Extremal Graph Theory for Minors, Improper Colourings and Gonality

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Abstract

This thesis explores a broad range of topics in extremal graph theory.

The first area of focus is graph minors. In particular, we focus on the extremal function for the class of graphs excluding a given minor. We show that every graph with at least $5n-8$ edges contains a Petersen graph as a minor, and that this is tight for infinitely many graphs. As a corollary, we show that every graph with no Petersen minor is 9-colourable. This is best possible. We also establish an extremal function for general excluded minors. We show that every graph with average degree at least $i+5.8105q$ contains every graph with at most q edges and i isolated vertices as a minor. This improves on a recent result of Reed and Wood [*Combin. Probab. Comput.*, 2015].

The second area of focus is that of improper graph colourings. An (improper) graph colouring has *defect* d if each monochromatic subgraph has maximum degree at most d, and has *clustering* c if each monochromatic component has at most c vertices. In particular, we study defective and clustered list-colourings for graphs with given maximum average degree. We prove that every graph with maximum average degree less than $\frac{2d+2}{d+2}k$ is k-choosable with defect d. This improves upon a similar result by Havet and Sereni [*J. Graph Theory*, 2006]. For clustered choosability of graphs with maximum average degree m, no $(1-\epsilon)m$ bound on the number of colours was previously known. The above result with $d=1$ solves this problem. It implies that every graph with maximum average degree m is $\lfloor\frac{3}{4}m+1\rfloor$-choosable with clustering 2. This extends a result of Kopreski and Yu [*Discrete Math.*, 2017] to the setting of choosability. We then prove two results about clustered choosability that explore the trade-off between the number of colours and the clustering. In particular, we prove that every graph with maximum average degree m is $\lfloor\frac{7}{10}m+1\rfloor$-choosable with clustering 9, and is $\lfloor\frac{2}{3}m+1\rfloor$-choosable with clustering $O(m)$. As an example, the later result implies that every biplanar graph is 8-choosable with bounded clustering. This is the first non-trivial result for the clustered choosability version of the earth-moon problem.

The final topic is that of divisorial gonality for graphs. By considering graphs as discrete analogues of Riemann surfaces, Baker and Norin [*Adv. Math.*, 2007] developed a concept of linear systems of divisors for graphs. Building on this idea, a concept of gonality for graphs has been defined and has generated much recent interest. We show that there are connected graphs of treewidth 2 of arbitrarily high gonality. We also show that there exist connected graphs G and H such that $H\subseteq G$ and H has strictly lower gonality than G. These results resolve three open problems posed in a recent survey by Norin [*Surveys in Combinatorics*, 2015].

Declaration

This thesis contains no material which has been accepted for the award of any other degree or diploma at any university or equivalent institution and that, to the best of my knowledge and belief, this thesis contains no material previously published or written by another person, except where due reference is made in the text of the thesis.

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Kevin Hendrey
27.09.18

Publications During Enrolment

This thesis is a combination of several pieces of work (published, submitted and in preparation). Some of these works are joint work other authors, as detailed below.

- Chapter 2 is based on a paper published in *J. Combinatorial Theory, Series B* [67]. This is joint work with David Wood.
- Chapter 3 is joint work with Sergey Norin. A paper including content from this chapter is currently in preparation.
- Chapter 4 is based on a submitted article [66]. This is joint work with David Wood.
- Chapter 5 is based on a sole-author paper published in *SIAM J. Disc. Math.* [65].

I am extremely appreciative of my collaborators.

For Mum and Dad.

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Contents

Chapter 1

Introduction

Bollobás [16] writes "In extremal graph theory one is interested in the values of various graph invariants, such as order, size, connectivity, minimum degree, maximum degree, chromatic number and diameter, and also on the values of these invariants which ensure that the graph has certain properties". This thesis examines a range of topics in extremal graph theory, focussing on graph minors, notions of improper graph colouring, and the parameters of treewidth, number of edges and gonality.

There are three main threads which link these topics together: Hadwiger's conjecture and the parameters of maximum average degree and treewidth. The famous conjecture of Hadwiger proposes a strong link between graph colouring and graph minor theory, and highlights a large gap in our knowledge of both fields. The parameter of maximum average degree provides the most obvious connection between graph minors and graph colouring, since for every graph H the maximum average degree of the class of graphs with no H minor is bounded, which in turn provides a bound on the chromatic number.

In Chapter 2, we present our first contribution to the field, which is to determine a tight upper bound on the maximum average degree of graphs with no Petersen graph minor. In fact, we determine the exact extremal function for Petersen minors, and characterise the extremal graphs. As an immediate corollary, we prove that the maximum chromatic number of a graph with no Petersen graph minor equals 9.

In Chapter 3, we consider extremal functions for general excluded minors. For sparse graphs H, we improve the best known upper bounds on the maximum average degree of H-minor-free graphs, in terms of the number of vertices and edges of H.

Improper colourings provide one possible approach for attacking Hadwiger's conjecture. Indeed, for one form of improper colouring that we examine, namely that of defective colouring, a weaker version of Hadwiger's conjecture has been proven. Our main contribution to this area, which we present in Chapter 4, is to provide upper bounds on both the clustered chromatic number and the defective chromatic number of graph classes with given maximum average degree. Our results apply to all graph classes with bounded maximum average degree, and thus cover a broader range of graph classes than has previously been studied.

Treewidth has applications in a wide range of areas of graph theory, and is central to much of our understanding of graph minor theory. The treewidth of a graph also provides an upper bound on the parameter of gonality, which is known to be tight for a variety of graph classes. This naturally leads to questions about the extent to which the parameter of gonality might be related to graph minor theory. In a recent survey on graph minor theory, Norin [102] poses three questions about the nature of this relationship. In Chapter 5, we resolve each of these questions, determining that graph gonality is not as closely linked to treewidth or graph minor theory as other results in the area may have lead us to expect.

1.1 Graph Minors

A graph H is a *minor* of a graph G if a graph isomorphic to H can be obtained from G by the following operations: vertex deletion, edge deletion and edge contraction. The theory of graph minors is at the forefront of research in graph theory. In the seminal work of Robertson and Seymour, they prove the following theorem, considered by many to be among the deepest results in mathematics [111–114, 119–137].

Theorem 1 (Graph Minor Theorem). *Every infinite set of graphs contains a pair of distinct graphs H and G such that H is a minor of G.*

Their proof of this astounding result appeals to the axiom of choice, and is non-constructive. The importance of the result is best demonstrated through its application to minor-closed classes. A graph class is *minor-closed* if every minor of every graph in the class is in the class (equivalently, if the class is closed under the operations of edge contraction, edge deletion and vertex deletion). The Graph Minor Theorem tells us that every minor-closed class can be characterised in terms of a finite set of excluded minors. For every graph H, there is a polynomial time algorithm for determining whether a given graph G contains H as a minor [131]. Thus, for every minor-closed class, there is a polynomial time algorithm for determining membership in that class (given a graph G, simply check one by one whether G contains one of the finite number of excluded minors for the class). Since the proof of the Graph Minor Theorem is non-constructive, we are in the surprising position of knowing that these algorithms exist without knowing how to construct them.

One of the most famous examples of a minor-closed class is the class of planar graphs, which consists of all graphs that can be embedded in the plane so that edges do not cross. Predating the Graph Minor Theorem, the following theorem shows that planar graphs can be described in terms of only two excluded minors.

Theorem 2 (Kuratowski–Wagner [90, 153]). *A graph is planar if and only if it does not contain a K_5 minor and it does not contain a $K_{3,3}$ minor.*

The Graph Minor Theorem tells us that it is not some peculiarity of the plane which makes a theorem of this type possible. For example, for any surface, there is an associated minor-closed class consisting of all graphs which can be embedded in the surface such that edges do not cross. Theorem 1 implies that there is Kuratowski–Wagner type theorem for every surface, which was unknown prior to the work of Robertson and Seymour.

1.2 Graph Colouring and Hadwiger's Conjecture

In extremal graph theory, we are interested in exploring the relationship between different graph properties and parameters. In doing so, we are often forced to deepen our understanding of the concepts we are trying to relate, which is why the area is of such great importance. An excellent starting point for further exploring graph minors is its relationship to graph colouring. A *colouring* of a graph assigns each vertex a colour so that adjacent vertices are assigned different colours. The *chromatic number* of a graph G is the minimum number of colours in a colouring of G.

Given a minor-closed class, it is interesting to ask what is the maximum chromatic number of a graph in the class. The fact that such a maximum exists (unless the class contains all graphs) was first proved by Wagner [154]. The Four Colour Theorem [10, 11, 110] famously answers this question for planar graphs.

Theorem 3 (Four Colour Theorem [10, 11, 110]). *Every planar graph is 4-colourable.*

The proof of this simple sounding result eluded graph theorists for many decades, and is one of the earliest examples of a computer assisted proof. Much research has gone into generalising this result. For all other surfaces, the maximum chromatic number has long been known (see [56, 109]). Thus, the Four Colour Theorem completes our knowledge of the maximum chromatic number of all surfaces. Generalising further, it is interesting to consider what the Four Colour Theorem tells us about other minor-closed graph classes. Wagner [153] characterised the class of K_5-minor-free graphs in terms of planar graphs. Given this characterisation, the Four Colour Theorem implies that every K_5-minor-free graph is 4-colourable. Note that this is the broadest minor-closed class of 4-colourable graphs, since K_5 is not 4-colourable. Hadwiger's Conjecture proposes the following deep generalisation of this result.

Conjecture 4 (Hadwiger's Conjecture [58])**.** *Every graph with no K_t-minor is $(t-1)$-colourable.*

This conjecture is widely regarded as one of the most important open problems in graph theory. The *Hadwiger number* of a graph G is the maximum number of vertices in a complete graph minor of G. Thus, Hadwiger's conjecture asserts that the Hadwiger number of a graph is an upper bound on the chromatic number.

For low values of t, the conjecture is trivial. When $t = 2$, K_t-minor-free graphs have no edges, and when $t = 3$, the conjecture states that every forest is 2-colourable. The first non-trivial case, when $t = 4$, was proved by Dirac [45] and by Hadwiger himself [58]. As mentioned above, the $t = 5$ case is equivalent to the Four Colour Theorem [153]. Robertson, Seymour and Thomas [115] proved the $t = 6$ case. For all $t \geqslant 7$, the conjecture remains open. However, the conjecture is known to hold when restricted to various graph classes, for example complements of Kneser graphs [158], line graphs [105], quasi-line graphs [32] and several families of link graphs [73]. In fact the conjecture is true for almost every graph: the chromatic number of the random graph $G(n, \frac{1}{2})$ asymptotically almost surely is at most the Hadwiger number [17].

Regardless of whether the conjecture is true, it highlights a wide gap in our knowledge of both graph colouring and graph minor theory. It seems unlikely that the conjecture will be resolved in the affirmative without some breakthrough in both our understanding of graph colouring and our understanding of complete graph minors. If there are counterexamples to the conjecture, it may well be that a computer search will one day discover one without granting us any profound insight. However, the following weakening of Hadwiger's conjecture is also open, and is less vulnerable to such attacks.

Conjecture 5 (Weak Hadwiger's Conjecture)**.** *There exists a constant c such that every graph with no K_t minor is ct-colourable.*

A possible approach for tackling Conjectures 4 and 5 is to find some stronger result from which they might follow. In this direction, one idea would be to consider a strengthening of the notion of colouring. We now introduce a widely used example of such a strengthening, namely list colouring.

1.2.1 List Colouring and Choosability

List colourings are example of a generalisation of a graph colouring. A *list-assignment* for a graph G is a function L that assigns a set $L(v)$ of colours to each vertex $v \in V(G)$. A list-assignment L is a *k-list-assignment* if $|L(v)| \geqslant k$ for each vertex $v \in V(G)$. An *L-colouring* is a colouring of G such that each vertex $v \in V(G)$ is assigned a colour in $L(v)$.

A graph is *k-choosable* if it is L-colourable for every k-list-assignment L. The *choosability* of a graph is the minimum integer k such that it is k-choosable.

A k-colouring of a graph G is equivalent to an L-colouring where L is the k-list-assignment that assigns every vertex of G the same set of k-colours. Thus, the choosability of a graph is always at least the chromatic number. In some cases, the two values are the same (for example, complete graphs). However, Alon [4] showed that the choosability of a graph with minimum degree d is at least $(\frac{1}{2} + o(1)) \log_2 d$, in contrast to the case of chromatic number where even bipartite graphs can have arbitrarily high minimum degree.

The list colouring version of Hadwiger's conjecture is false, since there are K_t-minor-free graphs which are not $(t-1)$-colourable for $t \geqslant 5$ [152]. However, the weak version of Hadwiger's conjecture is open even in the setting of choosability. In this setting, Barát, Joret and Wood [15] have shown that if there is some constant c such that every graph with no K_t-minor is ct-choosable, then $c \geqslant \frac{4}{3}$.

For large values of t, the best known upper bounds on the choosabilty of K_t-minor-free graphs exhibit the same asymptotic behaviour as the best known upper bounds on the chromatic numbers of these graphs. The upper bounds in both cases are derived using the extremal function for excluded minors, which we now introduce.

1.3 Extremal Functions for Specific Minors

The *extremal function for excluded minors* gives the maximum number of edges in a graph with n vertices which excludes a specified graph H as a minor. We denote this number by $\mathrm{ex_m}(n, H)$, by analogy to the well known extremal function for excluded subgraphs $\mathrm{ex}(n, H)$, which gives the maximum number of edges in an n-vertex graph with no H subgraph.

Evaluating $\mathrm{ex_m}(n, H)$ gives an upper bound on the chromatic number for H-minor-free graphs, as per the following well known lemma (see Section 1.9 for a proof).

Lemma 6. *Let H be a graph such that $\mathrm{ex_m}(n, H) < cn$ for some positive integer c. Then every H-minor-free graph is $2c$-choosable, and if $|V(H)| \leqslant 2c$ then every H-minor-free graph is $(2c-1)$-colourable.*

Thus, Hadwiger's conjecture has generated much interest in the extremal function for excluded complete minors. Trivially, $\mathrm{ex_m}(n, K_2) = 0$ and $\mathrm{ex_m}(n, K_3) = n - 1$, since K_2-minor-free graphs have no edges and K_3 minor free graphs are forests. We also know $\mathrm{ex_m}(n, K_t)$ for $t \in \{4, 5, \ldots, 9\}$, as per Table 1. When $t \geqslant 10$, the precise extremal function is unknown, but as we discuss in Section 1.4, the asymptotic behaviour of $\mathrm{ex_m}(n, K_t)$ for large t is known precisely.

Table 1

Excluded Minor (H)	$\mathrm{ex_m}(n, H)$	Citation
K_4	$2n-3$	Wagner 1937 [153] and Dirac 1964 [46]
K_5	$3n-6$	Wagner 1937 [153] and Dirac 1964 [46]
K_6	$4n-10$	Mader 1968 [94]
K_7	$5n-15$	Mader 1968 [94]
K_8	$6n-20$	Jørgensen 1994 [75]
K_9	$7n-27$	Song, Thomas 2006 [143]

When $t \leqslant 9$, not only is $\mathrm{ex_m}(n, K_t)$ known for all values of n, the structure of the extremal graphs (the K_t-minor-free graphs with the maximum number of edges) is also known. To describe these extremal graphs, we introduce the notion of cockades.

Let $\mathcal{S}$ be a set of graphs, and let k be a non-negative integer. Firstly, every graph in $\mathcal{S}$ is an $(\mathcal{S}, k)$-*cockade* (or equivalently a k-*cockade* of graphs in $\mathcal{S}$). A graph $G \notin \mathcal{S}$ is an $(\mathcal{S}, k)$-*cockade* if and only if there are $(\mathcal{S}, k)$-cockades G_1 and G_2, each with strictly fewer vertices than G, such that $G_1 \cup G_2 = G$ and $G_1 \cap G_2 \cong K_k$. For a graph G, a (G, k)-*cockade* is just a $(\{G\}, k)$-cockade.

Cockades are often useful for constructing extremal graphs for a minor-closed class. The following well known result (see Section 1.9 for a proof) gives a sufficient condition for a cockade to be H-minor-free.

Lemma 7. *For every $(t + 1)$-connected graph H and every H-minor-free graph G_0, every (G_0, t)-cockade is H-minor-free.*

As a simple example, a graph is a $(K_2, 1)$-cockade if and only if it is a tree, and this is exactly the class of extremal K_3-minor-free graphs. In fact, for $t \in \{2, 3, 4\}$, the class of extremal K_t-minor-free graphs is exactly the class of $(K_{t-1}, t - 2)$-cockades [46]. The extremal graphs for K_5-minor-free graphs are exactly the 3-cockades of edge-maximal planar graphs [153]. A graph is *apex* if it can be transformed into a planar graph by deleting at most one vertex. The extremal graphs for K_6-minor-free graphs are exactly the 4-cockades of edge-maximal apex graphs as well as the graph $K_{2,2,2,3}$ [74]. A graph is *2-apex* if it can be transformed into a planar graph by deleting at most two vertices. The class of extremal K_7-minor-free graphs consists of all 5-cockades of edge-maximal 2-apex graphs, as well as the single graph $K_{2,2,2,3}$ [74].

For $t \in \{2, 3, 4, 5, 6, 7\}$ the extremal function for K_t minors can be written as $(t-2)n - \binom{t-1}{2}$. This is false for $t = 8$, due to the K_8-minor-free graph $K_{2,2,2,2,2}$. An n-vertex graph with no K_8 minor has $6n - 20$ edges if and only if it is a $(K_{2,2,2,2,2}, 5)$-cockade [75]. The extremal function for K_t is also greater than $(t-2)n - \binom{t-1}{2}$ when $t = 9$, where the extremal graphs are $K_{2,2,2,3,3}$ and all $(K_{1,2,2,2,2,2}, 6)$-cockades [143].

The fact that $K_{2,2,2,3}$, $K_{2,2,2,2,2}$ and $K_{2,2,2,2,2,1}$ are extremal graphs for K_7, K_8 and K_9 minors respectively, is a special case of the following lesser known result due to Cera et al. [28, 29], who determined $\mathrm{ex_m}(n, K_t)$ for an infinite family of cases when t is close to n. In particular, $\mathrm{ex_m}(n, K_t) = \mathrm{ex}(n, K_{2t-n})$ if $t \in [\frac{5n+9}{8}, \max(\frac{2n-1}{3}, n-24)] \cup [\frac{2n+3}{3}, n]$. It is an open problem to determine the set of ordered pairs (n, t) for which $\mathrm{ex_m}(n, K_t) = \mathrm{ex}(n, K_{2t-n})$.

Computing the extremal function is one of the most fundamental problems to consider for any minor-closed graph class. As such, the study of extremal functions has not focussed solely on complete graphs, and the extremal function is known for various other graphs, including the bipartite graphs $K_{3,3}$ [59] and $K_{2,t}$ [33], the octahedron $K_{2,2,2}$ [42], and the complete graph on eight vertices minus an edge K_8^- [142].

1.3.1 Petersen Minors

In Chapter 2, we calculate the extremal function when the excluded minor is the Petersen graph (see Figure 2), denoted by $\mathcal{P}$. The class of $\mathcal{P}$-minor-free graphs is interesting for several reasons. As an extension of the 4-colour theorem, Tutte [148] conjectured that every bridgeless graph with no $\mathcal{P}$-minor has a nowhere zero 4-flow. Edwards, Robertson, Sanders, Seymour and Thomas [54, 117, 118, 138, 140] have announced a proof that every bridgeless cubic $\mathcal{P}$-minor-free graph is edge 3-colourable, which is equivalent to Tutte's conjecture in the cubic case. Alspach, Goddyn and Zhang [6] showed that a graph has the circuit cover property if and only if it has no $\mathcal{P}$-minor. It is recognised that determining the structure of $\mathcal{P}$-minor-free graphs is a key open problem in graph minor theory (see [43, 95] for example). Our results are a step in this direction.

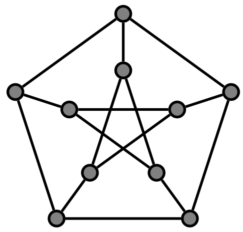


Figure 2

Our primary result is the following:

Theorem 8 (§2.1)**.**

$$\mathrm{ex_m}(n, \mathcal{P}) = \begin{cases} \binom{n}{2} & \text{if } n \leqslant 9, \\ 5n - 14 & \text{if } n \in \{11, 12\}, \\ 5n - 9 & \text{if } n \equiv 2 \pmod 7, \\ 5n - 12 & \text{otherwise.} \end{cases}$$

For $n \equiv 2 \pmod 7$, we in fact completely characterise the extremal graphs (see Theorem 10 below). Theorem 8 and Lemma 6 with $c = 5$ imply the following Hadwiger-type theorem for $\mathcal{P}$-minors, which is best possible for $\mathcal{P}$-minor-free graphs with K_9 subgraphs.

Theorem 9. *Every $\mathcal{P}$-minor-free graph is 9-colourable.*

For $n \geqslant 13$, the upper bound in Theorem 8 is implied by the following result, which also shows that $(K_9, 2)$-cockades are the unique extremal examples of $\mathcal{P}$-minor-free graphs when $n \equiv 2 \pmod 7$. Indeed, this theorem characterises $\mathcal{P}$-minor-free graphs that are within two edges of extremal when $n \equiv 2 \pmod 7$.

Theorem 10 (§2.1)**.** *Every graph with $n \geqslant 3$ vertices and $m \geqslant 5n - 11$ edges contains a Petersen minor or is a $(K_9, 2)$-cockade minus at most two edges.*

Since $(K_9, 2)$-cockades have connectivity 2, it is interesting to ask for the maximum number of edges in more highly connected $\mathcal{P}$-minor-free graphs. Hence, we define $\mathrm{ex_m}(n, H, k)$ to be the maximum number of edges in a k-connected, n-vertex H-minor-free graph. First note that Theorem 10 implies that 3-connected $\mathcal{P}$-minor-free graphs, with the exception of K_9, have at most $5n - 12$ edges. The following result shows that this bound is tight for all but finitely many values, and also provides the lower bound in Theorem 8 for an infinite family of cases.

Theorem 11 (§2.3)**.** *If $n \geqslant 13$ and $n \notin \{16, 17, 22\}$, then $\mathrm{ex_m}(n, \mathcal{P}, 3) = 5n - 12$.*

Exploring the relationship between connectivity and the extremal function further, we find that there are 5-connected graphs with almost as many edges as the extremal function.

Theorem 12 (§2.3)**.** *For $n \geqslant 10$, $\mathrm{ex_m}(n, \mathcal{P}, 5) \geqslant 5n - 15$.*

We now show that there are infinitely many 6-connected $\mathcal{P}$-minor-free graphs. Since $K_{3,3}$ is a minor of $\mathcal{P} - v$ for each vertex v, the Petersen graph is not apex and every apex graph is $\mathcal{P}$-minor-free. An n-vertex graph G obtained from a 5-connected planar triangulation by adding one universal vertex is 6-connected, $\mathcal{P}$-minor-free, and has $4n - 10$ edges. We know of no infinite families of 6-connected $\mathcal{P}$-minor-free graphs with more edges. We also know of no infinite families of 7-connected $\mathcal{P}$-minor-free graphs. Indeed, it is possible that every sufficiently large 7-connected graph contains a $\mathcal{P}$-minor. The following conjecture is even possible.

Conjecture 13. *Every sufficiently large 6-connected $\mathcal{P}$-minor-free graph is apex.*

This is reminiscent of Jørgensen's conjecture [75], which asserts that every 6-connected K_6-minor-free graph is apex. Jørgensen's conjecture has recently been proved for sufficiently large graphs [78, 79]. In this respect, K_6 and $\mathcal{P}$ possibly behave similarly. Indeed, they are both members of the so-called Petersen family [93, 116, 139]. Note however, that the extremal functions of K_6 and $\mathcal{P}$ are different, since $\operatorname{ex_m}(n, K_6) = 4n - 10$ [94].

1.4 Extremal Functions for General Minors

The extremal function for a graph H is the maximum number of edges in an n-vertex graph not containing H as a minor. It is unlikely that there is a simple way of computing this function for an arbitrary graph H. Even computing the function for relatively small graphs is difficult, and for $t \geqslant 10$, the extremal function for K_t remains unknown.

What we can do is find general upper bounds on the extremal function for arbitrary graphs. The following function which is closely related to the extremal function has received much attention. For a graph H, let $f(H)$ be the infimum of all non-negative real numbers c such that every graph with average degree at least c contains H as a minor. Equivalently, $f(H)$ is the supremum over all positive integers n of $(2 \operatorname{ex_m}(n, H))/n$ (except when H has no edges, in which case $\operatorname{ex_m}(n, H)$ is generally undefined). Thus, an upper bound on $f(H)$ is equivalent to a linear upper bound on $\operatorname{ex}_m(n, H)$. For example, the results in Table 1 show that $f(K_t) = 2t - 4$ for $t \leqslant 9$.

Tight bounds on the function f are known for various families of graphs, including general complete graphs K_t [40, 84, 85, 145, 146], unbalanced complete bipartite graphs $K_{s,t}$ [86–89, 99], disjoint unions of complete graphs [147], disjoint unions of cycles [38, 60], general dense graphs [100] and general sparse graphs [61, 106].

As with the extremal function itself, the study of $f(K_t)$ is in part motivated by Hadwiger's conjecture. An immediate corollary of Lemma 6 is that every K_t-minor-free graph is $(\lfloor f(K_t) \rfloor + 1)$-choosable, and $\lfloor f(K_t) \rfloor$-colourable whenever $f(K_t) \geqslant t$. Given the results in Table 1, Lemma 6 implies that, for $t \in \{4, 5, \ldots, 9\}$, every graph with no K_t minor is $(2t - 5)$-colourable and $(2t - 4)$-choosable. This bound is weaker than Hadwiger's conjecture, but if it held for all values of t, then the Weak Hadwiger Conjecture would be true, even in the stronger setting of choosability. In fact, any linear upper bound on $f(K_t)$ would prove the choosabilty version of the Weak Hadwiger Conjecture.

It turns out that there is no linear upper bound on $f(K_t)$. Kostochka [84, 85] showed that, for large t, there are random graphs with no K_t-minor with average degree of the order of $t\sqrt{\log t}$, and the same was shown independently by de la Vega [40] based on the work of Bollobás, Catlin and Erdős [17] at around the same time. Kostochka [84, 85] and Thomason [145] independently also showed the upper bound $f(K_t) \leqslant O(t\sqrt{\log t})$. Finally, Thomason [146] determined the constant $\alpha = 0.638\ldots$ such that $f(K_t) = (\alpha + o(1))t\sqrt{\ln t}$. For large t, every extremal K_t-minor-free graph consist of essentially disjoint copies of quasi-random graphs [98, 146].

Highly connected K_t-minor-free graphs exhibit different behaviour. Norin and Thomas [101, 144] have announced a proof that, for $t \geqslant 5$ and $n \gg t$, every t-connected K_t-minor-free graph on n-vertices has a set of $t-5$ vertices whose deletion leaves a planar graph, and hence $\mathrm{ex}_{\mathrm{m}}(n, K_t, t) = (t-2)n - \binom{t-1}{2}$.

Since every t-vertex graph H is a minor of K_t, $f(H) \leqslant O(t\sqrt{\log t})$. Intuitively, if $|E(H)|$ is small, then we should be able to improve this bound. This leads to the following definition. Given non-negative integers t and q, the *extremal function for general graphs*, denoted $f(t,q)$, is the maximum value of $f(H)$ over all graphs H with at most t vertices and at most q edges.

1.4.1 Dense Graphs

We call a class of graphs *dense* if there are positive constants c and τ such that for every integer n, all n-vertex graphs in the class have at least $cn^{1+\tau}$ edges. Much is known about the behaviour of the extremal function for excluded minors from a dense graph class.

Myers and Thomason [100] introduce a new graph invariant γ, in terms of which they find the following approximation for the extremal function.

Theorem 14 ([100])**.** *The following holds for every t-vertex graph H, where $\alpha = 0.638\ldots$ and* err *is an error function such that* $\max_{|V(H)|=t} |\operatorname{err}(H)| = o(1)$.

$$f(H) = (\alpha\gamma(H) + \operatorname{err}(H))t\sqrt{\ln t}.$$

Myers and Thomason note that calculating γ is non-trivial. However, they show that if H is a t-vertex graph with $t^{1+\tau}$ edges, then $\gamma(H) \leqslant \sqrt{\tau}$ and $\gamma(H) \approx \sqrt{\tau}$ for almost every graph and for every regular graph.

Theorem 14 implies the following.

Corollary 15 ([100])**.** *If $c, \tau > 0$, then $f(t, ct^{1+\tau}) = (\alpha\sqrt{\tau} + o(1))t\sqrt{\ln t}$, where $o(1) \to 0$ as $t \to \infty$.*

For any family of graphs on which γ is bounded below by some constant independent of t, this result gives the asymptotic behaviour of the extremal function. However, note that on classes of graphs where $\gamma(G) = o(t)$, Theorem 14 says little.

1.4.2 Sparse Graphs

We call a class of graphs *sparse* if the average degree of every graph in the class is bounded from above by a constant. Note that for a sparse graph class, $\gamma(G)$ will approach 0 as the size of G increases, so Theorem 14 is not applicable. The extremal function for sparse minors has previously been studied by Reed and Wood [106], who proved the following theorem.

Theorem 16 ([106])**.** *There is some d_0 such that if H is a graph with average degree $d \geqslant d_0$, then $f(H) \leqslant 3.895\sqrt{\ln(d)}t$, where $t := |V(H)|$.*

This theorem is tight up to a constant factor for numerous graphs, due to the above-mentioned result of Myers and Thomason [100], and if $2c \geqslant d_0$, then $f(t, ct) = \Theta(\sqrt{\ln(c)}t)$. However, the value of d_0 might be extremely large. Determining the behaviour of $f(t,q)$ when Theorem 16 is not applicable is an interesting problem.

Reed and Wood [106] also provided the first non-trivial results here, proving a number of inequalities of the form $f(H) \leqslant \alpha|V(H)| + \beta|E(H)|$, where α and β are explicit constants.

In particular, they note that the minimum possible value of α in such an inequality is 1, since $f(\overline{K_t}) = t - 1$. Thus, focussing on minimising the value of β in such an inequality, they prove the following two theorems.

Theorem 17 ([106]). $f(t, q) \leqslant t + 6.291q$.

Theorem 18 ([106]). *For every graph H with exactly i isolated vertices and q edges,*

$$f(H) \leqslant i + 6.929q.$$

In Chapter 3, we improve on these results with the following theorem.

Theorem 19 (§3.1). *For every graph H with exactly i isolated vertices and q edges,*

$$f(H) \leqslant i + 5.8105q.$$

In terms of the function $f(t, q)$, this implies the following corollary.

Corollary 20. $f(t, q) \leqslant t + 5.8112q$.

It is an interesting open problem to determine the least real number α such that, for all integers t and q, $f(t, q) \leqslant t + \alpha q$. Csóka et al. [38], proved that for every positive integer k, $f(kK_3) = 4k - 2$. As Reed and Wood [106] observe, this implies that $\alpha \geqslant \frac{1}{3}$, which is the best known lower bound.

It is also interesting to ask for the least real number β such that $f(H) \leqslant i + \beta q$ for every graph H with exactly i isolated vertices and q edges. Since kK_3 has no isolated vertices and exactly $3k$ edges, the same construction shows that $\beta \geqslant 4$, which again is the best known lower bound.

1.5 Improper Graph Colourings

Another possible approach for attacking Hadwiger's conjecture is through weakening the notion of colouring. An improper colouring of a graph G is simply a map from the vertex set of G to another set, where the elements of the second set are referred to as *colours*. In a (proper) graph colouring, there is the added restriction that adjacent vertices must map to different colours, but there are several interesting varieties of improper colourings where this restriction is in some way relaxed. We shall focus on three in particular. For simplicity, we occasionally use "colouring" to mean improper colouring. For the remainder of the thesis, we refer to a colouring in which no pair of adjacent vertices are assigned the same colour as a "proper colouring".

1.5.1 Vertex Arboricity

The first variety of improper colouring, which we shall touch on only briefly, is the notion of vertex arboricity. A proper colouring of a graph G can be thought of as a partition of $V(G)$ such that each part induces an edgeless subgraph, equivalently a subgraph with no K_2-minor. One way of generalising this is to instead ask for a partition of $V(G)$ such that each part induces a K_t-minor-free subgraph for some larger value of t. We are interested in the $t = 3$ case, but there has been much research into this topic for larger values of t (see for example [44]). The minimum integer k such that there exist a partition of $V(G)$ into k sets such that each set induces a K_3-minor-free subgraph (equivalently a forest), is called the *vertex arboricity* of G. G is *d-degenerate* if every subgraph of G has a vertex of degree at most d. Chartrand and Kronk [30] proved that every d-degenerate graph has vertex arboricity at most $\lceil \frac{d+1}{2} \rceil$. By Theorem 8, every $\mathcal{P}$-minor-free graph is 9-degenerate. Hence, we have the following result, which is best possible for $\mathcal{P}$-minor-free graphs with K_9 subgraphs.

Theorem 21. *Every $\mathcal{P}$-minor-free graph has vertex arboricity at most 5.*

Other classes of graphs for which the maximum vertex arboricity is known include planar graphs [30], locally planar graphs [141], triangle-free locally planar graphs [141], for each $k \in \{3,4,5,6,7\}$ the class of planar graphs with no k-cycles [68, 104], planar graphs of diameter 2 [1], K_5-minor-free graphs of diameter 2 [69], and $K_{4,4}$-minor-free graphs [76].

1.5.2 Defective and Clustered Colouring

We now define defective and clustered graph colouring, which is the focus of Chapter 4. Given an improper colouring of a graph G, the *monochromatic subgraph* of G is the spanning subgraph consisting of all edges between vertices of the same colour. A *monochromatic component* of G is a connected component of the monochromatic subgraph. An improper graph colouring has *defect* k if each monochromatic component has maximum degree at most k; that is, each vertex v is adjacent to at most k vertices of the same colour as v. An improper graph colouring has *clustering* k if each monochromatic component has at most k vertices. Of course, a colouring is proper if and only if it has defect 0 or clustering 1.

Defective and clustered graph colouring has been widely studied on a variety of graph classes, including bounded maximum degree [5, 63], planar [37, 39, 52], bounded genus [12, 31, 36, 37, 55, 157], excluding a minor [51, 53, 91, 103, 149], excluding a topological minor [51, 103], excluding an immersion [149]. See [155] for a survey on defective and clustered colouring.

The *defective chromatic number* of a graph class $\mathcal{G}$ is the minimum integer k such that for some integer d, every graph in $\mathcal{G}$ is k-colourable with defect d. The *clustered chromatic number* of a graph class $\mathcal{G}$ is the minimum integer k such that for some integer c, every graph in $\mathcal{G}$ is k-colourable with clustering c.

For the defective chromatic number, a variant of Hadwiger's conjecture is known to hold, as follows.

Theorem 22 ([53])**.** *For $t \geqslant 2$, the defective chromatic number of the class of K_t-minor-free graphs equals $t-1$. In particular, every K_t-minor-free graph is $(t-1)$-colourable with defect $O(t^2 \log t)$.*

The $O(t^2 \log t)$ defect bound in Theorem 22 was improved to $O(t)$ by Van den Heuvel and Wood [149].

For clustered chromatic number, a proof of the analogous conjecture has been announced by Dvořák and Norin [51], who present a proof of the first few cases: for $t \in \{2,3,\dots,9\}$, the clustered chromatic number for the class of K_t-minor-free graphs equals $t-1$ [51]. Also, Kawarabayashi and Mohar[77] showed that the clustered chromatic number of the class of K_t-minor-free graphs is $O(t)$, resolving the Weak Hadwiger conjecture for clustered colouring. The best published upper bound on the clustered chromatic number of the class of K_t-minor-free graphs is now $2t-2$ [51, 149].

There is a natural way to generalise clustered and defective colouring to clustered and defective list-colouring, and it is in this setting that we make our contributions to the field.

1.5.3 Defective Choosability

A graph G is *k-choosable with defect d* if G has an L-colouring with defect d for every k-list-assignment L of G. The *maximum average degree* of a graph G, denoted $\mathrm{mad}(G)$, is the maximum average degree of a subgraph of G. Defective choosability with respect to maximum average degree was previously studied by Havet and Sereni [62], who proved the following theorem.

Theorem 23 ([62])**.** *For $d \geqslant 0$ and $k \geqslant 2$, every graph G with $\text{mad}(G) < k + \frac{kd}{k+d}$ is k-choosable with defect d.*

We improve on Theorem 23 as follows:

Theorem 24 (§4.1)**.** *For $d \geqslant 0$ and $k \geqslant 1$, every graph G with $\text{mad}(G) < \frac{2d+2}{d+2}k$ is k-choosable with defect d.*

Note that the two theorems are equivalent for $k = 2$. But for $k \geqslant 3$, the assumption in Theorem 24 is weaker than the corresponding assumption in Theorem 23, thus Theorem 24 is stronger than Theorem 23.

Theorem 23 can be restated as follows: every graph G with $\text{mad}(G) = m$ is k-choosable with defect $\lfloor \frac{k(m-k)}{2k-m} \rfloor + 1$, whereas Theorem 24 says that G is k-choosable with defect $\lfloor \frac{m}{2k-m} \rfloor$. Both results require that $2k > m$, and the minimum value of k for which either theorem is applicable is $k = \lfloor \frac{m}{2} \rfloor + 1$. In this case, Theorem 24 gives a defect bound of $\lfloor \frac{m}{2k-m} \rfloor$, which is an order of magnitude less than the defect bound of $(1 + o(1))\frac{k^2}{2k-m}$ in Theorem 23. Note that Havet and Sereni [62] gave a construction to show that no lower value of k is possible. That is, for $m \in \mathbb{R}^+$, the defective chromatic number of the class of graphs with maximum average degree m equals $\lfloor \frac{m}{2} \rfloor + 1$; also see [155].

See [18–25, 80, 81] for results about defective 2-colourings of graphs with given maximum average degree, where each of the two colour classes has a prescribed degree bound. Also note that Dorbec et al. [47] proved a result analogous to Theorems 23 and 24 (with weaker bounds) for defective colouring of graphs with given maximum average degree, where in addition, a given number of colour classes are stable sets.

1.5.4 Clustered Choosability

A graph G is *k-choosable with clustering c* if G has an L-colouring with clustering c for every k-list-assignment L of G. Prior to the present work, the following theorem, due to Kopreski and Yu [83], is the only known non-trivial result for clustered colourings of graphs with given maximum average degree[1].

Theorem 25 ([83])**.** *Every graph G is $\lfloor \frac{3}{4} \text{mad}(G) + 1 \rfloor$-colourable with defect 1, and thus with clustering 2.*

There are no existing non-trivial results for clustered choosability of graphs with given maximum average degree. The closest such result, due to Dvořák and Norin [51], says that for constants $\alpha, \gamma, \epsilon > 0$, if a graph G has at most $(k+1-\gamma)|V(G)|$ edges, and every n-vertex subgraph of G has a balanced separator of order at most $\alpha n^{1-\epsilon}$, then G is k-choosable with clustering some function of α, γ and ϵ. Note that the number of colours here is roughly half the average degree of G. This result determines the clustered chromatic number of several graph classes, but for various other classes (that contain expanders) this result is not applicable because of the requirement that every subgraph has a balanced separator.

Theorem 24 with $d = 1$ implies the above result of Kopreski and Yu [83] and extends it to the setting of choosability:

Theorem 26. *Every graph G is $\lfloor \frac{3}{4} \text{mad}(G) + 1 \rfloor$-choosable with defect 1, and thus with clustering 2.*

[1]Kopreski and Yu [83] actually proved the following stronger result: For $a \geqslant 1$ and $b \geqslant 0$, every graph G with $\text{mad}(G) < \frac{4}{3}a + b$ is $(a+b)$-colourable, such that a colour classes have defect 1, and b colour classes are stable sets.

As an example of Theorem 26, it follows from Euler's formula that toroidal graphs have maximum average degree at most 6, implying every toroidal graph is 5-choosable with defect 1 and clustering 2, which was first proved by Dujmović and Outioua [48]. Previously, Cowen, Goddard and Jesurum [36] proved that every toroidal graph is 5-colourable with defect 1.

The following two theorems are our main results for clustered choosability. The first still has an absolute bound on the clustering, while the second has fewer colours but at the expense of allowing the clustering to depend on the maximum average degree.

Theorem 27 (§4.4)**.** *Every graph G is $\lfloor \frac{7}{10} \operatorname{mad}(G) + 1 \rfloor$-choosable with clustering* 9*.*

Theorem 28 (§4.5)**.** *Every graph G is $\lfloor \frac{2}{3} \operatorname{mad}(G)+1 \rfloor$-choosable with clustering $57 \lfloor \frac{2}{3} \operatorname{mad}(G) \rfloor + 6$.*

Theorem 28 says that the clustered chromatic number of the class of graphs with maximum average degree m is at most $\lfloor \frac{2m}{3} \rfloor + 1$. This is the best known upper bound. The best known lower bound is $\lfloor \frac{m}{2} \rfloor + 1$; see [155]. Closing this gap is an intriguing open problem.

1.5.5 Generalisation

The above results generalise via the following definition. For a graph G and integer $n_0 \geqslant 1$, let $\operatorname{mad}(G, n_0)$ be the maximum average degree of a subgraph of G with at least n_0 vertices, unless $|V(G)| < n_0$, in which case $\operatorname{mad}(G, n_0) := 0$. The next two results generalise Theorems 24 and 28 respectively with $\operatorname{mad}(G)$ replaced by $\operatorname{mad}(G, n_0)$, where the number of colours stays the same, and the defect or clustering bound also depends on n_0.

Theorem 29 (§4.1)**.** *For integers $d \geqslant 0$, $n_0 \geqslant 1$ and $k \geqslant 1$, every graph G with $\operatorname{mad}(G, n_0) < \frac{2d+2}{d+2} k$ is k-choosable with defect $d' := \max\{\lceil \frac{n_0 - 1}{k} \rceil - 1, d\}$.*

Theorem 30 (§4.5)**.** *For integers $d \geqslant 0$, $n_0 \geqslant 1$ and $k \geqslant 1$, every graph G with $\operatorname{mad}(G, n_0) < \frac{3}{2}k$ is k-choosable with clustering $c := \max\{\lceil \frac{n_0 - 1}{k} \rceil, 57k - 51\}$.*

Note that Theorem 29 with $n_0 = 1$ is equivalent to Theorem 24, and Theorem 30 with $n_0 = 1$ and $k = \lfloor \frac{2}{3} \operatorname{mad}(G) \rfloor + 1$ is equivalent to Theorem 28.

Graphs on surfaces provide motivation for this extension. The *Euler genus* of the orientable surface with h handles is $2h$. The *Euler genus* of the non-orientable surface with k cross-caps is k. The *Euler genus* of G is the minimum Euler genus of a surface in which G embeds. Graphs with Euler genus g can have average degree as high as $\Theta(\sqrt{g})$, the complete graph being one example. But such graphs necessarily have bounded size. In particular, Euler's formula implies that every n-vertex m-edge graph with Euler genus g satisfies $m < 3(n + g)$. Thus, for $\epsilon > 0$, if $n \geqslant \frac{6}{\epsilon} g$ then G has average degree $\frac{2m}{n} < 6 + \epsilon$. In particular, $\operatorname{mad}(G, 6g) < 7$.

Using this observation, Theorems 29 and 30 respectively imply that graphs with bounded Euler genus are 4-choosable with bounded defect and are 5-choosable with bounded clustering. Both these results are actually weaker than known results. In particular, several authors [12, 31, 36, 157] have proved that graphs with bounded Euler genus are 3-colourable or 3-choosable with bounded defect. And Dvořák and Norin [51] proved that graphs with bounded Euler genus are 4-choosable with bounded clustering. The proof of Dvořák and Norin [51] uses the fact that graphs of bounded Euler genus have strongly sub-linear separators. The advantage of our approach is that it works for graph classes that do not have sub-linear separator theorems. Graphs with given g-thickness are such a class [49]. We explore this direction in Section 4.6.

1.5.6 Clustered Choosability and Maximum Degree

Alon et al. [5] and Haxell, Szabó and Tardos [63] studied clustered colourings of graphs with given maximum degree. Haxell, Szabó and Tardos [63] proved that every graph with maximum degree Δ is $\lceil \frac{1}{3}(\Delta+1) \rceil$-colourable with bounded clustering. Moreover, for some Δ_0 and $\epsilon > 0$, every graph with maximum degree $\Delta \geqslant \Delta_0$ is $\lfloor \left(\frac{1}{3} - \epsilon\right) \Delta \rfloor$-colourable with bounded clustering. For both these results, the clustering bound is independent of Δ.

Clustered choosability of graphs with given maximum degree has not been studied in the literature (as far as we are aware). As a by-product of our work for graphs with given maximum average degree we prove the following results for clustered choosability of graphs with given maximum degree.

Theorem 31 (§4.3)**.** *Every graph G with maximum degree $\Delta \geqslant 3$ is $\lceil \frac{1}{3}(\Delta+2) \rceil$-choosable with clustering $\lceil \frac{19}{2}\Delta \rceil - 17$.*

Theorem 32 (§4.4)**.** *Every graph G with maximum degree Δ is $\lceil \frac{2}{5}(\Delta+1) \rceil$-choosable with clustering 6.*

$\Delta = 5$ is the first case in which the above results for clustered choosability are weaker than the known results for clustered colouring. In particular, Haxell, Szabó and Tardos [63] proved that every graph with maximum degree 5 is 2-colourable with bounded clustering, whereas Theorems 31 and 32 only prove 3-choosability. It is open whether every graph with maximum degree 5 is 2-choosable with bounded clustering.

1.6 Treewidth

A central concept in the area of graph minor theory is the parameter of treewidth, which measures how tree-like a graph is. Indeed, a connected graph has treewidth 1 if and only if it is a tree.

A *tree decomposition* of a graph G consists of a tree T together with a function B which assigns each vertex $t \in V(T)$ a set $B(t)$ of vertices of G (henceforth a *bag*), such that:

1) every vertex of G is in at least one bag,

2) for every edge of G there is at least one bag containing both of its endpoints,

3) for every vertex $v \in V(G)$, the set of vertices of T whose bags contain v induces a connected subgraph of T.

The *width* of a tree decomposition is the maximum of $|B(t)|-1$ for $t \in V(T)$. The *treewidth* $\text{tw}(G)$ of a graph G is the minimum width of a tree decomposition of G.

Treewidth is an extremely important concept in graph minor theory. One simple connection is not too difficult to see: if H is a minor of G, then the treewidth of G is at least the treewidth of H. In fact, for each of the three graph minor operations, modifying the tree decomposition in a natural way suffices to show this. This means in particular that the set of graphs with treewidth at most t forms a minor-closed class. As we have already mentioned, this class can alternatively be defined as the set of graphs which are subgraphs of (K_{t+1}, t)-cockades. This means that many of the questions we have considered for other minor-closed classes are trivial for this class: the extremal graphs are (K_{t+1}, t)-cockades, which have $tn - \binom{t+1}{2}$ edges, and have chromatic number and choosability exactly $t+1$. In fact, a graph has treewidth at most t if and only if it is a subgraph of a (K_{t+1}, t)-cockade. In general, the exact set of excluded minors is unkown.

The relationship between graph minor theory and treewidth is stronger than this, due to the following theorem of Robertson and Seymour [122].

Theorem 33 ([122]). *For every planar graph H, there is a number k such that every graph G with $tw(G) \geqslant k$ contains H as a minor.*

Robertson and Seymour further note that such a number can only be found for planar graphs, since the $k \times k$ rectangular grid has treewidth k and has no non-planar minor. As an immediate corollary, a minor-closed class has bounded treewidth if and only if it has an excluded minor which is planar. This result was central to the proof of Theorem 1. In fact, Robertson and Seymour first prove Theorem 1 for bounded treewidth graphs, and then prove it for graphs with unbounded treewidth (via the Graph Minor Structure Theorem).

In a similar vein, dividing a problem into the case of graphs with unbounded treewidth and the case of graphs with bounded treewidth has proven to be a useful technique for other problems in graph minor theory. For example, this was the approach used by Kawarabayashi et al. to prove that Jørgensen's conjecture is true for sufficiently large graphs [78, 79]. The same approach might well work in the case of Conjecture 13.

Treewidth is an extremely useful concept in a wide range of areas (see [107] for a survey), but our own interest in treewidth is due to its importance in graph minor theory, and its connection to the concept of graph gonailty, which we introduce below.

1.7 Gonality

The final topic of this thesis is the notion of divisorial gonality for graphs. This concept comes to graph theory from the field of algebraic geometry, where the gonality of a curve is an important and well studied concept. Recently, Baker and Norin [14] developed a framework for translating concepts from algebraic geometry to analogous concepts about graph theory and proved an analogue of the well-known Riemann-Roch Theorem. Their work has led to intensive and fruitful research in this area (see [7, 9, 27, 35, 72, 82] for example). Within this framework, a concept of gonality for graphs has been defined and studied [8, 13, 34, 41, 150, 151].

To understand what gonality is, consider the following simple chip firing game, played on a graph G. First, assign a non-negative number of chips to each vertex. Making a move in the game consists of selecting a non-empty subset $A \subseteq V(G)$ and for every edge $vw \in E(G)$ with one endpoint $v \in A$ and one endpoint $w \notin A$, moving one chip from v to w. In order for a move to be *legal*, there must be a non-negative number of chips on every vertex after the move is performed. An initial configuration is *winning* if for every vertex $v \in V(G)$, it is possible to transfer a chip to v via some (possibly empty) sequence of legal moves. The gonality of a graph G, denoted by $\text{gon}(G)$, is defined as the minimum number of chips required for a winning chip configuration in G.

The study of graph gonality is in part motivated by possible relationships to other graph parameters. In a recent survey, Norin [102] discusses the potential relevance of gonality to graph minor theory. As we have mentioned, treewidth is a central concept in graph minor theory. Van Dobben de Bruyn and Gijswijt [151] have shown that the treewidth $\text{tw}(G)$ of a graph G is a lower bound for its gonality, and we know of no connected graph that has been shown to have gonality greater than its treewidth prior to the present work. In his survey, Norin raises the following questions.

Question 1. *Is there some function f such that for every connected graph G, $\text{gon}(G) \leqslant f(\text{tw}(G))$?*

Question 2. *Is $\text{gon}(H) \leqslant \text{gon}(G)$ for every connected graph G and every connected minor H of G?*

Question 3. *Is* $\mathrm{gon}(H) \leqslant \mathrm{gon}(G)$ *for every connected graph* G *and every connected subgraph* H *of* G*?*

In Section 5.2 we answer Question 1 in the negative, proving the following stronger result.

Theorem 34 (§5.2)**.** *For all integers* $k \geqslant 2$ *and* $l \geqslant k$*, there exists a* k*-connected graph* G *with* $\mathrm{tw}(G) = k$ *and* $\mathrm{gon}(G) \geqslant l$*.*

In terms of relating connectivity, treewidth and gonality, this result is best possible, as we discuss in Section 5.2. We also show that the answer to Question 2 is "no", by considering a special class of graphs of treewidth 2 known as *fans*, consisting of graphs constructed by adding a universal vertex to a path. We precisely determine the gonality of all fans, and in so doing prove that gonality is unbounded on this class. In Section 5.3, we present a class of graphs that has unbounded gonality, while each is a subgraph of some connected graph of gonality 2, thus answering Question 3 in the negative. However, in the special case where the subgraph H has a universal vertex, we show in Section 5.2 that the answer to Question 3 is "yes".

1.8 Standard Definitions

All graphs in this thesis are simple, finite and undirected unless otherwise specified. The *components* of a graph G are the maximal connected subgraphs of G. For $S \subseteq V(G)$, the graph $G - S$ is the graph obtained from G by deleting every vertex in S. The graph $G[S] := G - (V(G) \setminus S)$ is the subgraph of G induced by S. If $G[S]$ is a complete graph, S is a *clique*. If $S \subseteq E(G)$, the graph $G - S$ is the graph with vertex set $V(G)$ and edge set $E(G) \setminus S$. For simplicity, we write $G - x$ for $G - \{x\}$. For any subgraph H of G, we write $G - H$ for $G - V(H)$.

For each vertex $v \in V(G)$, the set $N_G(v) := \{w \in V(G) : vw \in E(G)\}$ is the *neighbourhood* of v, and $N_G[v] := \{v\} \cup N_G(v)$ is the *closed neighbourhood*. Similarly, for each subgraph C of G, the set $N_G(C)$ is the set of vertices in $G - C$ that are adjacent in G to some vertex of C, and $N_G[C] := V(C) \cup N_G(C)$. When there is no ambiguity, we write $N(v)$, $N[v]$, $N(C)$ and $N[C]$ respectively for $N_G(v)$, $N_G[v]$, $N_G(C)$ and $N_G[C]$. A vertex v is *universal* in G if $N_G[v] = V(G)$, and *isolated* if $N_G(v) = \emptyset$.

For a subset $A \subseteq V(G)$ and vertex $v \in V(G)$, let $N_A(v) := N_G(v) \cap A$ and $\deg_A(v) := |N_A(v)|$. We sometimes refer to $|V(G)|$ as $|G|$.

The operation of *contracting* the edge $vw \in E(G)$ consists of deleting v and w and adding a new vertex adjacent to $(N_G(v) \cup N_G(w)) \setminus \{v, w\}$. We denote the graph obtained from G by contracting an edge e by G/e, and the graph obtained by contracting each of a set S of edges by G/S. A graph H is a *minor* of a graph G if a graph isomorphic to H can be obtained from some subgraph of G by contracting edges, and G is H*-minor-free* if H is not a minor of G.

Given a graph H, the operation of *subdividing* an edge consists of deleting the edge and replacing it with a new *subdivision vertex*, whose neighbours are the endpoints of the deleted edge. A graph H' is a *subdivision* of H if it is obtained from H through a sequence of edge subdivisions. H is a *topological minor* of a graph G if some subgraph of G is isomorphic to a subdivision of H.

For a positive integer k, G is k*-connected* if $|V(G)| > k$ and for every subset $S \subseteq V(G)$ of size less than k, $G - S$ is connected. We denote by $\delta(G)$, $a(G)$ and $\Delta(G)$ respectively the minimum degree, the average degree and the maximum degree of G. For convenience, we define all three of these values to be 0 for the graph with no vertices. We denote by

$G\dot{\cup}H$ the disjoint union of two graphs G and H, and by kG the disjoint union of k copies of G. We denote by $G+H$ the graph formed from $G\dot{\cup}H$ by adding an edge between every vertex of H and every vertex of G. For every positive integer t, $[t]$ is the set $\{1,2,\dots,t\}$, and $[0]:=\emptyset$.

Given a colouring, the *monochromatic subgraph* of G is the spanning subgraph consisting of those edges whose endpoints have the same colour. The *defect* of a vertex is its degree in the monochromatic subgraph. Note that a colouring of defect k is also a colouring of defect $k+1$, but a vertex of defect k in a coloured graph is not a vertex of defect $k+1$.

G is *k-choosable with defect d* if G has an L-colouring with defect d for every k-list-assignment L of G. Similarly, G is *k-choosable with clustering c* if G has an L-colouring with clustering c for every k-list-assignment L of G.

1.9 Folklore

For completeness, we now provide proofs of two well known results appealed to previously in this chapter.

Lemma 6. *Let H be a graph such that $\mathrm{ex_m}(n,H)<cn$ for some positive integer c. Then every H-minor-free graph is $2c$-choosable, and if $|V(H)|\leqslant 2c$ then every H-minor-free graph is $(2c-1)$-colourable.*

Proof. Let G be an n-vertex H-minor-free graph, and let l be a $2c$-list assignment for G. We proceed by induction on n. The base case with $n\leqslant 2c-1$ is trivial. For $n\geqslant 2c$, $|E(G)|<c|V(G)|$, implying G has average degree less than $2c$. Thus G has a vertex v of degree at most $2c-1$. By induction, $G-v$ is l_{G-v}-colourable, where l_{G-v} is the restriction of l to $G-v$. Some colour in $l(v)$ is not used on the neighbours of v, and this colour can be assigned to v. Hence G is $2c$-choosable.

It remains to prove that G is $(2c-1)$-colourable under the assumption that $|V(H)|\leqslant 2c$. First suppose that $\deg(v)\leqslant 2c-2$. By induction, $G-v$ is $(2c-1)$-colourable. Some colour is not used on the neighbours of v, which can be assigned to v. Hence G is $(2c-1)$-colourable. Now assume that $\deg(v)=2c-1$. There is some pair of non-adjacent vertices x and y in $N(v)$, as otherwise G contains K_{2c} and hence H (since $|V(H)|\leqslant 2c$). Let G' be the graph obtained from G by contracting the edges vx and vy into a new vertex z. By induction, G' is $(2c-1)$-colourable. Colour each vertex of $G-\{v,x,y\}$ by the colour assigned to the corresponding vertex in G'. Colour x and y by the colour assigned to z. Since every vertex adjacent to x or y in $G-v$ is adjacent to z in G', this defines a $(2c-1)$-colouring of $G-v$. Now v has $2c-1$ neighbours, two of which have the same colour. Thus there is an unused colour on the neighbours of v, which can be assigned to v. Therefore G is $(2c-1)$-colourable. □

Lemma 7. *For every $(t+1)$-connected graph H and every H-minor-free graph G_0, every (G_0,t)-cockade is H-minor-free.*

Proof. Let G be a (G_0,t)-cockade. We proceed by induction on $|V(G)|+|E(G)|$. By assumption, G_0 is H-minor-free. Assume that there are (G_0,t)-cockades G_1 and G_2 distinct from G such that $G_1\cup G_2=G$ and $G_1\cap G_2\cong K_t$. Note that G_1 and G_2 are proper subgraphs of G, and hence by induction are H-minor-free. Suppose for contradiction that G contains an H-minor. Then there is a set of pairwise disjoint connected subgraphs of G such that if every edge inside one of these subgraphs is contracted and every vertex not in one of these subgraphs is deleted, then the graph obtained is a supergraph H' of H such that $|V(H')|=|V(H)|$. Each of these subgraphs will contract down to a separate vertex, so we call these subgraphs *prevertices*. There are exactly t vertices in $G_1 \cap G_2$, so the set S of prevertices that intersect $G_1 \cap G_2$ has size at most t. Since H is $(t+1)$-connected, each prevertex not in S is in the same connected component of $G - S$. Without loss of generality, each prevertex not in S is a subgraph of G_1. Now, there is no path of G between two non-adjacent vertices of G_1 that is internally disjoint from G_1. Hence, by deleting every vertex of $G_2 \setminus G_1$ and then contracting the remaining edges of the prevertices and deleting the remaining vertices that are not in any prevertex, we obtain H', contradicting the assumption the G_1 contains no H-minor. $\square$

Chapter 2

Petersen Minors

2.1 Overview

In this chapter we determine the extremal function for Petersen graph minors. Recall that $\mathcal{P}$ is the Petersen graph (see Figure 2), and that $\mathrm{ex_m}(n, \mathcal{P})$ is the maximum number of edges in an n-vertex $\mathcal{P}$-minor-free graph. Our main result is the following.

Theorem 8.

$$\mathrm{ex_m}(n, \mathcal{P}) = \begin{cases} \binom{n}{2} & \text{if } n \leqslant 9, \\ 5n - 14 & \text{if } n \in \{11, 12\}, \\ 5n - 9 & \text{if } n \equiv 2 \pmod 7, \\ 5n - 12 & \text{otherwise.} \end{cases}$$

For $n \equiv 2 \pmod 7$, we in fact completely characterise the extremal graphs which are within two edges of extremal.

Theorem 10. *Every graph with $n \geqslant 3$ vertices and $m \geqslant 5n - 11$ edges contains a Petersen minor or is a $(K_9, 2)$-cockade minus at most two edges.*

The bulk of this chapter (Section 2.2), concerns the proof of Theorem 10, which is the main element of the proof of Theorem 8. Theorem 10 is also used in the proof of Theorem 11 in Section 2.3, and Theorems 10 and 11 together suffice to prove Theorem 8 for all but finitely many values of n. The final element of the proof of Theorem 8 is a computational result, Lemma 36, which we also appeal to in the proof of Theorem 10.

2.2 Proof of Theorem 10

We now sketch the proof of Theorem 10. Assume to the contrary that there is some counterexample to Theorem 10, and select a minor-minimal counterexample G. Define $\mathcal{L}$ to be the set of vertices v of G such that $\deg(v) \leqslant 9$ and there is no vertex u with $N[u] \subsetneq N[v]$. For a vertex $v \in V(G)$, a subgraph $H \subseteq G$ is *v-suitable* if it is a component of $G - N[v]$ that contains some vertex of $\mathcal{L}$.

Section 2.2.1 shows some elementary results that are used throughout the other sections. In particular, it shows that $\delta(G) \in \{6, 7, 8, 9\}$, and hence that $\mathcal{L} \neq \emptyset$. Sections 2.2.2 and 2.2.3 respectively show that that no vertex of G has degree 7 and that no vertex of G has degree 8. Sections 2.2.4 and 2.2.5 show that for every $v \in \mathcal{L}$ with degree 6 or 9 respectively there is some v-suitable subgraph, and that for each $v \in \mathcal{L}$ with degree 6 or 9 and every v-suitable subgraph C of G there is some v-suitable subgraph C' of G such that $N(C') \setminus N(C) \neq \emptyset$.

Pick $u \in \mathcal{L}$ and a u-suitable subgraph H of G such that $|V(H)|$ is minimised. By the definition of u-suitable, there is some $v \in \mathcal{L} \cap V(H)$. Let C be a v-suitable subgraph of G containing u, and let C' be a v-suitable subgraph of G such that $N(C') \setminus N(C) \neq \emptyset$. Section 2.2.6 shows that C' selected in this way is a proper subgraph of H, contradicting our choice of H.

The basic idea of our proof is similar to proofs used for example in [143] and [2], with the major points of difference conceptually being the use of skeletons, defined in Section 2.2.1, to rule out certain configurations, and the proof in Section 2.2.1 that the minimal counterexample is 4-connected.

We use the following notation throughout the proof. For $i \in \mathbb{N}$, we denote by $V_i(G)$ the set of vertices in G with degree i, and by $V_{\geqslant i}(G)$ the set of vertices of G of degree at least i.

For a tree T and $v, w \in V(T)$, let vTw be the path in T from v to w. A vertex of T is *high degree* if it is in $V_{\geqslant 3}(T)$. For a path P with endpoints x and y, $\mathrm{int}(P) := xy$ if $E(P) = \{xy\}$ and $\mathrm{int}(P) := V(P) \setminus \{x, y\}$ otherwise.

A subset S of $V(G)$ is a *fragment* if $G[S]$ is connected. Distinct fragments X and Y are *adjacent* if some vertex in X is adjacent to some vertex in Y.

2.2.1 Basic Results

To prove Theorem 10, suppose for contradiction that G is a minor-minimal counterexample to Theorem 10. That is, G is a graph with the following properties:

(i) $|V(G)| \geqslant 3$,

(ii) $|E(G)| \geqslant 5|V(G)| - 11$,

(iii) G is not a spanning subgraph of a $(K_9, 2)$-cockade,

(iv) $\mathcal{P}$ is not a minor of G,

(v) Every proper minor H of G with at least three vertices satisfies $|E(H)| \leqslant 5|V(H)| - 12$ or is a spanning subgraph of a $(K_9, 2)$-cockade.

If H is a $(K_9, 2)$-cockade or K_2, then $|E(H)| = 5|V(H)| - 9$. Hence, (v) immediately implies:

(vi) Every proper minor H of G with at least two vertices satisfies $|E(H)| \leqslant 5|V(H)| - 9$.

Lemma 10.1. *G has at least 10 vertices.*

Proof. Since $5n - 11 > \binom{n}{2}$ for $n \in \{3, 4, \ldots, 8\}$, every graph satisfying (i) and (ii) has at least 9 vertices. Every 9-vertex graph is a spanning subgraph of a $(K_9, 2)$-cockade. □

A *separation* of a graph H is a pair (A, B) of subsets of $V(H)$ such that both $A \setminus B$ and $B \setminus A$ are non-empty and $H = H[A] \cup H[B]$. The *order* of a separation (A, B) is $|A \cap B|$. A *k-separation* is a separation of order k. A *$(\leqslant k)$-separation* is a separation of order at most k. A graph is *k-connected* if it has at least $k + 1$ vertices and no separation of order less than k.

Let x, y and z be distinct vertices of a graph H. A K_3-minor *rooted at* $\{x, y, z\}$ is a set of three pairwise-disjoint, pairwise-adjacent fragments $\{X, Y, Z\}$ of H such that $x \in X$, $y \in Y$, $z \in Z$. The following lemma is well known and has been proved, for example, by Wood and Linusson [156].

Lemma 10.2. *Let x, y and z be distinct vertices of a graph H. There is a K_3-minor of H rooted at $\{x, y, z\}$ if and only if there is no vertex $v \in V(H)$ for which the vertices in $\{x, y, z\} \setminus \{v\}$ are in distinct components of $H - v$.*

Lemma 10.3. *G is 4-connected.*

Proof. By Lemma 10.1, $|V(G)| \geqslant 10$. Suppose for contradiction that there is a $(\leqslant 3)$-separation (A, B) of G. Note that $A \setminus B$ and $B \setminus A$ are both non-empty by definition. We separate into cases based on $|A \cap B|$ and on whether $|A \setminus B|$ is a singleton. Note that while Case 1 is redundant, it is useful to know that Case 1 does not hold when proving that Cases 2 and 4 do not hold.

Case 1. *There is a $(\leqslant 3)$-separation (A, B) of G such that $|A \setminus B| = \{v\}$:*

By Lemma 10.1, $|B| \geqslant 9$. Now by (vi) we have

$$|E(G)| \leqslant |E(G[B])| + \deg(v) \leqslant 5(|V(G)| - 1) - 9 + 3 = 5|V(G)| - 11.$$

By (ii), equality holds throughout. In particular $\deg(v) = 3$ and $|E(G[B])| = 5|B| - 9$ so $G[B]$ is a $(K_9, 2)$-cockade by (v). For every edge e incident to v, we have $E(G/e) = E(G[B])$ by (vi). Hence, $|A \cap B|$ is a clique, and is therefore contained in a subgraph $H \cong K_9$ of $G[B]$. Then $\mathcal{P} \subseteq H \cup G[A] \subseteq G$ contradicting (iv).

Case 2. *There is a $(\leqslant 1)$-separation (A, B) of G:*

If either $|A \setminus B| = 1$ or $|B \setminus A| = 1$ then we are in Case 1. Otherwise, $|A| \geqslant 2$ and $|B| \geqslant 2$, so by (v) we have $|E(G[A])| \leqslant 5|A| - 9$, with equality if and only if $G[A] \cong K_2$ or $G[A]$ is a $(K_9, 2)$-cockade, and the same for B. Now

$$|E(G)| = |E(G[A])| + |E(G[B])| \leqslant 5(|V(G)| + 1) - 9 - 9 = 5|V(G)| - 13,$$

contradicting (ii).

Case 3. *There is a 2-separation (A, B) of G:*

If there is a component C of $G - (A \cap B)$ such that $N(C) \neq A \cap B$, then G has a $(\leqslant 1)$-separation, and we are in Case 2. Otherwise, let C_B be a component of $G - A$ and let G_A be the graph obtained from G by contracting $G[N[C_B]]$ down to a copy of K_2 rooted at $A \cap B$ and deleting all other vertices of B. Let G_B be defined analogously. If $|E(G_A)| \leqslant 5|A| - 12$, then

$$|E(G)| \leqslant |E(G_A)| + |E(G_B)| - 1 \leqslant 5(|V(G)| + 2) - 12 - 9 - 1 = 5|V(G)| - 12,$$

contradicting (ii). Hence, $|E(G_A)| \geqslant 5|A| - 11$, and by (v), G_A is a spanning subgraph of a $(K_9, 2)$-cockade H_A. By symmetry, G_B is a spanning subgraph of a $(K_9, 2)$-cockade H_B. Then G is a spanning subgraph of the $(K_9, 2)$-cockade formed by gluing H_A and H_B together on $A \cap B$, contradicting (iii).

Case 4. *There is a 3-separation (A, B) of G:*

First, suppose that $G[A]$ does not contain a K_3 minor rooted at $A \cap B$. Then there exists a vertex v such that the vertices in $A \cap B$ are in distinct components of $G[A] - v$ by Lemma 10.2. Recall that $|A \setminus B| > 1$, so there is a vertex $w \neq v$ in $A \setminus B$. Let C be the component of $G[A] - v$ containing w. Then there is a $(\leqslant 2)$-separation (A', B') of G where $A' \setminus B' = V(C) \setminus (A \cap B)$, so we are in either Case 2 or Case 3. Hence, there is a K_3 minor of $G[A]$ rooted at $A \cap B$, and by the same argument a K_3 minor of $G[B]$ rooted at $A \cap B$. Let G_A be obtained from G by contracting $G[B]$ down to a triangle on $A \cap B$, and let G_B be obtained from G by contracting $G[A]$ down to a triangle on $A \cap B$. Suppose $|E(G_A)| \geqslant 5|A| - 11$. Since G satisfies (v), we have that G_A is a spanning subgraph of a $(K_9, 2)$-cockade, and so G_A is a $(K_9, 2)$-cockade minus at most two edges. Since $A \cap B$ is a clique of G_A, there is some set S of nine vertices in A, containing $A \cap B$, such that $G_A[S]$ is K_9 minus at most two edges. Let C be a component of $G - A$, and note that $N(C) = A \cap B$, or else we are in Case 2 or Case 3. Now it is quick to check that the graph obtained from $G[S \cup V(C)]$ by contracting C to a single vertex contains $\mathcal{P}$ as a subgraph, contradicting (iv). Hence, $|E(G_A)| \leqslant 5|A| - 12$, and by symmetry $|E(G_B)| \leqslant 5|B| - 12$. Now

$$|E(G)| \leqslant |E(G_A)| + |E(G_B)| - 3 \leqslant 5(|V(G)| + 3) - 12 - 12 - 3 = 5|V(G)| - 12,$$

contradicting (ii). □

Lemma 10.4. $\delta(G) \in \{6, 7, 8, 9\}$ *and every edge is in at least five triangles.*

Proof. Suppose for contradiction that some edge vw is in t triangles with $t \leqslant 4$. Now

$$|E(G/vw)| \geqslant |E(G)| - t - 1 \geqslant 5|V(G)| - 12 - t \geqslant 5|V(G/e)| - 11.$$

Since G satisfies (v), G/vw is a spanning subgraph of some $(K_9, 2)$-cockade H. By Lemma 10.3, G is 4-connected, which implies G/vw is 3-connected, so G/vw is K_9 minus at most two edges. It follows from (ii) that G is a 10-vertex graph with at most six non-edges, and so $\mathcal{P} \subseteq G$ by Lemma 36 (a) (this is the only reference to Lemma 36 in the proof of Theorem 10).

Hence, every edge of G is in at least five triangles. By Lemma 10.3, G has no isolated vertex, and $\delta(G) \geqslant 6$.

Let e be an edge of G. By (vi), $|E(G - e)| \leqslant 5|V(G)| - 9$, so $|E(G)| \leqslant 5|V(G)| - 8$, and hence $\delta(G) \leqslant 9$. □

Recall that $\mathcal{L}$ is the set of vertices v of G such that $\deg(v) \leqslant 9$ and there is no vertex u with $N[u] \subsetneq N[v]$. By Lemma 10.4, every vertex of minimum degree is in $\mathcal{L}$, and $\mathcal{L} \neq \emptyset$.

The following result is the tool we use for finding v-suitable subgraphs.

Lemma 10.5. *If* (A, B) *is a separation of* G *of order* $k \leqslant 6$ *such that there is a vertex* $v \in B \setminus A$ *with* $A \cap B \subseteq N(v)$*, then there is some vertex* $u \in (A \setminus B) \cap \mathcal{L}$.

Proof. We may assume that every vertex in $A \cap B$ has a neighbour in $A \setminus B$.

Let u be a vertex in $A \setminus B$ with minimum degree in G. Suppose for a contradiction that $\deg_G(u) \geqslant 10$. It follows that every vertex in $A \setminus B$ has degree at least 10 in $G[A]$. Hence, $G[A]$ has at most six vertices of degree less than 10, so $G[A]$ is not a spanning subgraph of a $(K_9, 2)$-cockade. Now $|A| \geqslant |N[u]| \geqslant 11$, so by (v),

$$\sum_{w \in A \cap B} \deg_{G[A]}(w) = 2|E(G[A])| - \sum_{w \in A \setminus B} \deg_{G[A]}(w) \leqslant 2(5|A| - 12) - 10|A \setminus B| = 10k - 24. \tag{2.1}$$

Let X be the set of edges of G with one endpoint in $A \cap B$ and the other endpoint in $A \setminus B$. It follows from Lemma 10.3 that there are a pair of disjoint edges e_1 and e_2 in X, since deleting the endpoints of an edge $e_1 \in X$ from G does not leave a disconnected graph and $|A \setminus B| \geqslant |N[u]| - k \geqslant 5$. By Lemma 10.4, e_1 is in at least five triangles. Each of these triangles contains some edge in $X \setminus \{e_1, e_2\}$, so $|X| \geqslant 7$. By (2.1),

$$\delta(G[A \cap B]) \leqslant \frac{1}{k} \sum_{w \in A \cap B} \deg_{G[A \cap B]}(w) = \frac{1}{k}\left(\left(\sum_{w \in A \cap B} \deg_{G[A]}(w)\right) - |X|\right) \leqslant \frac{1}{k}(10k - 31).$$

Since $k \leqslant 6$, some vertex $x \in A \cap B$ has degree at most 4 in $G[A \cap B]$. Let $G' := G[A \cup \{v\}]/vx$. Then $|E(G')| \geqslant |E(G[A])| + (k-5)$. Recall that every vertex in $A \setminus B$ has degree at least 10 in $G[A]$. Further, every vertex in $A \cap B$ is incident with some edge in X, and hence has at least six neighbours in A by Lemma 10.4. Hence $|E(G')| \geqslant \frac{1}{2}(10|A \setminus B| + 6k) + (k-5) \geqslant \frac{1}{2}(10|A| - 4k) + k - 5 \geqslant 5|A| - 11$. Then G' is a spanning subgraph of a $(K_9, 2)$-cockade by (v), and so $G[A]$ is a spanning subgraph of a $(K_9, 2)$-cockade, a contradiction.

Hence, $\deg_G(u) \leqslant 9$. Suppose for contradiction that $N[w] \subsetneq N[u]$ for some vertex w. Then $w \in N(u)$ and $\deg_G(w) < \deg_G(u)$, so $w \in A \cap B$. But $N[w] \subseteq N[u]$, so $w \notin N(v)$, which contradicts the assumption that $A \cap B \subseteq N(v)$. Therefore $u \in \mathcal{L}$, as required. □

For an induced subgraph H of G, a subtree T of $G[N[H]]$ is a *skeleton* of H if $V_1(T) = N(H)$.

Lemma 10.6. *Let S be a fragment of G, let T be a skeleton of $G[S]$, and let v and w be distinct vertices of T. If $vw \notin E(T)$ and $T \neq vTw$, then there is a path P of $G[N[S]]-\{v,w\}$ from vTw to $T - vTw$ with no internal vertex in T.*

Proof. $G - \{v, w\}$ is connected by Lemma 10.3, so there is a path in $G - \{v, w\}$ from vTw to $T - vTw$. Let P be a vertex-minimal example of such a path with endpoints x in vTw and y in $T - vTw$.

Suppose to the contrary that there is some internal vertex z of P in T. Then either z is in vTw and the subpath of P from z to y contradicts the minimality of P, or z is in $T - vTw$ and the subpath of P from x to z contradicts the minimality of P.

Suppose to the contrary that there is some vertex z in $P - N[S]$. The subpath P' of P from x to z has one end in S and one end in $G - N[S]$, so there is some internal vertex z' of P' in $N(S)$. But $N(S) \subseteq V(T)$, so z' is an internal vertex of P in T, a contradiction. □

Lemma 10.7. *If (A, B) is a separation of G such that $N(A \setminus B) = A \cap B$, $|A \setminus B| \geqslant 2$ and $G[A \setminus B]$ is connected, then there is a skeleton of $G[A \setminus B]$ with at least two high degree vertices.*

Proof. There is at least one subtree of $G[A]$ in which every vertex of $A \cap B$ is a leaf, since we can obtain such a tree by taking a spanning subtree of $G[A \setminus B]$ and adding the vertices in $A \cap B$ and, for each vertex in $A \cap B$, exactly one edge $e \in E(G)$ between that vertex and some vertex of $A \setminus B$. We can therefore select T a subtree of $G[A]$ such that $A \cap B \subseteq V_1(T)$ and such that there is no proper subtree T' of T such that $A \cap B \subseteq V_1(T')$. There is no vertex v in $V_1(T) \setminus B$, since for any such vertex $T - v$ is a proper subtree of T and $A \cap B \subseteq V_1(T - v)$, a contradiction. Hence, $V_1(T) = A \cap B$. If $|V_{\geqslant 3}(T)| \geqslant 2$ then we are done, so we may assume there is a unique vertex w in $V_{\geqslant 3}(T)$.

Suppose that for some $x \in A \cap B$ there is some vertex in $\mathrm{int}(xTw)$. By Lemma 10.6, there is a path P of $G[A] - \{x, w\}$ from xTw to $T - xTw$ with no internal vertex in T. Let y be the endpoint of P in xTw and let z be the other endpoint. Then $T' := (T \cup P) - \mathrm{int}(zTw)$ is a skeleton of $G[A \setminus B]$ that has a vertex of degree exactly 3. Since $|V_1(T')| = |A \cap B| \geqslant 4$ by Lemma 10.3, T' has at least two high degree vertices, (namely y and w).

Suppose instead that $V(T) = \{w\} \cup (A \cap B)$. By Lemma 10.3 G is 4-connected, so $(A \setminus B, B \cup \{w\})$ is not a separation of G, so there is some vertex y in $A \setminus (B \cup \{w\})$ adjacent to some vertex x in $A \cap B$. Let P_1 be a minimal length path from y to $A \cap B$ in $G - \{x, w\}$ (and hence in $G[A] - \{x, w\}$), and let z be the endpoint of P_1 in $A \cap B$. Let P_1' be the path formed by adding the vertex x and the edge xy to P_1. Since $G[A \setminus B]$ is connected, we can select a minimal length path P_2 of $G[A \setminus B]$ from P_1 to w. Then $(T \cup P_1' \cup P_2) - \{xw, zw\}$ is a skeleton of $G[A \setminus B]$ that has a degree 3 vertex, and therefore at least two high degree vertices, (namely the endpoints of P_2). □

For any graph H a *table* of H is an ordered 6-tuple $\mathcal{X} := (X_1, \ldots, X_6)$ of pairwise disjoint fragments of H such that X_5 is adjacent to X_1, X_2 and X_6, and X_6 adjacent to X_3 and X_4. For any subset S of $V(H)$, $\mathcal{X}$ is *rooted* at S if $|X_i \cap S| = 1$ for $i \in \{1,2,3,4\}$ and $X_5 \cap S = X_6 \cap S = \emptyset$.

Lemma 10.8. *If (A, B) is a separation of G such that $N(A \setminus B) = A \cap B$, $|A \cap B| \geqslant 4$, $|A \setminus B| \geqslant 2$ and $G[A \setminus B]$ is connected, then there is a table of $G[A]$ rooted at $A \cap B$.*

Proof. By Lemma 10.7, there is some skeleton T of $G[A \setminus B]$ such that $|V_{\geqslant 3}(T)| \geqslant 2$. Let w and x be distinct vertices in $V_{\geqslant 3}(T)$. Let w_1, w_2 and w' be three neighbours of w in T, and let x', x_3 and x_4 be three neighbours of x in T, labelled so that w' and x' are both in $V(xTw)$. For $i \in \{1,2\}$ let X_i be the vertex set of a path from w_i to a leaf of T in the component subtree of $T - w$ that contains w_i, and for $i \in \{3,4\}$ let X_i be the vertex set of a path from x_i to a leaf of T in the component subtree of $T - x$ that contains x_i. Since $V_1(T) = A \cap B$, $|X_i \cap B| = 1$ for $i \in \{1,2,3,4\}$. Let $X_5 := V(wTx')$ and let $X_6 := \{x\}$. Then $\mathcal{X} := (X_1, \ldots, X_6)$ satisfies our claim. $\square$

2.2.2 Degree 7 Vertices

In this section we show that $V_7(G) = \emptyset$.

Claim 10.9. *If $v \in V_7(G)$, then there is no isolated vertex in $G - N[v]$.*

Proof. Suppose for contradiction that there is some isolated vertex u in $G - N[v]$. By Lemma 10.4, $|N(u)| \geqslant 6$. By Lemma 10.1, there is some component C of $G - N[v]$ not containing u. Since $|N(C)| \geqslant 4$ by Lemma 10.3 and $|N(u) \cup N(C)| \leqslant |N(v)| = 7$, there is some vertex v_1 in $N(u) \cap N(C)$. Let v_1, v_2 and v_3 be distinct vertices in $N(C)$, and let v_4 and v_5 be distinct vertices in $N(u) \setminus \{v_1, v_2, v_3\}$. Let v_6 and v_7 be the remaining vertices of $N(v)$. By Lemma 10.4, for $i \in \{1, 2, \ldots, 7\}$, $N(v_i) \cap N(v) \geqslant 5$. If some vertex in $\{v_2, v_3\}$, say v_2, is not adjacent to some vertex in $\{v_4, v_5\}$, say v_5, then v_2 and v_5 are both adjacent to every other vertex in $N(v)$, and in particular v_2v_4 and v_3v_5 are edges in G. Hence, there are two disjoint edges between $\{v_2, v_3\}$ and $\{v_4, v_5\}$. Without loss of generality, $\{v_2v_4, v_3v_5\} \subseteq E(G)$. We now consider two cases depending on whether $v_6v_7 \in E(G)$.

Case 1. $v_6v_7 \in E(G)$:

Since v_1 is adjacent to all but at most one of the other neighbours of v, either $v_1v_6 \in E(G)$ or $v_1v_7 \in E(G)$, so without loss of generality $v_1v_6 \in E(G)$. Since v_7 is adjacent to all but at most one of the other neighbours of v, either $\{v_7v_2, v_7v_5\} \subseteq E(G)$ or $\{v_7v_3, v_7v_4\} \subseteq E(G)$, so without loss of generality $\{v_7v_2, v_7v_5\} \subseteq E(G)$. Let G' be obtained from G by contracting C to a single vertex. Then $\mathcal{P} \subseteq G'$ (see Figure 3a), contradicting (iv).

Case 2. $v_6v_7 \notin E(G)$:

Then v_6 and v_7 are both adjacent to every other neighbour of v. Let G' be obtained from G by contracting C to a single vertex. Then $\mathcal{P} \subseteq G'$ (see Figure 3b), contradicting (iv). $\square$

The following is the main result of this section.

Lemma 10.10. $V_7(G) = \emptyset$.

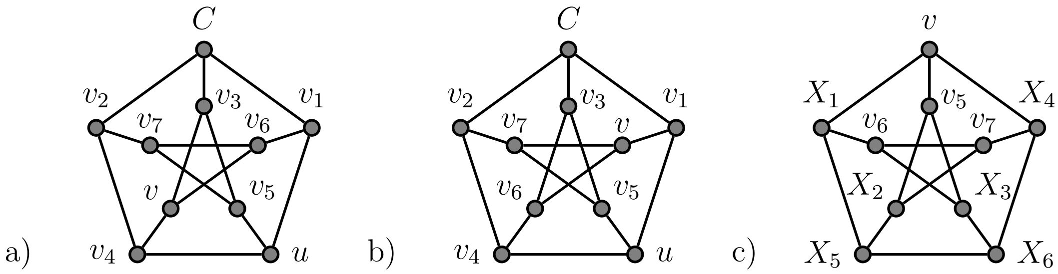


Figure 3

Proof. Suppose for contradiction that there is some vertex $v \in V_7(G)$. By Lemma 10.1, there is a non-empty component C of $G - N[v]$. By Lemma 10.3, $|N(C)| \geqslant 4$ and by Claim 10.9, $|V(C)| \geqslant 2$. Hence, by Lemma 10.8 with $A := N[C]$ and $B := V(G-C)$, there is a table $\mathcal{X} := (X_1, \ldots, X_6)$ of $G[N[C]]$ rooted at $N(C)$.

Let $\{v_1, \ldots, v_7\} := N(v)$, with $v_i \in X_i$ for $i \in \{1,2,3,4\}$. By Lemma 10.4, $|N(v_i) \cap N(v)| \geqslant 5$ for $i \in \{1,2,\ldots,7\}$. We consider two cases depending on whether $v_5v_6v_7$ is a triangle of G.

Case 1. *$v_5v_6v_7$ is a triangle of G:*

Let Q be the bipartite graph with bipartition $V := \{v_1, v_2, v_3, v_4\}$, $W := \{v_5, v_6, v_7\}$ and $E(Q) := \{xy : xy \notin E(G), x \in V, y \in W\}$. Then $\Delta(Q) \leqslant 1$, so without loss of generality $E(Q) \subseteq \{v_1v_5, v_2v_6, v_3v_7\}$. Let G' be obtained from G by contracting $G[X_i]$ to a single vertex for each $i \in \{1,2,\ldots,6\}$. Then $\mathcal{P} \subseteq G'$ (see Figure 3c), contradicting (iv).

Case 2. *$v_5v_6v_7$ is not a triangle of G:*

We may assume without loss of generality that $v_5v_6 \notin E(G)$. Then v_5 and v_6 are both adjacent to every other neighbour of v. At most one neighbour of v is not adjacent to v_7, so v_7 has some neighbour in $\{v_1, v_2\}$, say v_2, and some neighbour in $\{v_3, v_4\}$, say v_4. Let G' be obtained from G by contracting $G[X_i]$ to a single vertex for each $i \in \{1,2,\ldots,7\}$. Then $\mathcal{P} \subseteq G'$ (see Figure 3c), contradicting (iv). $\square$

2.2.3 Degree 8 Vertices

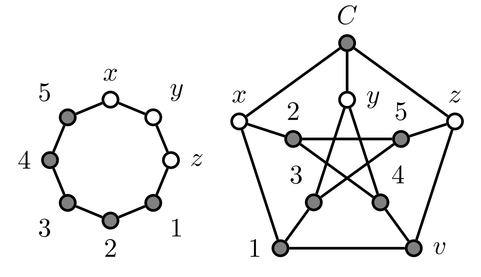
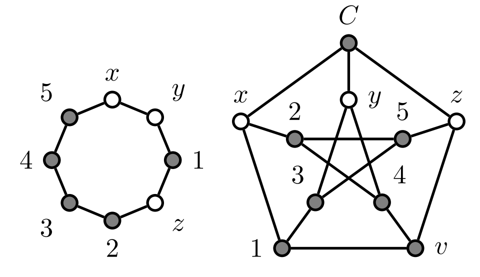
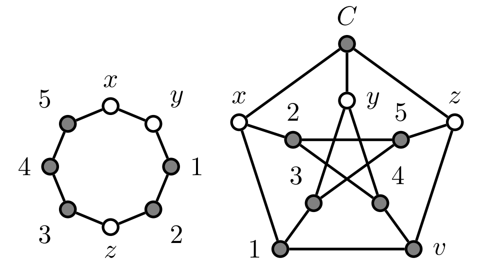
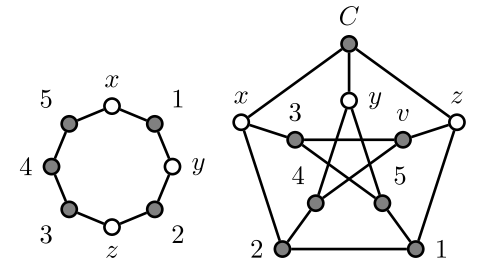
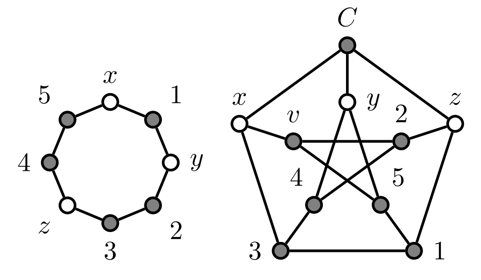
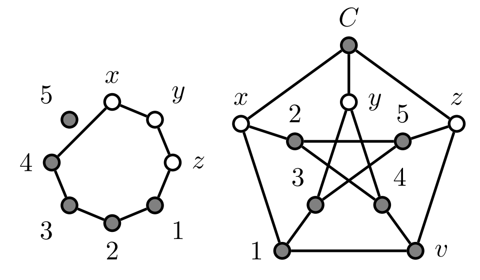
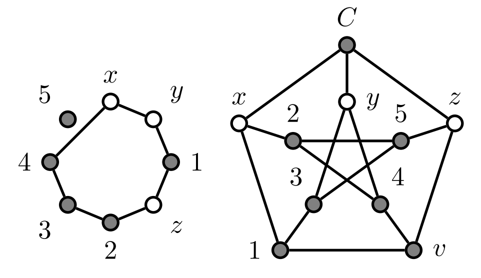
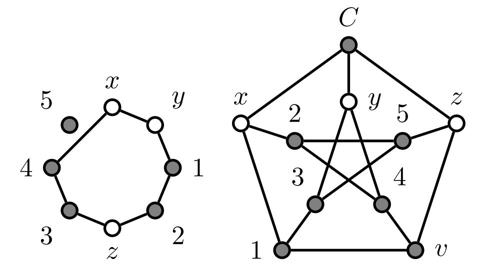
We now prove that $V_8(G) = \emptyset$. Note that the following lemma applies to any graph, not just G. This means we can apply it to minors of G, which we do in Claims 10.20 and 10.22.

Claim 10.11. *If H is a graph that contains a vertex v such that $\deg(v) = 8$, $|N(v') \cap N(v)| \geqslant 5$ for all $v' \in N(v)$, and C is a component of $H \setminus N[v]$ with $|N_H(C)| \geqslant 3$, then $\mathcal{P}$ is a minor of H unless all of the following conditions hold:*

1. *$\overline{K_3}$ is an induced subgraph of $H[N(v) \setminus N(C)]$,*
2. *$\overline{C_4}$ is an induced subgraph of $H[N(v)]$,*
3. *$H[N(C)] \cong K_3$.*

Proof. By assumption, $\delta(H[N(v)]) \geqslant 5$. Let H' be an edge-minimal spanning subgraph of $H[N(v)]$ such that $\delta(H') \geqslant 5$. Every edge e in H' is incident to some vertex of degree 5, since otherwise $\delta(H' - e) \geqslant 5$, contradicting the minimality of H'. Hence, the vertices of degree at most 1 in $\overline{H'}$ form a clique in $\overline{H'}$. Now $\Delta(\overline{H}) \leqslant 2$, since $|V(H')| = \deg(v) = 8$ and $\delta(H') \geqslant 5$. It follows that $\overline{H'}$ is the disjoint union of some number of cycles, all on at least three vertices, and a complete graph on at most two vertices. Let x, y and z be three vertices in $N(C)$, and let $1, 2, \ldots, 5$ be the remaining vertices of $N(v)$. Colour x, y, and z white and colour $1, 2, \ldots, 5$ black. In Table 4 we examine every possible graph $\overline{H'}$, up to colour preserving isomorphism. We use cycle notation to label the graphs, with an ordered pair representing an edge and a singleton representing an isolated vertex. In each case we find $\mathcal{P}$ as a subgraph of the graph G' obtained from G by contracting C to a single vertex, except in the unique case where K_3 is an induced subgraph of $\overline{H'} - \{x, y, z\}$, C_4 is an induced subgraph of $\overline{H'}$ and $\{x, y, z\}$ is an independent set of vertices in $\overline{H'}$.

Table 4

 (wwwbbbbb)	 (wwbwbbbb)
 (wwbbwbbb)	 (wbwbwbbb)
 (wbwbbwbb)	 (wwwbbbb)(b)
 (wwbwbbb)(b)	 (wwbbwbb)(b)

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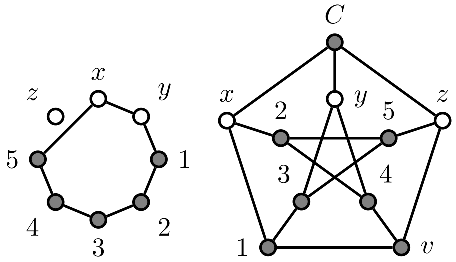
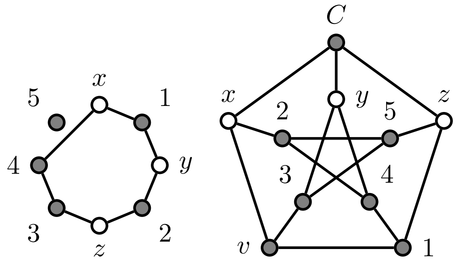
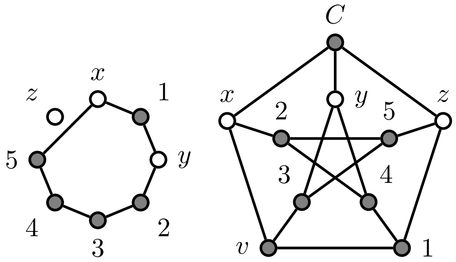
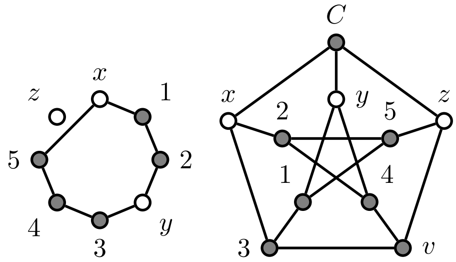
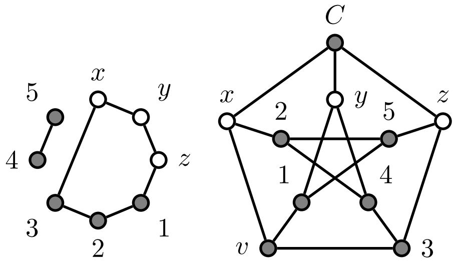
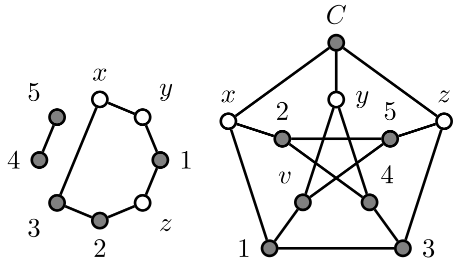
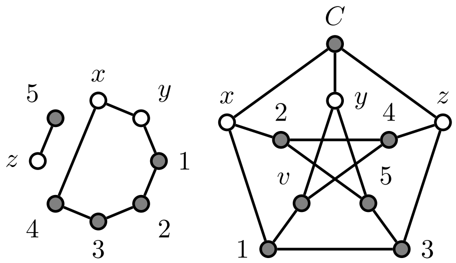
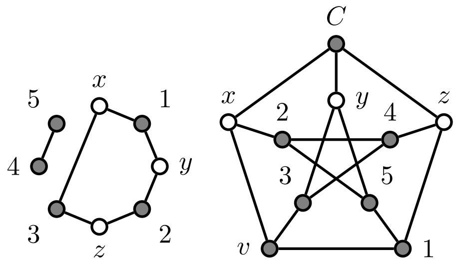
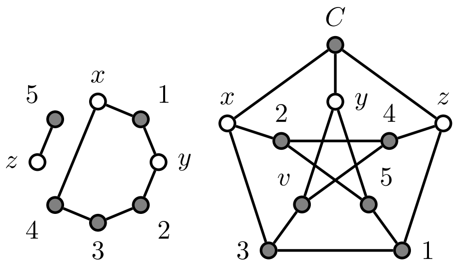
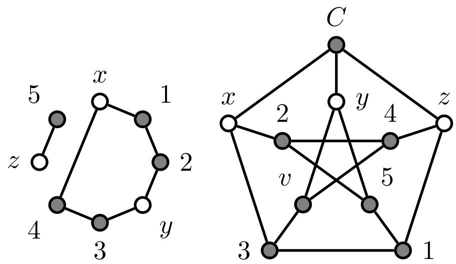
 (wwbbbbb)(w)	 (wbwbwbb)(b)
 (wbwbbbb)(w)	 (wbbwbbb)(w)
 (wwwbbb)(bb)	 (wwbwbb)(bb)
 (wwbbbb)(wb)	 (wbwbwb)(bb)
 (wbwbbb)(wb)	 (wbbwbb)(wb)

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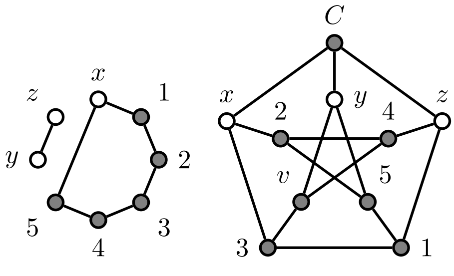
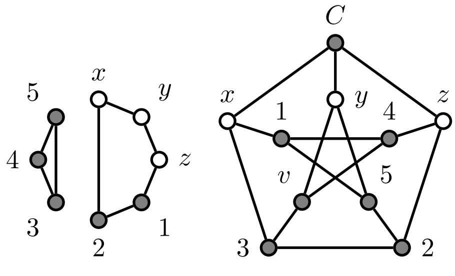
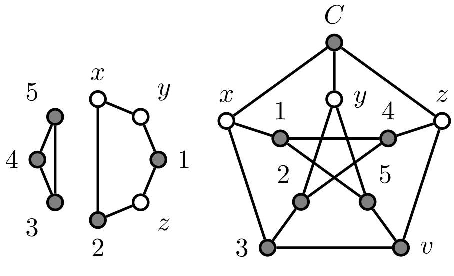
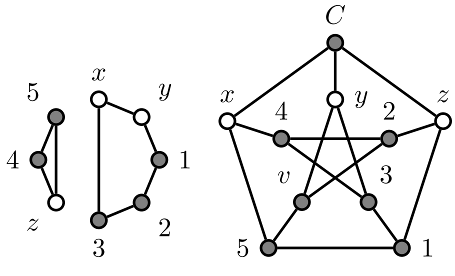
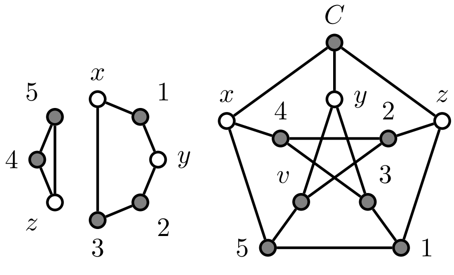
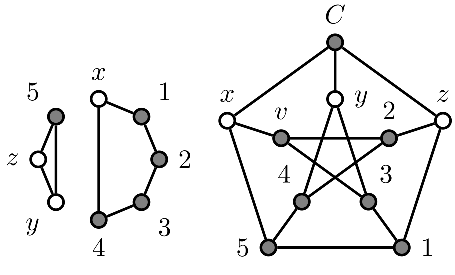
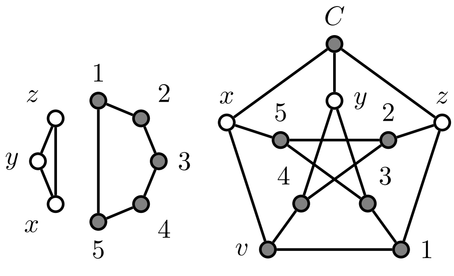
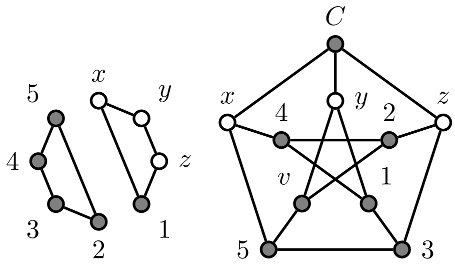
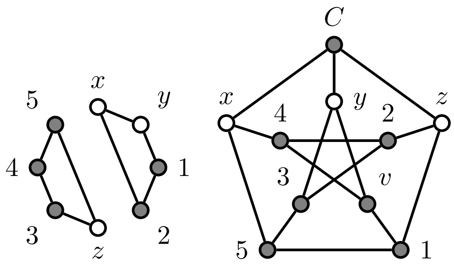
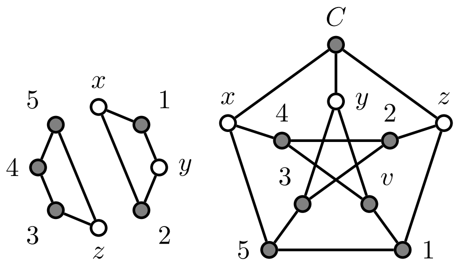
 (wbbbbb)(ww)	 (wwwbb)(bbb)
 (wwbwb)(bbb)	 (wwbbb)(wbb)
 (wbwbb)(wbb)	 (wbbbb)(wwb)
 (bbbbb)(www)	 (wwwb)(bbbb)
 (wwbb)(wbbb)	 (wbwb)(wbbb)

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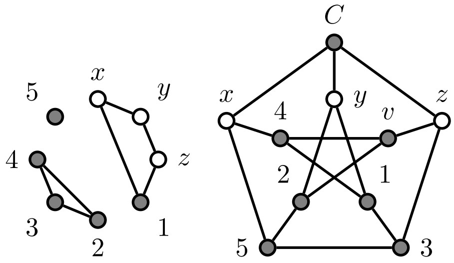
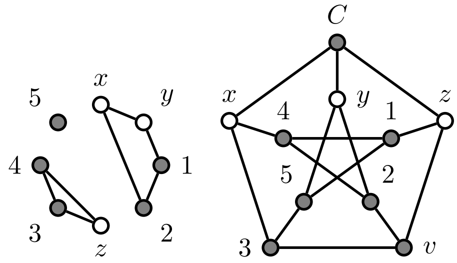
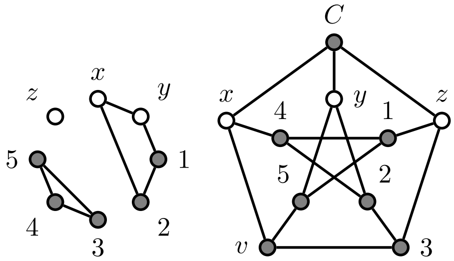
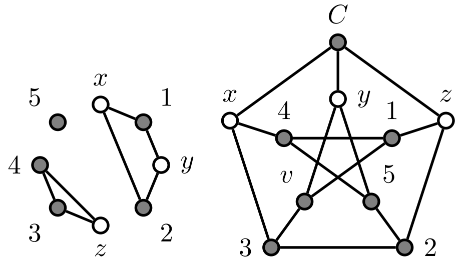
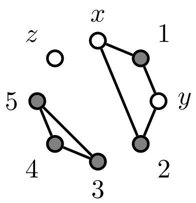
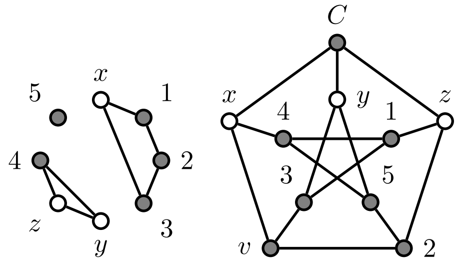
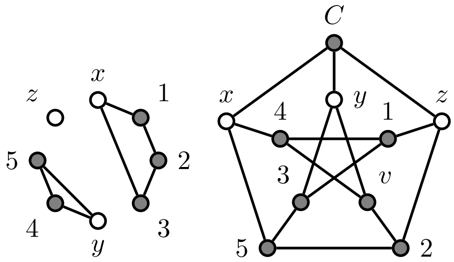
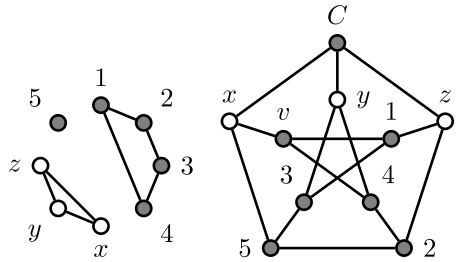
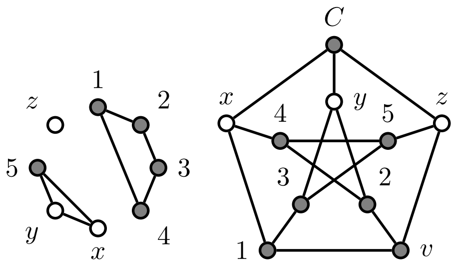
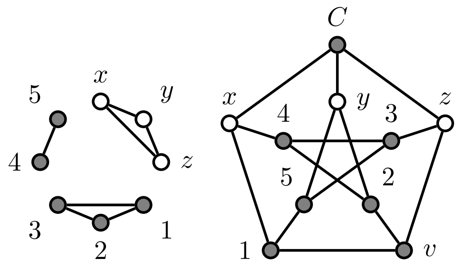
 (wwwb)(bbb)(b)	 (wwbb)(wbb)(b)
 (wwbb)(bbb)(w)	 (wbwb)(wbb)(b)
 1., 2. and 3. hold. (wbwb)(bbb)(w)	 (wbbb)(wwb)(b)
 (wbbb)(wbb)(w)	 (bbbb)(www)(b)
 (bbbb)(wwb)(w)	 (www)(bbb)(bb)

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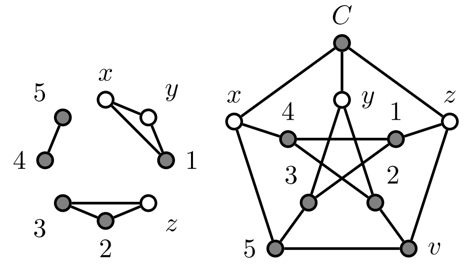
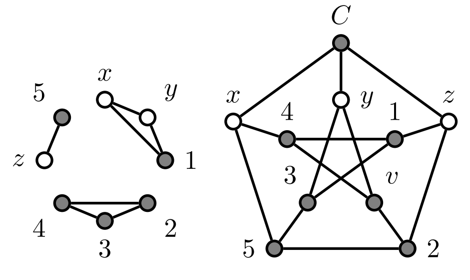
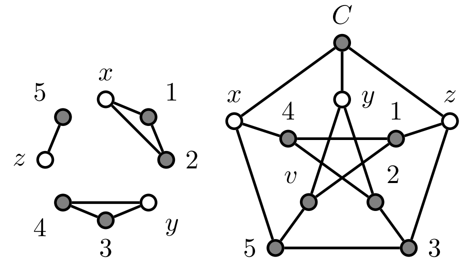
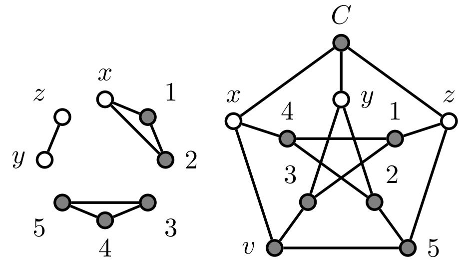
 (wwb)(wbb)(bb)	 (wwb)(bbb)(wb)
 (wbb)(wbb)(wb)	 (wbb)(bbb)(ww)

It follows that if $N(C) = \{x, y, z\}$, then the claim holds. Suppose to the contrary that $\mathcal{P}$ is not a minor of H and $|N(C)| \geqslant 4$. As Table 4 shows, $\overline{H'}$ contains both K_3 and C_4 as induced subgraphs. Since $\Delta(\overline{H'}) \leqslant 2$, no vertex of $\overline{H'}$ is in more than one cycle, so there is a unique triangle in $\overline{H'}$. For any subset $S \subseteq N(C)$ of size 3, S is a set of independent vertices in $\overline{H'}$, disjoint from the unique triangle of $\overline{H'}$ by the case analysis in Table 4. Hence, $N(C)$ is an independent set of at least four vertices in $\overline{H'}$, disjoint from the unique triangle of $\overline{H'}$. However, given the structure of H, there is no such set, a contradiction. $\square$

The following is the main result of this section.

Lemma 10.12. $V_8(G) = \emptyset$.

Proof. Suppose to the contrary that $v \in V(G)$ has degree 8. By Lemma 10.4, $|N(v') \cap N(v)| \geqslant 5$ for all $v' \in N(v)$. By Lemma 10.1, $G - N[v]$ has some non-empty component C. By Lemma 10.3, $|N(C)| \geqslant 4$, so $G[N(C)] \ncong K_3$. Hence, by Claim 10.11, $\mathcal{P}$ is a minor of G, contradicting (iv). $\square$

2.2.4 Degree 6 Vertices

In this section we focus on vertices of degree 6 in G. Recall that for a given vertex v of our minimal counterexample G, a subgraph H of G is *v-suitable* if it is a component of $G - N[v]$ that contains some vertex of $\mathcal{L}$. The main result of this section is that if $v \in V_6(G)$, then for any v-suitable subgraph H there is a v-suitable subgraph H' such that $N(H') \setminus N(H) \neq \emptyset$ (see Lemma 10.17).

Claim 10.13. *If $v \in V_6(G)$, then $N[v]$ is a clique.*

Proof. By definition, v is dominant in $G[N[v]]$. Let w be a vertex in $N(v)$. Then w is adjacent to each of the five other vertices in $N(v)$, by Lemma 10.4 applied to the edge vw. $\square$

This result is useful because it means that for an induced subgraph H of $\mathcal{P}$ on seven or fewer vertices, $H \subseteq G[N[v]]$. Throughout this section we show that certain statements about the structure of G imply $\mathcal{P}$ is a minor of G, and are therefore false. When illustrating this, the vertices of $N[v]$ will be coloured white, for ease of checking.

Claim 10.14. *If $v \in V_6(G)$ and C is a component of $G - N[v]$ with $|N(C)| \geqslant 5$, then $|V(C)| = 1$.*

Proof. Suppose for contradiction that $|V(C)| > 1$. By Lemma 10.7 with $A := N[C]$ and $B := V(G - C)$, there is a skeleton T of C with at least two high degree vertices. The handshaking lemma implies

$$\sum_{i=3}^{\infty}(i-2) \cdot |V_i(T)| = |V_1(T)| - 2. \tag{2.2}$$

Note that $|V_1(T)| = |N(C)|$ and $|N(C)| \in \{5, 6\}$, so $|V_1(T)| - 2 \in \{3, 4\}$. Hence either $|V_{\geqslant 3}(T)| \in \{3, 4\}$ (Case 2 below), $V_3(T) = \emptyset$ and $|V_4(T)| = 2$ (Cases 3 and 4 below), or $|V_3(T)| = 1$ and $|V_{\geqslant 4}(T)| = 1$ (Case 5).

Case 1. $|V(C)| = 2$:

Since C is connected, the two vertices w and x of C are adjacent. By Lemma 10.4 applied to wx, w and x have at least five common neighbours, $v_1, \ldots, v_5$. By Lemma 10.1, $|V(G - N[v] - C)| \geqslant 1$, so there is some component $C' \neq C$ of $G - N[v]$. By Lemma 10.3, $|N(C')| \geqslant 4$. Both $N(C')$ and $\{v_1, \ldots, v_5\}$ are subsets of $N(v)$ and $|N(v)| = 6$, so $|N(C') \cap \{v_1, \ldots, v_5\}| \geqslant 3$. Assume without loss of generality that $N(C') \supseteq \{v_1, v_2, v'\}$, where v' is neither v_3 nor v_4. Let v'' be the unique vertex in $N(v) \setminus \{v_1, v_2, v_3, v_4, v'\}$. Let G' be obtained from G by contracting C' to a single vertex. Then $\mathcal{P} \subseteq G'$ by Claim 10.13 (see Figure 5), contradicting (iv).

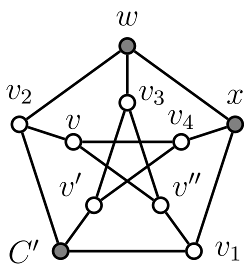


Figure 5

Case 2. *C has a skeleton T with at least three high degree vertices*:

By repeatedly contracting edges of $T \cap C$, we can obtain a minor T' of T such that T' is a tree, $V_1(T') = N(C)$, there are at least three vertices in $V_{\geqslant 3}(T')$ and $|V_{\geqslant 3}(T'/e)| \leqslant 2$ for every edge $e \in E(T' - V_1(T'))$. Contracting an edge of $T' - V_1(T')$ can only reduce $|V_{\geqslant 3}(T')|$ by 1, and only if both endpoints of the edge are in $|V_{\geqslant 3}(T')|$. Hence, there are exactly three vertices of $T' - V_1(T)$, and each has degree at least 3 in T'. Now $T' - V_1(T)$ is a tree on three vertices, and hence is a path wxy. Since w, x and y all have degree at least 3 in T', there are distinct vertices $v_1, \ldots, v_5$ such that w is adjacent to v_1 and v_2 in T', y is adjacent to v_4 and v_5 in T', and x is adjacent to v_3 in T'. Let v_6 be the remaining vertex in $N(v)$, and recall that $G[N[v]]$ is a complete subgraph of G by Claim 10.13. Let E be the set of edges that were contracted to obtain T', and let $G' := G/E$. Then $\mathcal{P} \subseteq G'$ (see Figure 6), contradicting (iv).

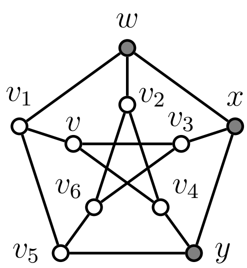


Figure 6

Case 3. *There is a skeleton T of C with $|V_4(T)| = 2$ and with some $y \in V_2(T)$*:

Let w and x be the vertices in $V_4(T)$.

First, suppose that y is in xTw. Then by Lemma 10.6, there is a path P of $G[N[C]]$ from xTw to $T - xTw$ with no internal vertex in T. Let a be the endpoint of P in xTw and let b

be the other endpoint. Without loss of generality, $w \notin V(xTb)$. Let $R := (T \cup P) - \mathrm{int}(xTb)$. Then R is a skeleton of C and $V_{\geqslant 3}(R) = \{x, w, a\}$, so we are in Case 2.

Suppose instead that y is not in xTw. Without loss of generality, y is in the component of $T - \mathrm{int}(xTw)$ containing x. Let z be the leaf of T such that y is in xTz. By Lemma 10.6, there is a path P of $G[N(C)] - \{x, z\}$ from xTz to $T - xTz$ with no internal vertex in T. Let a be the endpoint of P in xTz and let b be the other endpoint. If $w \notin V(xTb)$ or $w = b$, then let $R := (T \cup P) - \mathrm{int}(xTb)$. Otherwise, let $R := (T \cup P) - \mathrm{int}(wTb)$. In either case, R is a skeleton of C and $V_{\geqslant 3}(R) = \{x, w, a\}$, so we are in Case 2.

Case 4. *There is a skeleton T of C with $|V_4(T)| = 2$ and $V_2(T) = \emptyset$:*

Since T is a skeleton of C, $|V_1(T)| = |N(C)| \leqslant 6$. It then follows from (2.2) that $V(T) \setminus V_1(T) = V_4(T)$, and $|V_1(T)| = 6$. We may assume that we are not in Case 1, so there is some vertex in $C - V_4(T)$. Since C is connected, there is some vertex y in $C - V_4(T)$ adjacent to some vertex x in $V_4(T)$. Let w be the other vertex of $V_4(T)$. By Lemma 10.3, there is a path of $G - x$ from y to T. Let P be a vertex-minimal example of such a path, and note that $\mathrm{int}(P)$ is disjoint from T. Also, since $N(C) \subseteq V(T)$, every vertex of P is in $N[C]$. Let P' be the path formed from P by adding x and the edge xy, and let b be the other endpoint of P'.

Suppose that either $b = w$ or $w \notin V(bTx)$. Let $R := (T \cup P') - \mathrm{int}(bTx)$. Then R is a skeleton of C with $|V_4(T)| = 2$ and $y \in V_2(T)$, so we are in Case 3.

Suppose instead that $w \in \mathrm{int}(bTx)$. Note that $V(T) = \{x, w\} \cup V_1(T)$, and hence $xTw = xw$. Hence, by Lemma 10.4, x and w have at least five common neighbours. If some common neighbour z of x and w is in C, then $R := (T \cup wzx) - \mathrm{int}(xTw)$ is a skeleton of C with $|V_4(R)| = 2$ and $z \in V_2(R)$ and we are in Case 3. We may therefore assume that $N(x) \cap N(w) \subseteq N(C)$. Let $v_1, \ldots, v_5$ be distinct vertices in $N(x) \cap N(w)$, and let v_6 be the remaining vertex of $N(C)$. Let w_1, w_2 and w_3 be distinct neighbours of w in $\{v_1, \ldots, v_6\} \setminus \{b\}$, with $w_1 = v_6$ if possible. Since $\{v_1, \ldots, v_5\} \subseteq N(x)$ and at least one of w and x is adjacent to v_6, x has two neighbours x_1 and x_2 in $\{v_1, \ldots, v_6\} \setminus \{b, w_1, w_2, w_3\}$. Let $V(R) := \{x, w, v_1, \ldots, v_6\} \cup V(P)$ and $E(R) := \{ww_1, ww_2, ww_3, xx_1, xx_2, xw\} \cup E(P')$. Then R is a skeleton of C with $V_4(R) = \{x, w\}$ and $y \in V_2(R)$, and we are in Case 3.

Case 5. *There is a skeleton T of C with exactly one vertex $x \in V_3(T)$ and exactly one vertex $w \in V_{\geqslant 4}(T)$:*

Since $\deg_T(x) = 3$ there are distinct leaves v_1 and v_2 such that $w \notin V(v_1Tv_2)$. Let $v_3, v_4, \ldots, v_k$ be the remaining leaves of T, where $k = |N(C)|$. Let C' be the component of $C - w$ containing x, and note that $N(C') \subseteq N(C) \cup \{w\}$. Since G is 4-connected by Lemma 10.3, there is some vertex in $N(C') \cap (N(C) \setminus \{v_1, v_2\})$, and hence some path P of $G[N[C] \setminus \{w, v_1, v_2\}]$ from x to $N(C) \setminus \{v_1, v_2\}$. Let P' be a subpath of P of shortest possible length while having an endpoint a in the component $T - w$ containing x and an endpoint b in some other component of $T - w$. Note that $P' \subseteq G[N[C] - \{w, v_1, v_2\}]$ and no internal vertex of P' is in T. Let $R := (T \cup P') - \mathrm{int}(bTw)$, and note that R is a skeleton of C. If $a \neq x$, then $V_{\geqslant 3}(R) = \{a, x, w\}$, and we are in Case 2. If $a = x$ and $w \in V_5(T)$, then $V_4(R) = \{x, w\}$, and we are in Case 3 or Case 4. Hence, we may assume $x = a$ and $w \in V_4(T)$, meaning $|N(C)| = 5$. We now consider two subcases, depending on whether $xw \in E(T)$.

Case 5a. $wx \notin E(T)$:

By Lemma 10.6, there is a path Q of $G[N[C]] - \{x, w\}$ from xTw to $T - xTw$ with no internal vertex in T. Let c be the endpoint of Q in xTw, and let d be the other endpoint.

Suppose first that Q intersects P'. Let Q' be the subpath of Q from c to P' that is internally disjoint from P', and let d' be the endpoint of Q' in P'. Let $S := (R \cup Q') - \mathrm{int}(d'Rx)$. Then S is a skeleton of C with $V_{\geqslant 3}(S) = \{x, c, w\}$, and we are in Case 2.

Suppose instead that Q is disjoint from P'. If $x \notin V(dTw)$, then let $S := (T \cup Q) - \mathrm{int}(dTw)$. Otherwise, let $S := (R \cup Q) - \mathrm{int}(dRx)$. Then S is a skeleton of C with $V_{\geqslant 3}(S) = \{x, c, w\}$, and we are in Case 2.

Case 5b. $xTw = xw$:

By Lemma 10.4 applied to the edge xw, $|N(x) \cap N(w)| \geqslant 5$.

Suppose there is some vertex $y \in (N(x) \cap N(w)) \setminus N(C)$. If $y \in (N(x) \cap N(w)) \setminus V(T)$, then let $S := (T \cup xyw) - xw$. Then S is a skeleton of C with exactly one vertex $x \in V_3(S)$ and exactly one vertex $w \in V_{\geqslant 4}(S)$ and $xw \notin E(S)$, so we are in Case 5a. If $y \in N(x) \cap N(w) \cap V(xTv_i - v_i)$ for some $i \in \{1, 2\}$, then let S be the graph obtained from R by adding the edge wy and deleting the edge wx. If $y \in N(x) \cap N(w) \cap V(xTv_i - v_i)$ for some $i \in \{3, 4, 5\}$, then let S be the graph obtained from T by adding the edge xy and deleting the edge wx. Then S is a skeleton of C with $V_{\geqslant 3}(S) = \{x, y, w\}$, and we are in Case 2.

Suppose instead that $N(x) \cap N(w) \subseteq N(C)$. Since $|N(C)| = 5$, we have $N(x) \cap N(w) = N(C)$. We may assume we are not in Case 1, so by Lemma 10.3, there is some vertex y in $C - \{x, w\}$ adjacent to some vertex in $N(C)$. Since $\{x, w\}$ is complete to $N(C)$, assume without loss of generality that $v_5 \in N(y)$. Since C is connected, there is a path Q of C from y to $\{w, x\}$. Choose Q to be of shortest possible length, so that $\mathrm{int}(Q)$ is disjoint from $\{x, w\}$, and without loss of generality assume x is an endpoint of Q (since $\{x, w\}$ is complete to $N(C)$). Let S be the skeleton with $V(S) := \{w, v_1, \dots, v_5\} \cup V(Q)$ and $E(S) := \{wv_1, wv_2, wv_3, wx, xv_4, yv_5\} \cup E(Q)$. By Lemma 10.6, there is a path Q' of $G[N[C]] - \{x, v_5\}$ from xSv_5 to $S - xSv_5$, internally disjoint from S. Let c be the endpoint of Q' in xSv_5 and let d be the other endpoint. If $d \in \{v_1, v_2, v_3\}$, then let $S' := (S \cup Q') - dw$. Then S' is a skeleton of C with $V_{\geqslant 3}(S') = \{w, x, c\}$, and we are in Case 2. If either $d = w$ and there is some vertex in $\mathrm{int}(Q')$, or $d = v_4$, then let $S' := (S \cup Q') - dx$. Then S' is a skeleton of C with exactly one vertex $c \in V_3(S')$ and exactly one vertex $w \in V_{\geqslant 4}(S')$, and $cw \notin E(S)$, so we are in Case 5a. If $d = w$ and there is no vertex in $\mathrm{int}(Q')$, then either $c \in N(x) \cap N(w)$, contradicting the assumption that $N(x) \cap N(w) \subseteq N(C)$, or $|V(ySc \cup Q')| < |V(Q)|$, contradicting our choice of Q. ☐

Claim 10.15. *If $v \in V_6(G)$ and C is a component of $G - N[v]$, then $V(C) \neq \emptyset$ and $|N(C)| = 4$.*

Proof. By Lemma 10.1, $V(G) \setminus N[v]$ is non-empty, so $V(C) \neq \emptyset$. Hence $|N(C)| \geqslant 4$ by Lemma 10.3. Suppose for contradiction that $|N(C)| \geqslant 5$. Then $|V(C)| = 1$ by Claim 10.14. Hence, by Lemma 10.4, $|N(C)| \geqslant 6$, so $N(C) = N(v)$.

Suppose that there is some component C' of $G - N[v]$ with $|N(C')| = 4$. By Lemma 10.4, $|V(C')| \geqslant 3$. Hence, by Lemma 10.8 with $A := N[C']$ and $B := V(G - C')$, there is a table $\mathcal{X} := (X_1, \dots, X_6)$ of $G[N[C']]$ rooted at $N(C')$. For $i \in \{1, 2, 3, 4\}$, let v_i be the unique vertex in $X_i \cap N(C')$. Let v_5 and v_6 be the remaining vertices of $N(v)$. By Claim 10.13, $G[N[v]] \cong K_7$. Let G' be obtained from G by contracting $G[X_i]$ to a single vertex for each $i \in \{1, 2, \dots, 6\}$. Then $\mathcal{P} \subseteq G'$ (see Figure 7a), contradicting (iv).

Suppose instead that every component C' of $G - N[v]$ satisfies $|N(C')| \geqslant 5$. Then by Claim 10.14 every component of $G - N[v]$ is an isolated vertex and by Lemma 10.4 each component C' of $G - N[v]$ satisfies $N(C') = N(v)$. Now by Lemma 10.1 there are at least three distinct components C, C' and C'' of $G - N[v]$. Hence, by Claim 10.13, $\mathcal{P} \subseteq G$ (see Figure 7b), contradicting (iv). ☐

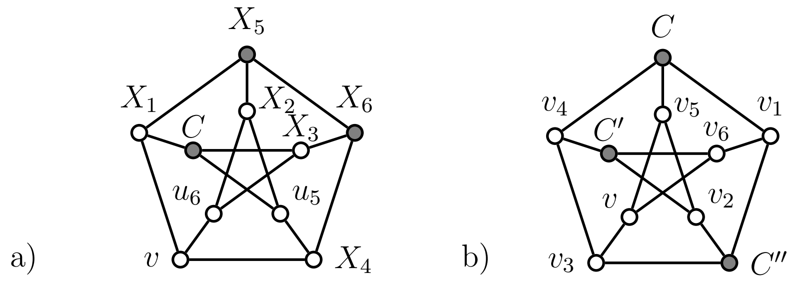


Figure 7

Claim 10.15 and Lemma 10.5 immediately imply the following corollary, which we use in the final step of the proof, in Section 2.2.6.

Corollary 10.16. *For every vertex $v \in V_6(G)$, there is at least one v-suitable subgraph.*

We now prove the main result of this section.

Lemma 10.17. *If $v \in V_6(G)$ and H is a v-suitable subgraph of G, then there is some v-suitable subgraph H' of G such that $N(H') \setminus N(H) \neq \emptyset$.*

Proof. By Claim 10.15, $|N(H)| = 4$. Suppose for contradiction that there exist distinct vertices $w, x \in N(v)$ such that $N[x] \subseteq N[v]$ and $N[w] \subseteq N[v]$. Let $G' := G - \{v, w, x\}$. By (ii),

$$|E(G')| \geqslant |E(G)| - 3 - 3(4) \geqslant (5|V(G)| - 11) - 15 = 5|V(G')| - 11.$$

By (v), G' is a $(K_9, 2)$-cockade minus at most two edges. Every $(K_9, 2)$-cockade has at least nine vertices of degree exactly 8, so $|V_8(G')| \geqslant 5$. Then some vertex in $V(G') \setminus N[v]$ has degree exactly 8 in G, contradicting Lemma 10.12.

Hence there is at most one vertex w in $N(v)$ such that $N[w] \subseteq N[v]$, so there is some vertex x in $N(v) \setminus N(H)$ with some neighbour y in $G - N[v]$. Let H' be the component of $G - N[v]$ that contains y. The vertex x is in $N(H')$, so $N(H') \setminus N(H) \neq \emptyset$. By Claim 10.15 and Lemma 10.5, H' is v-suitable, as required. □

2.2.5 Degree 9 Vertices

In this section, we focus on vertices in $V_9(G) \cap \mathcal{L}$. For each such vertex v, the minimum degree of $G[N(v)]$ is at least 5, by Lemma 10.4 applied to each edge incident to v. Let H_v be the complement of an edge-minimal spanning subgraph of $G[N(v)]$ with minimum degree 5.

The main result of this section, Lemma 10.23, states that for each component C of $G - N[v]$, there is some v-suitable subgraph C' with a neighbour not in the neighbourhood of C. We argue for this claim directly when each component C' of $G - N[v]$ has $|N(C')| = 4$. Otherwise, we first look at the case where the maximum distance between two vertices of degree 3 in H_v is at most 2. Then we consider the case where there are two vertices of degree 3 at distance at least 3 in H_v. A useful technique is that a graph obtained by contracting some edge in $G[N(v)]$ must violate some condition of Claim 10.11.

Claim 10.18. *If $v \in V_9(G) \cap \mathcal{L}$, then $\Delta(H_v) = 3$ and the vertices of H_v with degree at most 2 form a clique.*

Proof. Since $|V(H_v)| = |N(v)| = 9$, if a vertex u has degree greater than 3 in H_v, then u has degree less than 5 in $\overline{H_v}$, a contradiction. If two non-adjacent vertices x and y in H_v

both have degree at most 2 in H_v, then $\overline{H_v} - xy$ is a spanning subgraph of $G[N(v)]$ with minimum degree at least 5, contradicting the definition of H_v. Thus the vertices of degree at most 2 form a clique of size at most 3, so there is indeed a vertex of degree 3 in H_v. $\square$

The following claim guarantees that $|V(G)| \geqslant 11$ if we find a vertex $v \in V_9(G) \cap \mathcal{L}$, and hence that the components of $G - N[v]$ are non-empty.

Claim 10.19. *If $v \in V_9(G) \cap \mathcal{L}$, then $V(G - N[v]) \neq \emptyset$.*

Proof. By (iv), $\mathcal{P} \nsubseteq G[N[v]]$, so $G[v] \ncong K_{10}$. Hence, there is some vertex $w \in N(v)$ such that $N[w] \neq N[v]$. By the definition of $\mathcal{L}$, there is some vertex $x \in N[w] \setminus N[v]$ and $x \in V(G - N[v])$. $\square$

A graph is *cubic* if every vertex has degree exactly 3.

Claim 10.20. *If $v \in V_9(G) \cap \mathcal{L}$, then there are vertices x and y in $V_3(H_v)$ such that $\mathrm{dist}_{H_v}(x, y) \geqslant 3$, unless either $|N(C)| = 4$ for every component C of $G - N[v]$ or $H_v \cong K_{3,3} \dot\cup K_3$.*

Proof. Suppose for contradiction that $\mathrm{dist}_{H_v}(x, y) \leqslant 2$ whenever $\{x, y\} \subseteq V_3(H_v)$, there is some component C of $G - N[v]$ such that $|N(C)| \neq 4$ and $H_v \ncong K_{3,3} \dot\cup K_3$. By Claim 10.19, $V(C) \neq \emptyset$, so by Lemma 10.3, $|N(C)| \geqslant 5$. Let $S := V_0(H_v) \cup V_1(H_v) \cup V_2(H_v)$. By Claim 10.18, S is a clique, so $|S| \leqslant 3$. Since $|V(H_v)| = 9$, the number of vertices of odd degree in H_v is even and $V(H_v) \setminus S = V_3(H_v)$, we have $S \neq \emptyset$. We consider five cases depending on S and whether there is any triangle in H_v.

Case 1. $|S| = 3$:

In this case, $S = V_2(H_v)$ and $H_v[S] \cong K_3$, and there is no edge in H_v from a vertex in S to a vertex not in S. Hence, $H_v - S$ is a 6-vertex cubic graph. By assumption, $H_v \ncong K_{3,3}$. There is only one other 6-vertex cubic graph, so H_v is the graph depicted in Figure 8. Then $\mathcal{P} \subseteq G[N[v]]$ (see Figure 8b), contradicting (iv).

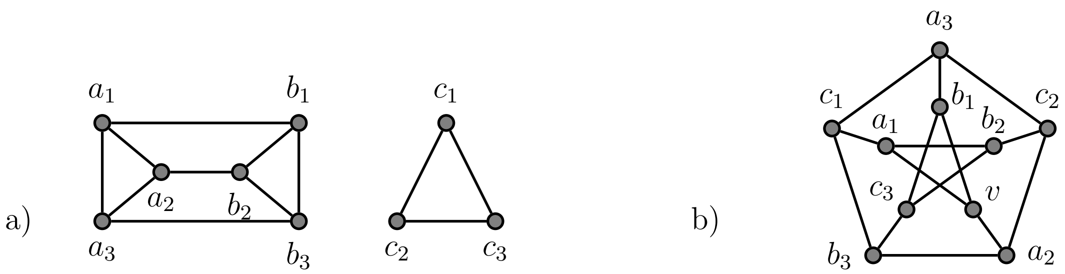


Figure 8

Case 2. $|S| = 2$:

Since $|V(H_v)|$ is odd, there are an odd number of vertices of even degree in H_v. Since S is a clique, $\delta(H_v) \geqslant 1$. Hence, by Claim 10.18, there is a unique vertex $x \in V_2(H_v)$, and since $|S| = 2$, there is some vertex $v_1 \in V_3(H_v)$ adjacent to x in H_v. Let v_2 and v_3 be the other neighbours of v_1 in H_v, and note that $\{v_2, v_3\} \subseteq V_3(H_v)$. Since $\mathrm{dist}_{H_v}(v_1, y) \leqslant 2$ for every vertex y in $V_3(H_v)$, each of the four remaining vertices of $H_v - S$ is adjacent to $\{v_2, v_3\}$. Since v_2 and v_3 each have only three neighbours in H_v, $v_2v_3 \notin E(H_v)$. Let G' be obtained from G by deleting every edge in $G \cap H_v$ and then contracting v_2v_3. Now $v \in V_8(G')$. Let v' be a vertex in $N_{G'}(v)$. If $v' \in S$, then $|N_{G'}(v') \cap N_{G'}(v)| \geqslant 8 - \deg_{H_v}(v') - 1 \geqslant 5$. If v' is in $H_v - (S \cup \{v_2, v_3\})$, then $|N_{G'}(v') \cap N_{G'}(v)| = 8 - \deg_{H_v}(v') = 5$. If v' is the new vertex of G', then $|N_{G'}(v') \cap N_{G'}(v)| = 8 - |N_{H_v}(v_2) \cap N_{H_v}(v_3)| - 1 = 6$. Hence, $|N_{G'}(v') \cap N_{G'}(v)| \geqslant 5$ for any vertex $v' \in N_{G'}(v)$. Finally, $|N_{G'}(C)| \geqslant |N_G(C)| - 1 \geqslant 4$, so $G'[N_{G'}(C)] \ncong K_3$. Hence $\mathcal{P}$ is a minor of G by Claim 10.11, contradicting (iv).

Case 3. *There is some triangle $v_1v_2v_3$ of H_v and $S = V_0(H_v) = \{x\}$:*

Let $\{v_4, v_5, \ldots, v_8\}$ be the other vertices of H_v, where $v_4v_1 \in E(H_v)$. For every vertex y in $H_v - S$ we have $\mathrm{dist}_{H_v}(v_1, y) \leqslant 2$ by assumption, so y is either adjacent to v_1 or adjacent to a neighbour of v_1. Since $\{v_2, v_3, v_4\} \subseteq V_3(H_v)$, we may assume without loss of generality that $\{v_2v_5, v_3v_6, v_4v_7, v_4v_8\} \subseteq E(H_v)$. Since $\Delta(H_v) = 3$ and $\mathrm{dist}_{H_v}(v_i, v_j) \leqslant 2$ for $i \in \{2, 3\}$ and $j \in \{7, 8\}$, H_v is the graph depicted in Figure 9a. Then $\mathcal{P} \subseteq G[N_G[v]]$ (see Figure 9b), contradicting (iv).

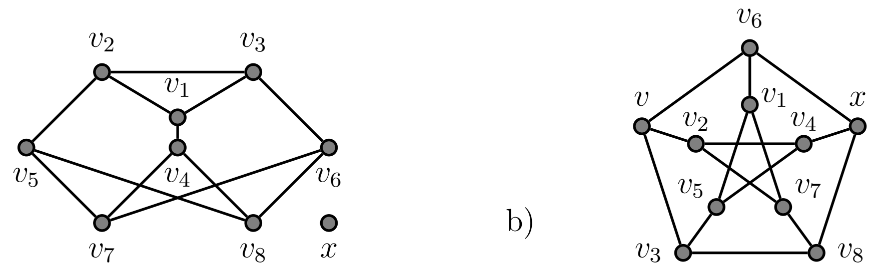
a)  b)

Figure 9

Case 4. *There is no triangle of H_v and $S = V_0(H_v) = \{x\}$:*

So $H_v - x$ is a cubic, triangle-free graph, with diameter 2 and exactly eight vertices. We now show that there is exactly one such graph, namely the Wagner graph. Let v_1 be a vertex of $H_v - S$, and let v_2, v_3 and v_4 be its neighbours in H_v. Since H_v contains no triangle, $\{v_2, v_3, v_4\}$ is an independent set in H_v. Let $\{v_5, v_6, v_7, v_8\}$ be the remaining vertices of $H_v - S$. If v_2, v_3 and v_4 all share some common neighbour, say v_5, in H_v, then there are six edges in $H_v[\{v_1, \ldots, v_5\}]$, and at most three other edges in H_v incident to some vertex in $\{v_1, \ldots, v_5\}$. By the handshaking lemma, $E(H_v - S) = E(H_v) = 12$, since $S = V_0(H_v)$ and $V(H_v - S) = V_3(H_v)$. Hence $v_6v_7v_8$ is a triangle of H_v, a contradiction. If for every pair $i, j \in \{2, 3, 4\}$ v_i and v_j share a neighbour in H_v distinct from v_1, then $|N_{H_v}[v_2] \cup N_{H_v}[v_3] \cup N_{H_v}[v_4]| \leqslant 3(4) - 3(2) + 1 = 7$ by inclusion-exclusion, contradicting the assumption that $\mathrm{dist}_{H_v}(v_1, y)$ for each of the 8 vertices y in $V_3(H_v)$. Hence, without loss of generality, v_2 and v_3 have no common neighbour in H_v, and $\{v_2v_5, v_2v_6, v_3v_7, v_3v_8\} \subseteq E(H_v)$. Without loss of generality $v_8 \in N_{H_v}(v_4)$, since $\{v_5, v_6, v_7, v_8\} \cap N_{H_v}(v_4) \neq \emptyset$. Since $v_7v_3v_8$ is a path in H_v and H_v contains no triangle, the other vertex adjacent to v_8 is either v_5 or v_6, so without loss of generality $v_8v_6 \in E(H_v)$. Since $v_5v_2v_6$ and $v_4v_8v_6$ are paths in H_v, the remaining vertex adjacent to v_6 is v_7. Since $V_3(H_v) = V(H_v) \setminus \{x\}$ and $x \in V_0(H_v)$, the remaining two vertices adjacent to v_5 are v_7 and v_4. Hence H_v is the Wagner Graph, plus a single isolated vertex, as illustrated in Figure 10a. Then $\mathcal{P} \subseteq G[N_G[v]]$ (see Figure 10b), contradicting (iv).

Case 5. *$S = \{x\}$ and $x \notin V_0(H_v)$:*

The number of vertices of odd degree in H_v is even, so $x \in V_2(H_v)$. By contracting an edge of H_v incident to x, we obtain a cubic graph on eight vertices with diameter at most 2. In Cases 3 and 4 we showed that there are only two such graphs (one with and one without a triangle), so H_v is a copy of one of these in which exactly one edge is subdivided exactly

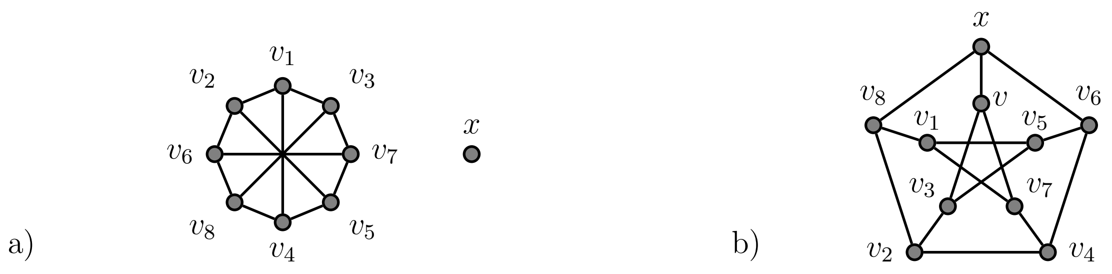


Figure 10

once. It is quick to check that the only such graph in which $\mathrm{dist}(x', y') \leqslant 2$ whenever x' and y' both have degree 3 is the graph depicted in Figure 11a. Then $\mathcal{P} \subseteq G[N_G[v]]$ (see Figure 11b), contradicting (iv). □

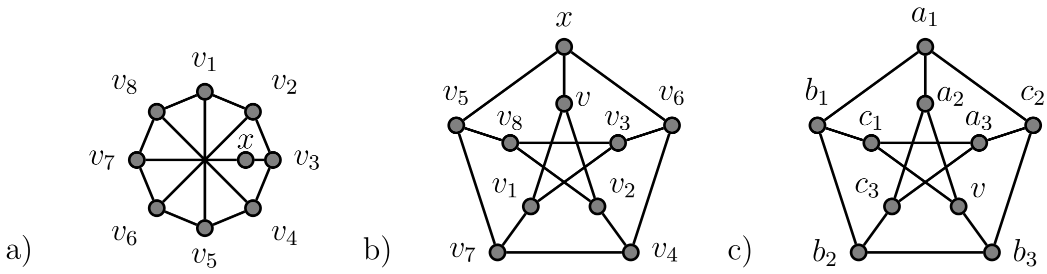


Figure 11

Claim 10.21. *If $v \in V_9(G) \cap \mathcal{L}$ and $H_v \cong K_{3,3} \dot{\cup} K_3$, then for each component C of $G - N[v]$, there is some v-suitable subgraph C' with $N(C') \setminus N(C) \neq \emptyset$.*

Proof. Let $\{a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3\} := V(H_v)$, with $a_i b_j \in E(H_v)$ for $i, j \in \{1, 2, 3\}$, and $c_i c_j \in E(H_v)$ for distinct $i, j \in \{1, 2, 3\}$. Suppose for contradiction that there is a path P of G from a_i to b_j with no internal vertex in $N[v]$ for some $i, j \in \{1, 2, 3\}$. Without loss of generality, $i = j = 1$. Let G' be obtained from G by contracting all but one edge of P. Then $\mathcal{P} \subseteq G'$ (see Figure 11c), contradicting (iv). Hence, there is no such path P. In particular, no vertex v' in $\{a_1, a_2, a_3, b_1, b_2, b_3\}$ is adjacent to every vertex of $N(v) \setminus \{v'\}$. Hence, since $v \in \mathcal{L}$, for each $v' \in \{a_1, a_2, a_3, b_1, b_2, b_3\}$ there is some component C of $G - N[v]$ such that $v' \in N(C)$. However, there is no component C such that $N(C)$ contains some vertex in $\{a_1, a_2, a_3\}$ and some vertex in $\{b_1, b_2, b_3\}$. Hence, for each component C of $G - N[v]$ there is a component C' of $G - N[v]$ with $N(C') \setminus N(C) \neq \emptyset$. Suppose for contradiction that C' is not v-suitable. By Lemma 10.5 $|N(C')| \geqslant 7$. Since $\overline{G[N(C')]} \subseteq H_v$, there is some vertex in $\{a_1, a_2, a_3\} \cap N(C')$ and some vertex in $\{b_1, b_2, b_3\} \cap N(C')$, a contradiction. Hence C' satisfies our claim. □

Claim 10.22. *If $v \in V_9(G) \cap \mathcal{L}$ and there are two vertices x and y in $V_3(H_v)$ such that $\mathrm{dist}_{H_v}(x, y) \geqslant 3$ and there is some component C of $G - N[v]$ with $|N(C)| \geqslant 5$, then for each component C' of $G - N[v]$ there is a v-suitable subgraph C'' with $N(C'') \setminus N(C') \neq \emptyset$.*

Proof. By Claim 10.19, $V(C) \neq \emptyset$. Choose x and y, if possible, so that

$$N_{H_v}(x) \cup N_{H_v}(y) \subseteq V_3(H_v). \tag{2.3}$$

Let $G' := G/xy$, let x' be the new vertex of G', and let $H' = H_v - \{x, y\}$. Note that $\deg_{G'}(v) = 8$. Since $|N_G(C)| \geqslant 5$, we have $|N_{G'}(C)| \geqslant 4$, and hence $G'[N_{G'}(C)] \ncong K_3$. By Claim 10.11 and (iv), G' does not satisfy $|N_{G'}(v') \cap N_{G'}(v)| \geqslant 5$ for all $v' \in N_{G'}(v)$.

Now $\{x, y\} \subseteq V_3(\overline{H_v})$ and $\mathrm{dist}_{\overline{H_v}}(x, y) \geqslant 3$, so $|N_{H_v}(x) \cap N_{H_v}(y)| = 6$. Also, since $\overline{G[N(v)]} \subseteq H_v$, there is no common neighbour of x and y in $G[N(v)]$, so x' is dominant in $G'[N_{G'}[v]]$, and $|N_{G'}(x') \cap N_{G'}(v)| = 7 > 5$.

By Claim 10.18, $\Delta(H_v) = 3$. If $v' \in N_{H_v}(x) \cup N_{H_v}(y)$, then since v' is not adjacent to both x and y in $\overline{H_v}$, we have $|N_{G'}(v') \cap N_{G'}(v)| \geqslant |N_{\overline{H_v}}(v')| \geqslant 8 - 3 = 5$.

Hence, the unique vertex z in $H' - (N_{H_v}(x) \cup N_{H_v}(y))$ satisfies $|N_{G'}(z) \cap N_{G'}(v)| \leqslant 4$. Thus z has at most three neighbours in $G'[N(v) \setminus \{x, y\}]$ and hence $|N_{H'}(z)| \geqslant 7-1-3 = 3$. Since $\Delta(H') \leqslant \Delta(H_v) = 3$, we have $\deg_{H_v}(z) = 3$.

There are an even number of vertices, including z, with odd degree in H'. We have $\deg_{H'}(v') \leqslant \Delta(H_v) - 1 = 2$ for the six vertices v' in $N_{H_v}(x) \cup N_{H_v}(y) = V(H' - z)$, so there are an odd number of vertices in $V_1(H')$. Each vertex in $V_1(H')$ has degree at most 2 in H_v since x and y have no common neighbour in H_v. So $V_1(H')$ is a clique of H_v by Claim 10.18, and hence a clique of H'. Since $|V_1(H')|$ is odd, there is a unique vertex w in $V_1(H')$. By the same argument, the vertices of $V_0(H') \cup V_1(H')$ form a clique of H'. No vertex in $V_0(H')$ is adjacent in H' to w, so $V_0(H') = \emptyset$. Hence, $V_1(H') = \{w\}$, $V_3(H') = \{z\}$ and $V_2(H') = V(H' - \{w, z\})$.

Now w is one of the six vertices of $N_{H_v}(x) \cup N_{H_v}(y)$, and $\deg_{H_v}(w) \leqslant \deg_{H'}(w) + 1 \leqslant 2$. In particular x and y do not satisfy (2.3), so no such pair satisfy (2.3). This means, there are no two vertices x' and y' in $V_3(H_v)$ that satisfy (2.3) such that $\mathrm{dist}_{H_v}(x', y') \geqslant 3$.

We consider four cases depending on whether H' is connected and on the components of $G - N[v]$.

Case 1. *H' is not connected*:

Since each connected component of H' has an even number of vertices of odd degree, z and w are in the same component, and each other component is a cycle. Since $|V(H') \setminus N_{H'}[z]| = 3$, there is a unique component D of H' not containing z and D is a triangle. Since $|V_2(H')| = 5$, there is some vertex x_0 of degree 2 not in D and not adjacent to w. Assume without loss of generality that x_0 is adjacent to x in H_v. Since $x \in V_3(H_v)$, there is some vertex y_0 in D such that $y_0x \notin E(H_v)$. Now y_0 is adjacent to no neighbour of x_0 in H_v, so $\mathrm{dist}_{H_v}(x, y) \geqslant 3$. But the vertices adjacent to $\{x_0, y_0\}$ in H_v are all in $V_3(H_v)$ since w is adjacent to neither x_0 nor y_0 in H_v. Therefore x_0 and y_0 satisfy (2.3), a contradiction.

For the remaining cases, H' is a connected graph such that $|V_1(H')| = |V_3(H')| = 1$ and every other vertex has degree 2. Hence, H' is composed of a path P from z to w and a cycle Q of size at least 3 containing z, with $V(P \cap Q) = \{z\}$. Let z_0 be the neighbour of z in the path from z to w, and let z_1 and z_2 be the other neighbours of z in H'.

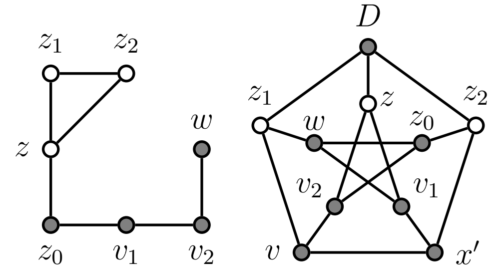
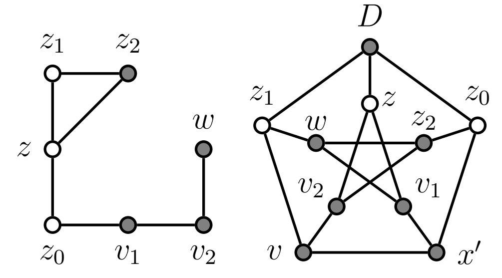
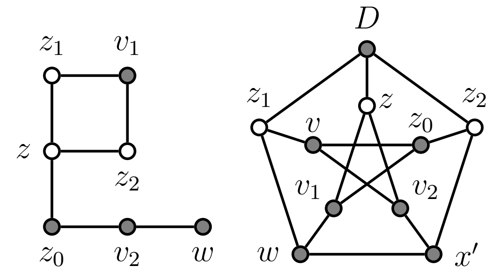
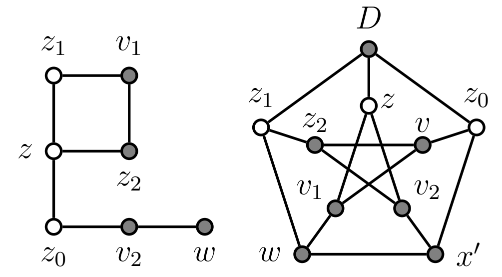
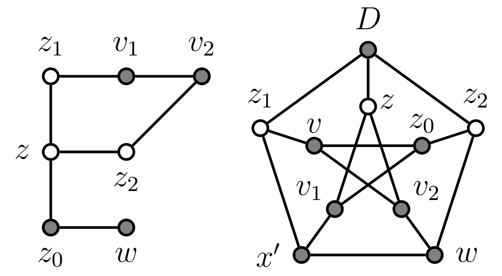
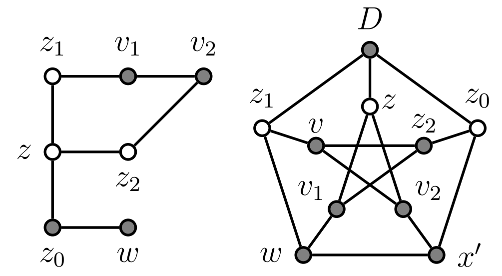
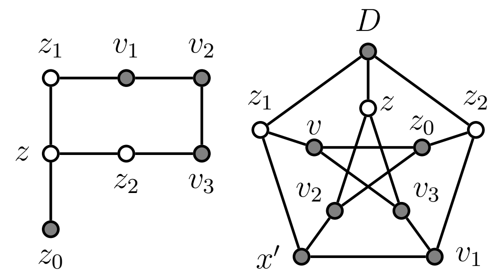
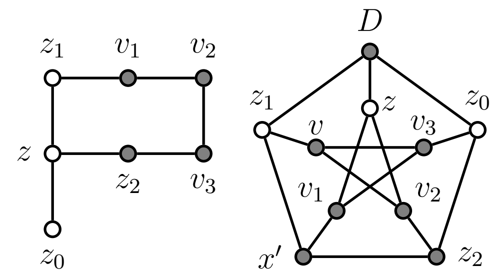
Case 2. *H' is connected and there is some component D of $G - N[v]$ such that $z \in N(D)$ and $|N(D) \cap N_{H'}(z)| \geqslant 2$*:

At least one vertex is in $\{z_1, z_2\} \cap N(D)$, so without loss of generality $z_1 \in N(D)$. Either z_0 or z_2 is also in $N(D)$. Since $V(Q) \subseteq V(H') \setminus \{w\}$, we have $3 \leqslant |V(Q)| \leqslant 6$. Let $G'' := G'/E(D)$. The diagrams in Table 12 demonstrate that $\mathcal{P} \subseteq G''$, contradicting (iv).

Case 3. *H' is connected and there is some component D of $G - N[v]$ such that $z_0 \in N(D)$, $N(D) \cap \{z_1, z_2\} \neq \emptyset$ and $N(D) \cap \{x, y\} \neq \emptyset$*:

Without loss of generality, $z_1 \in N(D)$. Note that $\{z_1, z_0, x'\} \subseteq N_{G'}(D)$, and let $G'' := G'/E(D)$. The diagrams in Table 13 demonstrate that $\mathcal{P} \subseteq G''$, contradicting (iv).

Table 12

 $\lvert V(Q)\rvert = 3,\ z_2 \in N(D)$	 $\lvert V(Q)\rvert = 3,\ z_0 \in N(D)$
 $\lvert V(Q)\rvert = 4,\ z_2 \in N(D)$	 $\lvert V(Q)\rvert = 4,\ z_0 \in N(D)$
 $\lvert V(Q)\rvert = 5,\ z_2 \in N(D)$	 $\lvert V(Q)\rvert = 5,\ z_0 \in N(D)$
 $\lvert V(Q)\rvert = 6,\ z_2 \in N(D)$	 $\lvert V(Q)\rvert = 6,\ z_0 \in N(D)$

Case 4. *H' is connected and there is no component D of $G-N[v]$ such that either $z \in N(D)$ and $|N(D) \cap N_{H'}(z)| \geqslant 2$ or $z_0 \in N(D)$, $N(D) \cap \{z_1, z_2\} \neq \emptyset$ and $N(D) \cap \{x, y\} \neq \emptyset$:*

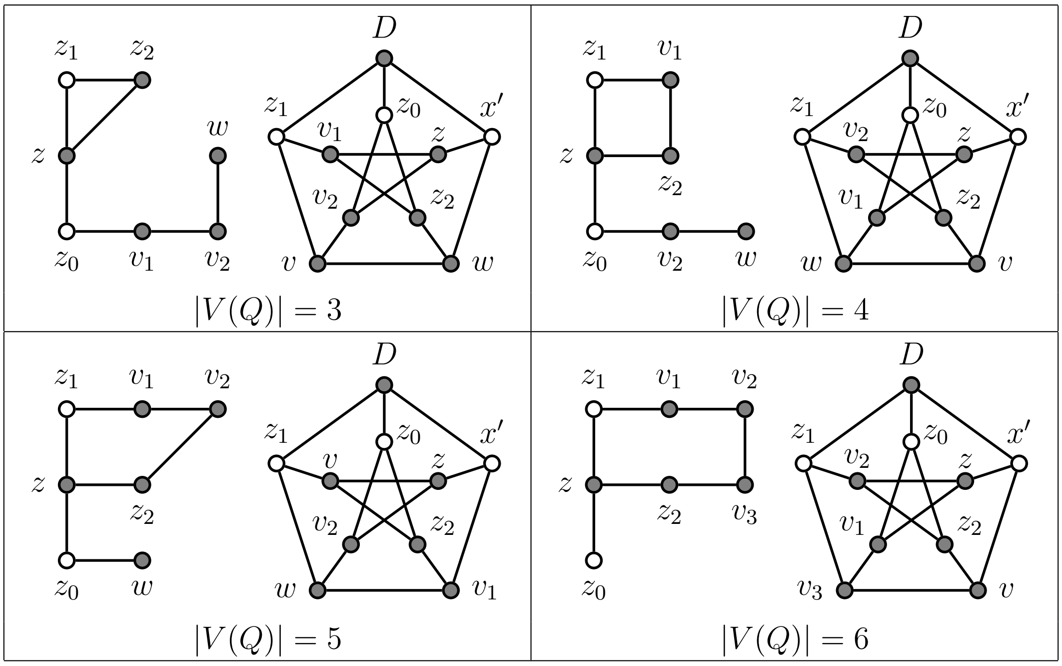
Recall that $|N_{G'}(z) \cap N_{G'}(v)| \leqslant 4$. Hence z has at least three non-neighbours in $G'[N(v)]$. Since $G'[N(v) \setminus \{x, y\}] \subseteq H'$ and x' is dominant in $G'[N_{G'}(v)]$, z is non-adjacent in G' to each vertex in $N_{H'}(z)$. Hence, for every vertex $z' \in N_{H'}[z]$ there is a component $C_{z'}$ of $G - N[v]$ such that $z' \in N(C_{z'})$, since $v \in \mathcal{L}$.

By Lemma 10.5, each component $C_{z'}$ of $G - N[v]$ satisfying $|N(C_{z'})| \leqslant 6$ is v-suitable.

Recall that C' is an arbitrary component of $G - N[v]$. We now show that, for some $z' \in N_{H_v}[z]$, $C_{z'}$ is v-suitable and $N(C_{z'}) \setminus N(C') \neq \emptyset$, as required.

Suppose first that there is no component D of $G-N[v]$ such that $z \in N(D)$ and $|N(D) \cap N_{H'}(z)| \geqslant 1$. Then $|N(C_z)| \leqslant 6$. Furthermore, $z \notin N(C_{z_0})$ and either $N(C_{z_0}) \cap \{z_1, z_2\} = \emptyset$ or $N(C_{z_0}) \cap \{x, y\} = \emptyset$ since Case 3 does not apply, so $|N(C_{z_0})| \leqslant 6$. Hence, C_z and C_{z_0}

Table 13



are both v-suitable. By assumption, $N(C)$ does not contain both z and z_0, so $z' \notin N(C)$ for some vertex $z' \in \{z, z_0\}$. Hence, $N(C_{z'}) \setminus N(C') \neq \emptyset$, and the claim holds.

Now assume that there is some component D of $G - N[v]$ such that $z \in N(D)$ and $|N(D) \cap N_{H'}(z)| \geqslant 1$. Since Case 2 does not apply, $|N(D) \cap N_{H'}(z)| = 1$. Let $\{z', z''\} := N_{H'}(z) \setminus N(D)$. If $|N(C_{z'})| \leqslant 6$ and $|N(C_{z''})| \leqslant 6$ (in which case $C_{z'}$ and $C_{z''}$ are both v-suitable), and $\{z', z''\} \nsubseteq N(C')$, then the claim holds. So we may assume that either $D' := C'$ satisfies $\{z', z''\} \subseteq N(D')$ or some $D' \in \{C_{z'}, C_{z''}\}$ satisfies $|N(D')| \geqslant 7$. Now D' is distinct from D since $N(D') \cap \{z', z''\} \neq \emptyset$, and $|N_{G'}(D')| \geqslant 3$ since $|N(D')| \geqslant 4$ by Lemma 10.3. Let G'' be obtained from G' by contracting D onto z. Then $v \in V_8(G'')$, $|N_{G''}(v) \cap N_{G''}(v')| \geqslant 5$ for every vertex $v' \in N_{G''}(v)$, and $|N_{G''}(D')| = |N_{G'}(D')| \geqslant 3$. Furthermore, there is at most one cycle in $\overline{G''[N(v)]}$, namely Q, so $\overline{K_3}$ and $\overline{C_4}$ are not both induced subgraphs of $G''[N(v)]$. Hence by Claim 10.11, $\mathcal{P} \subseteq G''$, contradicting (iv). $\square$

We finally reach the main result of this section.

Lemma 10.23. *If $v \in V_9(G) \cap \mathcal{L}$ and C is a component of $G - N[v]$, then there is some v-suitable subgraph C' such that $N(C') \setminus N(C) \neq \emptyset$.*

Proof. Suppose first that each component C' of $G - N[v]$ has $|N(C')| = 4$. Then every component of $G - N[v]$ is v-suitable by Lemma 10.5. Suppose for contradiction that there is no v-suitable subgraph C' such that $N(C') \setminus N(C) \neq \emptyset$. Then $N(C') \subseteq N(C)$ for every component C' of $G - N[v]$, so there are at least five vertices in $N(v)$ with no neighbour outside of $N[v]$. Since $v \in \mathcal{L}$, each of these vertices is dominant in $G[N[v]]$. Let G' be obtained from G by contracting C onto some vertex x of $N(C)$ and then deleting all other components of $G - N[v]$. There are at most three non-dominant vertices in G', so $|E(G')| \geqslant \binom{10}{2} - 3 = 42 = 5|V(G')| - 8$, contradicting (vi).

Suppose instead that there is some component C' of $G - N[v]$ with $|N(C')| \geqslant 5$. By Claims 10.20 and 10.21, we may assume that there are two vertices x and y in $V_3(H_v)$ such that $\mathrm{dist}_{H_v}(x, y) \geqslant 3$. The result then follows directly from Claim 10.22. $\square$

Lemma 10.23 immediately implies the following corollary, which we use in Section 2.2.6.

Corollary 10.24. *For every vertex $v \in V_9(G)$ there is at least one v-suitable subgraph.*

2.2.6 Final Step

We now complete the proof sketched at the start of this chapter.

Proof of Theorem 10. Let G be the minimum counterexample defined at the start of Section 2.2.1. By Lemmas 10.4, 10.10 and 10.12, $\mathcal{L} \subseteq V_6(G) \cup V_9(G)$, so for every vertex $v \in \mathcal{L}$ there is some v-suitable subgraph of G by Corollaries 10.16 and 10.24. Choose $v \in \mathcal{L}$ and H a v-suitable subgraph of G so that $|V(H)|$ is minimised. Let u be a vertex of $\mathcal{L}$ in H. Since $u \in V(H)$ and H is a component of $G - N[v]$, u is not adjacent to v, so v is in some component C of $G - N[u]$. Since $v \in \mathcal{L}$, C is u-suitable. By Lemmas 10.17 and 10.23, there is some u-suitable subgraph C' of G with $N(C') \setminus N(C) \neq \emptyset$.

Now $N(C') \subseteq N(u)$, so $v \notin N(C')$. Since $N(C') \setminus N(C) \neq \emptyset$, we have that C and C' are distinct (and thus disjoint), so $v \notin N[C']$ and C' is disjoint from $N[v]$. Hence $G[V(C') \cup (N(C') \setminus N(C)) \cup \{u\}]$ is a connected subgraph of $G - N[v]$, and thus a subgraph of H. But $u \in V(H) \setminus V(C')$, so $|V(C')| < |V(H)|$, contradicting our choice of v and H. This contradiction shows that in fact there are no counterexamples to Theorem 10. $\square$

2.3 Wrapping Up

We finish off this chapter by presenting two results relating to the function $\mathrm{ex_m}(n, \mathcal{P}, k)$ (which gives the maximum number of edges in a k-connected, n-vertex graph with no $\mathcal{P}$ minor), as well as a computational lemma and finally the proof of Theorem 8.

We first prove a sufficient condition for a graph to be $\mathcal{P}$-minor-free. For a graph G, we call a subset S of $V(G)$ *special* if $|S| \leqslant 3$, every component of $G - S$ has size at most 6, and every 6-vertex component of $G - S$ contains a vertex that is non-adjacent in G to every vertex in S.

Lemma 35. *If G is graph with a special set of vertices, then G is $\mathcal{P}$-minor-free.*

Proof. It is quick to check that the operations of deleting vertices, deleting edges and contracting edges preserve the property of having a special set, so every minor of G has a special set.

Suppose for contradiction that $\mathcal{P}$ has a special set S. Since $|S| \leqslant 3$, $|V(\mathcal{P} - S)| \geqslant 7$, so $\mathcal{P} - S$ is disconnected by the definition of a special set. It follows that $S = N(v)$ for some vertex v of $\mathcal{P}$, and so $\mathcal{P} - S$ has a 6-vertex component. However, the maximum distance between a pair of vertices in $\mathcal{P}$ is 2, so every vertex in this component is adjacent to some vertex in $N(v)$. This contradicts the definition of a special set, so $\mathcal{P}$ has no special set and hence is not a minor of G. $\square$

The following theorem relies on Theorem 10, and is used to prove Theorem 8.

Theorem 11. *If $n \geqslant 13$ and $n \notin \{16, 17, 22\}$, then $\mathrm{ex_m}(n, \mathcal{P}, 3) = 5n - 12$.*

Proof. The upper bound is an immediate corollary of Theorem 10, since a 3-connected subgraph of a $(K_9, 2)$-cockade has at most nine vertices.

If $n \geqslant 13$ and $n \notin \{16, 17, 22\}$, then there is at least one pair of non-negative integers c_5 and c_6 such that $n = 5c_5 + 6c_6 + 3$. To see this, first note that $n - 3 = 5k + t$, for some pair of non-negative integers k and t such that $t \leqslant 4$. Since $n = 5k + t + 3 = 5(k - t) + 6t + 3$ and $k - t \geqslant 0$ for the specified values of n, we may set $c_5 := k - t$ and $c_6 := t$.

Let G be the $(n - 3)$-vertex graph consisting of c_5 disjoint copies of K_5 and c_6 disjoint copies of K_6. Let G' be the n-vertex graph obtained from G by marking exactly one vertex in each K_6 component of G and adding a set S of three new vertices, each adjacent to every unmarked vertex of G and to the other vertices in S. By construction, S is a special set for G', so G' is $\mathcal{P}$-minor-free by Lemma 35. G' is 3-connected since every separating set of vertices in G' either contains every vertex in S or every unmarked vertex of some component of G. The number of edges in G' is

$$3 + 3(5(c_5 + c_6)) + |E(G)| = 3 + 15c_5 + 15c_6 + (10c_5 + 15c_6) = 5n - 12. \qquad \square$$

We now show that there are 5-connected $\mathcal{P}$-minor-free graphs with almost as many edges as $(K_9, 2)$-cockades.

Theorem 12. *For $n \geqslant 10$, $\mathrm{ex_m}(n, \mathcal{P}, 5) \geqslant 5n - 15$.*

Proof. Consider the class $\mathcal{C}$ of all graphs G with a vertex cover of size at most 5. C is minor-closed, and $\mathcal{P}$ is not in $\mathcal{C}$. Let $G := K_5 + \overline{K_{n-5}}$. Then G is 5-connected with $|E(G)| = 5n - 15$, and G is in $\mathcal{C}$ and thus is $\mathcal{P}$-minor-free. $\square$

Part (a) of the following computational lemma was used in the proof of Theorem 10. Parts (b) and (c) allow us to compute $\mathrm{ex_m}(11, \mathcal{P})$ and $\mathrm{ex_m}(12, \mathcal{P})$ respectively.

Lemma 36.

a) Every 10-vertex graph with 39 edges has a subgraph isomorphic to $\mathcal{P}$,

b) Every 11-vertex graph with 42 edges has a subgraph isomorphic to $\mathcal{P}$,

c) Every 12-vertex graph with 47 edges has a subgraph isomorphic to $\mathcal{P}$.

This lemma was verified by computer search. The program nauty [96] was used to generate all non-isomorphic 10-vertex graphs with 39 edges, 11-vertex graphs with 42 edges and 12-vertex graphs with 47 edges and minimum degree at least 6. A program written by Michael Gill was used to find $\mathcal{P}$-subgraphs in all graphs generated. Note that given (b), checking all 12-vertex graphs with 47 edges and minimum degree at least 6 suffices to prove (c), since a 12-vertex graph with 47 edges and minimum degree at most 5 contains an 11-vertex graph with 42 edges as a subgraph. The time taken to verify Lemma 36 was under half an hour.

We now determine the extremal function for Petersen minors.

Proof of Theorem 8.

Case 1. *If $n \leqslant 9$, then* $\mathrm{ex_m}(n, \mathcal{P}) = \binom{n}{2}$.

K_n is a $\mathcal{P}$-minor-free graph with $\binom{n}{2}$ edges, and there are no n-vertex graphs with more edges.

Case 2. *If $n \equiv 2 \pmod 7$, then* $\mathrm{ex_m}(n, \mathcal{P}) = 5n - 9$.

Let $n = 7t + 2$. Note that if $t \in \{0, 1\}$, then $\binom{n}{2} = 5n - 9$. If $t \geqslant 1$, then there is some $(K_9, 2)$-cockade of order n (for example the graph $K_2 + tK_7$). Since $|V(\mathcal{P})| > 9$, K_9 is $\mathcal{P}$-minor-free, and since $\mathcal{P}$ is 3-connected, every $(K_9, 2)$-cockade is $\mathcal{P}$-minor-free by Lemma 7. The number of edges in an n-vertex $(K_9, 2)$-cockade is $t\binom{9}{2} - (t - 1) = 35t + 1 = 5n - 9$. By Theorem 10, this is the extremal number of edges.

Case 3. *If $n \in \{11, 12\}$, then* $\mathrm{ex_m}(n, \mathcal{P}) = 5n - 14$.

The upper bound is given by Lemma 36.

The graph $K_2 + (K_7 \dot{\cup} K_2)$ is an 11-vertex graph with 41 edges, and since it is a subgraph of $(K_9, 2)$-cockade, it is $\mathcal{P}$-minor-free.

The graph $K_3 + (K_4 \dot{\cup} K_5)$ is a 12-vertex graph with 46 edges, and contains a special set, since $K_4 \dot{\cup} K_5$ has no component with more than five vertices. Hence, by Lemma 35, it is $\mathcal{P}$-minor-free.

Case 4. *If* $n \in \{10, 17, 22\}$, *then* $\mathrm{ex_m}(n, \mathcal{P}) = 5n - 12$.

The upper bound is given by Theorem 10, since there are no $(K_9, 2)$-cockades on 10, 17 or 22 vertices. The graph $H_1 := K_2 + (K_1 \dot{\cup} K_7)$ is a 10-vertex graph with 38 edges. The graph $H_2 := K_2 + (K_1 \dot{\cup} 2K_7)$ is a 17-vertex graph with 73 edges. The graph $H_3 := K_2 + (K_6 \dot{\cup} 2K_7)$ is a 22-vertex graph with 98 edges. Since $H_1 \subseteq H_2 \subseteq H_3$ and H_3 is a subgraph of a $(K_9, 2)$-cockade, all three graphs are $\mathcal{P}$-minor-free.

Case 5. *If* $n \geqslant 13$, $n \notin \{17, 22\}$ *and* $n \not\equiv 2 \pmod 7$, *then* $\mathrm{ex_m}(n, \mathcal{P}) = 5n - 12$.

By definition, $\mathrm{ex_m}(n, \mathcal{P}) \geqslant \mathrm{ex_m}(n, \mathcal{P}, 3)$, so the lower bound follows from Theorem 11. Since $n \not\equiv 2 \pmod 7$, there are no $(K_9, 2)$-cockades of order n, and the upper bound follows from Theorem 10. $\square$

Chapter 3

Sparse Minors

3.1 Overview

Recall that $f(H)$ is the infimum of all non-negative real numbers c such that every graph with average degree at least c contain H as a minor. The following theorem is the main result of this chapter.

Theorem 19. *For every graph H with exactly i isolated vertices and q edges,*

$$f(H) \leqslant i + 5.8105q.$$

Our proof closely follows the proof of a similar result by Reed and Wood [106]. It relies on the following result, which we prove in Section 3.3.

Theorem 37. *If G is an n-vertex graph with average degree at least $5.8105t$ and H is a 2-degenerate graph on at most t vertices, then H is a minor of G.*

This result in turn follows from the following lemmas, which we prove in Section 3.4 and Section 3.5 respectively.

Lemma 38 (§3.4)**.** *Let t and n be positive integers with $n \geqslant t$ and let α and β be positive real numbers. Let G be an n-vertex graph such that*

$$|E(G)| \geqslant \frac{\alpha^2(n^2 - t^2)}{2} + \left(\frac{\alpha^2}{2} + (\alpha + 1)\beta t - 1\right)(n - t) + \binom{t}{2}.$$

G has a minor G_0 such that either $G_0 \cong K_t$ or $\delta(G_0) \geqslant \alpha|V(G_0)| + \beta t - 1$.

Lemma 39 (§3.5)**.** *Let n be a positive integer and let G be an n-vertex graph with minimum degree δ and let H be a t-vertex, 2-degenerate graph. If $\delta(G) \geqslant 0.421344n + 0.735998t - 1$, then H is a topological minor of G.*

Lemma 39 is our major innovation, and it is this lemma that allows us to improve on the results of Reed and Wood.

3.2 Sparse Minors in Graphs of High Average Degree

The following lemma is well known, but we prove it here for completeness.

Lemma 40. *Let G be a graph with minimum degree δ and let T be a tree with $t \geqslant 2$ vertices. If $\delta \geqslant t - 1$, then $T \subseteq G$.*

Proof. We proceed by induction on t. If $t = 2$, the result is trivial.

Suppose $t \geqslant 3$. Let v be a leaf of T. By induction, $T - v$ is a subgraph of G, since $T - v$ is a $t - 1$ vertex tree and $\delta > t - 2$. This mean there is an injective function ϕ from $V(T)$ to $V(G)$ such that for every pair of adjacent vertices x and y in H, $\phi(x)$ and $\phi(y)$ are adjacent in G.

Let w be the neighbour of v in T. Now

$$\deg_G(\phi(w)) \geqslant \delta \geqslant t - 1 > |\phi(V(T - v)) \setminus \{\phi(w)\}|,$$

so $\phi(w)$ has some neighbour $v' \notin \phi(T - v)$. Setting $\phi(v) := v'$, we find $T \subseteq G$. $\square$

Proof of Theorem 19. We prove by on induction on $|V(H)| + |V(G)|$ that if G is a graph with average degree at least $5.8105q + i$ and H is a graph with exactly q edges and i isolated vertices, then H is a minor of G. When $|V(H)| = 0$ (and, in particular, when $|V(H)| + |V(G)| = 0$), the result is trivial.

Suppose that $i \geqslant 1$. Let v be an isolated vertex of H, let w be a vertex of minimum degree in G, and let $G' := G - w$. Now,

$$a(G') = \frac{2(|E(G)| - \delta(G))}{|V(G)| - 1} \geqslant \frac{a(G)n - 2a(G)}{n - 1} \geqslant a(G) - 1.$$

Thus, G' has average degree at least $5.8105q + i - 1$ and $H - v$ has q edges and $i - 1$ isolated vertices. By induction $H - v$ is a minor $G - w$, and hence H is a minor of G.

Suppose instead that $i = 0$ and some component C of H is a tree with at least two vertices. Let $c := |V(C)|$. If G has a vertex v of degree at most $\frac{1}{2}a(G)$, then $a(G - v) \geqslant a(G)$ and so H is a minor of $G - v$ by induction. Hence, we may assume that $\delta(G) \geqslant \frac{1}{2}a(G)$. Since $a(G) \geqslant 5.81128|E(C)|$, we have $\delta(G) \geqslant c - 1$. Thus, G has a subgraph T isomorphic to C by Lemma 40. Let $G' := G - T$.

$$a(G') \geqslant \frac{2(|E(G)| - n|V(T)|)}{n - |V(T)|} \geqslant \frac{a(G) - 2n|V(T)|}{n} = a(G) - 2|V(T)|.$$

Since C is a tree with at least one edge, $H - C$ is a graph with at most $q - \frac{c}{2}$ edges (and no isolated vertices by assumption). Since G' has average degree at least $a(G) - 2c \geqslant 5.81128q - 2c > 5.81128(q - \frac{c}{2})$, $H - C$ is a minor of G' by induction, and hence H is a minor of G

Now suppose that no component of H is a tree. For every component C of H, let H_C be a graph obtained from a spanning subtree of H by adding one edge of $H[C]$ that is not already in the subtree. Since this graph is one edge away from being 1-degenerate, it is 2-degenerate. Let H_0 be the union of the subgraphs H_C of all components C of H. Since every component of H_0 is 2-degenerate, H_0 is 2-degenerate. Let H' be obtained from H by subdividing every edge not in H_0. Then H' is a 2-degenerate graph with $t + (q - |E(H_0)|)$ vertices, and by our choice of H_0, $t + (q - |E(H_0)|) = q$. Hence, H' is a minor of G by Theorem 37, and so H is a minor of G. $\square$

3.3 2-Degenerate Minors in Graphs of High Average Degree

We now prove Theorem 37 from Lemma 38 and Lemma 39. Theorem 37 gives the best known upper bound for the extremal function for general 2-degenerate minors.

Theorem 37. *If G is an n-vertex graph with average degree at least $5.8105t$ and H is a 2-degenerate graph on at most t vertices, then H is a minor of G.*

Proof. We proceed by induction on n. If $n \leqslant 5.8105t$ then no n-vertex graph has average degree $5.8105t$, so the result holds vacuously.

Now suppose that $n > 5.8105t$ and G is a graph with average degree at least $5.8105t$. Let v be a vertex of minimum degree in G, and note that $\deg(v) \leqslant 5.8105t$. If v is an isolated vertex, then $G - v$ has average degree at least $5.8105t$, and the result holds by induction. Let $G' := G[N[v]]$, let $n' := |V(G')| = \delta(G) + 1$ and let $\delta' := \delta(G')$. Let w be a vertex distinct from v with degree δ' in G'. If $\delta' \leqslant \frac{1}{2}(5.8105t)$, then G/vw has average degree

$$\frac{2|E(G/vw)|}{n-1} = \frac{2(|E(G)| - \delta')}{n-1} \geqslant \frac{5.8105tn - 5.8105t}{n-1} = 5.8105t,$$

and the result holds by induction.

Now suppose $\delta' > 2.90525t$. Note that $n' > \delta' > 2.90525t$. Also, since $\delta(G) \leqslant 5.8105t$, $5.8105t + 1 \geqslant n'$ and $-(n't - t)/5.8105 \geqslant -t^2$. Let $\alpha := 0.421344$ and let $\beta := 0.735998$. Now

$$\begin{aligned}
|E(G')| &\geqslant \frac{1}{2}(2.90525tn') \\
&\geqslant \left(\frac{\alpha^2}{2}(5.8105) + (\alpha+1)\beta - \frac{0.63487}{5.8105}\right) tn' \\
&\geqslant \frac{\alpha^2}{2}(5.8105t + 1)n' + (\alpha+1)\beta n't - 0.63487\left(\frac{n't - t}{5.8105}\right) - \frac{\alpha^2 n}{2} - \frac{0.63487t}{5.8105} \\
&\geqslant \frac{\alpha^2 n'^2}{2} + (\alpha+1)\beta n't - (\frac{\alpha^2}{2} + (\alpha+1)\beta - \frac{1}{2})t^2 - \frac{\alpha^2 n'}{2} - 0.11t \\
&\geqslant \frac{\alpha^2(n'^2 - t^2)}{2} + (\alpha+1)\beta t(n' - t) + \frac{t^2}{2} - \frac{\alpha^2 n'}{2} - 0.11t + (1-\alpha^2)(2.90525t - n') \\
&\geqslant \frac{\alpha^2(n'^2 - t^2)}{2} + \left(\frac{\alpha^2}{2} + (\alpha+1)\beta t - 1\right)(n' - t) + \binom{t}{2}.
\end{aligned}$$

Hence, by Lemma 38, G' has a minor G_0 such that either $G_0 \cong K_t$ or $\delta(G_0) \geqslant \alpha|V(G_0)| + \beta t - 1$. If $G_0 \cong K_t$, then $H \subseteq G_0$ since $|V(H)| = t$, so H is a minor of G. If $\delta(G_0) \geqslant \alpha|V(G_0)| + \beta t - 1$ then the result then follows from Lemma 39. □

3.4 Minors of High Minimum Degree

The following lemma allows us to find a minor with high minimum degree (at least some linear function of the number of vertices), in a graph of high average degree, as per the proof of Theorem 37.

Lemma 38. *Let t and n be positive integers with $n \geqslant t$ and let α and β be positive real numbers. Let G be an n-vertex graph such that*

$$|E(G)| \geqslant \frac{\alpha^2(n^2 - t^2)}{2} + \left(\frac{\alpha^2}{2} + (\alpha+1)\beta t - 1\right)(n - t) + \binom{t}{2}.$$

G has a minor G_0 such that either $G_0 \cong K_t$ or $\delta(G_0) \geqslant \alpha|V(G_0)| + \beta t - 1$.

Proof. We proceed by induction on n. If $n = t$, then $|E(G)| \geqslant \binom{t}{2}$. Hence $K_t \cong G$, and we are done.

Now suppose $n > t$. Let v be a vertex of degree $\delta(G)$. If $\delta(G) = 0$, then $|E(G - v)| = |E(G)|$, and so by induction $G - v$ has a minor G_0 such that either $G_0 \cong K_t$ or $\delta(G_0) \geqslant \alpha|V(G_0)| + \beta t - 1$. Since $G - v$ is a minor of G, G_0 is a minor of G.

If $\delta(G) \geqslant 1$, let $d := \delta(G[N[v]])$ and let w be a vertex distinct from v with degree d in $G[N(v)]$ (note that if $\deg(v) = d$, then $G[N[v]] \cong K_{d+1}$). Let $G' := G/vw$. Now

$$\begin{aligned}
|E(G')| &= |E(G)| - d \\
&\geqslant \frac{\alpha^2(n^2 - t^2)}{2} + \left(\frac{\alpha^2}{2} + (\alpha+1)\beta t - 1\right)(n-t) + \binom{t}{2} - d \\
&= \frac{\alpha^2((n-1)^2 - t^2)}{2} + \left(\frac{\alpha^2}{2} + (\alpha+1)\beta t - 1\right)(n-1-t) + \binom{t}{2} \\
&\quad + (\alpha^2 n + (\alpha+1)\beta t - 1) - d.
\end{aligned}$$

If $d \leqslant \alpha^2 n + (\alpha+1)\beta t - 1$, then by induction G' (and hence G) has a minor G_0 such that either $G_0 \cong K_t$ or $\delta(G_0) \geqslant \alpha|V(G_0)| + \beta t - 1$.

Now suppose $d > \alpha^2 n + (\alpha+1)\beta t - 1$, and let $G_0 := G[N[v]]$. Recall that $\deg(v) = \delta(G)$. We may assume $\delta(G) < \alpha n + \beta t - 1$ since G is a minor of itself. Hence, $|V(G_0)| < \alpha n + \beta t$, and so

$$\delta(G_0) = d \geqslant \alpha^2 n + (\alpha+1)\beta t - 1 \geqslant \alpha|V(G_0)| + \beta t - 1. \qquad \square$$

3.5 2-Degenerate Minors in Graphs of High Minimum Degree

In this section we prove Lemma 39, which is a major component of the proof of Theorem 37 and hence of Theorem 19. This lemma is the main innovation of this chapter, and is what enables us to improve on the results of Reed and Wood [106].

If a graph H' is a *subdivision* of a graph H then every edge e of H will be represented by a path in H'. We define the *depth* of e in H' to be the number of internal (subdivision) vertices in the path representing e (so every non-subdivided edge has depth 0).

Lemma 39. *Let n be a positive integer and let G be an n-vertex graph with minimum degree δ and let H be a t-vertex, 2-degenerate graph. If $\delta(G) \geqslant 0.421344n + 0.735998t - 1$, then H is a topological minor of G.*

To prove Lemma 39, suppose for contradiction that $\delta \geqslant 0.421344n + 0.735998t - 1$ and H is an edge-maximal, 2-degenerate, t-vertex graph that is not a topological minor of G. Note that every 2-degenerate graph is a subgraph of an edge-maximal 2-degenerate graph, and that if G contains H as a topological minor then G contains every subgraph of H as a topological minor.

Define $H_0, H_1, H_2, \dots, H_t$ to be induced subgraphs of H and $(v_1, v_2, \dots, v_t)$ an ordering of $V(H)$ so that $H_t = H$, and for $i \in [t]$ the vertex v_i has degree at most 2 in H_i and $H_{i-1} = H_i - v_i$. Thus, H_0 is the empty graph. Note that such an ordering is possible since H is 2-degenerate. Also, since H is edge-maximal, for $i \geqslant 3$ v_i has degree exactly 2 in H_i, or else adding an edge from v_i to either v_1 or v_2 will produce a 2-degenerate graph with more edges.

If H' is a subdivision of H_i and $j \in [i]$, the *depth* of v_j in H' is the sum of the depths of the edges incident to v_j in H_j. The *depth* of H' is the maximum depth of a vertex of H_i in H'. Let $\mathcal{H}$ be the union over $i \in [t]$ of the sets of all subdivisions of H_i.

For $i \in [t]$ and $H' \in \mathcal{H}$ a subdivision of H_i and $j \in [i]$, define H'_j to be the subdivision of H_j given by the combination of vertices and paths of H' corresponding to the vertices and edges of H_j. For every non-negative integer k, define $d_k(H')$ to be the maximum $j \in [i]$ such that H'_j has depth at most k. Fix $H' \in \mathcal{H}$ to be a subdivision which lexicographically maximises $(d_0(H'), d_1(H'), d_2(H'), \dots)$, subject to the restriction that G has a subgraph isomorphic to H'.

Claim 39.1. *For every subgraph G' of G with at least one vertex,*

$$d_0(H') \geqslant 2(\delta(G') + 1) - |V(G')|.$$

Proof. By our choice of H', $d_0(H')$ is the maximum $i \in [t]$ such that H_i is a subgraph of G. We prove by induction on i that, if $\delta(G') \geqslant \frac{1}{2}(|V(G')| + i - 2)$, then H_i is a subgraph of G'. The result is trivial for $i = 1$, since G' has at least one vertex, and also for $i = 2$, since $\delta(G') \geqslant \frac{1}{2}(|V(G')| + i - 2) > 0$.

Now suppose $i \geqslant 3$. By induction, $H_{i-1} \subseteq G'$, meaning there is an injective function ϕ from $V(H_{i-1})$ to $V(G)$ such that for every pair of adjacent vertices x and y in H_{i-1}, $\phi(x)$ and $\phi(y)$ are adjacent in G'.

Since $i \geqslant 3$ and H is an edge-maximum 2-degenerate graph, v_i has exactly two neighbours v_a and v_b in H_i. Now $|N(\phi(v_a)) \cap N(\phi(v_b))| \geqslant 2\delta(G') - |V(G')| \geqslant i - 2$. In particular, since $\{\phi(v_a), \phi(v_b)\} \cap (N(\phi(v_a)) \cap N(\phi(v_b))) = \emptyset$ and $|\phi(V(H_{i-1})) \setminus \{\phi(v_a), \phi(v_b)\}| = t - 3$, there is some common neighbour v' of $\phi(v_a)$ and $\phi(v_a)$ that is not in $\phi(V(H_{i-1}))$. Setting $\phi(v_i) := v'$, we find $H_i \subseteq G'$. Hence, if $\delta(G') \geqslant \frac{1}{2}(|V(G')| + i - 2)$, then H_i is a subgraph of G, which means $d_0(H') \geqslant 2(\delta(G') + 1) - |V(G')|$. □

The next claim is the basis for the proof of Lemma 39. It provides a recursive formula for $d_k(H')$, the number of consecutive vertices, starting from v_1, that have depth at most k in H'. The proof of this claim rests on the assumption that no subdivision of H is isomorphic to a subgraph of G. Using Claim 39.2, we prove Claim 39.3. This allows us to show that $d_n(H') \geqslant t$, which means H' is a subdivision of H that is a subgraph of G, a contradiction.

Claim 39.2. *For every positive integer k,*

$$d_k(H') \geqslant \frac{1}{k+1}\left(\frac{(5 \cdot 2^k - 1)(\delta + 1) - (2^{k+1})n}{2^{k+1} - 1} + \sum_{i=1}^{k-1} d_i(H')\right).$$

Proof. Let $m := d_k(H')$. By the definition of H', G has some subgraph isomorphic to H', so there is an injective function ϕ from $V(H'_m)$ to $V(G)$ such that for every pair of adjacent vertices x and y in H'_m, $\phi(x)$ and $\phi(y)$ are adjacent in G. By assumption, $n \geqslant 1$ and since H is not a topological minor of G, $t \geqslant 1$. Hence, $\delta > 0.4 + 0.7 - 1 > 0$, so G has a subgraph isomorphic to K_2, and hence $d_0(H') \geqslant 2$ by our choice of H'. By definition, $d_k(H') \geqslant d_0(H')$, so $m \geqslant 2$.

Since H is not a topological minor of G, $m \leqslant t - 1$. Since H is an edge-maximal 2-degenerate graph, v_{m+1} has exactly two neighbours s and t in H_{m+1}. Let $s' := \phi(s)$, let $t' := \phi(t)$ and let $U := \phi(V(H'_m))$.

If i is a non-negative integer and $l \in [d_i(H')]$, then we have $|V(H'_l) \setminus V(H'_{l-1})| \leqslant i + 1$, since H'_l is a subdivision of depth at most i. Hence,

$$|U| = |V(H'_m)| \leqslant d_0(H') + \sum_{i=1}^{k}(i+1)(d_i(H') - d_{i-1}(H')) = (k+1)m - \sum_{i=0}^{k-1} d_i(H'). \quad (3.1)$$

If H'' is a graph obtained from H'_m by adding a path (internally disjoint from H'_m) with at least one internal vertex and at most $k + 1$ internal vertices, then H'' is isomorphic to subdivision of H_{m+1} of depth at most k. By our choice of H', H'' is not isomorphic to a subgraph of G. Hence there is no path P from s' to t' in G such that P is internally disjoint from U and $3 \leqslant |V(P)| \leqslant k + 3$.

Let $T := N(t') \setminus U$, let $S_0 := N(s') \setminus U$ and for $i \in [k-1]$, let $S_i := N[S_{i-1}] \setminus U$. Let $S_k := V(G) \setminus (U \cup T)$, and for $i \in [k]$, let $S'_i := S_i \setminus S_{i-1}$. Suppose for contradiction that some vertex in S_{k-1} is adjacent to some vertex in $T \cup \{t'\}$. Then there is a path P from s' to t' in G of length at most $k+2$, with at least one internal vertex and with no internal vertex in U, a contradiction. Hence, for $i \in \{0, 1, \ldots, k-1\}$, we have $N(S_i) \subseteq (U \cup S'_{i+1}) \setminus T$. Let j be the maximum value in $[k]$ such that $|S'_j| \leqslant 2^{k-j}|S_k \setminus S_0|/(2^k - 1)$. The existence of j is guaranteed, since otherwise

$$|S_k \setminus S_0| = \sum_{i=1}^{k} |S'_k| > \sum_{i=1}^{k} \frac{2^{k-i}}{2^k - 1}|S_k \setminus S_0|,$$

a contradiction.

Let $G' := G[S_{j-1}]$, $n' := |S_{j-1}|$. Suppose that $n' = 0$. In particular, $|S_0| = 0$. This means $\delta + 1 - |U| \leqslant 0$, since $|S_0| = |N[s'] \setminus U|$. Hence, by (3.1),

$$0 \geqslant \delta + 1 - \left((k+1)m - \sum_{i=0}^{k-1} d_i(H') \right).$$

By Claim 39.1, $d_0(H') \geqslant 2(\delta + 1) - n$. Hence,

$$\begin{aligned} m &\geqslant \frac{1}{k+1} \left(3(\delta+1) - n + \sum_{i=1}^{k-1} d_i(H') \right) \\ &= \frac{1}{k+1} \left(\frac{(6 \cdot 2^k - 3)(\delta+1) - (2^{k+1} - 1)n}{2^{k+1} - 1} + \sum_{i=1}^{k-1} d_i(H') \right) \\ &\geqslant \frac{1}{k+1} \left(\frac{(5 \cdot 2^k - 1)(\delta+1) - (2^{k+1})n}{2^{k+1} - 1} + \sum_{i=1}^{k-1} d_i(H') \right), \end{aligned}$$

as required.

Suppose instead that $n' \geqslant 0$, and let $\delta' := \delta(G')$. By Lemma 39.1, $d_0(H') \geqslant 2\delta' + 2 - n' \geqslant 2(\delta - |U| - |S'_j|) + 2 - |S_{j-1}|$. By our choice of j,

$$\begin{aligned} 2|S'_j| + |S_{j-1}| &\leqslant (|S_k| - \sum_{i=j+1}^{k} |S'_i|) + |S'_j| \leqslant |S_0| + |S_k \setminus S_0|(\frac{2^{k-j}}{2^k - 1} + 1 - \sum_{i=j+1}^{k} \frac{2^{k-i}}{2^k - 1}) \\ &= |S_0| + (\frac{2^k}{2^k - 1})|S_k \setminus S_0|. \end{aligned}$$

Note that $|S_k \setminus S_0| = n - |S_0| - |T| - |U|$ and $|S_0| = |N[s'] \setminus U| \geqslant \delta + 1 - |U|$. Similarly, $|T| \geqslant \delta + 1 - |U|$. Hence, by (3.1),

$$\begin{aligned} 0 &\geqslant -d_0(H') + 2(\delta - |U| - |S'_j|) + 2 - |S_{j-1}| \\ &\geqslant -d_0(H') + 2\delta + 2 - 2|U| - (|S_0| + (\frac{2^k}{2^k - 1})|S_k \setminus S_0|) \\ &\geqslant -d_0(H') + 2\delta + 2 - 2|U| - |S_0| - (\frac{2^k}{2^k - 1})(n - |S_0| - |T| - |U|) \\ &\geqslant -d_0(H') + 2\delta + 2 - 2|U| - (\frac{2^k}{2^k - 1})(n - |U|) + \frac{2^k + 1}{2^k - 1}(\delta + 1 - |U|) \\ &\geqslant -d_0(H') + \frac{3 \cdot 2^k - 1}{2^k - 1}(\delta + 1) - (\frac{2^k}{2^k - 1})n - \frac{2^{k+1} - 1}{2^k - 1} \left((k+1)m - \sum_{i=0}^{k-1} d_i(H') \right) \end{aligned}$$

$$\geqslant \frac{3\cdot 2^k-1}{2^k-1}(\delta+1)-(\frac{2^k}{2^k-1})n-\frac{2^{k+1}-1}{2^k-1}\left((k+1)m-\sum_{i=1}^{k-1}d_i(H')\right)+\frac{2^k}{2^k-1}(2\delta+2-n)$$
$$\geqslant \frac{5\cdot 2^k-1}{2^k-1}(\delta+1)-(\frac{2^{k+1}}{2^k-1})n+\frac{2^{k+1}-1}{2^k-1}\left(\sum_{i=1}^{k-1}d_i(H')\right)-\frac{2^{k+1}-1}{2^k-1}(k+1)d_k(H'),$$

from which the result follows. □

We now convert the recursive formula in Claim 39.2 into a form which is easier to use.

Claim 39.3.

$$d_k(H')\geqslant \frac{(5\cdot 2^k-1)(\delta+1)-(2^{k+1})n}{(k+1)(2^{k+1}-1)}+\sum_{i=1}^{k-1}\frac{(5\cdot 2^i-1)(\delta+1)-(2^{i+1})n}{(i+1)(i+2)(2^{i+1}-1)}$$

Proof. For $j\geqslant 1$, define

$$a_j := \frac{(5\cdot 2^j-1)(\delta+1)-(2^{j+1})n}{2^{j+1}-1}.$$

By Claim 39.2, we have

$$d_k(H')\geqslant \frac{1}{k+1}\left(a_k+\sum_{i=1}^{k-1}d_i(H')\right). \tag{3.2}$$

We now prove by induction on i that

$$d_i(H')\geqslant \frac{a_i}{i+1}+\left(\sum_{j+1}^{i-1}\frac{a_j}{(j+1)(j+2)}\right). \tag{3.3}$$

When $i=1$, (3.3) follows immediately from (3.2). Now suppose (3.3) holds for all $i<k$. Hence, by (3.2), we have

$$\begin{aligned}
d_k(H') &\geqslant \frac{a_k}{k+1}+\frac{1}{k+1}\sum_{i=1}^{k-1}\left(\frac{a_i}{i+1}+\sum_{j=1}^{i-1}\frac{a_j}{(j+1)(j+2)}\right)\\
&= \frac{a_k}{k+1}+\frac{1}{k+1}\left(\left(\sum_{i=1}^{k-1}\frac{a_i}{i+1}\right)+\sum_{j=1}^{k-2}\sum_{i=j+1}^{k-1}\frac{a_j}{(j+1)(j+2)}\right)\\
&= \frac{a_k}{k+1}+\frac{1}{k+1}\left(\left(\sum_{i=1}^{k-1}\frac{a_i}{i+1}\right)+\sum_{j=1}^{k-2}\frac{a_j(k-1-j)}{(j+1)(j+2)}\right)\\
&= \frac{a_k}{k+1}+\frac{1}{k+1}\left(\frac{a_{k-1}}{k}+\sum_{i+1}^{k-2}\frac{a_i(i+2)+a_i(k-1-i)}{(i+1)(i+2)}\right)\\
&= \frac{a_k}{k+1}+\sum_{i+1}^{k-1}\frac{a_i}{(i+1)(i+2)}.
\end{aligned}$$

Thus the result holds by induction. □

We are now ready to prove Lemma 39. Recall that we are assuming for contradiction that $\delta\geqslant 0.421344n+0.735998t-1$ and that H is not a topological minor of G. By our choice of H', this means that $d_k(H')\leqslant t-1$ for every non-negative integer k.

Proof of Lemma 39. First, note that H' is a subdivision of depth less than n, since G has a subgraph isomorphic to H' and hence H' has at most n vertices. This means that for $k \geqslant n$, we have $d_k(H') = d_n(H')$, since $|V(H')| \leqslant |V(G)| = n$. Thus, by Claim 39.3, we have

$$\begin{aligned}
d_n(H') &\geqslant \lim_{k\to\infty} \frac{(5\cdot 2^k - 1)(\delta+1) - (2^{k+1})n}{(k+1)(2^{k+1}-1)} + \sum_{i=1}^{k-1} \frac{(5\cdot 2^i - 1)(\delta+1) - (2^{i+1})n}{(i+1)(i+2)(2^{i+1}-1)} \\
&> \left(\sum_{i=1}^{100} \frac{(5\cdot 2^i - 1)(\delta+1) - (2^{i+1})n}{(i+1)(i+2)(2^{i+1}-1)}\right) + \left(\frac{5(\delta+1)}{2} - \frac{2^{101}n}{2^{101}-1}\right) \sum_{i=101}^{\infty} \frac{1}{(i+1)(i+2)} \\
&= \left(\sum_{i=1}^{100} \frac{(5\cdot 2^i - 1)(\delta+1) - (2^{i+1})n}{(i+1)(i+2)(2^{i+1}-1)}\right) + \left(\frac{5(\delta+1)}{2} - \frac{2^{101}n}{2^{101}-1}\right) \frac{1}{102} \\
&> 1.3587(\delta+1) - 0.57248n.
\end{aligned}$$

By definition, $d_n(H') \leqslant |V(H)|$, so $t > 1.3587(\delta+1) - 0.57248n$. This is a contradiction, since $\delta \geqslant 0.421344n + 0.735998t - 1 > \frac{1}{1.3587}(t + 0.57248n) - 1$ by assumption. $\square$

Chapter 4

Improper Colourings

4.1 Defective Choosability and Maximum Average Degree

In this section we prove the following theorem, via a stronger result (Theorem 29 below).

Theorem 24. *For $d \geqslant 0$ and $k \geqslant 1$, every graph G with $\text{mad}(G) < \frac{2d+2}{d+2}k$ is k-choosable with defect d.*

The following lemma is essentially a special case of an old result of Lovász [92].

Lemma 41. *If L is a list-assignment for a graph G, such that*

$$\deg_G(v) + 1 \leqslant |L(v)|(d+1)$$

for each vertex v of G, then G is L-colourable with defect d.

Proof. Colour each vertex v in G by a colour in $L(v)$ so that the number of monochromatic edges is minimised. Suppose that some vertex v coloured α is adjacent to at least $d+1$ vertices also coloured α. Since $\deg(v) < |L(v)|(d+1)$, some colour $\beta \in L(v) \setminus \{\alpha\}$ is assigned to at most d neighbours of v. Recolouring v by β reduces the number of monochromatic edges. This contradiction shows that no vertex v is adjacent to at least $d+1$ vertices of the same colour as v. Thus the colouring has defect d. □

Corollary 42. *Every graph G with $\Delta(G) + 1 \leqslant k(d+1)$ is k-choosable with defect d.*

The next lemma is a key idea of this chapter. It provides a sufficient condition for a partial list-colouring to be extended to a list-colouring of the whole graph.

Lemma 43. *Let L be a k-list-assignment of a graph G. Let A, B be a partition of $V(G)$, where $G[A]$ is L-colourable with defect d'. If $d \leqslant d'$ and for every vertex $v \in B$,*

$$(d+1)\deg_A(v) + \deg_B(v) + 1 \leqslant (d+1)k,$$

then G is L-colourable with defect d'.

Proof. Let ϕ be an L-colouring of $G[A]$ with defect d'. For each vertex $v \in B$, let $L'(v) := L(v) \setminus \{\phi(x) : x \in N_A(v\}$. Thus $|L'(v)| \geqslant k - \deg_A(v) \geqslant (\deg_B(v)+1)/(d+1)$. Lemma 41 implies that $G[B]$ is L-colourable with defect d. By construction, there is no monochromatic edge between A and B. Thus G is L-colourable with defect d'. □

We now prove our first main result, which is equivalent to Theorem 24 when $n_0 = 1$.

Theorem 29. *For integers $d \geqslant 0$, $n_0 \geqslant 1$ and $k \geqslant 1$, every graph G with* $\operatorname{mad}(G, n_0) < \frac{2d+2}{d+2}\, k$ *is k-choosable with defect $d' := \max\{\lceil \frac{n_0-1}{k} \rceil - 1, d\}$.*

Proof. We proceed by induction on $|V(G)|$. Let L be a k-list-assignment for G. For the base case, suppose that $|V(G)| \leqslant n_0 - 1$. For each vertex v of G, choose a colour in $L(v)$ so that each colour is used at most $\lceil \frac{|V(G)|}{k} \rceil$ times. We obtain an L-colouring with defect $\lceil \frac{n_0-1}{k} \rceil - 1$. Now assume that $|V(G)| \geqslant n_0$.

Let $v_1, \dots, v_p$ be a maximal sequence of vertices in G, such that if $A_i := \{v_1, \dots, v_{i-1}\}$ and $B_i := V(G) \setminus A_i$, then $(d+1) \deg_{A_i}(v_i) + \deg_{B_i}(v_i) \geqslant (d+1)k$.

First suppose that $p < |V(G)|$. Let $A := \{v_1, \dots, v_p\}$ and $B := V(G) \setminus A$. By induction, $G[A]$ is L-colourable with defect d'. By the maximality of $v_1, \dots, v_p$, for every vertex $v \in B$, we have $(d+1) \deg_A(v) + \deg_B(v) + 1 \leqslant (d+1)k$. By Lemma 43, G is L-colourable with defect d', and we are done.

Now assume that $p = |V(G)|$. Thus, each vertex v_i satisfies $d \deg_{A_i}(v_i) + \deg_G(v_i) \geqslant (d+1)k$. Hence

$$(d+2)|E(G)| = \sum_{i=1}^{|V(G)|} d \deg_{A_i}(v_i) + \deg_G(v_i) \geqslant (d+1)k|V(G)|.$$

Since $|V(G)| \geqslant n_0$, we have $\operatorname{mad}(G, n_0) \geqslant \frac{2|E(G)|}{|V(G)|} \geqslant \frac{2d+2}{d+2}\, k$, which is a contradiction. □

4.2 Using Independent Transversals

This section introduces a useful tool, called “independent transversals”, which have been previously used for clustered colouring by Alon et al. [5] and Haxell, Szabó, and Tardos [63]. Haxell [64] proved the following result.

Lemma 44 ([64])**.** *Let G be a graph with maximum degree at most Δ. Let $V_1, \dots, V_n$ be a partition of $V(G)$, with $|V_i| \geqslant 2\Delta$ for each $i \in [n]$. Then G has a stable set $\{v_1, \dots, v_n\}$ with $v_i \in V_i$ for each $i \in [n]$.*

Lemma 45. *Let $\Delta \geqslant 3$ and let G be a graph of maximum degree at most Δ. If H is a subgraph of G with $\Delta(H) \leqslant 2$, then there is a stable set $S \subseteq V(H)$ of vertices of degree 2 in H with the following properties:*

1. *every subpath of H with at least $3\Delta - 6$ vertices that contains a vertex with degree 1 in H contains at least one vertex in S,*
2. *every subpath of H with at least $5\Delta - 9$ vertices that contains a vertex with degree 1 in H contains at least two vertices in S,*
3. *every connected subgraph C of H with at least $\lceil \frac{19}{2}\Delta \rceil - 16$ vertices contains at least three vertices in S.*

Proof. Consider each cycle component C of H with $|C| \geqslant 8\Delta - 12$. Say $|C| = (2\Delta - 3)a + b$, where $a \geqslant 4$ and $b \in [0, 2\Delta - 4]$. Partition C into subpaths $A_1B_1A_2B_2 \dots A_aB_a$ where $|A_i| = 2\Delta - 4$ and $|B_i| \in [1, 1 + \lceil \frac{b}{a} \rceil]$ for $i \in [a]$. Note that $|B_i| \leqslant 1 + \lceil \frac{b}{a} \rceil \leqslant \lceil \frac{1}{2}\Delta \rceil$.

Now consider each path component P of H with $|P| \geqslant 2\Delta - 4$. Say $|P| = (2\Delta - 3)a + b - 1$, where $a \geqslant 1$ and $b \in [0, 2\Delta - 4]$. Partition P into subpaths $B_0A_1B_1 \dots A_aB_a$ where $|A_i| = 2\Delta - 4$ for $i \in [a]$, $|B_i| = 1$ for $i \in [a-1]$, and $|B_i| \leqslant \lceil \frac{b}{2} \rceil$.

Let $\mathcal{A}$ be the set of all such paths A_i taken over all the components of H. Let $G' := G[\bigcup_{A \in \mathcal{A}} V(A)] - E(H)$. Then $\mathcal{A}$ gives a partition of $V(G')$ into paths, each of which has exactly $2\Delta - 4$ vertices, and $\Delta(G') \leqslant \Delta - 2$. By Lemma 44, G' has a stable set S that contains exactly one vertex in each path in $\mathcal{A}$. By construction, every vertex in S has degree 2 in H and S is a stable set in H, so S is a stable set in G.

Let P be a path in H that contains a vertex of degree 1 in H. Then H is subpath of some component path P' of H. If P contains at least $3\Delta - 6$ vertices, then $|P| = (2\Delta - 3)a + b - 1$ where $a \geqslant 1$ and $b \in [0, 2\Delta - 4]$. Now, using our previous notation, $|B_0A_1| \leqslant \Delta - 2 + 2\Delta - 4 = 3\Delta - 6 \leqslant |P|$ and $|A_aB_a| \leqslant \Delta - 2 + 2\Delta - 4 = 3\Delta - 6 \leqslant |P|$, so P is not a proper subpath of B_0A_1 or of B_aA_a. Hence P contains every vertex of A_i for some $i \in \{1, a\}$, so P contains a vertex in S.

If P contains at least $5\Delta - 9$ vertices, then $|P'| = (2\Delta - 3)a + b - 1$ where $a \geqslant 2$ and $b \in [0, 2\Delta - 4]$. Now, $|B_0A_1B_1A_2| \leqslant \Delta - 2 + 2(2\Delta - 4) + 1 = 5\Delta - 9 \leqslant |P|$ and $|A_{a-1}B_{a-1}A_aB_a| \leqslant 5\Delta - 9 \leqslant |P|$, so P is not a proper subpath of $B_0A_1B_1A_2$ or of $A_{a-1}B_{a-1}A_aB_a$. Hence P contains every vertex A_i and of A_{i+1} for some $i \in \{1, a-1\}$, so P contains two vertices in S.

Suppose for contradiction there is a connected subgraph C of H on $\lceil \frac{19}{2}\Delta \rceil - 16$ vertices with at most two vertices in S. By the definition of S, there are at most two paths $A_i \in \mathcal{A}$ with $V(A_i) \subseteq V(C)$. If C is contained in some path component of H, then C is a proper subpath of $A_jB_jA_{j+1}B_{j+1}A_{j+2}B_{j+2}A_{j+3}$ for some $j \in \{0, \dots, a-3\}$, where we take A_0 and A_{a+1} to be the empty path for simplicity (so $|A_0B_0| = |B_0| \leqslant \Delta - 2$ and $|B_aA_{a+1}| = |B_a| \leqslant \Delta - 2$). Now $|A_jB_jA_{j+1}B_{j+1}A_{j+2}B_{j+2}A_{j+3}| \leqslant 4(2\Delta - 4) + 3 \leqslant \lceil \frac{19}{2}\Delta \rceil - 16$.

If C is contained in some cycle component of H, we may assume without loss of generality that C is a subpath of the path $A_1B_1A_2B_2A_3B_3A_4$, and does not contain every vertex of A_1 and does not contain every vertex of A_4. Thus, $|V(C)| \leqslant |A_1B_1A_2B_2A_3B_3A_4| - 2 \leqslant 4(2\Delta - 4) + 3\lceil \frac{1}{2}\Delta \rceil - 2 \leqslant \lceil \frac{19}{2}\Delta \rceil - 17$, a contradiction. $\square$

4.3 Clustered Choosability and Maximum Degree

This section proves our first result about clustered choosability of graphs with given maximum degree (Theorem 31). The preliminary lemmas will also be used in subsequent sections.

Lemma 46. *If L is a list-assignment for a graph G, such that $\deg_G(v) + 2 \leqslant 3|L(v)|$ for each vertex v of G, and ϕ is an L-colouring of G that minimises the number of monochromatic edges, then ϕ has defect 2. Moreover, for each vertex v with defect 2 under ϕ, there is a colour $\beta_v \in L(v) \setminus \{\phi(v)\}$, such that at most two neighbours of v are coloured β_v under ϕ.*

Proof. Suppose that some vertex v coloured α is adjacent to at least three vertices also coloured α. Since $\deg(v) < 3|L(v)|$, some colour $\beta \in L(v) \setminus \{\alpha\}$ is assigned to at most two neighbours of v. Recolouring v by β reduces the number of monochromatic edges. This contradiction shows that every vertex has defect at most 2.

Consider a vertex v coloured α with defect 2. Suppose that v has at least three neighbours coloured β for each $\beta \in L(v) \setminus \{\alpha\}$. Thus $\deg(v) \geqslant 2 + 3(|L(v)| - 1)$, implying $\deg(v) + 1 \geqslant 3|L(v)|$, which is a contradiction. Thus some colour $\beta \in L(v) \setminus \{\alpha\}$ is assigned to at most two neighbours of v. $\square$

Given a colouring ϕ of a graph G, let $G[\phi]$ denote the monochromatic subgraph of G given ϕ. The idea for the following lemma is by Haxell, Szabó, and Tardos [63, Lemma 2.6], adapted here for the setting of list-colourings.

Lemma 47. *If H is a bipartite graph with bipartition (X, Y) and L is a list-assignment for H such that $|L(v)| = 2$ for all $v \in X$ and $|L(v)| = 1$ for all $v \in Y$ and every L-colouring*

ϕ has defect 2, then H has an L-colouring ϕ such that every connected subgraph of $H[\phi]$ at most two vertices in X.

Proof. We begin by orienting the edges of H so that for every vertex $v \in V(H)$ and every colour $c \in L(v)$, v has at most one out-neighbour w with $c \in L(w)$ and v has at most one in-neighbour w with $c \in L(w)$. Let $L(H)$ be the union of the lists of all vertices of H. For each colour $c \in L(H)$, let H_c be the subgraph of H induced by the vertices $w \in V(H)$ with $c \in L(w)$. There is an L-colouring which assigns each vertex of H_c the colour c, so $\Delta(H_c) \leqslant 2$. Also, since every edge of H has an endpoint $y \in Y$ and $|L(y)| = 1$, evey edge of H is in $E(H_c)$ for at most one $c \in L(H)$. For each $c \in L(H)$, orient the edges of H_c so that no vertex has more than one in-neighbour or out-neighbour (possible since $\Delta(H_c) \leqslant 2$). Orient all remaining edges of H arbitrarily.

We now construct an L-colouring ϕ. First, colour each vertex in Y with the unique colour in its list. Now run the following procedure, initialising $i := 1$.

1: If $i > |X|$, then exit.

2: Select $v_i \in X \setminus \{v_i : i \in [i-1]\}$ and select $\phi(v_i) \in L(v_i)$ arbitrarily. Increment i by 1 and go to **3**.

3: If there is a directed path $v_{i-1}yx$ such that $x \in X \setminus \{v_i : i \in [i-1]\}$ and $\phi(v_{i-1}) = \phi(y)$ and $\phi(v_{i-1}) \in L(x)$, let $v_i := x$, select $\phi(v_i) \in L(v_i) \setminus \{\phi(v_{i-1})\}$, increment i by 1 and go to **3**. Otherwise go to **4**.

4: If there is a directed path xyv_{i-1} such that $x \in X \setminus \{v_i : i \in [i-1]\}$ and $\phi(v_{i-1}) = \phi(y)$ and $\phi(v_{i-1}) \in L(x)$, let $v_i := x$, select $\phi(v_i) \in L(v_i) \setminus \{\phi(v_{i-1})\}$, increment i by 1 and go to **3**. Otherwise go to **1**.

Suppose for contradiction that some component C of $H[\phi]$ has at least three vertices in S. Since ϕ is an L-colouring, C has a directed subpath $x_1y_1x_2y_2x_3$ such that $\{x_1, x_2, x_3\} \subseteq X$. If x_1 was the first vertex in $\{x_1, x_2\}$ to be coloured, then x_2 was coloured next and $\phi(x_2) \neq \phi(x_1)$, a contradiction. If x_2 was the first vertex in $\{x_2, x_3\}$ to be coloured, then x_3 was coloured next and $\phi(x_3) \neq \phi(x_2)$, a contradiction. Hence, x_2 was coloured before x_1 and after x_3. But then x_1 was coloured immediately after x_2 and $\phi(x_1) \neq \phi(x_2)$, a contradiction. □

We now prove our first result for clustered choosability of graphs with given maximum degree.

Theorem 31. *Every graph G with maximum degree $\Delta \geqslant 3$ is $\lceil \frac{1}{3}(\Delta + 2) \rceil$-choosable with clustering $\lceil \frac{19}{2}\Delta \rceil - 17$.*

Proof. Let $k := \lceil \frac{\Delta+2}{3} \rceil$. Let L be a k-list-assignment for G. Let ϕ be an L-colouring of G that minimises the number of monochromatic edges. By Lemma 46, ϕ is an L-colouring with defect 2. Moreover, for each vertex v with defect 2 under ϕ, there is a colour $\beta_v \in L(v) \setminus \{\phi(v)\}$, such that at most two neighbours of v are coloured β_v under ϕ. Let $L'(v) := \{\phi(v), \beta_v\}$ for each vertex v with defect 2.

Let M be the monochromatic subgraph of G. Thus $\Delta(M) \leqslant 2$. By Lemma 45, there is a set $S \subseteq V(M)$, such that S is stable in G, every vertex in S has defect 2 under ϕ, and the following hold:

1. every subpath of M with at least $3\Delta - 6$ vertices that contains a vertex with degree 1 in M contains at least one vertex in S,

2. every subpath of M with at least $5\Delta - 9$ vertices that contains a vertex with degree 1 in M contains at least two vertices in S, and

3. every connected subgraph C of M on at least $\lceil \frac{19}{2}\Delta \rceil - 16$ vertices contains at least three vertices in S.

Define a subpath of M to have *type 1* if it contains no vertex in S and at least one vertex of degree at most 1 in M. Define a subpath of M to have *type 2* if it contains at most one vertex in S and at least one vertex of degree at most 1 in M. Note that every path of type 1 is also of type 2, and every path of type 2 or 1 that contains a vertex of degree 0 in M has only one vertex. By the definition of S, every path of type 1 has at most $3\Delta - 7$ vertices and every path of type 2 has at most $5\Delta - 10$ vertices.

Let $\mathcal{T}$ be the set of connected components of $M - S$. Let H be the bipartite graph with $V(H) := \{S, \mathcal{T}\}$, where $s \in S$ is adjacent to $T \in \mathcal{T}$ if and only if s is adjacent to T in G, and the colour of the vertices of T is in $L'(s)$. Define L'_H so that $L'_H(s) := L'(s)$ for every $s \in S$, and $L'_H(T)$ is the singleton containing the colour assigned to the vertices of T for every $T \in \mathcal{T}$.

Let ϕ'_H be an arbitrary L'_H-colouring of H, and let ϕ' be the corresponding L-colouring of G. Note that every vertex of $v \in S$ is assigned a colour in $L'(v)$ and every other vertex is assigned its original colour in ϕ. Since S is a stable set and by the definition of L', the number of monochromatic edges given ϕ' is at most the number of monochromatic edges given ϕ. Hence by our choice of ϕ, no L-colouring yields fewer monochromatic edges than ϕ'. Hence the monochromatic subgraph M' of G given ϕ' satisfies $\Delta(M') \leqslant 2$. Let M'_H be the graph obtained from M' by contracting each $T \in \mathcal{T}$ to a single vertex. Then M'_H is isomorphic to the monochromatic subgraph of H given ϕ'_H. Since M'_H is a minor of M', we have $\Delta(M'_H) \leqslant 2$. Hence, every L'_H-colouring of H has defect 2.

By Lemma 47, H has an L'_H-colouring ϕ'_H such that no component of the monochromatic subgraph has more than two vertices in S. Let ϕ' be the corresponding L-colouring of G, and note that no component of the monochromatic subgraph M' of G given ϕ' has more than two vertices in S. Vertices of $G - S$ keep their colour from ϕ, and vertices $v \in S$ get a colour from $L'(v)$, so ϕ' is an L-colouring that minimises the number of monochromatic edges.

Suppose for contradiction that some vertex in $V(G - S)$ has degree 2 in M and is adjacent in M' to some vertex $s \in S$ which is not its neighbour in M (so $\phi'(s) \neq \phi(s)$). Then the L'-colouring obtained from ϕ by recolouring s with $\phi'(s)$ is not 2-defective, a contradiction.

It follows that the largest possible monochromatic component C of M' is obtained either from three disjoint paths in M of type 1 linked by two vertices in S, or is obtained from a path of type 1 and a path of type 2 linked by a vertex of S, or is a subgraph of M that contains at most two vertices in S. In each case, we have $|V(C)| \leqslant \lceil \frac{19}{2}\Delta \rceil - 17$. □

4.4 Clustered Choosability with Absolute Bounded Clustering

This section proves our results for clustered choosability of graphs with given maximum average degree (Theorem 27) or given maximum degree (Theorem 32), where the clustering is bounded by an absolute constant. The following lemma is the heart of the proof. With $I = \emptyset$, it immediately implies Theorem 32.

Lemma 48. *If I is a stable set of vertices in a graph G and L is a list-assignment for G such that $5|L(v)| \geqslant 2(\deg(v) + 1)$ for all $v \in V(G - I)$ and $5|L(v)| \geqslant 2\deg(v) + 1$ for all*

$v \in I$, then G has an L-colouring with clustering 9. Furthermore, if $I = \emptyset$, then G has an L-colouring with clustering 6.

Proof. Let $\mathcal{C}$ be the class of L-colourings ϕ that minimise the number of monochromatic edges. Given $\phi \in \mathcal{C}$ and $v \in V(G)$, let $L(\phi, v)$ be the set of colours $c \in L(v)$ such the colouring ϕ' obtained from ϕ by recolouring v with c is in $\mathcal{C}$. Note that in particular $\phi(v) \in L(\phi, v)$, and that a colour $c \in L(v)$ is in $L(\phi, v)$ if and only if $|\{w \in N(v) : \phi(w) = c\}| = \deg_{G[\phi]}(v)$.

Claim 48.1. *If $\phi \in \mathcal{C}$, then $\Delta(G[\phi]) \leqslant 2$.*

Proof. Let v be a vertex of maximum degree in $G[\phi]$. If for some colour $c \in L(v)$ we have $|\{w \in N_G(v) : \phi(w) = c\}| < \deg_{G[\phi]}(v)$, the colouring ϕ' obtained from ϕ by changing the colour of v to c satisfies $|E(G[\phi'])| < |E(G[\phi])|$, contradicting the assumption that $\phi \in \mathcal{C}$. Hence, $\deg_G(v) \geqslant \deg_{G[\phi]}(v)|L(v)|$. By assumption $|L(v)| \geqslant \frac{1}{5}(2\deg_G(v)+1)$, and the result follows. ☐

Claim 48.2. *If $\{\phi, \phi'\} \subseteq \mathcal{C}$, $v \in V(G - I)$ and $\deg_{G[\phi]}(v) = \deg_{G[\phi']}(v) = 2$, then $|L(\phi, v) \cap L(\phi', v)| \geqslant 2$.*

Proof. Suppose for contradiction that $|L(\phi, v) \cap L(\phi', v)| \leqslant 1$. Note that $L(\phi, v) \cup L(\phi', v) \subseteq L(v)$. Given that $|L(\phi, v)| + |L(\phi', v)| = |L(\phi, v) \cup L(\phi', v)| + |L(\phi, v) \cap L(\phi', v)| \leqslant |L(v)| + 1$, we have $|L(\phi, v)| \leqslant (|L(v)| + 1)/2$ without loss of generality. Since $\phi \in \mathcal{C}$, for every colour $c \in L(v)$, v has at least two neighbours in G coloured c by ϕ (or else recolouring v with c would yield a colouring ϕ' with $|E(G[\phi'])| < |E(G[\phi])|$). For every colour $c \in L(v) \setminus l(\phi, v)$, v has at least three neighbours coloured c by ϕ. Hence, $\deg(v) \geqslant 3|L(v)| - (|L(v)| + 1)/2$, meaning $|L(v)| \leqslant \frac{1}{5}(2\deg(v) + 1)$, a contradiction. ☐

Choose $\phi_0 \in \mathcal{C}$ and $S \subseteq V(G - I)$ such that S is a stable set in $G[\phi_0]$, every vertex in S has degree 2 in $G[\phi]$, and subject to this $|S|$ is maximised. let $S := \{s_1, s_2, \dots, s_t\}$. For $i \in [t]$, define ϕ_i recursively so that $\phi_i(v) = \phi_{i-1}(v)$ for $v \in V(G) \setminus \{s_i\}$ and $\phi_i(s_i) \in (L(\phi_0, s_i) \cap L(\phi_{i-1}, s_i)) \setminus \{\phi_0(s_i)\}$. Such L-colourings exist by Claim 48.2.

Define $L'(v) := \{\phi_0(v), \phi_t(v)\}$ for all $v \in V(G)$.

Claim 48.3. *If ϕ is an L'-colouring of G and $s \in S$, then $|N_{G[\phi]}(s) \setminus S| = 2$.*

Proof. Note that $L'(v) = \{\phi_0(v)\}$ for $v \in V(G) \setminus S$. Hence $|N_{G[\phi]}(s) \setminus S| = |N_{G[\phi_0]}(s) \setminus S| = 2$ if $\phi(v) = \phi_0(v)$. Now suppose that $\phi(s) = \phi_t(s)$. By construction, $\phi_t(s) \in L(\phi_0, s)$, so the colouring ϕ' obtained from ϕ_0 by changing the colour of s to $\phi_t(s)$ is in $\mathcal{C}$. Now $\Delta(G[\phi']) \leqslant 2$ by Claim 48.1, so no vertex $s' \in S$ is adjacent to s in $G[\phi']$, since s' already has two neighbours in $G[\phi] - S$ and hence in $G[\phi'] - S$. Since $|E(G[\phi'])| = |E(G[\phi_0])|$, we have $\deg_{G[\phi']}(s) = \deg_{G[\phi_0]}(s) = 2$. Hence $|N_{G[\phi]}(s) \setminus S| = |N_{G[\phi']}(s) \setminus S| = \deg_{G[\phi']}(s) = 2$. ☐

Claim 48.4. *If ϕ is an L'-colouring of G, then $\phi \in \mathcal{C}$.*

Proof. Suppose for contradiction that for some $\{v, w\} \subseteq S$, $vw \in E(G[\phi])$. Since S is a stable set in $G[\phi_0]$, either $\phi(v) = \phi_t(v)$ or $\phi(w) = \phi_t(w)$.

If $\phi(v) = \phi_t(v)$ and $\phi(w) = \phi_t(w)$, then v has three neighbours in $G[\phi_t]$ by Claim 48.3. But since $\phi_i(s_i) \in l(\phi_{i-1}, s_i)$ for $i \in [t]$, we have $\phi_t \in \mathcal{C}$, a contradiction.

Hence, without loss of generality, $\phi(v) = \phi_0(v)$ and $\phi(w) = \phi_t(w)$. Now $\phi_t(w) \in l(\phi_0, w)$, so the colouring ϕ' obtained from ϕ_0 by recolouring w with $\phi_t(w)$ is in $\mathcal{C}$. Note $vw \in E(G[\phi'])$ by assumption. By Claim 48.3, $|N_{G[\phi']}(v) \setminus S| = |N_{G[\phi_0]}(v) \setminus S| = 2$, so $\deg_{G[\phi']}(v) = 3$, contradicting Claim 48.1.

Now $|E(G[\phi])| = |E(G[\phi] - S)]| + 2|S|$ by Claim 48.3. But $G[\phi] - S = G[\phi_0] - S$, so $|E(G[\phi])| = |E(G[\phi_0])|$, and $\phi \in \mathcal{C}$. ☐

Let $\mathcal{T}$ be the set of components of $G[\phi_0] - S$. Let H be the bipartite graph with bipartition $(S, \mathcal{T})$ such that $s \in S$ is adjacent to $T \in \mathcal{T}$ if s is adjacent to T in G and the colour assigned to the vertices of T by ϕ_0 is in $L'(s)$. Let L'_H be the natural restriction of L' to H. Note that an L'_H-colouring ϕ_H of H corresponds to an L'-colouring of G, and $H[\phi_H]$ is a minor of $G[\phi]$, which means $\Delta(H[\phi_H]) \leqslant 2$ by Claims 48.1 and 48.4. Hence, by Lemma 47, H has an L'_H-colouring ϕ_H such that no component of $H[\phi_H]$ has more than two vertices in S. Let ϕ be the corresponding L'-colouring of G. Note that each component of $G[\phi]$ has at most two vertices in S.

Suppose for contradiction that some component C of $G[\phi]$ has at least ten vertices. Now $\Delta(G[\phi]) \leqslant 2$ by Claims 48.1 and 48.4, so C is a cycle or a path. Hence C has an induced subpath $P := p_1 p_2 \dots p_8$ such that every vertex of P has degree 2 in $G[\phi]$. Since I is a stable set in G, at most one vertex in each of $\{p_1, p_2\}$, $\{p_4, p_5\}$ and $\{p_7, p_8\}$ is in I, so $C - I$ contains a stable set S_C of size 4 such that every vertex of S_C has degree 2 in $G[\phi]$. Define $S' := (S \setminus V(C)) \cup S_C$. Since $|S \cap V(C)| \leqslant 2$, we have $|S'| > |S|$. However $S' \subseteq V(G - I)$, S' is a stable set in $G[\phi]$, and every vertex of S' has degree 2 in $G[\phi]$, contradicting our choice of ϕ_0 and S.

Finally, suppose for contradiction that $I = \emptyset$ and some component C of $G[\phi]$ has at least seven vertices. As before, C is either a cycle or a path, so there is a stable set S_C in C of size 3 such that every vertex in S_C has degree 2 in $G[\phi]$. Define $S' := (S \setminus V(C)) \cup S_C$. Since $|S \cap V(C)| \leqslant 2$, we have $|S'| > |S|$. However $S' \subseteq V(G - I)$, S' is a stable set in $G[\phi]$ and every vertex of S' has degree 2 in $G[\phi]$, contradicting our choice of ϕ_0 and S. □

Corollary 49. *Let (A, B) be a partition of the vertex set of a graph G, let $I \subseteq B$ be a stable set, and let L a list-assignment for G. If $5(|L(v)| - \deg_A(v)) \geqslant 2(\deg_B(v) + 1)$ for all $v \in B \setminus I$ and $5(|L(v)| - \deg_A(v)) \geqslant 2\deg_B(v) + 1$ for all $v \in I$, then every L-colouring of $G[A]$ with clustering 9 can be extended to an L-colouring of G with clustering 9.*

Proof. Let ϕ be an arbitrary L-colouring of $G[A]$, and let L' be the list-assignment for $G[B]$ such that, for every vertex v in B, $L'(v) := L(v) \setminus \{\phi(w) : w \in (N_G(v) \cap A)\}$. Note that every L'-colouring of $G[B]$ is also an L-colouring, and that for all $v \in B$, we have $|L'(v)| \geqslant |L(v)| - \deg_A(v)$. Hence, by Lemma 48, $G[B]$ has an L'-colouring ϕ' with clustering 9. By our choice of L', the combination of ϕ and ϕ' is an L-colouring of G with clustering 9. □

Theorem 27. *Every graph G is $\lfloor \frac{7}{10} \operatorname{mad}(G) + 1 \rfloor$-choosable with clustering 9.*

Proof. Let $k := \lfloor \frac{7}{10} \operatorname{mad}(G) \rfloor + 1$. We proceed by induction on $|V(G)|$. The claim is trivial if $|V(G)| \leqslant 8$. Assume that $|V(G)| \geqslant 9$. Let L be a k-list-assignment for G.

Let p be the maximum integer for which there are pairwise disjoint sets $X_1, \dots, X_p \subseteq V(G)$, such that for each $i \in [p]$, we have $|X_i| \in \{1, 2\}$, and if $A_i := X_1 \cup \dots \cup X_{i-1}$ and $B_i := V(G) \setminus A_i$, then at least one of the following conditions holds:

- $X_i = \{v_i\}$ and $5|L(v_i)| \leqslant 5\deg_{A_i}(v_i) + 2\deg_{B_i}(v_i)$, or
- $X_i = \{v_i, w_i\}$ and $v_i w_i \in E(G)$ and $5|L(v_i)| \leqslant 5\deg_{A_i}(v_i) + 2\deg_{B_i}(v_i) + 1$ and $5|L(w_i)| \leqslant 5\deg_{A_i}(w_i) + 2\deg_{B_i}(w_i) + 1$.

First suppose that $X_1 \cup \dots \cup X_p \neq V(G)$. Let $A := X_1 \cup \dots \cup X_p$ and $B := V(G) \setminus A$. We now show that Corollary 49 is applicable. By the maximality of p, each vertex $v \in B$ satisfies $5|L(v)| \geqslant 5\deg_A(v) + 2\deg_B(v) + 1$. Let I be the set of vertices $v \in B$ for which $5|L(v)| = 5\deg_A(v) + 2\deg_B(v) + 1$. By the maximality of p, I is a stable set. Since $\operatorname{mad}(G[A]) \leqslant \operatorname{mad}(G)$, by induction, $G[A]$ is L-colourable with clustering 9. By Corollary 49, G is L-colourable with clustering 9.

Now assume that $X_1 \cup \cdots \cup X_p = V(G)$. Let $R := \{i \in [p] : |X_i| = 1\}$ and $S := \{i \in [p] : |X_i| = 2\}$. Thus

$$\begin{aligned}
5k|V(G)| &\leqslant \sum_{i \in R}(3\deg_{A_i}(v_i) + 2\deg_G(v_i)) + \\
&\quad \sum_{i \in S}(3\deg_{A_i}(v_i) + 2\deg_G(v_i) + 1 + 3\deg_{A_i}(w_i) + 2\deg_G(w_i) + 1) \\
&\leqslant 3\sum_{i \in R}\deg_{A_i}(v_i) + 3\sum_{i \in S}(\deg_{A_i}(v_i) + \deg_{A_i}(w_i) + 1) + 2\sum_{v \in V(G)}\deg_G(v) \\
&= 7|E(G)|.
\end{aligned}$$

Hence $\frac{10}{7}k \leqslant \frac{2|E(G)|}{|V(G)|} \leqslant \operatorname{mad}(G)$, implying $k \leqslant \frac{7}{10}\operatorname{mad}(G)$, which is a contradiction. □

4.5 Clustered Choosability and Maximum Average Degree

This section proves our final results for clustered choosability of graphs with given maximum average degree (Theorems 28 and 30).

Lemma 50. *If I is a stable set in a graph G of maximum degree $\Delta \geqslant 3$, and L is a list-assignment of G, and $3|L(v)| \geqslant \deg_G(v)+1$ for each vertex $v \in I$, and $3|L(v)| \geqslant \deg_G(v)+2$ for each vertex $v \in V(G) \setminus I$, then G is L-colourable with clustering $19\Delta - 32$.*

Proof. Let ϕ be an L-colouring of G that minimises the number of monochromatic edges. By Lemma 46, ϕ is an L-colouring with defect 2. Moreover, for each vertex v with defect 2 under ϕ, there is a colour $\beta_v \in L(v) \setminus \{\phi(v)\}$, such that at most two neighbours of v are coloured β_v under ϕ. Let $L'(v) := \{\phi(v), \beta_v\}$ for each vertex v with defect 2.

Let M be the monochromatic subgraph of G. Thus $\Delta(M) \leqslant 2$. Each component of M is a cycle or path. Orient each cycle component of M to become a directed cycle, and orient each path component of M to become a directed path.

Let G' be obtained from G as follows: first delete all non-monochromatic edges incident to all vertices in I. Note that vertices in I now have degree at most 2. Now if vx is a directed monochromatic edge in G with $x \in I$ and x having defect 2, then contract vx into a new vertex v'. Note that $v \in V(G) \setminus I$ since I is a stable set. Note also that $\Delta(G') \leqslant \Delta(G) \leqslant \Delta$. Consider v' to be coloured by the same colour as v. Let $M_{G'}$ be the monochromatic subgraph of G'. Then $M_{G'}$ is obtained from M by the same set of contractions that formed G' from G, and $M_{G'}$ is an induced subgraph of G' with maximum degree at most 2.

By Lemma 45, there is a set $S \subseteq V(M_{G'})$, such that S is stable in G, every vertex in S has defect 2 under ϕ, and the following hold:

1. every subpath of $M_{G'}$ with at least $3\Delta - 6$ vertices that contains a vertex with degree 1 in M contains at least one vertex in S,

2. every subpath of $M_{G'}$ with at least $5\Delta - 9$ vertices that contains a vertex with degree 1 in M contains at least two vertices in S, and

3. every connected subgraph C of $M_{G'}$ with at least $\lceil \frac{19}{2}\Delta \rceil - 16$ vertices contains at least three vertices in S.

Let S be obtained from S' by replacing each vertex v' (arising from a contraction) by the corresponding vertex v in G. Thus $S \cap I = \emptyset$. By construction, S is a stable set in G, every vertex in S has defect 2 under ϕ, and each of the following hold:

1. every subpath of M with at least $6\Delta - 12$ vertices contains a vertex with degree 1 in M contains at least one vertex in S,
2. every subpath of M with at least $10\Delta - 18$ vertices contains a vertex with degree 1 in M contains at least two vertices in S,
3. every connected subgraph C of M with at least $19\Delta - 31$ vertices contains at least three vertices in S.

Define a subpath of M to have *type 1* if it contains no vertex in S and at least one vertex of degree at most 1 in M. Define a subpath of M to have *type 2* if it contains at most one vertex in S and at least one vertex of degree at most 1 in M. Note that every path of type 1 is also of type 2, and that any path of type 2 or 1 that contains a vertex of degree 0 in M has only one vertex. By the definition of S, every path of type 1 has at most $6\Delta - 13$ vertices and every path of type 2 has at most $10\Delta - 19$ vertices.

Let $\mathcal{T}$ be the set of connected components of $M - S$, and define a bipartite graph H with $V(H) := \{S, \mathcal{T}\}$, where $s \in S$ is adjacent to $T \in \mathcal{T}$ if and only if s is adjacent to T in G, and the colour of the vertices of T is in $L'(s)$. Define L'_H so that $L'_H(s) := L'(s)$ for every $s \in S$, and $L'_H(T)$ is the singleton containing the colour assigned to the vertices of T for every $T \in \mathcal{T}$.

Let ϕ'_H be an arbitrary L'_H-colouring of H, and let ϕ' be the corresponding L-colouring of G. Note that every vertex of $v \in S$ is assigned a colour in $L'(v)$ and every other vertex is assigned its original colour in ϕ. Since S is a stable set and by the definition of L', the number of monochromatic edges given ϕ' is at most the number of monochromatic edges given ϕ. Hence by our choice of ϕ, no L-colouring yields fewer monochromatic edges than ϕ'. Hence the monochromatic subgraph M' of G given ϕ' satisfies $\Delta(M') \leqslant 2$. Let M'_H be the graph obtained from M' by contracting each $T \in \mathcal{T}$ to a single vertex. Then M'_H is isomorphic to the monochromatic subgraph of H given ϕ'_H. Since M'_H is a minor of M', we have $\Delta(M'_H) \leqslant 2$. Hence, every L'_H-colouring of H has defect 2.

By Lemma 47, H has an L'_H-colouring ϕ'_H such that no component of the monochromatic subgraph has more than two vertices in S. Let ϕ' be the corresponding L-colouring of G, and note that no component of the monochromatic subgraph M' of G given ϕ' has more than two vertices in S. Vertices of $G - S$ keep their colour from ϕ, and vertices $v \in S$ get a colour from $L'(v)$, so ϕ' is an L-colouring which minimises the number of monochromatic edges.

Suppose for contradiction that some vertex in $V(G - S)$ has degree 2 in M and is adjacent in M' to some vertex $s \in S$ which is not its neighbour in M (so $\phi'(s) \neq \phi(s)$). Then the L'-colouring obtained from ϕ by recolouring s with $\phi'(s)$ is not 2 defective, a contradiction.

It follows that the largest possible monochromatic component C of M' is obtained either from three disjoint paths in M of type 1 linked by two vertices in S, or is obtained from a path of type 1 and a path of type 2 linked by a vertex of S, or is a subgraph of M that contains at most two vertices in S. In each case, we have $|V(C)| \leqslant 19\Delta - 32$. □

Lemma 51. *For a graph G, let A, B be a partition of $V(G)$ with $\Delta := \Delta(G[B]) \geqslant 3$, and let I be a stable set of G contained in B. Let L be a list-assignment for G and let c be an integer such that $c \geqslant 19\Delta - 32$, $G[A]$ is L-colourable with clustering c, $3|L(v)| \geqslant 3\deg_A(v) + \deg_B(v) + 1$ for each vertex $v \in I$, and $3|L(v)| \geqslant 3\deg_A(v) + \deg_B(v) + 2$ for each vertex $v \in B \setminus I$. Then G is L-colourable with clustering c.*

Proof. Let ϕ be an L-colouring of $G[A]$ with clustering c. For each vertex $v \in B$, let $L'(v) := L(v) \setminus \{\phi(x) : x \in N_A(v)\}$. Thus $|L'(v)| \geqslant |L(v)| - \deg_A(v)$, implying $3|L'(v)| \geqslant$

$\deg_B(v) + 1$ for each vertex $v \in I$, and $3|L'(v)| \geqslant \deg_B(v) + 2$ for each vertex $v \in B \setminus I$. Lemma 50 implies that $G[B]$ is L-colourable with clustering $19\Delta - 32$. By construction, there is no monochromatic edge between A and B. Thus G is L-colourable with clustering c. ☐

We now prove Theorem 30, which implies Theorem 28 when $n_0 = 1$.

Theorem 30. *For integers $d \geqslant 0$, $n_0 \geqslant 1$ and $k \geqslant 1$, every graph G with $\operatorname{mad}(G, n_0) < \frac{3}{2}k$ is k-choosable with clustering $c := \max\{\lceil \frac{n_0-1}{k} \rceil, 57k - 51\}$.*

Proof. We first prove the $k = 1$ case. Let G be a graph with $\operatorname{mad}(G, n_0) < \frac{3}{2}$. Every component of a graph with maximum average degree less than $\frac{3}{2}$ has at most three vertices. Thus every component of G has at most $\max\{n_0 - 1, 3\}$ vertices. Hence, every 1-list-assignment has clustering $\max\{n_0 - 1, 3\} \leqslant c$. Now assume that $k \geqslant 2$.

We proceed by induction on $|V(G)|$. Let L be a k-list-assignment for G. If $|V(G)| \leqslant n_0 - 1$, then colour each vertex v by a colour in $L(v)$, so that each colour is used at most $\lceil \frac{n_0-1}{k} \rceil$ times. We obtain an L-colouring with clustering $\lceil \frac{n_0-1}{k} \rceil$. Now assume that $|V(G)| \geqslant n_0$.

Let p be the maximum integer for which there are pairwise disjoint sets $X_1, \dots, X_p \subseteq V(G)$, such that for each $i \in [p]$, we have $|X_i| \in \{1, 2\}$, and if $A_i := X_1 \cup \dots \cup X_{i-1}$ and $B_i := V(G) \setminus A_i$, then at least one of the following conditions hold:

- $X_i = \{v_i\}$ and $3|L(v_i)| \leqslant 3\deg_{A_i}(v_i) + \deg_{B_i}(v_i)$, or
- $X_i = \{v_i, w_i\}$ and $v_i w_i \in E(G)$ and $3|L(v)| \leqslant 3\deg_{A_i}(v) + \deg_{B_i}(v) + 1$ and $3|L(w)| \leqslant 3\deg_{A_i}(w) + \deg_{B_i}(w) + 1$.

First suppose that $X_1 \cup \dots \cup X_p \neq V(G)$. Let $A := X_1 \cup \dots \cup X_p$ and $B := V(G) \setminus A$. Since $\operatorname{mad}(G[A], n_0) \leqslant \operatorname{mad}(G, n_0)$, by induction, $G[A]$ is L-colourable with clustering c. We now show that Lemma 51 is applicable. By the maximality of p, for each $v \in B$,

$$3k = 3|L(v)| \geqslant 3\deg_A(v) + \deg_B(v) + 1 \geqslant \deg_B(v) + 1.$$

Let $\Delta := 3k - 1$. Then $\Delta(G[B]) \leqslant 3k - 1 = \Delta$. Since $k \geqslant 2$, we have $\Delta \geqslant 5$ and $19\Delta - 32 = 19(3k - 1) - 32 = 57k - 51 \leqslant c$. Let I be the set of vertices $v \in B$ for which $3|L(v)| = 3\deg_A(v) + \deg_B(v) + 1$. By the maximality of p, I is a stable set. Lemma 51 thus implies that G is L-colourable with clustering c.

Now assume that $X_1 \cup \dots \cup X_p = V(G)$. Let $R := \{i \in [p] : |X_i| = 1\}$ and $S := \{i \in [p] : |X_i| = 2\}$. For $i \in R$, condition (A) holds, implying $3k \leqslant 2\deg_{A_i}(v_i) + \deg_G(v_i)$. For $i \in S$, condition (B) holds, implying $3k \leqslant 2\deg_{A_i}(v_i) + \deg_G(v_i) + 1$ and $3k \leqslant 2\deg_{A_i}(w_i) + \deg_G(w_i) + 1$. Thus

$$\begin{aligned}
3k|V(G)| &\leqslant \sum_{i\in R}(2\deg_{A_i}(v_i) + \deg_G(v_i)) + \\
&\quad \sum_{i\in S}(2\deg_{A_i}(v_i) + \deg_G(v_i) + 1 + 2\deg_{A_i}(w_i) + \deg_G(w_i) + 1) \\
&= 2\sum_{i\in R}\deg_{A_i}(v_i) + 2\sum_{i\in S}(\deg_{A_i}(v_i) + \deg_{A_i}(w_i) + 1) + \sum_{v\in V(G)}\deg_G(v) \\
&= 4|E(G)|.
\end{aligned}$$

Hence $\frac{3}{2}k \leqslant \frac{2|E(G)|}{|V(G)|} \leqslant \operatorname{mad}(G)$, and $|V(G)| \geqslant n_0$ implying $k \leqslant \frac{2}{3}\operatorname{mad}(G, n_0)$, which is a contradiction. ☐

4.6 Earth-Moon Colouring and Thickness

The union of two planar graphs is called an *earth-moon* (or *biplanar*) graph. The famous earth-moon problem asks for the maximum chromatic number of earth-moon graphs [3, 26, 57, 70, 71, 108]. It follows from Euler's formula that every earth-moon graph has maximum average degree less than 12, and is thus 12-colourable. On the other hand, there are 9-chromatic earth-moon graphs [26, 57]. So the maximum chromatic number of earth-moon graphs is 9, 10, 11 or 12.

Defective and clustered colourings provide a way to attack the earth-moon problem. First consider defective colourings of earth-moon graphs. Since the maximum average degree of every earth-moon graph is less than 12, Theorem 23 by Havet and Sereni [62] implies that every earth-moon graph is k-choosable with defect d, for $(k,d) \in \{(7,18),(8,9),(9,5),(10,3),(11,2)\}$. This result gives no bound with at most 6 colours. Ossona de Mendez et al. [103] went further and showed that every earth-moon graph is k-choosable with defect d, for $(k,d) \in \{(5,36),(6,19),(7,12),(8,9),(9,6),(10,4),(11,2)\}$. Examples show that 5 colours is best possible [103]: the defective chromatic number of earth-moon graphs equals 5. Theorem 24 implies that every earth-moon graph is k-choosable with defect d for $(k,d) \in \{(7,6),(8,3),(9,2),(11,1)\}$. These results improve the best known bounds when $k \in \{7,8,9,11\}$.

Now consider clustered colouring of earth-moon graphs. Wood [155] describes examples of earth-moon graphs that are not 5-colourable with bounded clustering. Thus the clustered chromatic number of earth-moon graphs is at least 6. Theorem 25 of Kopreski and Yu [83] proves that earth-moon graphs are 9-colourable with clustering 2. Other results for clustered colouring do not work for earth-moon graphs since they can contain expanders [49], and thus do not have sub-linear separators. Since every earth-moon graph has maximum average degree strictly less than 12, Theorems 26 and 28 imply the following:

Theorem 52. *Every earth-moon graph is:*

- 9-*choosable with clustering* 2.
- 8-*choosable with clustering* 405.

It is open whether every earth-moon graph is 6 or 7-colourable with bounded clustering.

Earth-moon graphs are generalised as follows. The *thickness* of a graph G is the minimum integer t such that G is the union of t planar subgraphs; see [97] for a survey. It follows from Euler's formula that graphs with thickness t are $(6t-1)$-degenerate and thus $6t$-colourable. For $t \geqslant 3$, complete graphs provide a lower bound of $6t-2$. It is an open problem to improve these bounds; see [70]. Ossona de Mendez et al. [103] studied defective colourings of graphs with given thickness, and proved the following result.

Theorem 53 ([103])**.** *The defective chromatic number of the class of graphs with thickness t equals $2t+1$. In particular, every such graph is $(2t+1)$-choosable with defect $2t(4t+1)$.*

Now consider clustered colourings of graphs with given thickness. Obviously, the clustered chromatic number of graphs with thickness t is at most $6t$, and Wood [155] proved a lower bound of $2t+2$. Since every graph with thickness t has maximum average degree strictly less than $6t$, Theorems 26 to 28 imply the following improved upper bounds.

Theorem 54. *Every graph with thickness t is:*

- $\lceil \frac{9}{2}t \rceil$-*choosable with defect 1 and clustering* 2,
- $\lceil \frac{21}{5}t \rceil$-*choosable with clustering* 9,

- *$4t$-choosable with clustering* $228t - 51$.

Thickness is generalised as follows; see [71, 103, 155]. For an integer $g \geqslant 0$, the *g-thickness* of a graph G is the minimum integer t such that G is the union of t subgraphs each with Euler genus at most g. Ossona de Mendez et al. [103] determined the defective chromatic number of this class as follows (thus generalising Theorem 53).

Theorem 55 ([103])**.** *For integers $g \geqslant 0$ and $t \geqslant 1$, the defective chromatic number of the class of graphs with g-thickness t equals $2t + 1$. In particular, every such graph is $(2t + 1)$-choosable with defect $2tg + 8t^2 + 2t$.*

Now consider clustered colourings of graphs with g-thickness t. Wood [155] proved that every such graph is $(6t + 1)$-choosable with clustering $\max\{g, 1\}$. Euler's formula implies that every n-vertex graph with g-thickness t has less than $3t(n + g - 2)$ edges (for $n \geqslant 3$), implying $\text{mad}(G, 4tg - 8t + 1) < 6t + \frac{3}{2}$. Hence, Theorem 30 implies the following improvement to this upper bound.

Theorem 56. *For $g \geqslant 0$ and $t \geqslant 1$, every graph with g-thickness t is $(4t + 1)$-choosable with clustering* $\max\{\lceil \frac{4tg-8t}{4t+1} \rceil, 228t + 6\}$.

This result highlights the utility of considering $\text{mad}(G, n_0)$.

4.7 Stack and Queue Layouts

This section applies our results to graphs with given stack- or queue-number. Again, previous results for clustered colouring do not work for graphs with given stack- or queue-number since they can contain expanders [49], and thus do not have sub-linear separators.

A *k-stack layout* of a graph G consists of a linear ordering $v_1, \dots, v_n$ of $V(G)$ and a partition $E_1, \dots, E_k$ of $E(G)$ such that no two edges in E_i cross with respect to $v_1, \dots, v_n$ for each $i \in [1, k]$. Here edges $v_a v_b$ and $v_c v_d$ *cross* if $a < c < b < d$. A graph is a *k-stack graph* if it has a k-stack layout. The *stack-number* of a graph G is the minimum integer k for which G is a k-stack graph. Stack layouts are also called *book embeddings*, and stack-number is also called *book-thickness*, *fixed outer-thickness* and *page-number*. Dujmović and Wood [50] showed that the maximum chromatic number of k-stack graphs is in $\{2k, 2k+1, 2k+2\}$.

A *k-queue layout* of a graph G consists of a linear ordering $v_1, \dots, v_n$ of $V(G)$ and a partition $E_1, \dots, E_k$ of $E(G)$ such that no two edges in E_i are nested with respect to $v_1, \dots, v_n$ for each $i \in [1, k]$. Here edges $v_a v_b$ and $v_c v_d$ are *nested* if $a < c < d < b$. The *queue-number* of a graph G is the minimum integer k for which G has a k-queue layout. A graph is a *k-queue graph* if it has a k-queue layout. Dujmović and Wood [50] showed that the maximum chromatic number of k-queue graphs is in the range $[2k + 1, 4k]$.

Consider clustered colourings of k-stack and k-queue graphs. Wood [155] noted the clustered chromatic number of the class of k-stack graphs is in $[k + 2, 2k + 2]$, and that the clustered chromatic number of the class of k-queue graphs is in $[k+1, 4k]$. The lower bounds come from standard examples, and the upper bounds hold since every k-stack graph has maximum average degree less than $2k + 2$, and every k-queue graph has maximum average degree less than $4k$. Theorems 26 to 28 thus imply the following improved upper bounds:

Theorem 57. *Every k-stack graph is:*

- $\lfloor \frac{3k+4}{2} \rfloor$*-choosable with defect* 1*, and thus with clustering* 2.
- $\lfloor \frac{7k+11}{5} \rfloor$*-choosable with clustering* 9.

- $\lfloor \frac{4k+6}{3} \rfloor$*-choosable with clustering at most* $76k + 53$.

Theorem 58. *Every* k*-queue graph is:*

- $3k$*-choosable with defect* 1*, and thus with clustering* 2.
- $\lfloor \frac{14k+4}{5} \rfloor$*-choosable with clustering* 9.
- $\lfloor \frac{8k+2}{3} \rfloor$*-choosable with clustering at most* $152k - 13$.

4.8 Open Problem

We conclude with the natural open problem that arises from this research. Theorem 28 says that the clustered chromatic number of the class of graphs with maximum average degree m is at most $\lfloor \frac{2m}{3} \rfloor + 1$. The best known lower bound is $\lfloor \frac{m}{2} \rfloor + 1$; see [155]. Closing this gap is an intriguing open problem.

Chapter 5

Gonality

In this chapter, we focus on graph gonality. In Section 1.7, we defined gonality in terms of a simple chip firing game. We now provide a more formal definition, and show that the two definitions are equivalent.

5.1 Preliminaries

A *divisor* of a graph G is a vector in $\mathbb{Z}^{V(G)}$. Let $\mathrm{Div}(G)$ denote the set of divisors of G and for $D \in \mathrm{Div}(G)$ and $v \in V(G)$, let $D(v)$ denote the value of D in position v. The *support* $\mathrm{supp}(D)$ of D is the set $\{v \in V(G) : D(v) \neq 0\}$. For every subgraph $H \subseteq G$, the *restriction* $D|_H$ of D to H is the divisor in $\mathrm{Div}(H)$ with $D|_H(v) := D(v)$ for all $v \in V(H)$. A divisor of G is *effective* if each entry is non-negative. Let $\mathrm{Div}_+(G)$ denote the set of effective divisors of G. The *degree* of a divisor D is given by

$$\deg(D) := \sum_{v \in V(G)} D(v).$$

Let $\mathcal{M}(G)$ denote the set of integer valued functions on $V(G)$. The *Laplacian operator* $\Delta : M(G) \to \mathrm{Div}(G)$ of G is given by

$$(\Delta(f))(v) = \sum_{w \in N(v)} (f(v) - f(w)).$$

Two divisors D and D' are *equivalent*, written $D \sim D'$, if there is some $S \in \mathcal{M}(G)$ such that $D' = D - \Delta(S)$.

Note that $\deg(\Delta(S)) = 0$ for every $S \in \mathcal{M}(G)$, and hence every pair of equivalent divisors have the same degree. The *rank* $r(D)$ of a divisor D is the maximum value of k such that for every effective divisor D' of degree k, there is some effective divisor equivalent to $D - D'$. Note that $r(D) \geqslant 1$ if and only if for every vertex $v \in V(G)$ there is some effective divisor D' equivalent to D with $D'(v) \geqslant 1$. The *gonality* of a graph, denoted $\mathrm{gon}(G)$, is the minimum degree of a divisor D of G with $r(D) \geqslant 1$.

In Section 1.7, we defined gonality in terms of a chip firing game. We now show that this definition is equivalent. There is a natural correspondence between initial configurations and vector in $\mathrm{Div}_+(G)$. Suppose that G has chip configuration corresponding to $D \in \mathrm{Div}_+(G)$, and a move is made by selecting a set $A \subseteq V(G)$. Define $\mathbf{1}_A \in \mathcal{M}(G)$ to be the function satisfying $\mathrm{supp}(\mathbf{1}_A) := A$ and $\mathbf{1}_A(v) := 1$ for all $v \in A$. By the definition of Δ, the new configuration corresponds to the divisor $D' = D - \Delta(\mathbf{1}_A)$. It follows that every winning configuration corresponds to a divisor of rank at least 1.

The following is well known. A simple proof of it can be found for example in [151].

Lemma 59. *If D and D' are equivalent effective divisors of G, then there is a unique chain of non-empty sets $A_1, A_2, \ldots, A_t$ and corresponding sequence of divisors $D_0, D_1, \ldots, D_t$ such that*

- $\emptyset \subsetneq A_1 \subseteq A_2 \subseteq \cdots \subseteq A_t \subsetneq V(G)$,
- $D_0 = D$ *and* $D_t = D'$ *and*
- *for all* $i \in [t]$, D_i *is effective and* $D_i = D_{i-1} - \Delta(\mathbf{1}_{A_i})$.

Now suppose D is a divisor with $r(D) \geqslant 1$. We may assume that D is effective, since it follows from the fact that $r(D) \geqslant 0$ that there is some effective divisor in the equivalence class containing D. Consider the initial configuration for G corresponding to D. Since $r(D) \geqslant 1$, for every vertex v there is some effective divisor $D' \sim D$ such that $D'(v) \geqslant 1$. Now Lemma 59 gives a chain of legal moves taking the chip configuration corresponding to D to the chip configuration corresponding to D'. Thus, the gonality of a graph can alternatively be defined as the minimum number of chips required for a winning chip configuration.

For several interesting families of graphs, the gonality has been precisely determined by van Dobben de Bruyn and Gijswijt [151]. In Sections 5.2 and 5.3, we make use of the following result.

Theorem 60 ([151])**.** *If n and m are positive integers with $n \geqslant m$ and G is the $n \times m$ rectangular grid graph, then* $\mathrm{gon}(G) = \mathrm{tw}(G) = m$.

5.2 Treewidth and Graphs with a Universal Vertex

In this section, we present a formula for the gonality of graphs with a universal vertex. Using this formula, we calculate the gonality of the family of graphs known as fans, consequently proving Theorem 34 and answering Question 2. We also show that the answer to Question 3 is "yes" in the special case where the subgraph H of G has a universal vertex.

In Section 1.7, we claimed that Theorem 34 was best possible. To see this, first note that the claim fails for $k = 1$, since every 1-connected graph of treewidth 1 is a tree, and hence has gonality 1. Further, the following elementary result (see [107] for a proof) immediately implies that a k-connected graph has treewidth at least k.

Lemma 61. *For every graph G,* $\mathrm{tw}(G) \geqslant \delta(G)$.

Due to Lemma 59, we can focus our analysis on pairs $\{D, D'\}$ of effective divisors such that $D' = D - \Delta(\mathbf{1}_A)$ for some set A. For this reason, for a graph G, an effective divisor $D \in \mathrm{Div}_+(G)$ and a vertex $v \in V(G)$, we are interested in the following set. We define the *clump* $\mathrm{clump}(D, v)$ of an effective divisor $D \in \mathrm{Div}_+(G)$ centred at v to be the intersection of all subsets $S \subseteq V(G)$ such that $v \in S$ and $D(w) \geqslant |N(w) \cap S|$ for every vertex w not in S (or equivalently $v \in S$ and $D - \Delta(\mathbf{1}_{S^C}) \in \mathrm{Div}_+(G)$). This set can equivalently be defined as the smallest set S such that $v \in S$ and $D - \Delta(\mathbf{1}_{S^C}) \in \mathrm{Div}_+(G)$. To see this, note that if $D - \Delta(\mathbf{1}_A) \in \mathrm{Div}_+(G)$ and $D - \Delta(\mathbf{1}_B) \in \mathrm{Div}_+(G)$, then $D - \Delta(\mathbf{1}_{A \cup B}) \in \mathrm{Div}_+(G)$. This fact and the following Lemma motivate our interest in this set.

Lemma 62. *If D is an effective divisor of a graph G and $A \subseteq V(G)$ is such that $D - \Delta(\mathbf{1}_A)$ is also an effective divisor, then for every subgraph H of G and every vertex $w \in V(H) \setminus A$, we have*

$$\mathrm{clump}(D|_H, w) \subseteq V(H) \setminus A.$$

Proof. Let w' be a vertex in $V(H) \cap A$. Since $(D - \Delta(\mathbf{1}_A))(w') \geqslant 0$, we have $D(w') \geqslant |N(w') \setminus A|$, so $D|_H(w') \geqslant |N_H(w') \setminus A|$. Let $S := V(H) \setminus A$. Then S is a subset of $V(H)$ such that $w \in S$ and for every vertex $w' \in V(H) \setminus S$, we have $D|_H(w') \geqslant |N_H(w') \cap S|$. By definition, $\mathrm{clump}(D|_H, w)$ is the intersection of all sets with these properties, so $\mathrm{clump}(D|_H, w) \subseteq V(H) \setminus A$. □

Let G be an arbitrary graph. We define the *clump width*, $\mathrm{clw}(D)$, of an effective divisor $D \in \mathrm{Div}_+(G)$ to be 0 if $\mathrm{supp}(D) = V(G)$ and otherwise to be the maximum size of a clump centred at a vertex not in $\mathrm{supp}(D)$. Formally,

$$\mathrm{clw}(D) := \max\{0, |\,\mathrm{clump}(D, w)| : D(w) = 0\}.$$

Lemma 63. *Let H be a graph with a universal vertex v and at least one other vertex, let $H' := H - v$ and let $D \in \mathrm{Div}_+(H')$. If $E \in \mathrm{Div}_+(H)$ is such that $E|_{H'} = D$ and $E(v) \geqslant \mathrm{clw}(D)$, then $r(E) \geqslant 1$.*

Proof. If $V(H') \subseteq \mathrm{supp}(E)$, then $E' := E - \Delta_H(\mathbf{1}_{V(H')})$ is an effective divisor of H with $E'(v) \geqslant 1$. If $V(H') \nsubseteq \mathrm{supp}(E)$, then let $w_0 \in V(H')$ be such that $E(w_0) = 0$. Now $E(v) \geqslant |\,\mathrm{clump}(E|_{H'}, w_0)|$ and $w_0 \in \mathrm{clump}(E|_{H'}, w_0)$, so $E(v) \geqslant 1$. Let $A := V(H) \setminus \mathrm{clump}(E|_{H'}, w_0)$, and consider the divisor $E' := E - \Delta_H(\mathbf{1}_A)$. We have $E'(v) = E(v) - |\,\mathrm{clump}(E|_{H'}, w_0)| \geqslant 0$ and for all $w' \in \mathrm{clump}(E|_{H'}, w_0)$, we have $E'(w') \geqslant E(w') + 1 \geqslant 1$ since $v \in N(w') \cap A$. Suppose $w' \in A \setminus \{v\}$. By the definition of $\mathrm{clump}(E|_{H'}, w_0)$, there is some set $S \subseteq V(H')$ such that $w' \notin S$, $w_0 \in S$ and for every vertex $w'' \in V(H') \setminus S$, we have $E(w'') \geqslant |N(w'') \cap S|$. Since $w' \notin S$, we have $E(w') \geqslant |N(w') \cap S|$, and since $\mathrm{clump}(E|_{H'}, w_0) \subseteq S$, we have $E(w') \geqslant |N(w') \cap \mathrm{clump}(E|_{H'}, w_0)|$. Hence $E'(w') \geqslant 0$. Therefore, E' is an effective divisor equivalent to E such that $E'(w_0) \geqslant 1$. □

An effective divisor D of a graph G is *v-reduced* if there is no non-empty subset $A \subseteq V(G)$ such that $v \notin A$ and $D - \Delta(\mathbf{1}_A)$ is an effective divisor. In the chip-firing game discussed in Section 5.1, every legal move from the chip configuration corresponding to a v-reduced effective divisor must contain v. The following result is due to Baker and Norine [14].

Lemma 64. *If G is a connected graph and D is an effective divisor of G, then for every vertex $v \in V(G)$, there exists a unique v-reduced effective divisor $D' \sim D$.*

The following lemma is our main tool for answering Questions 1 and 2.

Lemma 65. *Let H be a graph with a universal vertex v and let $H' := H - v$ and let. If $V(H') \neq \emptyset$, then*

$$\mathrm{gon}(H) = \min\{\deg(D) + \mathrm{clw}(D) : D \in \mathrm{Div}_+(H'), \mathrm{supp}(D) \neq V(H')\}.$$

Proof. Let D be a v-reduced effective divisor of H such that $r(D) \geqslant 1$ and $\deg(D) = \mathrm{gon}(H)$. Since D is v-reduced, $(D - \Delta_H(\mathbf{1}_{V(H')})) \notin Div_+(H)$, so $\mathrm{supp}(D|_{H'}) \neq V(H')$.

Let w_0 be a vertex of H' with $D(w_0) = 0$ and $|\,\mathrm{clump}(D|_{H'}, w_0)| = \mathrm{clw}(D)$. Since $r(D) \geqslant 1$, there exists $D' \in \mathrm{Div}_+(H)$ such that $D' \sim D$ and $D'(w_0) \geqslant 1$. Let $A_1, A_2, \ldots, A_t$ and $D_0, D_1, \ldots, D_t$ be defined as in Lemma 59, with $D_0 := D$ and $D_t := D'$. Since D_0 is v-reduced, $v \in A_1$. Since $D_t(w_0) > D_0(w_0)$, there is some $i \in [t]$ such that $w_0 \notin A_i$. By definition $A_1 \subseteq A_i$, so $w_0 \notin A_1$. By Lemma 62 with $A := A_1$, we have $\mathrm{clump}(D|_{H'}, w_0) \subseteq V(H') \setminus A_1$. Now,

$$0 \leqslant D_1(v) = D_0(v) - \Delta_H(\mathbf{1}_{A_1})(v) = D(v) - |N(v) \setminus A_1| \leqslant D(v) - |V(H') \setminus A_1|,$$

so $D(v) \geqslant |\operatorname{clump}(D|_{H'}, w_0)|$ and $\deg(D) \geqslant \deg(D|_{H'}) + |\operatorname{clump}(D|_{H'}, w_0)|$. Since $\operatorname{gon}(H) = \deg(D)$ and $D|_{H'} \in \operatorname{Div}_+(H')$, we have

$$\operatorname{gon}(H) \geqslant \min\{\deg(D) + \operatorname{clw}(D) : D \in \operatorname{Div}_+(H'), \operatorname{supp}(D) \neq V(H')\}.$$

The result now follows from Lemma 63.

□

Recall that Question 3 asks whether the gonality of a subgraph of a graph G is at most the gonality of G. We now show that the answer to Question 3 is "yes" when restricted to subgraphs with a universal vertex. This is useful for proving Theorem 34.

Lemma 66. *Let G be a connected graph, and let H be a subgraph of G. If H has a universal vertex v, then* $\operatorname{gon}(G) \geqslant \operatorname{gon}(H)$.

Proof. Let D be an effective divisor of G such that $\deg(D) = \operatorname{gon}(G)$ and $r(D) \geqslant 1$ and let $H' := H - v$. By Lemma 64, we may assume D is v-reduced. Hence, $D(v) \geqslant 1$, since $r(D) \geqslant 1$. Now if $V(H') \subseteq \operatorname{supp}(D)$, then $r(D|_H) \geqslant 1$ trivially, and so $\operatorname{gon}(G) \geqslant \operatorname{gon}(H)$.

If $V(H') \not\subseteq \operatorname{supp}(D)$, then let w_0 be a vertex of H' with $|\operatorname{clump}(D|_{H'}, w_0)| = \operatorname{clw}(D|_{H'})$ and $D(w_0) = 0$. Since $r(D) \geqslant 1$, there exists $D' \in \operatorname{Div}_+(G)$ such that $D' \sim D$ and $D'(w_0) \geqslant 1$. Let $A_1, A_2, \ldots, A_t$ and $D_0, D_1, \ldots, D_t$ be defined as in Lemma 59, with $D_0 := D$ and $D_t := D'$. Since D_0 is v-reduced, $v \in A_1$. Since $D_t(w_0) > D_0(w_0)$, there is some $i \in [t]$ such that $w_0 \notin A_i$. By definition $A_1 \subseteq A_i$, so $w_0 \notin A_1$. By Lemma 62 with $A := A_1$, we have $\operatorname{clump}(D|_{H'}, w_0) \subseteq V(H') \setminus A_1$. Now,

$$0 \leqslant D_1(v) = D_0(v) - \Delta_G(\mathbf{1}_{A_1})(v) = D(v) - |N(v) \setminus A_1| \leqslant D(v) - |V(H') \setminus A_1|,$$

so $D(v) \geqslant |\operatorname{clump}(D|_{H'}, w_0)|$ and $\deg(D) \geqslant \deg(D|_{H'}) + \operatorname{clw}(D|_{H'})$. Hence, by Lemma 65, $\operatorname{gon}(G) \geqslant \operatorname{gon}(H)$. □

The *fan* on n vertices is the n-vertex graph with a universal vertex v such that $G - v$ is a path. It is well-known that fans have treewidth 2. Lemma 66 provides a method for determining the gonality of a fan. The following lemma is the final tool we need.

Lemma 67. *Let G be a graph and let D be an effective divisor of G. If $v \in V(G)$ and $H := G[\{v\} \cup \{w \in V(G) : D(w) = 0\}]$, then the vertex set of the component of H that contains v is a subset of* $\operatorname{clump}(D, v)$.

Proof. By definition, $v \in \operatorname{clump}(D, v)$. Let t be a non-negative integer, and suppose for induction that every vertex at distance exactly t from v in H is in $\operatorname{clump}(D, v)$, and suppose w_0 is at distance exactly $t + 1$ from v in H. Since $w_0 \in V(H - v)$, we have $D(w_0) = 0$, and since w_0 is at distance $t + 1$ from v in H, w_0 has some neighbour w_1 in H at distance exactly t from v in H. By our inductive hypothesis, $w_1 \in \operatorname{clump}(D, v)$. Hence, for every subset $S \subseteq V(G)$ such that $v \in S$ and $D(w) \geqslant |N(w) \cap S|$ for every vertex w not in S, we have $|N(w_0) \cap S| \geqslant |\{w_1\}| > D(w_0)$, so $w_0 \in S$. Therefore $w_0 \in \operatorname{clump}(D, v)$, and by induction every vertex in the component of H that contains v is in $\operatorname{clump}(D, v)$. □

Theorem 68. *If G is the fan of n vertices, then* $\operatorname{gon}(G) = t + \lceil (n - 1 - t)/(t + 1) \rceil$, *where* $t = \lfloor (\sqrt{4n + 1} - 1)/2 \rfloor$.

Proof. Let v be the universal vertex in G, and let $G - v = p_1 p_2 \cdots p_{n-1}$. For every integer $t \geqslant 0$, define

$$f(t) := \min\{\operatorname{clw}(D) : D \in \operatorname{Div}_+(G - v), \deg(D) = t, \operatorname{supp}(D) \neq V(G - v)\}.$$

By Lemma 65, $\text{gon}(G) = \min\{t + f(t) : t \in \mathbb{Z}, t \geqslant 0\}$. Let D be an effective divisor of $G-v$, let $H(D) := G[\{w \in N(v) : D(w) = 0\}]$, and let w_0 be a vertex of $H(D)$. By Lemma 67, the vertex set of the component subpath P of $H(D)$ containing w_0 is a subset of $\text{clump}(D, w_0)$. Every neighbour w' of P in $G - v$ satisfies $D(w') \geqslant 1$, since P is a component of $H(D)$. Since $G - v$ is a path, $|N(w') \cap P| \leqslant 1$. Hence, $w_0 \in V(P)$ and $D(w') \geqslant |N(w') \cap V(P)|$ for every vertex $w' \in N(v)$, so $\text{clump}(D, w_0) = V(P)$ by the definition of $\text{clump}(D, w_0)$. Hence, $f(t) = \min\{\max\{|V(P)| : P \text{ is a component of } H(D)\} : D \in \text{Div}_+(G - v)\}$. Since $H(D)$ is entirely determined by $\text{supp}(D)$, we may restrict ourselves to divisors such that $\deg(D) = |\text{supp}(D)|$. In particular, $\text{gon}(G) = \min\{t + f(t) : t \in \{0, 1, \dots, n-1\}\}$. For all $t \in \{0, 1, \dots, n-1\}$ and all $D \in \text{Div}_+(G - v)$ with $\deg(D) = t$, the graph $H(D)$ has at most $t+1$ components and at least $n-1-t$ vertices, so $f(t) \geqslant \lceil (n-1-t)/(t+1) \rceil$. Let $A_t := \{p_k : k/(\lceil (n-1-t)/(t+1) \rceil + 1) \in [t]\}$ and let $\text{supp}(D) := A_t$. Then $D \in \text{Div}_+(G-v)$, $\deg(D) = t$ and

$$\max\{|V(P)| : P \text{ is a component of } H(D)\} = \lceil (n-1-t)/(t+1) \rceil.$$

Hence, $f(t) = \lceil (n-1-t)/(t+1) \rceil$. Consider the function g on the domain $(-1, \infty)$ given by $g(x) := x + (n-1-x)/(x+1)$, and note that for $t \in \{0, 1, \dots, n-1\}$, we have $t + f(t) = \lceil g(t) \rceil$. Now for $x \in (0, (\sqrt{4n+1} - 1)/2]$ we have $g(x) - g(x-1) \leqslant 0$ and for $x > (\sqrt{4n+1} - 1)/2$ we have $g(x) - g(x-1) > 0$. It follows that $g(t)$ and hence $t + f(t)$ is minimised for $t \in \{0, 1, \dots, n-1\}$ at $t = \lfloor (\sqrt{4n+1} - 1)/2 \rfloor$. ☐

Recall that Question 1 asks whether there is some function f such that for every connected graph G, $\text{gon}(G) \leqslant f(\text{tw}(G))$. We now prove Theorem 34 and answer Question 1 in the negative.

Theorem 34. *For all integers $k \geqslant 2$ and $l \geqslant k$, there exists a k-connected graph G with $\text{tw}(G) = k$ and $\text{gon}(G) \geqslant l$.*

Proof. Let $n := (l^2 + l)$, and let G be the graph formed by adding $k-1$ universal vertices $v_1, v_2, \dots, v_{k-1}$ to the path $P := p_1 p_2 \cdots p_{n-1}$. Let $T := P - p_{n-1}$, and let f be the function from $V(T)$ to the set of subsets of $V(G)$ such that $f(p_i) = \{p_i, p_{i+1}, v_1, v_2, \dots, v_{k-1}\}$ for $i \in [n-2]$. It is quick to check that T and f form a tree-decomposition of G of width k, so $\text{tw}(G) \leqslant k$. Since $\delta(G) = k$, we have $\text{tw}(G) = k$ by Lemma 61. Let $H := G - \{v_2, v_3, \dots, v_{k-1}\}$, and note that H is the fan on n vertices with universal vertex v_1. By Lemma 66, we have $\text{gon}(G) \geqslant \text{gon}(H)$. Now, $\lfloor (\sqrt{4n+1} - 1)/2 \rfloor = l$, so by Theorem 68, $\text{gon}(H) = l + \lceil (n-1-l)/(l+1) \rceil > l$. ☐

Recall that Question 2 asks whether $\text{gon}(H) \leqslant \text{gon}(G)$ for every connected graph G and every connected minor H of G. The following corollary answers Question 2 in the negative.

Corollary 69. *For every integer $l \geqslant 0$ there exist connected graphs G and H such that H is a minor of G, $\text{gon}(G) = 2$ and $\text{gon}(H) > l$.*

Proof. Let $n := (l^2 + l)$, let G be the $2 \times (n-1)$ rectangular grid and let H be the fan on n vertices. Then H can be obtained from G by contracting one of the rows of G to a single vertex, so H is a minor of G, and $\text{gon}(G) = 2$ by Theorem 60. By Theorem 68, we have $\text{gon}(H) > l$ (as in the previous proof). ☐

5.3 High Gonality Subgraphs

In this section we show that the answer to Question 3 is "no". In doing this, we provide an alternative proof that the answer to Questions 1 and 2 is "no".

Theorem 70. *For every positive integer g, there are connected graphs H and G such that* $\mathrm{gon}(H) > g$, $\mathrm{gon}(G) = 2$ *and* $H \subseteq G$.

Proof. Let $h := g + 1$ and let $k = 2g^2 + 4g + 1$. Let $V(H) := [k] \times \mathbb{Z}_{2h}$, and let $E(H) := \{((i, \mathbf{z}), (i, \mathbf{z} + \mathbf{1})) : i \in [k], \mathbf{z} \in \mathbb{Z}_{2h}\} \cup \{((i, \mathbf{h} - \mathbf{1}), (i+1, \mathbf{0})) : i \in [k-1]\}$. Let f map $V(H)$ to $\mathbb{Z}_{2h}^k$ so that

$$f((s, \mathbf{z}))(i) = \begin{cases} \mathbf{0} & \text{if } i < s, \\ \mathbf{z} & \text{if } i = s, \\ \mathbf{h} - \mathbf{1} & \text{if } i > s. \end{cases}$$

Now two vertices v and w are adjacent if and only if $f(v) = f(w)$ or $f(v)$ differs from $f(w)$ in exactly one coordinate and differs by exactly $\mathbf{1}$ in this coordinate. Let G be the $2 \times hk$ rectangular grid. We find $H \subseteq G$ by letting $\{(i, \mathbf{z}) \in V(H) : i \in [k], \mathbf{z} \in \{\mathbf{0}, \mathbf{1}, \ldots, \mathbf{h} - \mathbf{1}\}\}$ form the bottom row of G and the remaining vertices form the top row. By Theorem 60, $\mathrm{gon}(G) = 2$. We now show that $\mathrm{gon}(H) > g$.

If D is a divisor of H, let $\sigma(D)$ be given by

$$\sigma(D) := \sum_{v \in V(H)} D(v) f(v).$$

Note that for $v \in V(H)$, there are exactly two neighbours u and w of v such that $f(v) \neq f(u)$ and $f(v) \neq f(w)$. Furthermore, $f(u) + f(w) = 2f(v)$. Hence, if D and D' are divisors of H such that $D' := D - \Delta(\mathbf{1}_{\{v\}})$, then $\sigma(D) = \sigma(D')$. Every function $S \in \mathcal{M}(H)$ can be expressed as a sum of integer multiples of functions in $\{\mathbf{1}_{\{w\}} : w \in V(H)\}$, so if D_0 and D_1 are divisors of H and $D_0 \sim D_1$, then $\sigma(D_0) = \sigma(D_1)$.

Suppose for contradiction that D is an effective divisor of H with $\deg(D) \leqslant g$ and $r(D) \geqslant 1$. Let $I \subseteq [k]$ be the set of numbers $i \in [k]$ such that $D((i, \mathbf{z})) = 0$ for all $\mathbf{z} \in \mathbb{Z}_{2h}$, and note that $|I| \geqslant k - g$. In particular, there exist $s \in [k]$ such that $I' := \{s, s+1, \ldots, s+2g\}$ is an interval of integers in I, since $(k - g)/(g + 1) = 2g + 1$. For all $i, i' \in I$, with $i < i'$,

$$\begin{aligned} \sigma(D)(i') - \sigma(D)(i) &= \sum_{\mathbf{z} \in \mathbb{Z}_{2h}} (D((i', \mathbf{z}))\mathbf{z} - D((i, \mathbf{z}))\mathbf{z}) + \sum_{j=i+1}^{i'-1} \sum_{\mathbf{z} \in \mathbb{Z}_{2h}} D((j, \mathbf{z}))(\mathbf{h} - \mathbf{1}) \\ &= \sum_{j=i+1}^{i'-1} \sum_{\mathbf{z} \in \mathbb{Z}_{2h}} D((j, \mathbf{z}))(\mathbf{h} - \mathbf{1}). \end{aligned}$$

Hence, for all $i, i' \in I'$, with $i < i'$,

$$\sigma(D)(i') - \sigma(D)(i) = \sum_{j=i+1}^{i'-1} \sum_{\mathbf{z} \in \mathbb{Z}_{2h}} D((j, \mathbf{z}))(\mathbf{h} - \mathbf{1}) = \mathbf{0}.$$

Since $r(D) \geqslant 1$, there exists $D' \in \mathrm{Div}_+(H)$ such that $D'((s + g, \mathbf{0})) \geqslant 1$ and $D' \sim D$. Since $\deg(D') - D'((s + g, \mathbf{0})) \leqslant g - 1$, there is some $i_1 \in I' \cap [s + g - 1]$ and some

$i_2 \in I' \setminus [s+g]$ such that $D'((i_1, \mathbf{z})) = D((i_2, \mathbf{z})) = 0$ for all $\mathbf{z} \in \mathbb{Z}_{2h}$. Since $D' \sim D$, we have $\sigma(D')(i_2) - \sigma(D')(i_1) = \sigma(D)(i_2) - \sigma(D)(i_1) = \mathbf{0}$. We also have

$$\sigma(D')(i_2) - \sigma(D')(i_1) = \sum_{i=i_1+1}^{i_2-1} \sum_{\mathbf{z} \in \mathbb{Z}_{2h}} D'((i, \mathbf{z}))(\mathbf{h} - \mathbf{1}) = \deg(D'|_{H[X]})(\mathbf{h} - \mathbf{1}),$$

where $X := \{(j, \mathbf{z}) \in V(H) : i_1 + 1 \leqslant j \leqslant i_2 - 1\}$. Since $D'((s+g, \mathbf{0})) \geqslant 1$ and $\deg(D') \leqslant g$, $\deg(D'|_{H[X]}) \in [d]$. However $\mathbf{0} \notin \{d(\mathbf{h} - \mathbf{1}) : d \in [g]\}$, a contradiction. $\square$

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