Tutte-Whitney Polynomials for Directed Graphs and Maps

by

Kai Siong Yow

Thesis

Submitted by Kai Siong Yow

for fulfilment of the Requirements for the Degree of

Doctor of Philosophy (0190)

Supervisor: Prof. Graham Farr

Associate Supervisor: Dr. Kerri Morgan

Faculty of Information Technology
Monash University

January, 2019

Sow a thought, reap an action; sow an action, reap a habit; sow a habit, reap a character; sow a character, reap a destiny.

- Stephen R. Covey, *The 7 Habits of Highly Effective People*

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Tutte-Whitney Polynomials for Directed Graphs and Maps

Kai Siong Yow
KaiSiong.Yow@monash.edu
Monash University, 2019

Supervisor: Prof. Graham Farr
Graham.Farr@monash.edu
Associate Supervisor: Dr. Kerri Morgan
Kerri.Morgan@deakin.edu.au

Abstract

In this thesis, we investigate analogues of the Tutte polynomial for two types of directed graphs, namely Tutte invariants for alternating dimaps and greedoid polynomials for rooted digraphs.

Alternating dimaps were defined by Tutte (1948) as orientably embedded Eulerian directed graphs where, for each vertex, the sequence of edges incident with it is directed inwards and outwards alternately. Three reduction operations known as 1-reduction, ω-reduction and ω^2-reduction for alternating dimaps were introduced by Farr (2013). A *minor* of an alternating dimap can be obtained by reducing some of its edges using these reduction operations. Unlike classical minor operations for graphs, these reduction operations do not commute in general.

A *Tutte invariant* for alternating dimaps is a function P defined on every alternating dimap and taking values in a field such that P is invariant under isomorphism and obeys a certain linear recurrence relation involving reduction operations. This idea is motivated by the deletion-contraction recurrence for the Tutte polynomial. A number of different invariants including simple Tutte invariants, extended Tutte invariants and c-Tutte invariants, were defined by Farr.

It is well known that if a graph G is planar and G^* is the dual of G, then $T(G; x, y) = T(G^*; y, x)$. We prove an analogous relation for extended Tutte invariants for alternating dimaps.

As a result of non-commutativity of the reduction operations, Tutte invariants are not always well defined. We first determine necessary and sufficient conditions on their parameters for extended Tutte invariants to be well defined for all alternating dimaps of genus zero. We then determine the set of alternating dimaps of genus zero for which their extended Tutte invariants are well defined when all the parameters are independent. We also establish some excluded minor characterisations of those alternating dimaps of genus zero whose Tutte invariants are well defined.

The c-Tutte invariant for alternating dimaps is a special type of extended Tutte invariant involving two variables, which is similar to the Tutte polynomial. We determine the points at which the c-Tutte invariant is well defined for all alternating dimaps of genus

zero. For any embedded graph G, its associated alternating dimap $\mathrm{alt}_c(G)$ (respectively, $\mathrm{alt}_a(G)$) is obtained by replacing each edge of G by a pair of directed edges forming a clockwise face (respectively, anticlockwise face) of size two. Farr showed that the c-Tutte invariant $T_c(D; x, y)$ of an alternating dimap D is well defined for any alternating dimap of the form $\mathrm{alt}_c(G)$ where G is a plane graph, when it equals the Tutte polynomial of G. We extend this result and determine the class of alternating dimaps for which the c-Tutte invariant is well defined. It properly contains alternating dimaps of the form $\mathrm{alt}_c(G)$, where G is a plane graph. We then extend the relationship between Tutte polynomials of plane graphs and c-Tutte invariants of alternating dimaps, and show that every c-Tutte invariant of an alternating dimap in this class can be obtained from the Tutte polynomial of a plane graph.

Gordon and McMahon (1989) defined a two-variable greedoid polynomial for any greedoid. We focus on greedoids associated with rooted digraphs. We compute the greedoid polynomials for all rooted digraphs up to order six. We found that as the order of rooted digraphs increases, the ratio of the number of unique greedoid polynomials of rooted digraphs to the number of rooted digraphs shows a decreasing trend. The trend of this ratio contrasts with an analogous conjecture on the Tutte polynomial of graphs.

Gordon and McMahon proved that the greedoid polynomials of rooted digraphs have the multiplicative direct sum property. In addition, these polynomials are divisible by $1+z$ under certain conditions. A greedoid polynomial $f(D)$ of a rooted digraph D of order n *GM-factorises* if $f(D) = f(G) \cdot f(H)$ such that G and H are rooted digraphs of order at most n and $f(G), f(H) \neq 1$. We study the GM-factorability of greedoid polynomials of rooted digraphs, particularly those that are not divisible by $1 + z$. Our computational results for rooted digraphs up to order six reveal that as the order of rooted digraphs increases, the ratio of the number of unique greedoid polynomials of rooted digraphs that can be GM-factorised to the number of unique greedoid polynomials is increasing. We also give some examples and an infinite family of rooted digraphs that are not direct sums but their greedoid polynomials GM-factorise.

Tutte-Whitney Polynomials for Directed Graphs and Maps

Declaration

I declare that this thesis is my own work and has not been submitted in any form for another degree or diploma at any university or other institute of tertiary education. Information derived from the published and unpublished work of others has been acknowledged in the text and a list of references is given.

[redacted]

Kai Siong Yow

January 11, 2019

Acknowledgements

I must first thank my supervisor Prof. Graham Farr for giving me the opportunity to carry out research in graph theory. There were many occasions where other academics during our chats over coffee said that I am lucky to have Graham as my supervisor. I completely agree with them and I am certainly fortunate to have had Graham as my supervisor. I really appreciate his guidance and unconditional support throughout my candidature. Graham is one of the most dedicated researchers that I have ever met. He is patient with his students, respects others' opinions, and is very generous in sharing his knowledge. I am grateful for his advice, which helped to improve not only my presentation skills, writing skills and language, but also in becoming a better person. I also appreciate his effort for introducing me to Australian culture, by organising day trips and games, and inviting us for meals together with his family members during various occasions.

I also thank my associate supervisor Dr. Kerri Morgan who played an important role during my PhD journey. Kerri always suggests interesting ideas and encourages me to explore different research areas. She is a "KERRIng" and friendly supervisor. She provides assistance whenever necessary, to make sure that I am always on the right track. I really appreciate her effort, advice and guidance during the ups and downs of my journey.

I wish to thank members of the Discrete Mathematics Research Group (lead by Prof. Graham Farr and Prof. Ian Wanless). The weekly seminars and discussions are beneficial. It is my pleasure to be part of the research group. My sincere thanks go to Dr. Daniel Mathews, Prof. Ian Wanless and Prof. David Wood for allowing me to participate in their mathematical courses.

I extend my appreciation to my review panel members Dr. David Albrecht, Dr. Daniel Horsley and Assoc. Prof. Chung-Hsing Yeh (panel chair) for their insightful feedback during my milestone reviews. I also thank Prof. Susan McKemmish (former Associate Dean Graduate Research) and Danette Deriane (Graduate Research Student Services Coordinator) for their support all these years. I thank Prof. Gary Gordon (Lafayette College) for his useful feedback on drafts of the preprint that led to Chapter 7.

A special thank-you goes to my fiancée Dana Chor who accompanied me through different stages during my candidature. Your involvement is truly appreciated. I also thank my family members who never fail to trust me, and give me freedom to pursue my career path.

I owe a great debt of gratitude to my wonderful colleagues and friends at Monash University, Harald Bögeholz, James Collier, Hooman Reisi Dehkordi, Parthan Kasarapu, Ranjie Mo, Han Duy Phan, Ehsan Shareghi, Dinithi Sumanaweera, Srinibas Swain, Ying Yang, Xuhui Zhang and many others who have helped and advised me throughout this journey. Thanks for making this possible.

Last but not least, I thank the Ministry of Higher Education Malaysia and the University Putra Malaysia for their funding. I also wish to acknowledge the Faculty of Information Technology (Monash University) for providing additional funding under the International Postgraduate Research Scholarship and the Graduate Research Stipend scholarship.

Kai Siong Yow

Monash University
September 2019

CHAPTER 1

Introduction

Graph theory is a branch of combinatorics and one of the most popular areas in discrete mathematics with countless applications. The idea of graph theory was initiated by a mathematician from Switzerland, Leonhard Euler, in studying the Königsberg bridge problem in the 18th century [32, 4, 52, 20][1].

Graph theory is widely used in many disciplines including mathematics, information technology, engineering, biology and sociology [45, 9, 14, 68]. One of its famous applications in global human relationships is the so-called *Six Degrees of Separation*, a famous theory stating that the distance between two people is at most six steps in a chain of social connections. Graph theory also plays a crucial role in modelling many aspects of daily life. For instance, transportation systems, financial markets, social networks, connectivity of the World Wide Web and many other systems can be modelled by graphs [46].

A graph is a representation of a set of objects where certain pairs of the objects are linked. The objects are represented by *vertices* and the links that connect the pairs of vertices are called *edges*. For undirected graphs (*graphs*), two vertices are connected without considering the direction of the edges. For directed graphs (*digraphs*)[2], the edges are directed from one vertex to another. Undirected graphs have been studied more comprehensively than digraphs, as they are somewhat simpler. One of the foci of this thesis, alternating dimaps, is a class of digraphs with some intriguing properties.

The Tutte polynomial is a two-variable polynomial that gives us a variety of information about the enumeration of various substructures of undirected graphs. It has a well developed theory and a considerable amount of literature has been published over many years, see [88, 35, 27] for some recent surveys. The Tutte polynomial is defined for every undirected graph. Some partial evaluations of the Tutte polynomial include the chromatic polynomial, reliability polynomial and flow polynomial [13, 27]. In view of its rich structure, researchers often explore the relations between the Tutte polynomial and other graph polynomials. The Tutte polynomial is closely related to another two-variable

[1] The English translation of [32] is: *The solution of a problem to the geometry of position.*

[2] Prefix di- is used to indicate every edge in a graph is assigned with a direction.

polynomial, known as the Whitney rank generating function [89]. The Whitney rank generating function can be obtained by a slight modification of the two variables of the Tutte polynomial, and vice versa. As a result, they are sometimes referred to as Tutte-Whitney polynomials.

Over the years, several Tutte-like polynomials for directed graphs have been defined, including the greedoid polynomial [39], cover polynomial [16] and drop polynomial [17]. Chung and Graham [16] once commented: "for directed graphs, no analogue of the Tutte polynomial is known". Knowing that the Tutte polynomial is so important for undirected graphs, development of analogous polynomials for directed graphs is worth exploring. A survey of such polynomials can be found in [15].

In this thesis, we investigate Tutte-like polynomials for two types of directed graphs, namely (1) Tutte invariants for alternating dimaps and (2) the two-variable greedoid polynomial for rooted digraphs. These two polynomials are both mentioned in [15].

An *alternating dimap* is an orientably embedded Eulerian directed graph where, for each vertex, the sequence of edges incident with it is directed inwards and outwards alternately [78, 3, 36, 37]. Alternating dimaps are at least as diverse and rich in structure as orientably embedded undirected graphs, since any orientably embedded undirected graph G can be converted into an alternating dimap $\mathrm{alt}_c(G)$ (respectively, $\mathrm{alt}_a(G)$) by replacing each edge by a clockwise 2-cycle (respectively, anticlockwise 2-cycle) [78, 3, 37]. It is also worth noting that every orientably embedded Eulerian undirected graph with k components can be converted into an alternating dimap in 2^k ways: for each component, choose a reference edge arbitrarily, choose one of the two possible directions of it, and let the alternating property determine the direction of all other edges.

Farr [37] introduced three minor operations for alternating dimaps, namely 1-reductions, ω-reductions and ω^2-reductions. A *minor* of an alternating dimap is obtained by reducing some of its edges using these operations. Unlike classical minor operations, these operations do not always commute.

Given the breadth of the class of alternating dimaps, and the natural reduction operations for it, it is natural to ask whether a theory of Tutte polynomials may be developed for it. The first steps in this direction were taken by Farr in [37]. In this thesis, we develop the theory further, establishing when various types of Tutte invariant exist, with both positive and negative results.

A *Tutte invariant* for alternating dimaps is a function P defined on every alternating dimap and taking values in a field such that P is invariant under isomorphism and obeys a certain linear recurrence relation involving reduction operations. Farr [37] defined several Tutte invariants including extended Tutte invariants, the c-Tutte invariant and the a-Tutte invariant. We characterise these invariants in this thesis.

It is well known that if G is a planar graph and G^* is the dual graph of G, the Tutte polynomial of G satisfies $T(G; x, y) = T(G^*; y, x)$. We prove an analogous result for extended Tutte invariants of alternating dimaps.

Since the reduction operations do not always commute, an invariant defined recursively using reduction operations may not always be well defined. We investigate the properties of alternating dimaps that are required to obtain a well defined extended Tutte invariant.

First, we determine the form that an extended Tutte invariant must have, if it is to be well defined for all alternating dimaps of genus zero. It turns out that such invariants are of very restricted form. Then, we determine the structure of alternating dimaps of genus zero for which the most general possible extended Tutte invariant is well defined. We also establish some excluded minor characterisations for those alternating dimaps of genus zero.

The c-Tutte invariant $T_c(D; x, y)$ of an alternating dimap D was introduced in [37] and shown to be well defined for any alternating dimap of the form $\mathrm{alt}_c(G)$ where G is a plane graph, when it equals the Tutte polynomial of G. We show that it can be defined for some other alternating dimaps too. We determine the class of alternating dimaps for which the c-Tutte invariant is well defined. It properly contains alternating dimaps of the form $\mathrm{alt}_c(G)$, where G is a plane graph. This shows that the c-Tutte invariant properly extends the Tutte polynomial of a plane graph. Analogous results are established for a-Tutte invariants and $\mathrm{alt}_a(G)$.

Gordon and McMahon [39] defined a two-variable greedoid polynomial for any greedoid (see Section 2.3 for the definition of greedoids), which is an analogue of the Whitney rank generating function. They studied greedoid polynomials for rooted graphs and rooted digraphs. We compute the greedoid polynomials for all rooted digraphs up to order six. Bollobás, Pebody and Riordan conjectured that almost all graphs are determined by their chromatic or Tutte polynomials [6]. We found that greedoid polynomials of rooted digraphs up to order six behave in a completely different way.

One of the most natural things to study for any polynomial is its factors. Factorisation of some known graph polynomials reflects the structures of the respective graphs, e.g., the Tutte polynomial of a graph G factorises if and only if G is a direct sum [63]. Sometimes, the situation is more complex. One such example is the chromatic polynomial [65]. Gordon and McMahon showed that the greedoid polynomials of rooted digraphs have $1 + z$ among their factors under certain conditions. We address more general types of factorisation for these greedoid polynomials in this thesis.

A greedoid polynomial $f(D)$ of a rooted digraph D of order n *GM-factorises* if $f(D) = f(G) \cdot f(H)$ such that G and H are rooted digraphs of order at most n and $f(G), f(H) \neq 1$. We study the GM-factorability of greedoid polynomials of rooted digraphs, particularly those that are not divisible by $1 + z$. The GM-factorisation of greedoid polynomials of rooted digraphs is not as straightforward as the factorisation of the Tutte polynomial. We give some examples and an infinite family of rooted digraphs that are not direct sums but their greedoid polynomials GM-factorise.

1.1 Thesis Outline and Research Contributions

In this section, we give the outline of the rest of this thesis. The main contributions are summarised in the outlines of Chapters 5–7.

Chapter 2: Definitions and Notation

In Chapter 2, we give the definitions and notation used throughout this thesis.

Chapter 3: Alternating Dimaps

In Chapter 3, we give a survey of alternating dimaps, which is one of the foci of this thesis. We cover the edge types, trial operations and minor operations of alternating dimaps.

Chapter 4: Literature Review

In Chapter 4, we present an overview of the literature for topics including alternating dimaps, the Tutte polynomial and some known polynomials for directed graphs.

Chapter 5: Characterisations of Extended Tutte Invariants (this chapter is based on [92])

In Chapter 5, we describe the relationship between extended Tutte invariants for alternating dimaps and its two trials (Theorem 5.1). We determine necessary and sufficient conditions on the parameters of extended Tutte invariants, for them to be well defined for all alternating dimaps of genus zero (Section 5.1). We determine the set of alternating dimaps of genus zero for which their extended Tutte invariants are well defined, when no restriction is imposed on the parameters of the invariants (Section 5.2). We establish some excluded minor characterisations for alternating dimaps of genus zero when their extended Tutte invariants are well defined (Sections 5.1 and 5.2).

Chapter 6: Tutte Invariants That Extend the Tutte Polynomial (this chapter is based on [92])

In Chapter 6, we characterise the c-Tutte invariant[3] and determine those points at which it is well defined for all alternating dimaps of genus zero (Section 6.1). We show that the c-Tutte invariant is multiplicative over non-loop blocks and some specific loops for certain alternating dimaps (Theorem 6.17). We determine the class of alternating dimaps for which the c-Tutte invariant is well-defined (Theorem 6.24). We extend the relationship between the Tutte polynomial and the c-Tutte invariant, by showing that the Tutte polynomial of certain graphs and the c-Tutte invariant of some alternating dimaps are identical for a wider class of graphs (Theorem 6.25).

Chapter 7: Factorisation of Greedoid Polynomials of Rooted Digraphs (this chapter is based on [93])

In Chapter 7, we compute the greedoid polynomials of all rooted digraphs up to order six (Section 7.3 and Appendix A). We define more general types of factorisation of greedoid polynomials of rooted digraphs, and determine the proportion of greedoid polynomials of rooted digraphs up to order six that factorise (Section 7.2). We identify various types of rooted digraphs (with examples) that are not direct sums but their

[3]The c-Tutte invariant and the a-Tutte invariant are closely related. Any result on the former should have a corollary for the latter.

greedoid polynomials GM-factorise (Section 7.2). We give an infinite family of digraphs where their greedoid polynomials GM-factorise, and characterise the greedoid polynomials of rooted digraphs that belong to the family (Section 7.2.4).

Chapter 8: Conclusions and Future Work

In Chapter 8, we conclude our findings and suggest some topics for future research.

Appendix A

The appendix summarises commands and algorithms used to generate results in Chapter 7.

1.2 Publications

Publications arising from this thesis include:

K. S. Yow, G. E. Farr and K. J. Morgan, Tutte invariants for alternating dimaps. (submitted)

Chapter 5 and 6. See [92] for the version on arXiv.

K. S. Yow, K. J. Morgan and G. E. Farr, Factorisation of greedoid polynomials of rooted digraphs. (submitted)

Chapter 7. See [93] for the version on arXiv.

CHAPTER 2

Definitions and Notation

The terminology used in this thesis is mostly standard. All graphs are finite. Terminology of graphs, matroids and greedoids are given in Sections 2.1, 2.2 and 2.3, respectively.

2.1 Graphs

We usually consider a graph to be a *simple graph* that contains no parallel edges and loops, unless stated otherwise.

Let $G = (V, E)$ be a graph. The vertex set and the edge set of G are denoted by $V(G)$ and $E(G)$, respectively. The *order* of G is the number of vertices of G and the *size* of G is the number of edges of G. The number of connected components of G is denoted by $k(G)$. Let $u, v \in V(G)$. An edge (u, v) is usually written as uv, and the edge is *incident* with both u and v. In the context of digraphs, we use uv to represent an edge directed from u to v, hence $uv \neq vu$. We call v the *head* and u the *tail* of the edge.

The *degree* of a vertex $v \in V(G)$, denoted by $\deg_G(v)$ (or $\deg(v)$ where no ambiguity arises), is the number of edges that are incident with v.

The *indegree* (respectively, *outdegree*) of a vertex in a digraph is the number of incoming edges (respectively, outgoing edges) that are incident with the vertex. A digraph is *balanced* if for each vertex in the digraph, its indegree is equal to its outdegree.

Let G_1 and G_2 be two graphs. For $i \in \mathbb{N} \cup \{0\}$, an *i-union* of G_1 and G_2, denoted by $G_1 \cup_i G_2$, is obtained by identifying exactly i pairs of vertices u_j, v_j, $1 \leq j \leq i$ such that $\{u_1, u_2, \ldots, u_i\} \subseteq V(G_1)$ and $\{v_1, v_2, \ldots, v_i\} \subseteq V(G_2)$.

Two graphs G_1 and G_2 are *isomorphic*, written as $G_1 \cong G_2$, if there exists a bijection $\phi : V(G_1) \mapsto V(G_2)$ such that $uv \in E(G_1)$ if and only if $\phi(u)\phi(v) \in E(G_2)$.

A *walk* (of *length* k) in a graph is a non-empty alternating sequence $v_0e_0v_1e_1 \ldots e_{k-1}v_k$ of vertices and edges in the graph such that $e_i = v_iv_{i+1}$ for all $0 \leq i < k$. It is a *closed* walk if $v_0 = v_k$. A *trail* is a walk in which no edge is repeated. If a trail begins and ends at a same vertex, it is a *closed trail*. A *path* is a walk in which no vertex is repeated. If $P = v_0 \ldots v_k$ is a path and $k \geq 2$, then $C := P + v_kv_0$ is a *cycle* (or *closed path*).

An edge is a *coloop* (or *bridge*, or *isthmus*) if it is not in any cycle. Removal of a coloop increases the number of connected components of a graph.

An *Eulerian circuit* is a closed walk that visits every edge in a graph exactly once. A graph is *Eulerian* if it contains an Eulerian circuit.

A graph H is a *subgraph* of G (or G is a *supergraph of* H), written as $H \leq G$, if H is a graph such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If $V(H) = V(G)$, then H is a *spanning subgraph* of G. Let $X \subseteq V(G)$. The subgraph $G[X]$ of G *induced* by X is the subgraph with vertex set X and edge set containing precisely the edges of G which join two vertices in X.

A *cutvertex* in a graph G is a vertex v such that $k(G - v) > k(G)$. A connected graph is *biconnected* (or *2-connected*) if it contains at least three vertices and has no cutvertex. Generally, if a graph G contains more than $k \geq 0$ vertices and $G - X$ is connected for every set $X \subseteq V(G)$ with $|X| < k$, then G is *k-connected*.

A *cut* is a partition of the vertex set of a graph into two disjoint subsets. A *cutset* of a cut is the set of edges that have one endpoint in each of these subsets. A *bond* is a minimal non-empty cut in a graph. A connected graph is disconnected by removing a cutset.

A *block* of a graph is a maximal connected subgraph that contains no cutvertex. Therefore, a subgraph H of a graph G is a block if H is a maximal biconnected subgraph, or H is a bridge with its two endpoints, or H is an isolated vertex.

Let G be a graph. The *rank* $r(X)$ of a set of edges $X \subseteq E(G)$ is the number of vertices it meets minus the number of connected components (isolated vertices are excluded) of the subgraph.

An *acyclic graph* is a graph that contains no cycles. A forest is an acyclic graph. If a forest is connected, then it is a *tree*. A *spanning tree* T of a graph G is a spanning subgraph of G that is a tree. For every edge $e \in E(G) \setminus E(T)$, there is a unique cycle in $T + e$. The unique cycle is the *fundamental cycle* of e *with respect to* T. For every edge $f \in E(T)$, the forest $T - f$ has exactly two components. The set of edges of G between these components is a bond in G, which is the *fundamental cutset* of f *with respect to* T.

Let D be a digraph with a distingushed vertex v. We call v the *root* of D, and D a digraph *rooted at* v.

An *arborescence* [81] is a directed tree rooted at a vertex v such that every edge that is incident with v is an outgoing edge, and exactly one edge is directed into each of the other vertices. For every non-root vertex in an arborescence, there exists a unique directed path in the arborescence that leads from the root vertex to the non-root vertex. Occasionally, to highlight this property, people describe the root vertex as *Rome*[1] [81]. Some authors define arborescences by reversing the direction of each edge in our definition, giving a set of arborescences that is different to ours. In this scenario, each unique directed path in the arborescence directs into rather than away from the root vertex. In both definitions, for every pair of vertices there exists a one-to-one correspondence between the sets of

[1] From the proverb: All roads lead to Rome.

arborescences of the same size rooted at each vertex. To change from one definition to the other, simply reverse the direction for all the edges.

A *spanning arborescence* of a digraph D is a subdigraph of D that is an arborescence which includes every vertex of D. The *arborescence number* $a_v(D)$ of a digraph D *with respect to* $v \in V(D)$ is the number of spanning arborescences rooted at v in D.

Let D be a rooted digraph. A subdigraph F of D is *feasible* if F is an arborescence. We call the edge set of F a *feasbile set*. If the edge set of F is maximal, then it is a *basis*. The *rank* of a subset $X \subseteq E(D)$ is defined as $r(X) = \max\{|A| : A \subseteq X, A \text{ is feasible}\}$.

Suppose D, D_1 and D_2 are rooted digraphs, and $E(D_1), E(D_2) \subseteq E(D)$. The digraph D is the *direct sum* of D_1 and D_2, written as $D = D_1 \oplus D_2$, if $E(D_1) \cup E(D_2) = E(D)$, $E(D_1) \cap E(D_2) = \emptyset$ and the feasible sets of D are precisely the unions of feasible sets of D_1 and D_2.

A *surface* Σ is a topological space in which every point has a distinct neighbourhood that is homeomorphic to the plane. An *embedded graph* (or *map*) $G \subset \Sigma$ is a graph drawn on a surface Σ such that edges do not intersect except at their endpoints. The connected components of $\Sigma \setminus G$ are called the *faces* of G [30]. A *plane graph* is a graph that is embedded in the plane. A graph is *planar* if it has an embedding in the plane.

An embedded graph is *2-cell embedded* (or *cellularly embedded*) if each of its faces is homeomorphic to an open unit disc.

In any embedded graph, the *boundary* ∂g of a face g is the closed trail that bounds g. A face is *incident* with every vertex and every edge that belongs to its boundary. Two faces are *adjacent* if their boundaries share at least one common edge.

2.1.1 Duality

Duality is an involution operation, that is, a graph goes back to its original state by applying the operation twice. In other words, duality is an operation of order two. It was originally defined for maps and it is defined for matroids (see Section 2.2) as well. The *dual* graph G^* of a connected plane graph G is obtained by putting a vertex in each face of the embedding of G. Two vertices in G^* are adjacent if their corresponding faces shared a common edge in G. Consequently, there exists a one-to-one correspondence between edges in G and G^*. The dual of a plane graph is also a plane graph. It is routine to show that $(G^*)^* = G$.

One of the interesting aspects of plane duality is that it relates geometrically two types of edge sets, namely cycles and bonds [20]. The edge sets of cycles of G correspond to the edge sets of bonds of G^*, and vice versa.

2.1.2 Deletions and Contractions

Suppose G is a graph and $e \in E(G)$. The *deletion* of e from G, denoted by $G \backslash e$, is the graph obtained from G by deleting e. The *contraction* of e in G, denoted by G/e, is the graph obtained by removing e and identifying the two endvertices of e. Both of these operations are used in evaluating the Tutte polynomial of a graph.

Deletion and contraction operations are dual in planar graphs.

$$G^* \backslash e = (G/e)^*,$$

$$G^*/e = (G \backslash e)^*.$$

These two operations are commutative. Let $e, f \in E(G)$ and $e \neq f$, then

$$G \backslash e \backslash f = G \backslash f \backslash e,$$

$$G/e/f = G/f/e,$$

$$G \backslash e/f = G/f \backslash e.$$

2.1.3 The Tutte Polynomial

The *Tutte polynomial* [79, 80, 82, 87, 88] is a two-variable polynomial that contains a variety of information about other polynomials including the chromatic polynomial, reliability polynomial, Jones polynomial, flow polynomial, and the partition functions of the Ising model and Potts model. They are specialisations of the Tutte polynomial, and their relationships with the Tutte polynomial can be visualised through the Tutte plane[2].

There are three equivalent ways to define the Tutte polynomial of a graph G.

1. By the state sum expansion:

$$T(G; x, y) = \sum_{X \subseteq E(G)} (x-1)^{r(E)-r(X)} (y-1)^{|X|-r(X)}.$$

   The state sum expansion depends only on the rank function of a graph. It relates the Tutte polynomial to the *Whitney rank generating function*,

$$R(G; x, y) = \sum_{X \subseteq E(G)} x^{r(E)-r(X)} y^{|X|-r(X)},$$

   by coordinate transformation,

$$T(G; x, y) = R(G; x-1, y-1).$$

2. By the *deletion-contraction recurrence*:

$$T(G; x, y) = \begin{cases} 1, & \text{if } G \text{ is empty,} \\ x \cdot T(G/e; x, y), & \text{if } e \text{ is a coloop,} \\ y \cdot T(G \backslash e; x, y), & \text{if } e \text{ is a loop,} \\ T(G \backslash e; x, y) + T(G/e; x, y), & \text{otherwise.} \end{cases}$$

   The variables x and y in the recurrence are independent. Since deletion and contraction commute, this recurrence gives a well defined polynomial.

[2]The figure can be found in Welsh's book. Complexity: Knots, Colourings and Counting, page 140.

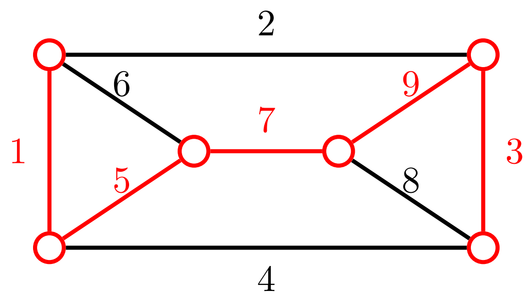


Figure 2.1: A graph G and one of its spanning trees T (highlighted in red)

3. By the notion of *basis activities*, using the fact that every connected graph has a spanning tree [53][3]:

   Let T be a spanning tree of G and $<$ be a total order on $E(G)$. An edge $e \in E(T)$ is *internally active* if e has the maximum order by the ordering of $<$ in the fundamental cutset in T. The *internal activity* $\text{int}(T)$ of T is the number of edges that are internally active in T. An edge $e \in E(G) \setminus E(T)$ is *externally active* if e has the maximum order by the ordering of $<$ in the fundamental cycle in T. The *external activity* $\text{ext}(T)$ of T is the number of edges that are externally active in T. Suppose $\mathcal{T}(G)$ be the set of spanning trees of G. Then,

$$T(G; x, y) = \sum_{T \in \mathcal{T}(G)} x^{\text{int}(T)} y^{\text{ext}(T)}. \tag{2.1}$$

   It has been shown that (2.1) is independent of the choice of $<$ [79]. In Figure 2.1, the edge order of the graph G is assigned arbitrarily. It is routine to check that edges $5, 7$ and 9 in $E(T)$ are internally active, and edge 6 is the only edge in $E(G) \setminus E(T)$ that is externally active, with respect to T.

   For a graph G that has $k > 1$ components $G_1, G_2, \ldots, G_k$,

$$T(G; x, y) = \prod_{i=1}^{k} T(G_i; x, y).$$

Two graphs G and H are *codichromatic* (or *Tutte equivalent*) if $T(G; x, y) = T(H; x, y)$.

The Tutte polynomial has made links between knot theory and statistical physics. Binary functions extend graphs and also have notions of deletion and contraction [33, 34].

By substituting appropriate values for variables x and y, the Tutte polynomial gives us a variety of information about a graph. For instance,

- $T(G; 1, 1)$ counts the number of spanning trees of a connected graph G.
- $T(G; 1, 2)$ counts the number of spanning subgraphs of G.
- $T(G; 2, 1)$ counts the number of forests of G.

[3] The English translation of [53] is: *Theory of Finite and Infinite Graph*, translated by Richard McCoart with commentary by W. T. Tutte.

The evaluation of the Tutte polynomial of a graph at a point in the complex (x, y)-plane is $\#P$-hard except on one special curve $(x-1)(y-1) = 1$, and some special points $(x, y) = (1,1), (-1,-1), (0,-1), (-1,0), (i,-i), (-i,i), (j,j^2), (j^2,j)$ where $j = e^{2\pi i/3}$ [35, 50].

The Tutte polynomial has been extended from graphs to matroids [77, 19, 12]. Welsh and Oxley [67] described the *Recipe Theorem*, a more general formula for using the deletion-contraction recurrence in matroids, extending earlier results of Tutte [76].

In this thesis, we study generalisations of the Tutte polynomial for some classes of directed maps and rooted directed graphs.

2.1.4 Minors

The notion of a minor plays an essential role in graph theory, especially in structural graph theory.

A graph H is a *minor* of a graph G, if a graph isomorphic to H can be obtained from G by some sequence of deletions and contractions. It obeys the three axioms of partial order, namely the reflexive, transitive and antisymmetric axioms.

A *subdivision* G' of a graph G is a graph obtained by replacing each edge of G by a new path whose internal vertices have degree exactly two in G'. A graph H is a *topological minor* of a graph G if a subdivision of H is isomorphic to a subgraph of G. (Note: H can be obtained from G by contracting some edges with at least one vertex of degree two, and deletion.) A topological minor is more restricted compared to an ordinary minor where the contraction operation is less restricted.

Some major results for graph minors include:

- *Kuratowski's Theorem* [20, 59]: a forbidden minor characterisation of planar graphs. It is a famous theorem which states that a finite graph is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$ as a subgraph.

- *Wagner's Theorem* [85]: states that a finite graph is planar if and only if its minors include neither K_5 nor $K_{3,3}$.

- *Robertson-Seymour Theorem* [20]: implies that an analogous forbidden minor characterisation exists for every property of graphs that is preserved under deletion and contraction. Equivalently, it says that for every minor-closed class of graphs, there exists a finite set of forbidden minors. This is one of the most remarkable results in graph theory in which Robertson and Seymour spent approximately two decades (1983–2004) on more than 500 pages in 20 different papers to prove a conjecture of Wagner.

2.1.5 Euler Genus and Euler Characteristic

Further details of definitions in this section can be found in [30].

Let Σ_1 and Σ_2 be two surfaces. The *connected sum* of Σ_1 and Σ_2 is obtained by deleting the interior of a disc in each surface and identifying the two boundaries. For instance, the connected sum of two tori is a 2-torus.

The *genus* $g(\Sigma)$ of a closed surface Σ is defined as follows:

$$g(\Sigma) := \begin{cases} 0, & \text{if } \Sigma \text{ is homeomorphic to the sphere,} \\ n, & \text{if } \Sigma \text{ is homeomorphic to the connected sum of } n \text{ tori,} \\ n, & \text{if } \Sigma \text{ is homeomorphic to the connected sum of } n \text{ real projective planes.} \end{cases}$$

If an embedded graph G is connected, the *genus* $g(G)$ of G is the genus of its surface. Otherwise, the genus of G is the sum of the genera of its components.

The *Euler genus* $\gamma(G)$ of a connected embedded graph G is defined as follows:

$$\gamma(G) := \begin{cases} 2g(G), & \text{if } G \text{ is orientable,} \\ g(G), & \text{if } G \text{ is non-orientable.} \end{cases}$$

The *Euler characteristic* $\chi(G)$ of an embedded graph G is

$$\chi(G) = |V(G)| - |E(G)| + |f(G)|.$$

The relationship between the Euler characteristic and the Euler genus is as follows:

$$\chi(G) = |V(G)| - |E(G)| + |f(G)| = 2k(G) - \gamma(G).$$

2.1.6 Kirchhoff's Matrix-Tree Theorem

The number of spanning trees of a connected graph G can be calculated by using *Kirchhoff's Matrix-Tree Theorem* (also known as *Kirchhoff-Trent Theorem* in some recent literature) [10, 75, 83]. A square matrix $L = L(G)$, namely the *Laplacian matrix*, is constructed in which the rows and columns of L are both indexed by the vertices of G, and $L(G) = (a_{ij})_{n\times n}$ where

$$a_{ij} = \begin{cases} \deg(i), & \text{if } i = j, \\ -(\text{the number of edges between } i \text{ and } j), & \text{if vertices } i \text{ and } j \text{ are adjacent,} \\ 0, & \text{otherwise.} \end{cases}$$

Deleting the row and the column of an arbitrary vertex in $L(G)$ gives a new matrix whose determinant is the number of spanning trees of G.

2.1.7 Bicubic Graphs and Bicubic Maps

A graph is *bipartite* if its vertex set can be partitioned into two distinct subsets, such that every edge has an endvertex in each part of the partition. A *cubic graph* (or *trivalent graph*) is a 3-regular graph, in which every vertex has degree three. A *bicubic graph* [11] is a 3-regular bipartite graph. Note that every bicubic graph contains no bridge [11].

A *proper colouring* of a graph is an assignment of colours to vertices such that every pair of adjacent vertices receive different colours. A graph that can be coloured by using

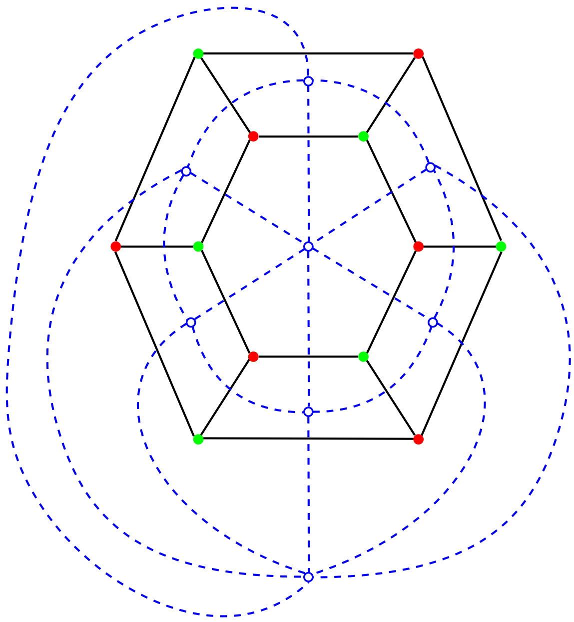


Figure 2.2: A bicubic map and its dual (in blue-dashed lines)

at most k colours is *k-colourable*. Since every cycle in a bipartite graph has even size, a bipartite graph and therefore a bicubic graph are both 2-colourable.

Suppose α is the number of vertices in the same colour class (either one) in a bicubic graph. The number of vertices and the number of edges in the bicubic graph are 2α and 3α, respectively. If a bicubic graph is embedded in the plane, it is a *bicubic map* [11] M. A 3-regular map is bicubic if and only if the number of sides of each of its faces is even. There are $\alpha + k(M) + 1$ faces (including the outer face) in M, where $k(M)$ is the number of connected components of M [11]. It can be seen easily that every face in the dual M^* of M is triangular, and every vertex in M^* has even degree.

An example of a bicubic map and its dual (in blue-dashed lines) is shown in Figure 2.2. In this example, the vertices of the bicubic map are coloured red and green.

2.2 Matroids

Matroids were introduced by Whitney [91] in studying abstractions of linearly independent and dependent subsets of the columns of matrices. Matroids can be defined using different axiom sets, including independence axioms, circuit axioms, basis axioms and rank axioms. They all are proved to be equivalent.

A *matroid* [86, 66] M over a finite *ground set* E is an ordered pair $(E, \mathcal{I})$ where $\mathcal{I} \subseteq 2^E$ is a collection of subsets of E satisfying the following three properties:

(I1) $\emptyset \in \mathcal{I}$.

(I2) If $X \in \mathcal{I}$ and $Y \subseteq X$, then $Y \in \mathcal{I}$.

(I3) If $X, Y \in \mathcal{I}$ and $|X| < |Y|$, then there is an element $y \in Y - X$ such that $X \cup \{y\} \in \mathcal{I}$.

The properties (I2) and (I3) are the *hereditary* and *independence augmentation properties*, respectively. The members of $\mathcal{I}$ are the *independent sets* of M. A subset of E that is not in $\mathcal{I}$ is *dependent*. If M is a matroid $(E, \mathcal{I})$, then we say M is a matroid *on* E.

Let M be a matroid on E, and $X \subseteq E$. A *base* in M is a maximal independent set of M. By contrast, a minimal dependent set of M is a *circuit*. The *rank* $r(M)$ of M is the size of the largest independent set. The *rank* $r(X)$ of X is the size of the largest independent subset of X.

Theorem 2.1. [66] *A function* $r : 2^E \mapsto \mathbb{N} \cup 0$ *is the rank function of a matroid on* E *if and only if* r *has the following properties:*

(R1) *If* $X \subseteq E$, *then* $0 \leq r(X) \leq |X|$.

(R2) *If* $X \subseteq Y \subseteq E$, *then* $r(X) \leq r(Y)$.

(R3) *If* $X, Y \subseteq E$, *then* $r(X \cup Y) + r(X \cap Y) \leq r(X) + r(Y)$.

Let M be a matroid with rank function r and $X \subseteq E(M)$. The subset X is independent if and only if $|X| = r(X)$. The subset X is a base if and only if $|X| = r(X) = r(M)$. The subset X is a circuit if and only if X is non-empty and $r(X - x) = |X| - 1 = r(X)$ for all $x \in X$.

Two matroids are *isomorphic* if there is a bijection between their underlying ground sets that preserves the rank.

One natural approach in studying the Tutte polynomial is to extend from graphs to matroids. Greedoids, which will be introduced in the following section, are a generalisation of matroids. We study a two-variable polynomial, which is an analogue of the Tutte polynomial, for an important class of greedoids in Chapter 7.

2.3 Greedoids

Greedoids were introduced by Korte and Lovász as collections of sets that generalise matroids [54]. Korte and Lovász observed that the optimality of some "greedy" algorithms including breadth-first search could be traced back to an underlying combinatorial structure that satisfies the greedoid, but not the matroid, framework. Björner and Ziegler [5] used two algorithmic constructions of a minimum spanning tree of a connected graph, i.e., Kruskal's and Prim's algorithms, to distinguish between greedoids and matroids. For each step in both algorithms, an edge with the minimum weight is added into the minimum spanning tree. The edge sets of the trees/forests that are obtained in each step form the feasible sets of a greedoid. Feasible sets obtained via Kruskal's algorithm remain feasible when removing any edge from the sets. However, this is not always true for feasible sets that are obtained via Prim's algorithm. Therefore, the greedoid that is obtained by using Kruskal's algorithm (but not Prim's algorithm) is in fact a matroid.

There are two equivalent ways to define greedoids, using set systems or hereditary languages [56, 57]. We define greedoids based on set systems. A *greedoid* over a finite ground set E is a pair (E, F) where $F \subseteq 2^E$ is a non-empty collection of subsets of E (called the *feasible sets*) satisfying:

(G1) For every non-empty $X \in F$, there is an element $x \in X$ such that $X - \{x\} \in F$.

(G2) For $X, Y \in F$ with $|X| < |Y|$, there is an element $y \in Y - X$ such that $X \cup \{y\} \in F$.

The *rank* $r(A)$ of a subset $A \subseteq E$ in a greedoid (E, F) is defined as $r(A) = \max\{|X| : X \subseteq A, X \in F\}$. Any greedoid is uniquely determined by its rank function.

Theorem 2.2. [55] *A function* $r : 2^E \mapsto \mathbb{N} \cup \{0\}$ *is the rank function of a greedoid* (E, F) *if and only if for all* $X, Y \subseteq E$ *and for all* $x, y \in E$ *the following conditions hold:*

(R1) $r(X) \leq |X|$.

(R2) *If* $X \subseteq Y$*, then* $r(X) \leq r(Y)$.

(R3') *If* $r(X) = r(X \cup \{x\}) = r(X \cup \{y\})$*, then* $r(X) = r(X \cup \{x\} \cup \{y\})$.

Important classes of greedoids are those associated with rooted graphs and rooted digraphs. These are called *branching greedoids* and *directed branching greedoids*, respectively.

Let G be a rooted undirected graph and $X \subseteq E(G)$. The *rank* $r(X)$ of X is defined as $r(X) = \max\{|A| : A \subseteq X, A \text{ is a rooted subtree}\}$. Let F be the set of subtrees of G containing the root vertex. Korte and Lovász [55] showed that (G, F) is a greedoid called the *branching greedoid* of G.

A *directed branching greedoid* over a finite set E of directed edges of a rooted digraph is a pair (E, F) where F is the set of feasible subsets of E. This was defined and shown to be a greedoid by Korte and Lovász [55].

In Chapter 7, we investigate a two-variable polynomial of directed branching greedoids.

CHAPTER 3

Alternating Dimaps

We adopt the terminology and notation of alternating dimaps in [78, 37].

A *dimap* is a directed graph that is drawn on an orientable surface such that edges do not intersect except at their endpoints.

An *alternating dimap* $D = (V, E) \subset \Sigma$ is a dimap that is 2-cell embedded in a disjoint union of orientable surfaces Σ (or 2-manifolds) where, for each vertex, the sequence of edges incident with it is directed inwards and outwards alternately in a cyclic order around the vertex. All vertices have even degree. An alternating dimap may have loops and multiple edges. If an alternating dimap contains no vertices, edges, or faces, it is the *empty alternating dimap*. The set of vertices, edges, faces and the number of connected components of D are denoted by $V(D)$, $E(D)$, $F(D)$, and $k(D)$, respectively. As D embeds in orientable surfaces, the genus $g(D)$ of D is given by the equation $|V(D)| - |E(D)| + |F(D)| = 2k(D) - 2g(D)$. As a consequence of the alternating property, every edge in a face of D is oriented in the same direction, to form either a clockwise face or an anticlockwise face. The number of clockwise faces and the number of anticlockwise faces of D are denoted by $\mathrm{cf}(D)$ and $\mathrm{af}(D)$, respectively. For simplicity, sometimes the clockwise faces are called c-faces, whereas the anticlockwise faces are called a-faces. An *in-star* is the set of edges directed into a vertex. The in-star that is directed into a vertex $v \in V(D)$ is denoted by $I_D(v)$ (or $I(v)$ when the context is clear). The number of in-stars of D is denoted by $\mathrm{is}(D)$. Every edge $e \in E(D)$ belongs to one clockwise face, one anticlockwise face and one in-star, which are denoted by $C_D(e)$, $A_D(e)$ and $I_D(e)$, respectively. Where no ambiguity arises, we may write $C(e)$ for $C_D(e)$, $A(e)$ for $A_D(e)$, and $I(e)$ for $I_D(e)$. The next edge after e going around $C(e)$ (respectively, $A(e)$) in the direction indicated by e is the *right successor* (respectively, *left successor*) of e.

An undirected embedded graph can be transformed into an alternating dimap by replacing each edge of the embedded graph by two oppositely-directed edges. These pairs of directed edges must all form either clockwise 2-cycles or anticlockwise 2-cycles. Based on this construction, the definition of alternating dimaps will be satisfied, even though the undirected graph is not necessarily an Eulerian graph.

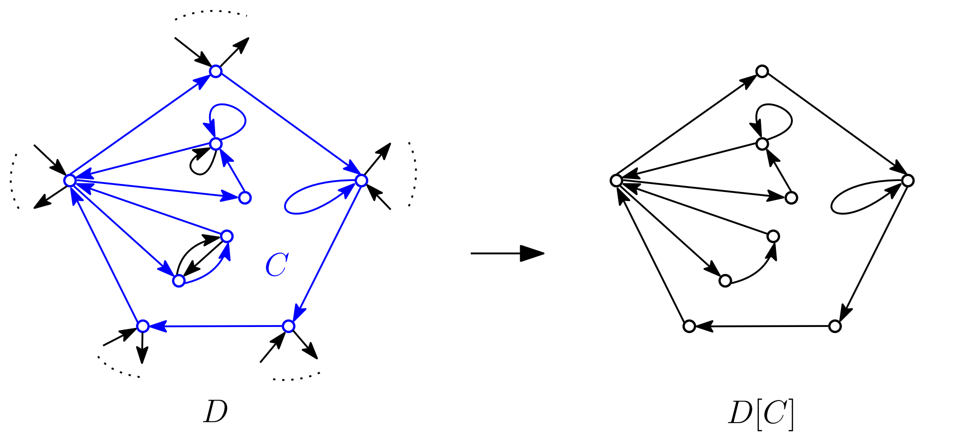


Figure 3.1: An alternating subdimap induced by a closed trail C (shown in blue) bounding a face in D

Every alternating dimap D defines three permutations[1], denoted by $\sigma_{D,1}, \sigma_{D,\omega}, \sigma_{D,\omega^2}$. Let $e \in E(D)$. The image of e under $\sigma_{D,1}$ is the next edge going around $I(v)$ in the clockwise direction. Correspondingly, the image of e under $\sigma_{D,\omega}$ (respectively, σ_{D,ω^2}) is the next edge going around $A(e)$ (respectively, $C(e)$) in the clockwise direction. The left successor of e is $\sigma_{D,\omega}^{-1}(e)$ in the permutation notation. Likewise, the right successor of e is $\sigma_{D,\omega^2}(e)$. Permutations are applied from right to left in the composition of permutations. Note that $\sigma_{D,1}\sigma_{D,\omega}\sigma_{D,\omega^2}$ gives the identity permutation on $E(D)$.

Two alternating dimaps D_1 and D_2 are *isomorphic*, written as $D_1 \cong D_2$, if there exists a bijection $\phi : V(D_1) \mapsto V(D_2)$ such that (i) $uv \in E(D_1)$ if and only if $\phi(u)\phi(v) \in E(D_2)$, and (ii) $uv \in E(D_1)$ has vw and vx as its right successor and left successor respectively, if and only if $\phi(u)\phi(v) \in E(D_2)$ has $\phi(v)\phi(w)$ and $\phi(v)\phi(x)$ as its right successor and left successor, respectively.

We say D' is an *alternating subdimap* of D, written as $D' \leq D$, if D' is an alternating dimap where $V(D') \subseteq V(D)$ and $E(D') \subseteq E(D)$.

Suppose $e \in E(D)$. We write D/e and $D \setminus e$ for the alternating dimap, or dimap, obtained from D by contracting e and by deleting e, respectively[2].

By the alternating property, every face in D is bounded either by an embedding of a clockwise closed trail or an anticlockwise closed trail. Let C be the closed trail bounding a face in D. The alternating subdimap *induced* by C, written as $D[C]$, is the alternating subdimap of D with the vertex set $V(C)$ and the edge set $E(C)$ (see Figure 3.1).

A *block* of an alternating dimap is a maximal connected alternating subdimap that contains no cutvertex.

The *closure* of a subset S of points in a topological space is the union of S and its boundary.

[1]The indices 1, ω and ω^2 are motivated by the cube roots of 1, namely 1, $\omega = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$ and $\omega^2 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$.

[2]A dimap (instead of an alternating dimap) $D \setminus e$ is obtained by deleting a non-loop edge e from an alternating dimap D. In this scenario, both of the endvertices of e have odd degree in $D \setminus e$, hence $D \setminus e$ is no longer an alternating dimap. See Section 3.2 for edge types in alternating dimaps.

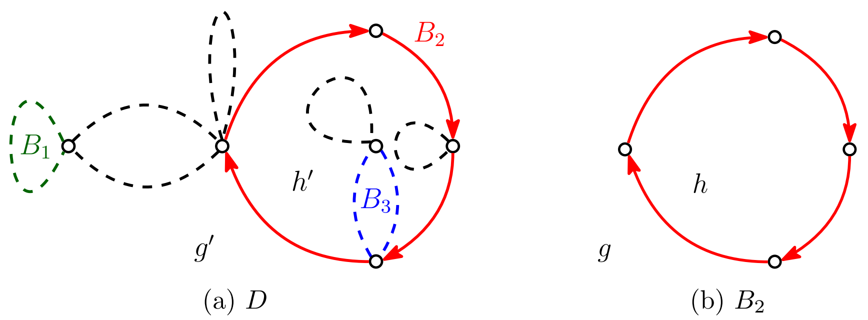


Figure 3.2: (a) Blocks within faces of the block B_2 in an alternating dimap D where B_2 is a directed cycle and other blocks are shown schematically using dashed lines, (b) The block B_2 in D

Suppose D contains blocks B_1 and B_2, and let $g \in F(B_2)$. The block B_1 is *within* the face g if the point set formed by the embeddings of $V(B_1)$ and $E(B_1)$ is a subset of the closure of g. In Figure 3.2(a), the block B_1 (highlighted in green) is within the face $g \in F(B_2)$, whereas the block B_3 (highlighted in blue) is within the face $h \in F(B_2)$ (and not within g). The faces $g, h \in F(B_2)$ are shown in Figure 3.2(b).

Let D_1 and D_2 be two alternating dimaps, let a_1 and a_2 be anticlockwise faces of D_1 and D_2 respectively, and let $v_1 \in V(\partial a_1)$ and $v_2 \in V(\partial a_2)$ be vertices of the faces a_1 and a_2. The *c-union* D of D_1 and D_2 *with respect to* a_1, v_1, a_2 and v_2, denoted by $D_1 \cup_c D_2$, is obtained by identifying v_1 and v_2 such that D_1 is within the anticlockwise face a_2 of D_2, and D_2 is within the anticlockwise face a_1 of D_1, in D. When the context is clear, we may just refer to the *c-union* D of D_1 and D_2. Note that the set of clockwise faces of D is the union of the sets of clockwise faces of D_1 and D_2 (hence the term c-union), and $|E(D)| = |E(D_1)| + |E(D_2)|$. An example of a c-union of two alternating dimaps is shown in Figure 3.3. The *a-union* is defined by appropriate modifications.

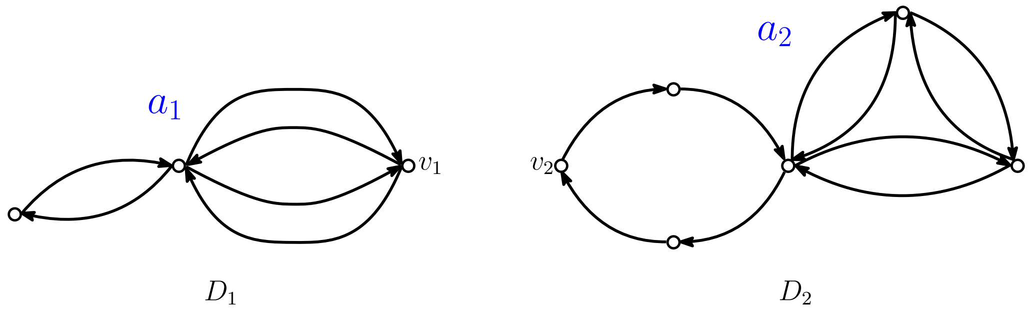
An *ordered alternating dimap* [37] is a pair $(D, <)$ where D is an alternating dimap and $<$ is a linear order on $E(D)$. An ordered alternating dimap can be obtained by assigning a fixed edge-ordering to an alternating dimap.

Let $\mathcal{G}$ be the set of plane graphs. Then, $\mathrm{alt}_c(\mathcal{G}) \coloneqq \{\mathrm{alt}_c(G) \mid G \in \mathcal{G}\}$ and $\mathrm{alt}_a(\mathcal{G}) \coloneqq \{\mathrm{alt}_a(G) \mid G \in \mathcal{G}\}$, where $\mathrm{alt}_c(G)$ and $\mathrm{alt}_a(G)$ of G are as defined in Chapter 1.

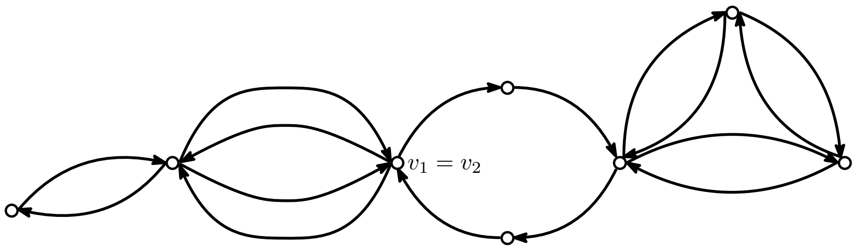
3.1 Outer Cycles

A *plane alternating dimap* $\mathrm{Pl}(D)$ of an alternating dimap D of genus zero is obtained from D by converting its embedding in the sphere into an embedding in the plane, by stereographic projection in the usual way [60].

The *face-rooted alternating dimap* D^g is an alternating dimap D in which the face g is distinguished from the other faces. If D^g has genus zero, then the plane alternating dimap $\mathrm{Pl}(D^g)$ of D^g is obtained according to the process described above such that g is the outermost region of $\mathrm{Pl}(D^g)$. Conversely, given a plane alternating dimap P with outermost region g, by reverse stereographic projection, we obtain a face-rooted alternating dimap $D^g = \mathrm{Pl}^{-1}(P)$ that has genus zero.



(a) Two alternating dimaps D_1 and D_2



(b) $D = D_1 \cup_c D_2$

Figure 3.3: A c-union of two alternating dimaps. For convenience, the anticlockwise faces $a_1 \in F(D_1)$ and $a_2 \in F(D_2)$ are both shown as outer regions

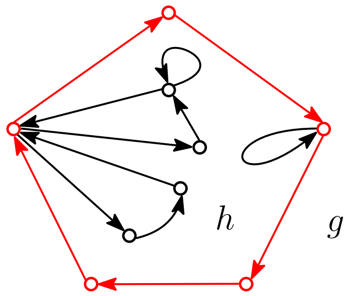


Figure 3.4: The plane alternating dimap $\mathrm{Pl}(H^g)$ of the face-rooted alternating dimap H^g where H is induced by the closed trail bounding h

Suppose C is the closed trail forming the boundary of a face h in an alternating dimap D, and $H = D[C]$ (see Figure 3.1). It is routine to show that H can be embedded on a sphere. The closed trail C can be partitioned into cycles, and each of these cycles encloses a face of opposite type to h. For any face $g \in F(H) \setminus \{h\}$, the face-rooted alternating dimap H^g can be used to obtain the plane alternating dimap $\mathrm{Pl}(H^g)$ (see Figure 3.4).

In this scenario, the *outer cycle* of h in D *with respect to* g is the cycle formed by the common edges between h and g in H. In Figure 3.4, the outer cycle of h in D with respect to g is coloured red. When the choice of g is clear from the context, we may refer just to the *outer cycle* of h in D. If an alternating subdimap $H = D[C]$ of genus zero is converted into a plane alternating dimap $P = \mathrm{Pl}(H^g)$, we use the outermost region g of P to determine the outer cycle of the face bounded by C in D.

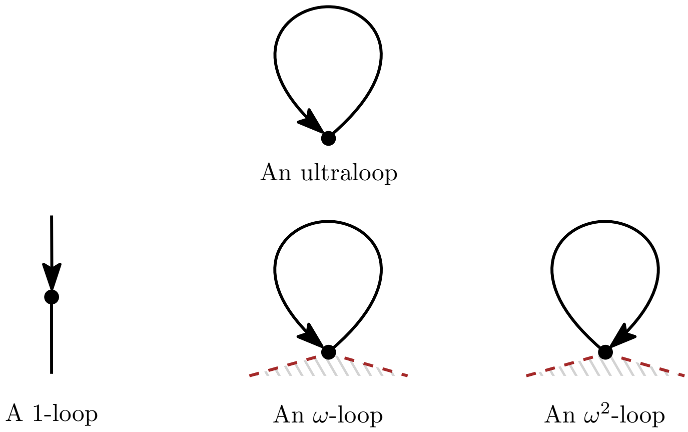


Figure 3.5: Loops

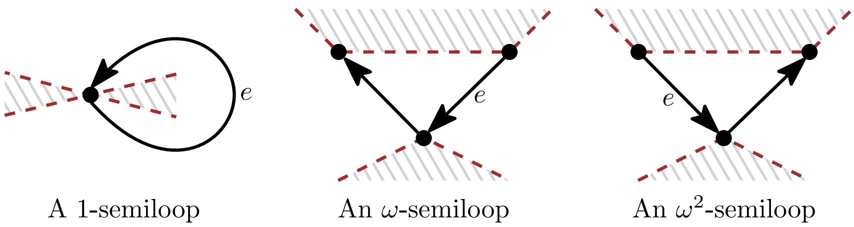


Figure 3.6: Semiloops

3.2 Triloops, Semiloops and Multiloops

There are a number of different types of special edges that have been defined in alternating dimaps including 1-loops, ω-loops and ω^2-loops [37]. An edge whose head has degree two is a *1-loop*. A single edge forming an a-face is an *ω-loop* whereas a single edge forming a c-face is an *ω^2-loop*. An *ultraloop* is concurrently a 1-loop, an ω-loop and an ω^2-loop. It is the only possible single-edge component in any alternating dimap. An illustration of these loops is given in Figure 3.5.

An edge is a *triloop* if it is a 1-loop, an ω-loop or an ω^2-loop. In other words, it is a μ-loop for some $\mu \in \{1, \omega, \omega^2\}$. If a μ-loop is not an ultraloop, then it is a *proper μ-loop*. A proper μ-loop is a *proper triloop*.

A *1-semiloop* is a standard loop. We consider two scenarios in defining ω-semiloops and ω^2-semiloops. If a loop e is its own right successor, e is an *ω-semiloop*. It is also an ω^2-loop under this circumstance. Note also that every ω^2-loop is an ω-semiloop. On the other hand, if e and its right successor are distinct, and they form a cutset of D or removal of them decreases the genus of D, then e is also an *ω-semiloop*. An ω-loop e is also an *ω^2-semiloop* if e is its own left successor. Note also that every ω-loop is an ω^2-semiloop. If e and its left successor are distinct, and they form a cutset of D or removal of them decreases the genus of D, then e is an *ω^2-semiloop*. In both cases, by removing the cutsets, the number of components of D is increased, or the genus of D is decreased. For $\mu \in \{1, \omega, \omega^2\}$, a μ-semiloop is a *proper μ-semiloop* if it is not a triloop. An illustration of the three different types of semiloop is given in Figure 3.6.

An edge is *proper* if it is not a semiloop (and hence not a triloop or an ultraloop).

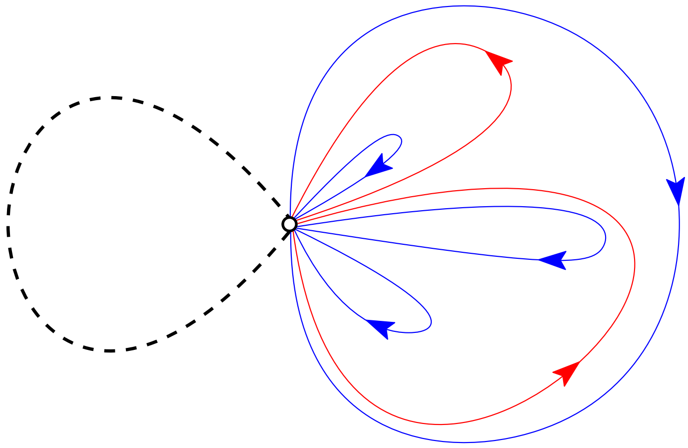


Figure 3.7: A c-multiloop within an anticlockwise face of an alternating dimap (the edges of the c-multiloop are coloured red and blue, and the alternating dimap is shown in dashed line)

Suppose e and f are two loops in an alternating dimap D. We can use our earlier definition (see page 19) of one block being within a face of another, and the fact that every loop is a block. The loop e is *within* a face g of f if the point set formed by the embedding of e is a subset of the closure of g.

Let D_1 be an alternating dimap of genus zero with a single vertex v_1 and $|E(D_1)| = m \geq 1$. Since every edge in D_1 is a loop, there exists a clockwise face or an anticlockwise face of size one in D_1. Suppose D_1 has an ω-loop e and let f_1 be the anticlockwise face of size one that is incident with e. Let D_2 be an alternating dimap of genus zero, $f_2 \in F(D_2)$ be an anticlockwise face and $v_2 \in V(\partial f_2)$. Suppose $D = D_1 \cup_c D_2$ with respect to f_1, v_1, f_2 and v_2. Then, D_1 is a *c-multiloop* of size m within the anticlockwise face f_2 of D_2, in D. An example of a c-multiloop is shown in Figure 3.7. An *a-multiloop* is defined by appropriate modifications.

A c-multiloop of size m within an anticlockwise face f_1 of an alternating dimap D of genus zero can also be constructed as follows. Add in one proper ω^2-loop e that is incident with a vertex $v \in V(\partial f_1)$. Denote by f_2 the clockwise face of size one incident with e. Then, $m-1$ proper ω-loops or ω^2-loops that are incident with v are added within the face f_2 such that these $m-1$ loops only intersect at v, and edges incident with v are directed inwards and outwards alternately in a cyclic order around v. The m edges that are added into D form a c-multiloop within f_1 of D. Note that at the time each of the $m-1$ loops h is added within f_2, if h is a proper ω-loop (respectively, ω^2-loop), it has an anticlockwise face (respectively, a clockwise face) of size one. If some other loops are added within the anticlockwise face (respectively, clockwise face) of size one of h, then h is no longer a proper ω-loop (respectively, ω^2-loop).

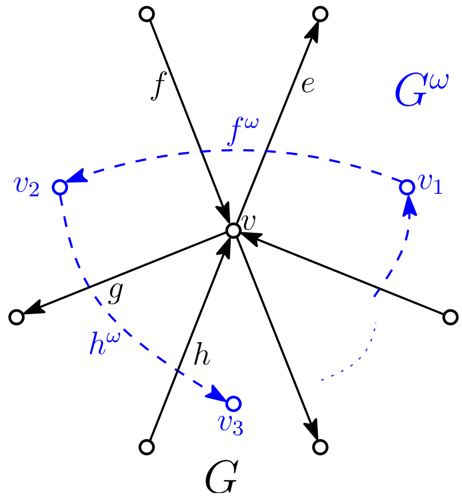


Figure 3.8: Construction of a trial map

3.3 Trial Operations

The concept of *triality* (or *trinity*) was introduced by Tutte when he studied the dissections of equilateral triangles [78]. We discuss trial operations in this section. See, for example [78, 81, 3, 37] for full details.

Given an alternating dimap D, we define the *trial* D^ω of D as follows.

- Vertices of D^ω correspond to the clockwise faces of D. We first place a vertex in each clockwise face of D.

- We construct edges of D^ω as follows. For all the clockwise faces that are incident with a common vertex $v \in V(D)$, let the vertices of D^ω assigned to these clockwise faces be $v_1, v_2, \ldots, v_n$, $n \in \mathbb{Z}^+$, in an anticlockwise order around v. Then, draw a directed edge from v_1 to v_2, from v_2 to v_3, and eventually from v_n to $v_{n+1} = v_1$ (see Figure 3.8), in such a way that each of these directed edges in D^ω first crosses an outgoing edge, followed by an incoming edge, of v (the outgoing edge and the incoming edge need not be distinct). Edges in D^ω are labelled as follows. Suppose v_i and v_j represent clockwise faces incident with v in D where $j \equiv i+1 \pmod{\mathrm{indeg}(v)}$. If an edge in D^ω that joins vertices v_i and v_j of D^ω crosses an incoming edge f of v in D, the edge will be denoted by f^ω.

Based on this construction, it is also clear that the map $e \mapsto e^\omega$ is a bijection from $E(G)$ to $E(G^\omega)$.

We usually write D^{ω^2} for $(D^\omega)^\omega$.

Note that the clockwise faces, anticlockwise faces and in-stars in D correspond to the in-stars, clockwise faces and anticlockwise faces in D^ω, respectively. In exponents of edge labels, the symbols ω multiply, and $\omega^3 = 1$, so e.g. $((e^\omega)^\omega)^\omega = e^{\omega^3} = e$. Hence, we have $D^{\omega^3} = D$.

Suppose e is an edge in an alternating dimap. The edge types of e^ω and e^{ω^2} are shown in Table 3.1.

e	e^{ω}	e^{ω^2}
ultraloop	ultraloop	ultraloop
proper 1-loop	proper ω-loop	proper ω^2-loop
proper 1-semiloop	proper ω-semiloop	proper ω^2-semiloop
proper edge	proper edge	proper edge

Table 3.1: The edge type of an edge e after trial operations on e (see Figures 3.5 and 3.6 for different edge types)

3.4 Reduction Operations

In this section, we give the definitions of three minor operations for alternating dimaps, namely 1-reductions, ω-reductions and ω^2-reductions [37].

Let D be an alternating dimap. Suppose $u, v \in V(D)$ and $e = uv \in E(D)$.

For $\mu \in \{1, \omega, \omega^2\}$, the alternating dimap that is obtained by reducing e in D using μ-reduction is denoted by $D[\mu]e$.

For the 1-*reduction*, if e is an ω-loop or an ω^2-loop, the edge e is deleted to obtain $D[1]e$. If e is not a loop (see Figure 3.9(a),(b)), then $D[1]e$ is obtained by contracting the edge e. Note that contracting an edge in an alternating dimap always preserves the alternating property of the alternating dimap. If e is a 1-semiloop that is incident with a vertex v, the alternating dimap $D[1]e$ is formed as follows (see Figure 3.10). Let $e, a_1, b_1, \ldots, a_s, b_s, e, c_1, d_1, \ldots, c_t, d_t$ be the cyclic order of the edges that are incident with v, starting from some edge e that is directed out from v. Observe that each a_i and each d_i is an incoming edge of v, and each b_i and each c_i is an outgoing edge of v. In $D[1]e$, the edge e is removed and the vertex v is split into two new vertices v_1 and v_2. Each a_i and each b_i is incident with v_2, while each c_i and each d_i is incident with v_1. The cyclic orderings of edges incident with v_1 and v_2 are induced by the cyclic ordering around v. Note that this reduction will either increase the number of components or reduce the genus.

Let $\ell = vn$ and $r = vm$ be the left successor and the right successor of $e = uv$ in D, respectively.

For the ω-*reduction*, if e is an ω-loop or an ω^2-loop, the edge e is deleted to obtain $D[\omega]e$. Otherwise, the alternating dimap $D[\omega]e$ is obtained by first deleting both of the edges e and ℓ, and a new edge $\ell' = un$ is created such that the position of the tail (respectively, head) of ℓ' in the cyclic ordering of edges incident with u (respectively, n) in $D[\omega]e$ is the same as the position of the tail (respectively, head) of e (respectively, ℓ) in the cyclic ordering of edges incident with u (respectively, n) in D (see Figure 3.9(a),(c)). If $\deg(v) = 2$, the vertex v is also removed from $D[\omega]e$.

For the ω^2-*reduction*, if e is an ω-loop or an ω^2-loop, the edge e is deleted to obtain $D[\omega^2]e$. Otherwise, the alternating dimap $D[\omega^2]e$ is obtained by first deleting both of the edges e and r, and a new edge $r' = um$ is created such that the position of the tail (respectively, head) of r' in the cyclic ordering of edges incident with u (respectively, m) in $D[\omega^2]e$ is the same as the position of the tail (respectively, head) of e (respectively, r) in

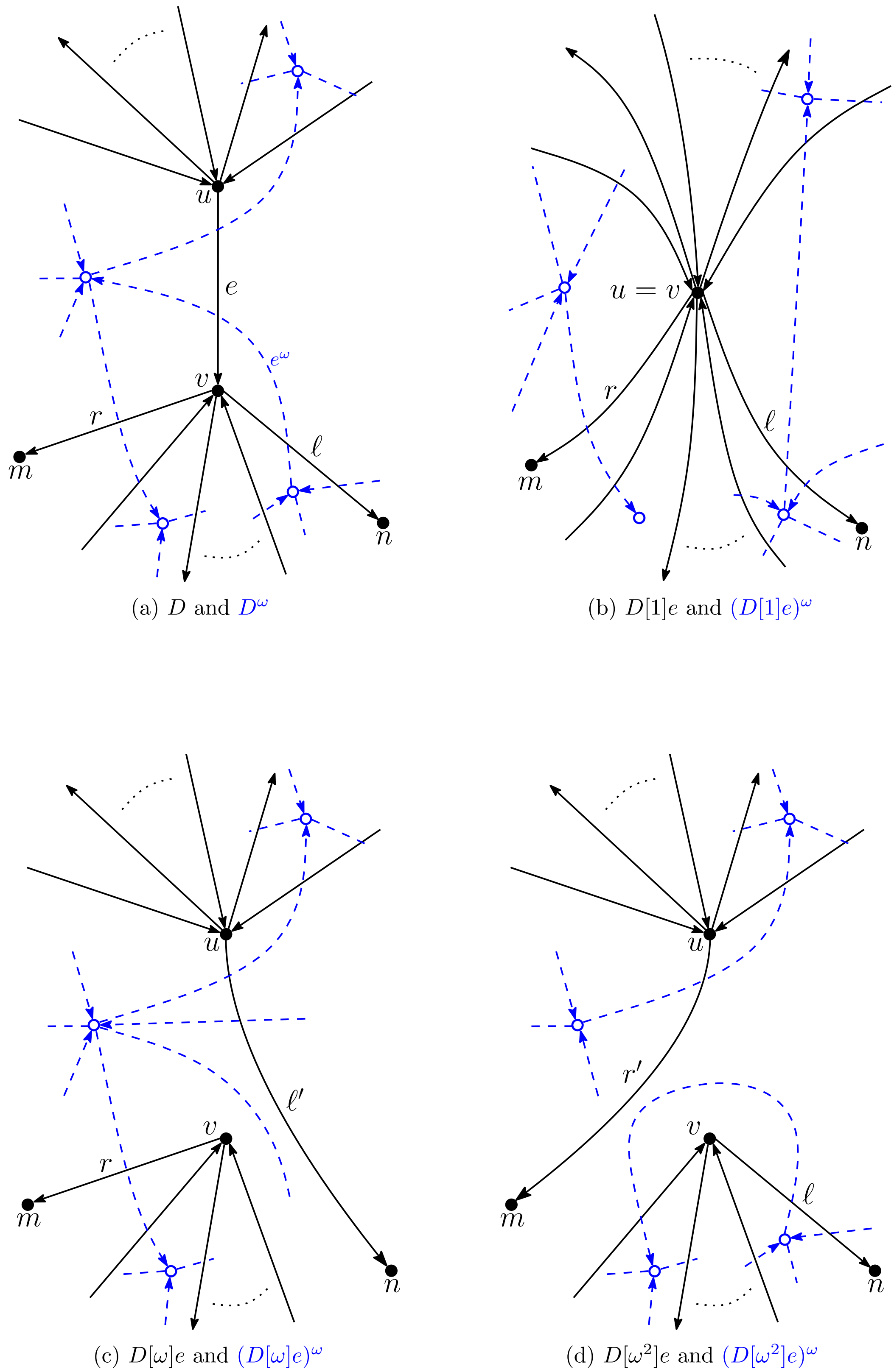


Figure 3.9: The three minor operations for alternating dimaps and their trials (in blue) [37]

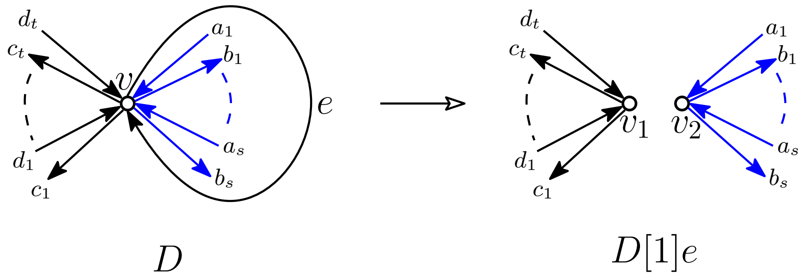


Figure 3.10: A 1-semiloop e in D is reduced by using the 1-reduction

the cyclic ordering of edges incident with u (respectively, m) in D (see Figure 3.9(a),(d)). If $\deg(v) = 2$, the vertex v is also removed from $D[\omega^2]e$.

We call these three operations the *reduction operations* or the *minor operations* for alternating dimaps.

A *minor* of an alternating dimap D is obtained by reducing some of its edges using a sequence of reduction operations.

For the reduction of a triloop $e \in E(D)$, we have $D[1]e = D[\omega]e = D[\omega^2]e$. Since the type of reduction operation is insignificant, we sometimes write $D[*]e$ when a triloop e is reduced.

For $\mu \in \{1, \omega, \omega^2\}$, if $(D, <)$ is an ordered alternating dimap, then the *μ-reduction* $(D, <)[\mu]$ of $(D, <)$ is the ordered alternating dimap $(D[\mu]e_0, <')$ where e_0 is the first edge in $E(D)$ under $<$ and the order $<'$ on $E(D) \setminus \{e_0\}$ is obtained by simply removing e_0 from the order $<$.

CHAPTER 4

Literature Review

In this chapter, we give an overview of certain topics, particularly alternating dimaps, the Tutte polynomial and some known polynomials for directed digraphs.

Alternating dimaps were introduced by Brooks, Smith, Stone and Tutte in 1940, when they studied the dissection of equilateral triangles into equilateral triangles. They found that each dissection of equilateral triangles can be represented using three directed graphs. As a result, they extended the concept of duality to a higher order, namely triality. Details are given in Sections 4.1 and 4.2.

Tutte introduced a two-variable polynomial for graphs, which plays an essential role in graph theory. The polynomial is now known as the Tutte polynomial. We give the background and some of the generalisations of the Tutte polynomial in Section 4.3.

To obtain minors of alternating dimaps, Farr introduced three reduction operations and showed that these operations do not commute in general. He also studied invariants of alternating dimaps, by establishing several recurrence formulae that involve these reduction operations. We present some of his results in Section 4.4.

A brief overview of arborescences is given in Section 4.5. Tutte justified further that triality is a generalisation of duality by showing that every alternating dimap has the same arborescence number as both of its trials.

Lastly, we discuss some polynomials for directed graphs. Our focus is the two-variable greedoid polynomial introduced by Gordon and McMahon [39]. This is an analogue of the Whitney rank generating function, which has a close relationship with the Tutte polynomial.

4.1 Squaring Squares

Plane alternating dimaps were introduced by Tutte [78] in 1948 as a tool for studying the dissection of equilateral triangles into equilateral triangles. This followed his collaboration with Brooks, Smith and Stone—known as the *Trinity Four*[1]—on dissecting rectangles into squares [10]. Their aim was to divide a rectangle into n squares of different sizes with

[1]They were members of the Trinity Mathematical Society at Cambridge University in the 1930s.

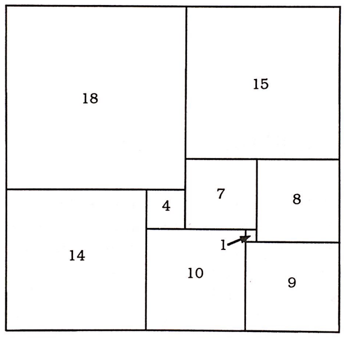


Figure 4.1: A 32×33 squared rectangle [83]

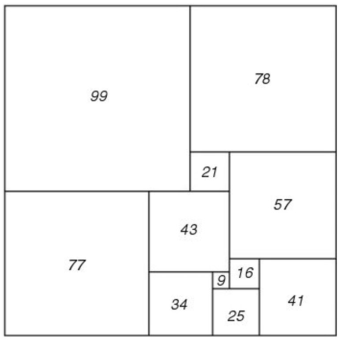


Figure 4.2: A 176×177 squared rectangle [1]

none of them overlapping any other. The rectangle has *order* n and is *perfect* if such a dissection exists. The process is known as *squaring the rectangle.*

The idea was initially raised by Stone based on a problem in the book "The Canterbury Puzzles" [22], written by H. E. Dudeney. In this problem, a square lid of a casket was subdivided into a rectangle and several squares of different sizes. Dudeney claimed that the solution is unique without any proof or reference. The Trinity Four discovered Lusin's Conjecture [61] which stated that a perfect square is infeasible. However, they stumbled across the fact that Moroń, a Polish mathematician, had given a perfect rectangle on the dissection of a 32×33 rectangle (see Figure 4.1) in Rouse Ball's book [71].

The Trinity Four found that all perfect rectangles have order of at least nine and there exist exactly two order-9 perfect rectangles. The first new perfect rectangle was found by Stone (see Figure 4.2). It has order 11 and dimension of 176×177. See [83, 1] for the method used. Most of their findings were of order 9, 10 and 11. Some simple perfect rectangles are available in Table 5.3 in their 1940s article [10, pp. 324–325].

The Trinity Four modelled a squared rectangle using a graph that is identical to an electrical network (see Figure 4.3). Each maximal horizontal line segment and each square of the squared rectangle were represented by a vertex and a directed edge, respectively. Another graph that represents the same squared rectangle was also constructed, by using the maximal vertical line segments of the squared rectangle (see Figure 4.4). The dotted lines in Figures 4.3 and 4.4 were added to complete the two networks. They named these networks *completed nets* (or *c-nets*) of the squared rectangle. The two c-nets of a squared

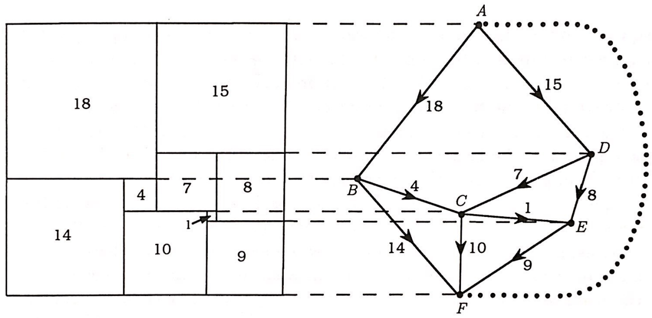


Figure 4.3: A horizontal electrical network [83]

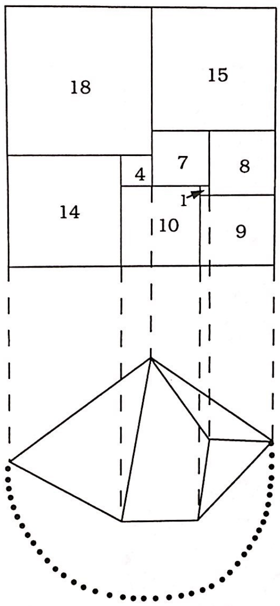


Figure 4.4: A vertical electrical network [83]

rectangle were used to describe the concept of duality. By putting both c-nets together in the plane, each vertex of a c-net lies in a face of another, as shown in Figure 4.5.

The first perfect square, which has order 55, was constructed by combining two perfect rectangles [73]. Duijvestijn [23] proved that the lowest order of a simple perfect square is 21 (see Figure 4.6), with the aid of the DEC-10 computer at the University of Twente, The Netherlands, in 1978. It is now known to be the unique perfect square of lowest order [24, 51].

4.2 Triangulating Triangles

The Trinity Four proposed several problems in [10] including dissections of equilateral triangles, which they studied in [78, 11]. In triangulating triangles, each dissection of a triangle was represented by a directed graph.

A *triangulation of order n* of any polygon (particularly equilateral triangles) is a dissection of the polygon into $n > 1$ equilateral triangles. The equilateral triangles obtained

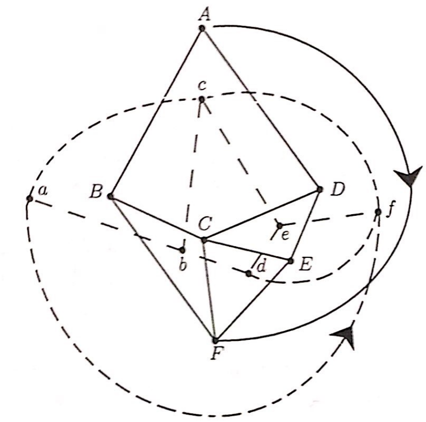


Figure 4.5: Dual c-nets [83]

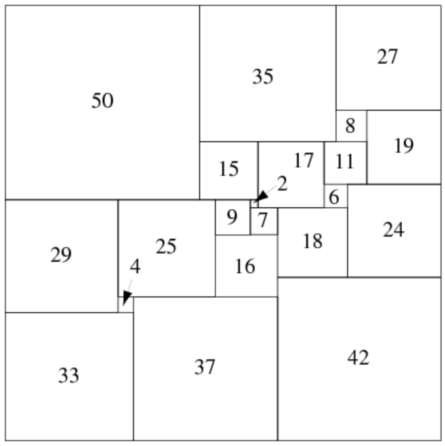


Figure 4.6: The lowest-order simple perfect squared square [23]

after the dissection are the *elements* of the triangulation. They do not overlap with each other and completely fill the original polygon. Tutte showed that it is not possible to dissect a triangle into equilateral triangles of unequal sizes [78]. However, it is possible to dissect a triangle into triangles and rhombuses where none of them are equal in size. There are two types of elements in a dissection of an equilateral triangle, namely *positive* elements and *negative* elements. Each positive element has an identical orientation to the original equilateral triangle whereas each negative element is rotated through an angle of 180° compared with the orientation of positive elements. The *size* of an element is determined by the side length of the element together with a sign. The size has a positive sign if the element is a positive element, and a negative sign otherwise. The triangulation is *perfect* if no two elements have the same size. It follows that each perfect triangulation has at most two elements of the same side length, and these two elements have different orientations.

Theorem 4.1. [78] *In any triangulation of an equialteral triangle some two elements have a side in common.*

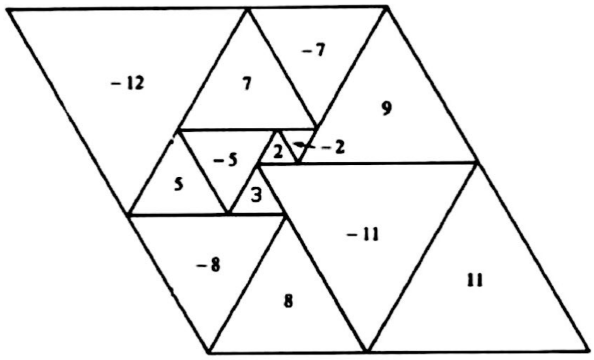
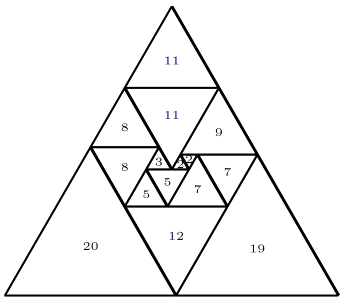
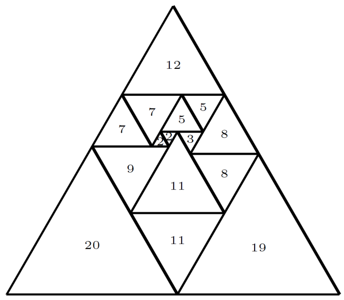


Figure 4.7: The simplest perfect parallelogram [83]



(a)



(b)

Figure 4.8: The smallest perfect equilateral triangle of size 15 in two different dissections

Through the triangulations of triangles, triality was shown to be a generalisation of duality. Networks constructed based on the dissection of equilateral triangles are found to appear in triples (more details will be given later). Tutte defined *triality* or *trinity* as the relationship among the three members.

Tutte found that the triangulations of parallelograms that have angles of 60° and 120° can be obtained easily through the triangulations of equilateral triangles, and vice versa. The simplest perfect parallelogram is shown in Figure 4.7. This parallelogram gives the two simplest perfect triangulations of equilateral triangles as shown in Figure 4.8, by using a transformation described in [78].

Drápal and Hämäläinen [21] proved by computer enumeration that the order of the smallest perfect equilateral triangle is 15 with two different dissections (see Figure 4.8). Their findings verified Tutte's Conjecture [78, 21] that the smallest perfect equilateral triangle has order 15.

Dissections of parallelograms into equilateral triangles are somehow analogous to dissections of rectangles into squares [78, 83]. Every square in a squared rectangle is first transformed into a rhombus that has angles of 60° and 120°, by shearing the upper side of the rectangle horizontally to the right (or left) with respect to the lower side. Each rhombus is dissected into two equilateral triangles by cutting along its short diagonal. Then, a triangulated parallelogram is obtained (see Figure 4.9). Each element in a dissected parallelogram is assigned a positive or a negative sign according to its orientation (the method is the same as the one in the triangulations of equilateral triangles). Using

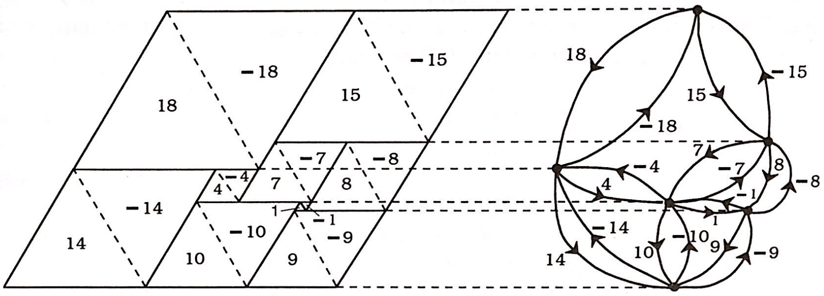


Figure 4.9: A sheared perfect rectangle [83]

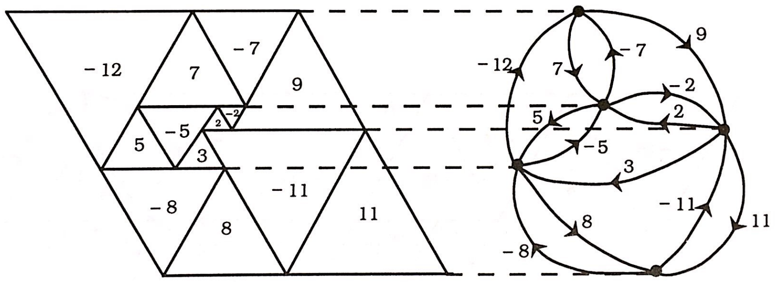


Figure 4.10: An unsymmetric triangulated parallelogram [83]

the differences of signs, Tutte said that the triangulation is actually *perfect* and concluded that every perfect rectangle shears into a perfect parallelogram. Note that there exist perfect triangulations of parallelograms that are not generated by perfect rectangles. One such example is shown in Figure 4.7.

The Trinity Four represented triangulated parallelograms by using electrical networks. The mechanism is similar to the electrical networks generated by perfect rectangles. Each horizontal line of a triangulated parallelogram is represented by a vertex. Each triangle is bounded by two horizontal lines, containing the apex and base, respectively. For each triangle, a directed edge is used to join the two vertices from its apex to base. The side length of each triangle, either positive or negative, is assigned as the weight of the corresponding directed edge (see Figure 4.9 and Figure 4.10[2]). The upper and lower sides of the parallelogram correspond to the positive and negative poles of a network, respectively. They observed that every vertex in the network has even degree and there is no crossing between edges. In addition, each triangle meets a given horizontal line either at its base or its apex, in an anticlockwise cyclic order. This implies that directed edges incident with each vertex in the network appear in the opposite direction to their neighbours. They called this planar map an *alternating dimap*, to highlight the alternation of the edge directions at each vertex [11, 83]. Note that the triangulated parallelogram in Figure 4.9 can be obtained by shearing the perfect rectangle in Figure 4.3. By comparing the two electrical networks in these two figures, Tutte found that an alternating dimap can be obtained from a graph by replacing each edge in the graph by two directed edges

[2]The figure is obtained from [83, p. 36] with a minor error. Both edges that have weight 2 and -2 should be directed to the opposite direction.

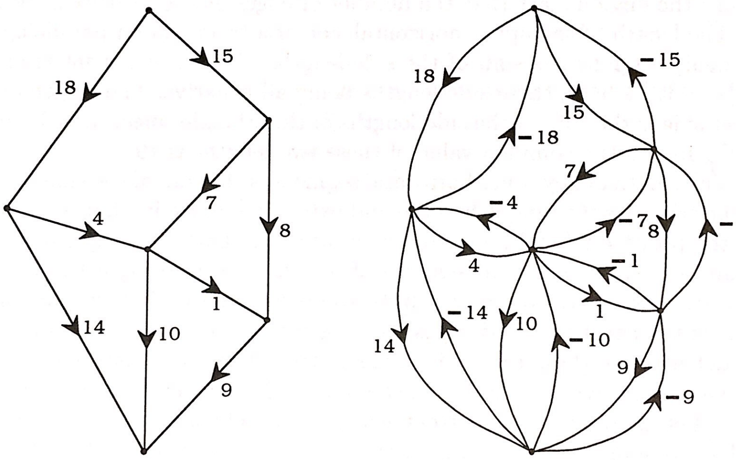


Figure 4.11: A comparison between two electrical networks [83]

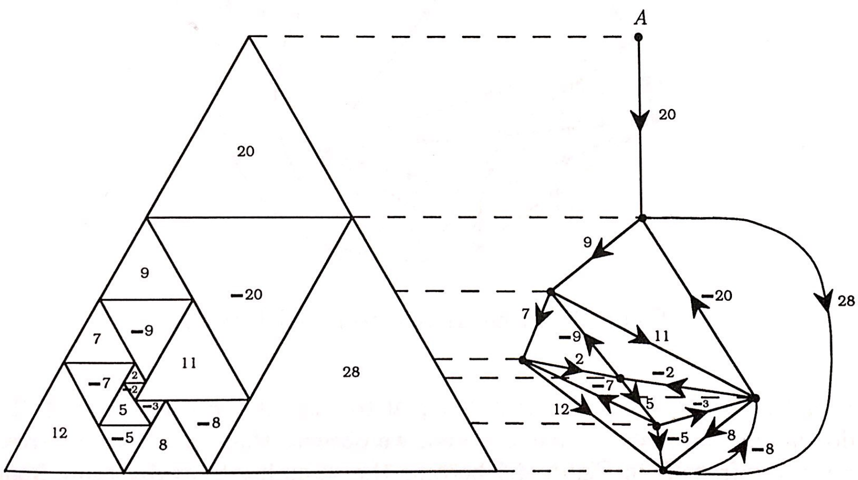


Figure 4.12: A triangulated triangle with one of its three networks [83]

that form either a clockwise face or an anticlockwise face (see Figure 4.11), as described in Chapter 3.

The Trinity Four found that every triangulated equilateral triangle gives three electrical networks (by rotating the triangle 120° twice). The constructions of these networks followed the one for parallelograms, except for one extra vertex A which was added to represent the apex of the triangle (see Figure 4.12). By identifying the two poles of each of these networks, followed by a minor modification on the weight of the outgoing edge of A, three alternating dimaps were obtained (see one of them in Figure 4.13, where the modified edge is indicated by a cross). Since the two electrical networks that were obtained through the dissection of a rectangle were called dual c-nets, they referred these three alternating dimaps as *trine* alternating dimaps. The relationship between trine alternating dimaps was called *triality* (or *trinity*), which is a generalisation of duality.

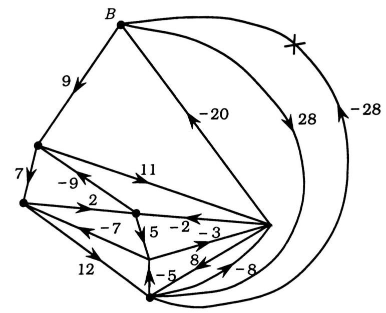


Figure 4.13: From triangle to parallelogram [83]

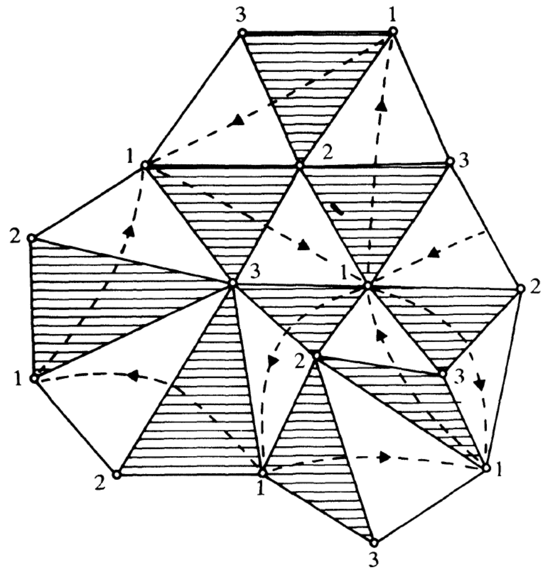


Figure 4.14: A 3-colourable Eulerian triangulation [81]

The Trinity Four referred to the electric flows in a network obtained from a triangulated triangle as *leaky*[3]. They showed that in a balanced directed graph, the number of spanning arborescences rooted at each vertex is the same [11]. Subsequently, they constructed three plane alternating dimaps, called a *triad*, by using a coloured bicubic map [11], using the fact that a cubic map can be face-coloured in three colours if and only if it is bicubic [47, 48, 49].

Tutte [81] introduced the concept of derived maps, which is then used to construct alternating dimaps. By using a finite connected graph M of size at least one that is drawn on the sphere, he constructed a derived map M' that has the following properties: (i) every face in M' is a triangle, (ii) it is 3-colourable, and (iii) the degree of each vertex is even. In other words, M' is an Eulerian triangulation (see the dual of the bicubic map in Figure 2.2). He also explained a way to convert from M' to M, and to generate the dual map M^* by swapping the vertices in M'. The dual map was studied by Smith and Tutte [72] in 1950. Tutte constructed alternating dimaps by considering a 3-colourable Eulerian triangulation map (shown in part in Figure 4.14) [81]. Tutte showed that trinity is a true generalisation of duality.

[3]Leaky electricity was named by Smith. In [83], Tutte proposed to rename it as *unsymmetrical electricity*.

Berman [3] studied the two derived alternating dimaps of an alternating dimap. He called the three alternating dimaps a *trinity of alternating dimaps*. He showed that duality is a special case of trinity.

4.3 Tutte-Whitney Polynomials

The Four-Colour Theorem states that no more than four colours are required to colour the regions of a map such that no two adjacent regions receive the same colour. Whitney [89] extended the notion of colouring and studied the vertex-colouring of the dual graph of a map. By using the inclusion-exclusion principle, he deduced the following formula to determine the number of ways to colour a graph G:

$$\chi(G;\lambda) = \sum_{p,s} (-1)^s G_{p,s} \lambda^p, \tag{4.1}$$

where $G_{p,s}$ is the number of subgraphs of G that have p components and s edges. Let X be a subgraph of G with r vertices, s edges and p components. He defined the rank and nullity of X as $i = r - p$ and $j = s - i = s - r + p$, respectively. By putting $G_{p,s} = m_{r-p,s-r+p} = m_{i,j}$ where $m_{i,j}$ is the number of subgraphs of G that have rank i and nullity j, and by appropriate substitutions into (4.1), he obtained

$$\chi(G;\lambda) = \sum_{i,j} (-1)^{i+j} m_{i,j} \lambda^{r-i} = \sum_{i} m_i \lambda^{r-i}, \tag{4.2}$$

where $m_i = (-1)^{i+j} m_{i,j}$. The polynomial (4.2) is known as the *chromatic polynomial*. The chromatic polynomial was generalised to the Whitney rank generating function [90], or equivalently the Tutte polynomial.

Tutte [79] generalised the chromatic polynomial by introducing a two-variable polynomial $\chi(G;x,y)$ for a graph G. He called this polynomial the *dichromate* of G. (The polynomial is now known as the *Tutte polynomial* $T(G;x,y)$. Tutte commented that: "This may be unfair to Hassler Whitney who knew and used analogous coefficients without bothering to affix them to two variables". We will refer this polynomial as the Tutte polynomial from now onwards.) The Tutte polynomial of a graph is defined in terms of the spanning trees and a fixed enumeration of the edges of the graph. He introduced *internal* and *external activities* (see Section 2.1.3 for definitions) of a spanning tree S of a connected graph G, which were denoted by $\mathrm{int}(S)$ and $\mathrm{ext}(S)$, respectively. The Tutte polynomial of G is defined as follows:

$$T(G;x,y) = \sum_{S \in \mathcal{T}(G)} x^{\mathrm{int}(S)} y^{\mathrm{ext}(S)}, \tag{4.3}$$

where $\mathcal{T}(G)$ is the set of spanning trees of G. Tutte also proved that (4.3) is independent of the choice of ordering of the edges.

Tutte introduced a recurrence relation, which is now known as the *deletion-contraction recurrence*. Note that the idea of this recurrence relation is an analogue of the *W-function* and *V-function* that he defined in [76]. He then showed that the Tutte polynomial is multiplicative over blocks.

Theorem 4.2. [79] *If a graph G consists of two connected graphs H_1 and H_2 having just one vertex in common, then*

$$T(G; x, y) = T(H_1; x, y) \cdot T(H_2; x, y). \tag{4.4}$$

Note that an analogous result to Theorem 4.2 holds for matroids. Let M, M_1 and M_2 be matroids. Brylawski [12] conjectured that if $T(M; x, y) = T(M_1; x, y) \cdot T(M_2; x, y)$, then M is the direct sum of M_1 and M_2. Merino, de Mier and Noy [63] proved Brylawski's conjecture. This implies that the Tutte polynomial of a matroid (respectively, a graph) factorises if and only if the matroid (respectively, graph) is a direct sum. In Chapter 7, we investigate the factorisation of a two-variable analogue of the Whitney rank generating function of one class of greedoids.

Tutte [80] defined the *dichromatic polynomial* of a graph G as

$$Q(G; x, y) = \sum_{S \subseteq E} x^{k(S)} y^{|E|-|A|+k(S)}, \tag{4.5}$$

and studied its combinatorial properties. He found that $Q(G; x, y)$ was completely determined by the properties of the Tutte polynomial, with the following relation:

$$T(G; x, y) = (x-1)^{-k(G)} Q(G; x-1, y-1).$$

Crapo [19] and Brylawski [12] generalised the Tutte polynomial from graphs to matroids. Crapo defined a two-variable rank generating function $R(M; x, y)$ for a matroid M and established its algebraic properties. Let M^* be the dual of a matroid M. Crapo proved that $R(M; x, y) = R(M^*; y, x)$. He also showed that $R(M; x, y) = T(M; x+1, y+1)$, where $T(M)$ is the Tutte polynomial of M. The result is an analogue of the relationship between the Tutte polynomial and the Whitney rank generating function of graphs. His results further imply that $T(M; x, y) = T(M^*; y, x)$. Brylawski studied invariants that can be obtained from the Tutte polynomial of a matroid. Several invariants including numbers of subsets, bases, spanning sets and independent sets of a matroid, as well as Möbius function were found.

Farr [33] extended the Whitney rank generating function by defining the *Whitney quasi-rank generating function* of an arbitrary function $f : 2^E \to \mathcal{R}$ as

$$R(f; x, y) = \sum_{X \subseteq E} x^{Qf(E)-Qf(X)} y^{(Qf)^*(E)-(Qf)^*(E \setminus X)},$$

where Qf is a *quasi-rank function* and $(Qf)^*$ is its dual, defined as follows:

$$Qf(S) = \log_2 \left(\frac{\sum_{T \subseteq E} f(T)}{\sum_{T \subseteq E \setminus S} f(T)} \right),$$

$$(Qf)^*(S) = |S| + \log_2 f(\emptyset) - \log_2 \sum_{T \subseteq S} f(T).$$

He developed a relationship between the Hadamard transform and duality for quasi-rank functions. He also introduced operations of deletion and contraction, and minors of binary functions. He gave three specialisations of $R(f; x, y)$, namely the weight enumerator of a code, the percolation probability of a clutter and Kung's generalisation of the chromatic polynomial [58]. Farr [34] showed that a generalisation of the partition function of the Potts model to binary functions is also a partial evaluation of the Whitney quasi-rank generating function.

The Tutte polynomial has also been extended to graphs embedded on surfaces. However, information that is encoded by topological Tutte polynomials is not as rich as in the Tutte polynomial. One example of a topological Tutte polynomial is the ribbon graph polynomial introduced by Bollobás and Riordan [7, 8]. The *ribbon graph polynomial* (or *Bollobás-Riordan polynomial*) $R(G)$ of an embedded graph G is a four-variable polynomial, which is defined by adding two extra variables in the state sum expansion of the Tutte polynomial, as follows:

$$R(G; x, y, z, w) = \sum_{A \subseteq E(G)} (x-1)^{r(G)-r(A)} y^{n(A)} z^{k(A)-f(A)+n(A)} w^{t(A)}$$

where $k(A)$, $r(A) = |V(A)| - k(A)$, $n(A) = |E(A)| - r(A)$, $f(A)$ and $t(A)$ are the number of connected components, rank, nullity, number of components of the boundary and orientability of A, respectively. If A is orientable, $t(A) = 0$. Otherwise, $t(A) = 1$. Bollobás and Riordan developed relationships between the ribbon graph polynomial and the Tutte polynomial. They also introduced a deletion-contraction recurrence for the ribbon graph polynomial, which was extended by Ellis-Monaghan and Moffatt using more specific edge types [30].

Another example of topological Tutte polynomials is the Penrose polynomial which was originally defined implicitly for plane graphs [69]. Ellis-Monaghan and Moffatt [29] extended the Penrose polynomial to graphs embedded in arbitrary surfaces. They also showed that the Penrose polynomial satisfies a deletion-contraction relation.

Ellis-Monaghan and Sarmiento [31] constructed a one-variable polynomial, the *generalised transition polynomial* for all Eulerian graphs using a weight system. Ellis-Monaghan and Moffatt [28] extended the generalised transition polynomial to the *topological transition polynomial*. They gave a linear recurrence relation to compute the topological transition polynomial, and showed that the polynomial has the duality property. Ellis-Monaghan and Moffatt [30] proved that the Penrose polynomial of an embedded graph is an evaluation of the topological transition polynomial.

The *topochromatic polynomial* of an embedded graph G is given by

$$Z(G; a, \boldsymbol{b}, c, w) = \sum_{A \subseteq E(G)} a^{k(A)} \left(\prod_{e \in A} b_e \right) c^{f(A)} w^{t(A)},$$

where a, c, w are indeterminates and b is a set of indeterminates indexed by $E(G)$, was defined by Moffatt [64] as an extension of the ribbon graph polynomial. Ellis-Monaghan and Moffatt [30] found a relation between the topological transition polynomial and the topochromatic polynomial.

4.4 Tutte Invariants for Alternating Dimaps

The reduction operations for alternating dimaps (see Section 3.4) were introduced by Farr [37]. These operations are used to obtain minors of alternating dimaps D. Through reduction operations, Farr established a relationship between triality and minors.

Theorem 4.3. [37] *If $e \in E(D)$ and $\mu, \nu \in \{1, \omega, \omega^2\}$ then*

$$D^\mu[\nu]e^\mu = (D[\mu\nu]e)^\mu.$$

It is well known that deletion and contraction operations always commute. A natural question arises here is whether the reduction operations obey the commutative rule. Farr proved that if one of the two edges in any reduction is a triloop, then the reduction operations commute.

Theorem 4.4. [37] *If f is a triloop and $\mu, \nu \in \{1, \omega, \omega^2\}$, then*

$$D[\mu]e[\nu]f = D[\nu]f[\mu]e.$$

Some examples reveal that reduction operations do not always commute, unless certain conditions are imposed. Farr [37] proved the following theorems.

Theorem 4.5. *Let D be an alternating dimap and $e, f \in E(D)$.*

a) If $f \neq \sigma_{D,\omega}(e)$, then
$$D[1]e[\omega]f = D[\omega]f[1]e.$$

b) If $f \neq \sigma_{D,1}(e)$, then
$$D[\omega]e[\omega^2]f = D[\omega^2]f[\omega]e.$$

c) If $f \neq \sigma_{D,\omega^2}(e)$, then
$$D[\omega^2]e[1]f = D[1]f[\omega^2]e.$$

A set of k reductions is *k-commutative* on an alternating dimap D if applying these reductions on D in any order preserves the same result. If every set of k reductions is k-commutative, D is *k-reduction-commutative*. If D is k-reduction-commutative for every k, then D is *totally reduction-commutative*. Farr showed that if D is totally reduction-commutative, then any minor of D has the same property.

An alternating dimap with a single vertex, $2k+1$ loops and exactly two faces is known as a *k-posy* and has genus k. For instance, a 1-posy is a toroidal alternating dimap with one vertex, three loops and two faces. (For the 1-posy, it may seem like there are "six different faces" incident with the vertex, formed by three anticlockwise faces and three clockwise faces alternately. In fact, there is only one anticlockwise face and one clockwise face, where each meets the vertex three times.) Farr proved that the genus of a non-empty alternating dimap is less than k if and only if none of its minors is a disjoint union of posies with total genus of k.

Farr extended the notion of Tutte invariants to alternating dimaps. He introduced recurrence formulae that involve reduction operations, including *simple Tutte invariants* and *extended Tutte invariants* [37]. In the former, the study was mainly focusing on ultraloops, proper 1-loops, proper ω-loops and proper ω^2-loops treating as special cases. In the latter, proper 1-semiloops, proper ω-semiloops and proper ω^2-semiloops were also treated as special cases. Farr gave a characterisation of simple Tutte invariants.

Theorem 4.6. [37] *The only simple Tutte invariants of alternating dimaps are:*

- $F(D) = 0$;
- $F(D) = 3^{|E(D)|}$;
- $F(D) = (-1)^{|V(D)|}$;
- $F(D) = (-1)^{\mathrm{af}(D)}$;
- $F(D) = (-1)^{\mathrm{cf}(D)}$.

Farr showed that extended Tutte invariants are much richer than the simple Tutte invariant. He introduced the c-Tutte invariant $T_c(D; x, y)$ and the a-Tutte invariant $T_a(D; x, y)$ for any alternating dimap D [37]. He proved that $T_c(D; x, y)$ and $T_a(D; x, y)$ are well defined for certain alternating dimaps, when these polynomials are equal to the Tutte polynomial of plane graphs.

Theorem 4.7. [37] *For any plane graph G,*

$$T(G; x, y) = T_c(\mathrm{alt}_c(G); x, y) = T_a(\mathrm{alt}_a(G); x, y).$$

Farr suggested some problems for further research in [37], including: (1) identify other excluded minor characterisations for alternating dimaps, (2) characterise extended Tutte invariants and (3) investigate Tutte invariants for ordered alternating dimaps. We address the first two problems in this thesis.

4.5 Arborescences

The *tree number* $t(G)$ of a graph G is the number of spanning trees of G. By the *Tree-Duality Theorem*, it is known that the tree number of G is equal to the tree number of its dual G^*,

$$t(G) = t(G^*).$$

Tutte showed that if a digraph H is balanced, then the number of spanning arborescences of H is independent of the root node [78, 81]. Where no ambiguity arises, he referred to this as *the arborescence number* of H. He found that the arborescence number of a balanced digraph remains unchanged by reversing the direction of each edge in the digraph.

To justify further that triality is a true generalisation of duality, Tutte [81] proved the *Tree-Trinity Theorem* (or *Tutte's trinity theorem*) which asserted that trine alternating dimaps M_1, M_2 and M_3 have the same arborescence number,

$$t(M_1) = t(M_2) = t(M_3).$$

Note that trine alternating dimaps are all balanced, as digraphs.

Recall that Kirchhoff's Matrix Tree Theorem (see Section 2.1.6) can be used to compute the tree number of a graph G. It can be modified to count the number of spanning arborescences in digraphs (not necessarily balanced). The *Kirchhoff matrix* for digraphs is constructed as follows. Let H be a digraph. A square matrix $L = L(H)$ is constructed in which the rows and columns of L are both indexed by the vertices of H, and $L(H) = (a_{ij})_{n\times n}$ where

$$a_{ij} = \begin{cases} (\text{indegree of } i) - (\text{number of loops incident with } i), & \text{if } i = j, \\ -\ (\text{number of directed edges from } j \text{ to } i), & \text{if } i \neq j. \end{cases}$$

Deleting the row and the column of an arbitrary vertex v in $L(H)$ gives a new matrix whose determinant is the number of spanning arborescences rooted at v in H. This method was used in studying the relation between circuits and trees in oriented graphs [84].

Berman used determinants to prove Tutte's trinity theorem. He introduced a new determinant formula to compute the number of spanning arborescences of a digraph [3].

4.6 Polynomials for Directed Graphs

Polynomials for directed graphs are less common than polynomials for undirected graphs. In this section, we review polynomials for directed graphs, emphasising the two-variable greedoid polynomial introduced by Gordon and McMahon.

Let G be a greedoid. Gordon and McMahon [39] defined a two-variable greedoid polynomial of G by

$$f(G;t,z) = \sum_{A\subseteq E(G)} t^{r(G)-r(A)} z^{|A|-r(A)}$$

which generalises the one-variable greedoid polynomial $\lambda(G;t)$ given by Björner and Ziegler in [5]. We call the two-variable greedoid polynomial $f(G;t,z)$ the *greedoid polynomial*. The greedoid polynomial is motivated by the Tutte polynomial of a matroid [79], and is an analogue of the Whitney rank generating function [89, 90]. Gordon and McMahon studied greedoid polynomials for branching greedoids and directed branching greedoids. They showed that $f(D;t,z)$ can be used to determine if a rooted digraph D is a rooted arborescence [39]. However, this result does not extend to unrooted trees [26].

Gordon and McMahon proved that the greedoid polynomials of rooted digraphs D have the multiplicative direct sum property, that is, if $D = D_1 \oplus D_2$, then $f(D;t,z) = f(D_1;t,z) \cdot f(D_2;t,z)$. They also proved that if $f(T_1;t,z) = f(T_2;t,z)$ where T_1 and T_2 are both rooted arborescences, then $T_1 \cong T_2$.

Gordon and McMahon gave a recurrence formula to compute $f(D;t,z)$ where D is a rooted digraph. The following proposition gives the formula, which involves the usual deletion-contraction operations.

Proposition 4.8. [39] *Let D be a digraph rooted at a vertex v, and e be an outgoing edge of v. Then*

$$f(D;t,z) = f(D/e;t,z) + t^{r(D)-r(D\setminus e)} f(D \setminus e;t,z).$$

A *greedoid loop* [62] in a rooted graph, or a rooted digraph, is an edge that is in no feasible set. It is either an ordinary (directed) loop, or an edge that belongs to no (directed) path from the root node.

The factorisation of greedoid polynomials of greedoids, rooted graphs and rooted digraphs were investigated. Let D be a rooted digraph and $\overline{D}$ be the greedoid with ground set $E(D)$ whose feasible sets are the edge sets of rooted arborescences. McMahon proved the following results:

Proposition 4.9. [62] *If G is a greedoid with k loops, then $(1+z)^k$ divides $f(G)$.*

Theorem 4.10. [62] *Let G be a rooted graph with $f(\overline{G}) = (1+z)^a h(t,z)$, where $1+z$ does not divide $h(t,z)$. Then a is the number of greedoid loops in G.*

Theorem 4.11. [62] *Let D be a rooted digraph with no greedoid loops. Then D has a directed cycle if and only if $1+z$ divides $f(\overline{D})$.*

Gordon and McMahon introduced another method to define the greedoid polynomial. They defined the greedoid polynomial based on external activities [41]. A *computation tree* $T(G)$ of a greedoid G is a rooted binary tree such that

- if G has no feasible element, then $T(G)$ is the trivial labelled rooted tree with a single vertex that is labelled by G.
- if e is feasible in G, then obtain $T(G)$ by forming the rooted labelled trees with two children G/e and $G \setminus e$, respectively.

Let m be the number of leaves of a computation tree $T(G)$. Suppose $\{G_k : 1 \le k \le m\}$ is the set of greedoid minors of G which label the leaves of $T(G)$. For each k, let F_k be the elements of G which are contracted in the unique path from the root to the leaf labelled by G_k, and $\mathrm{ext}_T(F_k)$ be the elements of G which correspond to loops in G_k. Note that F_k is a feasible set of G. Gordon and McMahon called $x \in \mathrm{ext}_T(F_k)$ *externally active* for F_k with respect to $T(G)$ (see [41] for examples of $T(G)$ and $\mathrm{ext}_T(F_k)$). They proved the following theorem.

Theorem 4.12. [41] *Let G be a greedoid, $T(G)$ be any computation tree and F be all feasible sets. Then,*

$$f(G;t,z) = \sum_F t^{r(G)-|F|}(1+z)^{|\mathrm{ext}_T(F)|}.$$

A subset $S \subseteq E(G)$ is *spanning* if S contains a basis. They also gave a graph-theoretic interpretation for the highest power of $1+z$ which divides $f(G)$, in the following theorem.

Theorem 4.13. [41] *Let G be the directed branching greedoid associated with a rooted digraph D with no greedoid loops or isolated vertices. If $f(G;t,z) = (1+z)^k h(t,z)$, where $1+z$ does not divide $h(t,z)$, then k is the minimum number of edges that need to be removed from D to leave an acyclic directed graph D' such that $E(D')$ spans G.*

Tedford [74] defined a three-variable greedoid polynomial $f(G;t,p,q)$ for any rooted graph G, which generalises the two-variable greedoid polynomial. He showed that $f(G;t,p,q)$ obeys a recursive formula. He also proved that $f(G;t,p,q)$ determines the number of greedoid loops in any rooted graph G. His main result shows that $f(G;t,p,q)$ distinguishes connected rooted graphs G that are loopless and have at most one cycle. He extended $f(G;t,p,q)$ from rooted graphs to general greedoids, and proved that the polynomial determines the number of loops for a larger class of greedoids.

Clouse defined three fundamental types of greedoid invariants and the greedoid Tutte polynomial $h(G;t,z) = f(G;t-1,z-1)$ in [18]. Clouse characterised each of these greedoid invariants in terms of the greedoid Tutte polynomial.

Gordon and McMahon defined a characteristic polynomial $p(G)$ for a greedoid G based on an evaluation of the two-variable greedoid polynomial $f(G;t,z)$ [40].

Definition 4.1. *Let G be a greedoid on the ground set E. The characteristic polynomial $p(G;\lambda)$ is defined by*

$$p(G;\lambda) = (-1)^{r(G)} f(G;-\lambda,-1).$$

They showed that the characteristic polynomial of a greedoid G can also be defined by using a Boolean expansion, a feasible set expansion and a deletion-contraction recursion, as follows.

Proposition 4.14 (Boolean expansion)**.**

$$p(G;\lambda) = \sum_{A \subseteq E(G)} (-1)^{|A|} \lambda^{r(G)-r(S)}.$$

Proposition 4.15 (Feasible set expansion)**.** *Let T_G be a computation tree for G and $\mathcal{F}_T$ denote the set of all feasible sets of G having no external activity. Then*

$$p(G;\lambda) = \sum_{F \in \mathcal{F}_T} (-1)^{|F|} \lambda^{r(G)-|F|}.$$

Proposition 4.16 (Deletion-contraction recursion)**.** *Let e be a feasible set in G. Then*

$$p(G;\lambda) = \lambda^{r(G)-r(G\setminus e)} p(G \setminus e;\lambda) - p(G/e;\lambda).$$

They also proved that the greedoid characteristic polynomial has the multiplicative direct sum property.

Gordon and McMahon then studied the characteristic polynomial for rooted graphs and rooted digraphs [42]. They completely determined the characteristic polynomial for rooted digraphs with the following results.

Lemma 4.17. *Suppose D is a rooted digraph with a directed cycle. Then $p(D;\lambda) = 0$.*

Lemma 4.18. *Let D be a rooted digraph consisting of a single feasible edge. Then $p(D;\lambda) = \lambda - 1$.*

They improved the efficiency of the deletion-contraction recursion for rooted digraphs.

Proposition 4.19. *Suppose e is a feasible edge of an acyclic rooted digraph D, where e is not a leaf.*

- *If e is in every basis, then $p(D;\lambda) = -p(D/e;\lambda)$.*
- *If e is not in every basis, then $p(D;\lambda) = p(D \setminus e;\lambda)$.*

Their main result counts the number of sinks in rooted digraphs.

Theorem 4.20. *Let D be a rooted digraph. If D contains no greedoid loops and no directed cycles, then $p(D;\lambda) = (-1)^{r(D)}(1-\lambda)^s$, where s is the number of sinks in D.*

Eaton and Tedford [25] defined multiply-rooted (directed) branching greedoids that have a set of root nodes. They studied the characteristic polynomial of multiply-rooted directed trees. They gave a combinatorial interpretation for this polynomial.

Gordon and Traldi [43] attempted to generalise the Tutte polynomial of undirected graphs to a polynomial invariant of directed graphs. They found that there exist more than one such generalisation, by using different definitions of the Tutte polynomial. They defined polynomials for rooted digraphs using three rank functions based on the Whitney rank generating function $f(G;t,z) = \sum_{A\subseteq E(G)} t^{r(E)-r(A)} z^{|A|-r(A)}$. Suppose D is a digraph rooted at v and $A \subseteq E(D)$. The three rank functions were defined as follows:

- $r_1(A) \equiv \max\{|T| : T \subseteq A \text{ is a rooted arborescence}\}$,
- $r_2(A) \equiv \max\{|T \cap A| : T \subseteq E(D) \text{ is a rooted arborescence}\}$,
- $r_3(A) \equiv \max\{|F| : F \subseteq A \text{ is a rooted forest of arborescences}\}$.

They associated the Whitney rank generating function to these rank functions. Several evaluations at certain coordinates for each of these rank functions were given, including the number of arborescences rooted at v in D.

Chung and Graham [16] commented that, for directed graphs, no analogue of the Tutte polynomial is known. They introduced the cover polynomial for directed graphs and studied its relationships to other graph polynomials [16]. The cover polynomial $C(D;x,y)$ of a directed graph D is a polynomial in two indeterminates x and y, that can be obtained using the following recursive formula:

$$C(D;x,y) = \begin{cases} 1, & \text{if } D \text{ is empty,} \\ x^n, & \text{if } D \text{ contains } n \text{ vertices and no edges,} \\ C(D\setminus e;x,y) + C(D/e;x,y), & \text{if } e \text{ is a non-loop edge,} \\ C(D\setminus e;x,y) + y\cdot C(D/e;x,y), & \text{if } e \text{ is a loop.} \end{cases}$$

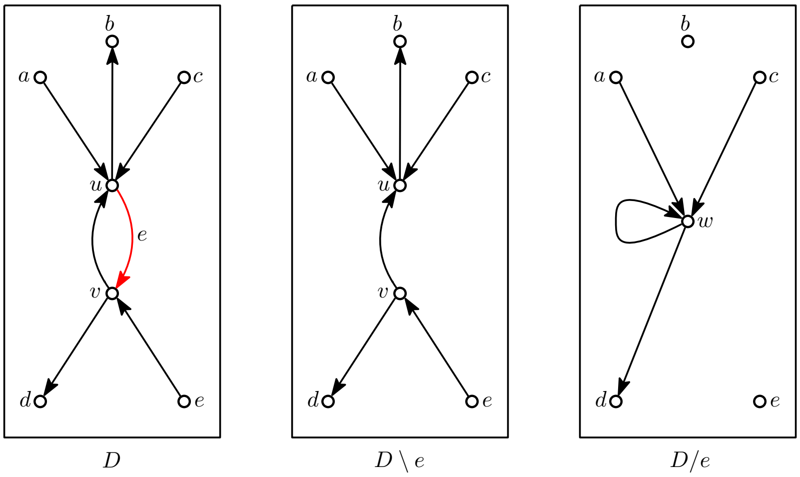


Figure 4.15: Deleting and contracting a non-loop edge from a digraph

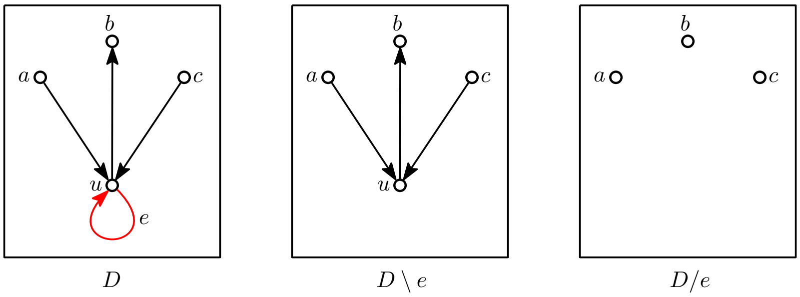


Figure 4.16: Deleting and contracting a loop from a digraph

Let $e \in E(D)$. In the recursive formula, the deletion operation is exactly the same as the usual deletion operation for undirected graphs, regardless of whether e is a loop or a non-loop edge. For $e = uv$ such that $u \neq v$, the contraction operation is obtained by replacing the two endpoints u and v by a new vertex w, and all edges of the form ux and yv in D are then removed. For $e = uu$ (a loop), the contraction operation is obtained by removing u and all the edges incident with u. These operations are illustrated in Figure 4.15 and Figure 4.16. They showed that $C(D; x, y)$ is a well defined invariant. They also gave some evaluations of $C(D; x, y)$ for specific values of x and y. One of their results that is analogous to a result on the Tutte polynomial is as follows:

Theorem 4.21. [16] *Suppose $D = (V, E)$ is formed by joining the disjoint digraphs $D_1 = (V_1, E_1)$ and $D_2 = (V_2, E_2)$ with all edges v_1v_2, $v_1 \in V_1$ and $v_2 \in V_2$. Then*

$$C(D) = C(D_1)C(D_2).$$

Awan and Bernardi [2] defined a trivariate polynomial, the B-polynomial, as a generalisation of the Tutte polynomial for directed graphs. Their goal is to extend the theory of the Tutte polynomial to digraphs. Let $D = (V, A)$ be a directed graph and q be any positive integer. For any function from V to $\mathbb{Z}$, suppose $f_A^{>}$ (respectively, $f_A^{<}$) is the set of arcs $(u, v) \in A$ such that $f(u) > f(v)$ (respectively, $f(u) < f(v)$). Then, the *B-polynomial* of D

$$B(D; q, y, z) = \sum_{f: V \to \{1,\dots,q\}} y^{|f_A^{>}|} z^{|f_A^{<}|}.$$

For any digraph D, several properties of $B(D; q, y, z)$ are given. One of these properties is that if D is the disjoint union of two digraphs D_1 and D_2, then $B(D; q, y, z) = B(D_1; q, y, z)B(D_2; q, y, z)$. They described a relation between the partition function of the Potts model of a graph G and the B-polynomial of the corresponding digraph that is obtained by replacing each edge uv of G by two arcs uv and vu. They proved that when D is acyclic, there exists a simple relation between $B(D; -q, y, 1)$ and $B(D; q, y, 1)$. They also defined the dual D^* of a connected planar digraph D and showed that there is a simple relation between $B(D; -1, y, z)$ and $B(D^*; -1, y, z)$.

CHAPTER 5

Characterisations of Extended Tutte Invariants

Farr introduced simple and extended Tutte invariants, and characterised simple Tutte invariants [37]. He commented that extended Tutte invariants are much richer than the simple Tutte invariant, as there exists a connection between extended Tutte invariants of certain alternating dimaps and the Tutte polynomial of planar graphs.

In this chapter, we extend Farr's work and investigate extended Tutte invariants for alternating dimaps of genus zero in two main directions. First, we characterise extended Tutte invariants that are well defined for all alternating dimaps of genus zero. We determine the restrictions that need to be imposed on the parameters of extended Tutte invariants, so that these invariants are well defined for all alternating dimaps of genus zero. Second, we assume that all the parameters of extended Tutte invariants are independent, and determine the set of alternating dimaps of genus zero for which their extended Tutte invariants are well defined. We also establish some excluded minor characterisations for these alternating dimaps of genus zero.

5.1 Arbitrary Alternating Dimaps, Dependent Parameters

The definition of extended Tutte invariants is given in Definition 5.2. For brevity, we use $P(D)$ as a shorthand for

$$P(D; w, x, y, z, a, b, c, d, e, f, g, h, i, j, k, l)$$

where D is an alternating dimap, throughout this chapter.

Throughout, $\mathcal{A}$ denotes a class of alternating dimaps.

Definition 5.1. *Let $\mathbb{F}$ be a field. A* multiplicative invariant (over $\mathbb{F}$) *for alternating dimaps in $\mathcal{A}$ is a function $P\colon \mathcal{A} \to \mathbb{F}$, such that P is invariant under isomorphism, $P(\emptyset) = 1$ and for the disjoint union of two alternating dimaps, G and H, $P(G \cup H) = P(G) \cdot P(H)$.*

Definition 5.2. *An* extended Tutte invariant *for alternating dimaps in $\mathcal{A}$ with respect to a parameter sequence $(w, x, y, z, a, b, c, d, e, f, g, h, i, j, k, l)$ of elements of a field $\mathbb{F}$ is a* *multiplicative invariant* P *over* $\mathbb{F}$ *such that for any alternating dimap* $D \in \mathcal{A}$ *and* $r \in E(D)$*,*

1. *if* r *is an ultraloop,*
$$P(D) = w \cdot P(D \setminus r), \tag{ETI1}$$

2. *if* r *is a proper 1-loop,*
$$P(D) = x \cdot P(D[1]r), \tag{ETI2}$$

3. *if* r *is a proper* ω*-loop,*
$$P(D) = y \cdot P(D[\omega]r), \tag{ETI3}$$

4. *if* r *is a proper* ω^2*-loop,*
$$P(D) = z \cdot P(D[\omega^2]r), \tag{ETI4}$$

5. *if* r *is a proper 1-semiloop,*
$$P(D) = a \cdot P(D[1]r) + b \cdot P(D[\omega]r) + c \cdot P(D[\omega^2]r), \tag{ETI5}$$

6. *if* r *is a proper* ω*-semiloop,*
$$P(D) = d \cdot P(D[1]r) + e \cdot P(D[\omega]r) + f \cdot P(D[\omega^2]r), \tag{ETI6}$$

7. *if* r *is a proper* ω^2*-semiloop,*
$$P(D) = g \cdot P(D[1]r) + h \cdot P(D[\omega]r) + i \cdot P(D[\omega^2]r), \tag{ETI7}$$

8. *otherwise,*
$$P(D) = j \cdot P(D[1]r) + k \cdot P(D[\omega]r) + l \cdot P(D[\omega^2]r). \tag{ETI8}$$

To define extended Tutte invariants for ordered alternating dimaps $(D, <)$, we have the following modifications:

1. For $\mu \in \{1, \omega, \omega^2\}$, each μ-reduction is replaced by $(D, <)[\mu]$.
2. The edge to be reduced is always the first edge e_0 in the linear order $<$ on $E(D)$, so the reference to edges is omitted for each reduction operation.

In this chapter, we require $\mathcal{A}$ to be the set of all alternating dimaps of genus zero. In Chapter 6, we will consider extended Tutte invariants that are only well defined for certain alternating dimaps.

It is well known that if a graph G is planar and G^* is the dual graph of G, then

$$T(G; x, y) = T(G^*; y, x).$$

We give an analogous relation for extended Tutte invariants, by using Theorem 4.3.

Theorem 5.1. *For any extended Tutte invariant P of an alternating dimap D,*

$$\begin{aligned}&P(D;w,x,y,z,a,b,c,d,e,f,g,h,i,j,k,l)\\&=P(D^{\omega};w,z,x,y,h,i,g,b,c,a,e,f,d,k,l,j)\\&=P(D^{\omega^2};w,y,z,x,f,d,e,i,g,h,c,a,b,l,j,k)\end{aligned}$$

Proof. Suppose P is an extended Tutte invariant and D is an alternating dimap. We now prove the first equality,

$$P(D;w,x,y,z,a,b,c,d,e,f,g,h,i,j,k,l)=P(D^{\omega};w,z,x,y,h,i,g,b,c,a,e,f,d,k,l,j),$$

by induction on $|E(D)|=m$. There exist eight different cases corresponding to the eight categories ((ETI1) to (ETI8)) in Definition 5.2. For the base case, suppose $m=0$. Clearly, the result follows.

Assume that $m>0$ and the result holds for every alternating dimap of size less than m. Let $r\in E(D)$ and r is first reduced.

i) r **is an ultraloop**.

$$\begin{aligned}&P(D;w,x,y,z,a,b,c,d,e,f,g,h,i,j,k,l)\\&=w\cdot P(D\setminus r;w,x,y,z,a,b,c,d,e,f,g,h,i,j,k,l) && \text{(by (ETI1))}\\&=w\cdot P((D\setminus r)^{\omega};w,z,x,y,h,i,g,b,c,a,e,f,d,k,l,j)\\& && \text{(by the inductive hypothesis)}\\&=w\cdot P(D^{\omega}\setminus r^{\omega};w,z,x,y,h,i,g,b,c,a,e,f,d,k,l,j)\\&=P(D^{\omega};w,z,x,y,h,i,g,b,c,a,e,f,d,k,l,j)\ \ \text{(by (ETI1), since } r^{\omega}\text{ is an ultraloop).}\end{aligned}$$

ii) r **is a proper** 1**-loop**.

$$\begin{aligned}&P(D;w,x,y,z,a,b,c,d,e,f,g,h,i,j,k,l)\\&=x\cdot P(D[1]r;w,x,y,z,a,b,c,d,e,f,g,h,i,j,k,l) && \text{(by (ETI2))}\\&=x\cdot P((D[1]r)^{\omega};w,z,x,y,h,i,g,b,c,a,e,f,d,k,l,j)\\& && \text{(by the inductive hypothesis)}\\&=x\cdot P(D^{\omega}[\omega^2]r^{\omega};w,z,x,y,h,i,g,b,c,a,e,f,d,k,l,j) && \text{(by Theorem 4.3)}\\&=x\cdot P(D^{\omega}[\omega]r^{\omega};w,z,x,y,h,i,g,b,c,a,e,f,d,k,l,j)\\& && \text{(since } r^{\omega}\text{ is a proper }\omega\text{-loop, } D^{\omega}[\omega^2]r^{\omega}=D^{\omega}[\omega]r^{\omega})\\&=P(D^{\omega};w,z,x,y,h,i,g,b,c,a,e,f,d,k,l,j) && \text{(by (ETI3) applied to } D^{\omega}).\end{aligned}$$

iii) r **is a proper** ω**-loop**.

$$\begin{aligned}&P(D;w,x,y,z,a,b,c,d,e,f,g,h,i,j,k,l)\\&=y\cdot P(D[\omega]r;w,x,y,z,a,b,c,d,e,f,g,h,i,j,k,l) && \text{(by (ETI3))}\end{aligned}$$

$$
\begin{aligned}
&= y \cdot P((D[\omega]r)^\omega; w, z, x, y, h, i, g, b, c, a, e, f, d, k, l, j) \\
&\hspace{18em} \text{(by the inductive hypothesis)} \\
&= y \cdot P(D^\omega[1]r^\omega; w, z, x, y, h, i, g, b, c, a, e, f, d, k, l, j) \qquad \text{(by Theorem 4.3)} \\
&= y \cdot P(D^\omega[\omega^2]r^\omega; w, z, x, y, h, i, g, b, c, a, e, f, d, k, l, j) \\
&\hspace{8em} \text{(since } r^\omega \text{ is a proper } \omega^2\text{-loop, } D^\omega[1]r^\omega = D^\omega[\omega^2]r^\omega) \\
&= P(D^\omega; w, z, x, y, h, i, g, b, c, a, e, f, d, k, l, j) \qquad \text{(by (ETI4) applied to } D^\omega).
\end{aligned}
$$

iv) r **is a proper** ω^2**-loop**.

$$
\begin{aligned}
&P(D; w, x, y, z, a, b, c, d, e, f, g, h, i, j, k, l) \\
&= z \cdot P(D[\omega^2]r; w, x, y, z, a, b, c, d, e, f, g, h, i, j, k, l) \qquad \text{(by (ETI4))} \\
&= z \cdot P((D[\omega^2]r)^\omega; w, z, x, y, h, i, g, b, c, a, e, f, d, k, l, j) \\
&\hspace{18em} \text{(by the inductive hypothesis)} \\
&= z \cdot P(D^\omega[\omega]r^\omega; w, z, x, y, h, i, g, b, c, a, e, f, d, k, l, j) \qquad \text{(by Theorem 4.3)} \\
&= z \cdot P(D^\omega[1]r^\omega; w, z, x, y, h, i, g, b, c, a, e, f, d, k, l, j) \\
&\hspace{8em} \text{(since } r^\omega \text{ is a proper 1-loop, } D^\omega[\omega]r^\omega = D^\omega[1]r^\omega) \\
&= P(D^\omega; w, z, x, y, h, i, g, b, c, a, e, f, d, k, l, j) \qquad \text{(by (ETI2) applied to } D^\omega).
\end{aligned}
$$

v) r **is a proper** 1**-semiloop**.

$$
\begin{aligned}
&P(D; w, x, y, z, a, b, c, d, e, f, g, h, i, j, k, l) \\
&= a \cdot P(D[1]r; w, x, y, z, a, b, c, d, e, f, g, h, i, j, k, l) \\
&\quad + b \cdot P(D[\omega]r; w, x, y, z, a, b, c, d, e, f, g, h, i, j, k, l) \\
&\quad + c \cdot P(D[\omega^2]r; w, x, y, z, a, b, c, d, e, f, g, h, i, j, k, l) \qquad \text{(by (ETI5))} \\
&= a \cdot P((D[1]r)^\omega; w, z, x, y, h, i, g, b, c, a, e, f, d, k, l, j) \\
&\quad + b \cdot P((D[\omega]r)^\omega; w, z, x, y, h, i, g, b, c, a, e, f, d, k, l, j) \\
&\quad + c \cdot P((D[\omega^2]r)^\omega; w, z, x, y, h, i, g, b, c, a, e, f, d, k, l, j) \\
&\hspace{18em} \text{(by the inductive hypothesis)} \\
&= a \cdot P(D^\omega[\omega^2]r^\omega; w, z, x, y, h, i, g, b, c, a, e, f, d, k, l, j) \\
&\quad + b \cdot P(D^\omega[1]r^\omega; w, z, x, y, h, i, g, b, c, a, e, f, d, k, l, j) \\
&\quad + c \cdot P(D^\omega[\omega]r^\omega; w, z, x, y, h, i, g, b, c, a, e, f, d, k, l, j) \\
&\hspace{14em} \text{(by Theorem 4.3 applied to each term)} \\
&= P(D^\omega; w, z, x, y, h, i, g, b, c, a, e, f, d, k, l, j) \\
&\hspace{6em} \text{(by (ETI6) applied to } D^\omega \text{, since } r^\omega \text{ is a proper } \omega\text{-semiloop).}
\end{aligned}
$$

vi) r **is a proper** ω**-semiloop**.

$$
\begin{aligned}
&P(D; w, x, y, z, a, b, c, d, e, f, g, h, i, j, k, l) \\
&= d \cdot P(D[1]r; w, x, y, z, a, b, c, d, e, f, g, h, i, j, k, l)
\end{aligned}
$$

$$
\begin{aligned}
&+ e \cdot P(D[\omega]r; w,x,y,z,a,b,c,d,e,f,g,h,i,j,k,l) \\
&+ f \cdot P(D[\omega^2]r; w,x,y,z,a,b,c,d,e,f,g,h,i,j,k,l) && \text{(by (ETI6))} \\
&= d \cdot P((D[1]r)^\omega; w,z,x,y,h,i,g,b,c,a,e,f,d,k,l,j) \\
&+ e \cdot P((D[\omega]r)^\omega; w,z,x,y,h,i,g,b,c,a,e,f,d,k,l,j) \\
&+ f \cdot P((D[\omega^2]r)^\omega; w,z,x,y,h,i,g,b,c,a,e,f,d,k,l,j) \\
& && \text{(by the inductive hypothesis)} \\
&= d \cdot P(D^\omega[\omega^2]r^\omega; w,z,x,y,h,i,g,b,c,a,e,f,d,k,l,j) \\
&+ e \cdot P(D^\omega[1]r^\omega; w,z,x,y,h,i,g,b,c,a,e,f,d,k,l,j) \\
&+ f \cdot P(D^\omega[\omega]r^\omega; w,z,x,y,h,i,g,b,c,a,e,f,d,k,l,j) \\
& && \text{(by Theorem 4.3 applied to each term)} \\
&= P(D^\omega; w,z,x,y,h,i,g,b,c,a,e,f,d,k,l,j) \\
& && \text{(by (ETI7) applied to } D^\omega \text{, since } r^\omega \text{ is a proper } \omega^2\text{-semiloop).}
\end{aligned}
$$

vii) r **is a proper** ω^2**-semiloop**.

$$
\begin{aligned}
&P(D; w,x,y,z,a,b,c,d,e,f,g,h,i,j,k,l) \\
&= g \cdot P(D[1]r; w,x,y,z,a,b,c,d,e,f,g,h,i,j,k,l) \\
&+ h \cdot P(D[\omega]r; w,x,y,z,a,b,c,d,e,f,g,h,i,j,k,l) \\
&+ i \cdot P(D[\omega^2]r; w,x,y,z,a,b,c,d,e,f,g,h,i,j,k,l) && \text{(by (ETI7))} \\
&= g \cdot P((D[1]r)^\omega; w,z,x,y,h,i,g,b,c,a,e,f,d,k,l,j) \\
&+ h \cdot P((D[\omega]r)^\omega; w,z,x,y,h,i,g,b,c,a,e,f,d,k,l,j) \\
&+ i \cdot P((D[\omega^2]r)^\omega; w,z,x,y,h,i,g,b,c,a,e,f,d,k,l,j) \\
& && \text{(by the inductive hypothesis)} \\
&= g \cdot P(D^\omega[\omega^2]r^\omega; w,z,x,y,h,i,g,b,c,a,e,f,d,k,l,j) \\
&+ h \cdot P(D^\omega[1]r^\omega; w,z,x,y,h,i,g,b,c,a,e,f,d,k,l,j) \\
&+ i \cdot P(D^\omega[\omega]r^\omega; w,z,x,y,h,i,g,b,c,a,e,f,d,k,l,j) \\
& && \text{(by Theorem 4.3 applied to each term)} \\
&= P(D^\omega; w,z,x,y,h,i,g,b,c,a,e,f,d,k,l,j) \\
& && \text{(by (ETI5) applied to } D^\omega \text{, since } r^\omega \text{ is a proper 1-semiloop).}
\end{aligned}
$$

viii) r **is a proper edge**.

$$
\begin{aligned}
&P(D; w,x,y,z,a,b,c,d,e,f,g,h,i,j,k,l) \\
&= j \cdot P(D[1]r; w,x,y,z,a,b,c,d,e,f,g,h,i,j,k,l) \\
&+ k \cdot P(D[\omega]r; w,x,y,z,a,b,c,d,e,f,g,h,i,j,k,l) \\
&+ l \cdot P(D[\omega^2]r; w,x,y,z,a,b,c,d,e,f,g,h,i,j,k,l) && \text{(by (ETI8))} \\
&= j \cdot P((D[1]r)^\omega; w,z,x,y,h,i,g,b,c,a,e,f,d,k,l,j) \\
&+ k \cdot P((D[\omega]r)^\omega; w,z,x,y,h,i,g,b,c,a,e,f,d,k,l,j)
\end{aligned}
$$

$$
\begin{aligned}
&+ l \cdot P((D[\omega^2]r)^\omega; w, z, x, y, h, i, g, b, c, a, e, f, d, k, l, j) \\
&\hspace{6cm}\text{(by the inductive hypothesis)} \\
&= j \cdot P(D^\omega[\omega^2]r^\omega; w, z, x, y, h, i, g, b, c, a, e, f, d, k, l, j) \\
&\quad + k \cdot P(D^\omega[1]r^\omega; w, z, x, y, h, i, g, b, c, a, e, f, d, k, l, j) \\
&\quad + l \cdot P(D^\omega[\omega]r^\omega; w, z, x, y, h, i, g, b, c, a, e, f, d, k, l, j) \\
&\hspace{6cm}\text{(by Theorem 4.3 applied to each term)} \\
&= P(D^\omega; w, z, x, y, h, i, g, b, c, a, e, f, d, k, l, j) \\
&\hspace{4cm}\text{(by (ETI8) applied to } D^\omega\text{, since } r^\omega \text{ is a proper edge).}
\end{aligned}
$$

The second equality of the theorem follows immediately, by applying the first equality to D^ω. □

We proceed to characterise the extended Tutte invariant.

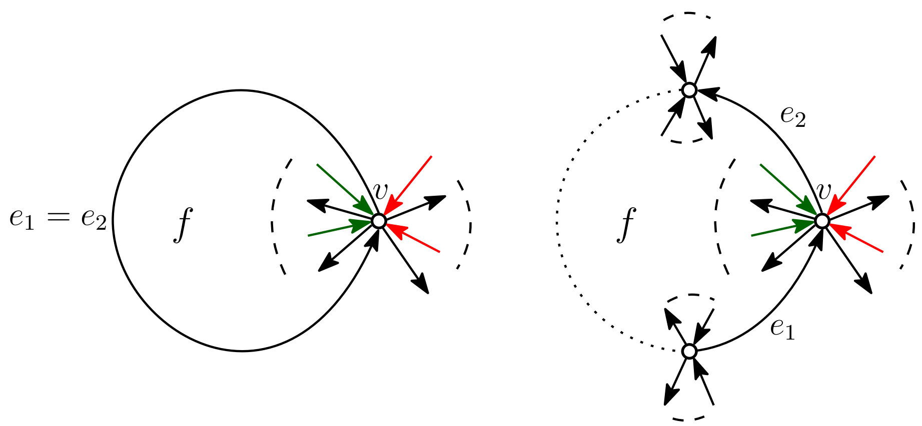
We first show that every alternating dimap of genus zero that has size two or three contains a triloop.

Lemma 5.2. *Let D be an alternating dimap of genus zero that has size m and $0 \le r \le m$. If $r = 2$ or 3, the reduced alternating dimap $D[\mu_1]e_1[\mu_2]e_2 \ldots [\mu_{m-r}]e_{m-r}$ has a triloop.*

Proof. Every alternating dimap D of genus zero that has size two or three contains a triloop (the triloop is proper if D is connected). There are r edges remaining in the reduced alternating dimap after $m - r$ reductions. □

We extend the result and prove that some sequence of reductions on a connected alternating dimap of genus zero gives a proper triloop in a reduced alternating dimap that has size at least three. Observe that not all alternating dimaps of genus zero that have size three contain a proper triloop.

Lemma 5.3. *If D is a connected alternating dimap of genus zero that has size at least three, then some minor of D with at least three edges contains a proper triloop.*



i) $e_1 = e_2$ ii) $e_1 \neq e_2$

Figure 5.1: The anticlockwise face f that has C as its outer cycle in the proof of Lemma 5.3

Proof. Let D be as stated. By inspection, $|E(D)| = 3$ implies that D contains a proper triloop, so no reduction operation is needed in this case. Now, suppose $|E(D)| > 3$ and there is no proper triloop in D.

Note that every non-empty alternating dimap contains at least two faces. Pick an arbitrary closed trail C that forms an anticlockwise face f (or a clockwise face with appropriate modifications in the following steps) in D. Let $H = D[C]$. Suppose R is the outer cycle of f and $v \in V(R)$. Let $e_1, e_2 \in E(R)$ be the edges that are directed into and out from the vertex v, respectively, and they partition $T = I(v) \setminus e_1$ in D into two sets (based on the cyclic order of T), (i) S_c that contains every edge directed into v that lies between e_1 and e_2 as we go from e_1 to e_2 in clockwise order around v, and (ii) $S_a = T \setminus S_c$ (see Figure 5.1, where edges in S_c and S_a are highlighted in green and red, respectively).

First, suppose $e_1 = e_2$ (as shown in Figure 5.1(i)). Since there exists no proper triloop in D, both the sets S_c and S_a are not empty. (S_c and S_a being non-empty implies that each of them has size of at least two, by the definition of alternating dimaps and using the fact that there exists no proper triloop in D.) By reducing every edge in S_c by ω^2-reductions, e_1 is now a proper ω-loop and there are at least three edges in the reduced alternating dimap.

Second, suppose $e_1 \neq e_2$ (as shown in Figure 5.1(ii)). By performing ω^2-reductions on every edge in S_c, and ω-reductions on every edge in S_a, the edge e_1 is then a proper 1-loop. Since there exists no proper triloop in D, the edge e_2 is not a proper 1-loop. Hence, there are at least three edges in the reduced alternating dimap.

Therefore, the result follows. □

A *derived polynomial* for an alternating dimap D is a polynomial in variables $w, x, y, z, a, b, c, d, e, f, g, h, i, j, k, l$ obtained as an extended Tutte invariant for $(D, <)$ where $<$ is a fixed edge-ordering on $E(D)$. The $m!$ permutations of the edge set of an alternating dimap of size m give $m!$ derived polynomials, where some of them may be identical.

We write $G_{n,m}$ for an alternating dimap G that consists of n vertices and m edges such that there exists at least one edge that is not a triloop.

Since there are two non-isomorphic alternating dimaps that may be denoted by $G_{2,3}$, we write $G^a_{2,3}$ and $G^c_{2,3}$ for the alternating dimap $G_{2,3}$ that consists of one anticlockwise face of size three and one clockwise face of size three, respectively. The possibilities for $G_{1,3}$ and $G_{2,3}$ are shown in Figure 5.2.

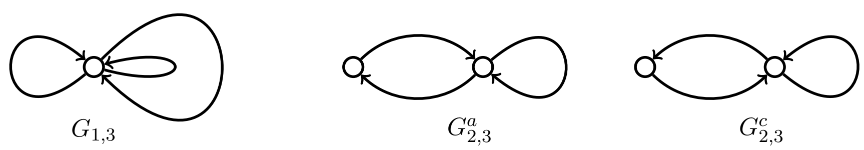


Figure 5.2: Alternating dimaps $G_{1,3}$ and $G_{2,3}$

We give the derived polynomials for an alternating dimap $G_{2,4}$ in the following lemma.

Lemma 5.4. *Let P be an extended Tutte invariant. There exist exactly 12 distinct derived polynomials for the alternating dimap $G_{2,4}$ as shown in Figure 5.3, namely*

E1. $P(D) = (jwyz \text{ or } j(aww + bwy + cwz)) + (kwxy \text{ or } k(gwy + hww + iwx)) + (lwxz \text{ or } l(dwz + ewx + fww)),$

E2. $P(D) = wxyz \text{ or } awwx + bwxy + cwxz,$

E3. $P(D) = wxyz \text{ or } dwyz + ewxy + fwwy,$

E4. $P(D) = wxyz \text{ or } gwyz + hwwz + iwxz.$

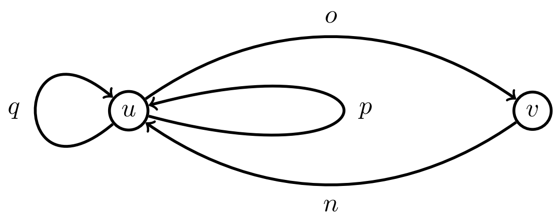


Figure 5.3: The alternating dimap $G_{2,4}$ in Lemma 5.4

Remark: We use "or" in (E1)–(E4) to show all the possible derived polynomials when the respective edge is first reduced. For instance, (E1) gives eight derived polynomials in total.

Proof. Since the alternating dimap $D \cong G_{2,4}$ has four edges, there exist $4! = 24$ possible edge-orderings. To obtain the derived polynomials as in (E1)–(E4), the first edge to be reduced in D is n, o, p and q, respectively. □

Lemma 5.5. *Let P be an extended Tutte invariant.*

a) The only distinct derived polynomials for the alternating dimap $G_{1,3}$ are

$$P(D) = wyz \quad \textit{and} \quad P(D) = aww + bwy + cwz.$$

b) The only distinct derived polynomials for the alternating dimap $G^a_{2,3}$ are

$$P(D) = wxz \quad \textit{and} \quad P(D) = dwz + ewx + fww.$$

c) The only distinct derived polynomials for the alternating dimap $G^c_{2,3}$ are

$$P(D) = wxy \quad \textit{and} \quad P(D) = gwy + hww + iwx.$$

Proof. Let P be an extended Tutte invariant. We first prove Lemma 5.5(a). Suppose $D \cong G_{1,3}$ (see Figure 5.2). Since D has three edges, there exist $3! = 6$ possible edge-orderings. By reducing D using all the possible edge-orderings, there exist two distinct derived polynomials $P(D) = wyz$ and $P(D) = aww + bwy + cwz$.

The proofs for two other cases follow a similar approach. □

Since the final edge to be reduced in each component of any non-empty alternating dimap D is always an ultraloop, we have $P(D) \equiv 0$ if $w = 0$. Hence, we assume that $w \neq 0$ hereinafter.

As shown in Lemma 5.3, every connected alternating dimap D of genus zero that has size at least three contains a proper triloop. By Definition 5.2, one variable x, y or z is produced if the proper triloop in D is reduced first. If that variable equals zero, a trivial solution will then be obtained. Hence, we first characterise extended Tutte invariants for alternating dimaps of genus zero under the assumption that $x, y, z \neq 0$.

Theorem 5.6. *Let $S = (w, x, y, z, a, b, c, d, e, f, g, h, i, j, k, l)$ be a parameter sequence such that $w, x, y, z \neq 0$. A function P is an extended Tutte invariant with respect to S for every alternating dimap D of genus zero if and only if*

$$P(D) = w^{k(D)} \cdot x^{\mathrm{is}(D)-k(D)} \cdot y^{\mathrm{af}(D)-k(D)} \cdot z^{\mathrm{cf}(D)-k(D)} \tag{5.1}$$

with

$$xyz = jyz + kxy + lxz, \tag{5.2}$$

$$yz = aw + by + cz, \tag{5.3}$$

$$xz = dz + ex + fw, \tag{5.4}$$

$$xy = gy + hw + ix. \tag{5.5}$$

Proof. Let S be as stated and D be an alternating dimap of genus zero. We first prove the forward implication. Note that all the derived polynomials must be equal for $P(D)$ to be an extended Tutte invariant. By (E1)–(E4) in Lemma 5.4 and as $w, x, y, z \neq 0$, we obtain (5.2)–(5.5) as desired.

We next show

$$P(D) = w^{k(D)} \cdot x^{\mathrm{is}(D)-k(D)} \cdot y^{\mathrm{af}(D)-k(D)} \cdot z^{\mathrm{cf}(D)-k(D)},$$

using induction on $|E(D)| = m$. There exist eight cases corresponding to the eight categories ((ETI1) to (ETI8)) in Definition 5.2. For the base case, suppose $m = 0$. Clearly, $P(D) = 1$ and the result follows. Assume that $m > 0$ and the result holds for every alternating dimap of genus zero that has size less than m. Let $r \in E(D)$.

i) r **is an ultraloop**. In $D \setminus r$, the number of components, in-stars, a-faces and c-faces are all reduced by 1. Thus,

$$\begin{aligned} P(D) &= w \cdot P(D \setminus r) \\ &= w \cdot w^{k(D\setminus r)} \cdot x^{\mathrm{is}(D\setminus r)-k(D\setminus r)} \cdot y^{\mathrm{af}(D\setminus r)-k(D\setminus r)} \cdot z^{\mathrm{cf}(D\setminus r)-k(D\setminus r)} \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by the inductive hypothesis)} \\ &= w \cdot w^{k(D)-1} \cdot x^{\mathrm{is}(D)-1-(k(D)-1)} \cdot y^{\mathrm{af}(D)-1-(k(D)-1)} \cdot z^{\mathrm{cf}(D)-1-(k(D)-1)} \\ &= w^{k(D)} \cdot x^{\mathrm{is}(D)-k(D)} \cdot y^{\mathrm{af}(D)-k(D)} \cdot z^{\mathrm{cf}(D)-k(D)}. \end{aligned}$$

ii) r **is a proper 1-loop**. The number of in-stars is reduced by 1 in $D[1]r$. Then,

$$\begin{aligned}
P(D) &= x \cdot P(D[1]r) \\
&= x \cdot w^{k(D[1]r)} \cdot x^{\mathrm{is}(D[1]r)-k(D[1]r)} \cdot y^{\mathrm{af}(D[1]r)-k(D[1]r)} \cdot z^{\mathrm{cf}(D[1]r)-k(D[1]r)} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by the inductive hypothesis)} \\
&= x \cdot w^{k(D)} \cdot x^{\mathrm{is}(D)-1-k(D)} \cdot y^{\mathrm{af}(D)-k(D)} \cdot z^{\mathrm{cf}(D)-k(D)} \\
&= w^{k(D)} \cdot x^{\mathrm{is}(D)-k(D)} \cdot y^{\mathrm{af}(D)-k(D)} \cdot z^{\mathrm{cf}(D)-k(D)}.
\end{aligned}$$

iii) r **is a proper ω-loop**. The number of a-faces is reduced by 1 in $D[\omega]r$. Then,

$$\begin{aligned}
P(D) &= y \cdot P(D[\omega]r) \\
&= y \cdot w^{k(D[\omega]r)} \cdot x^{\mathrm{is}(D[\omega]r)-k(D[\omega]r)} \cdot y^{\mathrm{af}(D[\omega]r)-k(D[\omega]r)} \cdot z^{\mathrm{cf}(D[\omega]r)-k(D[\omega]r)} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by the inductive hypothesis)} \\
&= y \cdot w^{k(D)} \cdot x^{\mathrm{is}(D)-k(D)} \cdot y^{\mathrm{af}(D)-1-k(D)} \cdot z^{\mathrm{cf}(D)-k(D)} \\
&= w^{k(D)} \cdot x^{\mathrm{is}(D)-k(D)} \cdot y^{\mathrm{af}(D)-k(D)} \cdot z^{\mathrm{cf}(D)-k(D)}.
\end{aligned}$$

iv) r **is a proper ω^2-loop**. The number of c-faces is reduced by 1 in $D[\omega^2]r$. Then,

$$\begin{aligned}
P(D) &= z \cdot P(D[\omega^2]r) \\
&= z \cdot w^{k(D[\omega^2]r)} \cdot x^{\mathrm{is}(D[\omega^2]r)-k(D[\omega^2]r)} \cdot y^{\mathrm{af}(D[\omega^2]r)-k(D[\omega^2]r)} \cdot z^{\mathrm{cf}(D[\omega^2]r)-k(D[\omega^2]r)} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by the inductive hypothesis)} \\
&= z \cdot w^{k(D)} \cdot x^{\mathrm{is}(D)-k(D)} \cdot y^{\mathrm{af}(D)-k(D)} \cdot z^{\mathrm{cf}(D)-1-k(D)} \\
&= w^{k(D)} \cdot x^{\mathrm{is}(D)-k(D)} \cdot y^{\mathrm{af}(D)-k(D)} \cdot z^{\mathrm{cf}(D)-k(D)}.
\end{aligned}$$

v) r **is a proper 1-semiloop**. In $D[1]r$, the number of components and in-stars are both increased by 1. In $D[\omega]r$ and $D[\omega^2]r$, the number of c-faces and a-faces are reduced by 1, respectively. Hence,

$$\begin{aligned}
yz \cdot P(D) &= yz \cdot \Big(a \cdot P(D[1]r) + b \cdot P(D[\omega]r) + c \cdot P(D[\omega^2]r)\Big) \\
&= yz \cdot \Big(a \cdot w^{k(D[1]r)} \cdot x^{\mathrm{is}(D[1]r)-k(D[1]r)} \cdot y^{\mathrm{af}(D[1]r)-k(D[1]r)} \cdot z^{\mathrm{cf}(D[1]r)-k(D[1]r)} \\
&\quad + b \cdot w^{k(D[\omega]r)} \cdot x^{\mathrm{is}(D[\omega]r)-k(D[\omega]r)} \cdot y^{\mathrm{af}(D[\omega]r)-k(D[\omega]r)} \cdot z^{\mathrm{cf}(D[\omega]r)-k(D[\omega]r)} \\
&\quad + c \cdot w^{k(D[\omega^2]r)} \cdot x^{\mathrm{is}(D[\omega^2]r)-k(D[\omega^2]r)} \cdot y^{\mathrm{af}(D[\omega^2]r)-k(D[\omega^2]r)} \cdot z^{\mathrm{cf}(D[\omega^2]r)-k(D[\omega^2]r)}\Big) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by the inductive hypothesis)} \\
&= yz \cdot \Big(a \cdot w^{k(D)+1} \cdot x^{\mathrm{is}(D)+1-(k(D)+1)} \cdot y^{\mathrm{af}(D)-(k(D)+1)} \cdot z^{\mathrm{cf}(D)-(k(D)+1)} \\
&\quad + b \cdot w^{k(D)} \cdot x^{\mathrm{is}(D)-k(D)} \cdot y^{\mathrm{af}(D)-k(D)} \cdot z^{\mathrm{cf}(D)-1-k(D)} \\
&\quad + c \cdot w^{k(D)} \cdot x^{\mathrm{is}(D)-k(D)} \cdot y^{\mathrm{af}(D)-1-k(D)} \cdot z^{\mathrm{cf}(D)-k(D)}\Big) \\
&= (aw + by + cz) \cdot w^{k(D)} \cdot x^{\mathrm{is}(D)-k(D)} \cdot y^{\mathrm{af}(D)-k(D)} \cdot z^{\mathrm{cf}(D)-k(D)} \\
P(D) &= w^{k(D)} \cdot x^{\mathrm{is}(D)-k(D)} \cdot y^{\mathrm{af}(D)-k(D)} \cdot z^{\mathrm{cf}(D)-k(D)} \qquad \text{(by (5.3))}.
\end{aligned}$$

vi) r **is a proper ω-semiloop**. In $D[\omega^2]r$, the number of components and a-faces are both increased by 1. In $D[1]r$ and $D[\omega]r$, the number of in-stars and c-faces are reduced by 1, respectively. Hence,

$$\begin{aligned}
xz\cdot P(D) &= xz\cdot\Big(d\cdot P(D[1]r)+e\cdot P(D[\omega]r)+f\cdot P(D[\omega^2]r)\Big)\\
&= xz\cdot\Big(d\cdot w^{k(D[1]r)}\cdot x^{\mathrm{is}(D[1]r)-k(D[1]r)}\cdot y^{\mathrm{af}(D[1]r)-k(D[1]r)}\cdot z^{\mathrm{cf}(D[1]r)-k(D[1]r)}\\
&\quad+e\cdot w^{k(D[\omega]r)}\cdot x^{\mathrm{is}(D[\omega]r)-k(D[\omega]r)}\cdot y^{\mathrm{af}(D[\omega]r)-k(D[\omega]r)}\cdot z^{\mathrm{cf}(D[\omega]r)-k(D[\omega]r)}\\
&\quad+f\cdot w^{k(D[\omega^2]r)}\cdot x^{\mathrm{is}(D[\omega^2]r)-k(D[\omega^2]r)}\cdot y^{\mathrm{af}(D[\omega^2]r)-k(D[\omega^2]r)}\cdot z^{\mathrm{cf}(D[\omega^2]r)-k(D[\omega^2]r)}\Big)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{(by the inductive hypothesis)}\\
&= xz\cdot\Big(d\cdot w^{k(D)}\cdot x^{\mathrm{is}(D)-1-k(D)}\cdot y^{\mathrm{af}(D)-k(D)}\cdot z^{\mathrm{cf}(D)-k(D)}\\
&\quad+e\cdot w^{k(D)}\cdot x^{\mathrm{is}(D)-k(D)}\cdot y^{\mathrm{af}(D)-k(D)}\cdot z^{\mathrm{cf}(D)-1-k(D)}\\
&\quad+f\cdot w^{k(D)+1}\cdot x^{\mathrm{is}(D)-(k(D)+1)}\cdot y^{\mathrm{af}(D)+1-(k(D)+1)}\cdot z^{\mathrm{cf}(D)-(k(D)+1)}\Big)\\
&= (dz+ex+fw)\cdot w^{k(D)}\cdot x^{\mathrm{is}(D)-k(D)}\cdot y^{\mathrm{af}(D)-k(D)}\cdot z^{\mathrm{cf}(D)-k(D)}\\
P(D) &= w^{k(D)}\cdot x^{\mathrm{is}(D)-k(D)}\cdot y^{\mathrm{af}(D)-k(D)}\cdot z^{\mathrm{cf}(D)-k(D)}\qquad\text{(by (5.4))}.
\end{aligned}$$

vii) r **is a proper ω^2-semiloop**. In $D[\omega]r$, the number of components and c-faces are both increased by 1. In $D[1]r$ and $D[\omega^2]r$, the number of in-stars and a-faces are reduced by 1, respectively. Hence,

$$\begin{aligned}
xy\cdot P(D) &= xy\cdot\Big(g\cdot P(D[1]r)+h\cdot P(D[\omega]r)+i\cdot P(D[\omega^2]r)\Big)\\
&= xy\cdot\Big(g\cdot w^{k(D[1]r)}\cdot x^{\mathrm{is}(D[1]r)-k(D[1]r)}\cdot y^{\mathrm{af}(D[1]r)-k(D[1]r)}\cdot z^{\mathrm{cf}(D[1]r)-k(D[1]r)}\\
&\quad+h\cdot w^{k(D[\omega]r)}\cdot x^{\mathrm{is}(D[\omega]r)-k(D[\omega]r)}\cdot y^{\mathrm{af}(D[\omega]r)-k(D[\omega]r)}\cdot z^{\mathrm{cf}(D[\omega]r)-k(D[\omega]r)}\\
&\quad+i\cdot w^{k(D[\omega^2]r)}\cdot x^{\mathrm{is}(D[\omega^2]r)-k(D[\omega^2]r)}\cdot y^{\mathrm{af}(D[\omega^2]r)-k(D[\omega^2]r)}\cdot z^{\mathrm{cf}(D[\omega^2]r)-k(D[\omega^2]r)}\Big)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{(by the inductive hypothesis)}\\
&= xy\cdot\Big(g\cdot w^{k(D)}\cdot x^{\mathrm{is}(D)-1-k(D)}\cdot y^{\mathrm{af}(D)-k(D)}\cdot z^{\mathrm{cf}(D)-k(D)}\\
&\quad+h\cdot w^{k(D)+1}\cdot x^{\mathrm{is}(D)-(k(D)+1)}\cdot y^{\mathrm{af}(D)-(k(D)+1)}\cdot z^{\mathrm{cf}(D)+1-(k(D)+1)}\\
&\quad+i\cdot w^{k(D)}\cdot x^{\mathrm{is}(D)-k(D)}\cdot y^{\mathrm{af}(D)-1-k(D)}\cdot z^{\mathrm{cf}(D)-k(D)}\Big)\\
&= (gy+hw+ix)\cdot w^{k(D)}\cdot x^{\mathrm{is}(D)-k(D)}\cdot y^{\mathrm{af}(D)-k(D)}\cdot z^{\mathrm{cf}(D)-k(D)}\\
P(D) &= w^{k(D)}\cdot x^{\mathrm{is}(D)-k(D)}\cdot y^{\mathrm{af}(D)-k(D)}\cdot z^{\mathrm{cf}(D)-k(D)}\qquad\text{(by (5.5))}.
\end{aligned}$$

viii) r **is a proper edge**. Observe that the number of in-stars, c-faces and a-faces, are all reduced by 1 in $D[1]r$, $D[\omega]r$ and $D[\omega^2]r$, respectively. Therefore, we have,

$$\begin{aligned}
xyz\cdot P(D) &= xyz\cdot\Big(j\cdot P(D[1]r)+k\cdot P(D[\omega]r)+l\cdot P(D[\omega^2]r)\Big)\\
&= xyz\cdot\Big(j\cdot w^{k(D[1]r)}\cdot x^{\mathrm{is}(D[1]r)-k(D[1]r)}\cdot y^{\mathrm{af}(D[1]r)-k(D[1]r)}\cdot z^{\mathrm{cf}(D[1]r)-k(D[1]r)}\\
&\quad+k\cdot w^{k(D[\omega]r)}\cdot x^{\mathrm{is}(D[\omega]r)-k(D[\omega]r)}\cdot y^{\mathrm{af}(D[\omega]r)-k(D[\omega]r)}\cdot z^{\mathrm{cf}(D[\omega]r)-k(D[\omega]r)}
\end{aligned}$$

$$
\begin{aligned}
&\quad + l \cdot w^{k(D[\omega^2]r)} \cdot x^{\mathrm{is}(D[\omega^2]r)-k(D[\omega^2]r)} \cdot y^{\mathrm{af}(D[\omega^2]r)-k(D[\omega^2]r)} \cdot z^{\mathrm{cf}(D[\omega^2]r)-k(D[\omega^2]r)}\Big) \\
&\hspace{20em} \text{(by the inductive hypothesis)} \\
&= xyz \cdot \Big(j \cdot w^{k(D)} \cdot x^{\mathrm{is}(D)-1-k(D)} \cdot y^{\mathrm{af}(D)-k(D)} \cdot z^{\mathrm{cf}(D)-k(D)} \\
&\quad + k \cdot w^{k(D)} \cdot x^{\mathrm{is}(D)-k(D)} \cdot y^{\mathrm{af}(D)-k(D)} \cdot z^{\mathrm{cf}(D)-1-k(D)} \\
&\quad + l \cdot w^{k(D)} \cdot x^{\mathrm{is}(D)-k(D)} \cdot y^{\mathrm{af}(D)-1-k(D)} \cdot z^{\mathrm{cf}(D)-k(D)}\Big) \\
&= (jyz + kxy + lxz) \cdot w^{k(D)} \cdot x^{\mathrm{is}(D)-k(D)} \cdot y^{\mathrm{af}(D)-k(D)} \cdot z^{\mathrm{cf}(D)-k(D)}
\end{aligned}
$$

$$
P(D) = w^{k(D)} \cdot x^{\mathrm{is}(D)-k(D)} \cdot y^{\mathrm{af}(D)-k(D)} \cdot z^{\mathrm{cf}(D)-k(D)} \qquad \text{(by (5.2))}.
$$

Conversely, we prove that

$$
P(D) = w^{k(D)} \cdot x^{\mathrm{is}(D)-k(D)} \cdot y^{\mathrm{af}(D)-k(D)} \cdot z^{\mathrm{cf}(D)-k(D)}
$$

with (5.2)–(5.5) is an extended Tutte invariant with respect to S for every alternating dimap D, as in Definition 5.2. We first show that $P(D)$ is a multiplicative invariant. Suppose G and H are two alternating dimaps. For the union of G and H, we have

$$
\begin{aligned}
&P(G \cup H) \\
&= w^{k(G\cup H)} \cdot x^{\mathrm{is}(G\cup H)-k(G\cup H)} \cdot y^{\mathrm{af}(G\cup H)-k(G\cup H)} \cdot z^{\mathrm{cf}(G\cup H)-k(G\cup H)} \\
&= w^{k(G)+k(H)} \cdot x^{\mathrm{is}(G)+\mathrm{is}(H)-k(G)-k(H)} \cdot y^{\mathrm{af}(G)+\mathrm{af}(H)-k(G)-k(H)} \cdot z^{\mathrm{cf}(G)+\mathrm{cf}(H)-k(G)-k(H)} \\
&= w^{k(G)} \cdot x^{\mathrm{is}(G)-k(G)} \cdot y^{\mathrm{af}(G)-k(G)} \cdot z^{\mathrm{cf}(G)-k(G)} \cdot w^{k(H)} \cdot x^{\mathrm{is}(H)-k(H)} \cdot y^{\mathrm{af}(H)-k(H)} \cdot z^{\mathrm{cf}(H)-k(H)} \\
&= P(G) \cdot P(H).
\end{aligned}
$$

Then, we show that $P(D)$ satisfies (ETI1) to (ETI8) by using induction on $|E(D)| = m$. When $m = 0$, we have $P(D) = 1$ and the result for $m = 0$ follows. So, suppose $m > 0$ and the result holds for every alternating dimap of genus zero that has size less than m. Let $r \in E(D)$.

i) r is an ultraloop.

$$
\begin{aligned}
P(D) &= w^{k(D)} \cdot x^{\mathrm{is}(D)-k(D)} \cdot y^{\mathrm{af}(D)-k(D)} \cdot z^{\mathrm{cf}(D)-k(D)} \\
&= w \cdot w^{k(D)-1} \cdot x^{\mathrm{is}(D)-1-(k(D)-1)} \cdot y^{\mathrm{af}(D)-1-(k(D)-1)} \cdot z^{\mathrm{cf}(D)-1-(k(D)-1)} \\
&= w \cdot w^{k(D\backslash r)} \cdot x^{\mathrm{is}(D\backslash r)-k(D\backslash r)} \cdot y^{\mathrm{af}(D\backslash r)-k(D\backslash r)} \cdot z^{\mathrm{cf}(D\backslash r)-k(D\backslash r)} \\
&= w \cdot P(D \setminus r) \qquad \text{(by the inductive hypothesis)}.
\end{aligned}
$$

ii) r is a proper 1-loop.

$$
\begin{aligned}
P(D) &= x \cdot w^{k(D)} \cdot x^{\mathrm{is}(D)-1-k(D)} \cdot y^{\mathrm{af}(D)-k(D)} \cdot z^{\mathrm{cf}(D)-k(D)} \\
&= x \cdot w^{k(D[1]r)} \cdot x^{\mathrm{is}(D[1]r)-k(D[1]r)} \cdot y^{\mathrm{af}(D[1]r)-k(D[1]r)} \cdot z^{\mathrm{cf}(D[1]r)-k(D[1]r)} \\
&= x \cdot P(D[1]r) \qquad \text{(by the inductive hypothesis)}.
\end{aligned}
$$

iii) r is a proper ω-loop.

$$\begin{aligned} P(D) &= y \cdot w^{k(D)} \cdot x^{\mathrm{is}(D)-k(D)} \cdot y^{\mathrm{af}(D)-1-k(D)} \cdot z^{\mathrm{cf}(D)-k(D)} \\ &= y \cdot w^{k(D[\omega]r)} \cdot x^{\mathrm{is}(D[\omega]r)-k(D[\omega]r)} \cdot y^{\mathrm{af}(D[\omega]r)-k(D[\omega]r)} \cdot z^{\mathrm{cf}(D[\omega]r)-k(D[\omega]r)} \\ &= y \cdot P(D[\omega]r) \qquad \text{(by the inductive hypothesis).} \end{aligned}$$

iv) r is a proper ω^2-loop.

$$\begin{aligned} P(D) &= z \cdot w^{k(D)} \cdot x^{\mathrm{is}(D)-k(D)} \cdot y^{\mathrm{af}(D)-k(D)} \cdot z^{\mathrm{cf}(D)-1-k(D)} \\ &= z \cdot w^{k(D[\omega^2]r)} \cdot x^{\mathrm{is}(D[\omega^2]r)-k(D[\omega^2]r)} \cdot y^{\mathrm{af}(D[\omega^2]r)-k(D[\omega^2]r)} \cdot z^{\mathrm{cf}(D[\omega^2]r)-k(D[\omega^2]r)} \\ &= z \cdot P(D[\omega^2]r) \qquad \text{(by the inductive hypothesis).} \end{aligned}$$

v) r is a proper 1-semiloop.

$$\begin{aligned} yz \cdot P(D) &= (aw + by + cz) \cdot w^{k(D)} \cdot x^{\mathrm{is}(D)-k(D)} \cdot y^{\mathrm{af}(D)-k(D)} \cdot z^{\mathrm{cf}(D)-k(D)} \\ & \qquad \text{(by (5.3))} \\ &= yz \cdot \Big(a \cdot w^{k(D)+1} \cdot x^{\mathrm{is}(D)+1-(k(D)+1)} \cdot y^{\mathrm{af}(D)-(k(D)+1)} \cdot z^{\mathrm{cf}(D)-(k(D)+1)} \\ &\quad + b \cdot w^{k(D)} \cdot x^{\mathrm{is}(D)-k(D)} \cdot y^{\mathrm{af}(D)-k(D)} \cdot z^{\mathrm{cf}(D)-1-k(D)} \\ &\quad + c \cdot w^{k(D)} \cdot x^{\mathrm{is}(D)-k(D)} \cdot y^{\mathrm{af}(D)-1-k(D)} \cdot z^{\mathrm{cf}(D)-k(D)}\Big) \\ &= yz \cdot \Big(a \cdot w^{k(D[1]r)} \cdot x^{\mathrm{is}(D[1]r)-k(D[1]r)} \cdot y^{\mathrm{af}(D[1]r)-k(D[1]r)} \cdot z^{\mathrm{cf}(D[1]r)-k(D[1]r)} \\ &\quad + b \cdot w^{k(D[\omega]r)} \cdot x^{\mathrm{is}(D[\omega]r)-k(D[\omega]r)} \cdot y^{\mathrm{af}(D[\omega]r)-k(D[\omega]r)} \cdot z^{\mathrm{cf}(D[\omega]r)-k(D[\omega]r)} \\ &\quad + c \cdot w^{k(D[\omega^2]r)} \cdot x^{\mathrm{is}(D[\omega^2]r)-k(D[\omega^2]r)} \cdot y^{\mathrm{af}(D[\omega^2]r)-k(D[\omega^2]r)} \cdot z^{\mathrm{cf}(D[\omega^2]r)-k(D[\omega^2]r)}\Big) \\ P(D) &= a \cdot P(D[1]r) + b \cdot P(D[\omega]r) + c \cdot P(D[\omega^2]r) \\ & \qquad \text{(by the inductive hypothesis).} \end{aligned}$$

vi) r is a proper ω-semiloop.

$$\begin{aligned} xz \cdot P(D) &= (dz + ex + fw) \cdot w^{k(D)} \cdot x^{\mathrm{is}(D)-k(D)} \cdot y^{\mathrm{af}(D)-k(D)} \cdot z^{\mathrm{cf}(D)-k(D)} \\ & \qquad \text{(by (5.4))} \\ &= xz \cdot \Big(d \cdot w^{k(D)} \cdot x^{\mathrm{is}(D)-1-k(D)} \cdot y^{\mathrm{af}(D)-k(D)} \cdot z^{\mathrm{cf}(D)-k(D)} \\ &\quad + e \cdot w^{k(D)} \cdot x^{\mathrm{is}(D)-k(D)} \cdot y^{\mathrm{af}(D)-k(D)} \cdot z^{\mathrm{cf}(D)-1-k(D)} \\ &\quad + f \cdot w^{k(D)+1} \cdot x^{\mathrm{is}(D)-(k(D)+1)} \cdot y^{\mathrm{af}(D)+1-(k(D)+1)} \cdot z^{\mathrm{cf}(D)-(k(D)+1)}\Big) \\ &= xz \cdot \Big(d \cdot w^{k(D[1]r)} \cdot x^{\mathrm{is}(D[1]r)-k(D[1]r)} \cdot y^{\mathrm{af}(D[1]r)-k(D[1]r)} \cdot z^{\mathrm{cf}(D[1]r)-k(D[1]r)} \\ &\quad + e \cdot w^{k(D[\omega]r)} \cdot x^{\mathrm{is}(D[\omega]r)-k(D[\omega]r)} \cdot y^{\mathrm{af}(D[\omega]r)-k(D[\omega]r)} \cdot z^{\mathrm{cf}(D[\omega]r)-k(D[\omega]r)} \\ &\quad + f \cdot w^{k(D[\omega^2]r)} \cdot x^{\mathrm{is}(D[\omega^2]r)-k(D[\omega^2]r)} \cdot y^{\mathrm{af}(D[\omega^2]r)-k(D[\omega^2]r)} \cdot z^{\mathrm{cf}(D[\omega^2]r)-k(D[\omega^2]r)}\Big) \\ P(D) &= d \cdot P(D[1]r) + e \cdot P(D[\omega]r) + f \cdot P(D[\omega^2]r) \\ & \qquad \text{(by the inductive hypothesis).} \end{aligned}$$

vii) r is a proper ω^2-semiloop.

$$\begin{aligned}
xy \cdot P(D) &= (gy + hw + ix) \cdot w^{k(D)} \cdot x^{\mathrm{is}(D)-k(D)} \cdot y^{\mathrm{af}(D)-k(D)} \cdot z^{\mathrm{cf}(D)-k(D)} && \text{(by (5.5))} \\
&= xy \cdot \Big(g \cdot w^{k(D)} \cdot x^{\mathrm{is}(D)-1-k(D)} \cdot y^{\mathrm{af}(D)-k(D)} \cdot z^{\mathrm{cf}(D)-k(D)} \\
&\quad + h \cdot w^{k(D)+1} \cdot x^{\mathrm{is}(D)-(k(D)+1)} \cdot y^{\mathrm{af}(D)-(k(D)+1)} \cdot z^{\mathrm{cf}(D)+1-(k(D)+1)} \\
&\quad + i \cdot w^{k(D)} \cdot x^{\mathrm{is}(D)-k(D)} \cdot y^{\mathrm{af}(D)-1-k(D)} \cdot z^{\mathrm{cf}(D)-k(D)} \Big) \\
&= xy \cdot \Big(g \cdot w^{k(D[1]r)} \cdot x^{\mathrm{is}(D[1]r)-k(D[1]r)} \cdot y^{\mathrm{af}(D[1]r)-k(D[1]r)} \cdot z^{\mathrm{cf}(D[1]r)-k(D[1]r)} \\
&\quad + h \cdot w^{k(D[\omega]r)} \cdot x^{\mathrm{is}(D[\omega]r)-k(D[\omega]r)} \cdot y^{\mathrm{af}(D[\omega]r)-k(D[\omega]r)} \cdot z^{\mathrm{cf}(D[\omega]r)-k(D[\omega]r)} \\
&\quad + i \cdot w^{k(D[\omega^2]r)} \cdot x^{\mathrm{is}(D[\omega^2]r)-k(D[\omega^2]r)} \cdot y^{\mathrm{af}(D[\omega^2]r)-k(D[\omega^2]r)} \cdot z^{\mathrm{cf}(D[\omega^2]r)-k(D[\omega^2]r)} \Big) \\
P(D) &= g \cdot P(D[1]r) + h \cdot P(D[\omega]r) + i \cdot P(D[\omega^2]r) && \text{(by the inductive hypothesis).}
\end{aligned}$$

viii) Otherwise, we have

$$\begin{aligned}
xyz \cdot P(D) &= (jyz + kxy + lxz) \cdot w^{k(D)} \cdot x^{\mathrm{is}(D)-k(D)} \cdot y^{\mathrm{af}(D)-k(D)} \cdot z^{\mathrm{cf}(D)-k(D)} && \text{(by (5.2))} \\
&= xyz \cdot \Big(j \cdot w^{k(D)} \cdot x^{\mathrm{is}(D)-1-k(D)} \cdot y^{\mathrm{af}(D)-k(D)} \cdot z^{\mathrm{cf}(D)-k(D)} \\
&\quad + k \cdot w^{k(D)} \cdot x^{\mathrm{is}(D)-k(D)} \cdot y^{\mathrm{af}(D)-k(D)} \cdot z^{\mathrm{cf}(D)-1-k(D)} \\
&\quad + l \cdot w^{k(D)} \cdot x^{\mathrm{is}(D)-k(D)} \cdot y^{\mathrm{af}(D)-1-k(D)} \cdot z^{\mathrm{cf}(D)-k(D)} \Big) \\
&= xyz \cdot \Big(j \cdot w^{k(D[1]r)} \cdot x^{\mathrm{is}(D[1]r)-k(D[1]r)} \cdot y^{\mathrm{af}(D[1]r)-k(D[1]r)} \cdot z^{\mathrm{cf}(D[1]r)-k(D[1]r)} \\
&\quad + k \cdot w^{k(D[\omega]r)} \cdot x^{\mathrm{is}(D[\omega]r)-k(D[\omega]r)} \cdot y^{\mathrm{af}(D[\omega]r)-k(D[\omega]r)} \cdot z^{\mathrm{cf}(D[\omega]r)-k(D[\omega]r)} \\
&\quad + l \cdot w^{k(D[\omega^2]r)} \cdot x^{\mathrm{is}(D[\omega^2]r)-k(D[\omega^2]r)} \cdot y^{\mathrm{af}(D[\omega^2]r)-k(D[\omega^2]r)} \cdot z^{\mathrm{cf}(D[\omega^2]r)-k(D[\omega^2]r)} \Big) \\
P(D) &= j \cdot P(D[1]r) + k \cdot P(D[\omega]r) + l \cdot P(D[\omega^2]r) && \text{(by the inductive hypothesis).}
\end{aligned}$$

This completes the backward implication, by induction. □

As seen in the proof of Theorem 5.6, variables x, y and z must be non-zero in order to complete the characterisation. We next consider cases where at least one of these three variables is zero, using different arguments. We shall first establish some excluded minor characterisations of alternating dimaps of genus zero.

We first show that every clockwise face of size greater than two in an alternating dimap can be reduced to a clockwise face of size exactly two, by a series of contractions.

Lemma 5.7. *Let D be an alternating dimap and put $k = \mathrm{cf}(D)$. If every clockwise face of D has size at least two, then D can be reduced to an alternating dimap that contains k clockwise faces of size exactly two, using a sequence of contraction operations.*

Proof. Let D be as stated. We proceed by induction on $|E(D)| = m$. For the base case, suppose $m = 2$. Since there is a clockwise face of size exactly two, the result for $m = 2$ follows.

For the inductive step, assume that $m > 2$ and the result holds for every alternating dimap of size less than m.

Suppose $k = \text{cf}(D)$, and $g \in F(D)$ is a clockwise face of size greater than two. Let $e \in E(\partial g)$. By contracting the edge e, the size of g will be reduced by one. As every edge in an alternating dimap belongs to one clockwise face and one anticlockwise face, we have $\text{cf}(D/e) = k$. By the inductive hypothesis, the alternating dimap D/e can be reduced to an alternating dimap that contains k clockwise faces of size exactly two, using a sequence of contraction operations. Therefore, alternating dimap D can be reduced to an alternating dimap that contains k clockwise faces of size exactly two by a sequence of contraction operations, namely, contraction of e followed by the aforementioned contraction sequence for D/e. $\square$

Lemma 5.8. *Let D be an alternating dimap and put $k = \text{af}(D)$. If every anticlockwise face of D has size at least two, then D can be reduced to an alternating dimap that contains k anticlockwise faces of size exactly two, using a sequence of contraction operations.*

Proof. The result follows by some appropriate modifications to the proof of Lemma 5.7. $\square$

We show that $G_{1,3}$, $G^a_{2,3}$ or $G^c_{2,3}$ (see Figure 5.2) is a minor for certain alternating dimaps, in the following lemmas.

Lemma 5.9. *Every alternating dimap of genus zero that contains a proper 1-semiloop has $G_{1,3}$ as a minor.*

Proof. Let D be an alternating dimap of genus zero that contains a proper 1-semiloop e. We proceed by induction on $|V(D)| = n$. For the base case, suppose $n = 1$. The alternating dimap $G_{1,3}$ can be obtained by repeatedly reducing some proper triloops.

For the inductive step, assume that $n > 1$ and the result holds for every D that has less than n vertices. If e belongs to a component that has exactly one vertex in D, from the base case, the alternating dimap D contains $G_{1,3}$ as a minor. So, suppose e belongs to a component P that has at least two vertices in D. This implies that there exists at least one non-loop edge f in P. By contracting f, we have $|V(D/f)| = n-1$. By the inductive hypothesis, the alternating dimap D/f contains $G_{1,3}$ as a minor. Since D/f is a minor of D, the result follows. $\square$

Since $G_{1,3}$ contains a proper ω-loop and a proper ω^2-loop, the following corollary follows from Definition 5.2.

Corollary 5.10. *If $y = 0$ or $z = 0$, and there exists a proper 1-semiloop in an alternating dimap D of genus zero, then $P(D) = 0$.* $\square$

Proof. Let P and D be as stated. By Lemma 5.9, D has $G_{1,3}$ as a minor. Since $G_{1,3}$ contains a proper ω-loop and a proper ω^2-loop and P is an extended Tutte invariant, by Definition 5.2, we complete the proof. $\square$

Lemma 5.11. *Every alternating dimap of genus zero that contains a proper ω-semiloop has $G^a_{2,3}$ as a minor.*

Proof. From Table 3.1, we can see that a proper ω-semiloop can be obtained from a proper 1-semiloop e by applying the trial operation on e once. Likewise, we have $(G_{1,3})^\omega = G^a_{2,3}$. Therefore, by triality and Lemma 5.9, we complete the proof. □

Since $G^a_{2,3}$ contains a proper 1-loop and a proper ω^2-loop, the following corollary follows from Definition 5.2.

Corollary 5.12. *If $x = 0$ or $z = 0$, and there exists a proper ω-semiloop in an alternating dimap D of genus zero, then $P(D) = 0$.*

Lemma 5.13. *Every alternating dimap of genus zero that contains a proper ω^2-semiloop has $G^c_{2,3}$ as a minor.*

Proof. By triality and Lemma 5.11 (or Lemma 5.9). □

Since $G^c_{2,3}$ contains a proper 1-loop and a proper ω-loop, the following corollary follows from Definition 5.2.

Corollary 5.14. *If $x = 0$ or $y = 0$, and there exists a proper ω^2-semiloop in an alternating dimap D of genus zero, then $P(D) = 0$.*

Lemma 5.15. *Every alternating dimap of genus zero that contains a proper edge has $G_{1,3}$, $G^a_{2,3}$ and $G^c_{2,3}$ as minors.*

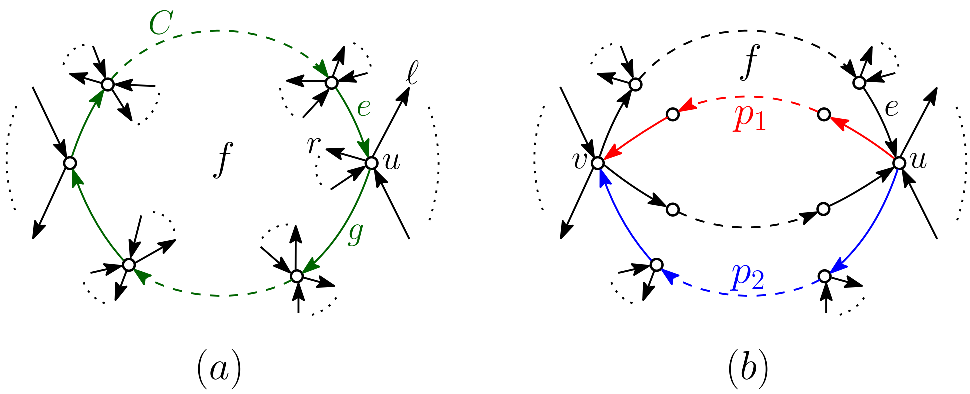


Figure 5.4: The alternating dimap D in the proof of Lemma 5.15

Proof. Suppose D is an alternating dimap of genus zero. Let $e \in E(D)$ be a proper edge that has $u \in V(D)$ as its head. Suppose f is a clockwise face (or an anticlockwise face with appropriate modifications) that contains e, and C is the outer cycle of f in D. We consider two cases as follows:

i) **There exists exactly one directed path between u and v, for all $u, v \in V(C)$** (see Figure 5.4(a)).

- Let ℓ and r be the left successor and the right successor of e, respectively. Suppose $\ell \in E(C)$. Given that there is exactly one directed path between every pair of vertices in $V(C)$, if $\ell \in E(C)$, the edge ℓ must be the next edge after e in C. This implies that e is a proper ω^2-semiloop instead of a proper edge. Hence, $\ell \notin E(C)$. Similar arguments show that $r \notin E(C)$. The fact that $\ell, r \notin E(C)$ implies that $\deg(u) \geq 6$. By contracting every edge in $E(C) \setminus e$ in D, the edge e becomes a proper 1-semiloop. By Lemma 5.9, we have $G_{1,3}$ as a minor of D.

- Let $g \in E(C)$ be the edge directed out from u. Suppose e and g partition $T = I(u) \setminus e$ in D into two sets (based on the cyclic order of T), (i) S_c that contains every edge directed into u that lies betwen g and e as we go from g to e in clockwise order around u, and (ii) $S_a = T \setminus S_c$. If we ω-reduce (respectively, ω^2-reduce) every edge in S_c (respectively, S_a), the edge e is now a proper ω-semiloop (respectively, proper ω^2-semiloop). By Lemma 5.11 (respectively, Lemma 5.13), we have $G^a_{2,3}$ (respectively, $G^c_{2,3}$) as a minor of D.

ii) **There exists more than one directed path between u and v, for some** $u, v \in V(C)$ (see Figure 5.4(b)). Let p_1 and p_2 be two of the paths that are directed from u to v, respectively.

- Contract every edge in p_1, and all but one edge in p_2; the remaining edge in p_2 is a proper 1-semiloop. By Lemma 5.9, we have $G_{1,3}$ as a minor of D.

- Recall that $e \in E(\partial f)$. Suppose $v \in V(\partial f)$ and $E(p_1) \subset E(\partial f)$. Let $h \in E(p_1)$ and h has v as its head. By Lemma 5.7, a face f' of size exactly two can be obtained from f, by a sequence of contraction operations. So, let $V(\partial f') = \{u, v\}$ and $E(\partial f') = \{e, h\}$. By ω^2-reducing every edge in $I(u) \setminus e$, we have $\deg(u) = 2$, and h is now a proper ω-semiloop. By Lemma 5.11, we have $G^a_{2,3}$ as a minor of D.

- Let g be an anticlockwise face in D, the proper edge $e \in E(\partial g)$ and $E(p_2) \subset E(\partial g)$. Suppose $h \in E(p_2)$ and h has v as its head. By Lemma 5.8, an anticlockwise face g' of size exactly two can be obtained from g. So, let $V(\partial g') = \{u, v\}$ and $E(\partial g') = \{e, h\}$. By ω-reducing every edge in $I(u) \setminus e$, we have $\deg(u) = 2$, and e is now a proper ω^2-semiloop. By Lemma 5.13, we have $G^c_{2,3}$ as a minor of D.

$\square$

Corollary 5.16. *If $x = 0$, $y = 0$ or $z = 0$, and there exists a proper edge in an alternating dimap D of genus zero, then $P(D) = 0$.*

Proof. By Lemma 5.15, if there exists a proper edge in an alternating dimap D of genus zero, then D contains $G_{1,3}$, $G^a_{2,3}$ and $G^c_{2,3}$ as minors. Since a proper 1-loop, a proper ω-loop and a proper ω^2-loop may each be found in at least one of these minors, by Definition 5.2, we have $P(D) = 0$ when at least one of x, y, z is zero. $\square$

By Lemmas 5.9, 5.11, 5.13 and 5.15, we obtain the following corollary.

Corollary 5.17. *Every alternating dimap of genus zero that contains a non-triloop edge has $G_{1,3}$ or $G_{2,3}$ as a minor.* □

Further from Theorem 5.6, we now discuss the the properties of alternating dimaps of genus zero that are required, in order to obtain a non-trivial invariant, when at least one of x, y, z is zero.

Theorem 5.18. *Let $S = (w, x, y, z, a, b, c, d, e, f, g, h, i, j, k, l)$ be a parameter sequence such that $w \neq 0$ and $x = y = z = 0$. A function P is an extended Tutte invariant with respect to S for every alternating dimap D of genus zero if and only if*

$$P(D) = \begin{cases} w^{k(D)}, & \text{if } \mathrm{is}(D)=\mathrm{af}(D)=\mathrm{cf}(D)=k(D), \\ 0, & \text{otherwise,} \end{cases}$$

with $a = f = h = 0$.

Proof. Let S be as stated and D be an alternating dimap of genus zero. We first prove the forward implication. Note that all the derived polynomials must be equal for $P(D)$ to be an extended Tutte invariant. By using Lemma 5.5 and the fact that $w \neq 0$ and $x = y = z = 0$, we obtain $a = f = h = 0$ as desired.

If $\mathrm{is}(D) = \mathrm{af}(D) = \mathrm{cf}(D) = k(D)$, then D is a disjoint union of ultraloops. By Definition 5.2, we have $P(D) = w^{k(D)}$.

We next show $P(D) = 0$ in the following cases.

i) $\mathrm{is}(D) \neq k(D) = \mathrm{af}(D) = \mathrm{cf}(D)$. Since $k(D) = \mathrm{af}(D) = \mathrm{cf}(D)$, each component of D has exactly one anticlockwise face and exactly one clockwise face. This implies that each componet of D is a directed cycle. The fact that $\mathrm{is}(D) \neq k(D)$ implies that at least one of the components A of D has order greater than one. Hence, A is a directed cycle of size greater than one. Since A is a component of D, there is a proper 1-loop in D. By (ETI2) and using $x = 0$, we have $P(D) = 0$.

ii) $\mathrm{af}(D) \neq k(D) = \mathrm{is}(D) = \mathrm{cf}(D)$. Since $k(D) = \mathrm{is}(D) = \mathrm{cf}(D)$, each component of D has exactly one vertex and exactly one clockwise face. The fact that $\mathrm{af}(D) \neq k(D)$ implies that at least one of the components of D has more than one anticlockwise face, and hence has a proper ω-loop. By (ETI3) and using $y = 0$, we have $P(D) = 0$.

iii) $\mathrm{cf}(D) \neq k(D) = \mathrm{is}(D) = \mathrm{af}(D)$. There is a proper ω^2-loop in D. By (ETI4) and using $z = 0$, we have $P(D) = 0$.

iv) $\mathrm{af}(D) \neq k(D)$, $\mathrm{cf}(D) \neq k(D) = \mathrm{is}(D)$. If every edge in D is a triloop, the result is trivial. Otherwise, $k(D) = \mathrm{is}(D)$ implies that each component of D has exactly one vertex. Since $\mathrm{af}(D) \neq k(D)$, $\mathrm{cf}(D) \neq k(D)$ and there is a non-triloop in D, there exists a proper 1-semiloop in D. By Corollary 5.10, we have $P(D) = 0$.

v) $\mathrm{is}(D) \neq k(D)$, $\mathrm{cf}(D) \neq k(D) = \mathrm{af}(D)$. If every edge in D is a triloop, the result is trivial. Otherwise, there exists a proper ω-semiloop in D. By Corollary 5.12, we have $P(D) = 0$.

vi) $\text{is}(D) \neq k(D)$, $\text{af}(D) \neq k(D) = \text{cf}(D)$. If every edge in D is a triloop, the result is trivial. Otherwise, there exists a proper ω^2-semiloop in D. By Corollary 5.14, we have $P(D) = 0$.

vii) $\text{is}(D) \neq k(D)$, $\text{af}(D) \neq k(D)$ and $\text{cf}(D) \neq k(D)$. If every edge in D is a triloop, the result is trivial. Otherwise, Corollaries 5.10, 5.12, 5.14 or 5.16 gives $P(D) = 0$.

Conversely, we show that

$$P(D) = \begin{cases} w^{k(D)}, & \text{if is(D)=af(D)=cf(D)=k(D)}, \\ 0, & \text{otherwise}, \end{cases}$$

with $a = f = h = 0$ is an extended Tutte invariant with respect to S for every alternating dimap D, as in Definition 5.2. Based on the proof in Theorem 5.6, it is clear that $P(D)$ is a multiplicative invariant.

We now show that $P(D)$ satisfies (ETI1) to (ETI8) by using induction on $|E(D)| = m$. When $m = 0$, we have $P(D) = 1$ and the result for $m = 0$ follows. So, suppose $m > 0$ and the result holds for every alternating dimap of genus zero that has size less than m.

Suppose $\text{is}(D) = \text{af}(D) = \text{cf}(D) = k(D)$. We have D as a disjoint union of ultraloops. Deletion of an ultraloop r from D reduces the number of components of D by 1. Thus,

$$\begin{aligned} P(D) &= w^{k(D)} \\ &= w \cdot w^{k(D)-1} \\ &= w \cdot w^{k(D\setminus r)} \\ &= w \cdot P(D \setminus r) && \text{(by the inductive hypothesis).} \end{aligned}$$

Note that for $\mu \in \{1, \omega, \omega^2\}$, it is possible to have $\text{is}(D[\mu]r) = \text{af}(D[\mu]r) = \text{cf}(D[\mu]r) = k(D[\mu]r)$ by μ-reducing r. Hence, we may have to consider more than one scenario for the remaining cases. Let $r \in E(D)$.

i) r is a proper 1-loop.

(a) $\text{is}(D[1]r) = \text{af}(D[1]r) = \text{cf}(D[1]r) = k(D[1]r)$.

$$\begin{aligned} P(D) &= 0 \\ &= x \cdot w^{k(D[1]r)} && \text{(since } x = 0) \\ &= x \cdot P(D[1]r) && \text{(by the inductive hypothesis).} \end{aligned}$$

(b) Otherwise,

$$\begin{aligned} P(D) &= 0 \\ &= x \cdot 0 \\ &= x \cdot P(D[1]r) && \text{(by the inductive hypothesis).} \end{aligned}$$

ii) r is a proper ω-loop.

(a) $\mathrm{is}(D[\omega]r) = \mathrm{af}(D[\omega]r) = \mathrm{cf}(D[\omega]r) = k(D[\omega]r)$.

$$\begin{aligned} P(D) &= 0 \\ &= y \cdot w^{k(D[\omega]r)} && \text{(since } y = 0\text{)} \\ &= y \cdot P(D[\omega]r) && \text{(by the inductive hypothesis).} \end{aligned}$$

(b) Otherwise,

$$\begin{aligned} P(D) &= 0 \\ &= y \cdot 0 \\ &= y \cdot P(D[\omega]r) && \text{(by the inductive hypothesis).} \end{aligned}$$

iii) r is a proper ω^2-loop.

(a) $\mathrm{is}(D[\omega^2]r) = \mathrm{af}(D[\omega^2]r) = \mathrm{cf}(D[\omega^2]r) = k(D[\omega^2]r)$.

$$\begin{aligned} P(D) &= 0 \\ &= z \cdot w^{k(D[\omega^2]r)} && \text{(since } z = 0\text{)} \\ &= z \cdot P(D[\omega^2]r) && \text{(by the inductive hypothesis).} \end{aligned}$$

(b) Otherwise,

$$\begin{aligned} P(D) &= 0 \\ &= z \cdot 0 \\ &= z \cdot P(D[\omega^2]r) && \text{(by the inductive hypothesis).} \end{aligned}$$

iv) r is a proper 1-semiloop. In $D[1]r$, the number of components and in-stars are both increased by 1. In $D[\omega]r$ and $D[\omega^2]r$, the number of c-faces and a-faces are reduced by 1, respectively.

(a) $\mathrm{is}(D[1]r) = \mathrm{af}(D[1]r) = \mathrm{cf}(D[1]r) = k(D[1]r)$.

$$\begin{aligned} P(D) &= 0 \\ &= a \cdot w^{k(D[1]r)} + b \cdot 0 + c \cdot 0 && \text{(since } a = 0\text{)} \\ &= a \cdot P(D[1]r) + b \cdot P(D[\omega]r) + c \cdot P(D[\omega^2]r) \\ & && \text{(by the inductive hypothesis).} \end{aligned}$$

(b) Otherwise,

$$\begin{aligned} P(D) &= 0 \\ &= a \cdot 0 + b \cdot 0 + c \cdot 0 \\ &= a \cdot P(D[1]r) + b \cdot P(D[\omega]r) + c \cdot P(D[\omega^2]r) \\ & && \text{(by the inductive hypothesis).} \end{aligned}$$

v) r is a proper ω-semiloop. In $D[\omega^2]r$, the number of components and a-faces are both increased by 1. In $D[1]r$ and $D[\omega]r$, the number of in-stars and c-faces are reduced by 1, respectively.

(a) $\mathrm{is}(D[\omega^2]r) = \mathrm{af}(D[\omega^2]r) = \mathrm{cf}(D[\omega^2]r) = k(D[\omega^2]r)$.

$$\begin{aligned} P(D) &= 0 \\ &= d \cdot 0 + e \cdot 0 + f \cdot w^{k(D[\omega^2]r)} && \text{(since } f = 0) \\ &= d \cdot P(D[1]r) + e \cdot P(D[\omega]r) + f \cdot P(D[\omega^2]r) \\ & && \text{(by the inductive hypothesis).} \end{aligned}$$

(b) Otherwise,

$$\begin{aligned} P(D) &= 0 \\ &= d \cdot 0 + e \cdot 0 + f \cdot 0 \\ &= d \cdot P(D[1]r) + e \cdot P(D[\omega]r) + f \cdot P(D[\omega^2]r) \\ & && \text{(by the inductive hypothesis).} \end{aligned}$$

vi) r is a proper ω^2-semiloop. In $D[\omega]r$, the number of components and c-faces are both increased by 1. In $D[1]r$ and $D[\omega^2]r$, the number of in-stars and a-faces are reduced by 1, respectively.

(a) $\mathrm{is}(D[\omega]r) = \mathrm{af}(D[\omega]r) = \mathrm{cf}(D[\omega]r) = k(D[\omega]r)$.

$$\begin{aligned} P(D) &= 0 \\ &= g \cdot 0 + h \cdot w^{k(D[\omega]r)} + i \cdot 0 && \text{(since } h = 0) \\ &= g \cdot P(D[1]r) + h \cdot P(D[\omega]r) + i \cdot P(D[\omega^2]r) \\ & && \text{(by the inductive hypothesis).} \end{aligned}$$

(b) Otherwise,

$$\begin{aligned} P(D) &= 0 \\ &= g \cdot 0 + h \cdot 0 + i \cdot 0 \\ &= g \cdot P(D[1]r) + h \cdot P(D[\omega]r) + i \cdot P(D[\omega^2]r) \\ & && \text{(by the inductive hypothesis).} \end{aligned}$$

vii) r is a proper edge.

$$\begin{aligned} P(D) &= 0 \\ &= j \cdot 0 + k \cdot 0 + l \cdot 0 \\ &= j \cdot P(D[1]r) + k \cdot P(D[\omega]r) + l \cdot P(D[\omega^2]r) \quad \text{(by the inductive hypothesis).} \end{aligned}$$

This completes the backward implication, by induction. □

Next, we have the following three results in which two of x, y, z are zero.

Theorem 5.19. *Let $S = (w, x, y, z, a, b, c, d, e, f, g, h, i, j, k, l)$ be a parameter sequence such that $w, z \neq 0$ and $x = y = 0$. A function P is an extended Tutte invariant with respect to S for every alternating dimap D of genus zero if and only if*

$$P(D) = \begin{cases} w^{k(D)} \cdot z^{\mathrm{cf}(D)-k(D)}, & \text{if } \mathrm{is}(D)=\mathrm{af}(D)=k(D), \\ 0, & \text{otherwise,} \end{cases}$$

with $a = c = d = f = h = 0$.

Proof. Let S be as stated and D be an alternating dimap of genus zero. We first prove the forward implication. Note that all the derived polynomials must be equal for $P(D)$ to be an extended Tutte invariant. By using Lemma 5.5 and the fact that $w, z \neq 0$ and $x = y = 0$, we obtain $a = c = d = f = h = 0$ as desired.

If $\mathrm{is}(D) = \mathrm{af}(D) = k(D)$, every edge in D is either an ultraloop or a proper ω^2-loop. We proceed by induction on the number m of edges of D. For the base case, suppose $m = 0$. This implies that D is empty. Therefore, we have $P(D) = w^0 \cdot z^{0-0} = 1$, and the result for $m = 0$ follows.

For the inductive step, assume that $m > 0$ and the result holds for every alternating dimap of genus zero that has size less than m. We consider two cases where an ultraloop or a proper ω^2-loop is first reduced in D. Let $r \in E(D)$.

i) r **is an ultraloop**. In $D \backslash r$, the number of components and c-faces are both reduced by 1. Thus,

$$\begin{aligned} P(D) &= w \cdot P(D \setminus r) \\ &= w \cdot w^{k(D\backslash r)} \cdot z^{\mathrm{cf}(D\backslash r)-k(D\backslash r)} && \text{(by the inductive hypothesis)} \\ &= w \cdot w^{k(D)-1} \cdot z^{\mathrm{cf}(D)-1-(k(D)-1)} \\ &= w^{k(D)} \cdot z^{\mathrm{cf}(D)-k(D)}. \end{aligned}$$

ii) r **is a proper ω^2-loop**. The number of c-faces is reduced by 1 in $D[\omega^2]r$. Thus,

$$\begin{aligned} P(D) &= z \cdot P(D[\omega^2]r) \\ &= z \cdot w^{k(D[\omega^2]r)} \cdot z^{\mathrm{cf}(D[\omega^2]r)-k(D[\omega^2]r)} && \text{(by the inductive hypothesis)} \\ &= z \cdot w^{k(D)} \cdot z^{\mathrm{cf}(D)-1-k(D)} \\ &= w^{k(D)} \cdot z^{\mathrm{cf}(D)-k(D)}. \end{aligned}$$

We next show $P(D) = 0$ in the following cases.

i) $\mathrm{is}(D) \neq k(D) = \mathrm{af}(D) = \mathrm{cf}(D)$. There is a proper 1-loop in D. By (ETI2) and using $x = 0$, we have $P(D) = 0$.

ii) $\mathrm{af}(D) \neq k(D) = \mathrm{is}(D) = \mathrm{cf}(D)$. There is a proper ω-loop in D. By (ETI3) and using $y = 0$, we have $P(D) = 0$.

iii) $\mathrm{af}(D) \neq k(D)$, $\mathrm{cf}(D) \neq k(D) = \mathrm{is}(D)$. If every edge in D is a triloop, there exist proper ω-loops and proper ω^2-loops in D. By (ETI3) and using $y = 0$, the result follows. Otherwise, there exists a proper 1-semiloop in D. By Corollary 5.10, we have $P(D) = 0$.

iv) $\mathrm{is}(D) \neq k(D)$, $\mathrm{cf}(D) \neq k(D)$ and $\mathrm{af}(D) = k(D)$. If every edge in D is a triloop, there exist proper 1-loops and proper ω^2-loops in D. By (ETI2) and using $x = 0$, the result follows. Otherwise, there exists a proper ω-semiloop in D. By Corollary 5.12, we have $P(D) = 0$.

v) $\mathrm{is}(D) \neq k(D)$, $\mathrm{af}(D) \neq k(D)$ and $\mathrm{cf}(D) = k(D)$. If every edge in D is a triloop, there exist proper 1-loops and proper ω-loops in D. By (ETI2) (respectively, (ETI3)) and using $x = 0$ (respectively, $y = 0$), the result follows. Otherwise, there exists a proper ω^2-semiloop in D. By Corollary 5.14, we have $P(D) = 0$.

vi) $\mathrm{is}(D) \neq k(D)$, $\mathrm{af}(D) \neq k(D)$ and $\mathrm{cf}(D) \neq k(D)$. If every edge in D is a triloop, there exist proper 1-loops, proper ω-loops and proper ω^2-loops in D. By (ETI2) (respectively, (ETI3)) and using $x = 0$ (respectively, $y = 0$), the result follows. Otherwise, Corollaries 5.10, 5.12, 5.14 or 5.16 gives $P(D) = 0$.

Conversely, we show that

$$P(D) = \begin{cases} w^{k(D)} \cdot z^{\mathrm{cf}(D)-k(D)}, & \text{if is(D)=af(D)=k(D)}, \\ 0, & \text{otherwise}, \end{cases}$$

with $a = c = d = f = h = 0$ is an extended Tutte invariant with respect to S for every alternating dimap D, as in Definition 5.2. Based on the proof in Theorem 5.6, it is clear that $P(D)$ is a multiplicative invariant.

We now show that $P(D)$ satisfies (ETI1) to (ETI8) by using induction on $|E(D)| = m$. When $m = 0$, we have $P(D) = 1$ and the result for $m = 0$ follows. So, suppose $m > 0$ and the result holds for every alternating dimap of genus zero that has size less than m.

Suppose $\mathrm{is}(D) = \mathrm{af}(D) = k(D)$. Every edge in D is either an ultraloop or a proper ω^2-loop. Let $r \in E(D)$.

i) r is an ultraloop.

$$\begin{aligned} P(D) &= w^{k(D)} \cdot z^{\mathrm{cf}(D)-k(D)} \\ &= w \cdot w^{k(D)-1} \cdot z^{\mathrm{cf}(D)-1-(k(D)-1)} \\ &= w \cdot w^{k(D\setminus r)} \cdot z^{\mathrm{cf}(D\setminus r)-k(D\setminus r)} \\ &= w \cdot P(D \setminus r) && \text{(by the inductive hypothesis).} \end{aligned}$$

ii) r is a proper ω^2-loop.

$$\begin{aligned} P(D) &= w^{k(D)} \cdot z^{\mathrm{cf}(D)-k(D)} \\ &= z \cdot w^{k(D)} \cdot z^{\mathrm{cf}(D)-1-k(D)} \end{aligned}$$

$$
\begin{aligned}
&= z \cdot w^{k(D[\omega^2]r)} \cdot z^{\mathrm{cf}(D[\omega^2]r)-k(D[\omega^2]r)} \\
&= z \cdot P(D[\omega^2]r) \qquad \text{(by the inductive hypothesis).}
\end{aligned}
$$

Note that for $\mu \in \{1, \omega, \omega^2\}$, it is possible to have $\mathrm{is}(D[\mu]r) = \mathrm{af}(D[\mu]r) = k(D[\mu]r)$ by μ-reducing r. Hence, we may have to consider more than one scenario for the remaining cases. Let $r \in E(D)$.

i) r is a proper 1-loop.

(a) $\mathrm{is}(D[1]r) = \mathrm{af}(D[1]r) = k(D[1]r)$.

$$
\begin{aligned}
P(D) &= 0 \\
&= x \cdot w^{k(D[1]r)} \cdot z^{\mathrm{cf}(D[1]r)-k(D[1]r)} && \text{(since } x = 0\text{)} \\
&= x \cdot P(D[1]r) && \text{(by the inductive hypothesis).}
\end{aligned}
$$

(b) Otherwise,

$$
\begin{aligned}
P(D) &= 0 \\
&= x \cdot 0 \\
&= x \cdot P(D[1]r) && \text{(by the inductive hypothesis).}
\end{aligned}
$$

ii) r is a proper ω-loop.

(a) $\mathrm{is}(D[\omega]r) = \mathrm{af}(D[\omega]r) = k(D[\omega]r)$.

$$
\begin{aligned}
P(D) &= 0 \\
&= y \cdot w^{k(D[\omega]r)} \cdot z^{\mathrm{cf}(D[\omega]r)-k(D[\omega]r)} && \text{(since } y = 0\text{)} \\
&= y \cdot P(D[\omega]r) && \text{(by the inductive hypothesis).}
\end{aligned}
$$

(b) Otherwise,

$$
\begin{aligned}
P(D) &= 0 \\
&= y \cdot 0 \\
&= y \cdot P(D[\omega]r) && \text{(by the inductive hypothesis).}
\end{aligned}
$$

iii) r is a proper 1-semiloop. In $D[1]r$, the number of components and in-stars are both increased by 1. In $D[\omega]r$ and $D[\omega^2]r$, the number of c-faces and a-faces are reduced by 1, respectively.

(a) $\mathrm{is}(D[1]r) = \mathrm{af}(D[1]r) = k(D[1]r)$ and $\mathrm{is}(D[\omega^2]r) = \mathrm{af}(D[\omega^2]r) = k(D[\omega^2]r)$.

$$
\begin{aligned}
P(D) &= 0 \\
&= a \cdot w^{k(D[1]r)} \cdot z^{\mathrm{cf}(D[1]r)-k(D[1]r)} + b \cdot 0 + c \cdot w^{k(D[\omega^2]r)} \cdot z^{\mathrm{cf}(D[\omega^2]r)-k(D[\omega^2]r)} \\
& \qquad\qquad \text{(since } a = c = 0\text{)}
\end{aligned}
$$

$$= a \cdot P(D[1]r) + b \cdot P(D[\omega]r) + c \cdot P(D[\omega^2]r)$$
(by the inductive hypothesis).

(b) Otherwise,

$$\begin{aligned} P(D) &= 0 \\ &= a \cdot 0 + b \cdot 0 + c \cdot 0 \\ &= a \cdot P(D[1]r) + b \cdot P(D[\omega]r) + c \cdot P(D[\omega^2]r) \end{aligned}$$
(by the inductive hypothesis).

iv) r is a proper ω-semiloop. In $D[\omega^2]r$, the number of components and a-faces are both increased by 1. In $D[1]r$ and $D[\omega]r$, the number of in-stars and c-faces are reduced by 1, respectively.

(a) $\text{is}(D[1]r) = \text{af}(D[1]r) = k(D[1]r)$ and $\text{is}(D[\omega^2]r) = \text{af}(D[\omega^2]r) = k(D[\omega^2]r)$.

$$\begin{aligned} P(D) &= 0 \\ &= d \cdot w^{k(D[1]r)} \cdot z^{\text{cf}(D[1]r)-k(D[1]r)} + e \cdot 0 + f \cdot w^{k(D[\omega^2]r)} \cdot z^{\text{cf}(D[\omega^2]r)-k(D[\omega^2]r)} \end{aligned}$$
(since $d = f = 0$)

$$= d \cdot P(D[1]r) + e \cdot P(D[\omega]r) + f \cdot P(D[\omega^2]r)$$
(by the inductive hypothesis).

(b) Otherwise,

$$\begin{aligned} P(D) &= 0 \\ &= d \cdot 0 + e \cdot 0 + f \cdot 0 \\ &= d \cdot P(D[1]r) + e \cdot P(D[\omega]r) + f \cdot P(D[\omega^2]r) \end{aligned}$$
(by the inductive hypothesis).

v) r is a proper ω^2-semiloop. In $D[\omega]r$, the number of components and c-faces are both increased by 1. In $D[1]r$ and $D[\omega^2]r$, the number of in-stars and a-faces are reduced by 1, respectively.

(a) $\text{is}(D[\omega]r) = \text{af}(D[\omega]r) = k(D[\omega]r)$.

$$\begin{aligned} P(D) &= 0 \\ &= g \cdot 0 + h \cdot w^{k(D[\omega]r)} \cdot z^{\text{cf}(D[\omega]r)-k(D[\omega]r)} + i \cdot 0 \qquad \text{(since } h = 0) \\ &= g \cdot P(D[1]r) + h \cdot P(D[\omega]r) + i \cdot P(D[\omega^2]r) \end{aligned}$$
(by the inductive hypothesis).

(b) Otherwise,

$$\begin{aligned} P(D) &= 0 \\ &= g \cdot 0 + h \cdot 0 + i \cdot 0 \end{aligned}$$

$$= g \cdot P(D[1]r) + h \cdot P(D[\omega]r) + i \cdot P(D[\omega^2]r)$$
(by the inductive hypothesis).

vi) r is a proper edge.

$$\begin{aligned} P(D) &= 0 \\ &= j \cdot 0 + k \cdot 0 + l \cdot 0 \\ &= j \cdot P(D[1]r) + k \cdot P(D[\omega]r) + l \cdot P(D[\omega^2]r) \quad \text{(by the inductive hypothesis).} \end{aligned}$$

This completes the backward implication, by induction. □

Triality leads to the following two corollaries, for $x = z = 0$ and $y = z = 0$, respectively.

Corollary 5.20. *Let $S = (w, x, y, z, a, b, c, d, e, f, g, h, i, j, k, l)$ be a parameter sequence such that $w, y \neq 0$ and $x = z = 0$. A function P is an extended Tutte invariant with respect to S for every alternating dimap D of genus zero if and only if*

$$P(D) = \begin{cases} w^{k(D)} \cdot y^{\mathrm{af}(D)-k(D)}, & \text{if } \mathrm{is}(D)=\mathrm{cf}(D)=k(D), \\ 0, & \text{otherwise,} \end{cases}$$

with $a = b = f = g = h = 0$. □

Corollary 5.21. *Let $S = (w, x, y, z, a, b, c, d, e, f, g, h, i, j, k, l)$ be a parameter sequence such that $w, x \neq 0$ and $y = z = 0$. A function P is an extended Tutte invariant with respect to S for every alternating dimap D of genus zero if and only if*

$$P(D) = \begin{cases} w^{k(D)} \cdot x^{\mathrm{is}(D)-k(D)}, & \text{if } \mathrm{af}(D)=\mathrm{cf}(D)=k(D), \\ 0, & \text{otherwise,} \end{cases}$$

with $a = e = f = h = i = 0$. □

Lastly, we investigate cases where exactly one of the three variables is zero.

Theorem 5.22. *Let $S = (w, x, y, z, a, b, c, d, e, f, g, h, i, j, k, l)$ be a parameter sequence such that $w, y, z \neq 0$ and $x = 0$. A function P is an extended Tutte invariant with respect to S for every alternating dimap D of genus zero if and only if*

$$P(D) = \begin{cases} w^{k(D)} \cdot y^{\mathrm{af}(D)-k(D)} \cdot z^{\mathrm{cf}(D)-k(D)}, & \text{if } \mathrm{is}(D)=k(D), \\ 0, & \text{otherwise,} \end{cases}$$

with $d = f = g = h = j = 0$ and $yz = aw + by + cz$.

Proof. Let S be as stated and D be an alternating dimap of genus zero. We first prove the forward implication. Note that all the derived polynomials must be equal for $P(D)$ to be an extended Tutte invariant. By using (E1)–(E4) in Lemma 5.4, and the fact that $w, y, z \neq 0$ and $x = 0$, we obtain $d = f = g = h = j = 0$ and $yz = aw + by + cz$ as desired.

If $\mathrm{is}(D) = k(D)$, then each component of D has a single vertex and every edge in D is either an ultraloop, a proper ω-loop, a proper ω^2-loop or a proper 1-semiloop. We proceed by induction on the number m of edges of D. For the base case, suppose $m = 0$. Clearly, $P(D) = 1$ and the result for $m = 0$ follows.

For the inductive step, assume that $m > 0$ and the result holds for every alternating dimap of genus zero that has size less than m. We consider four cases corresponding to the four possible edge types in D. Let $r \in E(D)$.

i) r **is an ultraloop**. In $D \setminus r$, the number of components, a-faces and c-faces, are all reduced by 1. Thus,

$$\begin{aligned} P(D) &= w \cdot P(D \setminus r) \\ &= w \cdot w^{k(D\setminus r)} \cdot y^{\mathrm{af}(D\setminus r)-k(D\setminus r)} \cdot z^{\mathrm{cf}(D\setminus r)-k(D\setminus r)} \quad \text{(by the inductive hypothesis)} \\ &= w \cdot w^{k(D)-1} \cdot y^{\mathrm{af}(D)-1-(k(D)-1)} \cdot z^{\mathrm{cf}(D)-1-(k(D)-1)} \\ &= w^{k(D)} \cdot y^{\mathrm{af}(D)-k(D)} \cdot z^{\mathrm{cf}(D)-k(D)}. \end{aligned}$$

ii) r **is a proper ω-loop**. The number of a-faces is reduced by 1 in $D[\omega]r$. Thus,

$$\begin{aligned} P(D) &= y \cdot P(D[\omega]r) \\ &= y \cdot w^{k(D[\omega]r)} \cdot y^{\mathrm{af}(D[\omega]r)-k(D[\omega]r)} \cdot z^{\mathrm{cf}(D[\omega]r)-k(D[\omega]r)} \\ &\qquad\qquad \text{(by the inductive hypothesis)} \\ &= y \cdot w^{k(D)} \cdot y^{\mathrm{af}(D)-1-k(D)} \cdot z^{\mathrm{cf}(D)-k(D)} \\ &= w^{k(D)} \cdot y^{\mathrm{af}(D)-k(D)} \cdot z^{\mathrm{cf}(D)-k(D)}. \end{aligned}$$

iii) r **is a proper ω^2-loop**. The number of c-faces is reduced by 1 in $D[\omega^2]r$. Thus,

$$\begin{aligned} P(D) &= z \cdot P(D[\omega^2]r) \\ &= z \cdot w^{k(D[\omega^2]r)} \cdot y^{\mathrm{af}(D[\omega^2]r)-k(D[\omega^2]r)} \cdot z^{\mathrm{cf}(D[\omega^2]r)-k(D[\omega^2]r)} \\ &\qquad\qquad \text{(by the inductive hypothesis)} \\ &= z \cdot w^{k(D)} \cdot y^{\mathrm{af}(D)-k(D)} \cdot z^{\mathrm{cf}(D)-1-k(D)} \\ &= w^{k(D)} \cdot y^{\mathrm{af}(D)-k(D)} \cdot z^{\mathrm{cf}(D)-k(D)}. \end{aligned}$$

iv) r **is a proper 1-semiloop**. In $D[1]r$, the number of components and in-stars are both increased by 1. In $D[\omega]r$ and $D[\omega^2]r$, the number of c-faces and a-faces are reduced by 1, respectively. Hence,

$$\begin{aligned} yz \cdot P(D) &= yz \cdot \Big(a \cdot P(D[1]r) + b \cdot P(D[\omega]r) + c \cdot P(D[\omega^2]r)\Big) \\ &= yz \cdot \Big(a \cdot w^{k(D[1]r)} \cdot y^{\mathrm{af}(D[1]r)-k(D[1]r)} \cdot z^{\mathrm{cf}(D[1]r)-k(D[1]r)} \\ &\quad + b \cdot w^{k(D[\omega]r)} \cdot y^{\mathrm{af}(D[\omega]r)-k(D[\omega]r)} \cdot z^{\mathrm{cf}(D[\omega]r)-k(D[\omega]r)} \\ &\quad + c \cdot w^{k(D[\omega^2]r)} \cdot y^{\mathrm{af}(D[\omega^2]r)-k(D[\omega^2]r)} \cdot z^{\mathrm{cf}(D[\omega^2]r)-k(D[\omega^2]r)}\Big) \\ &\qquad\qquad \text{(by the inductive hypothesis)} \\ &= yz \cdot \Big(a \cdot w^{k(D)+1} \cdot y^{\mathrm{af}(D)-(k(D)+1)} \cdot z^{\mathrm{cf}(D)-(k(D)+1)} \end{aligned}$$

$$
\begin{aligned}
&+ b \cdot w^{k(D)} \cdot y^{\mathrm{af}(D)-k(D)} \cdot z^{\mathrm{cf}(D)-1-k(D)} \\
&+ c \cdot w^{k(D)} \cdot y^{\mathrm{af}(D)-1-k(D)} \cdot z^{\mathrm{cf}(D)-k(D)}\Big) \\
&= (aw + by + cz) \cdot w^{k(D)} \cdot y^{\mathrm{af}(D)-k(D)} \cdot z^{\mathrm{cf}(D)-k(D)} \\
P(D) &= w^{k(D)} \cdot y^{\mathrm{af}(D)-k(D)} \cdot z^{\mathrm{cf}(D)-k(D)} \quad \text{(since } yz \neq 0 \text{ and } yz = aw + by + cz\text{).}
\end{aligned}
$$

We next show $P(D) = 0$ when $\mathrm{is}(D) \neq k(D)$.

i) $\mathrm{is}(D) \neq k(D) = \mathrm{af}(D) = \mathrm{cf}(D)$. There is a proper 1-loop in D. By (ETI2) and using $x = 0$, we have $P(D) = 0$.

ii) $\mathrm{is}(D) \neq k(D)$, $\mathrm{cf}(D) \neq k(D) = \mathrm{af}(D)$. If every edge in D is a triloop, there exist proper 1-loops and proper ω^2-loops in D. By (ETI2) and using $x = 0$, the result follows. Otherwise, there exists a proper ω-semiloop in D. By Corollary 5.12, we have $P(D) = 0$.

iii) $\mathrm{is}(D) \neq k(D)$, $\mathrm{af}(D) \neq k(D) = \mathrm{cf}(D)$. If every edge in D is a triloop, there exist proper 1-loops and proper ω-loops in D. By (ETI2) and using $x = 0$, the result follows. Otherwise, there exists a proper ω^2-semiloop in D. By Corollary 5.14, we have $P(D) = 0$.

iv) $\mathrm{is}(D) \neq k(D)$, $\mathrm{af}(D) \neq k(D)$ and $\mathrm{cf}(D) \neq k(D)$. If every edge in D is a triloop, there exist proper 1-loops, proper ω-loops and proper ω^2-loops in D. By (ETI2) and using $x = 0$, the result follows. Otherwise, Corollaries 5.12, 5.14 or 5.16 gives $P(D) = 0$.

Conversely, we show that

$$
P(D) = \begin{cases} w^{k(D)} \cdot y^{\mathrm{af}(D)-k(D)} \cdot z^{\mathrm{cf}(D)-k(D)}, & \text{if is(D)=k(D),} \\ 0, & \text{otherwise,} \end{cases}
$$

with $d = f = g = h = j = 0$ and $yz = aw + by + cz$ is an extended Tutte invariant with respect to S for every alternating dimap D, as in Definition 5.2. Based on the proof in Theorem 5.6, it is clear that $P(D)$ is a multiplicative invariant.

We now show that $P(D)$ satisfies (ETI1) to (ETI8) by using induction on $|E(D)| = m$. When $m = 0$, we have $P(D) = 1$ and the result for $m = 0$ follows. So, suppose $m > 0$ and the result holds for every alternating dimap of genus zero that has size less than m.

Suppose $\mathrm{is}(D) = k(D)$. Each component of D has a single vertex and every edge in D is either an ultraloop, a proper ω-loop, a proper ω^2-loop or a proper 1-semiloop. Let $r \in E(D)$.

i) r is an ultraloop.

$$
\begin{aligned}
P(D) &= w^{k(D)} \cdot y^{\mathrm{af}(D)-k(D)} \cdot z^{\mathrm{cf}(D)-k(D)} \\
&= w \cdot w^{k(D)-1} \cdot y^{\mathrm{af}(D)-1-(k(D)-1)} \cdot z^{\mathrm{cf}(D)-1-(k(D)-1)} \\
&= w \cdot w^{k(D\backslash r)} \cdot y^{\mathrm{af}(D\backslash r)-k(D\backslash r)} \cdot z^{\mathrm{cf}(D\backslash r)-k(D\backslash r)} \\
&= w \cdot P(D \setminus r) \qquad \text{(by the inductive hypothesis).}
\end{aligned}
$$

ii) r is a proper ω-loop.

$$\begin{aligned}
P(D) &= w^{k(D)} \cdot y^{\mathrm{af}(D)-k(D)} \cdot z^{\mathrm{cf}(D)-k(D)} \\
&= y \cdot w^{k(D)} \cdot y^{\mathrm{af}(D)-1-k(D)} \cdot z^{\mathrm{cf}(D)-k(D)} \\
&= y \cdot w^{k(D[\omega]r)} \cdot y^{\mathrm{af}(D[\omega]r)-k(D[\omega]r)} \cdot z^{\mathrm{cf}(D[\omega]r)-k(D[\omega]r)} \\
&= y \cdot P(D[\omega]r) && \text{(by the inductive hypothesis).}
\end{aligned}$$

iii) r is a proper ω^2-loop.

$$\begin{aligned}
P(D) &= w^{k(D)} \cdot y^{\mathrm{af}(D)-k(D)} \cdot z^{\mathrm{cf}(D)-k(D)} \\
&= z \cdot w^{k(D)} \cdot y^{\mathrm{af}(D)-k(D)} \cdot z^{\mathrm{cf}(D)-1-k(D)} \\
&= z \cdot w^{k(D[\omega^2]r)} \cdot y^{\mathrm{af}(D[\omega^2]r)-k(D[\omega^2]r)} \cdot z^{\mathrm{cf}(D[\omega^2]r)-k(D[\omega^2]r)} \\
&= z \cdot P(D[\omega^2]r) && \text{(by the inductive hypothesis).}
\end{aligned}$$

iv) r is a proper 1-semiloop.

$$\begin{aligned}
yz \cdot P(D) &= (aw + by + cz) \cdot w^{k(D)} \cdot y^{\mathrm{af}(D)-k(D)} \cdot z^{\mathrm{cf}(D)-k(D)} \\
&\qquad \text{(since } yz \neq 0 \text{ and } yz = aw + by + cz\text{)} \\
&= yz \cdot \Big(a \cdot w^{k(D)+1} \cdot y^{\mathrm{af}(D)-(k(D)+1)} \cdot z^{\mathrm{cf}(D)-(k(D)+1)} \\
&\quad + b \cdot w^{k(D)} \cdot y^{\mathrm{af}(D)-k(D)} \cdot z^{\mathrm{cf}(D)-1-k(D)} \\
&\quad + c \cdot w^{k(D)} \cdot y^{\mathrm{af}(D)-1-k(D)} \cdot z^{\mathrm{cf}(D)-k(D)}\Big) \\
&= yz \cdot \Big(a \cdot w^{k(D[1]r)} \cdot y^{\mathrm{af}(D[1]r)-k(D[1]r)} \cdot z^{\mathrm{cf}(D[1]r)-k(D[1]r)} \\
&\quad + b \cdot w^{k(D[\omega]r)} \cdot y^{\mathrm{af}(D[\omega]r)-k(D[\omega]r)} \cdot z^{\mathrm{cf}(D[\omega]r)-k(D[\omega]r)} \\
&\quad + c \cdot w^{k(D[\omega^2]r)} \cdot y^{\mathrm{af}(D[\omega^2]r)-k(D[\omega^2]r)} \cdot z^{\mathrm{cf}(D[\omega^2]r)-k(D[\omega^2]r)}\Big) \\
&= yz \cdot \Big(a \cdot P(D[1]r) + b \cdot P(D[\omega]r) + c \cdot P(D[\omega^2]r)\Big) \\
&\qquad \text{(by the inductive hypothesis)} \\
P(D) &= a \cdot P(D[1]r) + b \cdot P(D[\omega]r) + c \cdot P(D[\omega^2]r) \qquad \text{(since } yz \neq 0\text{).}
\end{aligned}$$

Note that for $\mu \in \{1, \omega, \omega^2\}$, it is possible to have $\mathrm{is}(D[\mu]r) = k(D[\mu]r)$ by μ-reducing r. Hence, we may have to consider more than one scenario for the remaining cases. Let $r \in E(D)$.

i) r is a proper 1-loop.

(a) $\mathrm{is}(D[1]r) = k(D[1]r)$.

$$\begin{aligned}
P(D) &= 0 \\
&= x \cdot w^{k(D[1]r)} \cdot y^{\mathrm{af}(D[1]r)-k(D[1]r)} \cdot z^{\mathrm{cf}(D[1]r)-k(D[1]r)} && \text{(since } x = 0\text{)} \\
&= x \cdot P(D[1]r) && \text{(by the inductive hypothesis).}
\end{aligned}$$

(b) Otherwise,

$$\begin{aligned} P(D) &= 0 \\ &= x \cdot 0 \\ &= x \cdot P(D[1]r) && \text{(by the inductive hypothesis).} \end{aligned}$$

ii) r is a proper ω-semiloop. In $D[\omega^2]r$, the number of components and a-faces are both increased by 1. In $D[1]r$ and $D[\omega]r$, the number of in-stars and c-faces are reduced by 1, respectively.

(a) $\mathrm{is}(D[1]r) = k(D[1]r)$ and $\mathrm{is}(D[\omega^2]r) = k(D[\omega^2]r)$.

$$\begin{aligned} P(D) &= 0 \\ &= d \cdot w^{k(D[1]r)} \cdot y^{\mathrm{af}(D[1]r)-k(D[1]r)} \cdot z^{\mathrm{cf}(D[1]r)-k(D[1]r)} + e \cdot 0 \\ &\quad + f \cdot w^{k(D[\omega^2]r)} \cdot y^{\mathrm{af}(D[\omega^2]r)-k(D[\omega^2]r)} \cdot z^{\mathrm{cf}(D[\omega^2]r)-k(D[\omega^2]r)} \\ &\qquad \text{(since } d = f = 0) \\ &= d \cdot P(D[1]r) + e \cdot P(D[\omega]r) + f \cdot P(D[\omega^2]r) \\ &\qquad \text{(by the inductive hypothesis).} \end{aligned}$$

(b) Otherwise,

$$\begin{aligned} P(D) &= 0 \\ &= d \cdot 0 + e \cdot 0 + f \cdot 0 \\ &= d \cdot P(D[1]r) + e \cdot P(D[\omega]r) + f \cdot P(D[\omega^2]r) \\ &\qquad \text{(by the inductive hypothesis).} \end{aligned}$$

iii) r is a proper ω^2-semiloop. In $D[\omega]r$, the number of components and c-faces are both increased by 1. In $D[1]r$ and $D[\omega^2]r$, the number of in-stars and a-faces are reduced by 1, respectively.

(a) $\mathrm{is}(D[1]r) = k(D[1]r)$ and $\mathrm{is}(D[\omega]r) = k(D[\omega]r)$.

$$\begin{aligned} P(D) &= 0 \\ &= g \cdot w^{k(D[1]r)} \cdot y^{\mathrm{af}(D[1]r)-k(D[1]r)} \cdot z^{\mathrm{cf}(D[1]r)-k(D[1]r)} \\ &\quad + h \cdot w^{k(D[\omega]r)} \cdot y^{\mathrm{af}(D[\omega]r)-k(D[\omega]r)} \cdot z^{\mathrm{cf}(D[\omega]r)-k(D[\omega]r)} + i \cdot 0 \\ &\qquad \text{(since } g = h = 0) \\ &= g \cdot P(D[1]r) + h \cdot P(D[\omega]r) + i \cdot P(D[\omega^2]r) \\ &\qquad \text{(by the inductive hypothesis).} \end{aligned}$$

(b) Otherwise,

$$\begin{aligned} P(D) &= 0 \\ &= g \cdot 0 + h \cdot 0 + i \cdot 0 \end{aligned}$$

$$= g \cdot P(D[1]r) + h \cdot P(D[\omega]r) + i \cdot P(D[\omega^2]r)$$
(by the inductive hypothesis).

iv) r is a proper edge. In $D[1]r$, $D[\omega]r$ and $D[\omega^2]r$, the number of in-stars, c-faces and a-faces are reduced by 1, respectively.

(a) $\mathrm{is}(D[1]r) = k(D[1]r)$.

$$\begin{aligned} P(D) &= 0 \\ &= j \cdot w^{k(D[1]r)} \cdot y^{\mathrm{af}(D[1]r)-k(D[1]r)} \cdot z^{\mathrm{cf}(D[1]r)-k(D[1]r)} + k \cdot 0 + l \cdot 0 \\ &\qquad \text{(since } j = 0) \\ &= j \cdot P(D[1]r) + k \cdot P(D[\omega]r) + l \cdot P(D[\omega^2]r) \\ &\qquad \text{(by the inductive hypothesis).} \end{aligned}$$

(b) Otherwise,

$$\begin{aligned} P(D) &= 0 \\ &= j \cdot 0 + k \cdot 0 + l \cdot 0 \\ &= j \cdot P(D[1]r) + k \cdot P(D[\omega]r) + l \cdot P(D[\omega^2]r) \\ &\qquad \text{(by the inductive hypothesis).} \end{aligned}$$

This completes the backward implication, by induction. □

Similarly, by using triality, we have

Corollary 5.23. *Let $S = (w, x, y, z, a, b, c, d, e, f, g, h, i, j, k, l)$ be a parameter sequence such that $w, x, z \neq 0$ and $y = 0$. A function P is an extended Tutte invariant with respect to S for every alternating dimap D of genus zero if and only if*

$$P(D) = \begin{cases} w^{k(D)} \cdot x^{\mathrm{is}(D)-k(D)} \cdot z^{\mathrm{cf}(D)-k(D)}, & \text{if } \mathrm{af}(D)=k(D), \\ 0, & \text{otherwise,} \end{cases}$$

with $a = c = h = i = l = 0$ and $xz = dz + ex + fw$. □

Corollary 5.24. *Let $S = (w, x, y, z, a, b, c, d, e, f, g, h, i, j, k, l)$ be a parameter sequence such that $w, x, y \neq 0$ and $z = 0$. A function P is an extended Tutte invariant with respect to S for every alternating dimap D of genus zero if and only if*

$$P(D) = \begin{cases} w^{k(D)} \cdot x^{\mathrm{is}(D)-k(D)} \cdot y^{\mathrm{af}(D)-k(D)}, & \text{if } \mathrm{cf}(D)=k(D), \\ 0, & \text{otherwise.} \end{cases}$$

with $a = b = e = f = k = 0$ and $xy = gy + hw + ix$. □

5.2 Restricted Alternating Dimaps, Independent Parameters

An extended Tutte invariant is *well defined* for an alternating dimap D if every edge-ordering of D gives an identical derived polynomial, when D is reduced using these edge-orderings.

In Section 5.1, we identified restrictions on the parameters that ensure extended Tutte invariants are well defined for all alternating dimaps of genus zero when the restrictions are satisfied (see Theorem 5.6). We now investigate the conditions on an alternating dimap that are required in order to obtain a well defined extended Tutte invariant for it, without any restriction on the parameters. The fact that no restriction is imposed on the parameters implies that the variables $w, x, y, z, a, b, c, d, e, f, g, h, i, j, k, l$ are all independent, and will be treated as indeterminates.

To formalise this distinction, we need a more specific extended Tutte invariant. The *complete* extended Tutte invariant of an alternating dimap takes values in a ring $\mathbb{E}[w, x, y, z, a, b, c, d, e, f, g, h, i, j, k, l]$, where $\mathbb{E}$ is a field. The ring is considered to be a subset of the field of fractions $\mathbb{F} \coloneqq \mathbb{E}(w, x, y, z, a, b, c, d, e, f, g, h, i, j, k, l)$ whose numerators and denominators are in $\mathbb{E}[w, x, y, z, a, b, c, d, e, f, g, h, i, j, k, l]$.

We will determine the domain of the complete extended Tutte invariant, which is the set of alternating dimaps for which it is well defined.

Let D be an alternating dimap. For $i \in \{1, 2, \ldots, |E(D)|\}$ and $\mu_i \in \{1, \omega, \omega^2\}$, a *reduction sequence* for a given edge-ordering $\mathcal{O} = e_1e_2 \ldots e_i$ of D is a sequence of reductions $\mathcal{R} = \mu_1, \mu_2, \ldots, \mu_i$. By reducing D using the edge-ordering $\mathcal{O}$ and the reduction sequence $\mathcal{R}$, we obtain the minor $D[\mu_1]e_1[\mu_2]e_2 \ldots [\mu_i]e_i$, which is denoted by $D[\mathcal{R}]\mathcal{O}$.

For a given edge-ordering, an extended Tutte invariant is constructed using a set of sequences of reductions of the edges, where the edges are reduced in the given order. For each sequence of reductions, a factor is introduced each time an edge is reduced in the sequence. For instance, the factor of x is introduced when a proper 1-loop is reduced in Definition 5.2. Note that if more than one reduction is performed on an edge (i.e., a non-triloop edge in Definition 5.2), the type of reduction operation determines the factor that will be introduced for each minor.

For $i \in \{1, 2, \ldots, |E(D)|\}$, suppose H is a minor of D that is obtained by reducing the first i edges of D. Then, the first i factors introduced by these i reductions form the *monomial of* H with respect to D and the reductions used. In Figure 5.5, the first two edges of an alternating dimap D are reduced in the extended Tutte invariant by using $\mathcal{O} = pqr$. Since only one edge p is reduced to obtain the minor $D[\omega]p$, the factor e is also the monomial of this minor. On the other hand, for the minor $D[\omega^2]p[*]q$, two factors f and w are obtained. Hence, we have fw as the monomial of $D[\omega^2]p[*]q$.

Proposition 5.25. *If an extended Tutte invariant is well defined for an alternating dimap D, then this holds for any minor of D.*

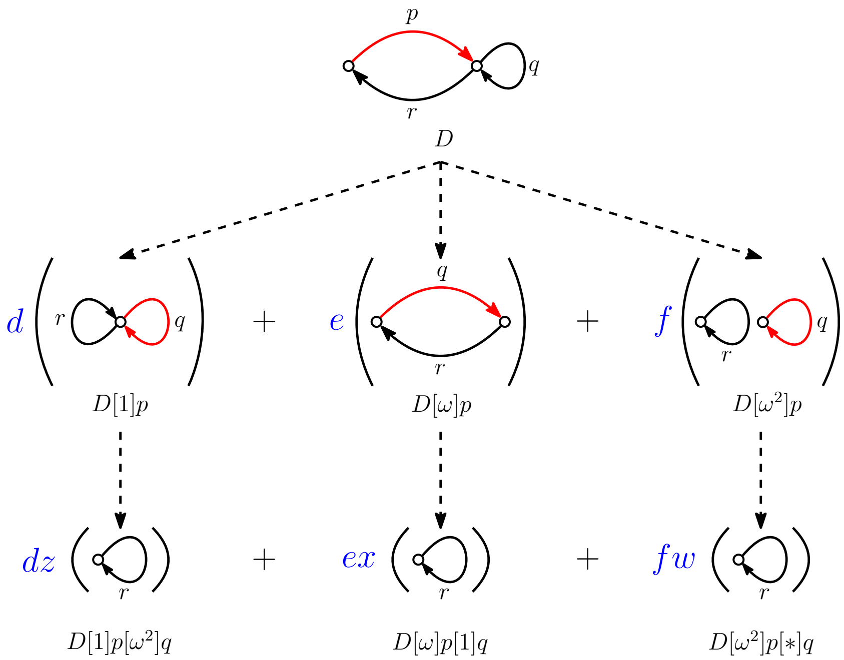


Figure 5.5: Reductions on the first two edges of an alternating dimap D

Proof. Let P be an extended Tutte invariant that is well defined for an alternating dimap D. By way of contradiction, suppose that there exists a minor D_1 of D such that P is not well defined for D_1. Since P is not well defined for D_1, there exist two edge-orderings $\mathcal{O}_1$, $\mathcal{O}_2$ for D_1 where two distinct derived polynomials P_1 and P_2 are obtained when D_1 is reduced using $\mathcal{O}_1$ and $\mathcal{O}_2$, respectively.

Since D_1 is a minor of D, for some $k \in \{1, 2, \ldots, |E(D)|\}$ there exists an edge-ordering $\mathcal{O} = e_1e_2\ldots e_k$ of some subset of $E(D)$ of size k and a reduction sequence $\mathcal{R} = \mu_1, \mu_2, \ldots, \mu_k$ such that $D_1 = D[\mathcal{R}]\mathcal{O}$. Let α be the monomial of D_1 with respect to D and these reductions. Suppose D contains $t \geq 1$ minors $D_1, D_2, \ldots, D_t$ after the first k edges are reduced.

i) If D_1 is reduced using $\mathcal{O}_1$, we have

$$P(D) = \alpha \cdot P_1 + \sum_{i=2}^{t} \beta_i \cdot P(D_i), \tag{5.6}$$

ii) If D_1 is reduced using $\mathcal{O}_2$, we obtain

$$P(D) = \alpha \cdot P_2 + \sum_{i=2}^{t} \beta_i \cdot P(D_i), \tag{5.7}$$

where in both cases, for $i \geq 2$, the factor β_i is the monomial of the respective minor D_i, and the remaining $|E(D)| - k$ edges of each D_i are reduced by using the same fixed edge-ordering.

Note that by reducing each D_i using a fixed edge-ordering, the summation over i in both (5.6) and (5.7) produces the same expression. Since $P_1 \neq P_2$, there exist two distinct derived polynomials for D. This implies that P is not well defined for D and we reach a contradiction. $\square$

Lemma 5.26. *The complete extended Tutte invariant is not well defined for the alternating dimap $G_{1,3}$.*

Proof. Let P be the complete extended Tutte invariant. Suppose $D \cong G_{1,3}$ has three edges r, s and t as shown in Figure 5.6.

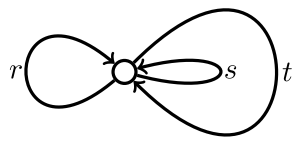


Figure 5.6: A labelled alternating dimap $G_{1,3}$

Since D has three edges, there exist $3! = 6$ possible edge-orderings. Let $\mathcal{O}_1 = rst$ and $\mathcal{O}_2 = trs$ be two of the possible edge-orderings of D.

If D is reduced using $\mathcal{O}_1$ (see Figure 5.7, where the edge to be reduced is in red), we obtain

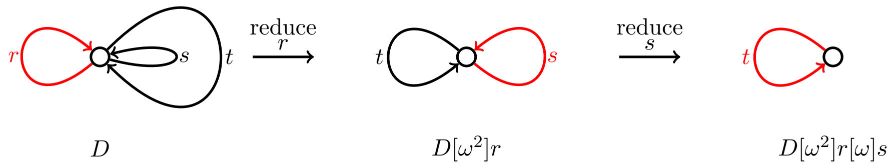


Figure 5.7: Reductions of $D \cong G_{1,3}$ using $\mathcal{O}_1 = rst$

$$\begin{aligned} P(D) &= z \cdot P(D[\omega^2]r) && (r \text{ is a proper } \omega^2\text{-loop in } D) \\ &= yz \cdot P(D[\omega^2]r[\omega]s) && (s \text{ is a proper } \omega\text{-loop in } D[\omega^2]r) \\ &= wyz && (\text{the final edge is always an ultraloop}). \end{aligned}$$

On the other hand, if D is reduced using $\mathcal{O}_2$ (see Figure 5.8), then

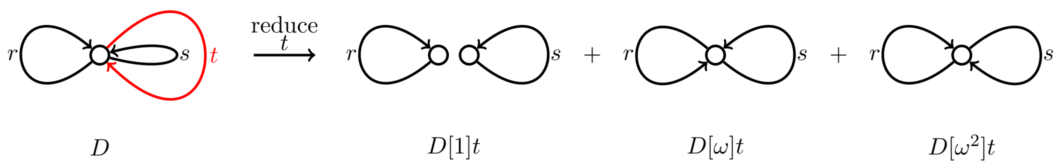


Figure 5.8: Reductions of $D \cong G_{1,3}$ on t using $\mathcal{O}_2 = trs$

$$\begin{aligned} P(D) &= a \cdot P(D[1]t) + b \cdot P(D[\omega]t) + c \cdot P(D[\omega^2]t) && (t \text{ is a proper 1-semiloop in } D) \\ &= aww + byw + czw && (\text{details are later in this paragraph}) \\ &= aww + bwy + cwz. \end{aligned}$$

In $D[1]t$, the edges r and s are both ultraloops. In $D[\omega]t$, the edges r and s are both proper ω-loops and the final edge to be reduced is always an ultraloop. In $D[\omega^2]t$, the edges r and s are both proper ω^2-loops and the final edge to be reduced is always an ultraloop.

Since the derived polynomials of D that are produced by the edge-orderings $\mathcal{O}_1$ and $\mathcal{O}_2$ are different, the result follows. □

Recall that the two alternating dimaps $G_{2,3}$ are shown in Figure 5.2.

Lemma 5.27. *The complete extended Tutte invariant is not well defined for either of the alternating dimaps $G_{2,3}$.*

Proof. Let P be the complete extended Tutte invariant. Since there exist two possibilities for $G_{2,3}$, we consider two alternating dimaps $G^a_{2,3}$ and $G^c_{2,3}$ separately. Suppose $D_1 \cong G^a_{2,3}$ and $D_2 \cong G^c_{2,3}$ both have three edges r, s and t as shown in Figure 5.9.

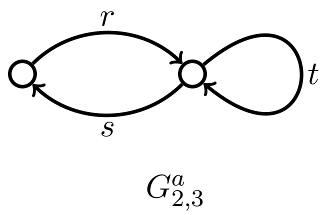


Figure 5.9: Two labelled alternating dimaps $G^a_{2,3}$ and $G^c_{2,3}$

Since they both have three edges, there exist $3! = 6$ possible edge-orderings for each of them.

First, let $\mathcal{O}_1 = srt$ and $\mathcal{O}_2 = rst$ be two of the possible edge-orderings of D_1.

If D_1 is reduced using $\mathcal{O}_1$ (see Figure 5.10), we obtain

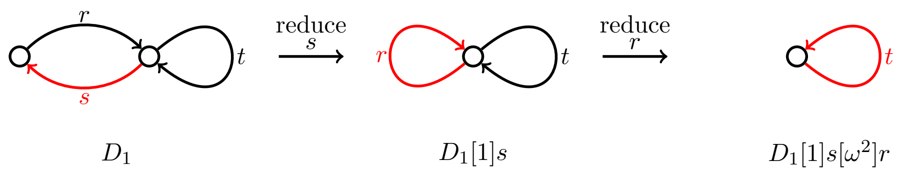


Figure 5.10: Reductions of $D_1 \cong G^a_{2,3}$ using $\mathcal{O}_1 = srt$

$$\begin{aligned} P(D_1) &= x \cdot P(D_1[1]s) && (s \text{ is a proper 1-loop in } D_1) \\ &= xz \cdot P(D_1[1]s[\omega^2]r) && (r \text{ is a proper } \omega^2\text{-loop in } D_1[1]s) \\ &= wxz && (\text{the final edge is always an ultraloop}). \end{aligned}$$

On the other hand, if D_1 is reduced using $\mathcal{O}_2$ (see Figure 5.11), then

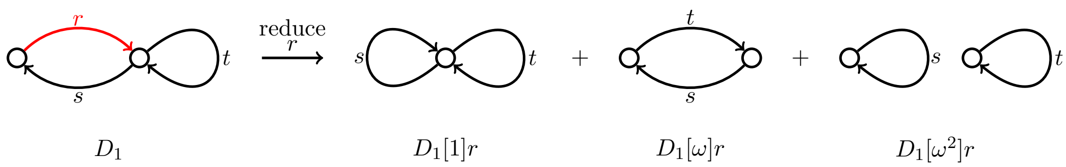


Figure 5.11: Reductions of $D_1 \cong G^a_{2,3}$ on r using $\mathcal{O}_2 = rst$

$$P(D_1) = d \cdot P(D_1[1]r) + e \cdot P(D_1[\omega]r) + f \cdot P(D_1[\omega^2]r)$$
$$(r \text{ is a proper } \omega\text{-semiloop in } D_1)$$

$$= dzw + exw + fww \qquad \text{(details are later in this paragraph)}$$
$$= dwz + ewx + fww.$$

In $D_1[1]r$, the edges s and t are both proper ω^2-loops and the final edge to be reduced is always an ultraloop. In $D_1[\omega]r$, the edges s and t are both proper 1-loops and the final edge to be reduced is always an ultraloop. In $D_1[\omega^2]r$, the edges s and t are both ultraloops.

Since the derived polynomials of D_1 that are produced by the edge-orderings $\mathcal{O}_1$ and $\mathcal{O}_2$ are different, the complete extended Tutte invariant is not well defined for the alternating dimap $G^a_{2,3}$.

Second, let $\mathcal{O}_3 = rst$ and $\mathcal{O}_4 = srt$ be two of the possible edge-orderings of D_2.

If D_2 is reduced using $\mathcal{O}_3$ (see Figure 5.12), we obtain

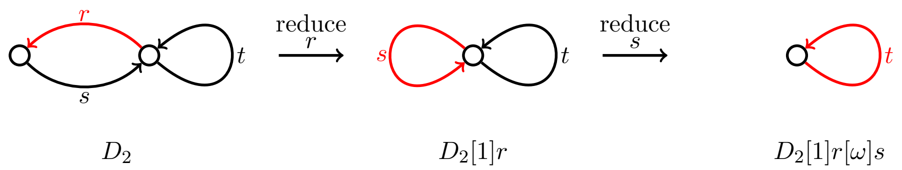


Figure 5.12: Reductions of $D_2 \cong G^c_{2,3}$ using $\mathcal{O}_3 = rst$

$$P(D_2) = x \cdot P(D_2[1]r) \qquad (r \text{ is a proper 1-loop in } D_2)$$
$$= xy \cdot P(D_2[1]r[\omega]s) \qquad (s \text{ is a proper } \omega\text{-loop in } D_2[1]r)$$
$$= wxy. \qquad \text{(the final edge is always an ultraloop)}$$

On the other hand, if D_2 is reduced using $\mathcal{O}_4$ (see Figure 5.13), then

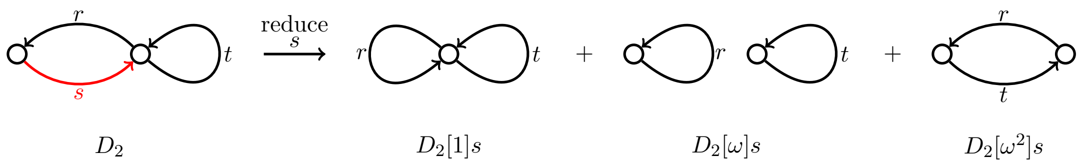


Figure 5.13: Reductions of $D_2 \cong G^c_{2,3}$ on s using $\mathcal{O}_2 = srt$

$$P(D_2) = g \cdot P(D_2[1]s) + h \cdot P(D_2[\omega]s) + i \cdot P(D_2[\omega^2]s)$$
$$(s \text{ is a proper } \omega^2\text{-semiloop in } D_2)$$
$$= gyw + hww + ixw \qquad \text{(details are later in this paragraph)}$$
$$= gwy + hww + iwx.$$

In $D_2[1]s$, the edges r and t are both proper ω-loops and the final edge to be reduced is always an ultraloop. In $D_2[\omega]s$, the edges r and t are both ultraloops. In $D_2[\omega^2]s$, the edges r and t are both proper 1-loops and the final edge to be reduced is always an ultraloop.

Since the derived polynomials of D_2 that are produced by the edge-orderings $\mathcal{O}_3$ and $\mathcal{O}_4$ are different, the complete extended Tutte invariant is not well defined for the alternating dimap $G^c_{2,3}$. □

We now give the conditions required in order to obtain a well defined complete extended Tutte invariant.

Theorem 5.28. *The complete extended Tutte invariant is well defined for an alternating dimap D of genus zero if and only if D contains only triloops.*

Proof. The forward implication is proved by contrapositive. Let D be an alternating dimap of genus zero that contains at least one non-triloop edge. By Corollary 5.17, the alternating dimap D contains $G_{1,3}$ or $G_{2,3}$ as a minor. By Lemma 5.26 and Lemma 5.27, the complete extended Tutte invariant is not well defined for $G_{1,3}$ and $G_{2,3}$, respectively. By Proposition 5.25, the complete extended Tutte invariant is then not well defined for D. Hence, the forward implication follows.

Conversely, suppose D is an alternating dimap of genus zero that contains only triloops. This implies that every edge in each connected component of D is of the same type. By Definition 5.2, each time a triloop is chosen and reduced, one factor w, x, y or z is introduced. Suppose $r \in E(D)$ is the first edge in a given edge-ordering. By using the given edge-ordering, the triloop r is deleted after the first reduction operation. All the other edges in $D \setminus r$ remain as triloops of the same type as they were in D. This is always true regardless of which edge is first reduced in D. In other words, the edge-ordering is inconsequential. In addition, the final edge to be reduced in each component is always an ultraloop (an improper triloop). Since the complete extended Tutte invariant for alternating dimaps is multiplicative, and multiplication is commutative, the complete extended Tutte invariant is well defined for D. □

CHAPTER 6

Tutte Invariants That Extend the Tutte Polynomial

Recall from [79] that the Tutte polynomial $T(G;x,y)$ of a graph G has the following deletion-contraction recurrence, for any $e \in E(G)$:

$$T(G;x,y) = \begin{cases} 1, & \text{if } G \text{ is empty,} \\ x \cdot T(G/e;x,y), & \text{if } e \text{ is a coloop,} \\ y \cdot T(G \setminus e;x,y), & \text{if } e \text{ is a loop,} \\ T(G \setminus e;x,y) + T(G/e;x,y), & \text{otherwise.} \end{cases}$$

In addition to the extended Tutte invariant in Definition 5.2, Farr [37] defined two other Tutte invariants, namely $T_c(D;x,y)$ and $T_a(D;x,y)$, for any alternating dimap D, which are analogues of the Tutte polynomial. Note that $T_c(D;x,y)$ and $T_a(D;x,y)$ are two special cases of extended Tutte invariants. Farr showed that the c-Tutte invariant (respectively, a-Tutte invariant) of an alternating dimap is well defined for any alternating dimap of the form $\text{alt}_c(G)$ (respectively, $\text{alt}_a(G)$), when it equals the Tutte polynomial of a plane graph G.

In this chapter, we discuss these two invariants for alternating dimaps that are 2-cell embedded on an orientable surface of genus zero, i.e., the plane or the sphere. We first characterise the c-Tutte invariant and the a-Tutte invariant, when they are well defined for all alternating dimaps of genus zero. We then determine the class of alternating dimaps for which these two invariants are well defined, without any restriction on the two parameters of these invariants. We show that each of these invariants properly extends the Tutte polynomial of a plane graph. Lastly, we discuss the factorisation of c-Tutte invariants.

6.1 Tutte Invariants with Dependent Paramaters for Arbitrary Alternating Dimaps

We first give the definitions of the c-Tutte invariant T_c and the a-Tutte invariant T_a.

Definition 6.1. *A* c-Tutte invariant *for alternating dimaps is a multiplicative invariant T_c such that, for any alternating dimap D and $e \in E(D)$,*

1. *if e is an ultraloop,*
$$T_c(D;x,y) = T_c(D \setminus e;x,y), \tag{TC1}$$

2. *if e is a proper 1-loop or a proper ω-semiloop,*
$$T_c(D;x,y) = x \cdot T_c(D[\omega^2]e;x,y), \tag{TC2}$$

3. *if e is a proper ω-loop or a proper 1-semiloop,*
$$T_c(D;x,y) = y \cdot T_c(D[1]e;x,y), \tag{TC3}$$

4. *if e is a proper ω^2-loop or a proper ω^2-semiloop,*
$$T_c(D;x,y) = T_c(D[\omega]e;x,y), \tag{TC4}$$

5. *otherwise,*
$$T_c(D;x,y) = T_c(D[1]e;x,y) + T_c(D[\omega^2]e;x,y). \tag{TC5}$$

Definition 6.2. *An* a-Tutte invariant *for alternating dimaps is a multiplicative invariant T_a such that, for any alternating dimap D and $e \in E(D)$,*

1. *if e is an ultraloop,*
$$T_a(D;x,y) = T_a(D \setminus e;x,y),$$

2. *if e is a proper 1-loop or a proper ω^2-semiloop,*
$$T_a(D;x,y) = x \cdot T_a(D[\omega]e;x,y),$$

3. *if e is a proper ω^2-loop or a proper 1-semiloop,*
$$T_a(D;x,y) = y \cdot T_a(D[1]e;x,y),$$

4. *if e is a proper ω-loop or a proper ω-semiloop,*
$$T_a(D;x,y) = T_a(D[\omega^2]e;x,y),$$

5. *otherwise,*
$$T_a(D;x,y) = T_a(D[1]e;x,y) + T_a(D[\omega]e;x,y).$$

Remark: For the reduction of a triloop $e \in E(D)$, we have $D[*]e = D[1]e = D[\omega]e = D[\omega^2]e = D\backslash e$.

Theorem 6.1. [37, Theorem 5.2] *For any plane graph G,*

$$T(G;x,y) = T_c(\mathrm{alt}_c(G);x,y) = T_a(\mathrm{alt}_a(G);x,y).$$

We now determine when the c-Tutte invariant is well defined for all alternating dimaps of genus zero, using our results on extended Tutte invariants.

Proposition 6.2. *The c-Tutte invariant is well defined for all alternating dimaps of genus zero if and only if*

$$x = \frac{1 \pm \sqrt{3}i}{2}, \ y = \frac{1 \mp \sqrt{3}i}{2}.$$

Proof. We first consider the definitions of the extended Tutte invariant (see Definition 5.2) and the c-Tutte invariant (see Definition 6.1). Since the variables x and y are used in both definitions, we use α and β instead of the variables x and y, respectively, that are used for the c-Tutte invariant. By comparing the recurrences in the two definitions, we see that the c-Tutte invariant is an extended Tutte invariant with parameters

$$x = f = \alpha, \ y = a = \beta, \ w = z = h = j = l = 1, \ b = c = d = e = g = i = k = 0. \tag{6.1}$$

i) Suppose $\alpha, \beta \neq 0$. In this case, the hypothesis of Theorem 5.6 is satisfied (since $w = z = 1 \neq 0$, $x = \alpha \neq 0$ and $y = \beta \neq 0$). By substituting the respective values in (6.1) into the necessary conditions in (5.2)–(5.5), and solving the equations, we obtain

$$\alpha\beta = \alpha + \beta = 1. \tag{6.2}$$

By solving (6.2), and using the fact that $\alpha = x$ and $\beta = y$ in the c-Tutte invariant, we have

$$x = \frac{1 \pm \sqrt{3}i}{2}, \ y = \frac{1 \mp \sqrt{3}i}{2}.$$

ii) Suppose $\alpha = 0$ and $\beta \neq 0$. In this case, the hypothesis of Theorem 5.22 is satisfied (since $w = z = 1 \neq 0$, $x = \alpha = 0$ and $y = \beta \neq 0$). However, by Theorem 5.22, the fact that $h = j = 1 \neq 0$ implies that we do not get a well defined extended Tutte invariant.

iii) Suppose $\alpha \neq 0$ and $\beta = 0$. In this case, the hypothesis of Corollary 5.23 is satisfied (since $w = z = 1 \neq 0$, $x = \alpha \neq 0$ and $y = \beta = 0$). However, by Corollary 5.23, the fact that $h = l = 1 \neq 0$ implies that we do not get a well defined extended Tutte invariant.

iv) Suppose $\alpha = \beta = 0$. In this case, the hypothesis of Theorem 5.19 is satisfied (since $w = z = 1 \neq 0$, $x = \alpha = 0$ and $y = \beta = 0$). However, by Theorem 5.19, the fact that $h = 1 \neq 0$ implies that we do not get a well defined extended Tutte invariant.

The backward implication follows, by Theorem 5.6. □

Corollary 6.3. *The only c-Tutte invariants that are well defined for all alternating dimaps D of genus zero are*

$$T_c(D; \frac{1 \pm \sqrt{3}i}{2}, \frac{1 \mp \sqrt{3}i}{2}) = \left(\frac{1 \pm \sqrt{3}i}{2}\right)^{\mathrm{is}(D) - \mathrm{af}(D)}.$$

□

Note that

$$x = \frac{1+\sqrt{3}i}{2},\ y = \frac{1-\sqrt{3}i}{2}$$

are the two primitive sixth roots of unity. These two points satisfy the equation $(x-1)(y-1) = 1$, so they lie on the hyperbola $H_1 := \{(x,y) : (x-1)(y-1) = 1\}$, on which $T(G;x,y)$ and hence $T_c(\mathrm{alt}_c(G);x,y)$ are easy to evaluate.

By using a similar approach, we have the following proposition for the a-Tutte invariant.

Proposition 6.4. *The a-Tutte invariant is well defined for all alternating dimaps of genus zero if and only if*

$$x = \frac{1 \pm \sqrt{3}i}{2},\ z = \frac{1 \mp \sqrt{3}i}{2}.$$

Proof. By comparing the definitions of the extended Tutte invariant (see Definition 5.2) and the a-Tutte invariant (see Definition 6.2), we have

$$x = h = \alpha,\ z = a = \beta,\ w = y = f = j = k = 1,\ b = c = d = e = g = i = l = 0.$$

Then, by using similar arguments as in the proof of Proposition 6.2, we obtain

$$x = \frac{1 \pm \sqrt{3}i}{2},\ z = \frac{1 \mp \sqrt{3}i}{2}.$$

□

Corollary 6.5. *The only a-Tutte invariants that are well defined for all alternating dimaps D of genus zero are*

$$T_a(D; \frac{1 \pm \sqrt{3}i}{2}, \frac{1 \mp \sqrt{3}i}{2}) = \left(\frac{1 \pm \sqrt{3}i}{2}\right)^{\mathrm{is}(D)-\mathrm{cf}(D)}.$$

□

6.2 Well Defined c-Tutte Invariants for Restricted Alternating Dimaps

The c-Tutte invariant and the a-Tutte invariant are closely related. Once a problem is solved for one of these invariants, it can then be solved for the other by some appropriate modifications, as evidenced in Section 6.1. Hence, we only focus on the c-Tutte invariant from now onwards.

A *c-cycle block* (respectively, an *a-cycle block*) of an alternating dimap D is a block that is a clockwise face (respectively, an anticlockwise face) of D that has the same number of vertices as edges. Such a block is a directed cycle of D.

A *c-simple alternating dimap* (see Figure 6.1) is a loopless alternating dimap of genus zero in which every block is either:

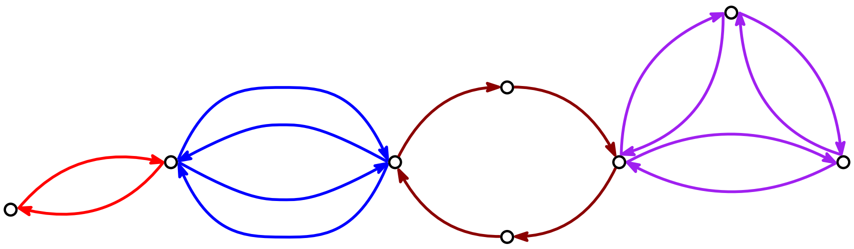


Figure 6.1: A c-simple alternating dimap with four blocks

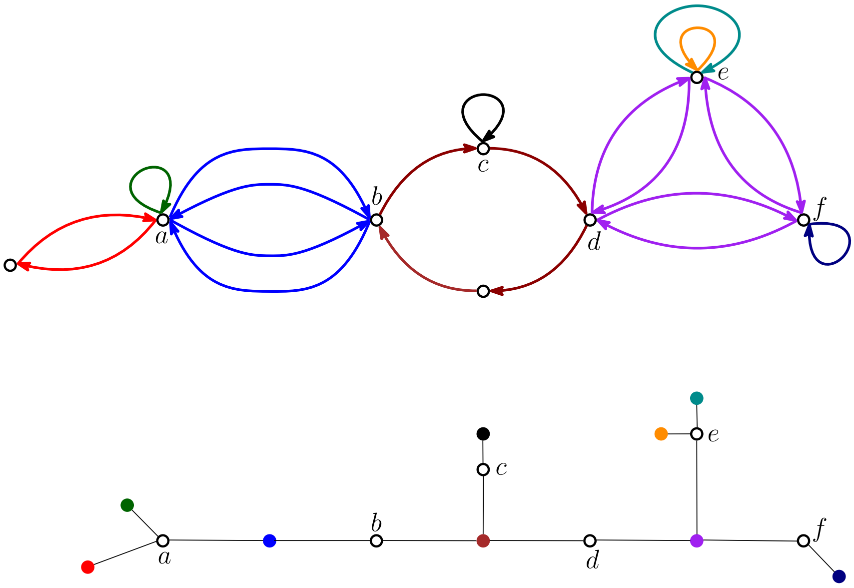


Figure 6.2: A c-alternating dimap and its c-block graph

i) a c-cycle block, or

ii) an element of $\mathrm{alt}_c(\mathcal{G})$,

and there exists no block within a clockwise face of any other block.

A *c-alternating dimap* is an alternating dimap of genus zero that can be obtained from a c-simple alternating dimap by adding some c-multiloops within some anticlockwise faces of the c-simple alternating dimap. Hence, a c-simple alternating dimap is merely a c-alternating dimap without any loops.

Let A denote the set of cutvertices and B denote the set of blocks of a c-alternating dimap H. We construct the *c-block graph* of H with vertex set $A \cup B$ as follows: $a_i \in A$ and $b_j \in B$ are adjacent if block b_j of H contains the cutvertex a_i of H. The construction of the c-block graph of a c-alternating dimap is the same as the construction of the *block graph* of a graph. Hence, the c-block graph of a connected c-alternating dimap is a tree. An example of a c-alternating dimap and the corresponding c-block graph is shown in Figure 6.2.

Lemma 6.6. *Let D be an alternating dimap. Every clockwise face of D has size exactly two if and only if there exists an undirected orientably embedded graph G such that $D \cong$ $alt_c(G)$.*

Proof. We first prove the forward implication. Given an alternating dimap D where all of its clockwise faces have size exactly two, we construct an undirected graph G as follows. Let $V(G) = V(D)$. For each clockwise face of D, a new edge of G that is incident with the same endvertices (i.e., the two vertices incident with this face, which may coincide) is added in G such that the new edge is within the clockwise face of D. Each clockwise face contains exactly one edge in this way, therefore edges in G do not intersect. Hence, we obtain an undirected embedded graph G such that $D \cong \mathrm{alt}_c(G)$.

Conversely, if there exists an undirected graph G such that an alternating dimap $D \cong \mathrm{alt}_c(G)$, every clockwise face of D has size exactly two, by the definition of $\mathrm{alt}_c(G)$. □

Proposition 6.7. *The c-union of two c-alternating dimaps is also a c-alternating dimap.*

Proof. Let D_1 and D_2 be c-alternating dimaps and $D = D_1 \cup_c D_2$. By the c-union construction (as defined on page 19) and the fact that the set of clockwise faces of D is the union of the sets of clockwise faces of D_1 and D_2, every non-loop block of D is either a c-cycle block or is an element of $\mathrm{alt}_c(\mathcal{G})$, and there exists no block within a clockwise face of any other block. In addition, D may contain some c-multiloops within some of its anticlockwise faces. The set of c-multiloops of D is also the union of the sets of c-multiloops of D_1 and D_2 Therefore, D is also a c-alternating dimap. □

Corollary 6.8. *The c-union of two c-simple alternating dimaps is also a c-simple alternating dimap.* □

Proposition 6.9. *Let D_1 and D_2 be alternating dimaps and $D = D_1 \cup_c D_2$. Then, the set of blocks in D is the union of the sets of blocks of D_1 and D_2.*

Proof. The result follows immediately, by the c-union construction. □

In the following lemmas, we show that the c-Tutte invariant is well defined for any alternating dimap (of genus zero) that is either a c-cycle block or a c-multiloop.

Lemma 6.10. *Let D be an alternating dimap that is a c-cycle block of size $m \geq 1$. Then,*

$$T_c(D; x, y) = x^{m-1}.$$

Proof. Suppose D is as stated. We proceed by induction on the number of edges of D. For the base case, suppose $m = 1$, so that D is an ultraloop. Then, we have $T_c(D; x, y) = x^0 = 1$. The result for $m = 1$ follows.

For the inductive step, assume that $m > 1$ and the result holds for every D that has less than m edges. Note that every edge that belongs to D is a proper 1-loop. Let $e \in E(D)$. By reducing e, we have

$$\begin{aligned} T_c(D; x, y) &= x \cdot T_c(D[\omega^2]e; x, y) && \text{(by (TC2))} \\ &= x \cdot x^{(m-1)-1} && \text{(by the inductive hypothesis)} \\ &= x^{m-1}. \end{aligned}$$

The result follows, by induction. □

Corollary 6.11. *Let D be an alternating dimap that is a c-cycle block of size $m \geq 1$ and S_t be a tree of size $t \geq 0$. Then,*

$$T_c(D; x, y) = T(S_{m-1}; x, y).$$

□

Lemma 6.12. *Let D be an alternating dimap of genus zero that is a c-multiloop of size m and put $k = \mathrm{cf}(D)$. Then,*

$$T_c(D; x, y) = y^{m-k}.$$

Proof. Suppose D is as stated. We proceed by induction on the number of edges of D. For the base case, suppose $m = 0$. Then, we have $T_c(D; x, y) = y^0 = 1$ and the result for $m = 0$ follows.

For the inductive step, assume that $m > 0$ and the result holds for every D that contains less than m edges.

Recall that every edge in an alternating dimap belongs to one clockwise face and one anticlockwise face. Each time an edge is reduced, the size of the alternating dimap is reduced by one. Let $e \in E(D)$ and $k = \mathrm{cf}(D)$. Note that e is either a proper 1-semiloop, a proper ω-loop or a proper ω^2-loop, in D. We consider these three cases as follows.

Suppose e is a proper 1-semiloop or a proper ω-loop. The number of clockwise faces remains unchanged after e is reduced. Hence,

$$\begin{aligned} T_c(D; x, y) &= y \cdot T_c(D[1]e; x, y) && \text{(by (TC3))} \\ &= y \cdot y^{(m-1)-k} && \text{(by the inductive hypothesis)} \\ &= y^{m-k}. \end{aligned}$$

Suppose e is a proper ω^2-loop. The reduction on e will now reduce the number of the clockwise faces by 1. Hence,

$$\begin{aligned} T_c(D; x, y) &= T_c(D[\omega]e; x, y) && \text{(by (TC4))} \\ &= y^{(m-1)-(k-1)} && \text{(by the inductive hypothesis)} \\ &= y^{m-k}. \end{aligned}$$

This completes the proof, by induction. □

Corollary 6.13. *Let D be an alternating dimap of genus zero that is a c-multiloop of size m and put $k = \mathrm{cf}(D)$, and L_t be a graph with t loops. Then,*

$$T_c(D; x, y) = T(L_{m-k}; x, y).$$

□

The following two lemmas show the general form of the c-Tutte invariant, when a c-cycle block or a c-multiloop is first reduced in certain alternating dimaps.

Lemma 6.14. *Let C_m be a c-cycle block of size $m \geq 1$ in a c-alternating dimap D. Then,*

$$T_c(D; x, y) = x^{m-1} \cdot T_c(D \setminus C_m; x, y).$$

Proof. Suppose C_m is as stated. We proceed by induction on m. For the base case, suppose $m = 1$. The c-cycle block C_1 is a proper ω^2-loop. By reducing the proper ω^2-loop, we have $T_c(D; x, y) = T_c(D[\omega]C_1; x, y) = x^{1-1} \cdot T_c(D \setminus C_1; x, y)$. Hence, the result for $m = 1$ follows.

For the inductive step, assume that $m > 1$ and the result holds for every $k < m$. Now, every edge in C_m is either a proper 1-loop or a proper ω-semiloop. Let $e \in E(C_m)$. By reducing e,

$$\begin{aligned} T_c(D; x, y) &= x \cdot T_c(D[\omega^2]e; x, y) && \text{(by (TC2))} \\ &= x \cdot x^{(m-1)-1} \cdot T_c(D[\omega^2]e \setminus C_{m-1}; x, y) && \text{(by the inductive hypothesis)} \\ &= x^{m-1} \cdot T_c(D \setminus C_m; x, y). \end{aligned}$$

This completes the proof, by induction. □

Lemma 6.15. *Let R_m be a c-multiloop of size m in an alternating dimap D of genus zero and put $k = \mathrm{cf}(R_m)$. Then,*

$$T_c(D; x, y) = y^{m-k} \cdot T_c(D \setminus R_m; x, y).$$

Proof. Let R_m be a c-multiloop with m edges in an alternating dimap D of genus zero. We proceed by induction on m. For the base case, suppose $m = 0$. Clearly, the result for $m = 0$ follows.

For the inductive step, assume that $m > 0$ and the result holds for every $\ell < m$. Let $e \in E(R_m)$ and $k = \mathrm{cf}(R_m)$. Note that e is either a proper 1-semiloop, a proper ω-loop or a proper ω^2-loop, in R_m. We consider these three cases as follows.

Suppose e is a proper 1-semiloop or a proper ω-loop. The number of clockwise faces of R_m remains unchanged after e is reduced. Hence,

$$\begin{aligned} T_c(D; x, y) &= y \cdot T_c(D[1]e; x, y) && \text{(by (TC3))} \\ &= y \cdot y^{(m-1)-k} \cdot T_c(D[1]e \setminus R_{m-1}; x, y) && \text{(by the inductive hypothesis)} \\ &= y^{m-k} \cdot T_c(D \setminus R_m; x, y). \end{aligned}$$

Suppose e is a proper ω^2-loop. The reduction on e will now reduce the number of clockwise faces by one. Hence,

$$\begin{aligned} T_c(D; x, y) &= T_c(D[\omega]e; x, y) && \text{(by (TC4))} \\ &= y^{(m-1)-(k-1)} \cdot T_c(D[\omega]e \setminus R_{m-1}; x, y) && \text{(by the inductive hypothesis)} \\ &= y^{m-k} \cdot T_c(D \setminus R_m; x, y). \end{aligned}$$

This complete the proof, by induction. □

We now show that the c-Tutte invariant is multiplicative over blocks for any c-simple alternating dimap.

Lemma 6.16. *Let D be a c-union of two c-simple alternating dimaps S_1 and S_2. Then,*

$$T_c(D; x, y) = T_c(S_1; x, y) \cdot T_c(S_2; x, y).$$

Proof. Suppose D is a c-union of two c-simple alternating dimaps S_1 and S_2. By Corollary 6.8, the alternating dimap D is also a c-simple alternating dimap. Thus, every block of D is either a c-cycle block or is an element of $\text{alt}_c(\mathcal{G})$.

We proceed by induction on the number p of c-cycle blocks of D. For the base case, suppose $p = 0$, so that there exists no c-cycle block in both S_1 and S_2. Thus, every block of S_1, S_2 and hence D is an element of $\text{alt}_c(\mathcal{G})$. Let $S_1 \cong \text{alt}_c(G_1)$ and $S_2 \cong \text{alt}_c(G_2)$. Then $D \cong \text{alt}_c(G)$ for some plane graph $G = G_1 \cup_i G_2$, where $i \in \{0, 1\}$. By Theorem 6.1,

$$T_c(D; x, y) = T(G; x, y),$$

$$T_c(S_i; x, y) = T(G_i; x, y), \;\; i \in \{1, 2\}.$$

Since the Tutte polynomial is multiplicative over blocks for any graph G, we have

$$T(G; x, y) = T(G_1; x, y) \cdot T(G_2; x, y).$$

Hence,

$$T_c(D; x, y) = T_c(S_1; x, y) \cdot T_c(S_2; x, y).$$

For the inductive step, assume that $p > 0$ and the result holds for any c-union that contains less than p c-cycle blocks. Without loss of generality, let C_m be a c-cycle block in S_1 that contains $m \geq 1$ edges. Since D is a c-union of S_1 and S_2, it contains C_m as one of its blocks. By first reducing every edge of C_m, we have

$$\begin{aligned}
T_c(D; x, y) &= x^{m-1} \cdot T_c(D \setminus C_m; x, y) && \text{(by Lemma 6.14 applied to } D\text{)}\\
&= x^{m-1} \cdot T_c(S_1 \setminus C_m; x, y) \cdot T_c(S_2; x, y) && \text{(by the inductive hypothesis)}\\
&= x^{m-1} \cdot \frac{T_c(S_1; x, y)}{x^{m-1}} \cdot T_c(S_2; x, y) && \text{(by Lemma 6.14 applied to } S_1\text{)}\\
&= T_c(S_1; x, y) \cdot T_c(S_2; x, y).
\end{aligned}$$

The result follows, by induction. □

We extend the result in Lemma 6.16, from c-simple alternating dimaps to c-alternating dimaps.

Theorem 6.17. *Let D be a c-union of two c-alternating dimaps S_1 and S_2. Then,*

$$T_c(D; x, y) = T_c(S_1; x, y) \cdot T_c(S_2; x, y).$$

Proof. Suppose D is a c-union of two c-alternating dimaps S_1 and S_2. By Propostion 6.7, the alternating dimap D is also a c-alternating dimap. Thus, every non-loop block of D is either a c-cycle block or is an element of $\mathrm{alt}_c(\mathcal{G})$, and there exists no block within a clockwise face of any other block. In addition, D contains p c-multiloops within some of its anticlockwise faces.

We proceed by induction on p. For the base case, suppose $p = 0$. Then, S_1, S_2 and D contain no c-multiloops. In other words, they all are c-simple alternating dimaps. By Lemma 6.16, we have $T_c(D; x, y) = T_c(S_1; x, y) \cdot T_c(S_2; x, y)$ and the result for $p = 0$ follows.

For the inductive step, assume that $p > 0$ and the result holds for every c-union that contains less than p c-multiloops. Without loss of generality, assume that S_1 and hence D contains a c-multiloop R_m. Let $k = \mathrm{cf}(R_m)$. By first reducing every edge of R_m, we have

$$
\begin{aligned}
T_c(D; x, y) &= y^{m-k} \cdot T_c(D \setminus R_m; x, y) && \text{(by Lemma 6.15 applied to } D\text{)} \\
&= y^{m-k} \cdot T_c(S_1 \setminus R_m; x, y) \cdot T_c(S_2; x, y) && \text{(by the inductive hypothesis)} \\
&= y^{m-k} \cdot \frac{T_c(S_1; x, y)}{y^{m-k}} \cdot T_c(S_2; x, y) && \text{(by Lemma 6.15 applied to } S_1\text{)} \\
&= T_c(S_1; x, y) \cdot T_c(S_2; x, y) && \text{(by (TC4)).}
\end{aligned}
$$

The result follows, by induction. □

Since there are a few non-isomorphic alternating dimaps that may be denoted by $G_{3,5}$, we write $G_{3,5,1}$ (see Figure 6.3) for the alternating dimap $G_{3,5}$ that is obtained by subdividing one of the edges of $\mathrm{alt}_c(G)$ where the plane graph G is a cycle of size exactly two.

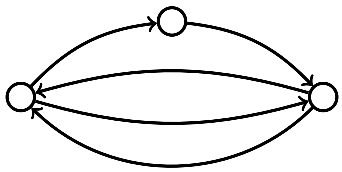


Figure 6.3: An alternating dimap $G_{3,5,1}$

We now present one definition and two results (with proofs) by Farr in 2013. These are unpublished and included by permission.

Definition 6.3 (Farr). *A* turner *is a directed closed trail (which may repeat vertices but not edges) in which every edge is followed either by its left successor or its right successor.*

Theorem 6.18 (Farr). *If G and $G \setminus X$ are alternating dimaps, with $X \subseteq E(G)$, then X is an edge-disjoint union of turners.*

Proof. Let G and X be as stated. Let $e_1 \in E(G \setminus X)$ and let e_2 be the left (respectively, right) successor of e_1 in $G \setminus X$, with e_1 has v as its head and e_2 has v as its tail. Let $E(v, e_1, e_2, X)$ be the set of edges of X between e_1 and e_2 in G going clockwise (resp., anticlockwise) around v from e_1 to e_2. Since G and $G \setminus X$ are both alternating dimaps, $|E(v, e_1, e_2, X)|$ is even. Label each edge of $E(v, e_1, e_2, X)$ directed into v as a *right-turn* (resp., *left-turn*). This done, each out-edge of v in $E(v, e_1, e_2, X)$ can be obtained from precisely one in-edge of v in $E(v, e_1, e_2, X)$ by following the instruction given by that in-edge's label. Doing this for each such triple (e_1, v, e_2) gives a label left-turn/right-turn to each edge of X (since every edge is an in-edge for some v). This labelling allows us to partition X into turners: for each edge in X, its turner is obtained by repeatedly using the label on the current edge to determine which edge (of the two out-edges next to it at its head) is its successor in the turner, a process which can never repeat an edge until it returns to where it started. Note that turners cannot cross. □

Corollary 6.19 (**Farr**). *If $H \leq G$ are alternating dimaps, then there exist $Y, Z \subseteq E(G)$ such that $G[\omega]Y[\omega^2]Z = H$.*

Proof. Let $X = E(G) \setminus E(H)$. By Theorem 6.18, X is an edge-disjoint union of turners. Consider the labelling of the edges of X given by the proof of that theorem. Let Y be the left-turn edges and Z be the right-turn edges. Then $G[\omega]Y[\omega^2]Z = H$. □

Fact 6.1. *A block of an alternating dimap is a c-cycle block if and only if the block has exactly one anticlockwise face and exactly one clockwise face.*

By using Corollary 6.19, we now show that certain alternating dimaps contain $G_{3,5,1}$ as a minor.

Lemma 6.20. *Every non-loop block of an alternating dimap that is neither a c-cycle block nor an element of $alt_c(\mathcal{G})$ contains $G_{3,5,1}$ as a minor.*

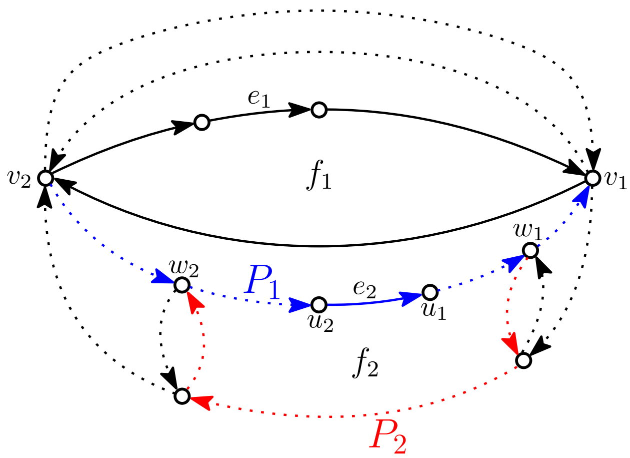


Figure 6.4: The block B in the proof of Lemma 6.20

Proof. Let B be a non-loop block of an alternating dimap that is neither a c-cycle block nor an element of $\mathrm{alt}_c(\mathcal{G})$. By Fact 6.1, the former implies that the number of a-faces or the number of c-faces of B is at least two. The existence of at least two a-faces (respectively, c-faces) in a block implies that the number of c-faces (respectively, a-faces) of the block is also at least two. Since B is a non-loop block, it contains no proper ω^2-loops. By Lemma 6.6, if B is an element of $\mathrm{alt}_c(\mathcal{G})$, every clockwise face of B has size exactly two. Hence, at least one of the c-faces f_1 of B has size greater than two.

Let $e_1 \in E(\partial f_1)$. Since B contains more than one c-face, there exists an edge e_2 such that $e_2 \notin E(\partial f_1)$ and $e_2 \in E(\partial f_2)$ where f_2 is another c-face in B. Since B contains no cutvertex, there exists a circuit C that contains both e_1 and e_2. Let u_1 be the head and u_2 be the tail of e_2. Pick the vertex u_1 and traverse C until the first vertex $v_1 \in V(\partial f_1)$ is met. Then, pick u_2 and traverse C in the opposite direction and stop once another vertex $v_2 \in V(\partial f_1)$ is met. Let P_1 (highlighted in blue in Figure 6.4) be the path in C that has v_1 and v_2 as its endvertices, P_1 does not use any edge that belongs to f_1 and $e_2 \in E(P_1)$. Now, observe that vertices v_1 and v_2 both have degree at least three (the vertices v_1 and v_2 both belong to f_1, and P_1 is incident with both of them). Let $w_1, w_2 \in V(\partial f_2)$ and P_2 (highlighted in red) be a w_1w_2-path in f_2 such that $E(P_2) = E(\partial f_2) \setminus E(P_1)$.

Note that f_1 can be contracted to a c-face f' of size three that contains three vertices (since every c-face of size greater than two can be contracted to a c-face of size exactly two, by Lemma 5.7). Then, by contracting every edge $g_3 \in E(P_1) \setminus \{e_2\}$, the path P_1 is reduced to a path of length one that is incident with two of the vertices $v_1', v_2' \in V(\partial f')$. Observe that P_2 now has v_1' and v_2' as both of its endvertices. Contract every edge in P_2 except one, leaving an edge e_3 that is incident with v_1' and v_2'. Suppose $E_1 = E(\partial f') \cup \{e_2, e_3\}$. Then, delete every edge $h \in E(B) \setminus E_1$ to obtain an alternating dimap $S \cong G_{3,5,1}$. Since $S \leq B$, by Corollary 6.19 there exist $Y, Z \subseteq E(B)$ such that $B[\omega]Y[\omega^2]Z = S$. □

We next show that certain alternating dimaps contain $G^c_{2,3}$ as a minor. The alternating dimap $G^c_{2,3}$ is shown in Figure 5.2.

Lemma 6.21. *Let D be an alternating dimap such that there exists a block within a clockwise face of some other block and they form a clockwise face of size greater than two. Then, D has $G^c_{2,3}$ as a minor.*

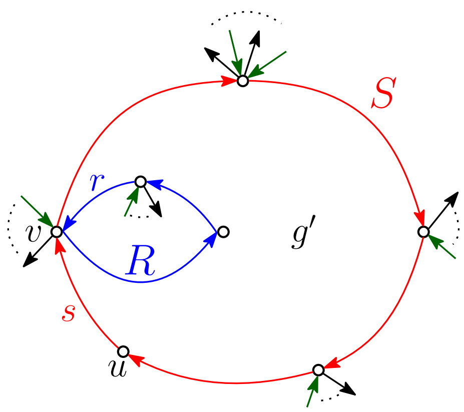


Figure 6.5: Two blocks B_1 and B_2 in the proof of Lemma 6.21

Proof. Suppose an alternating dimap D contains two blocks B_1 and B_2 such that these two blocks share exactly one common vertex v, and B_1 is within one of the c-faces g of B_2. Let $g' \in F(D)$ be the c-face of size greater than two that is formed by the boundary of g and some edges of B_1.

Since g' has size greater than two, it contains at least two vertices including v. The fact that v has degree greater than two in D implies that every edge $e_1 \in I(v)$ is not a proper 1-loop. Suppose $R = E(B_1) \cap E(\partial g')$ and $S = E(B_2) \cap E(\partial g')$. Note that

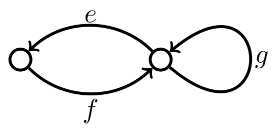


Figure 6.6: Alternating dimap $G^c_{2,3}$

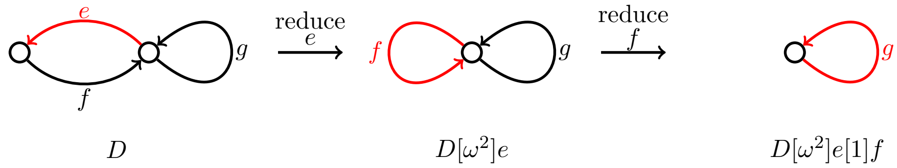


Figure 6.7: Reductions of $D \cong G^c_{2,3}$ using $\mathcal{O}_1 = efg$

S is the boundary of the face g in B_2. Let $r \in R \subseteq E(B_1)$ and $s \in S \subseteq E(B_2)$ and $r, s \in E(\partial g') \cap I(v)$. Since g' is formed by at least three edges, at least one of the two edges r and s is a non-loop edge (otherwise g' is a clockwise face of size two). Without loss of generality, let $s = uv$ be a non-loop edge. Since $s \in I(v)$ and is a non-loop edge, s is a non-triloop edge in D. By Corollary 5.17, the alternating dimap D has $G_{1,3}$ or $G_{2,3}$ as its minor.

To obtain $G^c_{2,3}$ as a minor in D, for each vertex $w \in V(\partial g')$ and for each edge $e_2 \in I(w) \setminus E(\partial g')$ (green edges in Figure 6.5), reduce e_2 using ω^2-reduction. Now, g' belongs to a component P that has exactly two blocks. Let $T = I(u) \cup I(v)$ in P. By contracting every edge $e_3 \in E(P) \setminus T$, we obtain a component $P' \cong G^c_{2,3}$. Since $P' \leq D$, by Corollary 6.19, there exist $Y, Z \subseteq E(D)$ such that $D[\omega]Y[\omega^2]Z = P'$.

Therefore, D has $G^c_{2,3}$ as a minor. □

Lemma 6.22. *The c-Tutte invariant is not well defined for the alternating dimap $G^c_{2,3}$.*

Proof. Suppose $D \cong G^c_{2,3}$ has three edges e, f and g as shown in Figure 6.6. Since D has three edges, there exist $3! = 6$ possible edge-orderings. Let $\mathcal{O}_1 = efg$ and $\mathcal{O}_2 = feg$ be two of the possible edge-orderings of D.

If D is reduced using $\mathcal{O}_1$ (see Figure 6.7), we obtain

$$\begin{aligned} T_c(D; x, y) &= x \cdot T_c(D[\omega^2]e; x, y) && (e \text{ is a proper 1-loop in } D) \\ &= xy \cdot T_c(D[\omega^2]e[1]f; x, y) && (f \text{ is a proper } \omega\text{-loop in } D[\omega^2]e) \\ &= xy && (\text{the final edge is always an ultraloop}). \end{aligned}$$

On the other hand, if D is reduced using $\mathcal{O}_2$ (see Figure 6.8), then

$$\begin{aligned} T_c(D; x, y) &= T_c(D[\omega]f; x, y) && (f \text{ is a proper } \omega^2\text{-semiloop in } D) \\ &= T_c(D[\omega]f \setminus e; x, y) && (e \text{ is an ultraloop in } D[\omega]f) \end{aligned}$$

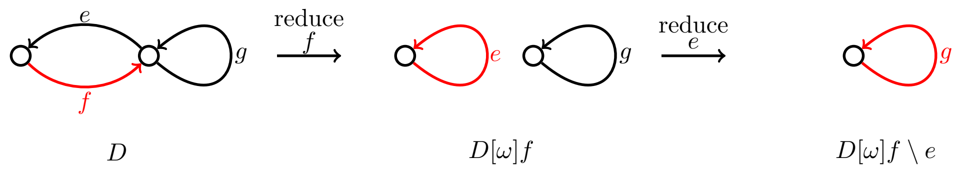


Figure 6.8: Reductions of $D \cong G^c_{2,3}$ using $\mathcal{O}_2 = feg$

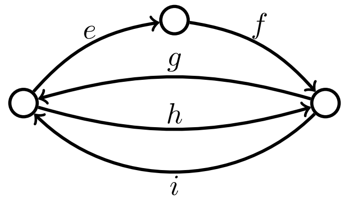


Figure 6.9: Alternating dimap $G_{3,5,1}$

$$= 1 \qquad \text{(the final edge is always an ultraloop).}$$

Since the derived polynomials of D that are produced by the edge-orderings $\mathcal{O}_1$ and $\mathcal{O}_2$ are different, the result follows. $\square$

By using a similar approach, we show that the c-Tutte invariant is not well defined for alternating dimap $G_{3,5,1}$.

Lemma 6.23. *The c-Tutte invariant is not well defined for the alternating dimap $G_{3,5,1}$.*

Proof. Suppose $D \cong G_{3,5,1}$ has five edges e, f, g, h and i as shown in Figure 6.9. Since D has five edges, there exist $5! = 120$ possible edge-orderings. Let $\mathcal{O}_1 = efghi$ and $\mathcal{O}_2 = feghi$ be two of the possible edge-orderings of D.

Note that the c-Tutte invariant is well defined for any element that belongs to $\mathrm{alt}_c(\mathcal{G})$. We also have $T_c(\mathrm{alt}_c(G); x, y) = T(G; x, y)$ for some plane graph G, by Theorem 6.1.

If D is reduced using $\mathcal{O}_1$ (see Figure 6.10), we obtain

$$\begin{aligned}
T_c(D; x, y) &= x \cdot T_c(D[\omega^2]e; x, y) && (e \text{ is a proper 1-loop in } D)\\
&= x \cdot (x + y) && (\text{since } D[\omega^2]e \text{ belongs to } \mathrm{alt}_c(\mathcal{G}))\\
&= x^2 + xy.
\end{aligned}$$

On the other hand, if D is reduced using $\mathcal{O}_2$ (see Figure 6.11), then

$$\begin{aligned}
T_c(D; x, y) &= T_c(D[1]f; x, y) + T_c(D[\omega^2]f; x, y) && (f \text{ is a proper edge in } D)\\
&= (x + y) + x^2 && (\text{since } D[1]f \text{ and } D[\omega^2]f \text{ both belong to } \mathrm{alt}_c(\mathcal{G}))\\
&= x^2 + x + y.
\end{aligned}$$

Since the derived polynomials of D that are produced by the edge-orderings $\mathcal{O}_1$ and $\mathcal{O}_2$ are different, the result follows. $\square$

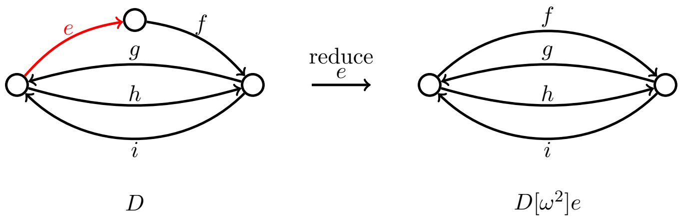


Figure 6.10: Reductions of $D \cong G_{3,5,1}$ on e using $\mathcal{O}_1 = efghi$

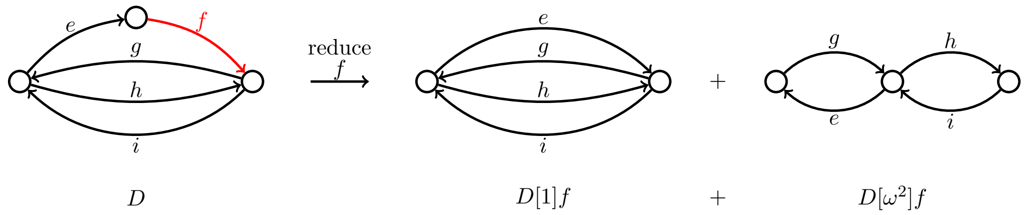


Figure 6.11: Reductions of $D \cong G_{3,5,1}$ on f using $\mathcal{O}_2 = feghi$

Theorem 6.24. *The c-Tutte invariant is well defined for an alternating dimap D if and only if D is a c-alternating dimap.*

Proof. The forward implication is proved by contradiction using two different cases.

Let D be an alternating dimap such that the c-Tutte invariant is well defined for D. Suppose D is not a c-alternating dimap. This implies that D is not a c-simple alternating dimap after every loop in D is removed. Thus, either it contains a block that is neither a c-cycle block nor an element of $\text{alt}_c(\mathcal{G})$, or there exists a block within a clockwise face of some other block.

First, assume that D contains a block B that is neither a c-cycle block nor an element of $\text{alt}_c(\mathcal{G})$. By Lemma 6.20, the block B contains $G_{3,5,1}$ as a minor. By Lemma 6.23, the c-Tutte invariant is not well defined for $G_{3,5,1}$. Since B is a block of D, the alternating dimap D contains $G_{3,5,1}$ as a minor. By Proposition 5.25, the c-Tutte invariant is not well defined for D. We reach a contradiction.

Secondly, suppose there exists a block B in D such that B contains another block B' within one of its clockwise faces. Note that B and B' form a clockwise face of size greater than two, else it is a c-multiloop. By Lemma 6.21, the alternating dimap D contains $G^c_{2,3}$ as a minor. By Lemma 6.22, the c-Tutte invariant is not well defined for $G^c_{2,3}$. By Proposition 5.25, we again get a contradiction. Hence, the forward implication follows.

It remains to show if D is a c-alternating dimap, then the c-Tutte invariant is well defined for D. Every non-loop block of D is either a c-cycle block or is an element of $\text{alt}_c(\mathcal{G})$, and D contains no block within a clockwise face of any other block. In addition, D may contain some c-multiloops within some of its anticlockwise faces. By Lemma 6.10, the c-Tutte invariant is well defined for every c-cycle block. By Theorem 6.1, the c-Tutte invariant is also well defined for alternating dimaps that belongs to $\text{alt}_c(\mathcal{G})$. By

Lemma 6.12, the c-Tutte invariant is also well defined for every c-multiloop. By Theorem 6.17, the c-Tutte invariant is multiplicative over non-loop blocks and c-multiloops for any c-alternating dimap. Hence, the c-Tutte invariant is well defined for D.

Therefore, the c-Tutte invariant is well defined for an alternating dimap D if and only if D is a c-alternating dimap. □

We now develop a relationship between plane graphs G and c-alternating dimaps D, when the Tutte polynomial of G and the c-Tutte invariant of D are both identical.

Theorem 6.25. *Let D be a c-alternating dimap, G be a plane graph and G' is obtained from G by deleting all the loops and bridges in G. Let R and S be the set of c-cycle blocks and c-multiloops in D, respectively. Let $E_R = \bigcup_{r\in R} E(r)$, $E_S = \bigcup_{s\in S} E(s)$ and $D' = D \setminus E_R \setminus E_S$. Then, $T(G;x,y) = T_c(D;x,y)$ if and only if $T_c(alt_c(G');x,y) = T_c(D';x,y)$ and G contains $\sum_{r\in R}(|r|-1)$ bridges and $\sum_{s\in S}(|s| - \mathrm{cf}(s))$ loops.*

Remark: $T(G;x,y) = T_c(D;x,y)$ if and only if $T_c(\mathrm{alt}_c(G);x,y) = T_c(D;x,y)$. So Theorem 6.25 is about situations of T_c-equivalence in alternating dimaps.

Proof. Let D, G, G', R, S, E_R and E_S be as stated. To prove the forward implication, we let B and L be the sets of bridges and loops in G, respectively. Suppose $|B| = p$ and $|L| = q$. Since the Tutte polynomial is multiplicative over blocks, we have

$$T_c(D;x,y) = T(G;x,y) = x^p \cdot y^q \cdot T(G';x,y).$$

By Theorem 6.17, the c-Tutte invariant is multiplicative over c-cycle blocks, elements of $\mathrm{alt}_c(\mathcal{G})$ and c-multiloops. By Definition 6.1, a factor of x is introduced when a proper 1-loop or a proper ω-semiloop is reduced. In any c-alternating dimap, proper 1-loops and proper ω-semiloops can only be found in c-cycle blocks. Note that if a plane graph H has a single edge, then $\mathrm{alt}_c(H)$ is also a c-cycle block of size two. By Lemma 6.14 and using the fact that the c-Tutte invariant is multiplicative over c-cycle blocks, we have $p = \sum_{r\in R}(|r|-1)$. Likewise, a factor of y is introduced when a proper ω-loop or a proper 1-semiloop is reduced. These two types of edges can only be found in c-multiloops. By Lemma 6.15 and using the fact that the c-Tutte invariant is multiplicative over c-multiloops, we have $q = \sum_{s\in S}(|s| - \mathrm{cf}(s))$. After each c-cycle block and each c-multiloop is reduced in D, we obtain $D' = D \setminus E_R \setminus E_S$. It can also be seen that D' is an element of $\mathrm{alt}_c(\mathcal{G})$. Since

$$x^p \cdot y^q \cdot T(G';x,y) = T_c(D;x,y) = x^p \cdot y^q \cdot T_c(D';x,y),$$

we have

$$T(G';x,y) = T_c(D';x,y). \tag{6.3}$$

Since G' is a plane graph, by Theorem 6.1 and (6.3), we obtain

$$T_c(\mathrm{alt}_c(G');x,y) = T(G';x,y) = T_c(D';x,y).$$

The backward implication follows immediately, by Theorem 6.1, Lemmas 6.14 and 6.15, and Theorem 6.17. □

Corollary 6.26. *If D is a c-alternating dimap, then there exists a plane graph H such that $T_c(D;x,y) = T_c(\mathrm{alt}_c(H);x,y) = T(H;x,y)$.*

Proof. Let D be a c-alternating dimap, R and S be the set of c-cycle blocks and c-multiloops in D, respectively. Let $E_R = \bigcup_{r\in R} E(r)$, $E_S = \bigcup_{s\in S} E(s)$ and $D' = D\backslash E_R\backslash E_S$. Every block in D' is an element of $\mathrm{alt}_c(G)$. This implies that every clockwise face of D' has size exactly two. By Lemma 6.6, there exists an undirected graph H' such that $D' \cong \mathrm{alt}_c(H')$. By adding in $\sum_{r\in R}(|r|-1)$ bridges and $\sum_{s\in S}(|s|-\mathrm{cf}(s))$ loops into H', we obtain a plane graph H and $T_c(D;x,y) = T_c(\mathrm{alt}_c(H);x,y) = T(H;x,y)$. □

By defining an *a-alternating dimap* with appropriate modifications, we have the following two corollaries, based on Theorem 6.24 and Theorem 6.25, respectively.

Corollary 6.27. *The a-Tutte invariant is well defined for an alternating dimap D if and only if D is an a-alternating dimap.* □

Corollary 6.28. *Let D be an a-alternating dimap, G be a plane graph and G' is obtained from G by deleting all the loops and bridges in G. Let R and S be the set of a-cycle blocks and a-multiloops in D, respectively. Let $E_R = \bigcup_{r\in R} E(r)$, $E_S = \bigcup_{s\in S} E(s)$ and $D' = D \setminus E_R \setminus E_S$. Then, $T(G;x,y) = T_a(D;x,y)$ if and only if $T_a(alt_a(G');x,y) = T_a(D';x,y)$ and G contains $\sum_{r\in R}(|r|-1)$ bridges and $\sum_{s\in S}(|s|-\mathrm{af}(s))$ loops.* □

6.3 Factorisation of c-Tutte Invariants

Factorisation of the Tutte polynomial of a graph reflects the structure of the graph [63]. It is natural to ask under what conditions the c-Tutte invariants of c-alternating dimaps factorise. We found that the factorisation of c-Tutte invariants also reflects the structure of the associated c-alternating dimaps.

It is clear that Theorem 6.1 leads to the following proposition.

Proposition 6.29. *Let G be a plane graph and $D \cong alt_c(G)$. Then, $T_c(D;x,y)$ factorises if and only if $T(G;x,y)$ factorises.* □

By Theorem 6.17, the c-Tutte invariant is multiplicative over non-loop blocks (c-cycle blocks and elements of $\mathrm{alt}_c(\mathcal{G})$) and c-multiloops for c-alternating dimaps. Using this and Theorem 6.25, we have the following corollaries.

Corollary 6.30. *Let D be a c-alternating dimap and R be its set of c-cycle blocks. Then, $\sum_{r\in R}(|r|-1) = p$ if and only if $x^p \mid T_c(D;x,y)$.* □

Corollary 6.31. *Let D be a c-alternating dimap and S be its set of c-multiloops. Then, $\sum_{s\in S}(|s|-\mathrm{cf}(s)) = q$ if and only if $y^q \mid T_c(D;x,y)$.* □

Corollary 6.32. *Let D be a c-alternating dimap, and R and S be the set of c-cycle blocks and c-multiloops in D, respectively. If $\sum_{r\in R}(|r|-1) \geq 2$ or $\sum_{s\in S}(|s|-\mathrm{cf}(s)) \geq 2$, then $T_c(D;x,y)$ factorises.* □

The converse of Corollary 6.32 does not necessarily hold, by Proposition 6.29.

We know that if D is a c-alternating dimap, then the c-Tutte invariant of D is the same as the Tutte polynomial of some graph by Theorem 6.25. A natural question that arises is whether this property still holds when D is not a c-alternating dimap. Before we present our counterexample, we have the following observation.

Proposition 6.33. *The polynomial $x^2 + 1$ is not the Tutte polynomial of any graph.*

Proof. Suppose $x^2 + 1$ is the Tutte polynomial of some connected graph G. The Tutte polynomial specialises to the chromatic polynomial $\chi(G; x)$ at $y = 0$ by the following relation:

$$\chi(G; x) = (-1)^{|V(G)|-k(G)} x^{k(G)} T(G; 1 - x, 0).$$

Since $x^2 + 1$ is the Tutte polynomial of some connected graph G, we have

$$\chi(G; x) = (-1)^{|V(G)|-1} x(x^2 - 2x + 2).$$

It is clear that $x - 1$ is not a factor of $\chi(G; x)$. The chromatic number of G is 1 implies that G has no edges. The multiplicity of the factor x is 1 in $\chi(G; x)$, so G has only one block. The only connected graph that has chromatic number 1 and a single block is a single vertex, and its chromatic polynomial is x. Hence, $\chi(G; x)$ is not a chromatic polynomial for any connected graph G. It follows that $x^2 + 1$ is not the Tutte polynomial of any connected graph. Since the Tutte polynomial is multiplicative over components and $x^2 + 1$ is irreducible over $\mathbb{Q}$, the result follows. □

Let D be the alternating dimap $G_{3,5,1}$, which is not a c-alternating dimap, as shown in Figure 6.9. We consider two ordered alternating dimaps $D_1 = (D, feghi)$ and $D_2 = (D, ifegh)$.

For D_1, it is routine to show that $T_c(D_1; x, y) = x^2 + x + y = T(G; x, y)$, where G is a triangle. This shows that there exists an ordered alternating dimap D whose underlying alternating dimap is not a c-alternating dimap, but $T_c(D; x, y) = T_c(\text{alt}_c(H); x, y)$ where H is plane graph.

We now give an ordered alternating dimap that has c-Tutte invariant x^2+1. For D_2, we have $T_c(D_2; x, y) = x^2+1$, which gives two imaginary roots i and $-i$. By Proposition 6.33, we know that $x^2 + 1$ is not the Tutte polynomial of any graph. This observation shows that for ordered alternating dimaps, there exist c-Tutte invariants that are not the Tutte polynomial of any graph.

CHAPTER 7

Factorisation of Greedoid Polynomials of Rooted Digraphs

Recall that Gordon and McMahon [39] defined a two-variable greedoid polynomial

$$f(G;t,z) = \sum_{A\subseteq E(G)} t^{r(G)-r(A)} z^{|A|-r(A)}$$

for any greedoid G, which we call the *greedoid polynomial*. They proved that the greedoid polynomials of rooted digraphs have the multiplicative direct sum property, that is, if a digraph $D = D_1 \oplus D_2$, then $f(D;t,z) = f(D_1;t,z)\cdot f(D_2;t,z)$. This raises the question of whether this is the only circumstance in which this polynomial can be factorised. Note that the Tutte polynomial of a graph G factorises if and only if G is a direct sum [63], but the situation for the chromatic polynomial is more complex [65]. Gordon and McMahon showed that the greedoid polynomial of a rooted digraph that is not necessarily a direct sum has $1+z$ among its factors under certain conditions (see Theorems 4.11 and 4.13). We address more general types of factorisation in this chapter.

Note that we focus on directed branching greedoids. Hence, all our digraphs are rooted.

7.1 Preliminaries

We compute the greedoid polynomials for all rooted digraphs (up to isomorphism unless otherwise stated) up to order six. All the labelled rooted digraphs (without loops and multiple edges, but cycles of size two are allowed) up to order six were provided by Brendan McKay[1] on 28 March 2018 (personal communication from McKay to Farr). We then study the factorability of these polynomials, particularly those that are not divisible by $1+z$

Two rooted digraphs are *GM-equivalent* if they both have the same greedoid polynomial. If a rooted digraph is a direct sum, then it is *separable*. Otherwise, it is *non-separable*.

A greedoid polynomial $f(D)$ of a rooted digraph D of order n *GM-factorises* if $f(D) = f(G)\cdot f(H)$ such that G and H are rooted digraphs of order at most n and $f(G), f(H) \neq 1$. Note that $f(G)$ and $f(H)$ are not necessarily distinct. A *GM-factor* of a rooted digraph D is a polynomial P where P divides $f(D)$ and $P \neq 1$. The polynomials $f(G)$ and $f(H)$

[1]More combinatorial data can be found at https://users.cecs.anu.edu.au/~bdm/data/.

are GM-factors of $f(D)$. Furthermore, any factor of $f(G)$ (or $F(H)$) is also a GM-factor. We also say a rooted digraph D *GM-factorises* if its greedoid polynomial GM-factorises. Every rooted digraph that is a direct sum has a GM-factorisation.

An irreducible GM-factor is *basic* if the GM-factor is either $1+t$ or $1+z$. Otherwise, the irreducible GM-factor is *nonbasic*. We are most interested in nonbasic GM-factors. A GM-factor is *primary* if it is irreducible, nonbasic and is not a GM-factor of any greedoid polynomial of rooted digraphs of smaller order. Such a factor appears as a GM-factor only in rooted digraphs with at least as many vertices as the current one. For $k \geq 1$, a non-separable digraph is *k-nonbasic* if its greedoid polynomial has k nonbasic GM-factors. A non-separable digraph is *totally k-nonbasic* if it is k-nonbasic and contains no basic GM-factors. Likewise, a non-separable digraph is *k-primary* if its greedoid polynomial has k primary GM-factors. A non-separable digraph is *totally k-primary* if it is k-primary and contains no basic GM-factors. It follows that if a non-separable digraph is (totally) k-primary, then the digraph is (totally) ℓ-nonbasic for some $\ell \geq k$.

Our results show that there exist non-separable digraphs that GM-factorise and their polynomials have neither $1+t$ nor $1+z$ as factors. In some cases (but not all), these non-separable digraphs of order n are GM-equivalent to a separable digraph of order at most n. We give the numbers of polynomials of this type of non-separable digraph. For rooted digraphs up to order six and $k \geq 2$, we found that there exist no $(k+1)$-nonbasic digraphs and no k-primary digraphs. We also provide the numbers of 2-nonbasic digraphs, totally 2-nonbasic digraphs, 1-primary digraphs and totally 1-primary digraphs. We then give the first examples of totally 2-nonbasic and totally 1-primary digraphs. Lastly, we give an infinite family of non-separable digraphs where their greedoid polynomials factorise into at least two non-basic GM-factors.

7.2 Results

The greedoid polynomials of all rooted digraphs (without loops and multiple edges, but cycles of size two are allowed) up to order six were computed by using Algorithm 1 (see Appendix A). This algorithm is based on the deletion-contraction recurrence in Proposition 4.8 that was introduced by Gordon and McMahon [39]. We then simplified and factorised all these greedoid polynomials using Wolfram Mathematica.

Throughout, numbers of rooted digraphs are up to isomorphism unless stated otherwise.

7.2.1 Separability and Non-separability

For each order, we determined the numbers of rooted digraphs, separable digraphs, non-separable digraphs, and non-separable digraphs of order n that are GM-equivalent to some separable digraph of order at most n (see Table 7.2, and the list of abbreviations in Table 7.1).

Note that the sequences of numbers of labelled rooted digraphs (T) and rooted digraphs (T-ISO) are not listed in The On-Line Encyclopedia of Integer Sequences (OEIS).

Abbreviation	Description
T	Number of labelled rooted digraphs
T-ISO	Number of rooted digraphs
S	Number of separable digraphs
NS	Number of non-separable digraphs
NSE	Number of non-separable digraphs of order n that are GM-equivalent to some separable digraph of order at most n

Table 7.1: Abbreviations for Table 7.2

n	T	T-ISO	S	NS	NSE
1	1	1	0	1	0
2	6	4	0	4	0
3	48	36	6	30	7
4	872	752	88	664	200
5	48040	45960	2404	43556	10641
6	9245664	9133760	150066	8983694	1453437

Table 7.2: Numbers of various types of rooted digraphs (up to order six)

We observe that the ratio of T-ISO to T shows an increasing trend. The ratio of NS to T-ISO is also increasing (for $n \geq 3$), as expected.

For each order, we also provide the number PU of unique greedoid polynomials and the ratio of PU to T-ISO, in Table 7.3.

n	T-ISO	PU	PU/T-ISO
1	1	1	1.0000
2	4	4	1.0000
3	36	22	0.6111
4	752	201	0.2673
5	45960	6136	0.1335
6	9133760	849430	0.0930

Table 7.3: Numbers PU of unique greedoid polynomials of rooted digraphs (up to order six) and the ratio of PU to T-ISO

Bollobás, Pebody and Riordan conjectured that almost all graphs are determined by their chromatic or Tutte polynomials [6]. However, this conjecture does not hold for matroids. The ratio of the number of unique Tutte polynomials of matroids to the number of non-isomorphic matroids approaches 0 as the cardinality of matroids increases, which can be shown using the bounds given in Exercise 6.9 in [13]. We believe that greedoid polynomials of rooted digraphs behave in a similar manner as matroids. According to our findings, the ratio of PU to T-ISO shows a decreasing trend. We expect that as n increases, this ratio continues to decrease. The question is, does this ratio eventually approach 0 or is it bounded away from 0? Further computation should give more insight on this question.

7.2.2 Factorability

For $n \in \{1, \ldots, 5\}$, we identified the numbers of greedoid polynomials that GM-factorise for rooted digraphs of order n. Details are given in Table 7.5 (see Table 7.4 for the list of abbreviations and Figure 7.1 for the corresponding Venn diagram).

Abbreviation	Description
	Number of greedoid polynomials of rooted digraphs that ...
PNF	...cannot be GM-factorised
PF	...can be GM-factorised
PFS	...can be GM-factorised and the digraph is separable
PFNS	...can be GM-factorised and the digraph is non-separable
PF	PFS ∪ PFNS
COMM	PFS ∩ PFNS
PFSU	PFS − COMM
PFNSU	PFNS − COMM

Table 7.4: Abbreviations for Figure 7.1 and Table 7.5

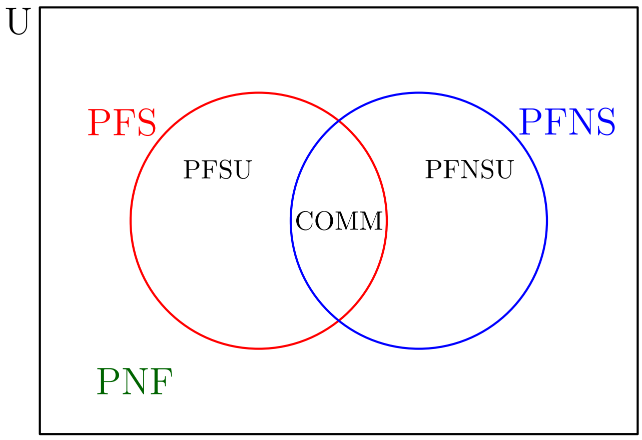


Figure 7.1: Venn diagram that represents the factorability of greedoid polynomials of rooted digraphs where U = PF ∪ PNF and PF = PFS ∪ PFNS

n	PNF	PF	PFS	PFNS	COMM	PFSU	PFNSU
1	1	0	0	0	0	0	0
2	3	1	0	1	0	0	1
3	6	16	6	13	3	3	10
4	37	164	41	145	22	19	123
5	1044	5092	444	4867	219	225	4648

Table 7.5: Factorability of greedoid polynomials of rooted digraphs (up to order five)

We found that the ratio of PF to PU shows an upward trend, and the ratio stands at 0.8299 when $n = 5$. It seems that most likely as n increases, the ratio will either approach 1 in which case almost all greedoid polynomials of rooted digraphs GM-factorise, or the ratio will approach a fixed α where $0.8299 \leq \alpha < 1$. We ask, what is the limiting proportion of greedoid polynomials of rooted digraphs that GM-factorise, as $n \to \infty$?

We categorised these polynomials into two classes, according to whether they are polynomials of separable or non-separable digraphs. Some of these polynomials are polynomials of both separable and non-separable digraphs. The number of such polynomials is given in column 6 (COMM) in Table 7.5. One such example for digraphs of order three is shown in Figure 7.2, where the two digraphs have the same greedoid polynomial $(1 + t)(1 + z)$.

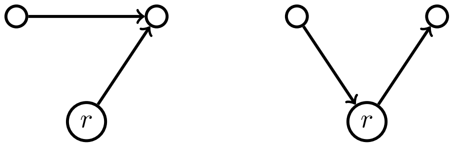


Figure 7.2: Digraphs that have the same greedoid polynomial where (a) is non-separable and (b) is separable

We are interested in non-separable digraphs that can be GM-factorised, especially those digraphs that have greedoid polynomials that are not the same as polynomials of any separable digraph. The numbers of greedoid polynomials of these digraphs are given in column PFNSU in Table 7.5, and examples of such rooted digraphs of order two and three are given in Figure 7.3 and Figure 7.4, respectively. It is easy to verify that the greedoid polynomial of the rooted digraph in Figure 7.3 is $(1 + t)(1 + z)$. The greedoid polynomials of rooted digraphs in Figure 7.4 are (from left to right starting from the first row) given in Table 7.6.

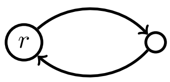


Figure 7.3: The non-separable digraph of order two that GM-factorises

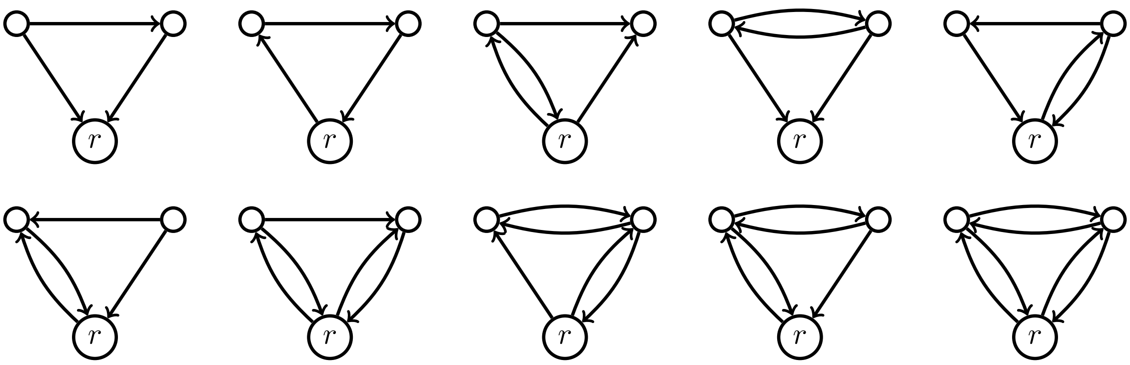


Figure 7.4: Ten of the 16 non-separable digraphs (one for each of the ten different greedoid polynomials) of order three that GM-factorise

Greedoid polynomials	Number of non-separable rooted digraphs of order three
1. $(1+z)^3$	2
2. $(1+z)(1+t+t^2+t^2z)$	3
3. $(1+z)(2+2t+t^2+z+tz+t^2z)$	2
4. $(1+z)^4$	1
5. $(1+z)^2(1+t+t^2+t^2z)$	3
6. $(1+t)(1+z)^3$	1
7. $(1+z)^2(2+2t+t^2+z+tz+t^2z)$	1
8. $(1+z)^2(3+2t+t^2+z+t^2z)$	1
9. $(1+z)^3(1+t+t^2+t^2z)$	1
10. $(1+z)^3(3+2t+t^2+z+t^2z)$	1

Table 7.6: Greedoid polynomials of non-separable digraphs of order three that GM-factorise and these polynomials are not the same as polynomials of any separable digraph of order three, and the numbers of associated non-separable digraphs (making 16 non-separable rooted digraphs altogether)

7.2.3 2-nonbasic and 1-primary Digraphs

We investigate greedoid polynomials that contain nonbasic and primary GM-factors. Details are given in Table 7.8 (see Table 7.7 for the list of abbreviations and Figure 7.5 for the corresponding Venn diagram). For rooted digraphs up to order six, each 1-primary digraph is a 2-nonbasic digraph, and each totally 1-primary digraph is a totally 2-nonbasic digraph.

Abbreviation	Description
2-NB	Number of 2-nonbasic digraphs
2-TNB	Number of totally 2-nonbasic digraphs
1-P	Number of 1-primary digraphs
1-TP	Number of totally 1-primary digraphs

Table 7.7: Abbreviations for Figure 7.5 and Table 7.8

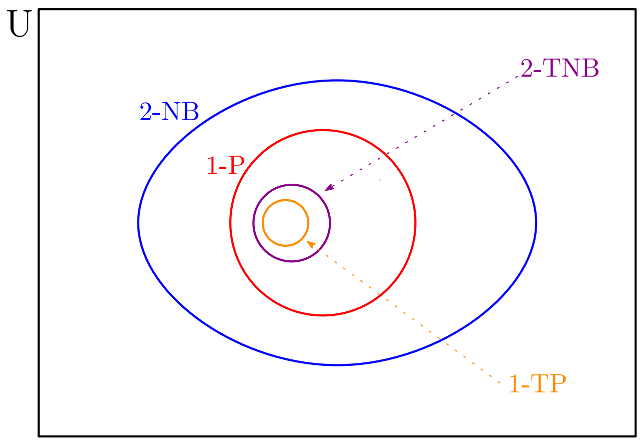


Figure 7.5: Venn diagram that represents four types of digraphs in Table 7.8 where U is the set of digraphs (up to order six) that can be GM-factorised

n	2-NB	2-TNB	1-P	1-TP
1	0	0	0	0
2	0	0	0	0
3	0	0	0	0
4	0	0	0	0
5	120	0	0	0
6	12348	15	1252	9

Table 7.8: Numbers of the four types of non-separable digraphs (up to order six) that can be GM-factorised

All rooted digraphs up to order four either have one nonbasic GM-factor or only basic GM-factors in their polynomials. There are 120 rooted digraphs of order five that have greedoid polynomials with at least two nonbasic GM-factors. The number of distinct greedoid polynomials of these 120 rooted digraphs is 34. Further examination showed that the number of nonbasic GM-factors in these polynomials is exactly two. Nonetheless, 117 of the 120 rooted digraphs have greedoid polynomials that contain at least one basic GM-factor, and the remaining three are separable digraphs (as shown in Figure 7.6).

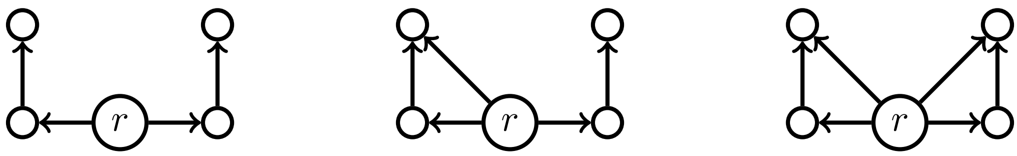


Figure 7.6: Three separable digraphs of order five that have two nonbasic GM-factors

Hence, there exist no totally 2-nonbasic digraphs of order five. In addition, none of the polynomials of these 120 rooted digraphs contains a primary GM-factor. This implies that none of the rooted digraphs up to order five are k-primary, for $k \geq 1$. Each of the GM-factors of greedoid polynomials of rooted digraphs up to order five is either basic, or is a GM-factor of some greedoid polynomials of rooted digraphs of smaller order.

There are 12348 rooted digraphs of order six that have greedoid polynomials with at least two nonbasic GM-factors. The number of distinct greedoid polynomials of these 12348 rooted digraphs is 837. A quick search showed that all these digraphs are 2-nonbasic. We found that 15 of these rooted digraphs are totally 2-nonbasic. One of the totally 2-nonbasic digraphs D_1 of order six is shown in Figure 7.7 and its greedoid polynomial is as follows:

$$f(D_1) = (1 + t + t^2 + t^2z)(2 + 2t + t^2 + t^3 + z + tz + t^2z + 3t^3z + 3t^3z^2 + t^3z^3).$$

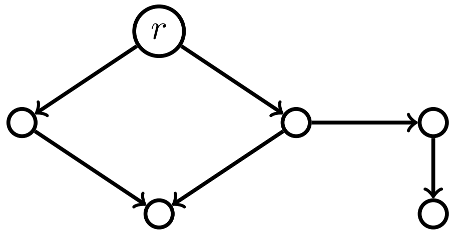


Figure 7.7: A totally 2-nonbasic digraph of order six

The nonbasic GM-factors of $f(D_1)$ are greedoid polynomials of rooted digraphs G and H that have order three and four, respectively (see Figure 7.8). We have $f(G) = 1+t+t^2+t^2z$ and $f(H) = 2+2t+t^2+t^3+z+tz+t^2z+3t^3z+3t^3z^2+t^3z^3$. However, D_1 is a non-separable digraph and hence not the direct sum of G and H.

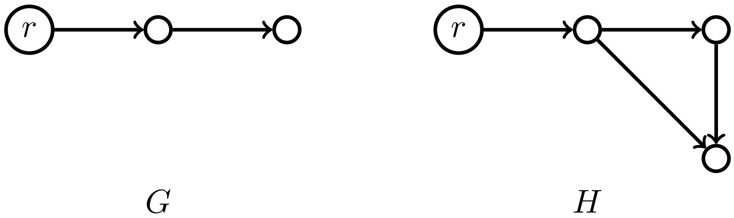


Figure 7.8: Rooted digraphs G and H

There are also 1252 rooted digraphs of order six that have greedoid polynomials with one primary GM-factor, and all these digraphs are non-separable. However, only nine of them are totally 1-primary digraphs. One of the totally 1-primary digraphs D_2 of order six is shown in Figure 7.9 and it has the following greedoid polynomial:

$$f(D_2) = (1+t+t^2+t^2z)(4+3t+t^2+t^3+4z+2tz+t^2z+4t^3z+z^2+6t^3z^2+4t^3z^3+t^3z^4).$$

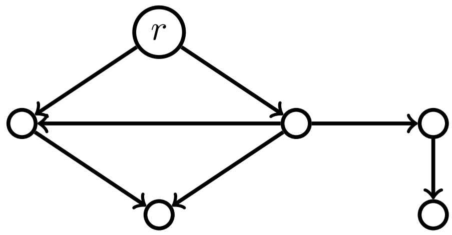


Figure 7.9: A totally 1-primary digraph of order six

The totally 1-primary digraph D_2 GM-factorises into one nonbasic GM-factor $1+t+t^2+t^2z$ and one primary GM-factor $4+3t+t^2+t^3+4z+2tz+t^2z+4t^3z+z^2+6t^3z^2+4t^3z^3+t^3z^4$. The GM-factor $1+t+t^2+t^2z$ is not primary as it is the greedoid polynomial of the rooted digraph G in Figure 7.8. Note that D_2 is also a totally 2-nonbasic digraph since every primary GM-factor is a nonbasic GM-factor.

The fact that a greedoid polynomial of a rooted digraph is not divisible by $1+z$ implies that the associated rooted digraph has neither a directed cycle nor a greedoid loop.

Our results show that there exist some non-separable digraphs (of order six) that GM-factorise into only nonbasic GM-factors, or both nonbasic and primary GM-factors. This implies that the multiplicative direct sum property, and the existence of greedoid loops and directed cycles, are not the only characteristics that determine if greedoid polynomials of rooted digraphs factorise.

7.2.4 An Infinite Family

Lastly, we show that there exists an infinite family of digraphs where their greedoid polynomials factorise into at least two nonbasic GM-factors. We first characterise greedoid polynomials of two classes of rooted digraphs.

Let P_{m,v_0} be a *directed path* $v_0v_1\dots v_m$ of size $m \geq 0$ *rooted* at v_0, and C_{m,v_0} be a *directed cycle* $v_0v_1\dots v_{m-1}v_0$ of size $m \geq 1$ *rooted* at v_0. For convenience, we usually write P_m for P_{m,v_0} and C_m for C_{m,v_0}.

Lemma 7.1.
$$f(P_m;t,z) = 1 + \frac{t(1-(t(1+z))^m)}{1-t(1+z)}.$$

Proof. We proceed by induction on the number m of edges of P_m. For the base case, suppose $m = 0$. We have $f(P_0) = 1$ and the result for $m = 0$ follows. Assume that $m > 0$ and the result holds for every rooted directed path of size less than m. Let e be the outgoing edge of the root vertex of P_m. By Proposition 4.8, we have

$$\begin{aligned}
f(P_m) &= f(P_m/e) + t^{r(P_m)-r(P_m\setminus e)} f(P_m \setminus e)\\
&= \left(1 + \frac{t(1-(t(1+z))^{m-1})}{1-t(1+z)}\right) + t^{m-0}(1+z)^{m-1}\\
&= 1 + \frac{t(1-(t(1+z))^m)}{1-t(1+z)}.
\end{aligned}$$

□

Suppose Q_m is an *undirected path* $v_0v_1\dots v_m$ of size $m \geq 0$ *rooted* at either v_0 or v_m. Then $f(P_m;t,z) = f(Q_m;t,z)$, since there is a rank-preserving bijection between $2^{E(P_m)}$ and $2^{E(Q_m)}$.

Lemma 7.2.
$$f(C_m;t,z) = (1+z)f(P_{m-1};t,z).$$

Proof. We proceed by induction on the number m of edges of C_m. For the base case, suppose $m = 1$. We have $f(C_1) = (1+z)f(P_0) = 1+z$ and the result for $m = 1$ follows. Assume that $m > 1$ and the result holds for every rooted directed cycle of size less than m. Let e be the outgoing edge of the root vertex of C_m. By Proposition 4.8, we have

$$\begin{aligned}
f(C_m) &= f(C_m/e) + t^{r(C_m)-r(C_m\setminus e)} f(C_m \setminus e)\\
&= (1+z)f(P_{m-2}) + t^{(m-1)-0}(1+z)^{m-1}\\
&= (1+z)\left(f(P_{m-2}) + t^{m-1}(1+z)^{m-2}\right)
\end{aligned}$$

$$= (1+z)f(P_{m-1}) \qquad \text{(by Lemma 7.1).}$$

□

Gordon gave a formula for the greedoid polynomials of rooted undirected cycles in [38]. Those polynomials are different to the ones given by Lemma 7.2.

We now give an infinite family of digraphs where their greedoid polynomials factorise into at least two nonbasic GM-factors, extending the example in Figure 7.7.

Lemma 7.3. *There exists an infinite family of non-separable digraphs D that have at least two nonbasic GM-factors, where*

$$f(D) = f(P_{k+1})\left(f(C_{k+1}) + f(P_{k+1}) + t^{k+2}(1+z)^{k+2}\right), \text{ for } k \geq 1.$$

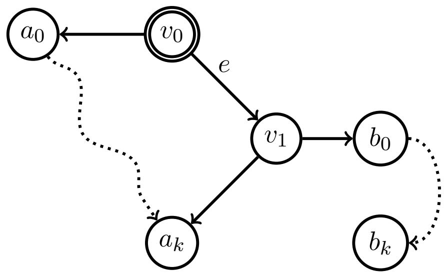
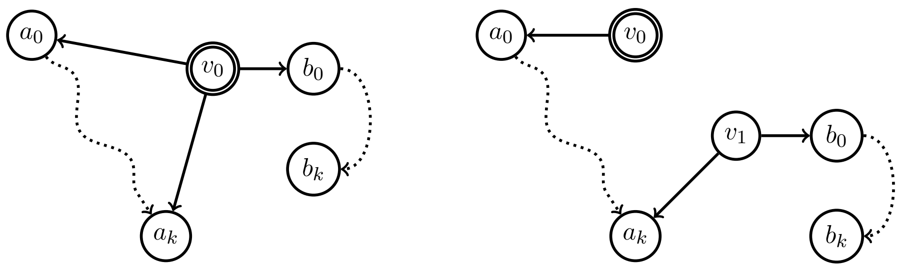


Figure 7.10: The digraph D in the proof of Lemma 7.3

Proof. Let D be the non-separable digraph rooted at vertex v_0 shown in Figure 7.10, where $a_0 \dots a_k$ and $b_0 \dots b_k$ are two directed paths in D of length $k \geq 1$ starting at a_0 and b_0, respectively. To compute the greedoid polynomial of D by using Proposition 4.8, we first choose the edge $e = v_0v_1$. By deleting and contracting e, we obtain the digraphs $D_1 = D/e$ and $D_2 = D \setminus e$ as shown in Figure 7.11.



(a) $D_1 = D/e$ (b) $D_2 = D \setminus e$

Figure 7.11: Two minors D/e and $D \setminus e$ of D

Note that D_1 is a separable digraph rooted at v_0. Let $R = \{v_0, a_0, \dots, a_k\} \subset V(D_1)$, $S = \{v_0, b_0, \dots, b_k\} \subset V(D_1)$ and $T = \{v_0, a_0, \dots, a_k\} \subset V(D_2)$. Suppose $A = D_1[R]$ and $B = D_1[S]$ are the subdigraphs of D_1 induced by R and S respectively, and $C = D_2[T]$ is the subdigraph of D_2 induced by T. Clearly, $B \cong C \cong P_{k+1}$. Hence we have $f(B) = f(C) = f(P_{k+1})$. Note that every edge $g \in E(D_2) \setminus E(C)$ is a greedoid loop, and

$|E(D_2) \setminus E(C)| = k+2$. By using the recurrence formula, we have

$$\begin{aligned} f(D) &= f(D/e) + t^{r(D)-r(D\setminus e)} f(D \setminus e) \\ &= f(A) \cdot f(B) + t^{(2k+3)-(k+1)} f(C) \cdot (1+z)^{k+2} \\ &= f(P_{k+1}) \left(f(A) + t^{k+2}(1+z)^{k+2} \right) \qquad \text{(since } f(B) = f(C) = f(P_{k+1})\text{).} \end{aligned}$$

It remains to show that $f(A)$ can be expressed in terms of $f(P_k)$ and $f(C_k)$. By taking $h = v_0 a_k \in E(A)$ (see Figure 7.12) as the outgoing edge in the recurrence formula, we have

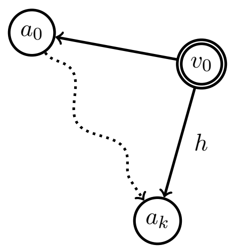


Figure 7.12: The subdigraph A of D_1 induced by R

$$\begin{aligned} f(A) &= f(A/h) + t^{r(A)-r(A\setminus h)} f(A \setminus h) \\ &= f(C_{k+1}) + t^{(k+1)-(k+1)} f(P_{k+1}) \qquad \text{(since } A/h \cong C_{k+1} \text{ and } A \setminus h \cong P_{k+1}\text{)} \\ &= f(C_{k+1}) + f(P_{k+1}). \end{aligned}$$

Therefore,

$$f(D) = f(P_{k+1}) \left(f(C_{k+1}) + f(P_{k+1}) + t^{k+2}(1+z)^{k+2} \right).$$

Clearly, both factors of $f(D)$ are nonbasic GM-factors. Since D is non-separable and $k \geq 1$, we complete the proof. □

We extend the infinite family in Lemma 7.3, and characterise the greedoid polynomials of a new infinite family, as follows.

Theorem 7.4. *There exists an infinite family of non-separable digraphs D that have at least two nonbasic GM-factors, where*

$$f(D) = f(P_{k+1}) \left(f(C_{k+1}) + f(P_{k+1}) + \frac{t^{k+2}(1+z)^{k+2}(1-(t(1+z))^{\ell})}{1-t(1+z)} \right), \text{ for } k, \ell \geq 1.$$

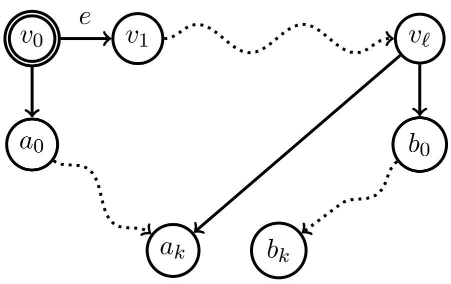


Figure 7.13: The digraph D in the proof of Theorem 7.4

Proof. Let D be the non-separable digraph rooted at vertex v_0 shown in Figure 7.13, where $L = v_0 \dots v_\ell$ is a directed path in D of length $\ell \geq 1$ starting at v_0. We proceed by induction on the length ℓ of L.

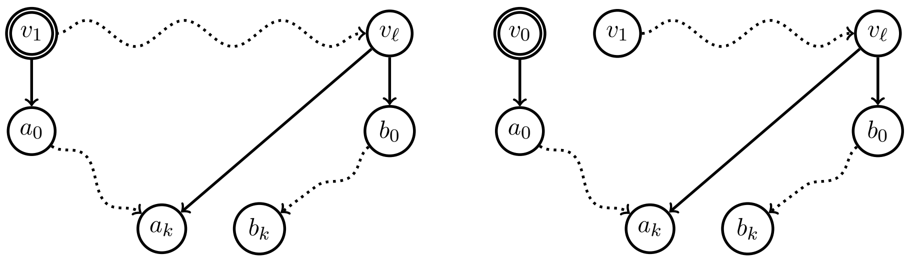
For the base case, suppose $\ell = 1$. By Lemma 7.3, we have

$$\begin{aligned} f(D) &= f(P_{k+1}) \left(f(C_{k+1}) + f(P_{k+1}) + t^{k+2}(1+z)^{k+2} \right) \\ &= f(P_{k+1}) \left(f(C_{k+1}) + f(P_{k+1}) + \frac{t^{k+2}(1+z)^{k+2}(1-(t(1+z))^{\ell})}{1-t(1+z)} \right), \end{aligned}$$

and the result for $\ell = 1$ follows.

Assume that $\ell > 1$ and the result holds for every $r < \ell$.

Let $e = v_0 v_1 \in E(D)$. By applying the deletion-contraction recurrence in Proposition 4.8 on e, we obtain the digraphs $D_1 = D/e$ and $D_2 = D \setminus e$ as shown in Figure 7.14.



(a) $D_1 = D/e$ (a) $D_2 = D \setminus e$

Figure 7.14: Two minors D/e and $D \setminus e$ of D

Note that D_1 is a non-separable digraph rooted at v_1. Since the directed path $v_1 \dots v_\ell$ in D_1 has length $\ell - 1$, we use the inductive hypothesis to obtain $f(D_1)$. Let $R = \{v_0, a_0, \dots, a_k\} \subset V(D_2)$, and $A = D_2[R]$ be the subdigraph of D_2 induced by R. Clearly, $A \cong P_{k+1}$. Hence, we have $f(A) = f(P_{k+1})$. Note that every edge $g \in E(D_2) \setminus E(A)$ is a greedoid loop, and $|E(D_2) \setminus E(A)| = k + \ell + 1$. By using the recurrence formula, we have

$$\begin{aligned} f(D) &= f(D/e) + t^{r(D)-r(D\setminus e)} f(D \setminus e) \\ &= f(P_{k+1}) \left(f(C_{k+1}) + f(P_{k+1}) + \frac{t^{k+2}(1+z)^{k+2}(1-(t(1+z))^{\ell-1})}{1-t(1+z)} \right) \\ &\quad + t^{(2k+\ell+2)-(k+1)} \left(f(P_{k+1}) \cdot (1+z)^{k+\ell+1} \right) \end{aligned}$$

$$= f(P_{k+1})\Bigg(f(C_{k+1}) + f(P_{k+1}) + \left(\frac{t^{k+2}(1+z)^{k+2}(1-(t(1+z))^{\ell-1})}{1-t(1+z)}\right)$$
$$+ t^{k+\ell+1}(1+z)^{k+\ell+1}\Bigg)$$
$$= f(P_{k+1})\left(f(C_{k+1}) + f(P_{k+1}) + \frac{t^{k+2}(1+z)^{k+2}(1-(t(1+z))^{\ell})}{1-t(1+z)}\right).$$

□

We observe that if every directed path has length at most one in a digraph D rooted at a vertex v, the greedoid polynomial of D is trivial. In this scenario, every vertex in D is either a sink vertex or a source vertex. If v is a sink vertex, then every edge in D is a greedoid loop. If v is a source vertex, every edge that is not incident with v is a greedoid loop.

In the following theorem, we give an infinite family of digraphs where each greedoid polynomial of these digraphs is a nonbasic GM-factor of the greedoid polynomial of some non-separable digraph. The proof follows similar approaches as in Lemma 7.3 and Theorem 7.4.

Theorem 7.5. *For any digraph G that has a directed path of length at least two, there exists a non-separable digraph D where $f(D)$ has $f(G)$ as a nonbasic GM-factor.*

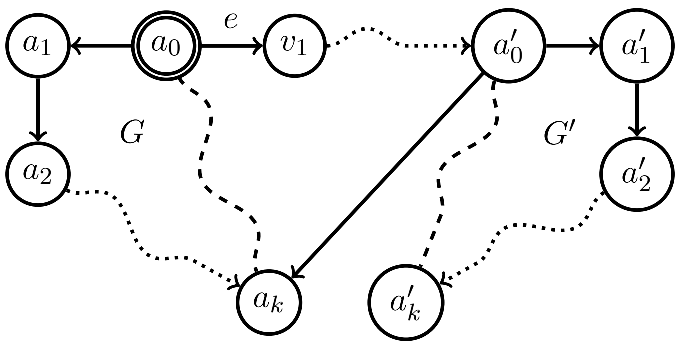


Figure 7.15: An illustration of the non-separable digraph D in Theorem 7.5

Proof. Let G be a digraph that has a directed path $K = a_0a_1 \dots a_k$ of length $k \geq 2$, and G' be a copy of G. The copy of K in G' is denoted by $K' = a'_0a'_1 \dots a'_k$.

We construct a non-separable digraph D_ℓ using G and G', as follows. We first create a directed path $L = a_0v_1 \dots v_{\ell-1}a'_0$ of length ℓ. We add a directed edge a'_0a_k, and assign v_0 as the root vertex of D_ℓ (see Figure 7.15).

To show that $f(G)$ is a nonbasic GM-factor of $f(D_\ell)$, we proceed by induction on the length ℓ of L.

For the base case, suppose $\ell = 1$. We apply the deletion-contraction recurrence in Proposition 4.8 on $e = a_0a'_0$. We denote a_0 the root vertex of the separable digraph D_1/e. We have

$$f(D_1) = f(D_1/e) + t^{r(D_1)-r(D_1\setminus e)}f(D_1 \setminus e)$$

$$= f(G + a_0a_k) \cdot f(G) + t^{(2r(G)+1)-r(G)} f(G) \cdot (1+z)^{|E(G)|+1}$$
$$= f(G) \left(f(G + a_0a_k) + t^{r(G)+1}(1+z)^{|E(G)|+1} \right).$$

Hence, the result for $\ell = 1$ follows.

Assume that $\ell > 1$ and the result holds for every $r < \ell$.

For the inductive step, we apply the deletion-contraction recurrence on $e = v_0v_1$. We have

$$\begin{aligned} f(D_\ell) &= f(D_\ell/e) + t^{r(D_\ell)-r(D_\ell \setminus e)} f(D_\ell \setminus e) \\ &= f(D_\ell/e) + t^{(2r(G)+\ell)-r(G)} f(G) \cdot (1+z)^{|E(G)|+\ell} \\ &= f(D_\ell/e) + t^{r(G)+\ell} f(G) \cdot (1+z)^{|E(G)|+\ell}. \end{aligned}$$

Note that $D_\ell/e \cong D_{\ell-1}$. By the inductive hypothesis, $f(D_\ell/e)$ has $f(G)$ as a nonbasic GM-factor. This implies that $f(D_\ell)$ has $f(G)$ as a nonbasic GM-factor. □

We now have the following corollary.

Corollary 7.6. *Let D be a non-separable digraph that belongs to the infinite family in Theorem 7.5. By replacing the edge $a_0'a_k \in E(D)$ by any digraph R such that every edge in $E(R)$ that is incident with a_k is an incoming edge of a_k, then $f(D)$ has $f(G)$ as a nonbasic GM-factor.* □

7.3 Computational Methods

All labelled rooted digraphs (without loops and multiple edges, but cycles of size two are allowed) up to order six were provided by Brendan McKay on 28 March 2018 (personal communication from McKay to Farr). Each digraph is given as a list of numbers on one line separated by a single space. The first number is the order of the digraph, the second number is the size of the digraph, and each pair of subsequent numbers represent a directed edge of the digraph. For instance, 3 2 2 0 2 1 represents a digraph of order 3 and size 2. The directed edges of the digraph are $(2, 0)$ and $(2, 1)$. Details are as follows:

$$\overbrace{3}^{\text{order}}\ \underbrace{2}_{\text{size}}\ \overbrace{2\ 0}^{\text{edge}}\ \underbrace{2\ 1}_{\text{edge}}.$$

We use a set of numbers $\{0, 1, \ldots, n-1\}$ to represent vertices for each digraph of order n, and an edge list to represent the edge set of each digraph, e.g., $[[0, 1]]$ represents a digraph with a single edge directed from vertex 0 to vertex 1. As there are 9,245,664 labelled rooted digraphs of order six, we split these digraphs into 52 files.

We use Python 3 (source code filename extension *.py*), Wolfram Mathematica 11 (source code filename extension *.nb*) and Bash Shell (Mac OS Version 10.13.4), in computing results for greedoid polynomials of rooted digraphs up to order six.

Algorithms of our programs are given in Appendix A. For brevity, we omit some elementary algorithms. Steps in obtaining our results are also summarised in Appendix A.

CHAPTER 8

Conclusions and Future Work

Several Tutte-like polynomials for directed graphs have been defined over many years. We focused on two of these polynomials in this thesis, namely Tutte invariants for alternating dimaps and two-variable greedoid polynomials for rooted digraphs. Research is an endless process. We now conclude our findings and suggest some open problems for future research, in Sections 8.1, 8.2 and 8.3.

We first have the following conjecture for alternating dimaps. This conjecture is inspired by the *Robertson-Seymour Theorem* [20, 70] (or graph minor theorem), which states that for every minor-closed class of graphs, there exists a finite set of forbidden minors.

Conjecture 8.1 (**Farr**). *For every infinite sequence of alternating dimaps G_k, $k \in \mathbb{Z}^+$, there exist i, j such that $i < j$ and G_i is a minor of G_j.*

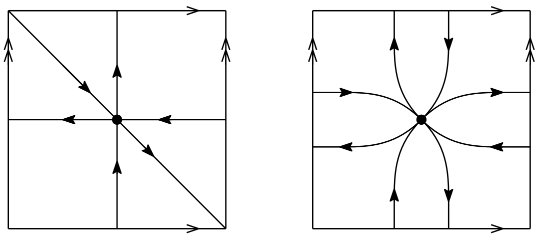
8.1 Characterisations of Extended Tutte Invariants

We described the relationship between extended Tutte invariants for alternating dimaps and their trials. We gave a full characterisation of extended Tutte invariants for all alternating dimaps of genus zero. With these characterisations, extended Tutte invariants of alternating dimaps can be obtained without performing reduction operations. We also showed that the extended Tutte invariant is well defined for an alternating dimap of genus zero if and only if the alternating dimap contains only triloops. We established some excluded minor characterisations of alternating dimaps of genus zero when their Tutte invariants are well defined.

Research in this area can be extended by investigating the situation where two alternating dimaps have the same extended Tutte invariant.

Problem 8.2. *Does there exist an efficient method to determine if two alternating dimaps have the same extended Tutte invariant, without performing reduction operations?*

Farr [37] has suggested to investigate extended Tutte invariants for ordered alternating dimaps. We could ask a similar question.



(a) (b)

Figure 8.1: (a) A 1-posy, (b) A minor of $\mathrm{alt}_c(K_4)$

Problem 8.3. *Does there exist an efficient method to determine if two ordered alternating dimaps have the same extended Tutte invariant, without performing reduction operations?*

Recall that in Chapter 5, we only discussed extended Tutte invariants for alternating dimaps that are embedded on an orientable surface that has genus zero. The reason is, we believe that more edge types may need to be defined if an alternating dimap is embedded on an orientable surface that has genus greater than zero. For instance, consider a k-posy for $k \geq 1$ (see [37] for more details), and a minor of $\mathrm{alt}_c(K_4)$ that is embedded on a torus (see Figure 8.1). (The minor of $\mathrm{alt}_c(K_4)$ that is shown in Figure 8.1 can be obtained by performing four 1-reductions and four ω-reductions on the edges of $\mathrm{alt}_c(K_4)$.) These alternating dimaps each have edges that are proper μ-semiloops for two or three distinct μ, a situation that cannot occur in the plane and which increases the ways in which a Tutte invariant may fail to be well defined.

Problem 8.4. *How do we properly define extended Tutte invariants for alternating dimaps that are embedded on an orientable surface with genus greater than zero?*

8.2 Tutte Invariants and the Tutte polynomial

The c-Tutte invariant for alternating dimaps is a special type of extended Tutte invariant involving two variables x and y, which is similar to the Tutte polynomial. We showed that the c-Tutte invariant is well defined for all alternating dimaps of genus zero if and only if $x = (1 \pm \sqrt{3}i)/2$ and $y = (1 \mp \sqrt{3}i)/2$. We proved that the c-Tutte invariant is multiplicative over non-loop blocks and c-multiloops for c-alternating dimaps. We then showed that the c-Tutte invariant is well defined for an alternating dimap if and only if the alternating dimap is a c-alternating dimap. We also extended the relationship between the Tutte polynomial and the c-Tutte invariant.

Since triality plays an important role for alternating dimaps, we could investigate the following problem.

Problem 8.5. *What is the relationship between the c-Tutte invariant of c-alternating dimaps and its two trials?*

Knowing that the c-Tutte invariant is an analogue of the Tutte polynomial under certain circumstances, a natural question that arises is:

Problem 8.6. *Does the c-Tutte invariant yield another option to compute the Tutte polynomial for abstract planar graphs?*

Since there are three equivalent ways to define the Tutte polynomial, we could investigate if there exist some equivalent definitions for the c-Tutte invariant.

Problem 8.7. *Can the c-Tutte invariant be defined by using a state sum expansion, or some notion of basis activities?*

Problem 8.8. *Is the c-Tutte invariant well defined for some alternating dimaps that are embedded on an orientable surface with genus greater than zero? If so, can we characterise them?*

Iain Moffatt (Royal Holloway) has also suggested the following problem in this area.

Problem 8.9. *Can we evaluate the c-Tutte invariant in terms of the ribbon graph polynomial of Bollobás and Riordan (cf. Theorem 6.1)?*

8.3 Factorisation of Greedoid Polynomials of Rooted Digraphs

We presented (i) the results from exhaustive computation of all small rooted digraphs and (ii) the first results of the GM-factorability of greedoid polynomials of rooted digraphs. We computed the greedoid polynomials for all rooted digraphs up to order six. We found that the multiplicative direct sum property, and the existence of greedoid loops and directed cycles, are not the only characteristics that determine if greedoid polynomials of rooted digraphs factorise. We showed that there exists an infinite family of non-separable digraphs where their greedoid polynomials GM-factorise. We also characterised the greedoid polynomials of rooted digraphs that belong to the family.

From Table 7.2, the ratio of PU to T-ISO show a decreasing trend. We expect that as n increases, this ratio continues to decrease. Hence, we have the following conjecture.

Conjecture 8.10. *Most rooted digraphs are not determined by their greedoid polynomials.*

From Tables 7.2 and 7.5, we can see that the ratio of PF to PU shows an upward trend and the ratio equals 0.8299 when $n = 5$.

Problem 8.11. *What is the limiting proportion of greedoid polynomials of rooted digraphs that GM-factorise as $n \to \infty$?*

Other potential problems in this area are as follows.

Problem 8.12. *Investigate the factorability of greedoid polynomials of rooted graphs, or even greedoids in general.*

Gordon and McMahon gave a graph-theoretic interpretation for the highest power of $1 + z$ for greedoid polynomials of rooted digraphs. We could investigate a similar problem for the other basic factor $1 + t$.

Problem 8.13. *Does there exist a graph-theoretic interpretation for the highest power of* $1+t$ *for greedoid polynomials of rooted digraphs?*

We found that the number of nonbasic factors and primary factors that exist in the greedoid polynomials of rooted digraphs up to order six is at most two and one, respectively. By Theorem 7.5, we can see that there exist (totally) k-nonbasic rooted digraphs for $k \geq 3$.

Problem 8.14. *For* $k \geq 2$*, does there exist a (totally)* k*-primary rooted digraph?*

For rooted digraphs of order six, there are 15 totally 2-nonbasic digraphs and nine totally 1-primary digraphs.

Problem 8.15. *For* $k \geq 1$*, can we characterise greedoid polynomials of totally* $(k+1)$*-nonbasic digraphs and totally* k*-primary digraphs?*

Since greedoid polynomials of rooted digraphs factorise under several scenarios, we could also investigate the following problem.

Problem 8.16. *Determine necessary and sufficient conditions for greedoid polynomials of rooted digraphs to factorise.*

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APPENDIX A

Commands and Algorithms

We summarised commands and algorithms of our programs in Chapter 7, as follows:

Part A

The relationships between files and programs in Part A is shown in Figure A.1.

1. Program name: *Greedoid_polynomial.py* (see Algorithms 1–8)
   Input: *dig_n.txt* that contains all digraphs of order n.
   Output:

   - *dig_n_edgeList.txt*: contains edge lists for each rooted digraph of order n.
   - *dig_n_poly.txt*: contains greedoid polynomials (not in their simplest form) for each rooted digraph of order n which are obtained by using the deletion-contraction recurrence in Proposition 4.8.
   - *dig_n_isomorphism.txt*: contains rooted digraphs of order n together with a set of root verticess such that each digraph that has its root vertex in the set is isomorphic to each other.
   - *dig_n_directSum.txt*: summarises whether each rooted digraph of order n is a direct sum (DS), not a direct sum (NDS), or is isomorphic (ISO) to some other rooted digraphs.
   - *dig_n_info.txt*: contains the number of rooted digraphs of order n that are direct sums, and the number of rooted digraphs of order n that need to be excluded so that all rooted digraphs of order n are non-isomorphic.

2. Program name: *dig_n_factorise.nb*
   Input: *dig_n_poly.txt*.
   Output: *dig_n_poly_factorised.txt* that contains the greedoid polynomials for rooted digraphs of order n in their factorised forms.

3. Program name: *Numbering_edgeList.py*
   Input: *dig_n_edgeList.txt*.

Output: *dig_n_edgeList_Numbering.txt* that includes the following numbering scheme for each line in the input file

n.z) edgeList

where n is the order of the digraph and $z \geq 1$.

4. Program name: *Numbering_block_poly_factorised.py*
   Input: *dig_n_poly_factorised.txt*.
   Output: *dig_n_poly_factorised_Numbering.txt* that includes the following numbering scheme for each line in the input file

   n.z.r) poly_factorised

   where n is the order of the digraph, $z \geq 1$ corresponds to the z^{th} rooted digraph in the *dig_n_edgeList_Numbering.txt*, and $0 \leq r \leq n-1$ represents the root vertex of the digraph.

5. Program name: *Numbering_block_directSum.py*
   Input: *dig_n_directSum.txt*.
   Output: *dig_n_directSum_Numbering.txt* that includes the following numbering scheme for each line in the input file:

   n.z.r) DS/NDS/ISO

   where n is the order of the digraph, $z \geq 1$ corresponds to the z^{th} rooted digraph in the *dig_n_edgeList_Numbering.txt*, and $0 \leq r \leq n-1$ represents the root vertex of the digraph.

6. Program name: *DirectSum_vs_notDirectSum.py* (see Algorithm 9)
   Input: *dig_n_directSum_Numbering.txt* and *dig_n_poly_factorised_Numbering.txt*.
   Output:

   - *dig_n_poly_directSum.txt*: contains greedoid polynomials for rooted digraphs of order n that are direct sums, with the respective numbering.
   - *dig_n_poly_notDirectSum.txt*: contains greedoid polynomials for rooted digraphs of order n that are not direct sums, with the respective numbering.

7. Program name: *DirectSum_vs_notDirectSum_unique.py* (similar to Algorithm 9)
   Input: *dig_n_directSum.txt* and *dig_n_poly_factorised.txt*.
   Output:

   - *dig_n_poly_directSum_unique.txt*: contains unique greedoid polynomials for rooted digraphs of order n that are direct sums.
   - *dig_n_poly_notDirectSum_unique.txt*: contains unique greedoid polynomials for rooted digraphs of order n that are not direct sums.

8. Language: Bash Shell.
   Input: *dig_n_poly_factorised.txt*.
   Output: *dig_n_unique_poly.txt* that contains all unique greedoid polynomials of rooted digraphs of order n.
   Command:

```
tr -d "\r" < dig_n_poly_factorised.txt | sort | uniq
> dig_n_unique_poly.txt
```

Remark: If we replace *uniq* by *uniq -c* in the above command, the last line of *dig_n_unique_poly.txt* gives the number of occurrences of "isomorphic", which is also the number of rooted digraphs of order n that need to be excluded so that all rooted digraphs of order n are non-isomorphic. This number should match with the one in *dig_n_info.txt*.

9. Language: Bash Shell.
   Input: *Combined_unique_poly_n-1.txt* and *dig_n_unique_poly.txt*.
   Output: *Combined_unique_poly_n.txt* that contains all unique greedoid polynomials of rooted digraphs up to order n.
   Command:
   ```
   cat Combined_unique_poly_n-1.txt dig_n_unique_poly.txt | sort | uniq
   > Combined_unique_poly_n.txt
   ```

   Remark: Since *Combined_unique_poly_1.txt* is literally *dig_1_unique_poly.txt*, we first use *dig_1_unique_poly.txt* and *dig_2_unique_poly.txt* as input files to obtain *Combined_unique_poly_2.txt*. For brevity, we only show *dig_n_unique_poly.txt* as the input in Figure A.1. A similar concept is used for both steps 13 and 18.

10. Program name: *Factorability_unique.py* (see Algorithm 10)
    Input: *dig_n_unique_poly.txt* and *Combined_unique_poly_n.txt*.
    Output: *dig_n_factorability_unique.txt* that contains the number of greedoid polynomials of rooted digraphs of order n that can be GM-factorised, and output each of these polynomials.

11. Program name: *Factorability_unique_directSum.py* (similar to Algorithm 10)
    Input: *dig_n_poly_directSum_unique.txt* and *Combined_unique_poly_n.txt*.
    Output: *dig_n_factorability_directSum_unique.txt* that contains the number of greedoid polynomials that can be GM-factorised for rooted digraphs of order n that are direct sums, and output each of these polynomials.

12. Program name: *Factorability_unique_notDirectSum.py* (similar to Algorithm 10)
    Input: *dig_n_poly_notDirectSum_unique.txt* and *Combined_unique_poly_n.txt*.
    Output: *dig_n_factorability_notDirectSum_unique.txt* that contains the number of greedoid polynomials that can be GM-factorised for rooted digraphs of order n that are not direct sums, and output each of these polynomials.

13. Language: Bash Shell.
    Input: *Combined_poly_directSum_n-1.txt* and *dig_n_poly_directSum.txt*.
    Output: *Combined_poly_directSum_n.txt* that contains all unique greedoid polynomials of rooted digraphs that are direct sums up to order n.
    Command:

```
cat Combined_poly_directSum_n-1.txt dig_n_poly_directSum.txt | sort
| uniq > Combined_poly_directSum_n.txt
```

14. Program name: *DirectSum_and_GM-equivalent.py* (see Algorithm 11)
    Input: *dig_n_poly_notDirectSum.txt* and *Combined_poly_directSum_n.txt*.
    Output: *dig_n_poly_ndsEquivalent.txt* that contains the number of rooted digraphs of order n that are not direct sums, but they are GM-equivalent to some rooted digraph up to order n that is a direct sum. Each such polynomial will be printed out without duplicates in the output file.

15. Language: Bash Shell.
    Input: *dig_n_poly_factorised.txt*.
    Output: *dig_n_nonbasic.txt* that contains all unique polynomials that have at least two nonbasic GM-factors (these nonbasic GM-factors might be identical) for rooted digraphs of order n.
    Command:
```
sed -e ‘s/(1 + t)[^(]*//; s/(1 + z)[^(]*//’ dig_n_poly_factorised.txt
| sort | uniq | grep ")\*(\|)\^\d\+\|([^+]* + [^+]*)"
> dig_n_nonbasic.txt
```

16. Language: Bash Shell.
    Input: *dig_n_nonbasic.txt*.
    Output: *dig_n_nonbasic_split.txt* that contains all unique nonbasic GM-factors that are split into separate lines for rooted digraphs of order n.
    Command:
```
sed -e ‘s/\*(/‘$’\n/g; s/(//; s/).*//’ dig_n_nonbasic.txt
| tr -d "\r" | sort | uniq > dig_n_nonbasic_split.txt
```

17. Language: Bash Shell.
    Input: *dig_n_poly_factorised.txt*.
    Output: *dig_n_factors.txt* that contains all unique GM-factors for greedoid polynomials of rooted digraphs of order n.
    Command:
```
sed -e ‘s/\*(/‘$’\n/g; /isomorphic/d; s/(//; s/).*//’
dig_n_poly_factorised.txt | tr -d "\r" | sort | uniq
> dig_n_factors.txt
```

18. Language: Bash Shell.
    Input: *all_factors_up_to_order_n-1.txt* and *dig_n_factors.txt*.
    Output: *all_factors_up_to_order_n.txt* that contains all unique GM-factors for greedoid polynomials of rooted digraphs up to order n.
    Command:
```
cat all_factors_up_to_order_n-1.txt dig_n_factors.txt | sort | uniq
> all_factors_up_to_order_n.txt
```

19. Language: Bash Shell.
    Input: *dig_n_nonbasic_split.txt* and *all_factors_up_to_order_n-1.txt*.
    Output: *dig_n_primary.txt* that contains all primary GM-factors for greedoid polynomials of rooted digraphs of order n.
    Command:
    ```
    comm -23 dig_n_nonbasic_split.txt all_factors_up_to_order_n-1
    > dig_n_primary.txt
    ```

20. Language: Bash Shell.
    Input: *dig_n_primary.txt* and *dig_n_poly_factorised_Numbering.txt*.
    Output: *dig_n_primaryPoly.txt* that contains all greedoid polynomials that have primary GM-factors for rooted digraphs of order n.
    Command:
    ```
    fgrep -f dig_n_primary.txt dig_n_poly_factorised_Numbering.txt
    > dig_n_primaryPoly.txt
    ```

21. Language: Bash Shell.
    Input: *dig_n_primaryPoly.txt*.
    Output: *dig_n_primaryNumbers_edgeList.txt* that contains the (edge list) numberings for all rooted digraphs of order n that have primary GM-factors in their greedoid polynomials.
    Command:
    ```
    awk '{print "^"$1}' dig_n_primaryPoly.txt | sed '/s/...$/)/'
    > dig_n_primaryNumbers_edgeList.txt
    ```

22. Language: Bash Shell.
    Input: *dig_n_primaryPoly.txt*.
    Output: *dig_n_primaryNumbers.txt* that contains the numberings for all greedoid polynomials that have primary GM-factors for rooted digraphs of order n.
    Command:
    ```
    awk '{print "^"$1}' dig_n_primaryPoly.txt > dig_n_primaryNumbers.txt
    ```

23. Language: Bash Shell.
    Input: *dig_n_primaryNumbers_edgeList.txt* and *dig_n_edgeList_Numbering.txt*
    Output: *dig_n_primaryGraphs.txt* that contains all edge lists for greedoid polynomials that have primary GM-factors for rooted digraphs of order n.
    Command:
    ```
    grep -f dig_n_primaryNumbers_edgeList.txt dig_n_edgeList_Numbering.txt
    > dig_n_primaryGraphs.txt
    ```

24. Language: Bash Shell.
    Input: *dig_n_primaryNumbers.txt* and *dig_n_directSum_Numbering.txt*
    Output: *dig_n_primaryVSdirectSum.txt* summarises whether each rooted digraph in *dig_n_primaryNumbers.txt* a direct sum or not a direct sum.
    Command:

```
grep -f dig_n_primaryNumbers.txt dig_n_directSum_Numbering.txt
> dig_n_primaryVSdirectSum.txt
```

25. Language: Bash Shell.
    Input: *dig_n_primaryVSdirectSum.txt*
    Output: *dig_n_primaryVSdirectSum_summary.txt* that contains rooted digraphs that are direct sums in *dig_n_primaryVSdirectSum.txt*.
    Command:

```
grep "\tDS" dig_n_primaryVSdirectSum.txt
> dig_n_primaryVSdirectSum_summary.txt
```

Part B

From Part A, we know that each rooted digraph (up to order six) that has $k \geq 1$ primary GM-factor in its greedoid polynomial is not a direct sum. We can now compute the number of greedoid polynomials of these digraphs that are not divisible by $1 + t$ or $1 + z$, which is also the number of totally k-primary digraphs.

26. Language: Bash Shell.
    Input: *dig_n_primaryPoly.txt*
    Output: *dig_n_totally_primaryPoly.txt* that contains greedoid polynomials that have primary GM-factors and are not divisible by $1 + t$ or $1 + z$ for rooted digraphs of order n that are not direct sums.
    Command:

```
grep -v "(1 + t)\|(1 + z)" dig_n_primaryPoly.txt
> dig_n_totally_primaryPoly.txt
```

By using a similar method, we compute the number of greedoid polynomials that contain at least two nonbasic GM-factors and are not divisible by $1+t$ or $1+z$, for rooted digraphs up to order six that are not direct sums.

27. Language: Bash Shell.
    Input: *dig_n_nonbasic.txt*, *dig_n_poly_factorised_Numbering.txt* and *dig_n_directSum_Numbering.txt*
    Output: *dig_n_totally_nonbasicPoly.txt* that contains greedoid polynomials that contain at least two nonbasic GM-factors and are not divisible by $1 + t$ or $1 + z$ for rooted digraphs of order n that are not direct sums.
    Command:

```
fgrep -f dig_n_nonbasic.txt dig_n_poly_factorised_Numbering.txt |
grep -v "(1 + t)\|(1 + z)" | awk '{print "^"$1}' | grep -f /dev/stdin
dig_n_directSum_Numbering.txt | grep "\tNDS"
> dig_n_totally_nonbasicPoly.txt
```

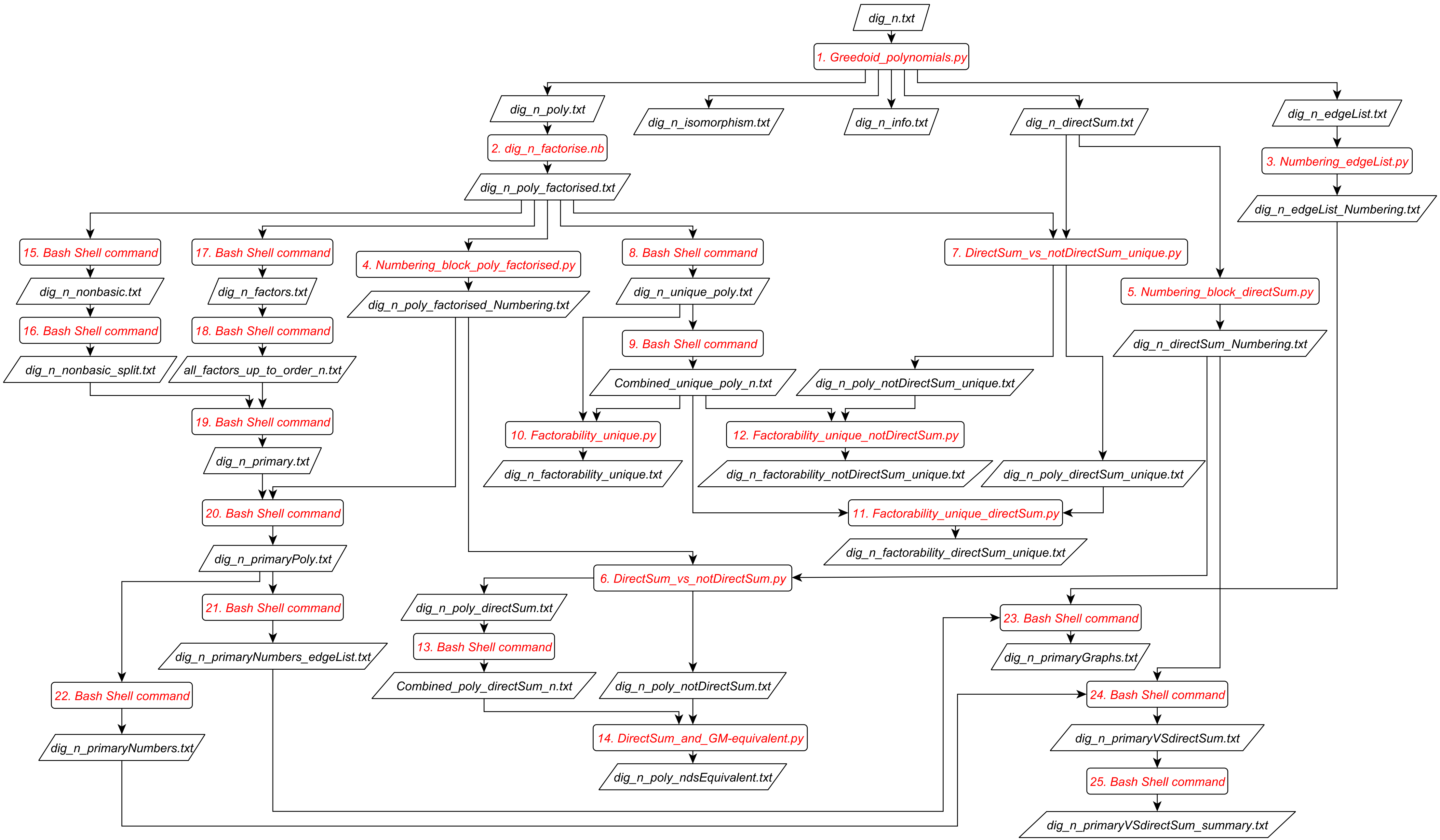

Figure A.1: Relationships between files and programs

Algorithm 1 GreedoidPolynomial

```
Input: dig_n.txt
Output: dig_n_edgeList.txt, dig_n_poly.txt, dig_n_isomorphism.txt, dig_n_info.txt and dig_n_directSum.txt
digraphFile ← open the input file
edgeListTable ← create an empty list
for line in digraphFile do
    order ← first number of line
    aList ← create a list that excludes the first two numbers in line
    edgeList ← create a list of lists that has ⌊aList/2⌋ empty lists
    for i in ⌊aList/2⌋ do
        j ← 2i
        Append aList[j] followed by aList[j + 1] to edgeList[i]
    Append edgeList to edgeListTable
Create output files: dig_n_edgeList.txt, dig_n_poly.txt, dig_n_isomorphism.txt, dig_n_info.txt and dig_n_directSum.txt
vertexList ← generate a vertex list numbered from 0 to order − 1
a ← 0
k ← 0
for item in edgeListTable do
    Write item to dig_n_edgeList.txt
    exclude ← create an empty list
    isomorphicTable ← IsomorphismTest(item,vertexList)
    if isomorphicTable is not empty then
        for element in isomorphicTable do
            ℓ ← length of element
            k ← k + ℓ − 1
            Write item and element to dig_n_isomorphism.txt
            for node in element do
                Append each node except the first to exclude
    cutVertexList ← CutVertices(item)
    for node in exclude do
        if node in cutVertexList then
            Remove node from cutVertexList
    c ← length of cutVertexList
    a ← a + c
    for vertex in vertexList do
        if vertex in cutVertexList then
            Write 'DS' to dig_n_directSum.txt
        else if vertex in exclude then
            Write 'ISO' to dig_n_directSum.txt
        else
            Write 'NDS' to dig_n_directSum.txt
    for vertex in vertexList do
        if vertex in exclude then
            Write isomorphic to dig_n_poly.txt
        else
            Write DeletionContraction(vertexList,item,vertex) to dig_n_poly.txt
Write both a and k to dig_n_info.txt
```

Algorithm 2 IsomorphismTest(edgeList,vertexList)

Input: An edge list and a vertex list of a digraph

Output: Return a table where each list in the table contains vertices of the digraph in which, when vertices in the list are assigned as the root vertex of the digraph, these rooted digraphs are isomorphic to each other

```
rootList_table ← create an empty list
isomorphic ← create an empty list
checked ← create an empty list
for v1 in vertexList do
    if v1 is not in checked then
        Append v1 to checked
        rootList ← create a list that contains v1
        vertexList_new ← create a list that excludes the first element up to v1 in vertexList
        for v2 in vertexList_new do
            if Isomorphism(edgeList,v1,v2) is True then
                Append v2 to both rootList and checked
        if length of rootList > 1 then
            Append rootList to rootList_table
            temp ← 0
            for element in rootList_table do
                if rootList is a subset of element then
                    temp ← temp + 1
            if temp = 1 then
                Append rootList to isomorphic
return isomorphic
```

Algorithm 3 Isomorphism(edgeList,$r1$,$r2$)

Input: An edge list and two vertices of a digraph

Output: Return *True* if the digraph rooted at $r1$ is isomorphic to the digraph rooted at $r2$, *False* otherwise

```
Import NetworkX package [44]
G1 ← append a loop incident with r1 in the digraph
G2 ← append a loop incident with r2 in the digraph
return nx.is_isomorphic(G1,G2)
```

Algorithm 4 CutVertices(edgeList)

Input: An edge list of a digraph

Output: Return a list of cutvertices of the digraph

```
Import NetworkX package [44]
G ← create an undirected multigraph using edgeList
return nx.articulation_points(G)
```

Algorithm 5 DeletionContraction(vertexList,edgeList,root)

Input: A vertex list, an edge list and a root vertex of a digraph
Output: Return the greedoid polynomial of the digraph

if length of *edgeList* $= 0$ **then**
  return 1
else if Outdegree(*edgeList*,*root*) $= 0$ **then**
  $r \leftarrow$ length of *edgeList*
  return $(1+z)^r$
else
  edgeList_del $\leftarrow$ create a copy of *edgeList*
  edgeList_con $\leftarrow$ create a copy of *edgeList*
  vertexList_con $\leftarrow$ create a copy of *vertexList*
  feasbile $\leftarrow$ FeasibleSet_SizeOne(*edgeList*,*root*)
  randomEdge $\leftarrow$ choose a random edge from *feasbile*
  edgeList_del $\leftarrow$ remove *randomEdge* from *edgeList_del*
  contractedGraph $\leftarrow$ contract *randomEdge* in *edgeList_con*
  edgeList_con $\leftarrow$ edge list of *contractedGraph*
  vertexList_con $\leftarrow$ vertex list of *contractedGraph*
  rank_ori $\leftarrow$ RankFunction(*vertexList*,*edgeList*,*root*)
  rank_del $\leftarrow$ RankFunction(*vertexList*,*edgeList_del*,*root*)
  $k \leftarrow$ *rank_ori* $-$ *rank_del*
  $d \leftarrow$ DeletionContraction(*vertexList*,*edgeList_del*,*root*)
  $c \leftarrow$ DeletionContraction(*vertexList_con*,*edgeList_con*,*root*)
  return $d * t^k + c$

Algorithm 6 Outdegree(edgeList,root)

Input: An edge list and a root vertex of a digraph
Output: Return the outdegree of the root vertex (loops are excluded)

outdegree $\leftarrow 0$
for *edge* in *edgeList* **do**
  if the initial vertex of *edge* is *root* and the endvertex of *edge* is not *root* **then**
    outdegree $\leftarrow$ *outdegree* $+ 1$
return *outdegree*

Algorithm 7 FeasibleSet_SizeOne(edgeList,root)

Input: An edge list and a root vertex of a digraph
Output: Return the feasible set of size one of the digraph

feasible $\leftarrow$ create an empty list
for *edge* in *edgeList* **do**
  if the initial vertex of *edge* is *root* and the endvertex of *edge* is not *root* **then**
    Append *edge* to *feasible*
return *feasible*

Algorithm 8 RankFunction(vertexList,edgeList,root)

Input: A vertex list, an edge list and a root vertex of a digraph
Output: Return the rank of the digraph

vertexList_new ← create a copy of *vertexList*
edgeList_new ← create a copy of *edgeList*
Remove *root* from *vertexList_new*
rootList ← create a list that contains *root*
k ← length of *edgeList*
for *root* in *rootList* **do**
    for *edge* in *edgeList* **do**
        if the initial vertex of *edge* is *root* and the endvertex of *edge* is in *vertexList_new* **then**
            Append the endvertex of *edge* to *rootList*
            Remove the endvertex of *edge* from *vertexList_new*
            Remove *edge* from *edgeList_new*
ℓ ← length of *edgeList_new*
return $k - \ell$

Algorithm 9 DirectSum_vs_NotDirectSum

Input: *dig_n_directSum_Numbering.txt* and *dig_n_poly_factorised_Numbering.txt*
Output: *dig_n_poly_directSum.txt* and *dig_n_poly_notDirectSum.txt*

dsFile ← open *dig_n_directSum_Numbering.txt*
polyFile ← open *dig_n_poly_factorised_Numbering.txt*
dsPolyFile ← create an output file *dig_n_poly_directSum.txt*
ndsPolyFile ← create an output file *dig_n_poly_notDirectSum.txt*
$k \leftarrow 1$
while $k \leq$ number of lines in *dsFile* **do**
    if k^{th} line in *dsFile* contains *'DS'* **then**
        Write the k^{th} line in *polyList* to *dsPolyFile*
    else if k^{th} line in *dsFile* contains *'NDS'* **then**
        Write the k^{th} line in *polyList* to *ndsPolyFile*
        $k \leftarrow k + 1$

Algorithm 10 Factorability_Unique

Input: *dig_n_unique_poly.txt* and *Combined_unique_poly_n.txt*
Output: *dig_n_factorability_unique.txt*

polyFile ← open *dig_n_unique_poly.txt*
combinedFile ← open *Combined_unique_poly_n.txt*
factorabilityFile ← create an output file *dig_n_factorability_unique.txt*
$k \leftarrow 0$
for *oriPoly* that has more than one factor in *polyFile* **do**
    for *poly1* in *combinedFile* **do**
        for *poly2* in *combinedFile* that excludes the first element up to the element before *poly1* **do**
            if $poly1 * poly2 = oriPoly$ **then**
                Write *oriPoly* to *factorabilityFile*
                $k \leftarrow k + 1$
                Break and move to the next element in *polyFile*
Write k to *factorabilityFile*

Algorithm 11 DirectSum_and_GM-equivalent

Input: *dig_n_poly_notDirectSum.txt* and *Combined_poly_directSum_n.txt*
Output: *dig_n_poly_ndsEquivalent.txt*

```
ndsPolyFile ← open dig_n_poly_notDirectSum.txt
combinedDsPolyFile ← open Combined_poly_directSum_n.txt
equivalentFile ← create an output file dig_n_poly_ndsEquivalent.txt
f ← 0
for ndsPoly in ndsPolyFile do
    for dsPoly in combinedDsPolyFile do
        if second column of ndsPoly = second column of dsPoly then
            Write the second column of ndsPoly to equivalentFile
            f ← f + 1
            Break and move to the next element in ndsPolyFile
Write f to equivalentFile
```