let $F := T - B$ and note that F is a forest where each component is a complete d-ary tree of height at most $\lceil h'/x \rceil$. Let $\mathcal{C}$ be the set of components of F. Root each component, $C \in \mathcal{C}$, at the vertex $r \in V(C)$ with minimum depth in T. The depth of r is $i\lceil h'/x \rceil$ for some $i \in \{0, \dots, x-1\}$. Define the *depth* of C to be i.

By the definition of c and x,

$$\log_{d+1}\left(\frac{k}{2d}\right) \geqslant \frac{h'}{x}.$$

This implies

$$k \geqslant 2d(d+1)^{\frac{h'}{x}} \geqslant 2d(d+1)^{\lceil \frac{h'}{x} \rceil - 1}.$$

Therefore, by Theorem 4.3, for every $C \in \mathcal{C}$, there exist a $(\leqslant k)$-subdivision, S_C, with $\phi(S_C) \leqslant 10$, because C has height at most $\lceil h'/x \rceil$. Anagram-free colour S_C using colours $\{10i+1, \dots, 10(i+1)\}$ where i is the depth of C. Let $S = B + \cup_{C \in \mathcal{C}} S_C$. Note that S is a $(\leqslant k)$-subdivision of T with a $10x$ colouring. We now show that this colouring of S is anagram-free.

Let P be a subpath of S. Let $j \in \{0, \dots, x-1\}$ be the minimum depth of component $C \in \mathcal{C}$ such that S_C has non-empty intersection with P. By the construction of S, P intersects exactly one $C' \in \mathcal{C}'$ of depth j. Therefore, P restricted to the colours of C' corresponds to a subpath of C' and, since C' is anagram-free, the restriction is not an anagram. Therefore, by Lemma 4.1, P is not an anagram. □

The following lemma generalises results for $(\leqslant k)$-subdivisions to k-subdivisions. Note that the k-subdivision a graph, G, is a subdivision of every $(\leqslant k)$-subdivision of G.

Lemma 4.8. *For every subdivision, S, of a graph G, $\phi(S) \leqslant \phi(G) + 4$.*

Proof. Fix an anagram-free $\phi(G)$-colouring of G and apply the colouring to the original vertices of S. The graph induced by the division vertices of S is a forest of paths. Colour all of these paths with an anagram-free colouring on four new colours. By Lemma 4.1, this colouring of S is anagram free. □

We combine the results of this section with those of Section 1.3.4 in the following upper and lower bounds.

Theorem 4.9. *The k-subdivision, S, of the complete d-ary tree of height h satisfies*

$$\sqrt{\frac{h}{\log_{\min\{d,(h(k+1))^2\}}(h(k+1))}} \leqslant \phi(S) \leqslant \frac{10h}{\log_{d+1}(k/2d)} + 14.$$

Proof. We first show the lower bound. The effective height of S is h because division vertices are not branch vertices. The height of S is $h(k+1)$ because there are h edges on root-leaf paths of complete trees of height h, and each edge has hk division vertices. If $\sqrt{d} \leqslant h(k+1)$, then the bound follows from Theorem 4.6. In the case $\sqrt{d} > h(k+1)$, we use Theorem 4.6 on a subtree of S with maximum degree $(h(k+1))^2$.

The upper bound follows from Corollary 4.7 and Lemma 4.8. □

Consider the bounds Theorem 4.9, restated in terms of k, for $d \leqslant (h(k+1))^2$, as follows

$$h^{-1}d^{h/\phi(S)^2} - 1 \leqslant k \leqslant 2d(d+1)^{10h/(\phi(S)-14)}. \tag{4.1}$$

For fixed d and $\phi(S)$, the upper and lower bounds on k are both dominated by exponentials in h. This shows that the subdivision and colouring in Theorem 4.3 is close to optimal.

4.2 Subdivisions of general graphs

In this section we construct subdivisions, of arbitrary graphs, with bounded anagram-free chromatic number. Let $t = t_1, t_2, \ldots$ be a sequence of positive integers. A subdivision, S, of a graph G, is a *t-sequence-subdivision of* G if there is a bijection, $\ell : V(G) \to [|V(G)|]$, that satisfies the following two conditions. The first condition is that there is a proper 2-colouring of G, with colours white and black, such that $\ell(u) > \ell(v)$ for every white vertex $u \in V(G)$ and black vertex $v \in V(G)$. The second condition is as follows. For every edge, $e \in E(G)$, define $w(e)$ to be the white vertex incident to e, and $b(e)$ to be the black vertex incident to e. Define the bijection, $\ell' : E(G) \to [|E(G)|]$, that orders edges in $E(G)$, first by the label of their white endpoint and second by the label of their black endpoint. That is, $\ell'(x) > \ell'(y)$ for edges $x, y \in E(G)$ if $\ell(w(x)) > \ell(w(y))$ or if $\ell(w(x)) = \ell(w(y))$ and $\ell(b(x)) > \ell(b(y))$. Note that ℓ' is determined entirely by ℓ. Now, the second condition on ℓ is that every edge, $e \in E(G)$, has $3t_{\ell'(e)}$ division vertices.

Let G be a graph, t be a sequence of positive integers, and S be a t-sequence-subdivision of G with corresponding vertex and edge labellings ℓ and ℓ'. Define functions X, Y, and Z such that for every edge, $e \in E(G)$, $X(e)$, $Y(e)$, and $Z(e)$ are pairwise disjoint paths in the division vertices of e with $|V(X(e))| = |V(Y(e))| = |V(Z(e))| = t_{\ell'(e)}$, $X(e)$ adjacent to the white end of e, and $Z(e)$ adjacent to the black end of e. Define the sets of these paths, $\mathcal{X} := X(E(G))$, $\mathcal{Y} := Y(E(G))$, and $\mathcal{Z} := Z(E(G))$. A vertex colouring of S is *discriminating* if the following conditions hold.

(1) The original vertices of S are coloured by the proper 2-colouring of G, determined by ℓ, and these two colours only occur on the original vertices.

(2) Every anagram in S contains at least one original vertex.

(3) For all $Q \in \{X, Y, Z\}$ there exists a nonempty set of colours, $C(Q)$, that occur only on the vertices of paths in $Q(E(G))$.

(4) For all $Q \in \{X, Y, Z\}$ and $q \in E(G)$,

$$\sum_{e \in E(G) : \ell'(e) < \ell'(q)} |V_{C(Q)}(Q(e))| < |V_{C(Q)}(Q(q))|.$$

Note that whether S has a discriminating vertex colouring depends on the sequence t. For example, the sequence $t_i = 1$, for all i, causes Condition (4) to fail for sufficiently large G.

Theorem 4.10. *Let S be a t-sequence-subdivision of a graph G with sequence t. Every discriminating vertex colouring of S is anagram-free.*

Proof. Let ℓ and ℓ' be the associated vertex and edge labellings of G. Let f be a discriminating vertex colouring of S.

Let P be a path in S and assume for the sake of contradiction that P is an anagram. By Condition (2), $V(P)$ contains at least one original vertex. Since G is properly 2-coloured, all subpaths of G that are anagrams have order at least 4. The 2-colouring of G is applied to the original vertices of S, so, by Lemma 4.1, P contains at least four original vertices. Therefore P has at least one subpath from each of $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$. Let $x, y, z \in E(G)$ be the edges maximizing ℓ' such that $V(P) \cap V(X(x)) \neq \emptyset$, $V(P) \cap V(Y(y)) \neq \emptyset$, and $V(P) \cap V(Z(z)) \neq \emptyset$.

A path, P', *partially intersects* P if $V(P') \nsubseteq V(P)$ and $V(P') \cap V(P) \neq \emptyset$. There are at most two paths in $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ that partially intersect P since every division vertex has

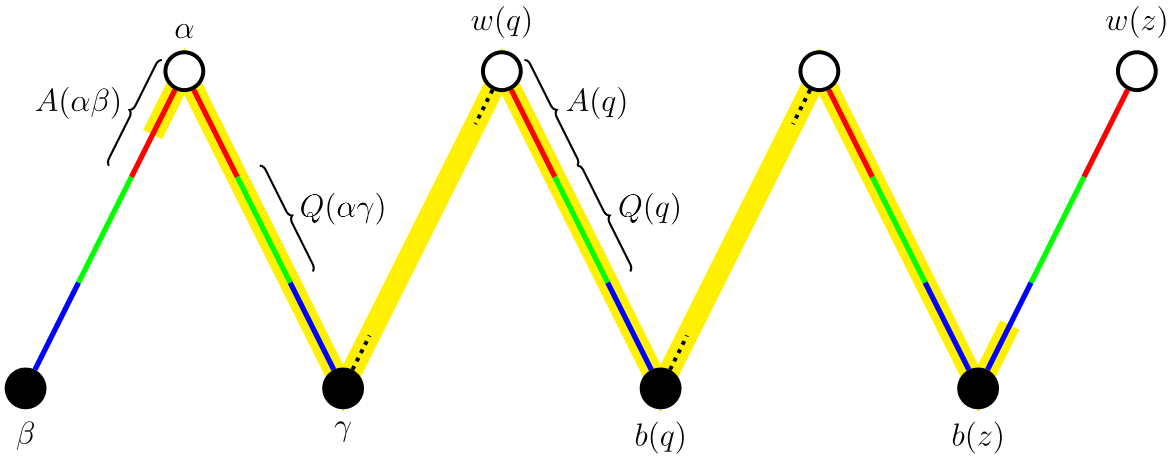


Figure 4.2: The path, P, in the case $A = X$ and $Q = Y$. Correspondingly, $\alpha\beta = x$ and $q = y$. The coloured lines correspond to division vertices, with the vertices in P highlighted in yellow. The contradiction is reached by noting that $|A(\alpha\beta)| > |A(q)|$ and $|Q(q)| > |Q(\alpha\gamma)|$.

degree 2 in S. Therefore at least one of $X(x)$, $Y(y)$, and $Z(z)$ is a subpath of P. Define $q \in \{x, y, z\}$ and $Q \in \{X, Y, Z\}$ such that $Q(q) \in \{X(x), Y(y), Z(z)\}$ is a subpath of P. Since f is a discriminating colouring

$$\sum_{e \in E(G): \ell'(e) < \ell'(q)} |V_{C(Q)}(Q(e))| < |V_{C(Q)}(Q(q))|.$$

Therefore, by the maximality of $\ell'(q)$, there are more vertices in $Q(q)$ coloured by $C(Q)$ than there are vertices coloured by $C(Q)$ in the rest of P. Thus $|V_{C(Q)}(Q(q))| > \frac{1}{2}|V_{C(Q)}(P)|$. Let L and R be the split of P. By Lemma 4.1, $V_{C(Q)}(L) = V_{C(Q)}(R) = \frac{1}{2}|V_{C(Q)}(P)|$ so both L and R intersect $Q(q)$. Therefore the midedge of P is an edge of $Q(q)$. Since the midedge of P is unique, exactly one of $X(x)$, $Y(y)$, and $Z(z)$ is a subpath of P.

Since G is properly 2-coloured, every subpath of G that is an anagram has a white endpoint and a black endpoint. Therefore one of the endmost original vertices of P is white, call this vertex α. Since P partially intersects exactly two of $X(x)$, $Y(y)$, and $Z(z)$, there is a black vertex $\beta \in N_G(\alpha)$ such that $\alpha\beta \in \{x, y, z\}$, where $N_G(\alpha)$ is the neighbourhood of α in G. Recall that both L and R contain at least two original vertices and the midpoint of P is in $Q(q)$. Therefore neither endpoint of q is an endmost original vertex of P, so $\alpha \neq w(q)$. Also, there is a black vertex, $\gamma \in N_G(\alpha)$, such that the division vertices of $\alpha\gamma$ are all in P. Since $\alpha\beta \in \{x, y, z\}$ and $\alpha\beta \neq q$, there is an $A \in \{X, Y, Z\}$ such $A(\alpha\beta) \in \{X(x), Y(y), Z(z)\}$, for some $A \neq Q$. See Figure 4.2 for a potential assignment of q, Q, and A. Now, $\ell'(\alpha\beta) > \ell'(q)$ because $A(q)$ is a subpath of P and $\alpha\beta$ maximises $\ell'(\alpha\beta)$ over edges, e, with a path, $A(e)$, which intersect P. Therefore $\ell(\alpha) > \ell(w(q))$, so $\ell'(\alpha\gamma) > \ell'(q)$. This contradicts the maximality of $\ell'(q)$ because $Q(\alpha\gamma)$ is a subpath of P. □

Theorem 4.11 follows from Theorem 4.10.

Theorem 4.11. *Every graph G has a $\big(\leqslant 6(2)^{2|E(G)|-1} - 1\big)$-subdivision, S, with $\phi(S) \leqslant 14$.*

Proof. Let G' be the 1-subdivision of G, and note that G' has a proper 2-colouring. Define the sequence t by $t_i = 2^{i-1}$ for $i \geqslant 1$. Let S be a t-sequence-subdivision of G'. Note that

G' has $2|E(G)|$ edges, $3t_{2|E(G)|} = 3(2)^{2|E(G)|-1}$, and each edge of G corresponds to a pair of edges of G'. It follows that S satisfies the bound on division vertices per edge required by the theorem. Let ℓ and ℓ' be the associated vertex and edge labellings of G'.

Let f be the vertex colouring of S defined as follows. Colour the original vertices of S with the proper 2-colouring of G' that corresponds to ℓ. Assign a disjoint set of four colours to each of $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$. Colour each of the paths in $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ with an anagram-free 4-colouring with their assigned set of four colours.

We now show that f is discriminating. Conditions (1) and (3) are satisfied trivially. Condition (2) is satisfied because each of the paths in $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ is anagram-free and they use their own set of colours so every anagram in S contains an original vertex. Condition (4) is satisfied because for all $Q \in \{X, Y, Z\}$ and $q \in E(G')$, $|V_{C(Q)}(Q(q))| = |V(Q(q))|$, and

$$\sum_{e\in E(G'):\ell'(e)<\ell'(q)} |V(Q(e))| = 2^{\ell'(q)-2} + \cdots + 1 = 2^{\ell'(q)-1} - 1 < 2^{\ell'(q)-1} = |V(Q(q))|.$$

Therefore f is an anagram-free 14-colouring of S. □

Theorem 4.11 can be extended to bound ϕ on subdivisions of graphs as a function of division vertices per edge.

Theorem 4.12. *For every graph G and $k \in \mathbb{Z}^+$ there exists a ($\leqslant 6(4)^{\lceil |E(G)|/k\rceil}$)-subdivision, S, of G with $\phi(S) \leqslant 2 + 12k$.*

Proof. Let $G_1, \dots, G_k$ be subgraphs that partition the edges of G such that $V(G_i) = V(G)$ and $|E(G_i)| \leqslant \lceil |E(G)|/k\rceil$, for all $i \in [k]$. Let G' be the 1-subdivision of G and fix the proper 2-colouring of G' which assigns the same colour to each division vertex of G'. Let G'_i be the 1-subdivision of G_i, for all $i \in [k]$. Apply the proof of Theorem 4.11 to obtain a subdivision, S_i, of each G'_i, skipping the 1-subdivision and 2-colouring steps because G'_i is already a bipartite graph with a proper 2-colouring inherited from G'. Note that $V(G'_i) \subseteq V(S_i)$ and $V(S_i) \cap V(S_j) = V(G')$, $\forall i, j \in [k]$. Let S be the union of $S_1, \dots, S_k$ and, since S is a subdivision of G', we call $V(G')$ the original vertices of S. By this construction, the 2-colouring of the original vertices of S is consistent across $S_i, \dots, S_k$. The anagram-free 14-colouring provided by the proof of Theorem 4.11 uses two colours for the original vertices and 12 colours for the division vertices. It follows that we may combine the colourings of $S_1, \dots, S_k$ to obtain a $(2+12k)$-colouring of S, where the division vertices of each S_i are coloured by a distinct set of 12 colours. We now show that this colouring is anagram-free.

Let P be a path in S. First consider the case where P is entirely contained in S_i, for some $i \in [k]$, or P contains fewer than four original vertices. By Theorem 4.11 P is not an anagram, so we are done.

In the remaining case, and without loss of generality, P contains an entire subdivision edge of S_1 and contains division vertices of S_2. Since the colouring of S_1 is discriminating, we have a partition of the division vertices of S_1 into sets of paths, denoted $\mathcal{X}_1$, $\mathcal{Y}_1$, and $\mathcal{Z}_1$. The same can be said of S_2, and we denote its corresponding sets of paths $\mathcal{X}_2$, $\mathcal{Y}_2$, and $\mathcal{Z}_2$. Because P contains a subdivision edge of S_1, P intersects paths from each of $\mathcal{X}_1$, $\mathcal{Y}_1$, and $\mathcal{Z}_1$. Also, P intersects at least one path from $\mathcal{W} \in \{\mathcal{X}_2, \mathcal{Y}_2, \mathcal{Z}_2\}$. Due to the structure of S, P partially intersects at most two paths from $\mathcal{X}_1 \cup \mathcal{Y}_1 \cup \mathcal{Z}_1 \cup \mathcal{W}$. It follows that there are two sets of paths, $\mathcal{A}, \mathcal{B} \in \{\mathcal{X}_1, \mathcal{Y}_1, \mathcal{Z}_1, \mathcal{W}\}$, which do not contain any paths that partially intersect P. As shown in the proof ofTheorem 4.10, for P to be an anagram its midpoint must be within a path in $\mathcal{A}$. The same holds for $\mathcal{B}$, and P has a unique midpoint, so P is not an anagram. □

We now apply Theorem 4.10 with the goal of lowering the bound on ϕ for sufficiently subdivided graphs.

Theorem 4.13. *Every graph G has a $\left(\leqslant 90\left(\frac{75}{9}+1\right)^{2|E(G)|-1}\right)$-subdivision, S, with $\phi(S) \leqslant 8$.*

Proof. Let G' be the 1-subdivision of G, and note that G' has a proper 2-colouring. Define the sequence t with $t_1 = 8$ and

$$t_n = 15 + \left\lfloor \frac{25}{3}\sum_{i=1}^{n-1} t_i \right\rfloor. \tag{4.2}$$

Let S be a t-sequence-subdivision of G'. It is straightforward to verify that $t_n \leqslant 15\left(1+\frac{75}{9}\right)^{n-1}$ so S satisfies the limit on division vertices per edge required by the theorem. Let ℓ and ℓ' be the associated vertex and edge labellings of G'.

Define the colouring $f : V(S) \to \{1,2,3,4,5,6,\text{white},\text{black}\}$ as follows. Original vertices are coloured white or black according to ℓ. For every $e \in E(G')$, define $P_e = v_1 \dots v_{3t_{\ell'(e)}}$ to be the division vertices of e. Let W be an anagram-free word on $\{1,2,3,4\}$ of length $3\ell'(e)$ and colour P_e as follows. For all $v_i \in V(P_e)$, if $W_i \in \{1,2,3\}$ then $f(v_i) := W_i$. Otherwise, $f(v_i) := 4$ if $v_i \in V(X(e))$, $f(v_i) := 5$ if $v_i \in V(Y(e))$, and $f(v_i) := 6$ if $v_i \in V(Z(e))$.

We now show that f is discriminating. Condition (1) is satisfied trivially. Condition (2) is satisfied because P_e is coloured by an anagram-free word for all $e \in E(G')$. Condition (3) is satisfied by $C(X) = \{4\}$, $C(Y) = \{5\}$, and $C(Z) = \{6\}$. We now show that Condition (4) is satisfied.

Let $Q \in \{X, Y, Z\}$ and $q \in E(G')$. The same symbol cannot occur twice in a row so $|V_{C(Q)}(Q(q))| \leqslant \frac{5}{9}|V(Q(q))|$, since $|V(Q(q))| \geqslant 8$. Therefore

$$\sum_{e\in E(G'):\ell'(e)<\ell'(q)} |V_{C(Q)}(Q(e))| \leqslant \frac{5}{9} \sum_{e\in E(G'):\ell'(e)<\ell'(q)} |V(Q(e))|.$$

Every anagram-free word of length 8 contains at least four distinct symbols. Therefore $|V_{C(Q)}(Q(q))| \geqslant \frac{1}{15}|V(Q(q))|$. By (4.2)

$$\frac{5}{9} \sum_{e\in E(G'):\ell'(e)<\ell'(q)} |V(Q(e))| = \frac{5}{9}\sum_{i=1}^{n-1} t_i < \frac{1}{15}\left(1 + \left\lfloor \frac{25}{3}\sum_{i=1}^{n-1} t_i \right\rfloor\right) = \frac{1}{15}t_n - \frac{14}{15}.$$

Therefore

$$\sum_{e\in E(G'):\ell'(e)<\ell'(q)} |V_{C(Q)}(Q(e))| \leqslant \frac{5}{9}\sum_{i=1}^{n-1} t_i < \frac{1}{15}t_n = \frac{1}{15}|V(Q(e))| \leqslant |V_{C(Q)}(Q(q))|.$$

Thus Condition (4) is satisfied so f is an anagram-free 8-colouring of S. □

Theorem 4.13 uses simple bounds on the density of symbols in anagram-free words. Better bounds on density would improve the base of the bound in Theorem 4.13.

4.2.1 Subdivisions of complete graphs

Nešetřil et al. [110] study π on subdivisions of the complete graph, and prove Theorem 1.3, which implies, $k \geqslant \log_c(n/2) - 1$ for every anagram-free c-colourable k-subdivision of K_n. We improve upon this bound with Theorem 4.14. Recall that $\mathcal{M}_{k,c}$ is the set of colour multisets on c symbols of size k, and that $\mathcal{M}_{\leqslant k,c}$ is the set of colour multisets of c symbols of size at most k.

Theorem 4.14. *Let S be a $(\leqslant k)$-subdivision of K_n. If S is anagram-free c-colourable then*

$$k \geqslant \left(c!\left(\frac{n}{c} - 1\right)\right)^{1/c} - c.$$

Proof. Suppose for the sake of contradiction that

$$k < \left(c!\left(\frac{n}{c} - 1\right)\right)^{1/c} - c. \tag{4.3}$$

Fix an anagram-free colouring of S. Colour each edge $e \in E(K_n)$ with the colour multiset of the subdivision vertices of e in S and colour each vertex of K_n with its colour in S. Note that there are

$$|\mathcal{M}_{\leqslant k,c}| = \sum_{i=0}^{k} \binom{i+c-1}{c-1} = \binom{k+c}{c} \leqslant \frac{(k+c)^c}{c!}$$

possibilities for the colour of each edge. Let $x := \lceil n/c \rceil$, and let G be a vertex-monochromatic K_x subgraph of K_n. Note that

$$|E(G)| = \frac{x}{2}(x-1) \geqslant \frac{x}{2}\left(\frac{n}{c} - 1\right).$$

Therefore, by (4.3),

$$|E(G)| \geqslant \frac{x}{2}\left(\frac{n}{c} - 1\right) > \frac{x}{2}\frac{(k+c)^c}{c!} \geqslant \frac{x}{2}|\mathcal{M}_{\leqslant k,c}| \geqslant \frac{x}{2}\#\text{colours}.$$

So there is a set of more than $x/2$ edges that have the same colour. Therefore there is a vertex, $v \in V(G)$, that is incident to at least two edges, $\alpha, \beta \in E(G)$, with the same colour. Let u be the other endpoint of α, P_α be the path induced by the division vertices of α, and P_β be the path induced by the division vertices of β. Then $uP_\alpha vP_\beta$ is an anagram in S. □

For fixed c, the number of division vertices in Theorem 4.14 grows as the c^{th} root in the number of edges of K_n. This is an improvement on the logarithmic bound from square-free colouring, however, the number of division vertices in the construction in Theorem 4.13 grows exponentially in the number of edges in K_n. The question of whether the number of division vertices per edge in an anagram-free c-colourable subdivision of K_n grows polynomially or exponentially is open.

Chapter 5

Colour schemes

This chapter proposes an axiomatic approach to graph colouring, as a framework in which to prove results that apply to many existing variants of graph colouring. The focus is entirely on vertex colourings of simple graphs, but this system could be reformulated for edge colouring, as well as other generalisations of graph colouring. We discuss many *variants of graph colouring* as well as particular *graph colourings*. To avoid the confusion that may arise from these similar phrases, we define a *coloured graph* to be a pair, (G, α), consisting of a graph G and colouring $\alpha : V(G) \to C$, where C is a set of colours. A *k-coloured graph* is a coloured graph where $|C| \leqslant k$. Keep in mind that a coloured graph may have any assignment of colours to vertices, it is not necessarily a proper colouring. Our focus is on finite graphs, however, we endeavour to define properties that retain their usefulness when applied to infinite graphs.

The notion of 'variant of graph colouring' is formalised by defining a *colour scheme* to be a set of coloured graphs which satisfies a few natural axioms, to be defined in Section 5.1. Many types of graph colouring found in the literature have a corresponding colour scheme. For example, the colour scheme corresponding to proper vertex colouring, denoted $\mathbf{P}$, contains a coloured graph (G, α) if and only if (G, α) has no monochromatic edges. Every variant of graph colouring corresponds to a set, $\mathbf{A}$, of coloured graphs which are considered valid, and in many cases $\mathbf{A}$ satisfies the colour scheme axioms. Representing a type of graph colouring as a set of coloured graphs has many advantages compared to the more commonly used rule-satisfaction formulation. In the literature, many results are stated in terms of chromatic numbers, which tend to be formulated as a rule that a colouring must satisfy. For example, a graph, G, has $\chi(G) \leqslant k$ if there exists a k-colouring of G with no monochromatic edge. This focus on chromatic numbers works within a variant, but is not suited to making general statements about multiple variants of graph colouring. With colour schemes, it is easier to compare multiple variants of graph colouring since we are able to use the tools and notation of set theory. A similar idea is used in the field of graph properties, which treats properties of graphs, such as being bipartite or planar, as sets of graphs closed under isomorphism, see Section 1.4.3. The new idea central to colour schemes is that every type of graph colouring can be formulated as a boolean property of coloured graphs.

A major goal of colour schemes is to allow for proofs of general results that apply to many variants of graph colouring. This is motivated by a concern that many results in graph colouring are specific to a particular colour scheme and tend not to be easily translatable to colour schemes with similar properties. Translating a proof between colour schemes often requires delving into the proof, taking the core insight, and rewriting it in terms of the new colour scheme. A more ideal situation would be to first identify the general properties that are required for the core insight and then to write the original proof in terms of general properties. To apply a proof to a new colour scheme one would only need to prove that the colour scheme satisfies the general properties required by the proof.

This chapter first defines and motivates the colour scheme axioms. We then derive many of the standard tools and operations of graph colouring from these axioms. The chapter ends by showing that many variants of graph colouring found in the literature are colour schemes. Chapter 6 investigates particular types of colour schemes, with a focus on extending the results about anagram-free colouring from Chapters 3 and 4. This investigation involves proving results about colour schemes that satisfy general properties, as well as the construction of new colour schemes with novel properties as answers to questions raised by the comparison of anagram-free colouring to square-free colouring.

5.1 Colour scheme axioms

We first define colour schemes and prove that many tools used in graph colouring follow from their axioms. A *colour scheme* is a set of coloured graphs, $\mathbf{A}$, that is closed under isomorphism and satisfies the following axioms.

Subgraphability : $\mathbf{A}$ is closed under taking coloured subgraphs. Formally, if $(G, \alpha) \in \mathbf{A}$ and H is a subgraph of G then $(H, \alpha|_H) \in \mathbf{A}$, where $\alpha|_H := \alpha|_{V(H)}$.

Recolourability : $\mathbf{A}$ is closed under fracturing. A coloured graph, (G, β), is a *fracture* of (G, α) if, for all vertices, $u, v \in V(G)$, $\beta(u) = \beta(v)$ implies $\alpha(u) = \alpha(v)$.

Universality : Every graph G has a colouring α such that $(G, \alpha) \in \mathbf{A}$.

Additivity : $\mathbf{A}$ is closed under taking disjoint unions of coloured graphs.

Locality : Every coloured graph $(G, \alpha) \notin \mathbf{A}$ has a finite subgraph not in $\mathbf{A}$.

These axioms intend to capture the notion of graph colouring in the sense of pattern avoidance or conflict minimization. Furthermore, we require that the patterns to be avoided are local and finite, in the sense that two vertices can only be in conflict if the distance between them is finite. The greatest justification for these axioms is that they work, as demonstrated by the results obtained in the following chapters and sections. However, to communicate some intuition, we give an overview of the justification for each axiom. In many cases this amounts to considering the degenerate behaviour that each axiom prevents.

The intuition behind Subgraphability is that the removal of vertices or edges from a graph with a valid colouring should not create the types of patterns that are avoidable by our notion of graph colouring. The following example provides a more concrete justification for Subgraphability. A *rainbow colouring of* G is a colouring, α, of G which assigns a unique colour to each vertex of G. Let $\mathbf{A}$ be a set of coloured graphs such that $(G, \alpha) \in \mathbf{A}$ if $|V(G)|$ is even or (G, α) contains a connected component with a rainbow colouring. $\mathbf{A}$ satisfies all of the axioms besides Subgraphability. However, the behaviour of $\mathbf{A}$, when restricted to finite, connected graphs, is determined by whether a graph has an even or odd number of vertices. This behaviour does not match our notion of graph colouring.

Recolourability has a role similar to that of Subgraphability. The intuition here is that conflicts are always caused by sets of monochromatic vertices. It follows that we can reassign colours in a validly coloured graph provided that no conflicts are created between vertices which were previously not in conflict. Without this axiom, the set of coloured graphs, $\mathbf{A}$, such that $(G, \alpha) \in \mathbf{A}$ if each of its connected components is coloured by at most x colours would be a colour scheme, for all $x \in \mathbb{Z}^+$. $\mathbf{A}$ is just counting the number of colours in a graph, so does not satisfy our notion of local pattern avoidance. Recolourability has many uses, including allowing us to take product colourings to combine colour schemes.

Universality ensures that every colour scheme has a notion of a chromatic number defined on every graph. The $\mathbf{A}$-*chromatic number*, $\chi_{\mathbf{A}}(G)$, of a graph, G, is the smallest integer k such that there exists a colouring, $\alpha : V(G) \to [k]$, with $(G, \alpha) \in \mathbf{A}$. Additivity and Locality enforce the notion that graph colouring is about avoiding conflicts on finite sets of vertices which are all within a finite distance of each other. Without Additivity, we could construct a colour scheme which essentially counts the number of connected components of a graph. Without Locality, the set of coloured finite graphs would be a colour scheme. In further support of Locality, de Bruijn and Erdős [46] prove that, for an infinite graph G and $k \in \mathbb{Z}^+$, if every finite subgraph, H, of G satisfies $\chi(H) \leqslant k$, then $\chi(G) \leqslant k$.

5.1.1 Subsets and subgraphs

The field of graph colouring has operations and tools that are used in many variants of graph colouring. The axioms of colour schemes are designed to ensure these tools are applicable to colour schemes. We first establish that chromatic numbers behave as expected, with respect to taking subgraphs.

Lemma 5.1. *Let G be a graph and $\mathbf{A}$ be a colour scheme. Then $\chi_{\mathbf{A}}(H) \leqslant \chi_{\mathbf{A}}(G)$ for all subgraphs H, of G.*

Proof. Firstly, $\chi_{\mathbf{A}}(G) = k$ implies that there exists $\alpha : V(G) \to [k]$ with $(G, \alpha) \in \mathbf{A}$. By Subgraphability, $(H, \alpha|_H) \in \mathbf{A}$. When α is restricted to H it still has codomain $[k]$, so $\chi_{\mathbf{A}}(H) \leqslant k$. □

The next lemma describes a notion that holds for many variants of graph colouring.

Lemma 5.2. *Let G be a graph and $\mathbf{A}$ be a colour scheme. For all finite graphs G, if $\chi_{\mathbf{A}}(G) = k$ then there exists $(G, \alpha) \in \mathbf{A}$ such that $|\alpha(V(G))| = x$, for all $x \in \{k, k+1, \dots, |V(G)|\}$.*

Proof. Let (G, α) be a k-coloured graph where each of the k colours occurs at least once. Let $U \in V(G)$ be a maximal set of vertices such that every $v \in U$ has a $u \notin U$ such that $\alpha(u) = \alpha(v)$, and note that $|U| = |V(G)| - k$. Let (G, β) be a recolouring of (G, α) such that $x - k$ of the vertices of U are given distinct new colours. By Recolourability, $(G, \beta) \in \mathbf{A}$, and (G, β) uses exactly x colours. □

By Lemma 5.2 and Universality, every colour scheme contains all rainbow coloured graphs. The rainbow colouring of a graph is unique up to relabelling colours, so we often refer to 'the rainbow colouring of G'.

Lemma 5.3. *Let α be the rainbow colouring of G. For all colour schemes $\mathbf{A}$, $(G, \alpha) \in \mathbf{A}$.*

Additivity enforces the notion of local pattern avoidance, but only does so in one direction. The reverse direction, that every connected component of a coloured graph in a colour scheme is also in the colour scheme, is not part of Additivity because it is implied by Subgraphability.

Lemma 5.4. *Let* $\mathbf{A}$ *be a colour scheme. Then* $(G, \alpha) \in \mathbf{A}$ *if and only* $(H, \alpha|_H) \in \boldsymbol{A}$ *for every connected connected component,* H*, of* G*.*

Proof. Let (G, α) be a coloured graph such that $(H, \alpha|_H) \in \mathbf{A}$ for every connected component, H, of G. Note that (G, α) is the disjoint union of each of its connected components. Therefore, by Additivity, $(G, \alpha) \in \mathbf{A}$.

We now show that the conditions are necessary. Let $(G, \alpha) \in \mathbf{A}$ and H be a connected component of G. Note that H is a subgraph of G. Therefore, by Subgraphability, $(H, \alpha|_H) \in \mathbf{A}$. □

Since colour schemes are sets, the standard subset, union, and intersection operations apply to colour schemes in the usual way. If one colour scheme is a subset of another we can infer a relationship between their chromatic numbers. For a set of coloured graphs, $\mathbf{A}$, and set of graphs, $\mathcal{G}$, $\mathbf{A}$ *restricted to* $\mathcal{G}$, is $\mathbf{A}|_\mathcal{G} := \{(G, \alpha) \in \mathbf{A} : G \in \mathcal{G}\}$. The following lemma is written in terms of restrictions of colour schemes, but in many cases we set $\mathcal{G}$ to be the set of all graphs.

Lemma 5.5. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be colour schemes, and let* $\mathcal{G}$ *be a set of graphs, with* $\mathbf{A}|_\mathcal{G} \subseteq \mathbf{B}|_\mathcal{G}$*. Then* $\chi_\mathbf{B}(G) \leqslant \chi_\mathbf{A}(G)$ *for all graphs* $G \in \mathcal{G}$*.*

Proof. Let $G \in \mathcal{G}$ and let $k := \chi_\mathbf{A}(G)$. There exists $\alpha : V(G) \to [k]$ with $(G, \alpha) \in \mathbf{A}|_\mathcal{G}$. Thus $\chi_\mathbf{B}(G) \leqslant k$, because $(G, \alpha) \in \mathbf{B}|_\mathcal{G}$. □

Lemma 5.5 demonstrates a subtle point about the expressiveness of statements about colour schemes as compared to statements about chromatic numbers. The statement '$\mathbf{A} \subseteq \mathbf{B}$' is stronger than '$\chi_\mathbf{B}(G) \leqslant \chi_\mathbf{A}(G)$' because, in general, the backwards implication of Lemma 5.5 does not hold. For example, take square-free colouring, denoted $\mathbf{Q}$, and exact-distance-2 colouring, denoted $\mathbf{E}$. A coloured graph, (G, α), is in $\mathbf{E}$ if and only if the endpoints of every path of length 2 in (G, α) have distinct colours. It is easy to verify that $\chi_\mathbf{E}(P_1) = 1$, $\chi_\mathbf{E}(P_2) = 1$ and $\chi_\mathbf{E}(P_n) = 2$ for all $n \geqslant 3$. The corresponding bounds on $\chi_\mathbf{Q}$ are $\chi_\mathbf{Q}(P_1) = 1$, $\chi_\mathbf{Q}(P_n) = 2$ for $n \in \{2, 3, 4\}$, and $\chi_\mathbf{Q}(P_n) = 3$ for all $n \geqslant 5$. It follows that $\chi_\mathbf{E}(P_n) \leqslant \chi_\mathbf{Q}(P_n)$ for all n. However, since the path with colour sequence 121 is in $\mathbf{Q}$ and not in $\mathbf{E}$, $\mathbf{Q}|_\mathcal{P}$ is not a subset of $\mathbf{E}|_\mathcal{P}$, where $\mathcal{P} = \{P_n : n \geqslant 1\}$.

Set notation provides a natural way to make statements about colour schemes that, in many cases, retain more information than corresponding statements about chromatic numbers. For example, take the following statement about star colouring and proper colouring. Let $\mathbf{P}$ be the colour scheme of all coloured graphs without monochromatic edges and $\mathbf{S}$ be the colour scheme that corresponds to star colouring. Recall that every star colouring is a proper colouring. Without colour schemes, this observation is likely to be expressed as $\chi_\mathbf{P}(G) \leqslant \chi_\mathbf{S}(G)$, for all graphs G. With colour schemes, it is natural to write the stronger result, which is $\mathbf{S} \subseteq \mathbf{P}$. The advantage of '$\mathbf{S} \subseteq \mathbf{P}$' is that it is as succinct as the statement about chromatic numbers and it contains more information. The statement '$\chi_\mathbf{P}(G) \leqslant \chi_\mathbf{S}(G)$' does not communicate the justification for the bound, that is, that every star colouring is a proper colouring. This example is fairly trivial, but in more complicated situations it is vital to have a natural way to keep track of stronger results.

5.1.2 Unions and intersections

By considering graph colouring in terms of sets of coloured graphs we are able to make use of standard set operations. Relatively simple colour schemes can be used as atoms to build more complex colour schemes. This approach is useful because many properties of the atoms are transferred to their progeny. We first show that new colour schemes can be generated by taking intersections of known colour schemes.

Lemma 5.6. *If* $\mathbf{A}$ *and* $\mathbf{B}$ *are colour schemes, then* $\mathbf{A} \cap \mathbf{B}$ *is a colour scheme.*

Proof. Let $(G, \alpha) \in \mathbf{A} \cap \mathbf{B}$ and (H, β) be a subgraph or a fracture of (G, α). By SUBGRAPHABILITY and RECOLOURABILITY, $(H, \beta) \in \mathbf{A}$ and $(H, \beta) \in \mathbf{B}$ so $(H, \beta) \in \mathbf{A} \cap \mathbf{B}$. Therefore $\mathbf{A} \cap \mathbf{B}$ satisfies SUBGRAPHABILITY and RECOLOURABILITY.

Let $(G, \alpha) \in \mathbf{A} \cap \mathbf{B}$ and $(H, \beta) \in \mathbf{A} \cap \mathbf{B}$. By ADDITIVITY, the disjoint union of (G, α) and (H, β) is in both $\mathbf{A}$ and $\mathbf{B}$. Therefore $\mathbf{A} \cap \mathbf{B}$ satisfies ADDITIVITY.

$\mathbf{A} \cap \mathbf{B}$ satisfies UNIVERSALITY because both $\mathbf{A}$ and $\mathbf{B}$ contain the rainbow colouring of every graph G.

Let $(G, \alpha) \notin \mathbf{A} \cap \mathbf{B}$. Without loss of generality $(G, \alpha) \notin \mathbf{A}$. By LOCALITY there is a finite $(G_a, \alpha_a) \notin \mathbf{A}$ which is a subgraph of (G, α). Since $(G_a, \alpha_a) \notin \mathbf{A} \cap \mathbf{B}$, $\mathbf{A} \cap \mathbf{B}$ satisfies LOCALITY. □

The intersection of two colour schemes corresponds to 'and' in the rule-satisfaction approach to defining variants of graph colouring. For example, $\mathbf{S} \cap \mathbf{E}$ contains the coloured graphs which are both star-coloured and exact distance-two coloured.

A colour scheme, $\mathbf{A}$, generated by intersections can have some of their properties inferred from the properties of the colour schemes used to generate $\mathbf{A}$. For example, the products of colourings in $\mathbf{A}$ and $\mathbf{B}$ are in $\mathbf{A} \cap \mathbf{B}$.

Lemma 5.7. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be colour schemes with* $(G, \alpha) \in \mathbf{A}$ *and* $(G, \beta) \in \mathbf{B}$*. Then* $(G, \psi) \in \mathbf{A} \cap \mathbf{B}$*, where* $\psi(v) := (\alpha(v), \beta(v))$ *for all* $v \in V(G)$*.*

Proof. Recolour (G, α) to (G, ψ) with ψ defined such that $\psi(v) := (\alpha(v), \beta(v))$ for all $v \in V(G)$. Then (G, ψ) is a fracture of (G, α) because, for all $u, v \in V(G)$, $\psi(u) = \psi(v)$ implies $(\alpha(u), \beta(u)) = (\alpha(v), \beta(v))$ which implies $\alpha(u) = \alpha(v)$. Therefore $(G, \psi) \in \mathbf{A}$, by RECOLOURABILITY. Similarly, (G, ψ) is a fracture of (G, β), so $(G, \psi) \in \mathbf{B}$. It follows that $(G, \psi) \in \mathbf{A} \cap \mathbf{B}$. □

Product colourings can be used to infer bounds on the chromatic number of a colour scheme generated by taking intersections. For brevity, and to reduce the use of subscripts, a colour scheme, $\mathbf{A}$, is *bounded* on a class of graphs, $\mathcal{G}$, if there exists a $c \in \mathbb{Z}^+$ such that $\chi_{\mathbf{A}}$ is bounded by c on $\mathcal{G}$. A colour scheme, $\mathbf{A}$, is *unbounded* on $\mathbf{G}$ if $\mathbf{A}$ is not bounded on $\mathcal{G}$.

Lemma 5.8. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be colour schemes and* $\mathcal{G}$ *be a set of graphs. Then* $\mathbf{A} \cap \mathbf{B}$ *is bounded on* $\mathcal{G}$ *if and only if both* $\mathbf{A}$ *and* $\mathbf{B}$ *are bounded on* $\mathcal{G}$*.*

Proof. Let $\chi_{\mathbf{A}}$ and $\chi_{\mathbf{B}}$ be bounded on $\mathcal{G}$ by a and b respectively. Let $G \in \mathcal{G}$. Let f_a be a $(\leqslant a)$-colouring of G in $\mathbf{A}$ and f_b be a $(\leqslant b)$-colouring of G in $\mathbf{B}$. By Lemma 5.7, there is a $(\leqslant ab)$-colouring of G that is in $\mathbf{A} \cap \mathbf{B}$. Therefore $\mathbf{A} \cap \mathbf{B}$ is bounded on $\mathcal{G}$.

Now to show that the conditions are necessary. Let $\chi_{\mathbf{A} \cap \mathbf{B}}$ be bounded on $\mathcal{G}$ by c. Let $G \in \mathcal{G}$ and let (G, f_c) be a $(\leqslant c)$-coloured graph such that $(G, f_c) \in \mathbf{A} \cap \mathbf{B}$. Since $(G, f_c) \in \mathbf{A}$ and $(G, f_c) \in \mathbf{B}$, both $\mathbf{A}$ and $\mathbf{B}$ are bounded on $\mathcal{G}$. □

Similar results can be obtained for unions, with the caveat that the union of two colour schemes is not the standard set union. This is required because the standard union of two colour schemes does not satisfy ADDITIVITY. As an example, take star colouring, denoted $\mathbf{S}$, and exact distance-two colouring, denoted $\mathbf{E}$, and consider the coloured paths 121 and 11. Note that $121 \in \mathbf{S} \setminus \mathbf{E}$ and $121 \in \mathbf{E} \setminus \mathbf{S}$ so, by SUBGRAPHABILITY, the disjoint union of 121 and 11 is not in $\mathbf{S}$ or $\mathbf{E}$. It follows that $\mathbf{S} \cup \mathbf{E}$ does not satisfy ADDITIVITY, since $121 \in \mathbf{S} \cup \mathbf{E}$ and $11 \in \mathbf{S} \cup \mathbf{E}$.

To obtain colour schemes from unions of colour schemes we must add the coloured graphs required to satisfy ADDITIVITY. The *additive closure* of a set of coloured graphs, $\mathbf{A}$, is the set of disjoint unions of coloured graphs in $\mathbf{A}$. The *closed union*, $\mathbf{A} \sqcup \mathbf{B}$, of two sets of coloured graphs $\mathbf{A}$ and $\mathbf{B}$ is the additive closure of $\mathbf{A} \cup \mathbf{B}$.

Lemma 5.9. *If $\mathbf{A}$ and $\mathbf{B}$ are colour schemes, then $\mathbf{A} \sqcup \mathbf{B}$ is a colour scheme.*

Proof. Let $(G, \alpha) \in \mathbf{A} \sqcup \mathbf{B}$ and (H, β) be a subgraph of a fracture of (G, α). Without loss of generality, let $(G', \alpha) \in \mathbf{A}$ be a connected component of (G, α). Let $(H', \beta|_{H'})$ be the subgraph of the fracture of (G', α) with $V(H') = V(G') \cap V(H)$ and $E(H') = E(G') \cap E(H)$. Note that $(H', \beta|_{H'}) \in \mathbf{A}$, by SUBGRAPHABILITY and RECOLOURABILITY. It follows that every connected component of (H, β) is in $\mathbf{A}$ or $\mathbf{B}$. Therefore $\mathbf{A} \sqcup \mathbf{B}$ satisfies SUBGRAPHABILITY and RECOLOURABILITY.

$\mathbf{A} \sqcup \mathbf{B}$ satisfies ADDITIVITY because it is the additive closure of $\mathbf{A} \cup \mathbf{B}$. $\mathbf{A} \sqcup \mathbf{B}$ satisfies UNIVERSALITY because $\mathbf{A}$ contains the rainbow colouring of every graph G.

Let $(G, \alpha) \notin \mathbf{A} \sqcup \mathbf{B}$. By ADDITIVITY, there is a connected component, $(H, \alpha|_H)$, of (G, α). It follows that $(H, \alpha|_H) \notin \mathbf{A} \cup \mathbf{B}$. By LOCALITY and Lemma 5.4, there exist two finite connected graphs, $(X, \alpha|_X) \notin \mathbf{A}$ and $(Y, \alpha|_Y) \notin \mathbf{B}$, both of which are subgraphs of (G, α). Let P be a uv-path in H with $u \in V(X)$ and $v \in V(Y)$. Let Z be the graph with $V(Z) := V(X) \cup V(Y) \cup V(P)$ and $E(Z) := E(X) \cup E(Y) \cup E(P)$. By this construction, $(Z, \alpha|_Z)$ is a finite connected subgraphs of H which, by SUBGRAPHABILITY, is not in $\mathbf{A}$ or $\mathbf{B}$. It follows that $\mathbf{A} \sqcup \mathbf{B}$ satisfies LOCALITY. □

The closure requirement tends to be just a technicality since colour schemes model variants of graph colouring for which all the interesting behaviour occurs on connected graphs. This technicality may be avoided by restricting the definition of colour schemes to connected graphs, however, doing so yields awkward phrasings or abuses of notation when taking subgraphs that are not necessarily connected. When restricted to connected graphs, the union of two colour schemes corresponds to 'or' in the rule-satisfaction formulation of graph colouring. In general, every connected component of a graph in $\mathbf{A} \sqcup \mathbf{B}$ is in $\mathbf{A}$ or $\mathbf{B}$.

Lemma 5.10. *Let $\mathbf{A}$ and $\mathbf{B}$ be colour schemes, $\mathcal{G}$ be a set of graphs, and $c \in \mathbb{Z}^+$. If $\chi_{\mathbf{A}}$ or $\chi_{\mathbf{B}}$ is bounded by c on $\mathcal{G}$, then $\chi_{\mathbf{A} \sqcup \mathbf{B}}$ is bounded by c on $\mathcal{G}$.*

Proof. Let $G \in \mathcal{G}$. Without loss of generality we may assume $\chi_{\mathbf{A}}$ is bounded on $\mathcal{G}$ by c. There is a fixed c-colouring, α, such that, $(G, \alpha) \in \mathbf{A}$. It follows that $(G, \alpha) \in \mathbf{A} \sqcup \mathbf{B}$. □

We now show that the sufficient conditions in Lemma 5.10 are not necessary. That is, we show that there exist colour schemes, $\mathbf{A}$ and $\mathbf{B}$, such that $\mathbf{A} \sqcup \mathbf{B}$ is bounded on a set of graphs, $\mathcal{G}$, with $\mathbf{A}$ and $\mathbf{B}$ unbounded on $\mathcal{G}$. Recall that a *subcycle* of a graph, G, is a subgraph of G which is isomorphic to a cycle. Let $\mathbf{D}$ be the set of coloured graphs (G, α) such that every subcycle of (G, α) with odd order is rainbow coloured. Similarly, let $\mathbf{V}$ be the set of coloured graphs (G, α) such that every subcycle of (G, α) with even order is rainbow coloured. We first establish that $\mathbf{D}$ and $\mathbf{V}$ satisfy the colour scheme axioms. SUBGRAPHABILITY and ADDITIVITY are satisfied because the subgraph and disjoint union operations cannot introduce new subcycles in a graph. RECOLOURABILITY is satisfied because rainbow subcycles are preserved by fracturing. UNIVERSALITY is satisfied because the rainbow colouring of a graph has the rainbow colouring on each of its subcycles. LOCALITY is satisfied because every non-admissible graph has a finite non-rainbow subcycle. Both $\mathbf{D}$ and $\mathbf{V}$ are unbounded on $\mathcal{S} := \{C_n : n \geqslant 3\}$ because they require that C_n has the rainbow colouring, for n odd or even, respectively. However, $\chi_{\mathbf{D}}(C_n) = 1$ for even n and $\chi_{\mathbf{V}}(C_n) = 1$ for odd n. It follows that $\chi_{\mathbf{D}\sqcup\mathbf{V}}(C_n) = 1$ for all n, so $\mathbf{D} \sqcup \mathbf{V}$ is bounded on $\mathcal{S}$.

As we have shown, in general, a bound on $\chi_{\mathbf{A}\sqcup\mathbf{B}}$ does not imply a bound on $\chi_{\mathbf{A}}$ or $\chi_{\mathbf{B}}$. However, we can prove the reverse implication of Lemma 5.10 for many commonly studied classes of graphs. In particular, we obtain a result for self-dominating sets of graphs. A sequence of graphs, $G_1, G_2, \ldots$, is *telescoping* if G_i is a subgraph of G_{i+1} for all $i \geqslant 1$. A set of graphs, $\mathcal{G}$, is *dominated* by a sequence of graphs $G_1, G_2, \ldots$ if, for every $G \in \mathcal{G}$, there exists $j \geqslant 1$ such that G is a subgraph of G_j. A set of graphs $\mathcal{G}$ is *self-dominating* if it is dominated by a telescoping sequence of graphs $G_1, G_2, \ldots$ such that $G_i \in \mathcal{G}$ for all $i \geqslant 1$. Note that self-dominating is not a particularly restrictive property because all sets of graphs which are closed under disjoint union are self-dominating. This includes many natural sets of graphs including the sets of forests, graphs of bounded maximum degree, and planar graphs. The sets of stars, paths, trees, and trees of bounded maximum degree are also self-dominating. The following lemma is stated in terms of telescoping sequences instead of self-dominating sets in order to retain its full generality.

Lemma 5.11. *Let $\mathbf{A}$ and $\mathbf{B}$ be colour schemes, $c \in \mathbb{Z}^+$, and $\mathcal{G}$ be a set of graphs dominated by a telescoping sequence of connected graphs $G_1, G_2, \ldots$. If $\chi_{\mathbf{A}\sqcup\mathbf{B}}$ is bounded by c on $G_1, G_2, \ldots$, then $\chi_{\mathbf{A}}$ or $\chi_{\mathbf{B}}$ is bounded by c on $\mathcal{G}$.*

Proof. We may assume that $\chi_{\mathbf{B}}$ is not bounded by c on $\mathcal{G}$. Note that $\chi_{\mathbf{B}}$ is not bounded by c on $G_1, G_2, \ldots$, by Lemma 5.1, because $G_1, G_2, \ldots$ dominates $\mathcal{G}$. Since G_i is connected, for all i, there is a c-colouring of G_i in $\mathbf{A} \sqcup \mathbf{B}$ because $\chi_{\mathbf{A}\sqcup\mathbf{B}}$ is bounded by c on $G_1, G_2, \ldots$. It follows that $\chi_{\mathbf{A}}$ is bounded by c on $G_1, G_2, \ldots$. Therefore there exists a c-colouring, α_i, with $(G_i, \alpha_i) \in \mathbf{A}$, for all $i \geqslant 1$. Since $G_1, G_2, \ldots$ dominates $\mathcal{G}$, $\chi_{\mathbf{A}}$ is bounded by c on $\mathcal{G}$, by Lemma 5.1. □

By Lemmas 5.6 and 5.9, the set of colour schemes is closed under intersection and closed union. In fact, the set of colour schemes is a distributive lattice with respect to these two operations. Lattices are well studied algebraic structures from the field of order theory. The structure of the set of colour schemes is discussed further in Section 6.4.

5.2 $\mathcal{G}$-dependence

In many variants of graph colouring, the validity of a coloured graph only depends on the sequences of colours along the paths in the graph. Some notable examples are square-free colouring, parity colouring, conflict-free colouring, and anagram-free colouring. When analysing these colour schemes, it is often useful to think in terms of the patterns along paths that they avoid. Also, we are able to define properties and prove theorems about colour schemes in terms of sequences of colours along paths that they prohibit.

In general, colour schemes can be studied in terms of the sets of coloured subgraphs which they avoid. For a set of graphs, $\mathcal{G}$, let $\mathrm{Col}(\mathcal{G})$ be the set of colourings of graphs in $\mathcal{G}$. For a coloured graph (G, ψ), and a set of graphs, $\mathcal{G}$, define its *coloured $\mathcal{G}$-subgraphs* as

$$\mathrm{Sub}_{\mathcal{G}}(G, \psi) := \{(H, \psi|_H) : H \text{ is a subgraph of } G\} \cap \mathrm{Col}(\mathcal{G}),$$

with equality in the intersection taken up to coloured graph isomorphism. In many cases, we set $\mathcal{G} = \mathcal{P}$, where $\mathcal{P}$ is the set of all paths. See Figure 5.1 for an example of a coloured graph and its set of coloured paths.

A set of coloured graphs, $\mathbf{A}$, is *$\mathcal{G}$-dependent* if there exists a set of coloured graphs, $\mathcal{B} \subseteq \mathrm{Col}(\mathcal{G})$, such that $(G, \alpha) \in \mathbf{A}$ if and only if $\mathrm{Sub}_{\mathcal{G}}(G, \alpha) \cap \mathcal{B} = \emptyset$, for all coloured

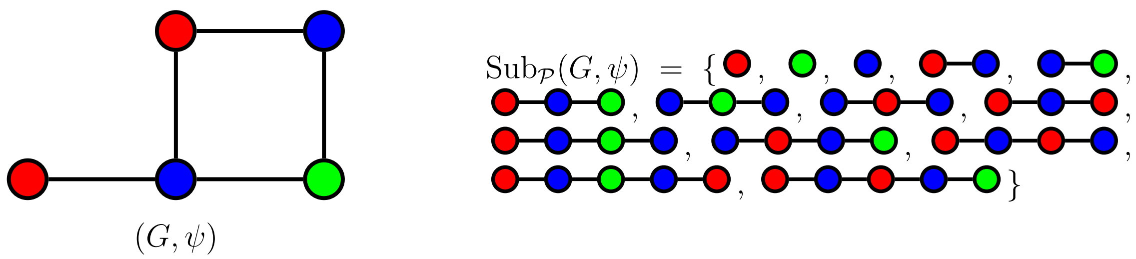


Figure 5.1: $\mathrm{Sub}_{\mathcal{P}}(G, \psi)$ of the coloured graph (G, ψ), where $\mathcal{P}$ is the set of paths.

graphs (G, α). For all such $\mathcal{B}$, we say that $\mathbf{A}$ is *determined with respect to* $\mathcal{G}$ by $\mathcal{B}$ and that $\mathcal{B}$ is a *determinant with respect to* $\mathcal{G}$ of $\mathbf{A}$. The *set of bad graphs with respect to* $\mathcal{G}$, denoted $\mathcal{B}_{\mathbf{A}}$, of a $\mathcal{G}$-dependent set of coloured graphs, $\mathbf{A}$, is $\mathrm{Col}(\mathcal{G}) \setminus \mathbf{A}$. Similarly, the *set of good graphs with respect to* $\mathcal{G}$, denoted $\mathcal{G}_{\mathbf{A}}$, of a $\mathcal{G}$-dependent set of coloured graphs, $\mathbf{A}$, is $\mathrm{Col}(\mathcal{G}) \cap \mathbf{A}$. The set of graphs, $\mathcal{G}$, that $\mathcal{B}$, $\mathcal{B}_{\mathbf{A}}$, and $\mathcal{G}_{\mathbf{A}}$ are taken with respect to is often clear from context, since we define $\mathcal{G}$ in the process of introducing a $\mathcal{G}$-dependent set of coloured graphs. In these cases, we implicitly take $\mathcal{B}$, $\mathcal{B}_{\mathbf{A}}$, and $\mathcal{G}_{\mathbf{A}}$ with respect to $\mathcal{G}$, unless indicated otherwise.

Lemma 5.12. *Let* $\mathbf{A}$ *be a colour scheme and* $\mathcal{G}$ *be the set of finite connected graphs. Then* $\mathbf{A}$ *is* $\mathcal{G}$*-dependent.*

Proof. Define the function f such that, for all $(G, \beta) \notin \mathbf{A}$, $f(G, \beta) := (H, \beta|_H)$ where $(H, \beta|_H)$ is a finite, connected, subgraph of (G, β), and $(H, \beta|_H) \notin \mathbf{A}$. Note that $f(G, \beta)$ exists for all $(G, \beta) \notin \mathbf{A}$ by Locality and Additivity, and, by Subgraphability, $f(G, \beta)$ is not a subgraph of any coloured graph in $\mathbf{A}$. Let $\mathcal{B} \subseteq \mathrm{Col}(\mathcal{G})$ be a set of coloured graphs such that $f(G, \beta) \in \mathcal{B}$ for all coloured graphs $(G, \beta) \notin \mathbf{A}$. It follows that $\mathcal{B}$ is a determinant of $\mathbf{A}$. □

Throughout our study of colour schemes we show two main applications of $\mathcal{G}$-dependence. The first, due to the results of Section 5.2.2, is that it is often easier to show that a set of coloured graphs is a colour scheme by first showing that it is $\mathcal{G}$-dependent. The second application is that many properties of $\mathcal{G}$-dependent colour schemes can be formulated as properties of sets of bad graphs, which are easier to study. Many results in Chapter 6 are obtained by studying the paths avoided by path-dependent colour schemes.

5.2.1 Determinants and subsets

In this section we establish some relationships between colour schemes and their determinants. Keep in mind that determinants and sets of bad graphs are implicitly taken with respect to $\mathcal{G}$ for sets of coloured graphs which are explicitly stated to be $\mathcal{G}$-dependent. We first show that the set of bad graphs is the maximal determinant of a $\mathcal{G}$-dependent set of coloured graphs.

Lemma 5.13. *Let* $\mathbf{A}$ *be a* $\mathcal{G}$*-dependent set of coloured graphs. Then* $\mathcal{B}_{\mathbf{A}}$ *is a determinant of* $\mathbf{A}$ *and* $\mathcal{B} \subseteq \mathcal{B}_{\mathbf{A}}$*, for all determinants* $\mathcal{B}$ *of* $\mathbf{A}$*.*

Proof. By the definition of $\mathcal{B}_{\mathbf{A}}$, $(G, \alpha) \in \mathcal{B}_{\mathbf{A}}$ if and only if $(G, \alpha) \notin \mathbf{A}$, for all $G \in \mathcal{G}$. Let $\mathcal{B}$ be a determinant of $\mathbf{A}$. It follows that $\mathcal{B} \subseteq \mathcal{B}_{\mathbf{A}}$ since $\mathcal{B} \subseteq \mathrm{Col}(\mathcal{G})$ and $\mathcal{B} \cap \mathbf{A} = \emptyset$. $\mathcal{B}_{\mathbf{A}}$ is a determinant of $\mathbf{A}$ because $\mathcal{B}_{\mathbf{A}} \subseteq \mathrm{Col}(\mathcal{G})$ and $\mathrm{Sub}_{\mathcal{G}}(H, \beta) \cap \mathcal{B}_{\mathbf{A}} = \emptyset$ if and only if $(H, \beta) \in \mathbf{A}$. □

Subset relationships between $\mathcal{G}$-dependent sets of coloured graphs can be established by comparing their determinants.

Lemma 5.14. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be* $\mathcal{G}$*-dependent sets of coloured graphs with determinants* $\mathcal{A}$ *and* $\mathcal{B}$ *respectively. If* $\mathcal{B} \subseteq \mathcal{A}$ *then* $\mathbf{A} \subseteq \mathbf{B}$.

Proof. Let $(G, \alpha) \in \mathbf{A}$ be a coloured graph. Since $\mathbf{A}$ is $\mathcal{G}$-dependent, $\text{Sub}_{\mathcal{G}}(G, \alpha) \cap \mathcal{A} = \emptyset$, implying that $\text{Sub}_{\mathcal{G}}(G, \alpha) \cap \mathcal{B} = \emptyset$. Therefore $(G, \alpha) \in \mathbf{B}$. □

Lemma 5.14 is useful because, for two $\mathcal{G}$-dependent colour schemes $\mathbf{A}$ and $\mathbf{B}$, we can show $\mathbf{A} \subseteq \mathbf{B}$ by comparing $\mathcal{B}_{\mathbf{A}}$ to a relatively small and easy to conceptualise determinant of $\mathbf{B}$. We now show that the subset relationship between determinants in Lemma 5.14 is necessary and sufficient for maximal determinants.

Lemma 5.15. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be* $\mathcal{G}$*-dependent sets of coloured graphs. Then* $\mathbf{A} \subseteq \mathbf{B}$ *if and only if* $\mathcal{B}_{\mathbf{B}} \subseteq \mathcal{B}_{\mathbf{A}}$.

Proof. We first show that $\mathcal{B}_{\mathbf{B}} \subseteq \mathcal{B}_{\mathbf{A}}$ is sufficient. Let $(G, \alpha) \in \mathbf{A}$ and recall that $\text{Sub}_{\mathcal{G}}(G, \alpha) \cap \mathcal{B}_{\mathbf{A}} = \emptyset$ by definition. It follows that $\text{Sub}_{\mathcal{G}}(G, \alpha) \cap \mathcal{B}_{\mathbf{B}} = \emptyset$. Thus $(G, \alpha) \in \mathbf{B}$ so $\mathbf{A} \subseteq \mathbf{B}$.

Now to show that $\mathcal{B}_{\mathbf{B}} \subseteq \mathcal{B}_{\mathbf{A}}$ is necessary. Take $\mathbf{A} \subseteq \mathbf{B}$ and assume for the sake of contradiction that $\mathcal{B}_{\mathbf{B}} \nsubseteq \mathcal{B}_{\mathbf{A}}$. Let $(G, \alpha) \in (\mathcal{B}_{\mathbf{B}} \setminus \mathcal{B}_{\mathbf{A}})$. By definition, $(G, \alpha) \in \mathbf{A}$ and $(G, \alpha) \notin \mathbf{B}$. This contradicts $\mathbf{A} \subseteq \mathbf{B}$. □

Lemmas 5.14 and 5.15 show the utility of $\mathcal{G}$-dependence, since they can be used to more easily establish hierarchies of subset relationships between colour schemes.

5.2.2 $\mathcal{G}$-dependence for axiom satisfaction

With the following lemmas, the easiest way to check whether a set of coloured graphs, $\mathbf{A}$, is a colour scheme is often to first show that $\mathbf{A}$ is $\mathcal{G}$-dependent, and then to inspect the properties of an appropriate determinant of $\mathbf{A}$.

Lemma 5.16. *Every* $\mathcal{G}$*-dependent set of coloured graphs satisfies* Subgraphability*, for every set of graphs* $\mathcal{G}$.

Proof. Let $\mathbf{A}$ be a $\mathcal{G}$-dependent set of coloured graphs. Let $(H, \alpha|_H)$ be a subgraph of $(G, \alpha) \in \mathbf{A}$. It follows that $(H, \alpha|_H) \in \mathbf{A}$, because $\text{Sub}_{\mathcal{G}}(H, \alpha|_H) \subseteq \text{Sub}_{\mathcal{G}}(G, \alpha)$ and $\text{Sub}_{\mathcal{G}}(G, \alpha) \cap \mathcal{B}_{\mathbf{A}} = \emptyset$. Therefore $\mathbf{A}$ satisfies Subgraphability. □

Additivity and Locality depend on properties of the set of graphs, $\mathcal{G}$, for which a set of coloured graphs is $\mathcal{G}$-dependent.

Lemma 5.17. *If every graph* $G \in \mathcal{G}$ *is connected, then every* $\mathcal{G}$*-dependent set of coloured graphs,* $\mathbf{A}$*, satisfies* Additivity.

Proof. Let (G, α) be the disjoint union of $(S, \alpha|_S) \in \mathbf{A}$ and $(T, \alpha|_T) \in \mathbf{A}$. By definition, $\text{Sub}_{\mathcal{G}}(S, \alpha|_S) \cap \mathcal{B}_{\mathbf{A}} = \emptyset$ and $\text{Sub}_{\mathcal{G}}(T, \alpha|_T) \cap \mathcal{B}_{\mathbf{A}} = \emptyset$. Every graph in $\mathcal{G}$ is connected so $\text{Sub}_{\mathcal{G}}(G, \alpha) = \text{Sub}_{\mathcal{G}}(S, \alpha|_S) \cup \text{Sub}_{\mathcal{G}}(T, \alpha|_T)$. Therefore $\text{Sub}_{\mathcal{G}}(G, \alpha) \in \mathbf{A}$ so $\mathbf{A}$ satisfies Additivity. □

Lemma 5.18. *If every graph* $G \in \mathcal{G}$ *is finite, then every* $\mathcal{G}$*-dependent set of coloured graphs,* $\mathbf{A}$*, satisfies* Locality.

Proof. Let $(G,\alpha) \notin \mathbf{A}$. By definition, $\mathrm{Sub}_{\mathcal{G}}(G,\alpha) \cap \mathcal{B}_{\mathbf{A}} \neq \emptyset$ so there is a $(B,\alpha|_B) \in \mathrm{Sub}_{\mathcal{G}}(G,\alpha) \cap \mathcal{B}_{\mathbf{A}}$. Note that B is finite, $(B,\alpha|_B) \notin \mathbf{A}$ and $(B,\alpha|_B)$ is a subgraph of (G,α). Therefore $\mathbf{A}$ satisfies LOCALITY. □

The conditions of Lemmas 5.16, 5.17 and 5.18 are inclusive enough to encompass all reasonable choices of $\mathcal{G}$ for a potential $\mathcal{G}$-dependent colour scheme.

To determine whether a $\mathcal{G}$-dependent set of coloured graphs, $\mathbf{A}$, satisfies RECOLOURABILITY and UNIVERSALITY we must look at the properties of a determinant of $\mathbf{A}$. RECOLOURABILITY is satisfied by a $\mathcal{G}$-dependent set of coloured graphs, $\mathbf{A}$, if there is a determinant, $\mathcal{B}$, of $\mathbf{A}$ such that $\mathrm{Col}(\mathcal{G}) \setminus \mathcal{B}$ is closed under fracturing. To state this property more intuitively, we say that (G,β) is a *meld* of (G,α) if (G,α) is a fracture of (G,β). Equivalently, (G,β) is a meld of (G,α) if, for all pairs of vertices, $u,v \in V(G)$, $\alpha(u) = \alpha(v)$ implies $\beta(u) = \beta(v)$.

Lemma 5.19. *Let* $\mathbf{A}$ *be a* $\mathcal{G}$*-dependent set of coloured graphs determined by a set of coloured graphs* $\mathcal{B}$*. If* $\mathcal{B}$ *is closed under melding then* $\mathbf{A}$ *satisfies* RECOLOURABILITY.

Proof. Let $(G,\alpha) \in \mathbf{A}$ and let (G,β) be a fracture of (G,α). Let H be a subgraph of G such that H is isomorphic to a graph in $\mathcal{G}$. By definition, $(H,\beta|_H)$ is a meld of $(H,\alpha|_H)$ because $(H,\beta|_H)$ is a fracture of $(H,\alpha|_H)$. Note that $(H,\alpha|_H) \notin \mathcal{B}$ because $(G,\alpha) \in \mathbf{A}$. It follows that $(H,\beta|_H) \notin \mathcal{B}$ because $\mathcal{B}$ is closed under melding. Therefore $(G,\beta) \in \mathbf{A}$, so $\mathbf{A}$ satisfies RECOLOURABILITY. □

Many sets, $\mathcal{B}$, may determine the same $\mathcal{G}$-dependent colour scheme and not all choices of $\mathcal{B}$ are closed under melding. For example, the set of coloured paths, $\mathcal{B}$, coloured by words of the form WW, where W is not a square, is a determinant of the colour scheme corresponding to square-free colouring, denoted $\mathbf{Q}$. Clearly $1212 \in \mathcal{B}$, but its meld, 1111, is not in $\mathcal{B}$. In general, it is not trivial to find a determinant of a $\mathcal{G}$-dependent set of coloured graphs which is closed under melding. However, it is easy to find such a determinant for a $\mathcal{G}$-dependent colour schemes, $\mathbf{A}$, since, by RECOLOURABILITY, $\mathcal{B}_{\mathbf{A}}$ is closed under melding.

UNIVERSALITY is easier to check, since it is sufficient for a determinant of $\mathbf{A}$ to not include any rainbow colourings. In fact, this condition is necessary for all determinants of colour schemes.

Lemma 5.20. *Let* $\mathbf{A}$ *be a* $\mathcal{G}$*-dependent set of coloured graphs determined by a set of coloured graphs* $\mathcal{B}$*.* $\mathbf{A}$ *satisfies* UNIVERSALITY *if every coloured graph* $(H,\alpha) \in \mathcal{B}$ *is not a rainbow colouring.*

Proof. Let $\mathcal{B}$ be a determinant of $\mathbf{A}$ with no rainbow colourings and let (G,α) be a rainbow colouring. Since every coloured graph in $\mathrm{Sub}_{\mathcal{G}}(G,\alpha)$ is a rainbow colouring, $\mathrm{Sub}_{\mathcal{G}}(G,\alpha) \cap \mathcal{B} = \emptyset$. It follows that $(G,\alpha) \in \mathbf{A}$, so $\mathbf{A}$ satisfies UNIVERSALITY. □

Lemma 5.21. *Let* $\mathbf{A}$ *be a* $\mathcal{G}$*-dependent colour scheme with determinant* $\mathcal{B}$*. Then* $\mathcal{B}$ *does not contain any rainbow colourings.*

Proof. Let $(G,\alpha) \in \mathbf{A}$ be a rainbow colouring. Since $\mathrm{Sub}_{\mathcal{G}}(G,\alpha) \cap \mathcal{B} = \emptyset$, $\mathcal{B}$ does not contain any rainbow colourings. □

5.2.3 Unions and intersections

We now show that many examples of $\mathcal{G}$-dependent colour schemes can be generated by taking unions and intersections of determinants.

Lemma 5.22. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be* $\mathcal{G}$*-dependent sets of coloured graphs. Then* $\mathbf{A} \cap \mathbf{B}$ *is* $\mathcal{G}$*-dependent and* $\mathcal{B}_{\mathbf{A}\cap\mathbf{B}} = \mathcal{B}_{\mathbf{A}} \cup \mathcal{B}_{\mathbf{B}}$.

Proof. Let (G, α) be a coloured graph. Take the case $(G, \alpha) \in \mathbf{A} \cap \mathbf{B}$. It follows that $\mathrm{Sub}_{\mathcal{G}}(G, \alpha) \cap \mathcal{B}_{\mathbf{A}} = \emptyset$ and $\mathrm{Sub}_{\mathcal{G}}(G, \alpha) \cap \mathcal{B}_{\mathbf{B}} = \emptyset$. Therefore $\mathrm{Sub}_{\mathcal{G}}(G, \alpha) \cap (\mathcal{B}_{\mathbf{A}} \cup \mathcal{B}_{\mathbf{B}}) = \emptyset$. Now take the case $(G, \alpha) \notin \mathbf{A} \cap \mathbf{B}$. It follows that $\mathrm{Sub}_{\mathcal{G}}(G, \alpha) \cap \mathcal{B}_{\mathbf{A}} \neq \emptyset$ or $\mathrm{Sub}_{\mathcal{G}}(G, \alpha) \cap \mathcal{B}_{\mathbf{B}} \neq \emptyset$. Therefore $\mathrm{Sub}_{\mathcal{G}}(G, \alpha) \cap (\mathcal{B}_{\mathbf{A}} \cup \mathcal{B}_{\mathbf{B}}) \neq \emptyset$.

We have shown that $(G, \alpha) \in \mathbf{A} \cap \mathbf{B}$ if and only if $\mathrm{Sub}_{\mathcal{G}}(G, \alpha) \cap (\mathcal{B}_{\mathbf{A}} \cup \mathcal{B}_{\mathbf{B}}) = \emptyset$, so $\mathbf{A} \cap \mathbf{B}$ is a $\mathcal{G}$-dependent set of coloured graphs determined by $\mathcal{B}_{\mathbf{A}} \cup \mathcal{B}_{\mathbf{B}}$. Note that $\mathcal{B}_{\mathbf{A}} \cup \mathcal{B}_{\mathbf{B}} = \mathrm{Col}(\mathcal{G}) \setminus \mathbf{A} \cup \mathrm{Col}(\mathcal{G}) \setminus \mathbf{B} = \mathrm{Col}(\mathcal{G}) \setminus (\mathbf{A} \cap \mathbf{B})$, so $\mathcal{B}_{\mathbf{A}\cap\mathbf{B}} = \mathcal{B}_{\mathbf{A}} \cup \mathcal{B}_{\mathbf{B}}$. □

In particular, the set of $\mathcal{G}$-dependent colour schemes is closed under intersection. The following analogous lemma for the intersection of $\mathcal{B}_{\mathbf{A}}$ and $\mathcal{B}_{\mathbf{B}}$ is not as strong, since $\mathbf{A} \sqcup \mathbf{B}$ is not necessarily a $\mathcal{G}$-dependent set of coloured graphs.

Lemma 5.23. *Let* $\mathcal{G}$ *be a set of finite connected graphs and let* $\mathbf{A}$ *and* $\mathbf{B}$ *be* $\mathcal{G}$*-dependent colour schemes.* $\mathcal{B}_{\mathbf{A}} \cap \mathcal{B}_{\mathbf{B}}$ *determines the* $\mathcal{G}$*-dependent colour scheme,* $\mathbf{S}$*, with* $\mathcal{B}_{\mathbf{S}} = \mathcal{B}_{\mathbf{A}} \cap \mathcal{B}_{\mathbf{B}}$ *and* $\mathbf{A} \sqcup \mathbf{B} \subseteq \mathbf{S}$.

Proof. Let $\mathcal{B} = \mathcal{B}_{\mathbf{A}} \cap \mathcal{B}_{\mathbf{B}}$. Let $\mathcal{S}$ be the set of coloured graphs determined by $\mathcal{B}$. $\mathbf{S}$ is $\mathcal{G}$-dependent so, by Lemma 5.16, $\mathbf{S}$ satisfies SUBGRAPHABILITY. $\mathbf{S}$ satisfies LOCALITY and ADDITIVITY, by Lemmas 5.17 and 5.18, because $\mathcal{G}$ is a set of finite connected graphs. $\mathcal{B}_{\mathbf{A}}$ and $\mathcal{B}_{\mathbf{B}}$ do not contain rainbow colourings so, by Lemma 5.20, $\mathbf{S}$ satisfies UNIVERSALITY. Furthermore, $\mathcal{B}_{\mathbf{A}}$ and $\mathcal{B}_{\mathbf{B}}$ are closed under melding so $\mathcal{B}_{\mathbf{A}} \cap \mathcal{B}_{\mathbf{B}}$ is closed under melding. Therefore $\mathbf{S}$ satisfies RECOLOURABILITY, by Lemma 5.19.

Let $(G, \alpha) \in \mathbf{A} \cup \mathbf{B}$. Then $\mathrm{Sub}(G, \alpha) \cap \mathcal{B}_{\mathbf{A}} = \emptyset$ or $\mathrm{Sub}(G, \alpha) \cap \mathcal{B}_{\mathbf{B}} = \emptyset$. Therefore $\mathrm{Sub}(G, \alpha) \cap \mathcal{B}_{\mathbf{A}} \cap \mathcal{B}_{\mathbf{B}} = \emptyset$. It follows that $(G, \alpha) \in \mathbf{S}$, implying $\mathbf{A} \cup \mathbf{B} \subseteq \mathbf{S}$. Since $\mathbf{S}$ is a colour scheme it is its own additive closure, so $\mathbf{A} \sqcup \mathbf{B} \subseteq \mathbf{S}$. Furthermore,

$$\mathcal{B}_{\mathbf{S}} = \mathrm{Col}(\mathcal{G}) \setminus \mathbf{S} \subseteq \mathrm{Col}(\mathcal{G}) \setminus (\mathbf{A} \cup \mathbf{B}) = (\mathrm{Col}(\mathcal{G}) \setminus \mathbf{A}) \cap (\mathrm{Col}(\mathcal{G}) \setminus \mathbf{B}) = \mathcal{B}_{\mathbf{A}} \cap \mathcal{B}_{\mathbf{B}} = \mathcal{B},$$

so $\mathcal{B} = \mathcal{B}_{\mathbf{S}}$, by the maximality of $\mathcal{B}_{\mathbf{S}}$. □

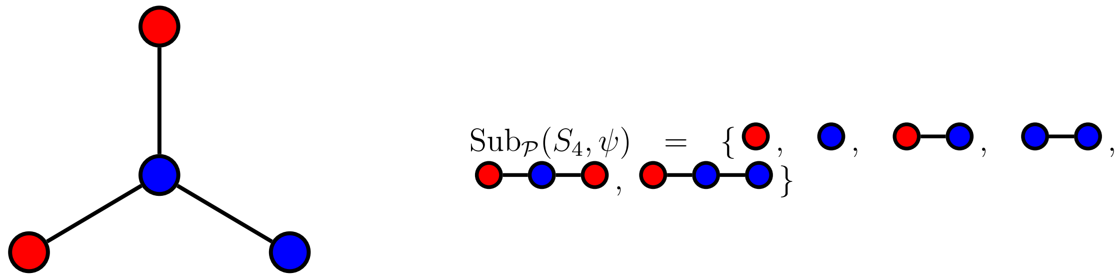
Lemma 5.23 can be strengthened to $\mathbf{A} \sqcup \mathbf{B} = \mathbf{S}$ when $\mathcal{G}$ is the set of finite, connected, graphs.

Lemma 5.24. *Let* $\mathcal{G}$ *be the set of all finite connected graphs and let* $\mathbf{A}$ *and* $\mathbf{B}$ *be* $\mathcal{G}$*-dependent colour schemes. Then* $\mathcal{B}_{\mathbf{A}\sqcup\mathbf{B}} = \mathcal{B}_{\mathbf{A}} \cap \mathcal{B}_{\mathbf{B}}$.

Proof. Let $\mathcal{B}_{\mathbf{S}} = \mathcal{B}_{\mathbf{A}} \cap \mathcal{B}_{\mathbf{B}}$ and $\mathbf{S}$ be the colour scheme determined by $\mathcal{B}_{\mathbf{S}}$. By Lemma 5.23, $\mathbf{A} \sqcup \mathbf{B} \subseteq \mathbf{S}$, so we only need to show $\mathbf{S} \subseteq \mathbf{A} \sqcup \mathbf{B}$.

Let $(G, \alpha) \in \mathbf{S}$. Let $(H, \alpha|_H)$ be a connected component of (G, α). Assume for the sake of contradiction that $(H, \alpha|_H) \notin \mathbf{A} \sqcup \mathbf{B}$. By LOCALITY, there is a finite subgraph of $(H, \alpha|_H)$, $(J, \alpha|_J)$, which is not in $\mathbf{A} \sqcup \mathbf{B}$. It follows that $(J, \alpha|_J) \in \mathcal{B}_{\mathbf{A}} \cap \mathcal{B}_{\mathbf{B}}$, which is a contradiction, because $\mathcal{B}_{\mathbf{S}}$ does not contain any finite connected subgraphs of (G, α). Therefore every connected component of (G, α) is in $\mathbf{A} \sqcup \mathbf{B}$.

By ADDITIVITY, $(G, \alpha) \in \mathbf{A} \sqcup \mathbf{B}$. It follows that $\mathbf{S} = \mathbf{A} \sqcup \mathbf{B}$. □



(S_4, ψ)

Figure 5.2: A colouring of the star of order 4, denoted (S_4, ψ), and $\text{Sub}_{\mathcal{P}}(S_4, \psi)$, where $\mathcal{P}$ is the set of paths.

Lemma 5.23 cannot be strengthened to $\mathbf{A} \sqcup \mathbf{B} = \mathbf{S}$ in general. $\mathbf{D}$ and $\mathbf{V}$ provides a simple counter-example. Recall that $\mathbf{D}$ avoids non-rainbow odd cycles and $\mathbf{V}$ avoids non-rainbow even cycles. $\mathbf{D}$ and $\mathbf{V}$ are cycle-dependent, where $\mathcal{B}_{\mathbf{D}}$ is the set of non-rainbow odd cycles, and $\mathcal{B}_{\mathbf{V}}$ is the set of non-rainbow even cycles. Note that $\mathcal{B}_{\mathbf{D}} \cap \mathcal{B}_{\mathbf{V}} = \emptyset$, but the 1-colouring of K_4 is not in $\mathbf{D} \sqcup \mathbf{V}$. Therefore $\mathbf{D} \sqcup \mathbf{V}$ is not determined by $\mathcal{B}_{\mathbf{D}} \cap \mathcal{B}_{\mathbf{V}}$.

While $\mathbf{D}$ and $\mathbf{V}$ provide a simple counter-example, the set of cycles is not a typical choice for $\mathcal{G}$ among colour schemes found in the literature. We present another counter-example, this time where $\mathcal{G}$ is the set of paths. Consider the colour schemes $\mathbf{E}_2$, the colour scheme of exact distance-2 colouring, and $\mathbf{P}$, the colour scheme of proper colouring. Let (S_4, ψ) be the colouring of the star of order 4 with two leaves coloured red and the remaining vertices coloured blue, see Figure 5.2. Note that the two red vertices are at distance 2 and the two blue vertices are adjacent, so $(S_4, \psi) \notin \mathbf{E}_2 \cup \mathbf{P}$. The only coloured path of order at most three in $\mathcal{B}_{\mathbf{E}_2} \cap \mathcal{B}_{\mathbf{P}}$ is 111, up to relabelling, so $\text{Sub}(S_4, \psi) \cap \mathcal{B}_{\mathbf{E}_2} \cap \mathcal{B}_{\mathbf{P}} = \emptyset$. It follows that (S_4, ψ) is in the colour scheme determined by $\mathcal{B}_{\mathbf{E}_2} \cap \mathcal{B}_{\mathbf{P}}$, so this colour scheme is not $\mathbf{E}_2 \sqcup \mathbf{P}$.

We have shown that unions and intersections of sets of bad graphs determine colour schemes. Arbitrary determinants of colour schemes can also be combined to generate new colour schemes, with a caveat in the case of Recolourability. Let the *closed intersection*, denoted $\mathcal{X} \sqcap \mathcal{Y}$, of two sets of coloured graphs, $\mathcal{X}$ and $\mathcal{Y}$, be the set of melds of $\mathcal{X} \cap \mathcal{Y}$.

Lemma 5.25. *Let $\mathcal{G}$ be a set of finite connected graphs and let* $\mathbf{A}$ *and* $\mathbf{B}$ *be $\mathcal{G}$-dependent colour schemes with determinants $\mathcal{X}$ and $\mathcal{Y}$. Then $\mathcal{X} \sqcap \mathcal{Y}$ determines a $\mathcal{G}$-dependent colour scheme.*

Proof. Let $\mathcal{B} = \mathcal{X} \sqcap \mathcal{Y}$. Let $\mathbf{S}$ be the set of coloured graphs determined by $\mathcal{B}$. $\mathbf{S}$ is $\mathcal{G}$-dependent so, by Lemma 5.16, $\mathbf{S}$ satisfies Subgraphability. $\mathbf{S}$ satisfies Locality and Additivity, by Lemmas 5.17 and 5.18, because $\mathcal{G}$ is a set of finite connected graphs. $\mathcal{X}$ does not contain rainbow colourings so, by Lemma 5.20, $\mathbf{S}$ satisfies Universality. By definition, $\mathcal{X} \sqcap \mathcal{Y}$ is closed under melding, so $\mathbf{S}$ satisfies Recolourability. □

No closures are required for unions of arbitrary determinants to determine colour schemes.

Lemma 5.26. *Let $\mathcal{G}$ be a set of finite connected graphs and let* $\mathbf{A}$ *and* $\mathbf{B}$ *be $\mathcal{G}$-dependent colour schemes with determinants $\mathcal{X}$ and $\mathcal{Y}$. Then $\mathcal{X} \cup \mathcal{Y}$ determines a $\mathcal{G}$-dependent colour scheme.*

Proof. Let $\mathcal{B} = \mathcal{X} \cup \mathcal{Y}$. Let $\mathbf{S}$ be the set of coloured graphs determined by $\mathcal{B}$. $\mathbf{S}$ is $\mathcal{G}$-dependent so, by Lemma 5.16, $\mathbf{S}$ satisfies SUBGRAPHABILITY. $\mathbf{S}$ satisfies LOCALITY and ADDITIVITY, by Lemmas 5.17 and 5.18, because $\mathcal{G}$ is a set of finite connected graphs. $\mathcal{X}$ and $\mathcal{Y}$ do not contain rainbow colourings so, by Lemma 5.20, $\mathbf{S}$ satisfies UNIVERSALITY.

Let $(G, \alpha) \notin \mathbf{S}$ and (G, β) be a meld of (G, α). Note that (G, α) is a fracture of (G, β), so for $\mathbf{S}$ to satisfy RECOLOURABILITY we need to show $(G, \beta) \notin \mathbf{S}$. Since $\mathbf{S}$ is determined by $\mathcal{X} \cup \mathcal{Y}$, (G, α) has a subgraph $(H, \alpha|_H)$ which, without loss of generality, is in $\mathcal{X}$. It follows that $(H, \alpha|_H) \notin \mathbf{A}$ and so, by RECOLOURABILITY, $(H, \beta|_H) \notin \mathbf{A}$. Therefore $(H, \beta|_H)$ has a subgraph, $(I, \beta|_I) \in \mathcal{X}$. Now, $(I, \beta|_I)$ is a subgraph of (G, β), so $(G, \beta) \notin \mathbf{S}$. □

5.2.4 Path-dependence

Many results in Chapter 6 extend theorems from Chapters 3 and 4 to the set of path-dependent colour schemes. A set of coloured graphs, $\mathbf{A}$, is *path-dependent* if it is $\mathcal{P}$-dependent, where $\mathcal{P}$ is the set of finite paths, and $\mathcal{B}_{\mathbf{A}}$ is called the *set of bad paths*. A colour scheme, $\mathbf{A}$, is *path-bounded* if $\mathbf{A}$ is bounded on paths. Define Sub := $\text{Sub}_{\mathcal{P}}$, where $\mathcal{P}$ is the set of finite paths. For brevity, we often denote a coloured path by P instead of (P, α) and in some cases treat P like a word.

An important feature of path-dependent colour schemes is that they have a unique smallest determinant. This is shown by the following lemma.

Lemma 5.27. *Let $\mathbf{A}$ be a path-dependent set of coloured graphs with determinants $\mathcal{A}$ and $\mathcal{B}$. Then $\mathcal{A} \cap \mathcal{B}$ is a determinant of $\mathbf{A}$.*

Proof. Let $(G, \alpha) \in \mathbf{A}$. $\text{Sub}(G, \alpha) \cap \mathcal{A} = \emptyset$ and $\text{Sub}(G, \alpha) \cap \mathcal{B} = \emptyset$. Therefore $\text{Sub}(G, \alpha) \cap \mathcal{A} \cap \mathcal{B} = \emptyset$.

Let $(G, \alpha) \notin \mathbf{A}$. $\text{Sub}(G, \alpha) \cap \mathcal{A} \neq \emptyset$ and $\text{Sub}(G, \alpha) \cap \mathcal{B} \neq \emptyset$. It follows that $\text{Sub}(G, \alpha) \cap (\mathcal{A} \cup \mathcal{B}) \neq \emptyset$. Pick a path, $P \in \text{Sub}(G, \alpha) \cap (\mathcal{A} \cup \mathcal{B})$, that minimizes $|V(P)|$. Note that $P \notin \mathbf{A}$ because at least one of $P \in \mathcal{A}$ or $P \in \mathcal{B}$. It follows that $\text{Sub}(P, \alpha|_P) \cap \mathcal{A} \neq \emptyset$ and $\text{Sub}(P, \alpha|_P) \cap \mathcal{B} \neq \emptyset$. By the minimality of $|V(P)|$, $P \in \mathcal{A}$ and $P \in \mathcal{B}$. Therefore $\text{Sub}(G, \alpha) \cap \mathcal{A} \cap \mathcal{B} \neq \emptyset$. □

Let $\mathcal{M}_{\mathbf{A}}$ denote the unique smallest determinant of a path-dependent colour scheme $\mathbf{A}$. We call $\mathcal{M}_{\mathbf{A}}$ the *set of minimally bad paths* of $\mathbf{A}$, which is justified by the following lemma.

Lemma 5.28. *Let $\mathbf{A}$ be a path-dependent set of coloured graphs. Then every $P \in \mathcal{M}_{\mathbf{A}}$ does not have any $Q \in \mathcal{M}_{\mathbf{A}}$ as a strict subpath.*

Proof. Assume for the sake of contradiction that there are two distinct coloured paths, $P, Q \in \mathcal{M}_{\mathbf{A}}$, such that P is a subpath of Q. Let (G, α) be a coloured graph such that $Q \in \text{Sub}(G, \alpha) \cap \mathcal{B}_{\mathbf{A}}$. It follows that $P \in \text{Sub}(G, \alpha) \cap \mathcal{B}_{\mathbf{A}}$. Therefore Q is redundant, so $\mathcal{M}_{\mathbf{A}} \setminus \{Q\}$ is a determinant of $\mathbf{A}$. This contradicts the minimality of $\mathcal{M}_{\mathbf{A}}$. □

For path-dependent colour schemes, we summarize Lemmas 5.16–5.20 with the following lemma.

Lemma 5.29. *Let $\mathbf{A}$ be a path-dependent set of coloured graphs determined by the set of coloured paths $\mathcal{B}$. If $\mathcal{B}$ is closed under melding and does not contain any rainbow colourings, then $\mathbf{A}$ is a colour scheme.*

5.3 Colour scheme zoo

Having armed ourselves with the tools of Sections 5.1 and 5.2, it is time to establish that many types of graph colouring found in the literature are colour schemes. The primary goal of this section is to show the types of colouring reviewed in Section 1.4.1 are colour schemes, and to compare some of their bounds, as shown in Table 5.1. See Section 1.4.1 for a more detailed review of the literature. The colour schemes listed in Table 5.1 are defined in this section or Chapter 6.

The rightmost two columns of Table 5.1 say whether there exists a c such that every tree or graph has a subdivision with an admissible c-colouring, which is a formalisation of the concept of being 'bounded on sufficiently subdivided graphs'. For a colour scheme, $\mathbf{A}$, $c \in \mathbb{Z}^+$, and set of graphs, $\mathcal{G}$, we say $\mathbf{A}$ is *bounded by c on choice-subdivisions* of $\mathcal{G}$ if, for all $G \in \mathcal{G}$, there exists a subdivision, H, of G such that $(H, \alpha) \in \mathbf{A}$ for some c-colouring α. Often we only care that there is a bound, so say that $\mathbf{A}$ is *bounded on choice-subdivisions* of $\mathcal{G}$ when there exists $c \in \mathbb{Z}^+$ such that $\mathbf{A}$ is bounded by c on choice-subdivisions of $\mathcal{G}$.

5.3.1 Distance and exact-distance colouring

A colouring of a graph G is a *distance-k colouring* if vertices at distance at most k receive distinct colours [92]. Let $\mathbf{P}_k$ denote the colour scheme that corresponds to distance-k colouring. By definition, for vertices, u and v, at distance n there is a path of length n with u and v as endpoints. It follows that $\mathbf{P}_k$ is path-dependent with determinant, $\mathcal{B}$, such that $P \in \mathcal{B}$ if the endpoints of P have the same colour and $|V(P)| \leqslant k+1$. The endpoints of $P \in \mathcal{B}$ share a colour, so $\mathcal{B}$ does not contain any rainbow colourings. Clearly $\mathcal{B}$ is closed under melding. Therefore $\mathbf{P}_k$ is a path-dependent colour scheme, by Lemma 5.29. Recall that $\chi_{\mathbf{P}_k}(G) \leqslant \Delta(G)\left(\Delta(G)-1\right)^{k-1}+1$, for all graphs G, because distance-k colouring is equivalent to proper colouring G^k. For the bound on trees in Table 5.1, Agnarsson and Halldórsson [2] show that, for all $k \in \mathbb{Z}^+$, there exists c such that every planar graph, G, is distance-k $\left(c\Delta(G)^{\lfloor k/2 \rfloor}\right)$-choosable and that this bound is tight.

A colouring of a graph G is an *exact distance-k colouring* if vertices at distance exactly k receive distinct colours [109]. Let $\mathbf{E}_k$ denote the colour scheme that corresponds to distance-k colouring. There are at most Δ^k vertices at distance exactly k from a vertex in a graph of maximum degree Δ, so $\chi_{\mathbf{E}_k}(G) \leqslant 1 + \Delta(G)^k$ for all graphs G [129]. Bousquet et al. [28] show that $\chi_{\mathbf{E}_k}(T) \leqslant \Delta(T) + k + 1$, for all trees T. $\mathbf{E}_2$ is unbounded on choice-subdivisions of trees because exact-distance-2 colourings of stars require a distinct colour on each leaf. For odd k, $\mathbf{E}_k$ is bounded on many classes of graphs, including planar graphs [110, 129].

5.3.2 Pattern-free colouring

A word, W, matches a pattern $p = x_1x_2\dots x_n$ if W can be divided into n non-empty blocks, denoted $W = B_1B_2\dots B_n$, such that $B_i = B_j$ if $x_i = x_j$, for all $i, j \in [n]$. A *p-free colouring* of a graph G is a colouring such that the sequence of colours along paths in G do not match the pattern p. Let $\mathbf{W}_p$ be the set of coloured graphs corresponding to p-free colouring. The definition of p-free colouring is in terms of avoiding a set of bad paths, so p-free colouring is path-dependent. Observe that the set of coloured paths that match p is closed under melding, and if a symbol in p occurs at least twice, no rainbow colouring is matched by p. It follows that, by Lemma 5.29, $\mathbf{W}_p$ is a colour scheme provided that p contains a symbol that occurs at least twice.

		Bounds on:				
		paths	trees of maximum degree Δ	graphs of maximum degree Δ	choice-subdivisions of trees	choice-subdivisions of graphs
$\mathbf{Q}$	Square-free	3	4	$c\Delta^2$	3	3
$\mathbf{Q}_k$	$(k \geqslant 3)$-power-free	2	4	$c\Delta^{k/(k-1)}$	3	3
$\mathbf{N}$	Anagram-free	4	∞	∞	8	8
$\mathbf{N}_3$	3-anagram-free	3	?	∞	8	8
$\mathbf{N}_k$	$(k \geqslant 4)$-anagram-free	2	4	∞	8	8
$\mathbf{P}$	Proper	2	2	$\Delta + 1$	2	2
$\mathbf{P}_k$	Distance-$(k \geqslant 2)$	$k+1$	$c\Delta^{\lfloor k/2 \rfloor}$	$c\Delta^k$	∞	∞
$\mathbf{E}$	Exact-distance-2	2	$c\Delta$	$c\Delta^2$	∞	∞
$\mathbf{S}$	Star	3	3	$c\Delta^{3/2}$	3	3
$\mathbf{L}$	Acyclic	2	2	$c\Delta^{4/3}$	2	3
$\mathbf{G}_k$	$(k \geqslant 2)$-frugal	3	$c\Delta^{(k+1)/k}$	$c\Delta^{(k+1)/k}$	∞	∞
$\mathbf{I}$	Parity	∞	∞	∞	∞	∞
$\mathbf{X}$	Centred	∞	∞	∞	∞	∞
$\mathbf{C}$	Conflict-free	∞	∞	∞	∞	∞
$\mathbf{D}$	Odd-cycle	1	1	∞	1	1
$\mathbf{V}$	Even-cycle	1	1	∞	1	∞
$\mathbf{F}$	Conflict-two	3	$c\Delta$	$c\Delta^2$	∞	∞
$\mathbf{T}$	Parity-two	3	$c\Delta$	$c\Delta^2$	∞	∞
$\mathbf{Y}$	Min-parity-two	3	$c\Delta$	$c\Delta^2$	∞	∞
$\mathbf{U}$	Updown	3	?	?	4	∞
$\mathbf{R}$	$xWx\overleftarrow{W}$-free	3	$c\Delta^2$	$c\Delta^2$	8	8
$\mathbf{M}_\varepsilon$	ε-uniform-free	∞	∞	∞	∞	∞
$\mathbf{I}_k$	k-parity	∞	∞	∞	∞	∞

Table 5.1: A list of colour schemes mentioned in this thesis, with some bounds on their chromatic numbers. The top two sections contain colour schemes and results from the literature or, in the case of anagram-free colouring, from other chapters. References for these bounds are provided in this section. The bottom section contains colour schemes introduced in this chapter. The bounds in terms of maximum degree, Δ, are not exact and are only intended to show the first order term. An '∞' indicates that the colour scheme is known to be unbounded. A '?' indicates it is not known whether the colour scheme is absolutely bounded on the class of graphs, and that it is also not known whether it is bounded by a function of maximum degree.

5.3.3 Square-free and k-power-free colouring

k-power-free colouring is pattern-free colouring for the pattern x^k, so it is a path-dependent colour scheme for $k \geqslant 2$. Square-free colouring is denoted $\mathbf{Q}$ and k-power-free colouring is denoted $\mathbf{Q}_k$. See Section 1.2 for a review of the bounds on $\mathbf{Q}_k$.

5.3.4 Anagram-free and k-anagram-free colouring

Anagram-free colouring is denoted $\mathbf{N}$ and k-anagram-free colouring is denoted $\mathbf{N}_k$. Clearly $\mathbf{N}_k$ is path-dependent because it is defined in terms of avoiding bad paths. The set of anagrams is closed under melding and a k-anagram contains at least k occurrence of each of its colours so, by Lemma 5.29, $\mathbf{N}_k$ is a colour scheme for $k \geqslant 2$. See Chapters 3 and 4 for bounds on anagram-free colouring.

5.3.5 Parity colouring

A colouring of a graph G is a *parity colouring* if every path in G contains a colour that occurs an odd number of times. The set of coloured graphs corresponding to parity colouring is denoted $\mathbf{I}$. Since $\mathbf{I}$ is defined in terms of avoiding bad paths it is path-dependent. The set of paths in which every colour occurs an even number of times is closed under melding, and a bad path contains at least two occurrences of each colour so, by Lemma 5.29, $\mathbf{I}$ is a colour scheme. Bunde et al. [34] show that $\mathbf{I}$ is unbounded on paths.

5.3.6 Centred colouring

A colouring of a graph G is *centred* if every connected subpgraph of G contains a unique colour. A *unique colour* of a coloured graph, (G, α), is a colour, c, such that $\alpha(v) = c$ for exactly one $v \in V(G)$. The set of centred coloured graphs is denoted $\mathbf{X}$, and the first thing to note is that $\mathbf{X}$ is not path-dependent. This is shown in Figures 5.3 and 5.4 as they have the same set of coloured paths, but only 5.4 is in $\mathbf{X}$.

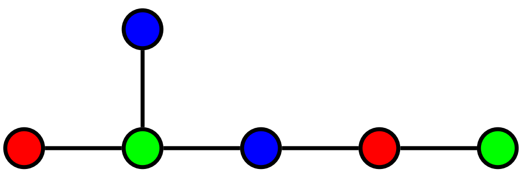


Figure 5.3: The coloured graph $(G_1, \alpha_1) \notin \mathbf{X}$.

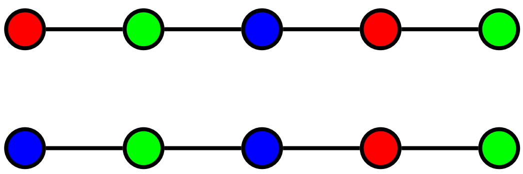


Figure 5.4: The coloured graph $(G_2, \alpha_2) \in \mathbf{X}$ and with $\mathrm{Sub}(G_1, \alpha_1) = \mathrm{Sub}(G_2, \alpha_2)$.

A *tree-dependent* colour scheme is $\mathcal{T}$-dependent, where $\mathcal{T}$ is the set of finite trees. Every connected graph has a spanning tree, so centred colouring is tree-dependent. Since $\mathbf{X}$ is tree-dependent, we can apply Lemmas 5.16–5.20 to establish that it is a colour scheme. All tree-dependent colour schemes satisfy Subgraphability, Additivity, and Locality because $\mathcal{T}$ is a set of connected finite graphs. Let $\mathcal{B}$ be the set of coloured trees that contain no unique colours and note that $\mathcal{B}$ determines $\mathbf{X}$. Clearly $\mathcal{B}$ is closed under melding and does not contain any rainbow colourings, so $\mathbf{X}$ is a colour scheme. Centred colouring is unbounded on paths [109].

5.3.7 Conflict-free colouring

A colouring of a graph G is *conflict-free* if every path in G contains a unique colour [61]. Let $\mathbf{C}$ denote the colour scheme corresponding to conflict-free colouring. First note that $\mathbf{C}$ is path-dependent because it is defined in terms of bad paths. The relationship between conflict-free colouring and centred colouring is interesting because, conflict-free colouring is centred colouring with its bad graphs restricted to paths. Formally, $\mathcal{B}_{\mathbf{C}} = \mathcal{B}_{\mathbf{X}}|_{\mathcal{P}}$, where $\mathcal{P}$ is the set of path. Since path-dependence implies tree-dependence, it follows that $\mathbf{X} \subseteq \mathbf{C}$, by Lemma 5.15.

The relationship between $\mathcal{B}_{\mathbf{C}}$ and $\mathcal{B}_{\mathbf{X}}$ allows for a quick proof that $\mathbf{C}$ is a colour scheme. Since $\mathbf{X}$ is a colour scheme, $\mathcal{B}_{\mathbf{X}}$ is closed under melding and does not contain any rainbow colourings. $\mathcal{B}_{\mathbf{C}} = \mathcal{B}_{\mathbf{X}}|_{\mathcal{P}}$, so $\mathcal{B}_{\mathbf{C}}$ is closed under melding and does not contain any rainbow colourings. It follows, by Lemma 5.29, that $\mathbf{C}$ is a colour scheme. Since $\mathbf{C}|_{\mathcal{P}} = \mathbf{X}|_{\mathcal{P}}$, conflict-free colouring is unbounded on paths.

5.3.8 Star colouring

A colouring of a graph G is a *star colouring* if every pair of colour classes induces a forest of stars [84]. The corresponding set of coloured graphs is denoted $\mathbf{S}$. An equivalent definition is that a star colouring is a proper colouring in which every path of order 4 contains three colours. This is the case because to induce a non-star requires a 2-coloured path of order 4. It follows that $\mathbf{S}$ is path-dependent, but, more precisely, $\mathbf{S}$ is $\{P_2, P_4\}$-dependent. Also note that $\mathcal{B}_{\mathbf{S}} = \mathcal{B}_{\mathbf{Q}}|_{\{P_2,P_4\}}$ so $\mathbf{Q} \subseteq \mathbf{S}$ by Lemma 5.15. Similarly to the case of conflict-free colouring, it follows that $\mathbf{S}$ is a colour scheme.

For the bounds in Table 5.1, Esperet and Parreau [60] show that $\chi_{\mathbf{S}}(G) \leqslant 2\sqrt{2}\Delta(G)^{3/2} + \Delta(G)$ for all graphs G. For every tree, T, with root r, the 3-colouring of T such that each $v \in V(T)$ with $\text{dist}(v, r) \equiv x \mod 3$ receives colour x, for $x \in \{1, 2, 3\}$, is a star 3-colouring of T. For the bound on choice-subdivisions, let S be the 3-subdivision of a graph, G, and colour each vertex $v \in V(S)$ by its distance to the closest original vertex to v in S. This colouring of S is a star 3-colouring.

5.3.9 Acyclic colouring

A colouring of a graph G is *acyclic* if every pair of colour classes induces a graph with no cycles [90]. Denote the corresponding set of coloured graphs by $\mathbf{L}$. Acyclic colouring is not path-dependent, as shown by Figures 5.5 and 5.6.

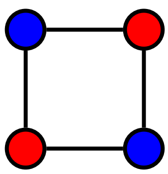


Figure 5.5: The coloured graph $(G_1, \alpha_1) \notin \mathbf{L}$.

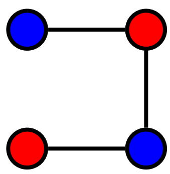


Figure 5.6: The coloured graph $(G_2, \alpha_2) \in \mathbf{L}$ and with $\text{Sub}(G_1, \alpha_1) = \text{Sub}(G_2, \alpha_2)$.

Acyclic colouring is defined in terms of avoiding monochromatic edges and 2-coloured cycles so $\mathbf{L}$ is $\mathcal{S}$-dependent where $\mathcal{S} := \{C_n : n \geqslant 3\} \cup \{P_2\}$. In particular, $\mathbf{L}$ is determined by the set of coloured graphs, $\mathcal{B}$, which consists of 2-coloured cycles and monochromatic paths of order 2. It is now straightforward to show that $\mathbf{L}$ is a colour scheme. The graphs in $\mathcal{S}$ are connected and finite, so Lemmas 5.17 and 5.18 apply. Every meld of $(G, \psi) \in \mathcal{B}$

is in $\mathcal{B}$ so Lemma 5.19 applies. None of the coloured graphs in $\mathcal{B}$ are rainbow colourings so Lemma 5.20 applies. Therefore $\mathbf{L}$ is a colour scheme.

For the bounds in Table 5.1, recall that Alon et al. [9] prove that $\chi_{\mathbf{L}}(G) = O(\Delta(G)^{4/3})$ and that Gonçalves et al. [67] optimise this result to $\chi_{\mathbf{L}}(G) \leqslant \frac{3}{2}\Delta(G)^{4/3} + c\Delta(G)$, for all graphs G. Since paths and trees contain no subcycles, the bounds on these classes of graphs follow from bounds on proper colouring. The bound on choice-subdivision of graphs follows because $\chi_{\mathbf{L}}(G) \leqslant \chi_{\mathbf{S}}(G)$, for all graphs G [62].

5.3.10 Frugal colouring

A proper colouring of a graph G is *k-frugal* if, for all $v \in V(G)$, each colour occurs at most k times in $N(v)$. Let $\mathbf{G}_k$ be the set of k-frugal coloured graphs. First note that $\mathbf{G}_k$ is S-dependent, where S is the set of stars, because k-frugal colouring is expressed in terms of the neighbourhood of each vertex of a graph. In particular, $\mathbf{G}_k$ is determined by the set of coloured stars, $\mathcal{B}$, with $G \in \mathcal{B}$ if G has a monochromatic edge or at least k leaves of G have the same colour. We now show that $\mathbf{G}_k$ is a colour scheme. Stars are connected finite graphs, so Lemmas 5.17 and 5.18 show that $\mathbf{G}_k$ satisfies Locality and Additivity. Fracturing cannot create new monochromatic edges or increase the multiplicity of a colour on the leaves, so $\mathbf{G}_k$ satisfies Recolourability. Clearly rainbow colourings are in $\mathbf{G}_k$, so $\mathbf{G}_k$ satisfies Universality. Therefore $\mathbf{G}_k$ is a star-dependent colour scheme. As previously noted, $\mathbf{G}_1 = \mathbf{P}_2$ and $\mathbf{G}_k \subseteq \mathbf{G}_{k+1}$, for all $k \geqslant 1$. Hind et al. [80] introduced k-frugal colouring and show that every graph with maximum degree $\Delta \geqslant e^{10^7}$ is $\log^8 \Delta$-frugal $(\Delta + 1)$-colourable. For the bound in Table 5.1, Kang and Müller [84] show that there exists a c such that $\chi_{\mathbf{G}_k}(G) \leqslant \frac{c}{k}\Delta(G)^{(k+1)/k}$ for all graphs G. Frugal colouring does not feature in the following sections, as we primarily focus on path-dependent colour schemes, but it serves as a notable example of a star-dependent colour scheme from the literature.

Chapter 6

Path-dependent colour schemes

Many types of graph colouring from the literature are colour schemes, and furthermore, many of these colour schemes are path-dependent. In this chapter, we apply the tools of Chapter 5 to study path-dependent colour schemes. This takes the form of general results about path-dependent colour schemes, as well as the construction of new path-dependent colour schemes, with novel properties.

Section 6.1 studies bounds for path-dependent colour schemes on choice-subdivisions of graphs. This investigation generalises analogous results in Chapter 4, and answers some questions raised by the similarity of ϕ and π on highly subdivided graphs. We find sufficient conditions for a colour scheme to be bounded on subdivisions of stars, as well as sufficient conditions for a colour scheme to be bounded on choice-subdivisions of graphs. Every path-dependent colour scheme, that we have identified in the literature, which is bounded on choice-subdivisions of trees is also bounded on choice-subdivisions of graphs. We ask whether this is true in general, and answer in the negative by constructing a counter-example. In Section 6.2 we study path-dependent colour schemes which are defined in terms of avoiding some set of permutations on paths. We find a sufficient condition for path-dependent colour schemes to be unbounded on trees. Conversely, we show that every path-dependent colour scheme with a sufficiently small set of minimally bad paths is bounded on graphs of bounded maximum degree. In Section 6.3 we study colour schemes which are subsets of $\mathbf{N}_k$, which is the colour scheme corresponding to k-anagram-free colouring. We define the families of colour schemes, $\mathbf{I}_k$, $\mathbf{M}_\varepsilon$, and $\mathbf{N}^k$, which have various interesting properties. Chapter 6 finishes with a summary of the relationships between colour schemes, and a discussion of the potential extensions and limitations of colour schemes.

6.1 Bounds on subdivisions

Given a path-bounded colour scheme, $\mathbf{A}$, it is natural to ask whether $\chi_{\mathbf{A}}$ is bounded on sufficiently subdivided graphs. This question received much interest for square-free colouring, see Section 1.2.6, and Chapter 4 answers the question for anagram-free colouring. The results from square-free colouring and anagram-free colouring motivate the question of whether every colour scheme that is bounded on subdivided stars is bounded on sufficiently subdivided graphs, since no counter-examples exist in the available literature. The intuition here is that every highly subdivided graph is, locally, a subdivision of a star or a path, and that this local behaviour may extend to all sufficiently subdivided graphs. This section explores this question and constructs novel colour schemes which answer it in the negative. Additionally, we give sufficient conditions for a colour scheme to be bounded on sufficiently subdivided graphs.

Recall that **D** and **V** are the colour schemes which avoid non-rainbow odd and even cycles, respectively. Both are bounded on subdivisions of trees, in fact, $\chi_{\mathbf{D}}(T) = 1$ and $\chi_{\mathbf{V}}(T) = 1$, for all trees T, because T has no subcycle. The 1-subdivision, H, of every graph G, has $\chi_{\mathbf{D}}(H) = 1$ because H has no subcycles of odd order. No analogous result holds for **V**, since, for every n, there is a N such that every edge 2-colouring of K_N contains a monochromatic graph isomorphic to K_n, by Ramsey's Theorem [117]. It follows that every subdivision of K_N contains a subgraph, S, isomorphic to a subdivision of K_n, with all edges subdivided either an even or odd number of times. S contains an even cycle of order at least $n-1$, so **V** is unbounded on choice-subdivisions of graphs. **V** is bounded on choice-subdivisions of trees, so it is a counter-example to the preceding question, however, **V** may be considered a trivial counter-example since it is not path-dependent. With this in mind, we restrict the questions in this section to path-dependent colour schemes.

6.1.1 Choice-subdivisions

Recall that, for class of graphs $\mathcal{G}$, a colour scheme, **A**, is *bounded on choice-subdivisions* of $\mathcal{G}$ if there exists $c \in \mathbb{Z}^+$ such that, for all $G \in \mathcal{G}$, there is a subdivision, H, of G such that $(H, \alpha) \in \mathbf{A}$ for some c-colouring, α, of H. The 'choice-subdivisions of $\mathcal{G}$' does not refer to a class of graphs, so our lemmas inferring bounds on colour schemes that are restricted to classes of graphs do not apply. However, some similar results hold.

Lemma 6.1. *Let $\mathcal{G}$ be a set of graphs and **A** be a colour scheme such that **A** is bounded by c on choice-subdivisions of $\mathcal{G}$. Then every colour scheme, **B**, such that $\mathbf{A} \subseteq \mathbf{B}$ is bounded by c on choice-subdivisions of $\mathcal{G}$.*

Proof. Let $G \in \mathcal{G}$ and H be a subdivision of G such that $(H, \alpha) \in \mathbf{A}$ for a c-colouring α. It follows that $(H, \alpha) \in \mathbf{B}$, so **B** is bounded by c on choice-subdivisions of $\mathcal{G}$. □

It follows from Lemma 6.1 that if **A** is bounded on choice-subdivisions of a set of graphs $\mathcal{G}$, then $\mathbf{A} \sqcup \mathbf{B}$ is bounded on choice-subdivisions of $\mathcal{G}$, for every colour scheme **B**. However, many results on classes of graphs do not transfer to choice-subdivisions of classes of graphs because being bounded on choice-subdivisions may require a particular number of subdivisions per edge. Recall Lemma 5.8, which says that $\mathbf{A} \cap \mathbf{B}$ is bounded on $\mathcal{G}$ if and only if **A** and **B** is bounded on $\mathcal{G}$. We now show that there is no analogous lemma for choice-subdivisions.

Theorem 6.2. *If $\mathcal{C}$ is the set of cycle graphs, then **D** and **V** are both bounded on choice-subdivisions of $\mathcal{C}$ but $\mathbf{D} \cap \mathbf{V}$ is not bounded on choice-subdivisions of $\mathcal{C}$.*

Proof. Both **D** and **V** are bounded by 1 on choice-subdivisions of $\mathcal{C}$ because every $G \in \mathcal{C}$ can be subdivided to an even or odd cycle, respectively. Note that $\mathbf{D} \cap \mathbf{V}$ is the cycle-dependent colour scheme which requires every cycle to have the rainbow colouring. It follows that $\mathbf{D} \cap \mathbf{V}$ is not bounded on choice-subdivisions of $\mathcal{C}$. □

A result analogous to Lemma 5.8 may exist for specific classes of graphs and colour schemes. We could also prove a stronger result with a stronger property than being bounded on choice-subdivisions. For example, a colour scheme **A** is *contiguously bounded by c on choice-subdivisions* of $\mathcal{G}$ if, for all $G \in \mathcal{G}$, there exists a k such that, for every $(\geqslant k)$-subdivision, H, of G, $(H, \alpha) \in \mathbf{A}$ for some c-colouring α, of H. This strengthening of choice-subdivisions may be useful for future work, but has some potentially problematic properties. For example, **D** and **V** are not contiguously bounded on choice-subdivisions of K_3. This is an unusual property, as we intuitively expect all notions of 'bounded' to be satisfied on all finite sets of graphs.

6.1.2 Subdivisions of stars

This section studies the question of which colour schemes are bounded on subdivisions of stars. The following results are also good examples of the types of properties that can be stated in terms of bad and minimally bad paths. Recall that $\mathcal{B}_{\mathbf{A}}$ and $\mathcal{M}_{\mathbf{A}}$ denote the bad and minimally bad paths of a path-dependent colour scheme $\mathbf{A}$, respectively.

A path-dependent colour scheme, $\mathbf{A}$, is *tame* if for every $P \in \mathcal{M}_{\mathbf{A}}$ and every $v \in V(P)$, there exists $P' \notin \mathcal{B}_{\mathbf{A}}$ obtained by recolouring v. We call this property 'tame' because it disallows minimally bad paths with wildcard characters.

Lemma 6.3. *A path-dependent colour scheme,* $\mathbf{A}$*, is tame if and only if every* $P \in \mathcal{M}_{\mathbf{A}}$ *has no unique colours.*

Proof. Let $\mathbf{A}$ be a tame path-dependent colour scheme. Assume for the sake of contradiction that $P \in \mathcal{M}_{\mathbf{A}}$ has a unique colour on $v \in V(P)$. The path, P', obtained by recolouring v with a new colour, is good, by the tameness of $\mathbf{A}$. Note that P' is a fracture of P so, by RECOLOURABILITY, $P \in \mathbf{A}$. This contradicts $P \in \mathcal{M}_{\mathbf{A}}$.

Now for the backwards implication. Let $\mathbf{A}$ be a path-dependent colour scheme such that every $P \in \mathcal{M}_{\mathbf{A}}$ contains no unique colour. Let $P \in \mathcal{M}_{\mathbf{A}}$ and $v \in V(P)$. Recolour v with a colour not in P to obtain P'. Since P' contains a unique colour, $P' \notin \mathcal{M}_{\mathbf{A}}$. Assume for the sake of contradiction that $P' \in \mathcal{B}_{\mathbf{A}}$. By definition, P' contains a strict subpath $Q \in \mathcal{M}_{\mathbf{A}}$. Now consider whether v is in Q. It cannot be that $v \in V(Q)$ because Q contains at least two occurrences of each of its colours. It also cannot be that $v \notin V(Q)$ because then Q would be a strict subpath of P, contradicting $P \in \mathcal{M}_{\mathbf{A}}$. In both cases we have a contradiction, so $\mathbf{A}$ is tame. □

Furthermore, tameness is equivalent to requiring that every bad path contains a minimally bad subpath with no unique colours.

Lemma 6.4. *Let* $\mathbf{A}$ *be a path-dependent colour scheme. Every* $P \in \mathcal{M}_{\mathbf{A}}$ *has no unique colours if and only if every* $Q \in \mathcal{B}_{\mathbf{A}}$ *has a subpath* $R \in \mathcal{M}_{\mathbf{A}}$ *with no unique colour.*

Proof. Let $\mathbf{A}$ be a path-dependent colour scheme such that every $P \in \mathcal{M}_{\mathbf{A}}$ has no unique colour. Every $Q \in \mathcal{B}_{\mathbf{A}}$ has a minimally bad subpath, $R \in \mathcal{M}_{\mathbf{A}}$, and $R \in \mathcal{M}_{\mathbf{A}}$ contains no unique colour.

Let $\mathbf{A}$ be a path-dependent colour scheme such that every every $Q \in \mathcal{B}_{\mathbf{A}}$ has a subpath $R \in \mathcal{M}_{\mathbf{A}}$ with no unique colour. By minimality, $P \in \mathcal{M}_{\mathbf{A}}$ only has itself as a minimally bad subpath so P does not have a unique colour. □

Lemma 6.3 implies that there is a set of coloured paths $\mathcal{T}$ such that $\mathcal{M}_{\mathbf{A}} \cap \mathcal{T} = \emptyset$ is equivalent to a path-dependent colour scheme, $\mathbf{A}$, being tame. Many properties can be stated in terms of $\mathcal{M}_{\mathbf{A}}$ avoiding some set of coloured paths. Let $\mathcal{C}$ be the set of coloured paths, up to a permutation of colours. We can factor out permutations due to the RECOLOURABILITY axiom of colour schemes. Let $\mathcal{C}_n \subseteq \mathcal{C}$ be the set of coloured paths of order n. To demonstrate, we list the first five entries of $\mathcal{C}_n$, up to reversing and relabelling, with the entries that cannot occur in the bad paths of a tame colour scheme highlighted.

$$\mathcal{C}_1 = \{\underline{1}\}$$
$$\mathcal{C}_2 = \{11, \underline{12}\}$$
$$\mathcal{C}_3 = \{111, 112, \underline{121}, \underline{123}\}$$
$$\mathcal{C}_4 = \{1111, 1112, 1121, 1122, 1123, 1212, \underline{1213}, 1221, 1223, \underline{1231}, \underline{1234}\}$$
$$\mathcal{C}_5 = \{11111, 11112, 11121, 11122, 11123, 11211, 11212, 11213, 11221, 11223, 11231, 11232, 11233, 11234, 12112, 12113, 12121, 12123, \underline{12131}, \underline{12132}, \underline{12134}, 12213, 12221, 12223, 12231, 12234, \underline{12312}, \underline{12314}, \underline{12321}, \underline{12341}, \underline{12345}\}$$

By inspecting $\mathcal{C}_n$ we can see that every tame path-dependent colour scheme is bounded on stars.

Theorem 6.5. *Let $\mathcal{A}$ be a tame path-dependent colour scheme. Then every star, S, has a 2-colouring, α, with $(S, \alpha) \in \mathbf{A}$.*

Proof. Let (G, α) be a coloured star with a unique colour on its root. By inspection, $121 \notin \mathcal{B}_{\mathbf{A}}$. Therefore $(G, \alpha) \in \mathbf{A}$. □

Many known path-dependent colour schemes are tame. An exception is distance-k colouring, denoted $\mathbf{P}_k$, for $k \geqslant 2$. The distance-2 chromatic number is not bounded on stars because $121 \in \mathcal{B}_{\mathbf{P}_2}$. As a corollary, $\chi_{\mathbf{P}_2}$ is unbounded on subdivisions of K_n. Note that $\mathbf{P}_2$ is path-bounded because $(123)^n \in \mathbf{P}_2$ for all $n \geqslant 1$. We now show that tameness is sufficient for path-bounded path-dependent colour schemes to be bounded on subdivisions of stars.

Theorem 6.6. *Every path-bounded and tame path-dependent colour scheme, $\mathbf{A}$, is bounded on subdivisions of stars.*

Proof. Let H be a subdivision of a star G. Let c be the constant bound for $\chi_{\mathbf{A}}$ for paths. Let α be the colouring of H defined as follows. Let r be the root of H. Colour the forest of paths $H - \{r\}$ with a c-colouring such that $(H - \{r\}, \alpha|_{H-\{r\}}) \in \mathbf{A}$, and colour r with a unique colour. By Lemma 6.3, every path, P, with $r \in V(P)$ is not in $\mathcal{M}_{\mathbf{A}}$. By construction, all paths, Q, in H which do not contain r are in $\mathcal{G}_{\mathbf{A}}$, so H does not have any bad subpaths. Therefore $(H, \alpha) \in \mathbf{A}$. □

Theorem 6.6 raises the question of whether tameness and path-boundedness are necessary for a colour scheme to be bounded on subdivisions of stars. This question is answered in the negative by $\mathbf{E}_k$, the colour scheme corresponding to exact distance-k colouring, for odd k. Clearly $\mathbf{E}_k$ is not tame since $(P_{k+1}, \alpha) \in \mathcal{M}_{\mathbf{E}_k}$ where α is a colouring such that the endpoints of P_{k+1} share a colour. $\mathbf{E}_k$ is bounded on planar graphs, which includes subdivisions of stars [129]. Perhaps a necessary condition for a path-dependent colour scheme to be bounded on subdivisions of stars could be obtained by a variant of tameness which disregards the minimally bad paths of even order, as we note that $\mathcal{M}_{\mathbf{E}_k}$ only contains paths of order $k + 1$.

We now construct three new colour schemes from $\mathbf{C}$, $\mathbf{I}$, and $\mathbf{P}_2$ to demonstrate the behaviour of colour schemes on subdivisions of stars. Recall that $\mathbf{C}$ is determined by the set of coloured paths that contain at least two occurrences of each colour, and that $\mathbf{I}$ is determined by the set of paths in which every colour occurs an even number of times. Define the colour schemes $\mathbf{F}$, $\mathbf{T}$ and $\mathbf{Y}$ such that $\mathbf{F}$ is determined by $\mathcal{B}_{\mathbf{P}_2} \cap \mathcal{B}_{\mathbf{C}}$, $\mathbf{T}$ is determined by $\mathcal{B}_{\mathbf{P}_2} \cap \mathcal{B}_{\mathbf{I}}$, and $\mathbf{Y}$ is determined by $\mathcal{B}_{\mathbf{P}_2} \sqcap \mathcal{M}_{\mathbf{I}}$. These colour schemes are related by $\mathbf{C} \subseteq \mathbf{F} \subseteq \mathbf{T} \subseteq \mathbf{Y}$ because $\mathcal{M}_{\mathbf{I}} \subseteq \mathcal{B}_{\mathbf{I}} \subseteq \mathcal{B}_{\mathbf{C}}$. Also, note that $\mathbf{P}_2 \subseteq \mathbf{F}$. The path coloured 1121 is in $\mathcal{M}_{\mathbf{T}}$, so $\mathbf{T}$ is not tame. Since $\mathbf{C}$ is bounded on stars, $\mathbf{T}$ and $\mathbf{F}$ are bounded on stars, which contrasts the behaviour of $\mathbf{P}_2$. However, $\mathbf{T}$ and $\mathbf{F}$ are unbounded on subdivisions of stars.

Theorem 6.7. $\mathbf{T}$ *and* $\mathbf{F}$ *are unbounded on subdivisions of stars.*

Proof. Fix a number of colours, c, and let (S, α) be a c-colouring of the (2^c)-subdivision of the star of order $c+2$. The root of S has degree $c+1$ so there exists a path $Q := v_1 r v_2 \in S$ with $\alpha(v_1) = \alpha(v_2)$, so $Q \in \mathcal{B}_{\mathbf{P}_2}$. Let P be the $v_1\ell$-path, where ℓ is the descendent of v_2 furthest from the root of S. $P \in \mathcal{B}_{\mathbf{P}_2}$, because Q is a subpath of P, and $P \in \mathcal{B}_{\mathbf{I}}$, because $|V(P)| = 2^c + 3$ and $\chi_{\mathbf{I}}(P_n) = \lceil \log_2(n+1) \rceil$ for all n [34]. Therefore $P \in \mathcal{B}_{\mathbf{P}_2} \cap \mathcal{B}_{\mathbf{I}}$ and since, $\mathcal{B}_{\mathbf{I}} \subseteq \mathcal{B}_{\mathbf{C}}$, we have $P \in \mathcal{B}_{\mathbf{P}_2} \cap \mathcal{B}_{\mathbf{C}}$. It follows that $\mathbf{T}$ and $\mathbf{F}$ are unbounded on subdivisions of stars. □

Perhaps the most interesting thing about these colour schemes is that $\mathbf{Y}$ is bounded on subdivisions of stars, even though it is defined in terms of colour schemes which are unbounded on subdivisions of stars.

Theorem 6.8. $\mathbf{Y}$ *is bounded on subdivisions of stars.*

Proof. A path is in $\mathcal{B}_{\mathbf{I}}$ if and only if it contains a parity path. It follows that every colour that occurs in $Q \in \mathcal{M}_{\mathbf{I}}$ occurs an even number of times, by the minimality of Q.

Let $P \in \mathcal{B}_{\mathbf{Y}}$ be a path with a unique colour. Since $\mathcal{B}_{\mathbf{P}_2} \cap \mathcal{M}_{\mathbf{I}}$ is a determinant of $\mathbf{Y}$, and $P \notin \mathcal{M}_{\mathbf{I}}$, there is a strict subpath, Q, of P with $Q \in \mathcal{B}_{\mathbf{P}_2} \cap \mathcal{M}_{\mathbf{I}}$. It follows that $P \notin \mathcal{M}_{\mathbf{Y}}$, so, by Lemma 6.3, $\mathbf{Y}$ is tame. $\mathbf{Y}$ is path-bounded because $\mathbf{P}_2$ is path-bounded, so, by Theorem 6.6, $\mathbf{Y}$ is bounded on subdivisions of stars. □

Another feature of $\mathbf{Y}$ is that $\mathbf{Y}$ has arbitrarily long minimally bad paths on 4 colours, which is not true of $\mathbf{P}_2$ or $\mathbf{I}$. For example, $323(123)^n 4(123)^n 42 \in \mathcal{M}_{\mathbf{Y}}$, for all $n \geqslant 1$, because $323(123)^n 4(123)^n 4 \in \mathcal{G}_{\mathbf{Y}}$ and $23(123)^n 4(123)^n 42 \in \mathcal{G}_{\mathbf{Y}}$.

6.1.3 Distinguishing subdivisions of trees and stars

Recall that $\mathbf{N}$ is the colour scheme corresponding to anagram-free colouring. Part of the intuition for Chapter 4 was that, since $\mathbf{N}$ is bounded on subdivisions of stars, it is potentially bounded on choice-subdivisions of graphs. This raises the question of whether all path-dependent colour schemes which are bounded on subdivisions of stars are bounded on choice-subdivisions of graphs. This question is answered in the negative, using the colour scheme $\mathbf{Y}$, defined in Section 6.1.2.

Theorem 6.9. *Every subdivision, H, of the complete $(c2^c+1)$-ary tree of height c, T, has $\chi_{\mathbf{Y}}(H) \geqslant c+1$.*

Proof. We proceed by induction on c. For the base case, all subdivisions, H, of the 3-ary tree of height 1 have $\chi_{\mathbf{Y}}(H) \geqslant 2$ because monochromatic edges are bad.

Let T be the complete $(c2^c+1)$-ary tree of height c, with root r, and H be a subdivision of T. By induction and SUBGRAPHABILITY, $\chi_{\mathbf{Y}}(H) \geqslant c$. Assume for the sake of contradiction that $\chi_{\mathbf{Y}}(H) = c$. Let $(H, \alpha) \in \mathbf{Y}$ be a c-coloured graph and, without loss of generality, let r be coloured red. Since there are $c2^c+1$ vertices adjacent to r, there is a monochromatic set of vertices, A, adjacent to r, of size at least 2^c+1.

Let $(S, \alpha|_S)$ be the subgraph of (H, α) obtained by deleting all vertices which do not have an ancestor or descendent in A. Let P be a path in $(S, \alpha|_S)$ that contains r as an interior vertex. Every vertex adjacent to r in $(S, \alpha|_S)$ has the same colour, so $P \in \mathcal{B}_{\mathbf{P}_2}$. Since $\mathbf{Y}$ is determined by $\mathcal{B}_{\mathbf{P}_2} \cap \mathcal{M}_{\mathbf{I}}$, we know $P \notin \mathcal{B}_{\mathbf{P}_2} \cap \mathcal{M}_{\mathbf{I}}$, so $P \notin \mathcal{M}_{\mathbf{I}}$. Therefore, P contains a colour that occurs an odd number of times.

Every rooted subtree, T_v, of every vertex in $v \in A$ contains at least one occurrence of each colour, because, by induction, every subdivision, H', of the complete $(c2^c + 1)$-ary tree of height $c - 1$ has $\chi_{\mathbf{Y}}(H') \geqslant c$. In particular, for every $v \in A$, T_v contains a red vertex, $t_v \in V(T_v)$. Let $\mathcal{R} \subseteq \mathrm{Sub}(S, \alpha|_S)$ be the set of rt_v-paths in S, for all $v \in A$. Each path in $P \in \mathcal{R}$ has an associated binary string of length c, where a 1 in position i indicates that the i^{th} colour occurs an odd number of times in P. There are 2^c such binary strings and $|\mathcal{R}| \geqslant 2^c + 1$, so there are two paths, P and Q, which have the same associated string. Let P' be the subpath of P with both endpoints deleted. One endpoint of P is r and both endpoints are red, so P', still has the same associated string as Q. The colours that occur an odd number of times in P' are the same colours that occur an odd number of times in Q. It follows that all colours occur an even number of times in $P'Q$, so $P'Q \in \mathcal{M}_{\mathbf{I}}$. Also, $P'Q$ contain r as an interior vertex, so $P'Q \in \mathcal{B}_{\mathbf{P}_2}$. It follows that $P'Q \in \mathcal{B}_{\mathbf{R}_2} \cap \mathcal{M}_{\mathbf{I}} \subseteq \mathcal{B}_{\mathbf{Y}}$, which is a contradiction, so $\chi_{\mathbf{Y}}(H) \geqslant c + 1$. □

Y is of particular interest because no colour schemes found in the literature are bounded on subdivisions of stars and unbounded on choice-subdivisions of trees.

6.1.4 Distinguishing subdivisions of graphs and trees

Thus far, the only colour scheme which is bounded on choice-subdivisions of trees but not on choice-subdivisions of graphs is **V**. It is unsurprising that there is a cycle-dependent colour scheme with this property, as trees are characterized by their lack of cycles. In this section we present a more surprising result, that there is a path-dependent colour scheme, **U**, which is bounded on choice-subdivisions of trees but not on choice-subdivisions of graphs.

We define **U** in terms of a determinant, which requires the following definitions. Recall that a coloured graph, (G, β), is a meld of (G, α) if, for all $u, v \in V(G)$, $\alpha(u) = \alpha(v)$ implies $\beta(u) = \beta(v)$. A *c-meld* of (G, α) is a meld, (G, β), of (G, α), such that (G, β) is coloured with at most c colours. For a coloured path P and set C, recall that $V_C(P)$ is the set of vertices in P assigned a colour in C. Then the *C-components of P* are the components of the forest induced by $V_C(P)$, and the *C-component sequence of P* is the sequence of orders of the C-components of P, listed in order of their occurrence in P. For example, the $\{1, 2\}$-components of $P := 12321321221323$ is the forest of paths 12, 21, 21221, and 2, and the $\{1, 2\}$-component sequence of P is $2, 2, 5, 1$. A sequence, $x_1, \ldots, x_n$, is *up-down* if there exists $i \in [n - 1]$ such that $x_1, \ldots, x_i$ is strictly increasing and $x_{i+1}, \ldots, x_n$ is strictly decreasing.

Let **U** be the path-determined set of coloured graphs determined by $\mathcal{B}$, where a path, P, is in $\mathcal{B}$ if every 4-meld of P has a subpath, Q, which violates one of the following properties.

(1) Q is distance-1 coloured (properly coloured).

(2) Q is distance-2 coloured, Q contains at least four colours, or $|V(Q)| \leqslant 5$.

(3) For all sets of three colours, C, that maximize $|V_C(Q)|$, the C-component sequence of Q is up-down.

We first check that $\mathbf{U}$ is a colour scheme. Every 4-meld of a meld of a path, P, is a 4-meld of P, so $\mathcal{B}$ is closed under melding. Now note that every path, P, with the distance-2 3-colouring $123123\ldots$, satisfies conditions (1), (2), and (3) because P is distance-2 coloured and its $\{1,2,3\}$-component sequence has length 1, and so is trivially up-down. A path of the form $123123\ldots$ is a 4-meld of every rainbow path, so $\mathcal{B}$ does not contain any rainbow paths. Therefore, by Lemma 5.29, $\mathbf{U}$ is a colour scheme.

The definition of $\mathbf{U}$ is somewhat artificial. Keep in mind that $\mathbf{U}$ is of interest because it satisfies the same natural axioms as many other types of graph colouring and has the surprising property of being bounded on choice-subdivisions of trees but not on choice-subdivisions of graphs.

Theorem 6.10. $\chi_{\mathbf{U}}$ *is bounded by* 4 *on choice-subdivisions of trees.*

Proof. Let T be a tree, rooted at an arbitrary vertex. Let $\ell : E(T) \to [|E(T)|]$ be a bijective labelling of $E(T)$ where edges are ordered by their distance from the root such that closer edges have larger label. Note that this labelling is up-down on each path in T.

Let S be the subdivision of T such that every edge, e, has $6 + 3\ell(e)$ division vertices. Let X_e be the path of division vertices of edge e, oriented such that the first vertex of X_e is adjacent to the parent vertex of e. Define the colouring ψ of S as follows. For all edges e, colour X_e with $123123\ldots123$ if e has even distance to the root and colour X_e with $321321\ldots321$ otherwise. Note that the neighbourhood of every original vertex is monochromatic and recall that edges adjacent to the root are at distance 0. Colour every original vertex with 4.

Let P be a path in (S,ψ), we now show that the identity 4-meld, that is, the meld of P that leaves P unaffected, satisfies Conditions (1), (2), and (3). Since (S,ψ) is properly coloured, P satisfies Condition (1). Now consider Condition (2). Since the division vertices of S are distance-2 coloured, Condition (2) is satisfied if P does not contain any original vertices. Every path in (S,ψ) around an original vertex is of the form 1234321 or 3214123, so, if P contains an original vertex, $|V(P)| \geqslant 6$ implies P contains 4 colours. Therefore P satisfies Condition (2).

Finally, consider Condition (3). The C-component sequence of a path with 3 colours or with a unique colour has length at most 2, so is trivially up-down. Therefore P satisfies Condition (3) if it contains at most one original vertex. Now consider the case where P contains at least two original vertices. Since every edge, $e \in E(T)$, has at least three occurrences of colours 1, 2, and 3, $C = \{1,2,3\}$ is the unique set of three colours that maximises $|V_C(P)|$. Let $X = x_1,\ldots,x_n$ be the $\{1,2,3\}$-component sequence of P. Note that $n := |X|$ is the number of edges of T that have division vertices in P. Let $Q = e_1\ldots e_n$ be the path in T such that $X_{e_i} \cap V(P) \neq \emptyset$ for all $i \in [n]$. Note that X_{e_i} is a subset of $V(P)$ for $i \in \{2,\ldots,n-1\}$. Therefore $x_i = \ell(e_i)$, for $i \in \{2, n-1\}$, and $x_i \leqslant \ell(e_i)$ otherwise. It follows that X is up-down, since every reduction of the start and end terms of an up-down sequence is up-down, so Condition (3) is satisfied. We have found a 4-meld of P that satisfies all three conditions, contradicting $P \in \mathcal{M}_{\mathbf{U}}$, so $(S,\psi) \in \mathbf{U}$. □

The proof of Theorem 6.10 demonstrates the purpose of each of the three conditions in the determinant of $\mathbf{U}$. Condition (1) forces paths to contain at least 3 colours. Condition (2) allows for the creation of 'gadgets' that make the colouring valid around original vertices, without impacting too heavily on the colouring of the division vertices. Condition (3) essentially requires that the graph be subdivided according to an up-down edge labelling, which is possible for trees but not for general graphs. As we will see, Condition (3) does the most work in making $\mathbf{U}$ unbounded on choice-subdivisions of graphs.

Theorem 6.11. *$\chi_{\mathbf{U}}$ is unbounded on choice-subdivisions of graphs.*

Proof. Assume for the sake of contradiction that $\chi_{\mathbf{U}}$ is bounded by c on choice-subdivisions of graphs. Fix an integer n such that $5\,(4^c) \leqslant \frac{n-1}{3c(c-1)} - \frac{7}{3}$. Let N be sufficiently large to guarantee a monochromatic clique of order n in every edge 2-colouring of K_N, the existence of which follows from Ramsey's Theorem [117]. Let $(H, \rho) \in \mathbf{U}$ be a c-coloured subdivision of K_N. Since $n > c$ and monochromatic edges are in $\mathcal{B}_{\mathbf{U}}$, H does not have a subgraph isomorphic to K_n. Therefore H has a ($\geqslant 1$)-subdivision of K_n, G, as a subgraph. Let $\psi := \rho|_G$. The remainder of the proof concerns the c-coloured ($\geqslant 1$)-subdivision of K_n, (G, ψ), which is in $\mathbf{U}$, by SUBGRAPHABILITY.

For all original vertices, $v \in V(G)$, let $D(v)$ be the set of uv-paths, for all $u \in V(G)$ such that $\operatorname{dist}(u, v) = 2$. Let S_v be the graph union, in G, of the paths $D(v)$. Note that S_v is isomorphic to a 1-subdivision of the star of order n. Let the *v-signature*, $s_v(\ell)$, of a leaf $\ell \in V(S_v)$ be the pair (x, y) where y is the colour of the parent of ℓ and x is the colour of ℓ. Let T_v be a maximal complete subtree of S_v such that all leaves share a v-signature. Let the *signature* of T_v be the v-signature shared by all of its leaves. There are $c(c-1)$ signatures, since monochromatic edges are in $\mathcal{B}_{\mathbf{U}}$, so T_v has at least $\frac{n-1}{c(c-1)}$ leaves. Let $\mathcal{V}$ be a maximal set of vertices such that, without loss of generality, T_v has signature $(1, 2)$ for every $v \in \mathcal{V}$. There is a tree, T_v, for every $v \in V(K_n)$ and there are $c(c-1)$ signatures, so $|\mathcal{V}| \geqslant \frac{n}{c(c-1)}$.

A *signature path of* G is a path, P, in G with $|V(P)| = 5$ and $V(P) \subseteq V(T_v)$, for some $v \in \mathcal{V}$. We now show that G has a cycle which contains many signature paths.

Claim 6.12. *There is a subgraph, S, of G such that S is a cycle and contains at least $p := \left\lfloor \frac{n-1}{3c(c-1)} - \frac{4}{3} \right\rfloor$ signature paths.*

Proof. The claim follows by appending the appropriate paths in G to find a cycle with the required properties. A *signature halfpath of* G is a root-to-leaf path of a tree in T_v for some $v \in \mathcal{V}$. A *signature endpath of* G is a path in G, with endpoints in $\mathcal{V}$, such that its first three and last three vertices are signature halfpaths. We will show that, for all $k \in \{0, \ldots, p\}$, there exists a set of paths $\mathcal{P}_k = \{P_1, \ldots, P_k\}$ that satisfy the following properties.

(1) The last vertex of P_i is the first vertex of P_{i+1}, for all $i \in [k-1]$.

(2) No pairs paths in $\mathcal{P}_k$ intersect, excepting intersections required by Property (1).

(3) P_i is a signature endpath, for all $i \in [k]$.

(4) P_i contains at most four original vertices, for all $i \in [k]$.

Since $\mathcal{P}_0$ is the empty set, it satisfies all conditions vacuously. We proceed by induction on k. Recall that G is a subdivision of H. For a path, P, in H, the *P-subdivision-path* is the path, Q, in G, such that Q is the subdivision of P in G. The *extension* of a signature halfpath, rv_1v_2, with $r \in \mathcal{V}$, is the rv-subdivision-path of G, where rv is the edge of G with v_1 as a division vertex. Note that v_1 is a division vertex because G is a ($\geqslant 1$)-subdivision of H. For $r \in \mathcal{V}$, let E_r be the set of extensions of signature subpaths of T_r, and note that, except for their shared endpoint, r, the paths in E_r are disjoint.

Let O_k be the set of original vertices that are in a path in $\mathcal{P}_k$. Let $v \in \mathcal{V} \setminus O_k$. Let u be the last vertex of P_k, in the case that $k \geqslant 1$, and let $u \in \mathcal{V} \setminus \{v\}$ otherwise. Every path in $\mathcal{P}_k$ starts and ends at original vertices because they are signature endpaths. It follows that every path, $P_i \in \mathcal{P}_k$, contributes an additional three original vertices, for $i \in \{2, \ldots, k\}$, and that P_1 contributes four original vertices, so $|O_k| \leqslant 3k+1$. Furthermore, there are at least $|E_u| - |O_k|$ extensions of signature paths of T_u that do not contain any vertices in $O_k \setminus \{u\}$. Recall $k \leqslant p := \left\lfloor \frac{n-1}{3c(c-1)} - \frac{4}{3} \right\rfloor$. It follows that

$$|E_u| - |O_k| \geqslant \frac{n-1}{c(c-1)} - 3k - 1 \geqslant \frac{n-1}{c(c-1)} - \left(\frac{n-1}{c(c-1)} - 4\right) - 1 \geqslant 3.$$

Therefore there is a $u' \notin O_k$ such that the uu'-subdivision-path is in E_u. Similarly, there is a $v' \notin O_k$ such that the vv'-subdivision-path is in E_v. Because $|E_v| - |O_k| \geqslant 3$, we can choose u', v and v' to be distinct vertices. Since G is a subdivision of a K_n, the $u'v'$-subdivision-path is in G. Let P_{k+1} be the $uu'v'v$-subdivision-path in G and $\mathcal{P}_{k+1} = \mathcal{P}_k \cup \{P_{k+1}\}$.

We now show that P_{k+1} satisfies the required conditions. Firstly, $P_i \in \mathcal{P}_{k+1}$, for all $i \in [k]$, satisfies the required conditions, by induction. By our construction, the start of P_{k+1} is the end of P_k, P_{k+1} does not intersect any other paths in $\mathcal{P}_{k+1}$, P_{k+1} is a signature endpath, and P_{k+1} contains at most four original vertices. Therefore $\mathcal{P}_i$ exists for all $i \in \{0, \dots, p\}$.

Let P be the concatenation of the paths in $\mathcal{P}_p$. Let x and y be the endpoints of P. We construct S by finding a xy-path which does not intersect P. As before,

$$|E_x| - |O_p| \geqslant \frac{n-1}{c(c-1)} - 3p - 1 \geqslant \frac{n-1}{c(c-1)} - \left(\frac{n-1}{c(c-1)} - 4\right) - 1 \geqslant 3.$$

Similarly, $|E_y| - |O_p| \geqslant 3$. Therefore there are distinct vertices $x', y' \notin O_p$ such that the xx'-subdivision-path is in E_x and the yy'-subdivision-path is in E_y. Let P' be $xx'y'y$-subdivision-path in G. We have shown that P' and P share endpoints and do not share any other vertices. P is a signature endpath containing at least p signature paths and P' is a signature endpath. Therefore the union of P and P' in G is a subcycle of G containing at least p signature paths. ☐

By Claim 6.12, there exists a subcycle, S, of G containing at least $\frac{n-1}{3c(c-1)} - \frac{7}{3}$ signature paths. Let $\mathcal{N} := V(S) \cap \mathcal{V}$ be the set of centre vertices of signature paths in S. Fix an orientation on S so we can use the words 'clockwise' and 'counter-clockwise'. For all $v \in \mathcal{N}$, let $L(v)$ be the set containing v and the three vertices clockwise of v, and $L^+(v)$ be $L(v)$ with the addition of the two vertices counter-clockwise of v.

We now show that, for all vertices $v \in \mathcal{N}$, paths P in S, and 4-melds, s, of P, $L^+(v) \subseteq V(P)$ implies $s(L(v))$ has four colours, provided that $s(P)$ satisfies Condition (2). Recall that $v \in \mathcal{N}$ is the middle vertex of a signature path in S. It follows that the path induced by $L^+(v)$ in S has colour sequence $W_v = 12\psi(v)21\psi(x_v)$, where x_v is the vertex three vertices clockwise of v. Note that the colour sequence 123213 violates Condition (2) since it is not distance-2 coloured, contains three colours and has length six. It follows that $s(\psi(v)) \neq s(\psi(x_v))$ in every 4-meld, s, of P that satisfies Condition (2).

Also, since 123211 violates Condition (1), $L^+(u) \cap L^+(v) = \emptyset$, for all distinct $u, v \in \mathcal{N}$, so $\operatorname{dist}(u, v) \geqslant 6$, for all distinct $u, v \in \mathcal{N}$. This simplifies the remainder of the proof as we can make statements about paths without worrying about degenerate overlaps.

For all $v \in \mathcal{N}$, let P_v be the xy-path of order $|V(S)|$, oriented clockwise in S, where x is one vertex clockwise of v and y is one vertex clockwise of x. Let $\mathcal{Q}$ be the set of paths with $P_v \in \mathcal{Q}$, for every $v \in \mathcal{N}$. Since $(G, \psi) \in \mathbf{U}$, every path $P_v \in \mathcal{Q}$ has at least one 4-meld satisfying Conditions (1), (2) and (3). Since there at most 4^c distinct 4-melds of a c-coloured graph, and n was defined to satisfy

$$5\,(4^c) \leqslant \frac{n-1}{3c(c-1)} - \frac{7}{3} \leqslant |\mathcal{Q}|,$$

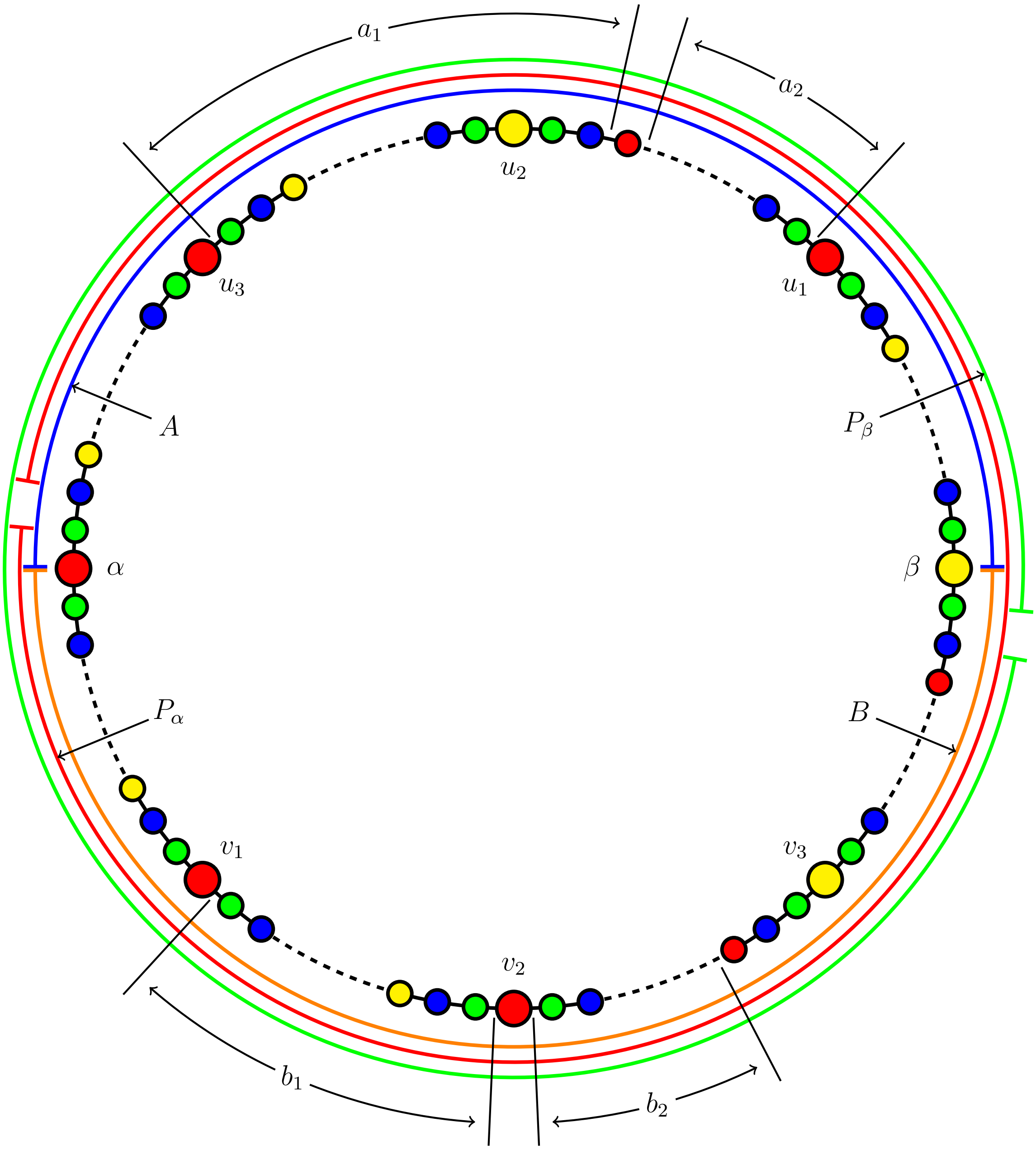


Figure 6.1: S with a 4-meld, s, such that, for all $v \in \mathcal{U}$, $s(P_v)$ satisfies Conditions (1), (2) and (3). The vertices α, β, u_1, u_2, and u_3 are in $\mathcal{U}$. The eight large vertices are in $\mathcal{N}$. Each vertex, $v \in \mathcal{N}$, is shown along with the vertices in $L^+(v)$. For every vertex, $v \in \mathcal{N}$, the set containing v and the three vertices clockwise of v is $L(v)$, which is known to contain all four colours in $(S, s(\psi|_S))$. The paths of order 5 with blue endpoints are signature paths in $\mathcal{G}$. Note that the sets of green and blue vertices have the same colour in (G, ψ), whereas the sets of red and yellow vertices may only have the same colour in $(G, s(\psi))$. The figure shows the case in which $C := \{\text{blue}, \text{green}, \text{yellow}\}$ maximises $|V_C(S, s(\psi|_S))|$ over sets of three colours. The numbers a_1, a_2, b_1, and b_2 indicate the sizes of four C-components shared by P_α and P_β. In general, the vertices of $\mathcal{U}$ are not necessarily arranged as shown here, and there may be more vertices from $\mathcal{N}$. The important property is, in all cases, A and B each contain at least five vertices from $\mathcal{N}$. This ensures that $s(A)$ and $s(B)$ each have two C-components which are shared by $s(P_\alpha)$ and $s(P_\beta)$.

there are five times as many paths in $\mathcal{Q}$ as there are distinct 4-melds of paths in $\mathcal{Q}$. Therefore, there exists a 4-meld, $s : [c] \to [4]$, and a set of at least five vertices, $\mathcal{U} \subseteq \mathcal{N}$, such that $s(P_v)$ satisfies Conditions (1), (2) and (3), for all $v \in \mathcal{U}$. Since $|\mathcal{N}| \geqslant 8$, $\mathcal{U} \subseteq \mathcal{N}$ and $|\mathcal{U}| \geqslant 5$, there exists two vertices, $\alpha, \beta \in \mathcal{U}$, such that both $\alpha\beta$-paths in S contain at least five vertices of $\mathcal{N}$. Let A and B be the two $\alpha\beta$-paths in S. Break the symmetry between A and B by having A be the $\alpha\beta$-path that contains the vertex immediately clockwise of α. Figure 6.1 shows the structure of P_α, P_β, A, and B in S.

Let C be a set of three colours that maximizes $|V_C(P_\alpha)|$, over all sets of three colours. Recall that $V(P_\alpha) = V(P_\beta)$, so C also maximizes $|V_C(P_\beta)|$. Let C_A, C_B, C_α and C_β be the C-component sequences of $s(A)$, $s(B)$, $s(P_\alpha)$ and $s(P_\beta)$ respectively. For a path, P, in S, let $o(P) := |X|$ where $X \subseteq \mathcal{N}$ is the largest set of vertices such that $L^+(v') \subseteq V(P)$ for all $v' \in X$. Note that $o(A) \geqslant 3$ and $o(B) \geqslant 3$ because both paths contain at least five vertices in $\mathcal{N}$. It follows that $|C_A| \geqslant 4$ and $|C_B| \geqslant 4$. Therefore at least two adjacent terms, a_1 and a_2, of C_A occur as adjacent terms in both C_α and C_β. Similarly, at least two adjacent terms, b_1 and b_2, of C_B occur as adjacent terms in both C_α and C_β. To further break symmetries, let a_1 and b_1 be the terms corresponding to C-components of A and B which are closer to α in A and B, respectively. It follows that these terms occur in the order a_1, a_2, b_2, b_1 in C_α and b_2, b_1, a_1, a_2 in C_β. Note that C_α and C_β are up-down, by Condition (3). Since C_α is up-down either $a_1 < a_2$ or $b_1 < b_2$, as only the two largest terms of an up-down sequence can be equal. Without loss of generality, take $a_1 < a_2$. Therefore, since C_β is up-down, we know that $b_2 < b_1 < a_1 < a_2$. This implies $a_1 < a_2$, $a_2 > b_2$ and $b_2 < b_1$, which is a contradiction of C_α being up-down. Therefore $\chi_{\mathbf{U}}$ is not bounded by c on choice-subdivisions of graphs. □

6.1.5 Sufficient conditions

The previous section demonstrates that, for path-dependent colour schemes, being bounded on choice-subdivisions of trees is not the same as being bounded on choice-subdivisions of graphs. This further motivates the search for non-trivial properties which are sufficient for a path-dependent colour scheme to be bounded on choice-subdivisions of graphs, a pair of which are presented in this section.

Let $\mathcal{F}$ be a determinant of a path-dependent colour scheme $\mathbf{A}$. A path, P, is $\mathcal{F}$-*adequate* with respect to $\mathbf{A}$ if there exists a subpath, $Q \in \mathcal{G}_{\mathbf{A}}$, of P such that, for all subpaths $R \in \mathcal{F}$ of P, Q is not a subpath of R. Since $\mathcal{F}$ is a determinant of $\mathbf{A}$, every good path of $\mathbf{A}$ is $\mathcal{F}$-adequate and every minimally bad path of $\mathbf{A}$ is not $\mathcal{F}$-adequate. The more interesting $\mathcal{F}$-adequate paths are the ones in $\mathcal{B}_{\mathbf{A}}$. As an example, consider $\mathbf{N}$, the colour scheme corresponding to anagram-free colouring. Recall that the split of a path P of even order is the pair of paths, L and R, such that, $LR = P$ and $|V(L)| = |V(R)|$. Also recall that the colour multiset function, M, counts the occurrences of colours on a path. Let $\mathcal{W}$ be the determinant of $\mathbf{N}$ such that $P \in \mathcal{W}$ if $M(L) = M(R)$ for the split, L and R, of P. The coloured path, $P := 212132123$, is bad, but also $\mathcal{W}$-adequate, because the good subpath, $Q := 213212$, of P is not a subpath of any subpath, $R \in \mathcal{W}$, of P.

Recall that P *restricted to* C is the word $f(v_1)f(v_2)\dots f(v_x)$, where $v_1, v_2, \dots, v_x$ are the vertices in $V_C(P)$, in the order defined by P, and f is the colouring of P. The *de-merge* of a path, P, by a set of colours, C, is the path coloured by the word P restricted to C. For example, the de-merge of 1214312343 by $\{1, 3\}$ is 113133. A path-dependent colour scheme, $\mathbf{A}$, is $\mathcal{F}$-*fixable* if $\mathcal{F}$ is a determinant of $\mathbf{A}$ and, for all paths $P \in \mathcal{M}_{\mathbf{A}}$ and sets of colours C, the de-merge of P by C is not an $\mathcal{F}$-adequate path of $\mathbf{A}$. This definition can be rephrased as follows.

Lemma 6.13. *A path-dependent colour scheme,* $\mathbf{A}$*, with determinant* $\mathcal{F}$*, is* $\mathcal{F}$*-fixable if and only if for all coloured paths* P*, the existence of a set of colours,* C*, such that the de-merge of* P *by* C *is* $\mathcal{F}$*-adequate, implies* $P \notin \mathcal{M}_{\mathbf{A}}$*.*

Proof. The proof follows by logical manipulation of the definition of $\mathcal{F}$-fixable. □

With Lemma 6.13, we can show that a path P is not minimally bad with respect to a $\mathcal{F}$-fixable path-dependent colour scheme by finding a $\mathcal{F}$-adequate de-merge of P. The intuition behind the name 'fixable' is that we can 'fix' the minimal badness of a path, P, by merging P with a $\mathcal{F}$-adequate path, Q, on a distinct set of colours. A *merge*, R, of two coloured paths $P = p_1 \dots p_i$ and $Q = q_1 \dots q_j$ is a path graph with $V(R) = V(P) \cup V(Q)$ which retains the order and colours of the vertices in P and Q. For example $v_1 v_2 v_a v_3 v_b v_c v_d v_4 v_e v_5$ is a merge of $v_1 v_2 v_3 v_4 v_5$ and $v_a v_b v_c v_d v_e$.

Lemma 6.14. *Let* $\mathbf{A}$ *be an* $\mathcal{F}$*-fixable colour scheme. Let* $P \in \mathcal{M}_{\mathbf{A}}$ *and* Q *be a* $\mathcal{F}$*-adequate path such that* P *and* Q *have no colours in common. Every merge of* P *and* Q *is not minimally bad.*

Proof. Let R be a merge of P and Q. Let C be the set of colours that appear in Q. The de-merge of R by C is Q, and Q is $\mathcal{F}$-adequate, so $R \notin \mathcal{M}_{\mathbf{A}}$. □

$\mathcal{F}$-fixable is a stronger property than tame, which is encouraging if we are to prove that $\mathcal{F}$-fixability is part of a pair of conditions which are sufficient for a path-dependent colour scheme to be bounded on choice-subdivisions of graphs.

Lemma 6.15. *Every path in* $\mathcal{M}_{\mathbf{A}}$ *contains at least two occurrences of each of its colours for every* $\mathcal{F}$*-fixable colour scheme* $\mathbf{A}$*.*

Proof. Assume for the sake of contradiction that $P \in \mathcal{M}_{\mathbf{A}}$ contains a unique colour. Let Q be the de-merge of P by its unique colour. Note that $|V(Q)| = 1$, so $Q \in \mathcal{G}_{\mathbf{A}}$. Therefore Q is $\mathcal{F}$-adequate, which contradicts the $\mathcal{F}$-fixability of $\mathbf{A}$. □

$\mathcal{F}$-fixability is one of the two conditions required by Theorem 6.18, which gives sufficient conditions for a colour scheme to be bounded on choice-subdivisions of graphs. This result extends Chapter 4, so we show that anagram-free colouring is $\mathcal{W}$-fixable.

Lemma 6.16. $\mathbf{N}$ *is* $\mathcal{W}$*-fixable, where* $\mathcal{W}$ *is the determinant of* $\mathbf{N}$ *such that* $P \in \mathcal{W}$ *if* $M(L) = M(R)$ *for the split,* L *and* R*, of* P*.*

Proof. Every minimally bad path of $\mathbf{N}$ is an anagram. Let $PQ \in \mathcal{M}_{\mathbf{N}}$ with $|V(P)| = |V(Q)|$ and C be a set of colours. Let RS be the de-merge of PQ by C with $|V(R)| = |V(S)|$. If RS is the empty path, then RS is not $\mathcal{W}$-adequate, so take the case $|V(RS)| \geqslant 1$. Since $M(P) = M(Q)$, $M(R) = M(S)$, so $RS \in \mathcal{W}$. Since, every good subpath of RS is a subpath of $RS \in \mathcal{W}$, RS is not $\mathcal{W}$-adequate. □

For the second condition of Theorem 6.18 we define a type of coloured path that bestows $\mathcal{F}$-adequacy on sufficiently small paths that contain it as a subpath. A path, $P \in \mathcal{G}_{\mathbf{A}}$, is $(\mathcal{F}, \varepsilon)$-*great* with respect to a path-dependent colour scheme, $\mathbf{A}$, if every path Q, such that P is a subpath of Q and $\varepsilon|V(Q)| \leqslant |V(P)|$, is $\mathcal{F}$-adequate. To take an example, using $\mathcal{W}$ as defined in Lemma 6.16, the path, $P := 1234$, is $(\mathcal{W}, \frac{4}{7})$-great with respect to $\mathbf{N}$ because P is not a subpath of any path in $\mathcal{W}$ of order at most 7. A path-dependent colour scheme, $\mathbf{A}$, is $(\mathcal{F}, \varepsilon, c)$-*great* if there are arbitrarily long $(\mathcal{F}, \varepsilon)$-great paths on c colours with respect to $\mathbf{A}$. Anagram-free colouring is $(\mathcal{W}, \frac{32}{33}, 5)$-great.

Lemma 6.17. $\mathbf{N}$ *is* $(\mathcal{W}, \frac{32}{33}, 5)$*-great, where* $\mathcal{W}$ *is the determinant of* $\mathbf{N}$ *such that* $P \in \mathcal{W}$ *if* $M(L) = M(R)$ *for the split,* L *and* R*, of* P*.*

Proof. Let $n \geqslant 15$ be an integer and $\varepsilon := \frac{32}{33}$. Let $P' \in \mathbf{N}$ be an anagram-free path of order $2n$ coloured by $\{1, 2, 3, 4\}$. Let P be the fracture of P' such that vertices with colour 4 in the first half of P' are coloured red in P. Let L and R be the split of P. Let Q be a coloured path with P as a subpath and $\varepsilon|V(Q)| \leqslant |V(P)|$.

We now show that there is no subpath, $S \in \mathcal{W}$, of Q with P as a subpath. Since P is good, this is all that is required to show that Q is $\mathcal{W}$-adequate. Let $L'R'$, with split L' and R', be a subpath of Q such that P is a subpath of $L'R'$. Since $|V(L'R')| > |V(LR)|$, it follows that L is a subpath of L' or R is a subpath of R'. Without loss of generality, let L be a subpath of L'. Recall that every anagram-free word on length 8 has at least four characters [39]. Therefore L, and thus L', contains at least $n/15$ red vertices. Note that $V(R') \cap V(L) = \emptyset$ and that R contains no red vertices. Since $|V(Q)| - |V(P)| \leqslant 2n\left(\frac{1}{\varepsilon} - 1\right) \leqslant \frac{n}{16} < \frac{n}{15}$, R' has fewer than $n/15$ red vertices. It follows that $M(L') \neq M(R')$, so $L'R' \notin \mathcal{W}$. □

$(\mathcal{F}, c, \varepsilon)$-great is the second sufficient condition, along with $\mathcal{F}$-fixable, for a path-dependent colour scheme to be bounded on choice-subdivisions of graphs. Since $\mathbf{N}$ is $\mathcal{W}$-fixable and $(\mathcal{W}, \frac{32}{33}, 5)$-great, the following theorem generalises Theorem 4.10, except for the value of the bound attained.

Theorem 6.18. *Let* $\mathbf{A}$ *be a path-bounded,* $\mathcal{F}$*-fixable and* $(\mathcal{F}, \varepsilon, c)$*-great path-dependent colour scheme for some determinant,* $\mathcal{F}$*, of* $\mathbf{A}$*,* $\varepsilon \in (0, 1)$*, and* $c \in \mathbb{Z}^+$*. Every graph,* G*, has a subdivision* H *such that* $\chi_{\mathbf{A}}(H) \leqslant 3c + 1$*.*

Proof. Denote the edges of G by $E(G) = \{e_1, \dots, e_m\}$. Let H be a subdivision of G such that, for all $i \in [m]$, there is a $(\mathcal{F}, \varepsilon)$-great path on c colours of order $|e_i|/3$ and

$$\varepsilon \sum_{j=1}^{i} |e_j| \leqslant |e_i|,$$

where $|e_i|$ is the number of division vertices of e_i. For every $i \in [m]$, let X_i, Y_i and Z_i be paths in H such that $X_iY_iZ_i$ is the path on the division vertices of e_i and $|V(X_i)| = |V(Y_i)| = |V(Z_i)| = |e_i|/3$. Let $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ be the sets containing X_i, Y_i and Z_i for all $i \in [m]$, respectively.

We define the $(3c+1)$-colouring, ψ, of H as follows. Assign disjoint sets of c colours, denoted C_x, C_y and C_z, to each of $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$. For all $i \in [m]$, colour the paths $X_i \in \mathcal{X}$, $Y_i \in \mathcal{Y}$, and $Z_i \in \mathcal{Z}$ with a $(\mathcal{F}, \varepsilon)$-great colouring with colours from C_x, C_y, and C_z, respectively. Fix a colour, $c' \notin C_x \cup C_y \cup C_z$, and colour all the original vertex of H with c'. Having defined (H, ψ), we now show that every coloured path, P, in (H, ψ) is good. Let $P \in \mathrm{Sub}(H, \psi)$, it is sufficient to show that P is not minimally bad.

The proof is split into three cases, depending on how many original vertices are in P. Consider the case where P contains no original vertices. Without loss of generality, P intersects X_i for some i. All subpaths of X_i are good and colours from C_x occur nowhere else in P, so the de-merge of P by C_x is a subpath of X_i. All subpaths of X_i are $\mathcal{F}$-adequate because X_i is good, so, since $\mathbf{A}$ is $\mathcal{F}$-fixable, P is not minimally bad. Now consider the case where P contains exactly one original vertex. The original vertex has a unique colour in P so, by Lemma 6.15, P is not minimally bad.

In the remaining case, P contains at least two original vertices. Since P contains all of the division vertices of an edge of G, at least one path from each of $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ is a subpath of P. Every vertex in every path in $\mathcal{X} \cup \mathcal{Y} \cup \mathcal{Z}$ has degree 2 in S, so P partially intersects at most two paths from $\mathcal{X} \cup \mathcal{Y} \cup \mathcal{Z}$. Therefore, without loss of generality, P entirely contains at least one path in $\mathcal{X}$ and does not partially intersect any paths in $\mathcal{X}$. Let X_k be the longest path of $\mathcal{X}$ that is a subpath of P and note that $|V(X_k)| = |e_k|/3$. Let Q be the de-merge of P by C_x, and note that X_k is a subpath of Q. Since X_k is $(\mathcal{F}, \varepsilon)$-great and

$$\varepsilon|V(Q)| \leqslant \frac{\varepsilon}{3}\sum_{j=1}^{k}|e_j| \leqslant \frac{|e_k|}{3} = |V(X_k)|,$$

Q is $\mathcal{F}$-adequate. Therefore, since $\mathbf{A}$ is $\mathcal{F}$-fixable, P is not minimally bad. □

Theorem 6.18 gives a bound of 16 for $\chi_{\mathbf{N}}$ on choice-subdivisions of graphs, while Theorem 4.13 gives a bound of 8. This is to be expected as the latter result uses more details about anagram-free colouring. An interesting feature of Theorem 6.18 is that it makes no reference to paths of the form $W\sigma(W)$, even though all our examples of path-dependent colour schemes which are bounded on choice-subdivisions of graphs are permutation-avoiding colour schemes. Furthermore, since all our examples of path-dependent colour schemes which are bounded on choice-subdivisions of graphs are supersets of $\mathbf{N}$, it is open as to whether the two conditions of Theorem 6.18 are necessary as well as sufficient.

6.2 Permutation-avoiding colouring

Square-free colouring and anagram-free colouring have similar definitions and distinctive behaviour, which motivates the study of further variations on their definitions. In this section we show that anagram-free colouring is unbounded on trees due to its avoidance of a small set of permutations, but that it is unbounded on graphs of bounded degree because it avoids a large set of permutations.

Previously we observed that square-free colouring and anagram-free colouring can both be defined as avoiding a set of words, $W\sigma W$, for $\sigma \in F$, where F is a set of permutations. With colour scheme notation, we now define this observation more precisely. For $k \geqslant 2$, a path-dependent set of coloured graphs is *k-permutation-avoiding* for a set of permutations, $\mathcal{S}$, if it is determined by a set of coloured paths, $\mathcal{B}$, such that $P \in \mathcal{B}$ if P has colour sequence $W_1W_2\dots W_k$ where, for all $i \in [k-1]$, $W_{i+1} = \sigma(W_i)$ for some $\sigma \in \mathcal{S}$. All k-permutation-avoiding sets of coloured graphs are colour schemes.

Lemma 6.19. *Every k-permutation-avoiding set of coloured graphs is a colour scheme.*

Proof. Let $\mathbf{A}$ be a k-permutation-avoiding path-dependent set of coloured graphs for a set of permutations $\mathcal{S}$. Using Lemma 5.29, we only need to show that $\mathcal{B}$ does not contain rainbow colourings and is closed under melding. $\mathcal{B}$ does not contain rainbow colourings because every colour on $P \in \mathcal{B}$ occurs k times. Let $P \in \mathcal{B}$ be a path with colour sequence $W_1W_2\dots W_k$. Let Q be a meld of P, and denote the colour sequence of Q by $W'_1W'_2\dots W'_k$. For all $i \in [k-1]$, $W_{i+1} = \sigma_i(W_i)$ for $\sigma_i \in \mathcal{S}$, so $W'_{i+1} = \sigma_i(W'_i)$. Therefore $Q \in \mathcal{B}$ so $\mathcal{B}$ is closed under melding. □

A colour scheme is *permutation-avoiding* if it is k-permutation avoiding for some $k \geqslant 2$. The colour scheme corresponding to anagram-free colouring, $\mathbf{N}$, is 2-permutation-avoiding for the set of all permutations. The colour scheme corresponding to square-free colouring, $\mathbf{Q}$, is 2-permutation-avoiding for the set of identity permutations. Anagram-free colouring, $\mathbf{N}$, is a notable permutation-avoiding colour scheme because every permutation-avoiding colour scheme, $\mathbf{A}$, satisfies $\mathbf{N} \subseteq \mathbf{A}$. We can isolate properties of $\mathbf{N}$ by studying permutation-avoiding colour schemes determined by subsets of determinants of $\mathbf{N}$.

In this section we prove general results about path-dependent colour schemes and apply them to a new permutation-avoiding colour scheme, $\mathbf{R}$. We construct $\mathbf{R}$ to show that two properties studied throughout this thesis, being bounded on trees and bounded on graphs of bounded maximum degree, are not correlated on permutation-avoiding colour schemes. Recall that $\mathbf{Q}$ is bounded on both classes of graphs and $\mathbf{N}$ is bounded on neither class. As shown in Section 3.3, $\mathbf{N}_4$ is bounded on trees but not bounded on graphs of bounded degree. Our new example, $\mathbf{R}$, completes the set as it is bounded on graphs of bounded maximum degree and not bounded on trees. In terms of $\mathbf{N}$, Theorem 6.20 shows that $\mathbf{N}$ is unbounded on trees because it avoids a particular small set of permutations, whereas Theorem 6.21 shows that there is no small set of permutations that cause a permutation-avoiding colour scheme to be unbounded on graphs of bounded maximum degree.

6.2.1 Bounds on trees

Recall that Theorem 3.3 shows that $\chi_{\mathbf{N}}$ is unbounded on trees. Recall that the proof of Theorem 3.3 is stronger than required, as it showed that every coloured tree has a path of the form $xWx\overleftarrow{W}$, where x is a character, W is a word, and $\overleftarrow{W}$ is W written backwards. As such, every colour scheme that avoids words of the form $xWx\overleftarrow{W}$ is unbounded on trees. To make this notion precise, we say that a path-dependent colour scheme, $\mathbf{A}$, is *reversible* if, for all $n \geqslant 1$, there exists a good path, P, with $|V(P)| \geqslant n$, that has colour sequence $xWx\overleftarrow{W}x$, for some word W and character x. Let $\mathbf{R}$ be the permutation-avoiding colour scheme which avoids the set of permutations, R, with $\sigma_n \in R$ for $\sigma_n : [n] \to [n]$ defined by $\sigma_n(1) := 1$ and $\sigma_n(i) := n + 2 - i$ for $i \in \{2, \ldots, n\}$. Let *$xWx\overleftarrow{W}$-free colouring* be the variant of colouring that corresponds to $\mathbf{R}$, and note that $\mathbf{R}$ is not reversible. We generalise Theorem 3.3 with the following theorem.

Theorem 6.20. *Every path-dependent colour scheme,* $\mathbf{A}$*, which is not reversible is unbounded on trees.*

Proof. $\mathbf{A}$, is not reversible so there exists $n \geqslant 1$ such that all coloured paths of order at least n and colour sequence of the form $xWx\overleftarrow{W}x$ are bad. Fix an integer c and let d and h be positive integers such that $d^{n(c+1)} \leqslant d^{h+1}c^{-h}$. Let T be the complete d-ary tree of height h with root r. Let L be the set of leaves of T, and fix an arbitrary c-colouring, $\psi : V(T) \to [c]$, of T.

There are at most c^h sequences of colours in root-to-leaf paths since each sequence has length $h + 1$ and they all start with $\psi(r)$. Since $|L| = d^h$ there is a set, $C \subseteq L$, of size at least d^h/c^h such that $S_v = S_w$ for all $v, w \in C$. We have found a large set of leaves with the same colour sequence on their root-to-leaf paths, as illustrated in Figure 6.2. Let R be the subtree of T induced by the set of all ancestors of leaves in C.

Define a *level of* R to be a maximal set of vertices of R that all have equal depth. R is coloured by level, as $\psi(u) = \psi(v)$ for every pair of vertices $u, v \in V(R)$ with the same depth. Let $\ell_0, \ell_1, \ldots, \ell_h$ be the sets of vertices corresponding to levels of R, where $\ell_0 = \{r\}$ and $\ell_h = C$. A level, ℓ_i, is *bad* if every vertex $v \in \ell_i$ has exactly one child in R. A level is *good* if it is not bad. Note that only level ℓ_h contains vertices with no children. Let g be the number of good levels of R and b be the number of bad levels of R. By definition, $h + 1 = g + b$. We now prove that there are at least $n(c + 1)$ good levels.

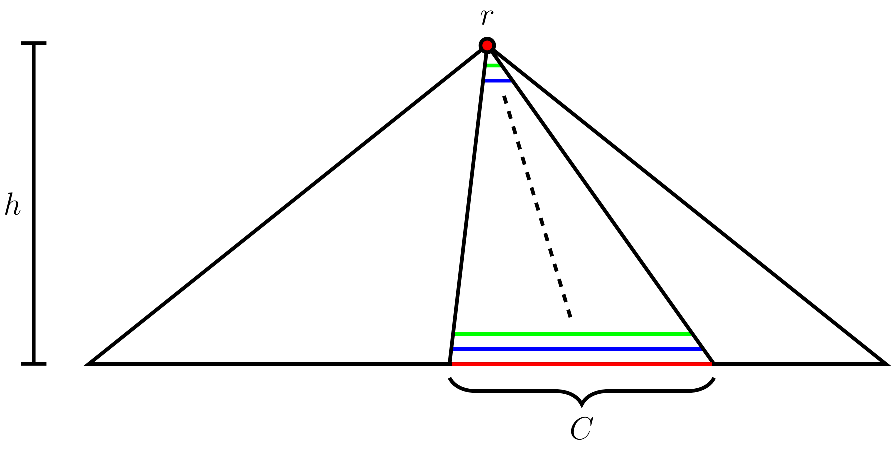


Figure 6.2: The complete d-ary tree of height h with a large set C of leaves which have the same root-to-leaf path colour sequence.

We bound the number of bad levels by considering the number of good levels required to attain $|\ell_h| \geqslant (d/c)^h$. If ℓ_i is bad, then $|\ell_i| = |\ell_{i+1}|$, and if ℓ_i is good, then $|\ell_i| < |\ell_{i+1}| \leqslant d|\ell_i|$. It follows that

$$|\ell_i| \leqslant d^{\#\text{preceding good levels}}.$$

Since ℓ_h is the final good level, it is preceded by $g-1$ good levels. Thus

$$(d/c)^h \leqslant |\ell_h| \leqslant d^{g-1}$$

so $d^{h+1}c^{-h} \leqslant d^g$. Recall that $d^{n(c+1)} \leqslant d^{h+1}c^{-h}$, so $n(c+1) \leqslant g$. There are at least $n(c+1)$ good levels, so there are integers, $a, b \in \{0, \dots, h\}$ such that good levels ℓ_a and ℓ_b have the same colour and $a + k = b$ for some $k \geqslant n$. Let $t \in \ell_a$ be a vertex with at least two descendants in ℓ_b. Let $u, v \in C$ be two distinct leaves of ℓ_b which are descendants of t. Note that $\text{dist}(u,t) = k$ and $\text{dist}(v,t) = k$. Let $p_0, p_1, \dots, p_k$ be the ut-path, $q_0, q_1, \dots, q_k$ be the vt-path, and P be the uv-path. Since R is coloured by level, $\psi(q_i) = \psi(p_i)$, for all $i \in \{0, \dots, k\}$. Also, $\psi(p_0) = \psi(q_0)$. It follows that

$$\psi\left(p_0 p_1 \dots, p_{k-1}\right) = \overleftarrow{\psi\left(q_{k-1} q_{k-2} \dots, q_0\right)}$$

so the colour sequence of P is $xWx\overleftarrow{W}x$ for $W := \psi\left(p_0 p_1 \dots, p_{k-1}\right)$ and $x := \psi(p_n)$. Since $k \geqslant n$ we know $|V(P)| \geqslant 2n+1$. Therefore, $P \in \mathcal{B}_{\mathbf{A}}$, so T has no c-colouring in $\mathbf{A}$. $\square$

$\mathbf{R}$ is bounded by 8 on choice-subdivisions of graphs, by Lemma 6.1, because $\mathbf{N} \subseteq \mathbf{R}$. Also, $\mathbf{R}$ is bounded by 3 on paths because the sequence $(123)^k$ avoids words of the form $xWx\overleftarrow{W}$, for all k.

6.2.2 Bounding by maximum degree

The bounds on the square-free chromatic number on graphs of bounded maximum degree are proved using variants of the Local Lemma and entropy compression. A notable feature of these proofs is that they use very few properties of square-free colouring. The main property they use is that, for fixed n and c, there are relatively few squares of order n on c colours. In terms of colour schemes, the required property is that $\mathbf{Q}$ has a determinant,

$\mathcal{B}$, which contains no more than c^{2n} c-coloured paths of order n. This property is sufficient to prove that $\mathbf{Q}$ is bounded on graphs of bounded degree. This observation leads to the following generalisation of Theorem 2.4.

Theorem 6.21. *Let* $\mathbf{A}$ *be a path-dependent colour scheme with determinant* $\mathcal{B}$ *and let* $P_{n,k}$ *be the set of* k*-coloured paths of order* n *in* $\mathcal{B}$*. If there exists* $\varepsilon \in (0,1)$ *such that* $|P_{n,k}| \leqslant k^{n(1-\varepsilon)}$*, for all* $n \geqslant 1$ *and* $k \geqslant 1$*, then* $\chi_{\mathbf{A}}(G) \leqslant (8\Delta(G))^{1/\varepsilon}$*, for every graph* G*.*

Proof. Let $k := (8\Delta)^{1/\varepsilon}$, $n := |V(G)|$ and $\mathcal{Q}$ be the set of paths in G. Colour each vertex of G independently and randomly with k colours. For each $P \in \mathcal{Q}$, let A_P be the event that $P \in \mathcal{B}$. Define the dependency graph, D, with vertex set $V(D) := \{A_P : P \in \mathcal{Q}\}$. An edge $\{A_P, A_Q\}$ is in $E(D)$ if the paths P and Q share a vertex in G. Observe that two events, A_P and A_Q, are mutually independent if P and Q have no common vertex, so D is a dependency graph. Partition $V(D)$ by the sets $S_i := \{A_P : P \text{ is a path of order } i\}$ for $i \in [n]$.

Let $\Delta := \Delta(G)$. We now show that, for all $i, j \in [n]$ and $P \in S_i$, $d(i,j) := ij\Delta^j$ is an upper bound on $|N(A_P) \cap S_j|$. To do so we consider a vertex, $v \in V(G)$, and bound the number of paths of order j that contain v. Note that there are at most Δ^{a-1} paths of order a in G which start at v. Therefore there are at most $\Delta^{b-1}\Delta^{j-b}$ paths in S_j with v as the b^{th} vertex. Since v can occur at any of j positions in a path of order j, there at most $j\Delta^j$ paths of order j going through v. It follows that a path on i vertices intersects at most $ij\Delta^j$ paths of order j, so $|N(A_P) \cap S_j| \leqslant ij\Delta^j$. Let $x_i = (3\Delta)^{-i}$, for all $i \in [n]$, and let $x_P := x_{|V(P)|}$ be the parameter, required by Lemma 2.3, corresponding to A_P for all $P \in \mathcal{Q}$. By definition,

$$\prod_{A_Q \in N(A_P)} (1 - x_Q) \geqslant \prod_{j=1}^{n} (1 - x_j)^{d(|V(P)|,j)}$$

for all $P \in \mathcal{Q}$. For all $i \in [n]$, $(1 - x_i) \geqslant e^{-\frac{5x_i}{4}}$ because $x_i \leqslant 3^{-1}$, so

$$\begin{aligned} x_i \prod_{j=1}^{n} (1 - x_j)^{d(i,j)} &\geqslant (3\Delta)^{-i} \prod_{j=1}^{n} e^{\frac{-5}{4} x_j d(i,j)} \\ &= (3\Delta)^{-i} \prod_{j=1}^{n} e^{\frac{-5}{4}(3\Delta)^{-j} ij\Delta^j} \\ &> (3\Delta)^{-i} \exp\left(\frac{-5i}{4} \sum_{j=1}^{\infty} \left(j/3^j\right)\right). \end{aligned}$$

Let $P \in \mathcal{Q}$ and $i = |V(P)|$. Note that

$$\mathbb{P}(A_P) = \frac{|P_{|P|,k}|}{k^{|P|}} \leqslant \frac{k^{|V(P)|(1-\varepsilon)}}{k^{|P|}} = k^{-|P|\varepsilon}.$$

The series $\sum_{j=1}^{\infty} j/3^j$ converges to $3/4$ so

$$x_P \prod_{A_Q \in N(A_P)} (1 - x_Q) \geqslant x_i \prod_{j=1}^{n} (1 - x_j)^{d(i,j)} > (3\Delta)^{-i} e^{-15i/16} > (8\Delta)^{-i} = k^{-|V(P)|\varepsilon} = \mathbb{P}(A_P).$$

Therefore $\mathbb{P}\left(\bigcap_{P \in \mathcal{Q}} \overline{A_P}\right) > 0$, by Lemma 2.3. It follows that G has k-colouring, ψ, such that $(G, \psi) \in \mathbf{A}$. $\square$

Recall from the previous section that $\mathbf{R}$ has a determinant with at most $c^{n/2}$ paths of order n on c colours. Therefore, by Theorem 6.21 with $\varepsilon = 1/2$, $\chi_{\mathbf{R}} \leqslant 64\Delta(G)^2$, for all graphs G. $\mathbf{R}$ has properties not found in the literature: it is a colour scheme which is unbounded on trees, bounded on choice-subdivisions of graphs, and bounded on graphs of maximum degree. Furthermore, Theorem 6.21 does not just apply to colour schemes which avoid a single permutation of each order. Every permutation-avoiding colour scheme is bounded on graphs of bounded maximum degree provided that it avoids sufficiently few permutations of each order.

Let $\mathbf{R}'$ be the permutation-avoiding colouring which avoids permutations WW and $xWx\overleftarrow{W}$. Since $\mathbf{R}'$ is determined by $\mathcal{B}_{\mathbf{R}} \cup \mathcal{B}_{\mathbf{Q}}$, $\mathbf{R}' = \mathbf{Q} \cap \mathbf{R}$, by Lemma 5.22. $\mathbf{R}'$ is interesting because $\mathbf{N} \subseteq \mathbf{R}' \subseteq \mathbf{Q}$ and it is close to a maximal example of a colour scheme that both satisfies this relationship and is unbounded on trees. $\mathbf{R}'$ is also bounded on graphs of bounded maximum degree, by Theorem 6.21.

6.3 Extensions of anagram-free colouring

In the previous section we studied anagram-free colouring by investigating a class of path-dependent colour schemes which are all supersets of $\mathbf{N}$. In this section we take the opposite approach by constructing path-dependent colour schemes which are subsets of $\mathbf{N}$. The motivation is to find path-bounded colour schemes which are non-trivial subsets of $\mathbf{N}$. While this goal is not achieved, the investigation still results in colour schemes with interesting properties which are closely related to $\mathbf{N}$.

6.3.1 ε-uniform-free colouring

Recall that $\mathbf{N}$ has a determinant, $\mathcal{B}$, such that $PQ \in \mathcal{B}$ if $M(P) = M(Q)$, where M is the colour multiset function. Our first relaxation of $\mathbf{N}$ comes from weakening the equality $M(P) = M(Q)$ by allowing P and Q to have almost the same number of occurrences of each colour.

Define the *density* of a colour c in a coloured path P to be

$$d_c(P) := \frac{|V_{\{c\}}(P)|}{|V(P)|}.$$

For the duration of this subsection, let $\mathcal{B}_\varepsilon$ be the set of coloured paths such that $PQ \in \mathcal{B}_\varepsilon$ if $|V(P)| = |V(Q)|$ and, for every colour c, either

$$d_c(P) = d_c(Q) = 0 \tag{6.1}$$

or

$$d_c(P) > 0, d_c(Q) > 0, \frac{d_c(P)}{d_c(Q)} \in \left(1 - \varepsilon, \frac{1}{1-\varepsilon}\right). \tag{6.2}$$

For $\varepsilon \in (0, 1) \subset \mathbb{R}$, let $\mathbf{M}_\varepsilon$ be the path-dependent set of coloured graphs determined by $\mathcal{B}_\varepsilon$. Call the corresponding type of graph colouring *ε-uniform colouring*. We first show that $\mathbf{M}_\varepsilon$ is a colour scheme.

Theorem 6.22. *For all $\varepsilon \in (0, 1)$, $\mathbf{M}_\varepsilon$ is a path-dependent colour scheme.*

Proof. We first show that $\mathcal{B}_\varepsilon$ is closed under melding. Let $PQ \in \mathcal{B}_\varepsilon$, a and b be two colours, and $P'Q'$ be the meld of PQ that identifies a and b to a new colour c. Take the case $d_a(P) > 0$, $d_a(Q) > 0$, $d_b(P) > 0$, $d_b(Q) > 0$, because the other cases all follow trivially. Note that $d_c(P') = (d_a(P) + d_b(P))$ and $d_c(Q') = (d_a(Q) + d_b(Q))$, and

$$d_a(Q)(1-\varepsilon) \leqslant d_a(P) \leqslant d_a(Q)/(1-\varepsilon),$$
$$d_b(Q)(1-\varepsilon) \leqslant d_b(P) \leqslant d_b(Q)/(1-\varepsilon).$$

It follows that $d_c(Q')(1-\varepsilon) \leqslant d_c(P') \leqslant d_c(Q')/(1-\varepsilon)$, so $P'Q' \in \mathcal{B}_\varepsilon$.

Clearly $\mathcal{B}_\varepsilon$ does not contain any rainbow colourings, as $d_k(P) = 0$ or $d_k(Q) = 0$ for every colour, k, in PQ that occurs at most once. Therefore, by Lemma 5.29, $\mathbf{M}_\varepsilon$ is a path-dependent colour scheme. □

We are interested in ε-uniform colouring because it sits between anagram-free colouring and conflict-free colouring in the hierarchy of colour schemes. We establish these relations in the following lemma.

Lemma 6.23. $\mathbf{C} \subseteq \mathbf{M}_\varepsilon \subseteq \mathbf{N}$ *for all* $\varepsilon \in (0, 1)$.

Proof. We first show $\mathbf{M}_\varepsilon \subseteq \mathbf{N}$. By Lemma 5.14, it suffices to show $\mathcal{M}_\mathbf{N} \subseteq \mathcal{B}_\varepsilon$. Let $PQ \in \mathcal{M}_\mathbf{N}$ with $|V(P)| = |V(Q)|$ and let c be a colour in PQ. Clearly $d_c(P) = d_c(Q)$, since PQ is an anagram. Therefore $PQ \in \mathcal{B}_\varepsilon$.

Now consider $\mathbf{C} \subseteq \mathbf{M}_\varepsilon$. By Lemma 5.14, it suffices to show $\mathcal{B}_\varepsilon \subseteq \mathcal{B}_\mathbf{C}$. Let $PQ \in \mathcal{B}_\varepsilon$ such that $|V(P)| = |V(Q)|$. By definition, for all colours, c, either $d_c(P) = d_c(Q) = 0$ or both $d_c(P)$ and $d_c(Q)$ are positive. In the latter case P and Q each contain one occurrence of c. It follows that every c that occurs in PQ occurs at least twice in PQ. Therefore $PQ \in \mathcal{B}_\mathbf{C}$. □

We investigate more closely where $\mathbf{M}_\varepsilon$ sits in the hierarchy of colour schemes by determining whether it behaves like $\mathbf{N}$ or $\mathbf{C}$. By Lemma 5.5, we know that $\mathbf{M}_\varepsilon$ is not bounded on trees since $\mathbf{M}_\varepsilon \subseteq \mathbf{N}$. On the other side, $\mathbf{C}$ is not bounded on paths. We now show that $\mathbf{M}_\varepsilon$ is not bounded on paths.

The following proof uses the space $[0,1]^k \subseteq \mathbb{R}^k$ with distance metric

$$\operatorname{dist}((x_1, \dots, x_k), (y_1, \dots, y_k)) := \max\{|x_1 - y_1|, \dots |x_k - y_k|\}.$$

Let $\varepsilon > 0$. A point $x \in [0,1]^k$ is *ε-permitted* by $S \subseteq [0,1]^k$ if there exist $a, b \in \left([0,1]^k \setminus S\right)$ such that $\operatorname{dist}(a, b) \geqslant \varepsilon$ and x is the average of a and b, that is, $x_i = (a_i + b_i)/2$ for all $i \in [k]$. A point $x \in [0,1]^k$ is *ε-blocked* by $S \subseteq [0,1]^k$ if it is not ε-permitted by S. Equivalently, a point $x \in [0,1]^k$ is *ε-blocked* by $S \subseteq [0,1]^k$ if for all $a, b \in [0,1]^k$ at least one of the following hold:

(1) $a \in S$ or $b \in S$,

(2) $\operatorname{dist}(a, b) < \varepsilon$,

(3) x is not the average of a and b.

Let $S \subseteq [0,1]^k$. An *ε-extension of S* is a set, $T \subseteq [0,1]^k$, such that all $t \in T$ are ε-blocked by S. An *ε-extension sequence* is a finite sequence $S_0, S_1, \dots, S_n$ such that S_i is an ε-extension of S_{i-1} for all $i \in [n]$.

Lemma 6.24. *For all* $\varepsilon > 0$ *and* $k \in \mathbb{Z}^+$, *there is an ε-extension sequence* $S_0, \dots, S_n$ *with* $S_0 = \emptyset$ *and* $S_n = [0,1]^k$.

Proof. We first show that the lemma holds for the case $k = 1$. Let $n = \left\lfloor \frac{2}{\varepsilon} \right\rfloor + 1$ and $S_i = \left[0, \frac{\varepsilon i}{2}\right) \cap [0,1]$ for all $i \in \{0, \dots, n\}$, taking $[0,0) := \emptyset$. Note that

$$S_n = \left[0, \frac{\varepsilon}{2}\left(\left\lfloor \frac{2}{\varepsilon} \right\rfloor + 1\right)\right) \cap [0,1] = [0,1].$$

To prove that, for all $i \in [n]$, S_i is an ε-extension of S_{i-1}, we show that every point in S_i is ε-blocked by S_{i-1}. Let $x \in S_i$ and $a, b \in [0,1]$, we show that x, a and b satisfy at least one of the three ε-blockage conditions. Condition (3) is satisfied if x is not the average a and b, so take the case that x is the average of a and b. Without loss of generality let $a \leqslant x$ and $b \geqslant x$. Condition (2) is satisfied if $\operatorname{dist}(a,b) < \varepsilon$, so take the case $\operatorname{dist}(a,b) \geqslant \varepsilon$. We now use the observation that $x < \frac{\varepsilon i}{2}$, which follows from $x \in S_i$. If $i = 1$ there is no a and b satisfying $\operatorname{dist}(a,b) \geqslant \varepsilon$, so S_1 is an ε-extension of S_0. In the case $i \geqslant 2$, we have $a \leqslant x - \frac{\varepsilon}{2} = \frac{\varepsilon(i-1)}{2}$ implying $a \in S_{i-1}$, satisfying Condition (1). Therefore x is ε-blocked by S_{i-1} so S_i is an ε-extension of S_{i-1}.

We proceed by induction on k. Let $S_0, \dots, S_n$ be an ε-extension sequence in $[0,1]^{k-1}$ with $S_0 = \emptyset$ and $S_n = [0,1]^{k-1}$. We use $S_0, \dots, S_n$ to construct a sequence of sets denoted

$$T = T_{0,0}, \dots, T_{0,n}, T_{1,0}, \dots, T_{1,n}, T_{2,0} \dots, T_{h-1,n}, T_{h,0} \dots, T_{h,n}$$

where $h = \left\lceil \frac{2^n}{\varepsilon} \right\rceil$. The intuition behind this construction is that the k^{th} dimension of $[0,1]^k$ can be filled with $h+1$ layering of $S_0 \dots, S_n$, each offset by $\frac{\varepsilon i}{2^n}$. For each term of T,

$$T_{i,j} := \left(\left(S_j \times \left[\frac{\varepsilon i}{2^n}, \frac{\varepsilon i}{2^n} + \frac{\varepsilon}{2^j}\right)\right) \cup \left([0,1]^{k-1} \times \left[0, \frac{\varepsilon i}{2^n}\right)\right)\right) \cap [0,1]^k$$

for $i \in \{0, \dots, h\}$ and $j \in \{0, \dots, n\}$, taking $[0,0) := \emptyset$. We now show that T is an ε-extension sequence. First note that

$$T_{i,0} = \left([0,1]^{k-1} \times \left[0, \frac{\varepsilon i}{2^n}\right)\right) \cap [0,1]^k$$
$$T_{i,n} = \left([0,1]^{k-1} \times \left[0, \frac{\varepsilon(i+1)}{2^n}\right)\right) \cap [0,1]^k$$

for all $i \in \{0, \dots, h\}$, because $S_0 = \emptyset$ and $S_n = [0,1]^{k-1}$. It follows that $T_{i-1,n} = T_{i,0}$ for all $i \in [h]$. Furthermore, $T_{i,0}$ is an extension of itself because $x \in T_{i,0}$ if and only if $x_k < \frac{\varepsilon i}{2^n}$. It follows that every $a, b \in [0,1]^k$ that average to $x \in T_{0,n}$ satisfy at least one of $a \in T_{i,0}$ and $b \in T_{i,0}$. Therefore $T_{i,0}$ is an extension of $T_{i-1,n}$ for all $i \in [h]$.

We now show that $T_{i,j}$ is an ε-extension of $T_{i,j-1}$ for all $i \in \{0, \dots, h\}$ and $j \in [n]$. Assume for the sake of contradiction that there exist $a, b \in ([0,1]^k \setminus T_{i,j-1})$ such that $\operatorname{dist}(a,b) \geqslant \varepsilon$ and $x \in T_{i,j}$ is the average of a and b. Now consider the bounds on a_k and b_k. Without loss of generality let $a_k \leqslant x_k$ and $b_k \geqslant x_k$. We know that $a_k \geqslant \frac{\varepsilon i}{2^n}$ because $[0,1]^{k-1} \times \left[0, \frac{\varepsilon i}{2^n}\right) \subseteq T_{i,j-1}$. We also know that $x_k < \frac{\varepsilon i}{2^n} + \frac{\varepsilon}{2^j}$ because $x \in T_{i,j}$. Therefore, $b_k < \frac{\varepsilon i}{2^n} + \frac{\varepsilon}{2^{j-1}}$, because x_k is the average of a_k and b_k.

So far we have shown $a_k \in \left[\frac{\varepsilon i}{2^n}, \frac{\varepsilon i}{2^n} + \frac{\varepsilon}{2^{j-1}}\right)$ and $b_k \in \left[\frac{\varepsilon i}{2^n}, \frac{\varepsilon i}{2^n} + \frac{\varepsilon}{2^{j-1}}\right)$, which is all we require for the remainder of the proof. Firstly, $(x_1, \dots, x_{k-1}) \in S_j$, $(a_1, \dots, a_{k-1}) \notin S_{j-1}$ and $(b_1, \dots, b_{k-1}) \notin S_{j-1}$ because, by definition,

$$\left(S_{j-1} \times \left[\frac{\varepsilon i}{2^n}, \frac{\varepsilon i}{2^n} + \frac{\varepsilon}{2^{j-1}}\right)\right) \cap [0,1]^k = \left(T_{i,j-1} \cap \left([0,1]^{k-1} \times \left[\frac{\varepsilon i}{2^n}, \frac{\varepsilon i}{2^n} + \frac{\varepsilon}{2^{j-1}}\right)\right)\right) \cap [0,1]^k.$$

Secondly, $\operatorname{dist}((a_1, \dots, a_{k-1}), ((a_1, \dots, b_{k-1})) \geqslant \varepsilon$ because $|a_k - b_k| < \varepsilon$ and $\operatorname{dist}(a,b) \geqslant \varepsilon$. Therefore $(x_1, \dots, x_{k-1})$ is ε-permitted by S_{j-1}, which is a contradiction of S_j being an ε-extension of S_{j-1}. □

We now prove that $\mathbf{M}_\varepsilon$ is unbounded on paths, using Lemma 6.24 and a connection between $\mathbf{M}_\varepsilon$ and ε-blocking sets. The idea behind the proof is to treat the vector of colour densities of a k-coloured path, P, as a point $d(P) \in [0,1]^k$. For two paths, P and Q, if $d(P)$ and $d(Q)$ are sufficiently close in $[0,1]^k$, then $PQ \in \mathcal{B}_{\mathbf{M}_\varepsilon}$, provided that $|V(P)| = |V(Q)|$. This observation is analogous to taking ε-extensions, and is used in the following theorem.

Theorem 6.25. *$\mathbf{M}_\varepsilon$ is unbounded on paths for all $\varepsilon > (0,1)$.*

Proof. Assume for the sake of contradiction that k is the smallest integer such that $\chi_{\mathbf{M}_\varepsilon}(P) \leqslant k$ for all paths P. Let ℓ be the maximum integer such that $\chi_{\mathbf{M}_\varepsilon}(P_\ell) < k$. It follows that every path, P, of order greater than ℓ has $d_c(P) > \frac{1}{2\ell}$ for all $c \in [k]$. Therefore, every k-coloured path, $AB \in \mathbf{M}_\varepsilon$ with $|V(A)| = |V(B)|$ and order at least 2ℓ, satisfies

$$d_c(B) + \frac{\varepsilon}{2\ell} < d_c(B) + \varepsilon d_c(A) < d_c(A) \text{ or}$$
$$d_c(A) + \frac{\varepsilon}{2\ell} < d_c(A) + \varepsilon d_c(B) < d_c(B)$$

for some $c \in [k]$. Equivalently, $\operatorname{dist}(d(A), d(B)) > \delta$, with $\delta := \varepsilon/(2\ell)$. By Lemma 6.24, let $S_0, \ldots, S_n$ be a δ-extension sequence with $S_0 = \emptyset$ and $S_n = [0,1]^k$.

Let $Q_i \in \mathbf{M}_\varepsilon$ be a k-coloured path with $|V(Q_i)| = \ell 2^i$, for $i \in \{0, \ldots, n\}$. We now show that $d(Q_i) \notin S_i$ for all $i \in \{0, \ldots, n\}$. Clearly $Q_0 \notin S_0 = \emptyset$. We proceed by induction on i. Since each $x \in S_i$ is δ-blocked by S_{i-1} we just need to show that $d(Q_i)$ is δ-permitted by S_{i-1}. Let $AB = Q_i$ such that $|V(A)| = |V(B)|$. By induction, $d(A) \notin S_{i-1}$ and $d(B) \notin S_{i-1}$. Note that $d(Q_i)$ is the average of $d(A)$ and $d(B)$. Finally, recall that, $\operatorname{dist}(A, B) > \delta$. Therefore $d(Q_i)$ is δ-permitted by S_{i-1} so $d(Q_i) \notin S_i$.

It follows that $d(Q_n) \notin [0,1]^k$, which is a contradiction. □

Note that $\mathbf{M}_\varepsilon$ can be further relaxed by removing the requirement that $|V(P)| = |V(Q)|$ in $\mathcal{B}_\varepsilon$. For $\varepsilon \in (0,1)$, let $\mathbf{M}'_\varepsilon$ be the path-dependent set of coloured graphs with determinant, $\mathcal{B}$, such that $PQ \in \mathcal{B}$ if, for every colour c, either

$$d_c(P) = d_c(Q) = 0 \tag{6.3}$$

or

$$d_c(P) > 0, d_c(Q) > 0, \frac{d_c(P)}{d_c(Q)} \in \left(1 - \varepsilon, \frac{1}{1-\varepsilon}\right). \tag{6.4}$$

Clearly $\mathcal{B}_\varepsilon \subseteq \mathcal{B}$ so $\mathcal{M}'_\varepsilon \subseteq \mathcal{M}_\varepsilon$. Also, $\mathbf{C} \subseteq \mathbf{M}'_\varepsilon$, which is easily shown with a modification to the proof of Lemma 6.23.

6.3.2 k-parity colouring

Recall that parity colouring, denoted $\mathbf{I}$, is the path-dependent colour scheme with determinant, $\mathcal{B}$, such that $P \in \mathcal{B}$ if every colour in P occurs an even number of times. Also, $\mathbf{I} \subseteq \mathbf{N}$ because each colour in an anagram occurs an even number of times, so $\mathcal{M}_\mathbf{N} \subseteq \mathcal{B}$. Throughout this subsection, let $\mathcal{B}_k$ be the set of coloured paths such that $P \in \mathcal{B}_k$ if each colour in P occurs $0 \mod k$ times. We extend $\mathbf{I}$ to the path-dependent set of coloured graphs, $\mathbf{I}_k$, with determinant $\mathcal{B}_k$. Clearly $\mathbf{I} = \mathbf{I}_2$. In general, $\mathbf{I}_k$ is a colour scheme.

Theorem 6.26. *For all integers $k \geqslant 2$, $\mathbf{I}_k$ is a path-dependent colour scheme.*

Proof. Let $P \in \mathcal{B}_k$, a and b be two colours, and P' be the meld of P that identifies a and b. By definition, $a \equiv 0 \mod k$ and $b \equiv 0 \mod k$, so $a + b \equiv 0 \mod k$. Therefore $P' \in \mathcal{B}_k$. Clearly $\mathcal{B}_k$ does not contain any rainbow colourings as $1 \not\equiv 0 \mod k$. Therefore, by Lemma 5.29, $\mathbf{I}_k$ is a path-dependent colour scheme. □

In general, $\mathbf{I}_k$ is not a subset of $\mathbf{N}$, but it does satisfy other subset relations. Firstly, the relation $\mathbf{I} \subseteq \mathbf{N}$ extends to $\mathbf{I}_k \subseteq \mathbf{N}_k$ because every colour in a k-anagram occurs a multiple of k times, implying $\mathcal{M}_{\mathbf{N}_k} \subseteq \mathcal{B}_k$. Also, $\mathbf{I}_k \subseteq \mathbf{I}_{kn}$, for all integers $k \geqslant 2$ and $n \geqslant 1$, because every colour in a path $P \in \mathcal{B}_a$ occurs $0 \mod kn$ times, so $P \in \mathcal{B}_{kn}$. We show that $\mathbf{I}_k$ is unbounded on paths. This generalises the result by Bunde et al. [34], that $\mathbf{I}$ is unbounded on paths.

Theorem 6.27. *For all integers $k \geqslant 2$, $\mathbf{I}_k$ is unbounded on paths.*

Proof. Let $W := w_1 w_2 \dots w_n$ be a word of length $n := k^c + 1$ on c symbols. Let W_i be the prefix W of length i, noting that W_0 is the empty word and $W_n = W$. For $i \in \{0, \dots, n\}$, let $s_i := (s_{i,1}, s_{i,2}, \dots, s_{i,c})$ where $s_{i,j}$ is the number of occurrences of j in W_i, $|W_i|_j$, modulo k. There are k^c vectors of length c on $\{0, 1, \dots, k-1\}$ so there exist two integers, $a, b \in \{0, \dots, n\}$, such that $s_a = s_b$. Without loss of generality, let $a < b$. Let B be the word such that $W_b = W_a B$. Let $j \in [c]$. Note that $|W_b|_j \equiv |W_a|_j + |B|_j \mod k$ and $|W_b|_j \equiv |W_a|_j \mod k$, so $|B|_j \equiv 0 \mod k$. Therefore every symbol in B occurs $0 \mod k$ times. It follows that every c-colouring of the path of order $k^c + 1$ is not in $\mathbf{I}_k$. □

For $k \geqslant 3$, the monochromatic path of order 2 is not in $\mathcal{B}_{\mathbf{I}_k}$, so $\mathbf{I}_k$ is not a subset of $\mathbf{N}$. Also note that, $\mathcal{B}_{\mathbf{I}_k} \subseteq \mathcal{B}_{\mathbf{C}}$ so $\mathbf{C} \subseteq \mathbf{I}_k$.

6.3.3 Powers of colour schemes

We now introduce a unary operation for generating colour schemes which is inspired by the relationship between distance-k colouring and G^k, the k^{th} power of a graph G. Recall that G^k is the graph with vertex set $V(G)$, where there is an edge between every pair of vertices $u, v \in V(G)$ with $\text{dist}(u, v) \leqslant k$ in G. We can define $\mathbf{P}_k$, the colour scheme corresponding to distance-k colouring, by saying that a coloured graph, (G, ψ), is in $\mathbf{P}_k$ if and only if $(G^k, \psi) \in \mathbf{P}$. We use this observation to generate extensions of anagram-free colouring that are particularly relevant to the results of Carmi et al. [35] from Section 1.3.5. However, we first define powers of colour schemes in full generality.

The relationship between $\mathbf{P}$ and $\mathbf{P}_k$ can be generalised to an operation for all sets of coloured graphs. For a set of coloured graphs, $\mathbf{A}$, let $\mathbf{A}^k$ be the set of coloured graphs with $(G, \alpha) \in \mathbf{A}^k$ if and only if $(G^k, \alpha) \in \mathbf{A}$. Call $\mathbf{A}^k$ the k^{th} *power of* $\mathbf{A}$. We now show that this operation yields a colour scheme.

Lemma 6.28. *For all colour schemes $\mathbf{A}$ and $k \in \mathbb{Z}^+$, $\mathbf{A}^k$ is a colour scheme.*

Proof. Let $(G, \alpha) \in \mathbf{A}^k$ and H be a subgraph of G. $(G^k, \alpha) \in \mathbf{A}$ and H^k is a subgraph of G^k so, by Subgraphability, $(H^k, \alpha|_H) \in \mathbf{A}$. It follows that $(H, \alpha|_H) \in \mathbf{A}^k$, so $\mathbf{A}^k$ satisfies Subgraphability.

Let $(G, \alpha) \in \mathbf{A}^k$, and let (G, β) be a fracture of (G, α). By definition, $(G^k, \alpha) \in \mathbf{A}$ and so, by Recolourability, $(G^k, \beta) \in \mathbf{A}$. It follows that $(G, \beta) \in \mathbf{A}^k$, so $\mathbf{A}^k$ satisfies Recolourability.

Let G be a graph. By Universality, $(G^k, \alpha) \in \mathbf{A}$ for some α. It follows that $(G, \alpha) \in \mathbf{A}^k$, so $\mathbf{A}^k$ satisfies Universality.

Let $(G, \alpha) \in \mathbf{A}^k$ and $(H, \beta) \in \mathbf{A}^k$. By definition, $(G^k, \alpha) \in \mathbf{A}$ and $(H^k, \beta) \in \mathbf{A}$. By Additivity, the disjoint union of (G^k, α) and (H^k, β) is in $\mathbf{A}$. It follows that $\mathbf{A}^k$ satisfies Additivity.

Let $(G, \alpha) \notin \mathbf{A}^k$. It follows that $(G^k, \alpha) \notin \mathbf{A}$ and, by Locality, G^k has a finite subgraph, H, with $(H, \alpha|_H) \notin \mathbf{A}$. There is a finite graph, I, with $V(H) \subseteq V(I)$ such that H is a subgraph of I^k, and I^k is a subgraph of G^k. By Subgraphability, $(I^k, \alpha|_I) \notin \mathbf{A}$. Therefore $(I, \alpha|_I)$ is a finite subgraph of (G, α) which is not in $\mathbf{A}^k$, so $\mathbf{A}^k$ satisfies Locality. □

Powers of the square-free colour scheme, $\mathbf{Q}$, form a descending chain of subsets which are all bounded on graphs of bounded maximum degree. Recall that $\mathbf{Q}$ is bounded on graphs of bounded maximum degree. Let $\mathcal{G}$ be a set of graphs of bounded maximum degree and $\mathcal{G}^k = \{G^k : G \in \mathcal{G}\}$. Note that, for fixed k, $\mathcal{G}^k$ has bounded maximum degree. It follows that $\mathbf{Q}^k$ is bounded on $\mathcal{G}^k$, and, more generally, that $\mathbf{Q}^k$ is bounded on graphs of bounded maximum degree. However, $\mathbf{Q}^2$ is not bounded on stars because S^2 is a complete graph, for every star S.

Conversely, high powers of $\mathbf{N}$ are not bounded on paths. For $k \in \mathbb{Z}^+$, let $\mathcal{P}^k$ be the set $\{P^k : P \in \mathcal{P}\}$. Carmi et al. [35] show that $\mathbf{N}$ is unbounded on a family of graphs, $\mathcal{G}$, such that every graph $G \in \mathcal{G}$ is a subgraph of a $H \in \mathcal{P}^3$. It follows that $\mathbf{N}^3$ is unbounded on paths. Whether $\mathbf{N}^2$ is bounded on paths is an open problem.

6.4 Hierarchies of colour schemes

Throughout this chapter, we have defined many new colour schemes as generalisations or combinations of existing colour schemes from the literature. We have also shown that many properties of a colour scheme can be determined by looking at how it relates to other colour schemes. For example, $\mathbf{A} \cap \mathbf{B}$ is bounded on trees if and only if $\mathbf{A}$ and $\mathbf{B}$ are bounded on trees, see Lemma 5.8. In this section we present a diagram of the relationships between many of these colour schemes, see Figure 6.3, and highlight some interesting properties of the structure of colour schemes.

A *lattice* is an algebraic structure consisting of a set, S, with two commutative and associative binary operations, $\wedge$ and $\vee$, that satisfy the following absorption laws

$$x \vee (x \wedge y) = x,$$
$$x \wedge (x \vee y) = x$$

for all $x, y \in S$. A *distributive lattice* is a lattice, $(S, \vee, \wedge)$, which also satisfies

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

for all $x, y, z \in S$. Let $\mathbb{S}$ denote the set of colour schemes.

Theorem 6.29. *$(\mathbb{S}, \sqcup, \cap)$ is a distributive lattice.*

Proof. Let $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{S}$ be colour schemes. $\mathbf{A} \sqcup (\mathbf{A} \cap \mathbf{B}) = \mathbf{A}$ because $\mathbf{A}$ is the additive closure of $\mathbf{A}$. $\mathbf{A} \cap (\mathbf{A} \sqcup \mathbf{B}) = \mathbf{A}$ because $\mathbf{A} \subseteq \mathbf{A} \sqcup \mathbf{B}$. It remains to show that $\mathbf{S} \cap (\mathbf{A} \sqcup \mathbf{B}) = (\mathbf{S} \sqcup \mathbf{A}) \cap (\mathbf{S} \sqcup \mathbf{B})$.

Let $(G, \alpha) \in \mathbf{S} \cap (\mathbf{A} \sqcup \mathbf{B})$. Let $(H, \alpha|_H)$ be a connected component of (G, α). Note that $(H, \alpha|_H) \in \mathbf{S} \cap (\mathbf{A} \cup \mathbf{B})$, which implies $(H, \alpha|_H) \in (\mathbf{S} \cup \mathbf{A}) \cap (\mathbf{S} \cup \mathbf{B})$. It follows that every connected component of (G, α) is in $(\mathbf{S} \cup \mathbf{A}) \cap (\mathbf{S} \cup \mathbf{B})$, so $(G, \alpha) \in (\mathbf{S} \sqcup \mathbf{A}) \cap (\mathbf{S} \sqcup \mathbf{B})$.

Let $(G', \alpha) \in (\mathbf{S} \sqcup \mathbf{A}) \cap (\mathbf{S} \sqcup \mathbf{B})$. Let $(H', \alpha|_{H'})$ be a connected component of (G', α). Note that $(H', \alpha|_{H'}) \in (\mathbf{S} \cup \mathbf{A}) \cap (\mathbf{S} \cup \mathbf{B})$, which implies $(H', \alpha|_{H'}) \in \mathbf{S} \cap (\mathbf{A} \cup \mathbf{B})$. It follows that every connected component of (G', α) is in $\mathbf{S} \cap (\mathbf{A} \cup \mathbf{B})$, so $(G', \alpha) \in \mathbf{S} \cap (\mathbf{A} \sqcup \mathbf{B})$. □

A lattice is distributive if and only if it is isomorphic to $(T, \cup, \cap)$ for some set of sets T [45, Chapter 6]. Because every colour scheme has a determinant with respect to the set of finite connected graphs, $\mathcal{G}$, $(\mathbb{S}, \sqcup, \cap)$ is isomorphic to $(\mathbb{S}', \cup, \cap)$ with $\mathbb{S}' := \{\mathbf{A}|_{\mathcal{G}} : \mathbf{A} \in \mathbb{S}\}$.

Lattice theory can serve as a guide towards proving some structural properties about colour schemes. For example, a lattice, $(S, \vee, \wedge)$, is *bounded* if $\vee$ and $\wedge$ both have identity elements [45, Chapter 5]. The identity of $\vee$ is called the *minimum* of $(S, \vee, \wedge)$, and the identity of $\wedge$ is called the *maximum* of $(S, \vee, \wedge)$. We note that $(\mathbb{S}, \sqcup, \cap)$ is a bounded lattice, and find its minimum and maximum. Let $\mathbb{S}_1$ be the set of all coloured graphs. $\mathbb{S}_1$ satisfies the first four axioms trivially and satisfies locality vacuously, so $\mathbb{S}_1$ is a colour scheme. Clearly $\mathbf{A} \subseteq \mathbb{S}_1$ for all $\mathbf{A} \in \mathbb{S}$ so $\mathbb{S}_1$ is the maximum of $(\mathbb{S}, \sqcup, \cap)$. Let $\mathbb{S}_0$ be the set of disjoint unions of rainbow coloured graphs.

Lemma 6.30. *$\mathbb{S}_0$ is a path-dependent colour scheme and $\mathbb{S}_0 \subseteq \mathbf{A}$, for all $\mathbf{A} \in \mathbb{S}$.*

Proof. A coloured graph, (G, α), is not in $\mathbb{S}_0$ if there are two vertices $u, v \in V(G)$ in the same connected component of G with $\alpha(u) = \alpha(v)$. Recall that a graph, H, is connected if there is a path between all pairs of vertices, $u, v \in V(H)$, and that all paths are finite. It follows that $\mathbb{S}_0$ is the path-dependent set of coloured graphs determined by $\mathcal{B}$, where $P \in \mathcal{B}$ if P is not rainbow coloured. $\mathcal{B}$ is closed under melding and does not contain rainbow colourings so, by Lemma 5.29, $\mathbb{S}_0$ is a path-dependent colour scheme.

Let $\mathbf{A}$ be a colour scheme. By UNIVERSALITY and RECOLOURABILITY, $\mathbf{A}$ contains every rainbow coloured graph. By ADDITIVITY, $\mathbf{A}$ contains all disjoint unions of rainbow coloured graphs. Therefore $\mathbb{S}_0 \subseteq \mathbf{A}$. □

For every set of finite connected graphs, $\mathcal{G}$, the $\mathcal{G}$-dependent colour schemes also corresponds to a lattice. Let $\mathbb{D}_{\mathcal{G}} \subseteq \mathbb{S}$ be the set of $\mathcal{G}$-dependent colour schemes. By Lemmas 5.22 and 5.23 , $\mathbb{D}_{\mathcal{G}}$ is closed under taking intersections and unions of sets of bad graphs. For two $\mathcal{G}$-dependent colour schemes, $\mathbf{A}$ and $\mathbf{B}$, let $\mathbf{A} \cap_{\mathcal{G}} \mathbf{B}$ be the $\mathcal{G}$-dependent colour scheme determined by $\mathcal{B}_{\mathbf{A}} \cup \mathcal{B}_{\mathbf{B}}$. Similarly, let $\mathbf{A} \cup_{\mathcal{G}} \mathbf{B}$ be the $\mathcal{G}$-dependent colour scheme determined by $\mathcal{B}_{\mathbf{A}} \cap \mathcal{B}_{\mathbf{B}}$. Note that $\cap_{\mathcal{G}} = \cap$, by Lemma 5.22. When $\mathcal{G}$ is the set of finite connected graphs $\cup_{\mathcal{G}} = \sqcup$, by Lemma 5.24, but, as demonstrated in Section 5.2.3, this does not hold in general. Now we can state that $(\mathbb{D}_{\mathcal{G}}, \cap_{\mathcal{G}}, \cup_{\mathcal{G}})$ is a distributive lattice, as its binary operations correspond to intersections and unions of the underlying sets of bad graphs. We now show that $(\mathbb{D}_{\mathcal{G}}, \cap_{\mathcal{G}}, \cup_{\mathcal{G}})$ is bounded, and find its maximum and minimum. Since $\mathbb{S}_1$ is determined by the empty set of bad graphs, $\mathbb{S}_1$ is $\mathcal{G}$-dependent and the minimum of $(\mathbb{D}_{\mathcal{G}}, \cap_{\mathcal{G}}, \cup_{\mathcal{G}})$. Note that, due to the symmetry broken by the distributive law, $\mathbb{S}_1$ is the maximum of $(\mathbb{S}, \sqcup, \cap)$ and the minimum of $(\mathbb{D}_{\mathcal{G}}, \cap_{\mathcal{G}}, \cup_{\mathcal{G}})$. To find the maximum of $(\mathbb{D}_{\mathcal{G}}, \cap_{\mathcal{G}}, \cup_{\mathcal{G}})$, let $\mathbf{Z}_{\mathcal{G}}$ be the $\mathcal{G}$-dependent set of coloured graphs determined by $\mathrm{Col}(\mathcal{G}) \setminus \mathcal{R}$, where $\mathcal{R}$ is the set of rainbow colourings in $\mathrm{Col}(\mathcal{G})$. Since $\mathbf{Z}_{\mathcal{G}}$ is closed under melding and does not contain any rainbow colourings, $\mathbf{Z}_{\mathcal{G}}$ is a colour scheme, by Lemmas 5.16–5.20, and so it is the maximum of $(\mathbb{D}_{\mathcal{G}}, \cap_{\mathcal{G}}, \cup_{\mathcal{G}})$. Since $\mathbb{S}_0$ is path-dependent, $\mathbb{S}_0 = \mathbf{Z}_{\mathcal{P}}$, so the lattice of path-dependent colour schemes has the same maximum and minimum as the lattice of colour schemes, except that the maximum and minimum are swapped. Note that $\mathcal{M}_{\mathbb{S}_0}$ contains paths of all orders $n \geqslant 2$, so the lattice of path-dependent colour schemes is the unique smallest lattice with this property.

Figure 6.3 is a section of the lattice of colour schemes that contains many of the colour schemes studied or mentioned throughout this thesis. Note that many elements of $(\mathbb{S}, \sqcup, \cap)$

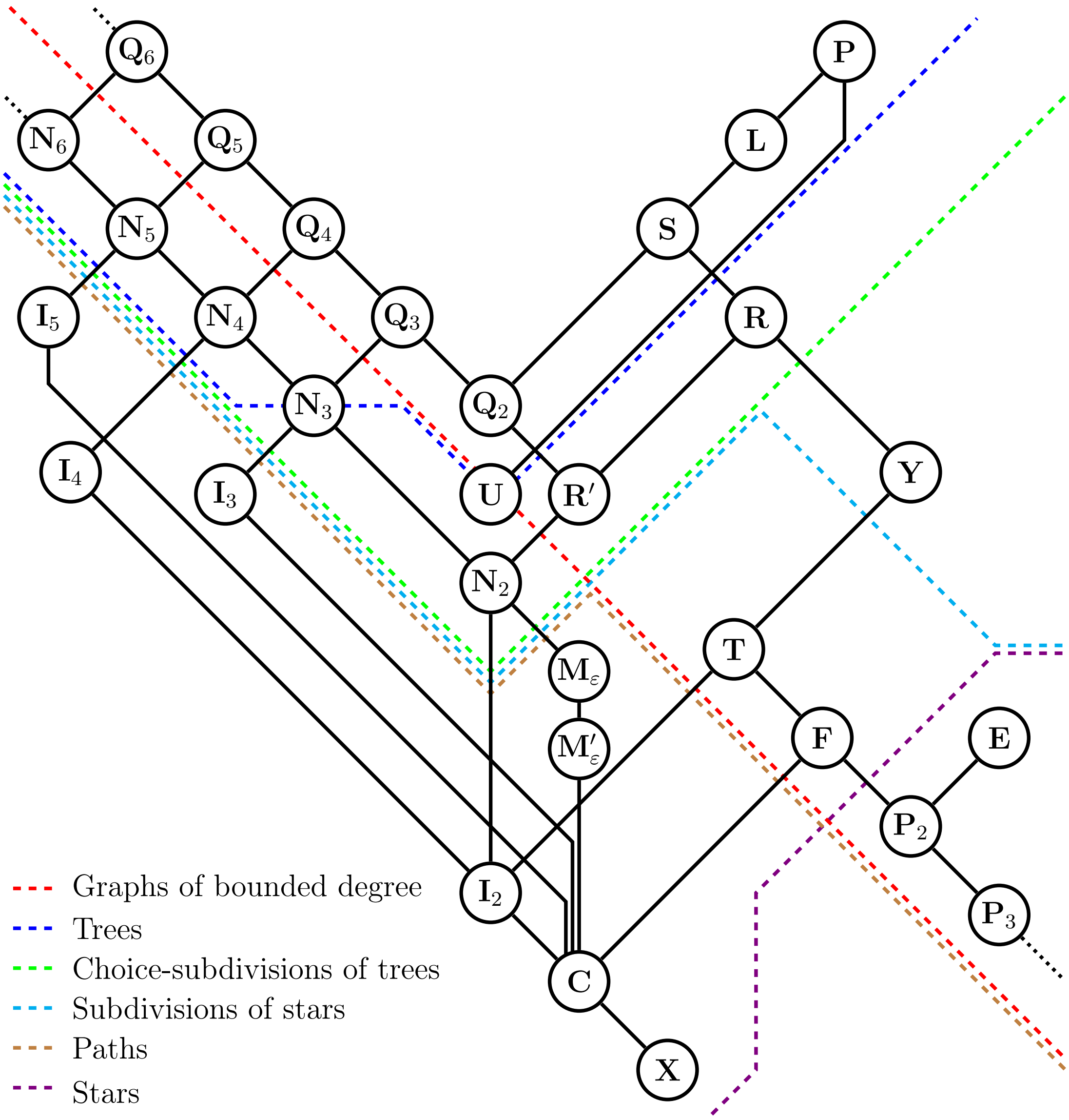


Figure 6.3: A representation of subset relationships between colour schemes, with $\mathbf{B} \subseteq \mathbf{A}$ if there is a line descending from $\mathbf{A}$ to $\mathbf{B}$. If a colour scheme, $\mathbf{A}$, is above a dashed line then $\mathbf{A}$ is bounded on the class of graphs corresponding to the line. If $\mathbf{A}$ is below a line it is not bounded on the corresponding class of graphs. In the case where the bounds on $\mathbf{A}$ are unknown, it is placed on the line. The dashed lines correspond to many of the classes of graphs studied throughout this thesis. The line for being bounded on choice-subdivisions of graphs is omitted, as, except for diverting around $\mathbf{U}$, it would be identical to the line for being bounded on choice-subdivisions of trees. Colour schemes which are above at least one of the top two lines are bounded on trees of bounded degree. The relationships between $\mathbf{M}_\varepsilon$ and $\mathbf{M}'_\varepsilon$ hold for fixed $\varepsilon \in (0,1)$. The relationships amongst $\mathbf{I}_k$ are barely shown in this representation, as recall that $\mathbf{C} \subseteq \mathbf{I}_k \subseteq \mathbf{I}_{kn}$, for all integers $k \geqslant 2$ and $n \geqslant 1$. All the colour schemes shown here, except $\mathbf{L}$ and $\mathbf{X}$, are path-dependent.

are omitted and that, for example, the representation does not intend to imply that $\mathbf{N}_3$ is the union of $\mathbf{I}_3$ and $\mathbf{N}_2$. While the lattice of colour schemes includes all unions and intersections of colour schemes, many of them are not shown in Figure 6.3. However, Figure 6.3 does encode the fact that $\mathbf{I}_3 \sqcup \mathbf{N}_2 \subseteq \mathbf{N}_3$ simply because $\mathbf{I}_3 \subseteq \mathbf{N}_3$ and $\mathbf{N}_2 \subseteq \mathbf{N}_3$. Figure 6.3 also omits $\mathbb{S}_1$ and $\mathbb{S}_0$, which would sit at the top and the bottom of the diagram, respectively.

The dashed lines that run through Figure 6.3 correspond to some of the classes of graphs studied throughout this thesis. If a colour scheme, $\mathbf{A}$, is above a dotted line, then it is bounded on the corresponding class of graph, if $\mathbf{A}$ is below the line then it is not bounded, and if $\mathbf{A}$ is on the line then it is unknown whether $\mathbf{A}$ is bounded. These lines have additional properties that hold for the entire lattice, not just the section shown here. A *chain* in a lattice is a set of totally ordered elements. Every line in Figure 6.3 cuts across $(\mathbb{S}, \sqcup, \cap)$ in the sense that every chain in $(\mathbb{S}, \sqcup, \cap)$ is crossed by each line at most once. This follows from the subset bounds on chromatic numbers from Lemmas 5.5 and 6.1. Trees, paths, stars, graphs of bounded degree, and subdivisions of stars are self-dominating sets of graphs so, by Lemmas 5.8, 5.10 and 5.11, their corresponding lines cut straight through the substructures of the lattice, in the sense that $\mathbf{A} \sqcup \mathbf{B}$ is above a line if and only if at least one of $\mathbf{A}$ and $\mathbf{B}$ are above the line, and $\mathbf{A} \cap \mathbf{B}$ is below a line if and only if at least one of $\mathbf{A}$ and $\mathbf{B}$ are below the line. Also, lines that correspond to sets of graphs that dominate the sets of graphs of other lines cannot cross. For example, every subdivision of a star is a tree so the line corresponding to trees never crosses below the line corresponding to subdivisions of stars.

The following colour schemes in Figure 6.3 are from the literature. $\mathbf{Q}_k$ and $\mathbf{P}_k$ correspond to k-power-free and distance-k colouring, respectively. $\mathbf{S}$, $\mathbf{L}$, $\mathbf{E}$ and $\mathbf{P}$ correspond to star, acyclic, exact distance-2, and proper colouring. The relationship $\mathbf{S} \subseteq \mathbf{L}$ holds because all proper 2-monochromatic cycles contain a 2-monochromatic path of order 4. $\mathbf{X}$ and $\mathbf{C}$ are centred and conflict-free colouring. Two of the colour schemes, namely $\mathbf{X}$ and $\mathbf{L}$, are not path-dependent. A section of the lattice of path-dependent colour schemes can be obtained by removing $\mathbf{X}$ and $\mathbf{L}$ from Figure 6.3.

Many colour schemes in Figure 6.3 are defined in this thesis. $\mathbf{N}_k$ and $\mathbf{I}_k$ correspond to k-anagram-free and k-parity colouring respectively. The relationships among $\mathbf{I}_k$ are established in Section 6.3. The colour schemes $\mathcal{M}_\varepsilon$ and $\mathcal{M}'_\varepsilon$ are relaxations of anagram-free colouring defined in Section 6.3. $\mathbf{U}$, $\mathbf{Y}$, $\mathbf{T}$, and $\mathbf{F}$, are colour schemes that feature in Section 6.1 and correspond to up-down, min-parity-two, parity-two, and conflict-two, respectively. The colour schemes $\mathbf{R}$ and $\mathbf{R}'$ correspond to $xWx\overleftarrow{W}$-avoiding colouring and $\{WW, xWx\overleftarrow{W}\}$-avoiding colouring.

We now establish the relations in Figure 6.3 which are not shown in previous sections. $\mathbf{R} \subseteq \mathbf{S}$ because $\mathbf{S}$ is determined by $\{11, 1212\}$, up to relabelling, and both bad paths are of the form $xWx\overleftarrow{W}$. The unintended relationship, $\mathbf{Y} \subseteq \mathbf{R}$, can be demonstrated by inspecting sets of bad paths, since both colour schemes are path-dependent. Every path in $\mathcal{M}_{\mathbf{R}}$ has the form $xWx\overleftarrow{W}$ which means that all paths of order at least four in $\mathcal{M}_{\mathbf{R}}$ have the form $xWyxy\overleftarrow{W}$. Recall that $\mathbf{Y}$ is determined by $\mathcal{M}_{\mathbf{I}_2} \sqcap \mathcal{B}_{\mathbf{P}_2}$. Clearly, every colour in $xWyxy\overleftarrow{W}$ occurs an even number of times and contains two vertices at distance 2 with the same colour. Therefore $\mathcal{M}_{\mathbf{R}} \subseteq \mathcal{M}_{\mathbf{I}_2} \sqcap \mathcal{B}_{\mathbf{P}_2} \subseteq \mathcal{B}_{\mathbf{Y}}$ so $\mathbf{Y} \subseteq \mathbf{R}$. No similar relation holds for $\mathbf{R}'$ because $\mathcal{M}_{\mathbf{R}'}$ contains words of the form WW, which are not necessarily in $\mathcal{B}_{\mathbf{Y}}$.

To verify the accuracy of Figure 6.3 we must also check that all the relationships between colour schemes are included in Figure 6.3. The transitivity of the subset relation significantly reduces the work required. Even so, we do not present an exhaustive analysis of all potential relations between colour schemes. Many relations, such as $\mathbf{T} \not\subseteq \mathbf{R}'$, follow from the observation that $(12\ldots i)^k \in \mathbf{P}_{i-1}$ and $(12\ldots i)^k \notin \mathbf{I}_k$, for all integers $i \geqslant 1$ and $k \geqslant 2$. We focus on $\mathbf{E}$ and $\mathbf{U}$ as they are particularly isolated colour schemes and serve as examples of the techniques involved.

Let (C_4, ψ) be the proper 2-colouring of the cycle of order 4. There is no relation between $\mathbf{E}$ and $\mathbf{U}$ because $11 \in \mathbf{E}$, $(C_4, \psi) \notin \mathbf{E}$, $11 \notin \mathbf{U}$, and $(C_4, \psi) \in \mathbf{U}$. To show that there are no additional relations for $\mathbf{E}$ in Figure 6.3 we use the fact that every remaining colour scheme is a superset of $\mathbf{X}$ and a subset of either $\mathbf{P}$ or $\mathbf{Q}_k$. First note that $121 \notin \mathbf{E}$ and $121 \in \mathbf{C}$, so no other colour scheme in Figure 6.3 is a subset of $\mathbf{E}$. No other colour scheme is a superset of $\mathbf{E}$ because $11 \in \mathbf{E}$ and $11 \notin \mathbf{P}$, and $(12\ldots i)^k \in \mathbf{P}_{i-1}$ and $(12\ldots i)^k \notin \mathbf{Q}_k$, for all integers $i \geqslant 1$ and $k \geqslant 2$.

Now consider $\mathbf{U}$. First note that $\mathbf{U} \subseteq \mathbf{P}$ because $11 \in \mathcal{B}_{\mathbf{U}}$. Firstly, $\mathbf{U}$ is unrelated to all colour schemes between $\mathbf{L}$ and $\mathbf{X}$ because $(C_4, \psi) \notin \mathbf{L}$ and $(C_4, \psi) \in \mathbf{U}$, and $1213121 \in \mathbf{X}$ and $1213121 \notin \mathbf{U}$. There are no relations between $\mathbf{U}$ and the colour schemes on the left because $(123)^k \in \mathbf{U}$ and $(123)^k \notin \mathbf{Q}_k$. There are no relations between $\mathbf{U}$ and colour schemes on the right because $(12\ldots i)^k \in \mathbf{P}_{i-1}$ and, for sufficiently large k, the C-component sequence of every 4-meld of $(12\ldots i)^k$ is periodic, except for some finite number, dependent on i, of terms.

The answers to many questions raised in this chapter are visible in Figure 6.3. For example, the crossing of the lines corresponding to graphs of bounded degree and trees shows that colour schemes may have one property but not the other. As another example, there is a colour scheme between every pair of lines that crosses the chain $\mathbf{S} - \mathbf{R} - \mathbf{Y} - \mathbf{T} - \mathbf{F} - \mathbf{P}_2$, which shows that none of the properties corresponding to these lines are equivalent. The diagram also highlights questions. For example, $\mathbf{N}_4$ is bounded on trees while $\mathbf{I}_4$ is unbounded on paths, which raises the question of whether there are any colour schemes with intermediate properties that lie between $\mathbf{N}_4$ and $\mathbf{I}_4$.

6.5 Extensions and exceptions

The formulation of colour schemes presented in Chapter 5 does not include every notion of graph colouring. Some variants, such as hypergraph colouring, choosability, and edge colouring, were excluded because I felt that, for the introduction of colour schemes, it is better to focus on one formulation rather than spread the work over multiple generalisations. Vertex colouring was selected because it is the richest source of variants of graph colouring. I expect that colour schemes can be extended to include these variants, and that there are many similar tools and results. Line graph constructions can be used to translate some results from vertex colourings to edge colourings.

Some variants of vertex colouring are not colour schemes because they violate the notion of graph colouring as local conflict avoidance. More specifically, these cases tend to violate Subgraphability or Additivity. Two notable examples are equitable colouring and harmonious colouring. An *equitable colouring* is a proper colouring in which the number of vertices in any two colour classes differs by at most one [88, 99]. A *harmonious colouring* is a proper colouring in which every pair of colours occur on at most one pair of adjacent vertices [52–56, 96]. In both of these cases, there is a non-admissible coloured graph which is the disjoint union of two admissible coloured graphs.

Other variants are excluded because they depend on relationships, other than equality, between their colours, which is a violation of Recolourability. Such variants are often types of graph colouring which require graphs to be coloured by integers which satisfy particular relationships. We have already seen an example, ordered colouring, which is a colouring of the vertices of a graph, G, by integers, such that each path in G contains its largest colour exactly once [37]. Two more examples are rank colouring and radio colouring. A *rank colouring* is a colouring such that every path between two vertices that share a colour, $i \in \mathbb{Z}$, contains a vertex with colour greater than i [93]. A *radio colouring* is a colouring, by integers, such that the colours on vertices at distance x differ by at least $3-x$, for $x \in \{1,2\}$. These colourings are not closed under a permutation of their colours.

Colour schemes are potentially generalisable to include colourings which depend on relationships between their colours, at the cost of additional complication. For example, we may require graphs to be coloured by elements of some algebraic structure and restrict the properties of the structure which can be used to evaluate whether the colours of two vertices are in conflict. To see the depth of the complication, note that RECOLOURABILITY would need to be extended to not introduce new conflicts under any extended notion of conflict.

Chapter 7

Open problems and generalisations

Many open problems are raised throughout this thesis, and the study of anagram-free colouring and colour schemes can be extended in a multitude of ways. This chapter summarises many of these open problems, and highlights some potential extensions and generalisations.

7.1 Anagram-free colouring

To start with narrow questions, we ask whether ϕ is bounded by 4 or 5 on cycles, as this would complete the characterization of ϕ on graphs of bounded degree. The answer is either 4 or 5 because ϕ is bounded by 4 on paths. Recall that Currie [41] answered a similar question for π, by proving $\pi(C_n) = 3$, for all $n \geqslant 3$ with the exception of $n = 5, 7, 9, 10, 14, 17$. I conjecture the analogous result, that $\phi(C_n) = 4$, for all $n \geqslant 3$ with only finitely many exceptions. This conjecture is supported by the results of Chapter 4, in which it is shown that ϕ is bounded on sufficiently subdivided graphs. Since ϕ is unbounded on graphs of maximum degree 3, determining $\phi(C)$, for all cycles C, answers all remaining questions about ϕ on graphs of bounded degree.

The relationship between ϕ and pathwidth is interesting because there are contrasting results on trees and graphs. Carmi et al. [35] show that ϕ is unbounded on planar graphs of pathwidth 3 and maximum degree 5. In contrast, Theorem 3.10 shows that ϕ is bounded by $4p + 1$ on trees of pathwidth at most p, and that there exists a tree, T, with $\phi(T) \geqslant p \geqslant \text{pw}(T)$. Furthermore, Theorem 4.10 shows that, for all p, there is a graph, G, with $\text{pw}(G) \geqslant p$ and $\phi(G) \leqslant 8$. I conjecture that ϕ is not bounded on graphs of pathwidth 2, and furthermore, that ϕ is not bounded on ladders.

Results from Chapter 3 and the literature motivate the study of upper bounds on $\phi(G)$ as a function of $|V(G)|$. Kamčev et al. [83] prove the polynomial bound $\phi(G) \leqslant 10|V(H)|^{3/2}|V(G)|^{1/2}$ for graphs, G, without H as a minor. Many of the lower bounds for ϕ are polynomials of $\log |V(G)|$. Planar graphs are a particularly interesting case, since Carmi et al. [35] and Theorem 3.2 provide two families of planar graphs for which $\phi(G)$ grows logarithmically in $|V(G)|$. This raises the question of whether $\phi(G) \leqslant c(\log |V(G)|)^k$, for some c and k, for all planar graphs, G. We also ask a related question for treewidth; does there exist a function, f, such that $\phi(G) \leqslant f(k)\,(\log |V(G)|)^{f(k)}$, for all graphs, G, of treewidth k?

Anagram-free colouring is primarily focused on 2-anagram-free vertex colourings, both in this thesis and the literature. As such, many questions about the edge colouring and k-anagram variants of anagram-free colouring are open. By Theorem 3.1, ϕ' is unbounded on trees of maximum degree 3, however, for $k \geqslant 3$, the only known lower bound on ϕ'_k is on trees of bounded pathwidth, see Theorem 3.14. By Theorem 3.13, ϕ_k and ϕ'_k are bounded on trees, for $k \geqslant 4$, which motivations the question of whether ϕ_3 or ϕ'_3 are bounded on trees. By Theorem 3.12, ϕ_k is unbounded on graphs of maximum degree $k+1$, but it is open as to whether there a fixed Δ such that, for all $k \geqslant 2$, ϕ_k is unbounded on graphs of maximum degree Δ. Regarding ϕ_k on outerplanar graphs, I conjecture that ϕ_k is bounded on outerplanar graphs, for $k \geqslant 4$. The support for this conjecture is Theorem 3.13, that $\phi_4(T) \leqslant 4$, for all trees T, and that outerplanar graphs have bounded treewidth [21]. More generally, I conjecture that ϕ_k and ϕ'_k are bounded on graphs of bounded treewidth, for $k \geqslant 4$.

Two notable open problems arise from Chapter 4, since Theorem 4.13 shows that ϕ is bounded by 8 on choice-subdivisions of graphs. The first open problem is to determine the smallest c such that every graph has an anagram-free c-colourable subdivision. Since π is bounded by 3 on sufficiently subdivided graphs [114], I conjecture that every graph, G, has a subdivision, H, with $\phi(H) \leqslant 4$. The second open problem is to optimize the number of division vertices per edge required to colour a subdivision with a constant number of colours. In the case of trees, Theorem 4.9 shows that the number of division vertices per edge is close to optimal. Theorem 4.14 shows that significant improvements can be made to bounds on the number of division vertices in c-colourable subdivisions of general graphs. The improvements are likely to take the form of better constructions, to reduce the upper bound, as well as stronger results on complete graphs, for better lower bounds. These questions can also be extended to ϕ_k. Using the results of Dekking [47], it is likely that the construction in Theorem 4.10 can be extended to prove a bound of 7 and 6 for ϕ_k on choice-subdivisions of graphs, for $k = 3$ and $k = 4$, respectively. Furthermore, I conjecture the stronger results, that $\phi_3(S) \leqslant 3$ and $\phi_4(S) \leqslant 2$ for every sufficiently subdivided graph, S. The edge colouring variant of Chapter 4 is completely open, except for the caveat that ϕ' is unbounded on stars. However, ϕ'_k is bounded on stars for $k \geqslant 3$, so there is the potential for analogous results.

Going further afield, there are extensions of anagram-free colouring which are barely touched on in this thesis. One such extension is anagram-free choice numbers. All the questions asked in this thesis can be extended to choosability in the same way that square-free colouring is extended to square-free choosability. In particular, we ask whether the anagram-free choice number is bounded on paths. Recall that Zhao and Zhu [135] define an extension of k-power-free colouring to non-integer values of k and show that $\pi_{(2+\varepsilon)\,\mathrm{ch}}(P_n) \leqslant 3$ for all $\varepsilon > 0$. The same extension can be studied for anagram-free colouring, in particular, is $\phi_{(2+\varepsilon)\,\mathrm{ch}}(P_n) \leqslant 4$ for all n? Another extension from the square-free colouring is anagram-free walks and trails. Given that ϕ is unbounded on many classes of graphs, it seems unlikely that further relaxing the requirements of anagram-free colouring would result in reasonable behaviour. However, the questions can still be asked. The closest work to this area is that of Czap et al. [44], who study anagram-free trails in the faces of planar graphs.

7.2 Colour schemes

The study of colour schemes can be improved upon and expanded in many ways. The tools of Chapter 5 and results in Chapter 6 are promising, and suggest that the colour scheme axioms are a good set of axioms to use in future work. One direction is to extend the colour scheme axioms to encompass a wider notion of graph colouring, such as edge colouring and choosability. Another direction is investigate currently studied colour schemes, determine some shared properties, and to prove general theorems about colour schemes in terms of these properties. Many of the results in Chapter 6 raise further questions and are likely able to be improved.

As mentioned in Section 6.1.1, our formulation of choice-subdivisions may have subtle problems. While this is the notion used in Chapter 4, as well as in the literature on square-free colouring, a stronger notion may be more useful for colour schemes in general. We could, instead, study subdivisions by requiring that the chromatic number in question be bounded on all ($\geqslant k$)-subdivisions of a graph G, with k chosen independently for each G. This is a stronger notion than the one studied in Section 6.1, so it may allow us to prove stronger results in this area.

Theorem 6.18 gives sufficient conditions for a path-dependent colour scheme to be bounded on choice-subdivisions of graphs or on subdivisions of stars. The natural question is whether these conditions are necessary. The resulting open problem is whether there is a counter-example or, whether it can be shown that the conditions are necessary. A related question is whether the results of Section 6.1 can be extended to include colour schemes which are not path-dependent. Also, given $\mathbf{U}$ is bounded on choice-subdivisions of trees and not on choice-subdivisions of graphs, there is the additional open problem of characterizing which path-dependent colour schemes are bounded on choice-subdivisions of trees.

The somewhat artificial definition of $\mathbf{U}$ raises the question of whether colour schemes with unusual properties can, or should, be excluded from the main study of colour schemes. Similar to the concept of 'Hausdorff space' in topology, defining $\mathbf{U}$ out of the set of 'nice' colour schemes could, potentially, reveal a more useful subset of colour schemes. The 4-subcolouring used at the start of the definition of $\mathbf{U}$ is particularly egregious, as the 4-subcolouring essentially circumvents restrictions put in place by Recolourability. Of course, there may be a much more natural colour scheme which fulfils the same role as $\mathbf{U}$. This motivates the search for more colour schemes which are bounded on choice-subdivisions of trees and not on choice-subdivision of graphs. The questions of whether $\mathbf{U}$ is bounded on trees or on graphs of bounded degree are also open.

There are many questions to ask about extensions of anagram-free colouring. Carmi et al. [35] show that the cube of anagram-free colouring, $\mathbf{N}^3$, defined in Section 6.3.3, is unbounded on paths and, conversely, $\mathbf{N}$ is bounded on paths. We conjecture that $\mathbf{N}^2$ is not bounded on paths. More generally, it is open whether there is any colour scheme, $\mathbf{A}$, with $\mathbf{A} \subseteq \mathbf{N}$, which is bounded on paths and non-trivially distinct from $\mathbf{N}$. Similarly, it is open whether there are any colour schemes between $\mathbf{I}_k$ and $\mathbf{N}_k$ with distinctive properties since, for $k \geqslant 4$, $\mathbf{N}_k$ is bounded on trees and $\mathbf{I}_k$ is unbounded on paths.

More widely, there are general questions to ask regarding bounds on colour schemes. Theorem 6.20 and 6.21 give sufficient conditions for a path-dependent colour scheme to be bounded on trees or bounded on graphs of bounded maximum degree. These results can be improved, with the eventual goal of finding necessary and sufficient conditions for a colour scheme to be bounded on trees or graphs of bounded degree. Many graph parameters exist, such as pathwidth, radius, and treewidth, which can be studied in the context of colour schemes. I conjecture that every path-dependent colour scheme which is bounded on trees is bounded on graphs of bounded treewidth. Theorem 6.21 demonstrates that variants of the Lovász Local Lemma can be used to prove upper bounds on path-dependent colour schemes which avoid sufficiently few patterns. We expect there to be similar results for colour schemes which are not necessarily path-dependent, and that entropy compression can be used to obtain better bounds. There are many more interesting questions in this area. A particularly daunting task is to find conditions which are necessary and sufficient for a colour scheme to be bounded on planar graphs.

The algebraic structure of colour schemes is another potential area of study. The union of two path-dependent colour schemes is not necessarily path-dependent, but, by Lemma 5.12, it must be $\mathcal{G}$-dependent for some set of connected finite graphs $\mathcal{G}$. Are there any further restrictions on $\mathcal{G}$? In general, what is the smallest set of graphs, $\mathcal{G}$, such that the union of two $\mathcal{H}$-dependent colour schemes is $\mathcal{G}$-dependent? Which path-dependent colour schemes are the union of two path-dependent colour schemes? A *sublattice* of a lattice is a subset of its elements that are closed under the binary operations of the original lattice. Are there any sets of graphs, $\mathcal{G}$, such that the $\mathcal{G}$-dependent colour schemes induce a non-trivial sublattice? The families of colour schemes $\mathbf{Q}_k$, $\mathbf{N}_k$, $\mathbf{I}_k$, $\mathbf{M}_\varepsilon$, and $\mathbf{P}_k$ imply the existence of chains of unbounded length. Is there a categorization of such chains?

Finally, we would like to see an expansion of the colour scheme zoo. Kang and Müller [84] combines k-frugal colouring with star and acyclic colouring, which corresponds to taking the intersection of $\mathbf{G}_k$ with $\mathbf{S}$ and $\mathbf{L}$. Pór and Wood [116] study types of graph colourings defined by avoiding subgraphs in the induced subgraphs of pairs of colour classes. Results such as these extend the colour scheme zoo and can be represented in a diagram similar to Figure 6.3. An increased number of examples of colour schemes increases our ability to intuit general properties and generate novel examples. We are also motivated to find more operations on colour schemes, as currently we only have the three binary operations of intersection, union and bad graph intersection, as well as the unary operation of raising colour schemes to the power of k. Additional operations allow for the generation of more examples and a better understanding of the structure of the lattice of colour schemes.

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