Transversal This, Transversal That

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Abstract

The study of transversals in Latin squares goes back more than two centuries when Euler used them to study mutually orthogonal Latin squares. Today, transversals are an incredibly interesting topic in their own right. A transversal of a Latin square is a selection of n entries so that each row, column and symbol is represented exactly once. Two famous conjectures of Ryser and Brualdi lie in the forefront of much of the study of transversals. In this thesis, we explore different avenues of approaching these problems by studying transversals in Latin squares and other Latin-like objects.

In 1990, Balasubramanian showed that every Latin square of even order contains an even number of transversals. Here, we extend this result to show that if the order of a Latin square is singly-even, then the number of transversals is necessarily a multiple of four. Between Balasubramanian's result and ours, we believe that these are the only modular restrictions on the number of transversals in a Latin square.

The permanent of a matrix plays a vital role in our results. We study the permanent of several classes of matrices and show relationships between the permanent of a matrix, its $\mathbb{Z}_p$-rank and its regularity. We also study a set of matrices that come from Latin squares and examine some underlying patterns in the permanent of these matrices.

We then shift our focus to a more generalised setting and explore transversals from two different avenues. First, we remove the restriction on the underlying squares so that they may contain an arbitrary number of symbols and ask how many symbols are required in the array before a transversal is guaranteed. Here, we provide the first non-trivial bound on this value. Second, we introduce the notion of covers in Latin squares where our goal is to minimise the number of entries needed to represent every row, column and symbol at least once rather than necessarily representing each exactly once. We compare and contrast covers and partial transversals as well as show that every Latin square contains a large minimal cover.

Combinatorics is a prime area for computational work and here, we give three separate computational results. First, we explore a generalisation of Brualdi's Conjecture where we show that every Latin array of order at most 11 contains a near transversal. Next, we search for a non-Desarguesian projective plane of order 11 and show that if any such plane exists, none of its underlying Latin squares may have high symmetry. Finally, we enumerate all transversal-free Latin arrays of small order.

Declaration

This thesis contains no material which has been accepted for the award of any other degree or diploma at any university or equivalent institution and that, to the best of my knowledge and belief, this thesis contains no material previously published or written by another person, except where due reference is made in the text of the thesis.

[redacted]

Darcy Best
27 March 2018

Publications During Enrolment

This thesis is a combination of several pieces of work (published, submitted and in preparation). Each of these works is joint work with various authors. In cases where I was not directly involved with a portion of an article, I have not included that part in this thesis.

- Chapter 3 is joint work with Saieed Akbari and Ian Wanless. A paper including content from this chapter is currently in preparation.
- Chapter 4 is joint work with Ian Wanless. A paper including content from this chapter is also currently in preparation.
- Chapter 5 is based on a part of a publication [9]. The entire paper was joint work with Kevin Hendrey, Ian Wanless, Tim Wilson and David Wood. However, the majority of the material presented in this chapter is based off joint work with Kevin Hendrey.
- Chapter 6 is based on a publication [10]. This work is joint work with Trent Marbach, Rebecca Stones and Ian Wanless.
- Chapter 7 contains computation that spans a few of the above publications. Each of these is joint work with Ian Wanless.

I am extremely appreciative of each of my collaborators.

To all those who pretended to
know what I was talking about.
And to the few that actually understood.

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Unfortunately, there are just too many to thank on this page, so I will use their initials here and leave it as an exercise to the reader to figure out who they are.

As always, much love goes to my family:

BB, TB, AS, BS, HS, MS

I am extremely indebted to several people that have helped me throughout this crazy journey—whether that help came in the form of working on research, day long chats about hockey or several late night chats about nothing and everything. Inevitably, I will miss someone here, and to you, I apologise.

CB, BC, KD, MG, wG, JH, KH, SH, TH, SJ, DL, AP, TW

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Contents

Chapter 1

Introduction

> *'Obvious' is the most dangerous word in mathematics.*
> – E. T. Bell

This thesis contains a combination of several novel ideas studied over the past four years. The bulk of the study centres around the idea of transversals in Latin squares and Latin-like objects: showing the existence of, the number of, or the structure of transversals in different cases. The work is a combination of published work [9], submitted work [10], and forthcoming publications.

The mathematical study of Latin squares finds its roots in the late eighteenth century when Euler [34] posed the famous thirty-six officer problem (translation from [33]):

> "A very curious question that has taxed the brains of many (has) inspired me to undertake the following research that has seemed to open a new path in Analysis and in particular in the area of combinatorics. This question concerns a group of thirty-six officers of six different ranks, taken from six different regiments, and arranged in a square in a way such that in each row and column there are six officers, each of a different rank and regiment."

Euler conjectured that there was no solution to this problem ("But after spending much effort to resolve this problem, we must acknowledge that such an arrangement is absolutely impossible, though we cannot give a rigorous proof."). He went on to conjecture that a solution to this problem does not exist even if you replaced "six" in the problem with any value of $n \equiv 2 \pmod 4$. The case of $n = 2$ is trivially true and more than a century later, Tarry [79] proved the conjecture to be true for $n = 6$. However, Bose, Shrikhande and Parker [13] later showed this conjecture is false in all other cases—namely that such a configuration exists for all $n \notin \{2, 6\}$.

The term *Latin square* comes from the fact that Euler used Latin letters (a, b, c, d, e, f) to represent the regiments of the officers and Greek letters $(\alpha, \beta, \gamma, \delta, \epsilon, \zeta)$ to represent the ranks, so by forgetting about the ranks, you have only Latin letters remaining in the square.

Euler introduced the notion of a *formule directrix*, a structure within a Latin square that he used to solve an infinite family of instances of the thirty-six officers problem. Today, we call these objects *transversals*. While transversals were useful in determining the lack of the existence of certain objects, their true usefulness came to the forefront with the advent of computers, which would allow one to utilise transversals to search for solutions to the generalised thirty-six officer problem by fixing one of the Latin squares, then using its transversals to find another suitable Latin square.

In 1967, Ryser [71] conjectured that every Latin square of odd order contains a transversal. Since that time, the study of transversals has taken off and has become a very interesting area of study in its own right without the direct connection to the thirty-six officer problem.

In this thesis, we will examine transversals not only in the context of Latin squares, but also other objects that are Latin-like.

We will begin our journey in Chapter 3 by studying properties of the permanent of specially structured matrices. These results will lead us into Chapter 4, where we will use the permanent results to extend an idea of Balasubramanian [7] to show that the number of transversals in Latin squares of order $n \equiv 2 \pmod 4$ is necessarily a multiple of 4. Along with this result, we will shed some light on some underlying patterns in the transversals of Latin squares of certain orders.

When transversals cannot be found in Latin squares, it is natural to wonder how close we can get to a transversal. A famous conjecture that has been attributed to Brualdi, Ryser and Stein [22, 31, 76] states that you can get extraordinarily close to a transversal no matter which Latin square you select. Chapters 5 and 6 will examine two different approaches to this question. First, in Chapter 5, we will loosen the restriction on the arrays in question and ask what the minimum number of symbols required in a Latin array is in order to guarantee the existence of a transversal. Here, we find the first non-trivial bound on this value. Then, in Chapter 6, we will introduce covers of Latin squares that, in some sense, resemble transversals. We will discuss some similarities and differences between covers and transversals.

Finally, in Chapter 7, we will finish off by showing three computational results. First, we will show that a general form of the famous conjecture mentioned above is true for $n \leqslant 11$. We then perform a search for projective planes in certain interesting orders and finish off by finding all Latin arrays of small orders that do not contain a transversal.

1.1 A Note on Notation

Throughout, we will use $\bullet$ in places where its value is irrelevant to our argument. For example, $(i, j, \bullet)$ is the entry in row i and column j, while $(\bullet, \bullet, k)$ is an arbitrary entry with symbol k.

Unless otherwise specified, we use $\mathbb{Z}_n$ as the symbol set and also use $\mathbb{Z}_n$ to index the rows and columns of our squares and matrices. Where convenient (such as when embedding a Latin square inside a larger one), we consider $\mathbb{Z}_n$ to be the set of integers $\{0, \dots, n-1\}$ rather than a set of congruence classes.

All matrices will be 0-based, so the top row of a matrix is row 0 and the bottom row is $n-1$.

Chapter 2

Background

Mathematics is the art of giving
the same name to different things.
– H. Poincaré

2.1 Latin Squares

We start our journey with a definition that lays the groundwork for the entire thesis.

Definition 2.1. A *Latin square* of order n is an $n \times n$ array of n symbols such that each symbol appears exactly once in each row and each column.

Here are three examples of Latin squares of various orders.

Example 2.2.

0	1	2	3
1	0	3	2
2	3	0	1
3	2	1	0

Order 4

0	1	2	3	4
1	2	3	4	0
2	3	4	0	1
3	4	0	1	2
4	0	1	2	3

Order 5

0	1	2	3	4	7	5	6
1	0	3	2	7	4	6	5
2	3	0	1	6	5	7	4
3	2	1	0	5	6	4	7
4	5	6	7	3	0	2	1
5	4	7	6	0	3	1	2
7	6	5	4	2	1	3	0
6	7	4	5	1	2	0	3

Order 8

We start with an easy result.

Proposition 2.3. *There exists a Latin square for every order n.*

Proof. Set the first row of the square to be $0, 1, \ldots, n-1$. Each of the other $n-1$ rows is simply the row above it shifted by 1 column to the left (and the leftmost column wrapping around to the rightmost column). $\square$

For $n = 5$, the Latin square of order 5 given in Example 2.2 is the square described in Proposition 2.3. Moreover, the construction in Proposition 2.3 is simply the Cayley table of the group $(\mathbb{Z}_n, +)$, which is a special case of the following.

Theorem 2.4. *The Cayley table of a finite group is a Latin square.*

Proof. This follows directly from the definition of a group. □

However, the converse of Theorem 2.4 is not true for most Latin squares. As is shown below, the number of Latin squares of order n is substantially larger than the number of groups of order n. For example, the Latin square of order 8 given in Example 2.2 is not the Cayley table of a group. To test if a Latin square is the Cayley table of a group, see [36, Chapter 2].

Instead of treating a Latin square as an $n \times n$ array, it is sometimes useful to consider a set of n^2 ordered triples, $L = \{(r, c, s)\} \subset \mathbb{Z}_n^3$, with the property that each pair of distinct triples in L agrees in at most one coordinate. Each triple in a Latin square is called an *entry* and consists of a row, column and symbol. By ignoring the symbol of a triple (r, c, s), we are left with an ordered pair (r, c) called a *cell*.

From a Latin square, L, we may create many other Latin squares. By independently permuting the rows, columns and symbols of L, we arrive at another Latin square. Two Latin squares, L and L', are *isotopic* if L can be transformed into L' using only these permutations. Furthermore, we may permute the role of the rows, columns and symbols in any of the $3! = 6$ ways. These permutations, called *conjugates*, are given a *type* based on the permutation. For example, the (123)-conjugate is just the square itself and the (213)-conjugate of L is simply the transpose of L since we are swapping the role of the rows and the columns. We say that L and L' are *paratopic* if any conjugate of L is isotopic to L'. Both isotopy and paratopy define equivalence classes of Latin squares. Determining the exact number of Latin squares is a computationally heavy task. The exact number of Latin squares up to equivalence is fully determined for $n \leqslant 11$ and can be found in Table 2.1 (see [51]). Note that the number of Latin squares is significantly more than the number of groups of order n.

Table 2.1: Number of Latin squares up to equivalence

n	Paratopy Classes	Isotopy Classes
1	1	1
2	1	1
3	1	1
4	2	2
5	2	2
6	12	22
7	147	564
8	283 657	1 676 267
9	19 270 853 541	115 618 721 533
10	34 817 397 894 749 939	208 904 371 354 363 006
11	2 036 029 552 582 883 134 196 099	12 216 177 315 369 229 261 482 540

There are other objects which are similar to Latin squares that we use throughout the thesis. A *row-Latin square* is an array where each symbol appears exactly once in each row, but no restrictions are placed on the number of times a symbol may appear in a column. A *column-Latin square* is defined similarly. A *Latin rectangle* is an $m \times n$ array where no symbol may appear more than once in any row or column, and the number of symbols is $\max(m, n)$. A *row-Latin rectangle* is an $m \times n$ array with n symbols where each symbol appears once in each row. A *column-Latin rectangle* is defined similarly. Any of the above items may be *generalised* by allowing an arbitrary number of symbols. When dealing with a *generalised Latin square* (resp., *generalised row-Latin square*), we use the term *Latin array* (resp., *row-Latin array*). Note that when clear from context, we may use Latin array for rectangles as well. An array can also be classified as *partial* if cells are allowed to be empty (meaning that they do not contain a symbol). Note that certain squares fall into several of these categories. For example, every Latin square is a special case of all of the above objects. When the exact type of array we are dealing with is not important, we use the term *Latin-like object*.

Example 2.5. Here are six examples of Latin-like objects: (1) a Latin square, (2) a row-Latin square, (3) a Latin rectangle, (4) a column-Latin rectangle, (5) a generalised Latin square and (6) a partial generalised Latin rectangle.

0	1	2	3
1	0	3	2
2	3	0	1
3	2	1	0

0	1	2	3
0	2	3	1
3	2	0	1
1	2	3	0

0	1	2	3
1	0	3	2
2	3	0	1

0	0	2	3
1	1	0	2
3	2	3	1

0	1	2	3
4	5	0	6
6	3	7	1
1	2	4	0

0		2	
4	0		1
3	2		0

2.2 Mutually Orthogonal Latin Squares

We now shift our attention to a generalisation of the thirty-six officer problem, by exploring the notion of orthogonality.

Definition 2.6. Two Latin squares, L_1 and L_2, are called *orthogonal* if the superimposition of L_1 onto L_2 provides n^2 distinct ordered pairs.

In the context of the thirty-six officer problem, the regiments represent one Latin square while the ranks represent the other Latin square. Here is an example of a pair of orthogonal Latin squares.

Example 2.7.

0	1	2	3
1	0	3	2
2	3	0	1
3	2	1	0

0	1	2	3
2	3	0	1
3	2	1	0
1	0	3	2

are orthogonal Latin squares since each of the 4^2 ordered pairs appears exactly once in the superimposition:

$$\begin{pmatrix} (0,0) & (1,1) & (2,2) & (3,3) \\ (1,2) & (0,3) & (3,0) & (2,1) \\ (2,3) & (3,2) & (0,1) & (1,0) \\ (3,1) & (2,0) & (1,3) & (0,2) \end{pmatrix}.$$

A set of Latin squares is called *mutually orthogonal* if any pair of distinct squares is orthogonal. We sometimes use the abbreviation *MOLS* for mutually orthogonal Latin squares. With the exception of $n = 2$ and $n = 6$, we know that there exists a pair of MOLS for every order (see [13, 34, 79]), so the question shifts to how many Latin squares can be mutually orthogonal. The following result is well-known and gives an upper bound on the size of the set.

Theorem 2.8. *If $M = \{L_1, \ldots, L_m\}$ is a set of mutually orthogonal Latin squares of order n, then $m \leqslant n-1$.*

Proof. Without affecting orthogonality, we may permute the symbols within each L_i independently so that the first row reads $(0, 1, 2, \ldots, n-1)$. This implies that in each of the pairs of Latin squares, the ordered pairs (i, i) are used by the first row. If we now examine the entry in the second row, first column of each square, then we know that it may not be a 0 (since there is already a 0 in the first column of each square), and the corresponding entry in each of the m Latin square must be different (since we already have the (i, i) ordered pairs from the top row). Thus, since we only have $n-1$ options to place in each square and they must be distinct, the result follows. □

A set of mutually orthogonal Latin squares which attains the upper bound of one less than the order is termed *complete*. This upper bound can be attained by the *Desarguesian set* as described in the next theorem.

Theorem 2.9 ([20])**.** *There exists a complete set of mutually orthogonal Latin squares for all prime-power orders.*

Proof. Let q be a prime-power and let $\mathbb{F}_q = \{0, \alpha_0, \ldots, \alpha_{q-1}\}$ be a finite field of order q. Consider the set $\{L_\alpha : \alpha \in \mathbb{F}_q \setminus \{0\}\}$. We label the rows and columns of each square by the entries in $\mathbb{F}_q$ and fill in the squares such that $L_\alpha(x, y) = \alpha \cdot x + y$ for all $x, y \in \mathbb{F}_q$. We will show that this set is a complete set of mutually orthogonal Latin squares.

We first show that each of these are Latin squares. Fix $\alpha \in \mathbb{F}_q \setminus \{0\}$. If two columns, y_0 and y_1, have the same symbol in row x, then $\alpha \cdot x + y_0 = \alpha \cdot x + y_1 \implies y_0 = y_1$. Similarly, if two rows, x_0 and x_1, have the same symbol in column y, then $\alpha \cdot x_0 + y = \alpha \cdot x_1 + y \implies x_0 = x_1$.

Now, we must show that L_α is orthogonal to L_β for $\alpha, \beta \in \mathbb{F}_q \setminus \{0\}$, $\alpha \neq \beta$. Suppose, for the sake of contradiction, that L_α and L_β are not orthogonal (for some $\alpha \neq \beta$), then there exists x_0, x_1, y_0 and y_1 with $(x_0, y_0) \neq (x_1, y_1)$ such that $\alpha x_0 + y_0 = \alpha x_1 + y_1$ and $\beta x_0 + y_0 = \beta x_1 + y_1$. Subtracting these two equations gives $x_0(\alpha - \beta) = x_1(\alpha - \beta)$. Since $\alpha \neq \beta$, we have that $x_0 = x_1$. But by substituting this into either of the original equations above, we get $y_0 = y_1$, which is a contradiction, so L_α and L_β must be orthogonal.

Thus, the set is a complete set of mutually orthogonal Latin squares. □

When n is prime, the set of MOLS given in Theorem 2.9 are quite structured. Any given row in any of the squares is simply a cyclic shift of the top row. However, when n is not prime (for example, $n = 4$ is shown below), much of that structure is removed.

Example 2.10. Here are the squares from the construction in Theorem 2.9 for $n = 3, 4$ and 5.

0	1	2
1	2	0
2	0	1

0	1	2
2	0	1
1	2	0

$n = 3$

0	1	2	3
1	0	3	2
2	3	0	1
3	2	1	0

0	1	2	3
2	3	0	1
3	2	1	0
1	0	3	2

0	1	2	3
3	2	1	0
1	0	3	2
2	3	0	1

$n = 4$

0	1	2	3	4
1	2	3	4	0
2	3	4	0	1
3	4	0	1	2
4	0	1	2	3

0	1	2	3	4
2	3	4	0	1
4	0	1	2	3
1	2	3	4	0
3	4	0	1	2

0	1	2	3	4
3	4	0	1	2
1	2	3	4	0
4	0	1	2	3
2	3	4	0	1

0	1	2	3	4
4	0	1	2	3
3	4	0	1	2
2	3	4	0	1
1	2	3	4	0

$n = 5$

Sets of MOLS are of vital interest for several reasons: MDS codes (*e.g.,* [74]), experiment design (see [42] for a good summary), generation of mutually unbiased bases (see [26] for an extensive overview and [50] for a non-standard use), tournament scheduling (see [47]) and projective planes (see [12]). In fact, a projective plane of order n exists if and only if a complete set of MOLS of order n exists [12]. To date, complete sets of MOLS are not known for any order that is not a prime-power, leading to the following two well-known conjectures.

Conjecture 2.11. A complete set of mutually orthogonal Latin squares of order n exists if and only if n is a prime-power.

When n is a prime, it is believed that the construction in Theorem 2.9 produces the only complete set of MOLS.

Conjecture 2.12. If n is a prime, then there is exactly one complete set of mutually orthogonal Latin squares (up to equivalence).

Conjecture 2.12 can easily be verified for $n \leqslant 7$. In Section 7.3, we perform a partial search in $n = 11$, which rules out non-Desarguesian complete sets that contain a square with high symmetry. A special way to generate a complete set of MOLS is to utilise orthomorphisms. A permutation ϕ is an orthomorphism if $x \mapsto \phi_x - x$ is a permutation. Two orthomorphisms, θ and ϕ, are orthogonal if $x \mapsto \theta_x - \phi_x$ is a permutation. A set of $n-1$ mutually orthogonal orthomorphisms that each have 0 as a fixed point can generate a complete set of MOLS. The construction is quite simple: for each orthomorphism, ϕ, create the Latin square by placing $\phi_j + i$ into the (i, j) cell. By taking the (321)-conjugate of each of these squares, we have a complete set of MOLS since each diagonal of weight 1 from different squares now meet in exactly one place (since $\theta - \phi$ is a permutation). It has been shown for prime orders $n \leqslant 17$ that the unique complete set of orthogonal orthomorphisms is the Desarguesian set (see [36] and the references therein for $n \leqslant 13$ and [48] for $n = 17$). It should be noted that for prime orders, a complete set of orthogonal orthomorphisms is equivalent to a Butson Hadamard matrix. For each orthomorphism, ϕ, we define a row of the Butson Hadamard matrix by setting the j^{th} column to $\exp(\frac{2\pi i}{n}\phi_j)$. Once adding an all-ones row, we have a Butson Hadamard matrix. We refer the reader to [48] for more details.

For the first two non-prime-power orders ($n \in \{6, 10\}$), we know that complete sets of mutually orthogonal Latin squares do not exist (see Tarry [79] for $n = 6$ and Lam *et al.* [56] for $n = 10$). The first infinite family of orders where complete sets cannot be attained was given as a consequence of the Bruck-Ryser Theorem [17], which states that if a complete set of mutually orthogonal Latin squares of order $n \equiv 1, 2 \pmod 4$ exists, then n must necessarily be the sum of two squared-integers. This immediately rules out many orders (for example, $n = 14$, 21 and 22).

With these facts, $n = 12$ is the smallest order for which the existence of a complete set of mutually orthogonal Latin squares is in question. In the orders where a complete set does not exist, the next natural question that arises is the maximum number of mutually orthogonal Latin squares. For $n = 6$, we cannot have a pair of orthogonal Latin squares, but even the next order ($n = 10$) is still unresolved. To date, there been many pairs of orthogonal Latin squares of order 10 found, but no triples of mutually orthogonal Latin squares have been found (see [60] for a detailed search).

2.3 Transversals

There are different ways to approach the search for mutually orthogonal Latin squares. One promising direction is through the use of transversals.

Definition 2.13. Let L be a Latin square of order n. A *transversal* is a set of n entries from L such that each row, column and symbol are represented exactly once.

We also discuss transversals of other Latin-like objects. Let L' be a generalised row-Latin rectangle of size $m \times n$. A *transversal* is a set of $\min(m, n)$ entries from L such that each row, column and symbol are represented at most once.

Note that all of the other Latin-like objects listed above (*e.g.,* column-Latin rectangles) follow a similar definition for transversals.

Example 2.14. The shaded entries in the following Latin square form a transversal.

0	1	2	3	4
4	0	1	2	3
3	4	0	1	2
2	3	4	0	1
1	2	3	4	0

It is possible for Latin squares to have several transversals. For example, the Latin square in Example 2.14 has 15 transversals. On the other hand, it is also possible for a Latin square to have no transversals. Euler [34] was the first to note that the Cayley table of $\mathbb{Z}_n$ contains no transversals when n is even. But here, we show a slightly different proof which utilises the following lemma, entitled the *Delta Lemma*. The idea behind this lemma was independently discovered by two sets of researchers in 2005.

Lemma 2.15 ([28, 35])**.** *Let L be a Latin square of order n. Define $\Delta : \mathbb{Z}_n^3 \to \mathbb{Z}_n$ where $\Delta(r, c, s) = r + c - s$. Then for any transversal T, we have*

$$\sum_{(r,c,s)\in T} \Delta(r, c, s) = \begin{cases} 0 & \text{if } n \text{ is odd and} \\ n/2 & \text{if } n \text{ is even.} \end{cases}$$

Proof.

$$\sum_{(r,c,s)\in T} \Delta(r, c, s) = \sum_{(r,c,s)\in T} (r + c - s) = \sum_{r\in\mathbb{Z}_n} r + \sum_{c\in\mathbb{Z}_n} c - \sum_{s\in\mathbb{Z}_n} s = \sum_{r\in\mathbb{Z}_n} r,$$

where the second equality is true since T is a transversal. The result follows from the final summation. $\square$

With the Delta Lemma, we can now prove the following.

Theorem 2.16 ([34])**.** *Let n be even. The Cayley table of $\mathbb{Z}_n$ has no transversals.*

Proof. For any entry (r, c, s) in L, we have that $\Delta(r, c, s) = r + c - (r + c) = 0$. Thus, for any set of entries $X \subseteq L$, $\sum_{(r,c,s)\in X} \Delta(r, c, s) = 0$. By Lemma 2.15, the result follows. $\square$

In Section 2.4, we give an alternative proof of Theorem 2.16. To date, every known example of a Latin square with no transversals is of even order, which lead Ryser to conjecture the following, which has now been verified for $n \leqslant 9$ (see [59]).

Conjecture 2.17 (Ryser's Conjecture, [71])**.** Every Latin square of odd order contains at least one transversal.

The usefulness of transversals stems from the following important theorem.

Theorem 2.18. *Let L be a Latin square of order n. L has an orthogonal mate if and only if L has n mutually disjoint transversals.*

For example, the two Latin squares below are orthogonal to one another. In the left square, 4 disjoint transversals are highlighted, which correspond directly to the 4 symbols in the other square.

0	1	2	3
1	0	3	2
2	3	0	1
3	2	1	0

0	1	2	3
2	3	0	1
3	2	1	0
1	0	3	2

The majority of computer searches done for sets of mutually orthogonal Latin squares directly utilise Theorem 2.18 by first finding all transversals in a specific Latin square, then locating a set of n of them which are mutually disjoint. Unfortunately, the number of transversals in a Latin square can be a poor heuristic in determining if an orthogonal mate exists. There are cases where a Latin square contains exactly n transversals which are disjoint, and other situations where a Latin square contains a very large number of transversals, but no orthogonal mates (for example, the turn square of order 14, see [59]). Moreover, large sets of disjoint transversals can exist without being extendible to a full set of n disjoint transversals. For example, Finney [39] gave an example of a Latin square of order 6 which contains 4 disjoint transversals. Since $n = 6$, we know that there cannot be six disjoint transversals since there is no pair of orthogonal Latin squares of order 6.

0	1	2	3	4	5
1	0	5	4	3	2
2	3	0	1	5	4
3	5	4	0	2	1
4	2	1	5	0	3
5	4	3	2	1	0

2.3.1 Counting Transversals

Very little is known about the exact number of transversals in Latin squares. In fact, even just for the Cayley table of $\mathbb{Z}_n$ (n odd), the number of transversals is only known for $n \leqslant 25$ (see [72, 82]). This is largely due to the fact that no efficient algorithms are known for counting transversals in Latin squares. In Section 7.1.1, we explore some heuristics which speed up the process immensely.

Asymptotically, upper bounds are known for the number of transversals. Let $T(n)$ be the maximum number of transversals amongst all Latin squares of order n. Combining two results of Glebov and Luria [43] (the lower bound) and Taranenko [78] (the upper bound), we have the following result.

Theorem 2.19 ([43, 78])**.**

$$T(n) = \left((1+o(1))\frac{n}{e^2}\right)^n$$

If we restrict ourselves to Cayley tables for groups, Eberhard *et al.* [27] have shown that any abelian group of odd order has

$$\frac{\left(e^{-1/2}+o(1)\right)n!^2}{n^{n-1}} \tag{2.1}$$

transversals.

When considering the minimum number of transversals in Latin squares, we know that for every even n, there is at least one Latin square with no transversal (for example, see Theorem 2.16). In fact, using a very similar argument as Theorem 2.16, Cavenagh and Wanless [18] were able to show that for n even, there are at least

$$n^{n^{3/2}(1/2-o(1))}$$

different transversal-free Latin squares of order n. However, nothing is known for odd orders.

Beyond simply counting the number of transversals, there has been some study on the number of transversals in Latin squares modulo different values. In 1990, Balasubramanian [7] showed the following.

Theorem 2.20. *Every Latin square of even order contains an even number of transversals.*

In Chapter 4, we strengthen this result when $n \equiv 2 \pmod 4$. We would like to note that in [7], Balasubramanian claims that Ryser had conjectured in [71] that the number of transversals in a Latin square is congruent to the order modulo 2 (Theorem 2.20 is the even half of this conjecture). However, we can find no written evidence of this conjecture (for more details, see [11]). The odd half of this "conjecture" is false for many values of $n \geqslant 7$ (see, for example, [3]).

Transversals have also been studied in the sense of edge-coloured bipartite graphs. In particular, from a Latin square of order n (or any Latin-like object), we can construct an edge-coloured bipartite graph with $2n$ vertices. The first n vertices represent the rows, while the remaining n vertices represent the columns. Connect a row vertex with a column vertex with an edge that is coloured based on the symbol in the corresponding row and column. In graph theoretic terms, we are searching for rainbow matchings: a set of k edges where each vertex is adjacent to at most one edge and each colour is used at most once. If $k = n$, then this is called a perfect rainbow matching, which corresponds directly to a transversal.

2.4 Partial Transversals

In the absence of a transversal in a Latin square, it is natural to wonder how close to a transversal we can get. To explore this question, we define the following two objects.

Definition 2.21. Let L be a Latin square of order n. A *partial transversal* of length k is a set of k entries from L such that each row, column and symbol are represented at most once in the set.

Definition 2.22. Let L be a Latin square of order n. A *diagonal* of weight w is a set of n entries from L such that (1) each row and each column are represented exactly once and (2) exactly w symbols are represented at least once.

The two different definitions have been used in different settings and each has their pros and cons. There is an obvious correlation between the two objects. In most situations where we are counting objects, we count diagonals with specific properties rather than partial transversals. Note that a partial transversal of length n, a diagonal of weight n and a transversal are all the same thing. A partial transversal of length $n-1$ is often called a *near transversal*.

With the notion of partial transversals, we may now provides an alternative (but similar) proof of Theorem 2.16.

Theorem 2.23. *Let T be a near transversal in the cyclic square of order n that does not include row i nor column j. Then the missing symbol from T is*

$$\begin{cases} i+j & \text{if } n \text{ is odd and} \\ n/2+(i+j) & \text{if } n \text{ is even.} \end{cases}$$

Proof. The symbol that is missing from T is

$$\sum_{s\in\mathbb{Z}_n} s - \sum_{(r,c,r+c)\in T} (r+c) = \sum_{s\in\mathbb{Z}_n} s - \sum_{(r,c,s)\in T} r - \sum_{(r,c,s)\in T} c$$
$$= \sum_{s\in\mathbb{Z}_n} s - \left(\left(\sum_{r\in\mathbb{Z}_n} r\right) - i\right) - \left(\left(\sum_{c\in\mathbb{Z}_n} c\right) - j\right) = i+j-\sum_{s\in\mathbb{Z}_n} s,$$

from which the result follows. □

Since the symbol in the (i,j) cell of the cyclic square is $i+j$, Theorem 2.23 shows that no near transversal is completable to a transversal when n is even (this is an alternate proof of Theorem 2.16) and that every near transversal is completable to a transversal when n is odd.

We know that there are many Latin squares that do not contain a transversal (see [18]). Thus, the best we can hope for is a partial transversal of length $n-1$.

Conjecture 2.24. Every Latin square of order n contains a partial transversal of length $n-1$.

Conjecture 2.24 is often called *Brualdi's Conjecture*, but its exact origin remains unknown and has been attributed to Brualdi, Stein and Ryser. In 1974, Dénes and Keedwell attributed the conjecture to Brualdi (page 103 of the 1st Edition of [22]); in 1975, Stein [76] concluded his article with seven conjectures (Conjecture 2.25), each of which imply Conjecture 2.24, but none of which are exactly Conjecture 2.24; in 1988, Erdős *et al.* [31], attributed the conjecture to Ryser (note that Stein was also an author of this article). This conjecture (as well as Ryser's Conjecture and Conjecture 2.28 below) has been generalised to rainbow matchings as well. We refer the reader to [1] for more details.

In 1975, Stein [76] studied transversals in equi-n-squares (an $n\times n$ array where each symbol is represented exactly n times, with no row or column restrictions). In this article, he made the following seven conjectures:

Conjecture 2.25 ([76])**.**

(1) Every equi-n-square has a near transversal.

(2) Every $n \times n$ array in which no symbol appears more than $n-1$ times has a transversal.

(3) Every $(n-1) \times n$ array in which no symbol appears more than n times has a transversal.

(4) Every $(n-1) \times n$ row-Latin rectangle has a transversal.

(5) Every $m \times n$ array (where $m < n$) in which no symbol appears more than n times has a transversal.

(6) Every $(n-1) \times n$ array in which each symbol appears exactly n times has a transversal.

(7) Every $m \times n$ array (where $m < n$) in which no symbol appears more than $m+1$ times has a transversal.

These conjectures are quite related to one another (with some just being special cases of others). Unfortunately, all but one of these conjectures has been disproven (with one of them remaining open). In 1998, Drisko [25] gave the following rectangle:

Construction 2.26 ([25])*.* Let m and n be such that $m < n \leqslant 2m-2$. Define A so that the first $m-1$ columns are $[0, 1, \ldots, m-1]^T$ and the remaining columns are $[1, 2, \ldots, m-1, 0]^T$. Then A has no transversals.

Proof. The proof is quite straightforward. We call the first $m-1$ columns the α *columns* and the other columns the β *columns*. Note that each row contains exactly two symbols (a and $a+1$ for some $a \in \mathbb{Z}_m$) and that each symbol appears in exactly two rows. Select some cell that contains a 0 to be in our transversal. The remaining symbol in that row (either 1 or $m-1$) only appears in one other row, so we must take it from there. Note that if we took the 0 from the α columns (resp., β columns), then this new symbol is also taken from the α columns (resp., β columns). We can repeat this process for each symbol and note that all of the cells chosen for the transversal must all come from α columns or all come from β columns. Since there are at most $m-1$ of each of these, but m symbols, we cannot possibly have a transversal. $\square$

Construction 2.26 is a directly counterexample to Conjecture 2.25(5), and is also a counterexample to (3), (6) and (7) by setting $m = n-1$. Furthermore, in 2017, Pokrovskiy and Sudakov [67] proved the following (by giving a constructive proof).

Theorem 2.27 ([67])**.** *For all sufficiently large n (say, $n \geqslant 10^{60}$), there exists an equi-n-square that does not have a partial transversal of length at least $n - \frac{1}{42}\log n$.*

This is a counterexample to Conjecture 2.25(1). Moreover, we can slightly extend Pokrovskiy and Sudakov's result to show that (2) is incorrect. Consider a sufficiently large n and construct the square from Theorem 2.27. Define A to be an $(n+1) \times (n+1)$ array and populate the top-left $n \times n$ subarray with a square from the proof of Theorem 2.27 and then populate the last row and column with $2(n+1)-1$ distinct symbols that do not appear in the top $n \times n$ subarray. Since at most two of these new symbols can be used in any partial transversal, this square does not have a partial transversal of length at least $n - \frac{1}{42}\log n + 2$, and thus, cannot contain a transversal. By a similar argument, we can pad an appropriate equi-n-square with either distinct symbols or the same symbols to provide different counterexamples to those that Drisko found a counterexample for (that is, (3), (5), (6) and (7)).

The only one of Stein's conjectures that remains unsolved is (4). While we are quite unsure about this conjecture, it seems much more promising if we also enforce the rectangle to be column-Latin.

Conjecture 2.28. Let R be an $(n-1)\times n$ Latin rectangle. Then R contains a transversal.

Note that this is a strengthened form of Brualdi's Conjecture, where we may choose which row (or column or symbol) is not include in our partial transversal.

In Latin squares, it is easy to find a partial transversal of length $\lceil n/2 \rceil$ (for each of the first $\lceil n/2 \rceil$ rows, greedily select any entry whose column and symbol are not represented yet). In 1969, Koksma [53] showed that there is a partial transversal of length at least $(2/3)n + 1/3$. Over the next decade, the $2/3$ coefficient was improved to $3/4$ and $9/11$ in [24] and [81], respectively.

In 1978, Brouwer *et al.* [14] and Woolbright [87] simultaneously proved that every Latin square contains a partial transversal of length at least $n - \sqrt{n}$. This result has been reproven in different settings. For example, Aharoni *et al.* [2] proved a similar result for matroids. Here, we give a fairly general version of the proof. The essence of the proof is not dramatically changed from those original proofs. In his thesis, Pula [68] proved the following for Latin squares with holes. We follow a similar proof style to his.

Theorem 2.29. *Let L be a generalised $m \times n$ partial column-Latin rectangle with at most h holes per column. If the longest partial transversal in L is of length t, then $t \geqslant (n-t)(m-t-h)$.*

Proof. Assume that there are k symbols in L and that the rows, columns and symbols are indexed by $\{0,\dots,m-1\}$, $\{0,\dots,n-1\}$ and $\{0,\dots,k-1\}$, respectively. Without loss of generality, assume that a partial transversal of length t appears on the top portion of the main diagonal (*i.e.,* $L(i,i) = i$ for $0 \leqslant i < t$). Partition the symbols into two categories: The set of symbols on T is called *small*, $\mathcal{S} = \{0,\dots,t-1\}$, and those symbols not on T is called *large*, $\mathcal{L} = \{t, t+1, \dots, k-1\}$. Build up a sequence of sets in the following way:

$$A_{-1} = \emptyset$$
$$A_i = \{r : L(r, i+t) \in A_{i-1} \cup \mathcal{L}\}\,, \text{ for } 0 \leqslant i < n-t$$

Claim: $A_i \subseteq \mathcal{S}$. Assume, on the contrary, that there is an entry $p \in A_i$ such that $p \in \mathcal{L}$ and that i is the smallest index where this occurs. If $L(p, t+i) \in \mathcal{L}$, then we may easily extend our partial transversal since row p, column $t+i$ and symbol $L(p,t+i)$ are not in the original partial transversal, which would contradict our assumption of maximality. We now work our way backwards through these sets in the following manner. Let $m_0 = L(p, t+i) \in A_{i-1}$ and $m_j = L(m_{j-1}, t+i-j) \in A_{\ell-1}$ for $0 < \ell \leqslant c$ where c is the first index such that $m_c \in \mathcal{L}$. Such a c must exist since $L(r,t) \in \mathcal{L}$ for all $r \in A_0$. We first assume that all m_i are distinct. We can make a transversal of length $t+1$ by replacing the entries in the left column with those entries in the right column. Note that $\bullet$ is some large symbol.

Original Entry			New Entry		
$(m_0,$	$m_0,$	$m_0)$	$(p,$	$t+i,$	$m_0)$
$(m_1,$	$m_1,$	$m_1)$	$(m_0,$	$t+i-1,$	$m_1)$
$(m_2,$	$m_2,$	$m_2)$	$(m_1,$	$t+i-2,$	$m_2)$
	$\vdots$			$\vdots$	
$(m_{c-2},$	$m_{c-2},$	$m_{c-2})$	$(m_{c-3},$	$t+i-c+2,$	$m_{c-2})$
$(m_{c-1},$	$m_{c-1},$	$m_{c-1})$	$(m_{c-2},$	$t+i-c+1,$	$m_{c-1})$
$(m_c,$	$m_c,$	$m_c)$	$(m_{c-1},$	$t+i-c,$	$m_c)$
	$-$		$(m_c,$	$t+i-c-1,$	$\bullet)$

One may easily check that this forms a partial transversal of length $t+1$, which contradicts the fact that the longest transversal is of length t. (The last entry contains a large symbol, so it does not appear on the partial transversal already. The set of rows used in the new partial transversal is $\mathcal{S} \cup \{p\}$. The columns used in the new entries were not used in the original partial transversal.) Now, if $m_a = m_b$ for some $a < b$, then we would be deleting some entries multiple times (for example, (m_a, m_a, m_a) would be deleted multiple times). If this is the case, we may just simply delete the portion of the table corresponding to the rows $(m_{a+1}, \ldots, m_b)$. By erasing this section, we get a sequence with fewer repetitions, so we may repeatedly remove these repetitions as above until we arrive at a sequence which has no repetitions. At this final stage, we may then do the replacements and arrive at a partial transversal of length $t+1$, which is a contradiction. This completes the proof of the claim.

We have shown that $A_i \subseteq \mathcal{S}$ and we will now show that $|A_i| \geqslant (i+1)(m-t-h)$ by induction. First, note that the claim is trivially true for $i = -1$. Now suppose that $i \geqslant 0$ and $|A_{i-1}| \geqslant i \cdot (m-t-h)$. Since $A_{i-1} \subseteq \mathcal{S}$, we may partition A_i into two sets:

$$A_i = \{r : L(r, t+i) \in A_{i-1}\} \cup \{r : L(r, t+i) \in \mathcal{L}\}.$$

Let ℓ be the number of entries in $\{r : L(r, t+i) \in A_{i-1}\}$. From this, we know that there are $|A_{i-1}| - \ell + d$ small entries not present in the $(t+i)$th column for some $d \geqslant 0$. This means that there are $t - (|A_{i-1}| - \ell + d)$ small symbols and at least $m - h - (t - (|A_{i-1}| - \ell + d))$ large symbols in the column. Thus, by induction, we have

$$\begin{aligned} |A_i| &= \ell + (m - h - (t - (|A_{i-1}| - \ell + d))) = |A_{i-1}| + m - t - h + d \\ &\geqslant |A_{i-1}| + m - t - h \geqslant i \cdot (m-t-h) + (m-t-h) = (i+1)(m-t-h). \end{aligned}$$

Since $A_{n-t-1} \subseteq \mathcal{S}$ and $|\mathcal{S}| = t$, we have that $t \geqslant (n-t)(m-t-h)$ as desired. □

The most used form of Theorem 2.29 is the following.

Corollary 2.30. *Every Latin square of order n contains a partial transversal of length at least $n - \sqrt{n}$.*

Proof. Simply let $m = n$ and $h = 0$ in Theorem 2.29 and solve for t. □

The bound of $n - \sqrt{n}$ was quickly improved by Shor [73] in 1982 to $n - O(\log^2 n)$. However, there was an error in his proof. In 2008, Hatami and Shor [46] fixed this error. In that paper, they state the result only for Latin squares, but their entire proof works for Latin arrays as well. However, it cannot easily be modified to work for row-Latin squares, squares with holes or rectangles as Theorem 2.29 could be. To date, this is still the best-known result.

Theorem 2.31 (Adaptation of [46, 73])**.** *Every Latin array of order n contains a partial transversal of length at least $n - 11.053\log^2 n$.*

In Section 7.2, we utilise ideas from [46] to prove a generalisation of Brualdi's Conjecture for $n \leqslant 11$.

We finish our introduction of transversals with an interesting result. If a Latin square is selected at random, it is quite likely that every maximal partial transversal is reasonably large.[1]

Theorem 2.32 ([10])**.** *Fix $\epsilon > 0$. With probability approaching 1 as $n \to \infty$, a Latin square of order n chosen uniformly at random has no maximal partial transversal of length less than $n - n^{2/3+\epsilon}$.*

Proof. Let L be a random Latin square of order n. Suppose that L has a maximal partial transversal T of deficit d. Let S be the $d \times d$ submatrix of L induced by the rows and columns that are not represented in T. By the maximality of T, we know that S contains none of the d symbols that are not represented in T. However, if this is the case and $d = n^{2/3+\epsilon}$, then [54, Theorem 2] would imply that $n^{1+3\epsilon} = d^3/n = O(n^{1+3\epsilon/2}\log n)$, which is a contradiction, so no such submatrix S exists in L. □

We have only brushed the surface of the topics studied in regards to transversals in Latin squares. We refer the reader to [82] for a comprehensive survey.

[1] I would like to note that even though I am an author on [10], I am including this as part of the introduction as I was not directly involved in its proof.

Chapter 3

Permanents

The search for something permanent
is one of the deepest of the instincts
leading men to philosophy.
– B. Russell

Before we examine transversals in Latin squares, we must first take a detour into permanents of $(0,1)$-matrices.

Definition 3.1. Let $A = [a_{ij}]$ be an $n \times n$ matrix. The *permanent* of A is

$$\operatorname{per}(A) = \sum_{\sigma \in S_n} \prod_{i=0}^{n-1} a_{i,\sigma_i}.$$

The permanent can be thought of as the unsigned version of the determinant since

$$\det(A) = \sum_{\sigma \in S_n} (-1)^{\epsilon(\sigma)} \prod_{i=0}^{n-1} a_{i,\sigma_i},$$

where $\epsilon : S_n \to \mathbb{Z}_2$ is the parity of a permutation in the usual sense. In fact, modulo 2, the permanent and the determinant are the same. This property is exploited several times throughout the chapter. However, while determinants are quite well-understood and simple to compute in polynomial time (for example, via Gaussian elimination), much less is known about permanents. In fact, Valiant [80] showed that even computing the permanent for $(0,1)$-matrices is #P-complete. In the same article, he showed that computing the permanent modulo k is #P-complete if and only if k is not a power of 2. For a general outline of what is known about permanents, we refer the reader to [15, 49, 62]. In this chapter, all matrices are square unless otherwise specified. We say that $M_n(S)$ is the ring of $n \times n$ matrices with each entry in S.

At several points in this section, we use Ryser's formula [70] to compute the permanent of a matrix.

Theorem 3.2 (Ryser's formula)**.** *Let $A = [a_{ij}] \in M_n(X)$ for some X. Then*

$$\operatorname{per}(A) = (-1)^n \sum_{S \subseteq \{0,\dots,n-1\}} (-1)^{|S|} \prod_{i=0}^{n-1} \sum_{j \in S} a_{ij}. \tag{3.1}$$

The results in this chapter focus on showing congruences for permanents under certain conditions. In almost all cases, we impose restrictions on the row and/or column sums of the matrix. Section 3.1 explores the connection between the $\mathbb{Z}_p$-nullity (mostly for $p = 2$) of the matrix and its permanent as well as give some general results. In Section 3.2, we focus on the permanent of adjacency matrices of certain multigraphs. Finally, Section 3.3 explores patterns in the parity of the permanent when deleting a row and column of a matrix.

While each result in this chapter focuses on the permanent of matrices, each one can be viewed in a purely graph theoretic sense.

Theorem 3.3. *Let G be a bipartite graph with bipartition $U \cup V$. If $|U| = |V| = n$, we can define the $n \times n$ matrix, $A = [a_{ij}]$, where $a_{ij} = 1$ if and only if vertex $i \in U$ is adjacent to vertex $j \in V$. The number of perfect matchings in G is equal to the permanent of A.*

Proof. Each $\sigma \in S_n$ such that $a_{i\sigma_i} = 1$ for all i is a perfect matching in G by matching vertex $i \in U$ with $\sigma_i \in V$ and also adds exactly 1 to $\operatorname{per}(A)$. $\square$

3.1 Permanents Based on Regularity and Nullity

One very important set of matrices is as follows.

Definition 3.4. Λ_n^k is the set of all $(0,1)$-matrices of order n which contain exactly k ones in each row and each column.

Throughout, we use J to be the all-ones matrix of appropriate size. Thus, this means that $JA = AJ = kJ$ for $A \in \Lambda_n^k$. These regular matrices are of use in counting since we can take a subset S of symbols in $\mathbb{Z}_n$ and create a matrix in Λ_n^k from a Latin square by setting the a_{ij} to either 1 or 0 depending on whether or not the symbol in the corresponding entry of the Latin square is in S. This is explored in Chapter 4.

Though many of our results directly give results about Λ_n^k, most of them are slightly more general and only require the row and column sums to be similar (*e.g.,* all even or all 2 modulo 4). We begin with an easy result.

Proposition 3.5. *Let $A \in M_n(\mathbb{Z})$. If all row sums are even, then* $\operatorname{per}(A) \equiv 0 \pmod 2$.

Proof. Since the sum of the entries in each row is even, the columns of A are linearly dependent over $\mathbb{Z}_2$. Thus, $\det(A) \equiv 0 \pmod 2$. By definition, we have that $\operatorname{per}(A) \equiv \det(A) \pmod 2$. $\square$

For odd orders, we can get a larger power of two in the permanent.

Theorem 3.6. *Let $n \equiv 1 \pmod 2$ and $A \in M_n(\mathbb{Z})$. If all row sums are a multiple of 4 and all column sums are even, then* $\operatorname{per}(A) \equiv 0 \pmod 4$.

Proof. We compute the permanent via (3.1):

$$\operatorname{per}(A) = (-1)^n \sum_{S \subseteq \{0,\ldots,n-1\}} (-1)^{|S|} \prod_{i=0}^{n-1} \sum_{j \in S} a_{ij}.$$

Let r_i be the sum of row i and c_j be the sum of column j. Fix a set S_0 and consider the terms corresponding to S_0 and its complement in the summation. For each pair,

$$(-1)^{n-|S_0|}\left(\prod_{i=0}^{n-1}\sum_{j\in S_0}a_{ij}-\prod_{i=0}^{n-1}\sum_{j\notin S_0}a_{ij}\right)=(-1)^{n-|S_0|}\left(\prod_{i=0}^{n-1}\sum_{j\in S_0}a_{ij}-\prod_{i=0}^{n-1}\left(r_i-\sum_{j\in S_0}a_{ij}\right)\right)$$
$$\equiv 2\prod_{i=0}^{n-1}\sum_{j\in S_0}a_{ij}\pmod 4. \tag{3.2}$$

Since each column of A has an even total,

$$\sum_{i=0}^{n-1}\sum_{j\in S_0}a_{ij}=\sum_{j\in S_0}c_j\equiv 0\pmod 2. \tag{3.3}$$

Since n is odd, $\sum_{j\in S_0}a_{ij}$ must be even for at least one value of i in (3.3). Thus, the product of the partial row sums must be even and (3.2) must be a multiple of 4. Summing over all such S_0, the result follows. □

Corollary 3.7. *Let* $n\equiv 1\pmod 2$ *and* $A\in\Lambda_n^{4k}$. *Then* $\mathrm{per}(A)\equiv 0\pmod 4$.

We now shift our focus to results that depend on the $\mathbb{Z}_p$-rank and $\mathbb{Z}_p$-nullity of the matrices (usually $p=2$). We define $\nu(A)$ to be the $\mathbb{Z}_2$-nullity of A.

Lemma 3.8. *Let* $A\in M_n(\mathbb{Z})$ *be such that* $\nu(A)=1$ *and each column sum is even. Then every binary* n*-vector with even sum is in the* $\mathbb{Z}_2$*-span of the columns of* A.

Proof. We work over the field $\mathbb{Z}_2$. Since all column sums are even, the same is true for any linear combination of columns of A. But $\nu(A)=1$, so the $\mathbb{Z}_2$-span of the columns has cardinality 2^{n-1}, which means it must include all vectors with even sum. □

Theorem 3.9. *Let* $n\equiv 0\pmod 2$ *and* $A\in M_n(\mathbb{Z})$ *be such that* $\nu(A)=1$ *and each row and column sum is even. Then* $\mathrm{per}(A)\equiv 2\pmod 4$.

Proof. For a particular set S in (3.1), let $x_i=\sum_{j\in S}a_{ij}$. Since $\sum_i x_i$ is even, we know that an even number of the x_i are odd, and hence an even number of them are even. Only the terms with 0 even x_i's matter if we want the permanent modulo 4 (as those with 2 or more vanish modulo 4).

Let $2k_i$ be the sum of the entries in row i. By Lemma 3.8, there is a subset S for which all x_i are odd. As $\nu(A)=1$, this subset is unique up to complementation. The contribution from it and its complement is, up to sign,

$$\prod_i x_i+\prod_i(2k_i-x_i)\equiv 2\prod_i x_i\equiv 2\pmod 4,$$

as desired. □

Theorem 3.10. *Let* $n\equiv 1\pmod 2$*,* $k\equiv 2\pmod 4$ *and* $A\in M_n(\mathbb{Z})$ *be such that* $\nu(A)=1$ *and each row and column sum of* A *is congruent to* $k\pmod 8$*, then* $\mathrm{per}(A)\equiv k\pmod 8$.

Proof. For a particular set S in (3.1), let $x_i=\sum_{j\in S}a_{ij}$. Since $\sum_i x_i$ is even, we know that an even number of the x_i are odd, and hence an odd number of them are even. Thus, only the terms with one even x_i matter if we want the permanent modulo 8 (as those with 3 or more vanish modulo 8).

By Lemma 3.8, for each j, there is a subset S for which x_j is even and all x_i are odd for $i \neq j$. By complementing if necessary, we assume that $|S|$ is odd, so that $\prod x_j$ has a positive coefficient. Even though our computations below are done modulo 8, we are only concerned about the value of each odd x_i modulo 4 since each term is multiplied by an even value. Let a be the number of x_i's that are 3 modulo 4 (the other $n-1-a$ odd x_i's are 1 modulo 4). Thus, we have

$$\prod_i x_i - \prod_i (k - x_i) \equiv x_j 3^a - (k - x_j)3^a \equiv 3^a(2x_j - k) \pmod{8}, \tag{3.4}$$

since $2 - x \equiv x \pmod{4}$ for odd x. Note that $3^2 \equiv 1 \pmod{8}$, so we only care about $a \pmod{2}$.

The sum of the columns corresponding to S give us that

$$2 \equiv k|S| \equiv x_j + (n-1-a) + 3a \equiv x_j + n - 1 + 2a \pmod{4},$$

and hence, $a \equiv (x_j + n + 1)/2 \pmod{2}$. We will now show that (3.4) is congruent to $3^{(n+1)/2}(-k) \pmod{8}$. If $x_j \equiv 0 \pmod{4}$, then the result is immediate. If $x_j \equiv 2 \pmod{4}$, then we use the fact that $4 - k \equiv k \pmod{8}$ and $3k \equiv -k \pmod{8}$ for $k \equiv 2 \pmod{4}$, so (3.4) is congruent to

$$3^{(n+1)/2+1}(4-k) \equiv 3^{(n+1)/2+1}k \equiv (3^{(n+1)/2})(3k) \equiv 3^{(n+1)/2}(-k) \pmod{8}.$$

Thus, we have that the contribution from S and its complement is independent of x_j. Summing over j, we get $\operatorname{per}(A) \equiv -nk3^{(n+1)/2} \pmod{8}$.

Lastly, note that whether $n \equiv 1 \pmod{4}$ or $n \equiv 3 \pmod{4}$, we have $-n3^{(n+1)/2} \equiv 1 \pmod{4}$. Thus, $\operatorname{per}(A) \equiv k \pmod{8}$ as desired. □

We finish this section with a very interesting connection between the $\mathbb{Z}_p$-rank of a matrix and its permanent. We prove the result for rectangular matrices rather than square ones. The permanent of a rectangular matrix is defined as the sum of the products of all diagonals that hit each row exactly once and each column at most once.

Theorem 3.11. *Let A be an $m \times n$ ($m \leqslant n$) integer matrix with $\mathbb{Z}_p$-rank r for a given prime p. Then $p^a \mid \operatorname{per}(A)$, where $a = \max(0, \lceil (m-(p-1)r)/p \rceil)$.*

Proof. To assist in this proof, we define the *fancy form*, $f(M)$, of a matrix M in the following way. Let $\{r_i\}$ be the row vectors of M. A row is called *totally-p* if it only contains entries which are multiples of p. Permute the rows of M so that all totally-p rows are at the top, in lexicographic order. Permute the rest of the rows of M into blocks of identical rows (formally, if $r_i = r_j$ for some $i < j$, then $r_i = r_{i+1}$). Now permute these blocks in such a way that the sizes of the blocks are in non-increasing order from top-to-bottom. If two blocks contain the same number of rows, sort them in lexicographic order. The resulting matrix is $f(M)$ and note that $\operatorname{per}(M) = \operatorname{per}(f(M))$ since we have only permuted the rows. A matrix in fancy form looks like this:

$$\left[\begin{array}{c} \text{Totally-}p \text{ rows} \\ \hline a_1 \text{ identical rows} \\ \hline a_2 \text{ identical rows} \\ \hline \vdots \\ \hline a_k \text{ identical rows} \end{array}\right],$$

where $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_k$.

First, note that if every entry in A is a multiple of p, then the $\mathbb{Z}_p$-rank is 0 and $p^m \mid \text{per}(A)$, so the result holds. Moreover, if $\text{per}(A) = 0$, then the result holds trivially. Now, let M be any $m \times n$ matrix with at least one entry that is not a multiple of p and $\text{per}(M) \neq 0$.

Let N be any $m \times n$ matrix where either (a) $f(N)$ has more totally-p rows than $f(M)$ or (b) $f(N)$ and $f(M)$ have the same number of totally-p rows, but the sizes of the blocks in $f(N)$ are larger than the blocks in $f(M)$ (using lexicographic order from top-to-bottom). We assume, for the sake of induction, that the result holds for all such N.

We attempt to find a special linear-combination of rows, $\alpha_{i_1} r_{i_1} + \alpha_{i_2} r_{i_2} + \cdots + \alpha_{i_\ell} r_{i_\ell}$ (where each $\alpha_i > 0$), that add to a totally-p vector. The following constraints are required:

- Row i can be used at most $p-1$ times (*i.e.*, $0 < \alpha_i \leqslant p-1$),
- If row i and row j are used ($i \neq j$), then $r_i \neq r_j$, and
- No totally-p row is used.

Case 1. If no such linear combination exists, then consider the set S, which contains one row from each block in $f(M)$. Note that S forms a linearly-independent set of vectors in $\mathbb{Z}_p^n$ that spans the rows of M (in $\mathbb{Z}_p^n$) since all rows are either totally-p or equal to a vector in S. Thus, if there are k blocks in $f(M)$ of size $a_1, a_2, \ldots, a_k$, then the $\mathbb{Z}_p$-rank of M is k. Each block contributes a factor of $a_i!$ to the permanent, and thus, at least $\lfloor a_i/p \rfloor$ factors of p. The $(m - \sum a_i)$ totally-p rows contribute at least one factor of p each. Thus, there are at least

$$\left(m - \sum a_i\right) + \sum \left\lfloor \frac{a_i}{p} \right\rfloor \geqslant \frac{m - \sum a_i}{p} + \frac{\sum(a_i - (p-1))}{p} = \frac{m - \sum(p-1)}{p} = \frac{m-(p-1)k}{p}$$

factors of p in the permanent. Since the number of factors of p is integral, we may round up to the nearest integer (or to zero if this value is negative) and the result follows.

Case 2. If a desired linear combination exists, then $\alpha_{i_1} r_{i_1} + \alpha_{i_2} r_{i_2} + \cdots + \alpha_{i_\ell} r_{i_\ell}$ is totally-p. We place the sum of these rows into row i_ℓ, the bottommost of $i_1, i_2, \ldots, i_\ell$.

$$\text{per}\begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ \vdots \\ \sum_{k=1}^{\ell} \alpha_{i_k} r_{i_k} \\ \vdots \end{pmatrix} = \alpha_{i_1}\,\text{per}\begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ \vdots \\ r_{i_1} \\ \vdots \end{pmatrix} + \alpha_{i_2}\,\text{per}\begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ \vdots \\ r_{i_2} \\ \vdots \end{pmatrix} + \cdots + \alpha_{i_\ell}\,\text{per}\begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ \vdots \\ r_{i_\ell} \\ \vdots \end{pmatrix}. \tag{3.5}$$

The matrix on the left-hand side, M', has one more totally-p row than M. So by induction, $p^{\lceil (m-(p-1)\,\text{rank}(M'))/p \rceil} \mid \text{per}(M')$. Note that $\text{rank}(M') = \text{rank}(M)$.

We now consider the matrices on the right-hand side. We use $M^{(i)}$ to denote the matrix with r_i in row i_ℓ. Note that $M^{(i_\ell)} = M$, and no other matrix is M. Consider $M^{(i)} \neq M$ on the right-hand side. $M^{(i)}$ contains the same number of totally-p rows as M, so we must examine the block structure. Note that $M^{(i)}$ has almost the same block structure as M, except the block that row i_ℓ was in has decreased in size by one (and possibly vanished) and the block that row i was in has increased in size by one. Since $i < i_\ell$, we may apply induction to $M^{(i)}$. So $p^{\lceil (m-(p-1)\,\text{rank}(M^{(i)}))/p \rceil} \mid \text{per}(M^{(i)})$. Since we have replaced a row of

M with a row that is already in the row span of M, it follows that $\operatorname{rank}(M) \geqslant \operatorname{rank}(M^{(i)})$ and $\lceil (m-(p-1)\operatorname{rank}(M^{(i)}))/p \rceil \geqslant \lceil (m-(p-1)\operatorname{rank}(M))/p \rceil$.

Thus, the permanent of each matrix in (3.5) not equal to M is divisible by p^a. Considering (3.5) modulo p^a, it follows that $\alpha_{i_\ell} \operatorname{per}(M) \equiv 0 \pmod{p^a}$. Since $\gcd(\alpha_{i_\ell}, p) = 1$, the desired result follows. $\square$

Since the rank and nullity sum to n, we have the following useful result.

Corollary 3.12. *Let $A \in M_n(\mathbb{Z})$. Then*

$$2^{\lceil \nu(A)/2 \rceil} \mid \operatorname{per}(A).$$

In the case when $\nu(A) = 2$ with even column sums, we can extract one more power of two. The following is only a strengthening of Corollary 3.12 for $\nu(A) = 2$.

Theorem 3.13. *Let $A \in M_n(\mathbb{Z})$ be such that $\nu(A) = 2$ and each column sum is even. Then $\operatorname{per}(A) \equiv 0 \pmod 4$.*

Proof. We use a similar idea (and notation) to Case 2 from the proof of Theorem 3.11 but consider two summations simultaneously. Note that since $p = 2$, we have that $\alpha_i = 1$ in all cases.

If there is at least one totally-p row in A, then consider deleting that row and the resulting $(n-1) \times n$ matrix B. Applying Theorem 3.11 to B, we see that $2 \mid \operatorname{per}(B)$ since its $\mathbb{Z}_2$-rank is $n-2$. Note that each diagonal in A is simply a diagonal in B that has been extended by one entry from the row we deleted. Since each entry in the deleted row is a multiple of 2, this contributes an extra factor of 2 to $\operatorname{per}(A)$. Thus, if there is a totally-p row, then the result holds. Moreover, if $a_2 \geqslant 2$, then the result follows immediately (since $a_1 \geqslant a_2 \geqslant 2$, the calculation from Case 1 gives the desired result), so we may assume that $a_2 = a_3 = \cdots = a_k = 1$, and that $a_1 \leqslant 3$ for similar reasons. The a_1 rows in the top block of $f(A)$ must be handled differently. We call these rows *special*.

Since $\nu(A) = 2$, there is a non-trivial proper subset, S, of rows whose sum is even. If $a_1 \geqslant 2$, we choose S to be two of the special rows. Since the column sums of A are all even, the rows in $\overline{S}$ also give an even sum.

Note that since there are no totally-p rows, both S and $\overline{S}$ have at least two elements each. We select some row in S and some non-special row in $\overline{S}$ to place the following summation (in a similar style to Case 2).

$$\operatorname{per}\begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ \sum_{r \in S} r \\ \vdots \\ \sum_{\rho \notin S} \rho \\ \vdots \end{pmatrix} = \sum_{r \in S} \sum_{\rho \notin S} \operatorname{per}\begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r \\ \vdots \\ \rho \\ \vdots \end{pmatrix}. \tag{3.6}$$

The two changed rows in the matrix on the left-hand side are totally-p rows, so its permanent is a multiple of 2^2. For the matrices on the right-hand side, we have three cases: (a) it is our original matrix, (b) it is not our original matrix and both r and ρ are special and (c) it is not our original matrix and at least one of r or ρ is not special.

If both r and ρ are special, then $a_1 = 3$. Since we placed the second summation in a non-special row, there are now 4 identical rows, which contribute at least 2^2 to its permanent (actually, at least 2^3). If at least one of r or ρ is not special, then they are not identical (since $a_i = 1$ for $i \neq 1$) and each contains a duplicate somewhere in the matrix, each independently contributing 2 to the permanent. Thus, since the permanent of every other matrix in (3.6) is a multiple of 4, the permanent of the original matrix must also be a multiple of 4. $\square$

3.2 Permanents of Adjacency Matrices of Multigraphs

This section assumes basic knowledge of graph theory. We refer the reader to [23] for basic terminology.

This section focuses on the permanent of adjacency matrices of multigraphs. The adjacency matrix, $A = [a_{ij}]$, of a multigraph G is defined such that a_{ij} is the number of edges between i and j. Note that loops are allowed in multigraphs and contribute 2 to the a_{ii} entry. Equivalently, these adjacency matrices are symmetric matrices in $M_n(\mathbb{Z})$ with all even diagonals. Note that adjacency matrices do not usually allow for a negative number of edges, however, since we are only concerned with congruences, the results below work with negative entries as well.

$$\begin{pmatrix} 0 & 1 & 2 & 1 \\ 1 & 2 & 0 & 0 \\ 2 & 0 & 0 & 1 \\ 1 & 0 & 1 & 4 \end{pmatrix} \xleftrightarrow{\substack{\text{adjacency matrix} \\ \text{of multigraph}}}$$

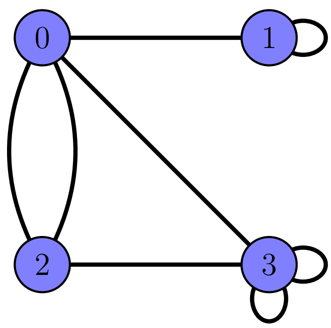


Figure 3.1: A multigraph and its corresponding adjacency matrix.

Definition 3.14. Let G be a multigraph on n vertices. An edge cover of G is a set of edges such that each vertex is incident to at least one edge in the set. A $(1, 2)$-*factor* of G is an edge cover where each connected component in the cover is either a single cycle or a single edge. A $(1, 2)$-factor can be partitioned into two parts: the cycles and the matching.

Because we are dealing with multigraphs, there are a couple of corner cases we wish to emphasise. Firstly, a loop is considered to be a cycle. Secondly, cycles of length 2 are allowed in our $(1, 2)$-factor and are different from selecting one of the edges to be part of the matching. For clarity, the nine $(1, 2)$-factors of the multigraph in Figure 3.1 are given in Example 3.16. The following is a connection between the permanent of the adjacency matrix and the number of $(1, 2)$-factors of the corresponding graph. This statement is well-known in terms of simple graphs but works the same way for multigraphs with a slight alteration to the proof as below.

Theorem 3.15. *Let G be a multigraph and A be the adjacency matrix of G. If G has t_c $(1, 2)$-factors with c cycles, then*

$$\operatorname{per}(A) = \sum_{c=0}^{n} 2^c t_c.$$

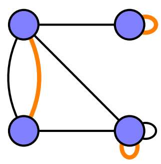
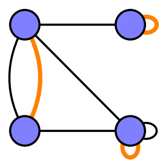
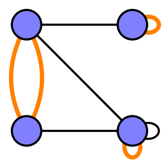
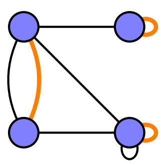
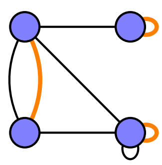
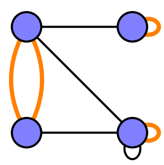
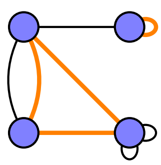
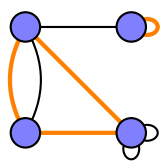
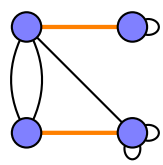
Proof. We show the correspondence in two steps. First, we count the number of directed $(1,2)$-factors, then show that this number is exactly the permanent of A.

A *directed* $(1,2)$*-factor* of G is a $(1,2)$-factor where each of the cycles is oriented in one of the two possible directions. For a directed $(1,2)$-factor, D, define $f(D)$ to be the subgraph found by ignoring the orientations on the cycles in D (this subgraph is a $(1,2)$-factor of G). For any $(1,2)$-factor of G with c cycles, exactly 2^c directed cycles map to it. Thus, G has exactly $\sum_{c=0}^{n} 2^c t_c$ directed $(1,2)$-factors.

Let D be a directed $(1,2)$-factor of G and define $g(D) = \sigma$ to be the permutation where (1) if vertex u is part of the matching, then σ_u is the vertex that u is matched with and (2) if u is part of the cycles, then σ_u is the next vertex is the directed cycle. Note that $\prod_{i=0}^{n-1} a_{i\sigma_i} \neq 0$.

Let $\sigma \in S_n$. We will show that $P = \prod_{i=0}^{n-1} a_{i\sigma_i}$ directed $(1,2)$-factors map to σ. First, note that if $P = 0$, then σ is not in the range of g. If $P \neq 0$, then we know that the edges (v, σ_v) are in G for every vertex v. Cycles (in the permutation sense) correspond to cycles in our directed $(1,2)$-factor. However, special care is needed for (permutation-)cycles of length 2. Let C be one of the directed cycles that is not of length 2: for each vertex, we may independently choose any one of the $a_{v\sigma_v}$ edges to leave vertex v. Each (permutation-)cycle of length 2 could correspond to either a matching or a directed cycle of length 2. Since A is symmetric, we have that $a_{v\sigma_v} = a_{\sigma_v v}$. Let $k = a_{v\sigma_v}$. There are $2\binom{k}{2}$ ways to form a cycle of length 2 (choose the two edges and the cycle's orientation) and there are k choices if we wish to make this edge part of the matching. Thus, in total, we have $2\binom{k}{2} + k = k^2 = a_{v\sigma_v}a_{\sigma_v v}$ independent choices for this (permutation-)cycle of length 2. Thus, in total, there are P directed $(1,2)$-factors that map to σ, which completes the proof. □

Example 3.16. All $(1,2)$-factors of the graph in Figure 3.1.

 2 cycles Contributes 2^2 to per(A)	 2 cycles Contributes 2^2 to per(A)	 3 cycles Contributes 2^3 to per(A)
 2 cycles Contributes 2^2 to per(A)	 2 cycles Contributes 2^2 to per(A)	 3 cycles Contributes 2^3 to per(A)
 2 cycles Contributes 2^2 to per(A)	 2 cycles Contributes 2^2 to per(A)	 0 cycles Contributes 2^0 to per(A)

Thus, per$(A) = 2^2 + 2^2 + 2^3 + 2^2 + 2^2 + 2^3 + 2^2 + 2^2 + 2^0 = 41$, which can also be verified directly.

Lemma 3.17. *Let* $n \equiv 1 \pmod 2$ *and* A *be the adjacency matrix of a multigraph on* n *vertices. Then* $\mathrm{per}(A) \equiv 0 \pmod 2$.

Proof. Since n is odd, the graph cannot contain any perfect matchings, so each $(1,2)$-factor must contain at least one cycle. Theorem 3.15 completes the proof. □

Theorem 3.18. *Let* A *be the adjacency matrix of a multigraph on* n *vertices that has* P *perfect matchings.*

- *If* $\nu(A) = 0$, *then* $\mathrm{per}(A) \equiv 1 \pmod 4$, $\det(A) \equiv (-1)^{n/2} \pmod 4$ *and* P *is odd.*
- *If* $\nu(A) = 1$, *then* $\mathrm{per}(A) \equiv \det(A) \pmod 4$ *and* P *is even.*
- *If* $\nu(A) \geqslant 2$, *then* $\mathrm{per}(A) \equiv \det(A) \equiv 0 \pmod 4$ *and* P *is even.*

Proof. Each of the statements about the number of perfect matchings will follow directly from Theorem 3.15 and the parity of $\mathrm{per}(A)$ determined below.

Let $\mathcal{A}_n$ denote the set of even permutations of $\{0, \dots, n-1\}$. Define

$$X = \sum_{\sigma \in \mathcal{A}_n} \prod_i A_{i\sigma(i)} \quad \text{and} \quad Y = \sum_{\sigma \notin \mathcal{A}_n} \prod_i A_{i\sigma(i)}.$$

Then $\mathrm{per}(A) = X + Y$ and $\det(A) = X - Y$ by definition.

Case 1. $\nu(A) = 0$, so we know $\mathrm{per}(A) \equiv \det(A) \equiv 1 \pmod 2$.
There is an odd number of perfect matchings, which means that n must be even (since there are no perfect matchings when n is odd). Theorem 3.15 tells us that modulo 4, we may ignore all $(1,2)$-factors with more than 1 cycle.

Let f_c be the number of $(1,2)$-factors that contain a single c-cycle and an $\frac{n-c}{2}$-matching (and f_0 is the number of perfect matchings). Each permutation counted in f_c has parity $\frac{n-c}{2} + c + 1 \equiv \frac{n+c}{2} + 1 \pmod 2$, which determines if it is counted in X or Y. Since n is even and $\det(A)$ is odd, [4, Theorem 1] gives us $\det(A) \equiv (-1)^{n/2} \pmod 4$.

- If $n \equiv 0 \pmod 4$, then $Y \equiv 2\sum_{c \equiv 0 \pmod 4} f_c \equiv 0 \pmod 2$. By definition, we have that $\mathrm{per}(A) - \det(A) = 2Y \equiv 0 \pmod 4$, so $\mathrm{per}(A) \equiv \det(A) \equiv 1 \pmod 4$.
- If $n \equiv 2 \pmod 4$, then $X \equiv 2\sum_{c \equiv 0 \pmod 4} f_c \equiv 0 \pmod 2$. By definition, we have that $\mathrm{per}(A) + \det(A) = 2X \equiv 0 \pmod 4$, so $\mathrm{per}(A) \equiv -\det(A) \equiv 1 \pmod 4$.

Which completes this case.

Case 2. $\nu(A) \geqslant 1$, so we know $\mathrm{per}(A) \equiv \det(A) \equiv 0 \pmod 2$.
The terms that contribute to X or to Y come in pairs that are symmetric about the main diagonal (since σ and σ^{-1} have the same parity) except for the involutions. Involutions that hit the main diagonal pick up a factor of 2, so only the involutions which miss the main diagonal matter. All the involutions which miss the main diagonal have the same sign. So one of X or Y is even.

If X is even, then $\mathrm{per}(A) + \det(A) = 2X \equiv 0 \pmod 4$ which gives us $\mathrm{per}(A) \equiv \det(A) \pmod 4$ since $\mathrm{per}(A)$ and $\det(A)$ are even. Similarly, if Y is even, then $\mathrm{per}(A) - \det(A) = 2Y \equiv 0 \pmod 4$ and hence $\mathrm{per}(A) \equiv \det(A) \pmod 4$. Thus we have proven that $\mathrm{per}(A) \equiv \det(A) \pmod 4$ whenever $\nu(A) \geqslant 1$. If $\nu(A) \geqslant 2$, then $\det(A) \equiv 0 \pmod 4$ (since A can be row reduced over $\mathbb{Z}_2$ to have two zero rows, it can be row reduced over $\mathbb{Z}$ to have two all even rows, which then means that $4 \mid \det(A)$). □

At this point, we can combine some previous results to give the following.

Corollary 3.19. *Let A be the adjacency matrix of a k-regular multigraph. If $\nu(A) = 1$, then* $\text{per}(A) \equiv \det(A) \equiv k \pmod 4$.

Proof. Note that for any symmetric matrix of rank r, there exists a principal submatrix of full rank ([44, Corollary 8.9.2] or [65, Theorem 5.19]). This can be accomplish by simply selecting r linearly independent rows (which exist since the rank is r) and consider the submatrix formed by those rows and the corresponding columns (since the matrix is symmetric). Since $\nu(A) = 1$, there is a principal $(n-1)\times(n-1)$ submatrix, A^*, whose $\mathbb{Z}_2$-rank is $n-1$. So we have $\det(A^*) \equiv 1 \pmod 2$, and thus $\text{per}(A^*) \equiv 1 \pmod 2$. Since A^* is the adjacency matrix of some multigraph, Lemma 3.17 tells us that $n-1$ must be even, and thus, n must be odd.

Since n is odd, the handshaking lemma tells us that k must be even and so we may combine Theorem 3.18 with either Theorem 3.6 or Theorem 3.10 to give the desired result. □

Moreover, the same observation gives us the following.

Corollary 3.20. *Let $n \equiv 0 \pmod 2$ and A be the adjacency matrix of a $2k$-regular multigraph with n vertices. Then* $\text{per}(A) \equiv 0 \pmod 4$.

Proof. Since all rows of A have even row sums, $\det(A) \equiv 0 \pmod 2$, so $\nu(A) \neq 0$. As stated above, $\nu(A) \neq 1$ since n is even, so $\nu(A) \geqslant 2$. By applying Theorem 3.18, the result follows. □

We may now combine Theorem 3.6 and Corollary 3.20 to give the following.

Corollary 3.21. *Let A be the adjacency matrix of a $4k$-regular graph. Then* $\text{per}(A) \equiv 0 \pmod 4$.

3.3 Permanental Minors

Given an $n \times n$ matrix A, we use $A(i \mid j)$ to denote the $(n-1)\times(n-1)$ matrix obtained by deleting row i and column j from A. We are concerned with the permanent of this submatrix in a similar way to minors when computing the determinant.

Theorem 3.22. *Let $A \in M_n(\mathbb{Z})$ for $n > 1$. Then*

- $\text{per}(A(i \mid j)) \equiv 0 \pmod 2$ *for all i, j iff $\nu(A) \geqslant 2$.*
- $\text{per}(A(i \mid j)) \equiv 1 \pmod 2$ *for all i, j iff $\nu(A) = 1$ and all row and column totals of A are even.*

Proof. It suffices to show analogous properties for determinants since the determinant and permanent agree modulo 2. All calculations in this proof will be working over $\mathbb{Z}_2$.

If $\nu(A) \geqslant 2$, then for all i, j we know that $\nu(A(i \mid j)) \geqslant 1$ so $\det(A(i \mid j)) \equiv 0 \pmod 2$.

If $\nu(A) = 0$, then A has an inverse so $\text{Adj}(A)$ has full rank and hence is not a multiple of J (given that $n > 1$). Hence not all minors of A are equal.

So suppose that $\nu(A) = 1$, and hence $\det(A) = 0$. Since $\nu(A) = 1$ there is at least one minor of A that equals 1.

If there is any row or column of A with odd sum, then expanding the determinant in that row/column shows that A has at least one minor which is zero, and hence not all minors are equal.

It remains to treat the case where each row and column sum of A is even. It suffices to show $\det(A(0 \mid 0)) \equiv \det(A(1 \mid 0)) \pmod 2$. But

$$\begin{aligned}\det(A(0 \mid 0)) + \det(A(1 \mid 0)) &= \det \begin{pmatrix} a_{01}+a_{11} & a_{02}+a_{12} & \cdots & a_{0,n-1}+a_{1,n-1} \\ a_{21} & a_{22} & \cdots & a_{2,n-1} \\ a_{31} & a_{32} & \cdots & a_{3,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-1,1} & a_{n-1,2} & \cdots & a_{n-1,n-1} \end{pmatrix} \\ &\equiv 0 \pmod 2,\end{aligned}$$

since the rows add to the zero vector. □

Since $\nu(A) \geqslant 1$ whenever all row totals are even, we have:

Corollary 3.23. *Let $A \in M_n(\mathbb{Z})$ such that all row and column sums are even. Then $\operatorname{per}(A(a \mid c)) \equiv \operatorname{per}(A(b \mid d)) \pmod 2$ for all a, b, c, d.*

Lemma 3.24. *Let $n \equiv 1 \pmod 2$ and $k \equiv 2 \pmod 4$. If $A \in \Lambda_n^k$, then*

$$\operatorname{per}(A(a \mid c)) + \operatorname{per}(A(b \mid c)) + \operatorname{per}(A(a \mid d)) + \operatorname{per}(A(b \mid d)) \equiv 0 \pmod 4$$

for any a, b, c, d.

Proof. First, if $a = b$ (or symmetrically, $c = d$), then Corollary 3.23 gives us $2(\operatorname{per}(A(a \mid c)) + \operatorname{per}(A(a \mid d))) \equiv 0 \pmod 4$. From here, it suffices to do the $a = c = 0$, $b = d = 1$ case. Define

$$B = \begin{pmatrix} a_{00}+a_{10}+a_{01}+a_{11} & a_{02}+a_{12} & \cdots & a_{0,n-1}+a_{1,n-1} \\ a_{20}+a_{21} & a_{22} & \cdots & a_{2,n-1} \\ a_{30}+a_{31} & a_{32} & \cdots & a_{3,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-1,0}+a_{n-1,1} & a_{n-1,2} & \cdots & a_{n-1,n-1} \end{pmatrix}.$$

Note that B has order $n-1$ and that its first row and column each sum to $2k \equiv 0 \pmod 4$, while its other rows and columns each sum to k. Also, by multilinearity of the permanent,

$$\begin{aligned}\operatorname{per}(B) &= \operatorname{per} \begin{pmatrix} a_{01}+a_{11} & a_{02}+a_{12} & \cdots & a_{0,n-1}+a_{1,n-1} \\ a_{21} & a_{22} & \cdots & a_{2,n-1} \\ a_{31} & a_{32} & \cdots & a_{3,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-1,1} & a_{n-1,2} & \cdots & a_{n-1,n-1} \end{pmatrix} \\ &\quad + \operatorname{per} \begin{pmatrix} a_{00}+a_{10} & a_{02}+a_{12} & \cdots & a_{0,n-1}+a_{1,n-1} \\ a_{20} & a_{22} & \cdots & a_{2,n-1} \\ a_{30} & a_{32} & \cdots & a_{3,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-1,0} & a_{n-1,2} & \cdots & a_{n-1,n-1} \end{pmatrix} \\ &= \operatorname{per}(A(0 \mid 0)) + \operatorname{per}(A(1 \mid 0)) + \operatorname{per}(A(0 \mid 1)) + \operatorname{per}(A(1 \mid 1)).\end{aligned}$$

Next, apply (3.1) to calculate $\mathrm{per}(B)$:

$$\mathrm{per}(B) = (-1)^{n-1} \sum_{S \subset \{0,\dots,n-2\}} (-1)^{|S|} \prod_{i=0}^{n-2} \sum_{j \in S} b_{ij}.$$

Fix a set S_0 and consider the terms corresponding to S_0 and its complement in the summation. For each pair,

$$\begin{aligned}
&(-1)^{n-1-|S_0|} \left(\prod_{i=0}^{n-2} \sum_{j \in S_0} b_{ij} + \prod_{i=0}^{n-2} \sum_{j \notin S_0} b_{ij} \right) \\
=&(-1)^{n-1-|S_0|} \left(\prod_{i=0}^{n-2} x_i + (2k - x_1) \prod_{i=1}^{n-2} (k - x_i) \right) \\
\equiv&(-1)^{n-1-|S_0|} \left(2 \prod_{i=0}^{n-2} x_i - k \sum_{i=1}^{n-2} \frac{1}{x_i} \prod_{i=0}^{n-2} x_i \right) \pmod 4. \qquad (3.7)
\end{aligned}$$

where

$$x_i = \sum_{j \in S_0} b_{ij}.$$

Now $\sum_i x_i$ is even, so an even number of the x_i's are even. If this number is non-zero, then (3.7) is clearly 0 modulo 4. So we may assume that every x_i is odd. But then (3.7) is 0 modulo 4 again, since each term in the sum is odd and there is an odd number of terms. □

Lemma 3.25. *Let* $n \equiv 1 \pmod 2$ *and* $k \equiv 2 \pmod 4$*. If* $A \in \Lambda_n^k$*, then* $\mathrm{per}(A) + 2\,\mathrm{per}(\overline{A}) \equiv 0 \pmod 4$*, where* $\overline{A} = J - A$*.*

Proof. By inclusion-exclusion,

$$\mathrm{per}(\overline{A}) = \sum_{i=0}^{n} (-1)^i (n-i)! \sigma_i(A) \equiv \sigma_{n-1}(A) - \mathrm{per}(A) \pmod 2,$$

where $\sigma_i(A)$ the sum of the permanents of all $i \times i$ submatrices of A. However, $\mathrm{per}(A)$ is even by Proposition 3.5, so $\mathrm{per}(A) + 2\,\mathrm{per}(\overline{A}) \equiv \mathrm{per}(A) + 2\sigma_{n-1}(A) \pmod 4$. Next, define an $n \times n$ matrix $C = [c_{ij}]$ by $c_{ij} = \mathrm{per}(A(i \mid j)) \pmod 4$. By Corollary 3.23 and Lemma 3.24, we know that each pair or rows (resp., columns) of C either agree in every position or disagree in every position. Hence, up to row and column permutations, C has the block form

$$\begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix}$$

where each block is constant, and each entry in $C_2 \cup C_3$ exceeds each entry in $C_1 \cup C_4$ by exactly 2. Note that C_2, C_3, C_4 may be empty matrices. Partition A into 4 blocks

$$\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}$$

whose dimensions match the corresponding block of C. Define $n_{r,i}$ to be the row total of row r in block A_i. Now consider calculating $\mathrm{per}(A) \pmod 4$ by taking an expansion along row r:

$$\mathrm{per}(A) \equiv \sum_{i=0}^{n-1} a_{ri} c_{ri} \pmod 4 \qquad (3.8)$$

The answer must be independent of r, which means that $n_{r,1} \pmod 2$ is constant for $0 \leqslant r < s$, where s is the number of rows in A_1. Analogous statements hold for each block. Now (at least) one of the 4 blocks have an even number of rows, so that block contains an even number of 1's. But then all four blocks must contain an even number of 1's since A has even row and column sums. Indeed, we can show using the invariance of (3.8) and the existence of a block with odd dimensions, that each block in A has even row and column totals.

It follows that the entries of C are even iff $\mathrm{per}(A) \equiv 0 \pmod 4$, and the entries of C are odd iff $\mathrm{per}(A) \equiv 2 \pmod 4$. Since $\sigma_{n-1}(A)$ is the sum of the n^2 entries of C, the result follows. □

Computationally, it seems that there are further patterns than we have mentioned above. However, we have not been able to prove these. Moreover due to the computational complexity of computing the permanent, not enough data has been collected to consider any of these formal conjectures.

It seems that in the case of adjacency matrices, Corollary 3.12 is not a tight bound. We have already seen this in Theorem 3.18, where an extra factor of 2 is guaranteed when $\nu = 2$. We also seem to get an extra factor of 2 for $\nu = 3$ and $\nu = 6$ (Corollary 3.12 guarantees 2^2 and 2^3, while it seems that for adjacency matrices that we get 2^3 and 2^4, respectively). This could indicate a larger pattern: if A is an adjacency matrix of a multigraph and $\nu(A) \equiv 2 \pmod 4$ or $\nu(A) \equiv 3 \pmod 4$, then it seems that we can get an extra factor on top of what is guaranteed from Corollary 3.12.

When we are concerned with k-regular (but not necessarily symmetric) matrices, it seems like there is more to say when k is even. Table 3.1 gives some data which is true for all matrices that we have searched.

$k = 4$ and $k = 8$			
n even		n odd	
$\nu(A)$	per(A)	ν	per(A)
$\nu = 1^*$	$2 \pmod 4$	$\nu = 1^*$	$0 \pmod 4$
$\nu = 2^*$	$0 \pmod 4$	$\nu = 2^*$	$0 \pmod 4$
$\nu \geqslant 3$	$0 \pmod 8$	$\nu = 3^*$	$0 \pmod 4$
		$\nu \geqslant 4$	$0 \pmod 8$

$k = 6$			
n even		n odd	
$\nu(A)$	per(A)	ν	per(A)
$\nu = 1$	$2 \pmod 8$	$\nu = 1^*$	$6 \pmod 8$
$\nu = 2$	$4 \pmod 8$	$\nu = 2$	$4 \pmod 8$
$\nu \geqslant 3$	$0 \pmod 8$	$\nu \geqslant 3$	$0 \pmod 8$

$k = 10$			
n even		n odd	
$\nu(A)$	per(A)	ν	per(A)
$\nu = 1$	$2 \pmod 8$	$\nu = 1^*$	$2 \pmod 8$
$\nu = 2$	$4 \pmod 8$	$\nu = 2$	$4 \pmod 8$
$\nu \geqslant 3$	$0 \pmod 8$	$\nu \geqslant 3$	$0 \pmod 8$

Table 3.1: Some patterns found by searching small k-regular matrices of order n. Entries marked with a * are proven in this chapter.

Chapter 4

Counting Transversals

In the binary system, we count
on our fists instead of on our fingers.
– Unknown

In Chapter 7, we discuss the difficulty of counting transversals efficiently. In this chapter, we examine other properties about the number and configuration of transversals.

4.1 Parity of the Number of Transversals

In this section, we lay out the ideas used first by Balasubramanian [7] then again by Akbari and Alipour [3] to count the number of transversals in even-ordered Latin squares modulo 2. For consistency with Akbari and Alipour, we define $E_m = E_m(L)$ to be the number of diagonals in L with exactly m symbols. In particular, $E_n(L)$ is the number of transversals in L. The key idea is to count the number of transversals using inclusion-exclusion.

Definition 4.1. Let $f(x_1, \ldots, x_n)$ be an arbitrary polynomial from $\mathbb{R}^n$ to $\mathbb{R}$. Then $\langle r \rangle f$ denotes the sum of the values of f at the $\binom{n}{r}$ vectors in $\mathbb{R}^n$ which have r coordinates equal to 1 and $n - r$ coordinates equal to 0.

The following result is a slight generalisation of both [7, Lemma 2] and [3, Theorem 2.1]. The proof given in [3] works in this more general case as well.

Lemma 4.2. *Let $f(x_1, \ldots, x_n)$ be an arbitrary polynomial from $\mathbb{R}^n$ to $\mathbb{R}$. Then the sum of the coefficients of monomials in f containing exactly m distinct variables is*

$$\sum_{r=0}^{m} (-1)^{m-r} \binom{n-r}{n-m} \langle r \rangle f.$$

For any transversal, $\{(r_i, c_i, s_i)\}$, we define three corresponding permutations: $\sigma_r(r_i) = c_i$, $\sigma_c(c_i) = s_i$ and $\sigma_s(s_i) = r_i$. The following proposition is immediate.

Proposition 4.3. *Let $T = \{(r_i, c_i, s_i)\}$ be a transversal of a Latin square L with corresponding permutations σ_r, σ_c and σ_s. Then $\sigma_r \circ \sigma_c \circ \sigma_s = id$.*

Proof. For any given symbol s_i, we have $s_i \xrightarrow{\sigma_s} r_i \xrightarrow{\sigma_r} c_i \xrightarrow{\sigma_c} s_i$. □

As an immediate corollary of Proposition 4.3, we have that $\epsilon(\sigma_r) + \epsilon(\sigma_c) + \epsilon(\sigma_s) = 0$, where $\epsilon : S_n \to \mathbb{Z}_2$ is the parity of a permutation in the usual sense (computation done modulo 2). Thus, for any given transversal, we give it one of four types: $T^{000}, T^{011}, T^{101}$ or T^{110} corresponding to the parity of σ_r, σ_c and σ_s, respectively. We now use these parities to aid in counting.

Definition 4.4. Let L be a Latin square of order n. The *parity of a transversal* is the parity of the transversal's corresponding σ_r. We define $E_n^{\pm}(L)$ to be the difference between the number of transversals in L with $\epsilon(\sigma_r) = 0$ (even transversals) and the number transversals of L with $\epsilon(\sigma_r) = 1$ (odd transversals).

Though the symbols in a Latin square L are normally $\mathbb{Z}_n$, we sometimes need to utilise the corresponding matrix with each symbol i replaced with a variable x_i in order to aid in our counting. This new matrix is denoted as $L[X]$.

Theorem 4.5. *Let L be a Latin square of order n. Then*

$$E_n(L) = \sum_{r=0}^{n} (-1)^{n-r} \langle r \rangle \operatorname{per}(L[X]) \tag{4.1}$$

and

$$E_n^{\pm}(L) = \sum_{r=0}^{n} (-1)^{n-r} \langle r \rangle \det(L[X]). \tag{4.2}$$

Proof. The terms in $\operatorname{per}(L[X])$ which consist of n distinct variables correspond to the transversals in L. We may then use Lemma 4.2 with $m = n$ and $f = \operatorname{per}$.

Similarly, the terms in $\det(L[X])$ correspond to the transversals of L up to sign. Any transversal which has even parity increases the sum by 1 and any transversal which has odd parity decreases the sum by 1. Using Lemma 4.2, we are done. □

The following lemma is a slight generalisation of [63, Lemma 1].

Lemma 4.6. *Let $A = [a_{ij}]$ be a $(0,1)$-matrix of even order. Define $A^* = [b_{ij}]$ by*

$$b_{ij} = \begin{cases} a_{ij} & \text{if the } i^{th} \text{ row has an even number of ones,} \\ 1 - a_{ij} & \text{otherwise.} \end{cases}$$

Then $\det(A) + \det(A^*)$ *is even.*

Proof. Let δ be a row vector of all-ones. Without loss of generality, we may assume that the first k rows of A have odd sum and the remaining rows of A have even sum (this may alter the sign of the determinant, but this does not matter modulo 2). Thus, we have

$$\pm\det(A^*) = \det \begin{pmatrix} \delta - A_0 \\ \delta - A_1 \\ \vdots \\ \delta - A_{k-1} \\ A_k \\ \vdots \\ A_{n-1} \end{pmatrix} = \det \begin{pmatrix} \delta - A_0 \\ A_0 - A_1 \\ \vdots \\ A_0 - A_{k-1} \\ A_k \\ \vdots \\ A_{n-1} \end{pmatrix}$$

$$= \det \begin{pmatrix} -A_0 \\ A_0 - A_1 \\ \vdots \\ A_0 - A_{k-1} \\ A_k \\ \vdots \\ A_{n-1} \end{pmatrix} + \det \begin{pmatrix} \delta \\ A_0 - A_1 \\ \vdots \\ A_0 - A_{k-1} \\ A_k \\ \vdots \\ A_{n-1} \end{pmatrix} = (-1)^k \det(A) + \det \begin{pmatrix} \delta \\ A_0 - A_1 \\ \vdots \\ A_0 - A_{k-1} \\ A_k \\ \vdots \\ A_{n-1} \end{pmatrix}.$$

Note that each of the row sums of the matrix on the right is now even, so its determinant is also even. Thus, considering the equality modulo 2, we have $\det(A^*) \equiv \det(A) \pmod 2$, and the result follows immediately. □

Note that Lemma 4.6 is usually utilised in the case where the row sums of a matrix are all the same. If all of the row sums are even, then the result shows nothing interesting. However, when each row sum is odd, Lemma 4.6 tells us that $\det(A) + \det(J - A) \equiv 0 \pmod 2$.

Balasubramanian [7] used Lemma 4.6 and (4.1) to show the following theorem. We give full details of the proof here as we use a similar technique below.

Theorem 4.7 ([7])**.** *Let $n \equiv 0 \pmod 2$ and L be a Latin square of order n. Then L has an even number of transversals.*

Proof. Note that $E_n(L) \equiv E_n^{\pm}(L) \pmod 2$. We pair up complementary terms in (4.2). That is, each term of the sum $\langle r\rangle \det(L[X])$ is paired with the unique term in $\langle n - r\rangle \det(L[X])$ for which the zero-one vectors sum to the all-ones vector. For each of these pairs of terms, we have one of two situations. If r is even, then $n-r$ is also even and so both of their determinants are even since their row sums are even (see proof of Proposition 3.5). Alternatively, if r is odd, then each row sum in $L[X]$ is odd, so $\det(L[X])+\det(J-L[X]) \equiv 0 \pmod 2$ by Lemma 4.6. Thus, each of the 2^{n-1} pairs contributes a multiple of two to the summation and the result follows. □

To proceed, we need a few extra linear algebraic results.

Lemma 4.8. *Let $n \equiv 0 \pmod 2$ and $k \equiv 0 \pmod 2$. If $A \in \Lambda_n^k$, then $\det(A) \equiv 0 \pmod 4$.*

Proof. Since $A \in \Lambda_n^k$, we have that $\det(A)$ is a multiple of $k \cdot \gcd(n, k)$ (see [63, Theorem 2]). The desired result follows directly since k and $\gcd(n, k)$ are both even. □

Lemma 4.9. *Let $n \equiv 2 \pmod 4$ and $k \equiv 1 \pmod 2$. If $A \in \Lambda_n^k$, then*

$$\det(A) \equiv -\det(J - A) \pmod 4.$$

Proof. Since $A \in \Lambda_n^k$, we have that $k \det(J - A) = (-1)^{n-1}(n - k) \det(A)$ (see [63, Lemma 1]). The result follows by noting that $k \equiv n - k \pmod 4$ and that k has a multiplicative inverse modulo 4. □

We now have enough framework to show our first result about counting transversals. The layout of the proof is very similar to that of Theorem 4.7.

Theorem 4.10. *Let $n \equiv 2 \pmod 4$ and L be a Latin square of order n, then*

$$E_n^{\pm}(L) \equiv 0 \pmod 4.$$

Proof. Similar to the above proofs, we pair up complementary terms in (4.2). That is, each term of the sum $\langle r \rangle \det(L[X])$ is paired with the unique term in $\langle n-r \rangle \det(L[X])$ for which the zero-one vectors sum to the all-ones vector. For each of these pairs of terms, we have one of two situations. If r is even, then $n-r$ is also even and we use Lemma 4.8 twice to show that both the terms are a multiple of four. Alternatively, if r is odd, then we use Lemma 4.9 to show that their sum is a multiple of four. Thus, each of the 2^{n-1} pairs contributes a multiple of four to the summation, so the result follows. □

Somewhat surprisingly, Theorem 4.10 gives us enough structure to show the main result of this chapter, which strengthens the result of Balasubramanian for Latin squares of singly-even order.

Theorem 4.11. *Let $n \equiv 2 \pmod 4$ and L be a Latin square of order n. Then*

$$E_n(L) \equiv 0 \pmod 4.$$

Proof. Let $T = \{\{(r_i, c_i, s_i)\}\}$ be the set of transversals of L. We define x, y, z and w to be the number of transversals of type T^{000}, T^{011}, T^{101} and T^{110}, respectively.

By definition, we have

$$x + y + z + w = E_n(L) \tag{4.3}$$

and

$$x + y - z - w = E_n^{\pm}(L). \tag{4.4}$$

Let L' be the (312)-conjugate of L (the Latin square obtained by applying the permutation (312) to each triple in L). There is a natural bijection between T and the set of transversals of L'. Each transversal in L' must be of the form $\{(c_i, s_i, r_i)\}$, where $\{(r_i, c_i, s_i)\} \in T$. The parity of each transversal in L' depends on the corresponding σ_c of the transversal in L, so we have

$$x - y + z - w = E_n^{\pm}(L'). \tag{4.5}$$

Similarly, if L'' is the (231)-conjugate of L, then each transversal of L'' must be of the form $\{(s_i, r_i, c_i)\}$, whose parity is equal to that of σ_s. So we have

$$x - y - z + w = E_n^{\pm}(L''). \tag{4.6}$$

The sum of (4.3) to (4.6) gives us

$$4x = E_n(L) + E_n^{\pm}(L) + E_n^{\pm}(L') + E_n^{\pm}(L''). \tag{4.7}$$

Theorem 4.10 applied to L, L' and L'' tells us that $E_n(L) \equiv 0 \pmod 4$. □

Based on computation of small squares, it seems that Theorem 4.7 and Theorem 4.11 are the only general modular restrictions on the number of transversals of a Latin square. We generated many random Latin squares and for each order $8 \leqslant n \leqslant 18$, we were able to find a Latin square with $k \pmod m$ transversals for each pair $k, m \leqslant 16$ that does not contradict Theorem 4.7 and Theorem 4.11. Note that for $n < 8$, there are some sporadic values of $k, m \leqslant 16$ where no Latin square of order n contains $k \pmod m$ transversals, however, we believe that these are just due to the small size of the Latin squares, not a modular restriction.

The proof of Theorem 4.11 leads us to the following interesting property.

Corollary 4.12. *Let $n \equiv 2 \pmod 4$ and L be a Latin square of order n and T be the set of transversals of L. The number of transversals of types T^{000}, T^{110}, T^{101} and T^{110} are all equal modulo 2.*

Proof. Define x, y, z and w as in Theorem 4.11. Examining (4.3) + (4.4) tells us that $2x + 2y = E_n(L) + E_n^{\pm}(L) \equiv 0 \pmod 4$ (by Theorem 4.10 and Theorem 4.11) which gives us that $x \equiv y \pmod 2$. Similarly, $(4.3)+(4.5)$ and $(4.3)+(4.6)$ tells us $x \equiv z \pmod 2$ and $x \equiv w \pmod 2$, respectively. □

It is important to remark that Theorem 4.11 is less general in one respect than the proof of Balasubramanian. Balasubramanian proved that the number of transversals in any even-ordered row-Latin square is even. Theorems 4.10 and 4.11 cannot be used in the general case since both the rows and columns of the $(0, 1)$-matrices need to be regular. Moreover, here are two examples of row-Latin squares whose number of transversals is not a multiple of 4. In the following example, the row-Latin square of order 2 has 2 transversals and the row-Latin square of order 6 has 6 transversals.

0	1
0	1

0	2	5	1	4	3
1	0	4	5	3	2
2	1	3	0	4	5
3	1	0	4	5	2
4	1	2	5	0	3
5	4	1	2	3	0

The ideas given in the proofs of Theorem 4.11 and Corollary 4.12 leave us with the following conjecture, which seems to hold computationally for small values.

Conjecture 4.13. Let $n \equiv 0 \pmod 4$ and L be a Latin square of order n. Let x, y, z and w be the number of transversals in L of types $T^{000}, T^{011}, T^{101}$ and T^{110}, respectively. Then

(a) $E_n(L) \equiv E_n^{\pm}(L) \pmod 4$ and

(b) $x \equiv y \equiv z \equiv w \pmod 2$

Note that Conjecture 4.13 is not true for many odd-ordered Latin squares.

Proposition 4.14. *The conditions (a) and (b) in Conjecture 4.13 are equivalent.*

Proof. Since we know that $E_n(L) \equiv 0 \pmod 2$ when n is even, showing (a) is equivalent to showing $E_n(L) + E_n^{\pm}(L) \equiv 0 \pmod 4$. We may then use the same idea as the proof of Corollary 4.12 to show the result. □

We should note that in the study of Latin squares, other notions of parity have previously been examined. Typically, the parity (or parities) of a Latin square are determined by permutations derived from its rows, columns and symbols. The parity of Latin squares has been quite useful in several areas of study. For example, Francetić *et al.* [40] used parity to study MOLS, while Lefevre *et al.* [57] used parity arguments to study the biembeddability of graphs from Latin squares on certain surfaces.

4.2 Patterns

While proving the results in the previous section, a number of patterns were discovered. In this section, we show the patterns that were unearthed and provide proofs for most of them.

4.2.1 Patterns from Depleted Latin Squares

In this section, we examine patterns and results found in *depleted Latin squares*: arrays formed by removing a row and/or a column of a Latin square. This depleted Latin square is a Latin array.

We define $t_{ij}(L)$ to be the number of transversals in the Latin array formed by deleting the ith row and jth column of L. When clear from context, the shorthand t_{ij} is used.

Theorem 4.15. *Let L be a row-Latin square of order n. Then for all i, j, k,*

$$t_{ij} \equiv t_{ik} \pmod 2.$$

Proof. Without loss of generality, we may assume that $i = 0$, $j = 0$, $k = 1$. Let $L[X] = [a_{ij}]$. We define $A_0[X]$ and $A_1[X]$ as

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ a_{1,0} & a_{1,1} & a_{1,2} & \cdots & a_{1,n-1} \\ & \vdots & & \ddots & \vdots \\ a_{n-1,0} & a_{n-1,1} & a_{n-1,2} & \cdots & a_{n-1,n-1} \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ a_{1,0} & a_{1,1} & a_{1,2} & \cdots & a_{1,n-1} \\ & \vdots & & \ddots & \vdots \\ a_{n-1,0} & a_{n-1,1} & a_{n-1,2} & \cdots & a_{n-1,n-1} \end{pmatrix},$$

respectively. Note that t_{ij} is the number of terms in $\operatorname{per}(A_0[X])$ which have exactly $n-1$ symbols. Thus, by Lemma 4.2,

$$t_{ij} = \sum_{r=0}^{n-1} (-1)^{n-1-r}(n-r)\langle r\rangle \operatorname{per}(A_0[X])$$

and

$$t_{ik} = \sum_{r=0}^{n-1} (-1)^{n-1-r}(n-r)\langle r\rangle \operatorname{per}(A_1[X]).$$

Furthermore, note that $\operatorname{per}(A_0[X]) + \operatorname{per}(A_1[X]) = \operatorname{per}(A[X])$, where

$$A[X] = \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ a_{10} & a_{11} & a_{12} & \cdots & a_{2,n-1} \\ & \vdots & & \ddots & \vdots \\ a_{n-1,0} & a_{n-1,1} & a_{n-1,2} & \cdots & a_{n-1,n-1} \end{pmatrix}.$$

Thus,

$$t_{ij} + t_{ik} = \sum_{r=0}^{n-1} (-1)^{n-1-r}(n-r)\langle r\rangle \operatorname{per}(A[X]) \equiv \sum_{r=0}^{n-1} (n-r)\langle r\rangle \det(A[X]) \pmod 2.$$

If n is odd, then we have two subcases. If r is even, then each row of A has an even sum, and so the determinant is even. If r is odd, then $n-r$ is even, and so each term in the summation is even.

If n is even, then we use a trick similar to Theorem 4.10 by pairing up complementary terms. Every $(0,1)$-vector in $\langle r\rangle$ has a complementary vector in $\langle n-r\rangle$. Note that the first row of A remains the same in each of these cases, but all remaining rows are complemented. We may use either the fact that the rows sums are even or Lemma 4.6 to give us our result. □

This immediately gives us a surprisingly simple result which is the lays the groundwork for the patterns found in the remainder of the section.

Theorem 4.16. *Let L be a Latin square of order n. Then for all i, j, k, ℓ,*

$$t_{ij} \equiv t_{k\ell} \pmod 2.$$

Proof. Since L is a row-Latin square, $t_{ij} \equiv t_{i\ell} \pmod 2$ by Theorem 4.15. Moreover, since the transpose (*i.e.,* the (213)-conjugate) of L is a row-Latin square, $t_{i\ell} \equiv t_{k\ell} \pmod 2$. □

This simple observation leads to several patterns relating to deleting a row and a column of a Latin square.

Corollary 4.17. *Let $n \equiv 0 \pmod 2$ and R be an $(n-1) \times n$ row-Latin rectangle. Then the number of transversals in R is even.*

Proof. Let L be some row-Latin square formed by adding one row to R. By definition, we have that

$$E_{n-1}(R) = t_{n-1,0}(L) + t_{n-1,1}(L) + \cdots + t_{n-1,n-1}(L).$$

Since each of these terms are congruent modulo 2 (by Theorem 4.15) and n is even, the result follows. □

For odd orders, Corollary 4.17 does not generalise as there are some rectangles that have an even number of transversals and other rectangles that have an odd number of transversals. For example, if any row is removed from the Cayley table of any cyclic group of odd order, the resulting Latin rectangle has an odd number of transversals since each near transversal in the cyclic group extends to a transversal (see Theorem 2.23) and the cyclic group has an odd number of transversals when n is odd.

We define $N_r = N_r(L)$ to be the number of diagonals of weight $n-1$ in L where one of the duplicate symbols appear in row r. The following two propositions follow directly from the definition of t_{ij}.

Proposition 4.18. *Let L be a Latin square of order n. Then for any row r,*

$$\sum_{c=0}^{n-1} t_{rc} = E_n + N_r.$$

Proof. Each transversal in the array formed by deleting row r and column c extends to either a transversal of L or a diagonal of weight $n-1$ depending on which symbol is in the cell (r, c). □

Proposition 4.19. *Let L be a Latin square of order n. Then*

$$\sum_{r=0}^{n-1} \sum_{c=0}^{n-1} t_{rc} = nE_n + 2E_{n-1}.$$

Proof. Across the whole summation, each transversal of L is counted n times (once for each entry in the transversal) and each diagonal with weight $n-1$ is counted twice (once for each entry containing the duplicated symbol). □

The next theorem is deceptively strong.

Theorem 4.20. *Let $n \equiv 1 \pmod 2$ and L be a Latin square of order n. Then for any r and c,*

$$t_{rc} \equiv E_n \pmod 2.$$

Proof. By Theorem 4.16 and since n is odd,

$$t_{rc} \equiv \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} t_{ij} \pmod 2.$$

Then Proposition 4.19 gives $t_{rc} \equiv nE_n + 2E_{n-1} \equiv E_n \pmod 2$, as desired. □

An interesting feature of Theorem 4.20 lies in the fact that a transversal of L can be inferred without necessarily locating one. In each of the other previous results, the number of diagonals with specific properties is all of similar form: congruent to 0 modulo m for some m. These results, while enlightening, do not give any insight into the existence of transversals. However, Theorem 4.20 gives a slightly different approach. In particular, if $t_{rc} \equiv 1 \pmod 2$ for any row and column, then there must exist a transversal in L even if none go through the cell (r, c).

Example 4.21. Consider L_5:

0	1	2	3	4
1	0	3	4	2
2	3	4	0	1
3	4	1	2	0
4	2	0	1	3

Every transversal in L_5 goes through the shaded entry. So if your search for a transversal chooses a diagonal that includes any other entry from the first column, the second row or one that contains the symbol 1, you will not be successful. However, the main diagonal is the sole transversal in $L_5(1 \mid 1)$, so $t_{00} = 1$. Thus, we can use Theorem 4.20 to deduce that at least one transversal exists in L_5 without completing an exhaustive search.

Proposition 4.22. *Let L be a Latin square of order n. Then $N_r \equiv 0 \pmod 2$ for any row r.*

Proof. Each term in $\sum_{c=0}^{n-1} t_{rc}$ is the same modulo 2. We examine two separate cases. If n is even, this sum is even. If n is odd, the sum is equivalent to t_{r0} modulo 2. In either case, the sum is equivalent to E_n modulo 2 (either by Theorem 4.7 or Theorem 4.20).

By Proposition 4.18, $\sum_{c=0}^{n-1} t_{rc} = E_n + N_r$ and the result follows. □

We finish the discussion of t_{ij} with a rather curious pattern found for small orders.

Conjecture 4.23. Let L be a Latin square of order n. Then $t_{ik} + t_{jk} + t_{i\ell} + t_{j\ell} \equiv 0 \pmod 4$ for all i, j, k, ℓ.

Conjecture 4.23 implies a very specific structure for the matrix $[t_{ij}]$. Modulo 4, every pair of rows is either identical or complementary. A similar property holds for columns.

4.2.2 Patterns from Inclusion-Exclusion

In this section, we look at patterns centred around the following equation, which was given in [3] and can be derived from Lemma 4.2. We use $R_i = R_i(L)$ as shorthand for $\langle i \rangle \operatorname{per}(L[X])$.

Theorem 4.24. *Let L be a Latin square of order n. Then*

$$E_m = \sum_{r=0}^{m} (-1)^{m-r} \binom{n-r}{n-m} R_r. \tag{4.8}$$

The following proposition is a list of identities which are either immediate from the definition of a Latin square or are proved in [7].

Proposition 4.25. *Let L be a Latin square of order n.*

(a) $R_1 = n$,

(b) $R_{n-1} = nd_n$, *where d_n is the number of derangements of order n,*

(c) $R_n = n!$,

(d) $R_{2i} \equiv 0 \pmod 2$,

(e) $R_i + R_{n-i} \equiv 0 \pmod 2$ *if* $n \equiv 0 \pmod 2$ *and*

(f) $R_{n/2} \equiv 0 \pmod 2$ *if* $n \equiv 0 \pmod 2$.

Balasubramanian used (d) and (e) to show Theorem 4.7, while Akbari and Alipour [3] showed several identities including the following result.

Theorem 4.26 ([3])**.** *Let L be a Latin square of order n. Then $E_{n-1} \equiv 0 \pmod 2$.*

We start with patterns and results about odd-ordered Latin squares, where the picture seems clearer. In Chapter 3, we found special properties about permanents of matrices. The following two corollaries come directly from those.

Corollary 4.27. *Let $n \equiv 1 \pmod 2$ and L be a Latin square of order n. Then $R_{4k} \equiv 0 \pmod 4$.*

Proof. Simply apply Corollary 3.7 on each individual matrix within R_{4k}. □

Corollary 4.28. *Let $n \equiv 1 \pmod 2$ and L be a Latin square of order n. Then*

$$R_{4i+2} + 2R_{n-(4i+2)} \equiv 0 \pmod 4.$$

Proof. Take the complementary pairs in R_{4i+2} and $R_{n-(4i+2)}$ and apply Lemma 3.25 on each pair. □

Based on computation of small ordered Latin squares, it seems that for odd-ordered Latin squares, Proposition 4.25, Corollary 4.27 and Corollary 4.28 are the only congruences satisfied by R_i.

We have the following strengthening of Theorem 4.26 for odd orders:

Theorem 4.29. *Let $n \equiv 1 \pmod 2$ and L be a Latin square of order n. Then $E_{n-1} \equiv 0 \pmod 4$.*

Proof. We compute E_{n-1} utilising (4.8). We pair up the complementary terms in this summation, $(n-r)R_r + rR_{n-r}$. Note that R_0 does not have a complementary term, but $R_0 = 0$, so we may ignore it. Within each of these pairs, we assume that r is even by symmetry. We examine two cases. First, if $r \equiv 0 \pmod 4$, then the second term vanishes modulo 4 and $(n-r)R_r \equiv 0 \pmod 4$ by Corollary 4.27. Alternatively, if $r \equiv 2 \pmod 4$, then R_r is even (by Proposition 4.25(d)), so $R_r \equiv -R_r \pmod 4$. We may use this to see that $(n-r)R_r \equiv R_r \pmod 4$. Thus, $(n-r)R_r + rR_{n-r} \equiv R_r + 2R_{n-r} \pmod 4$. We may now use Corollary 4.28. Each pair of complementary terms sum to a multiple of four, so the result follows. □

We now shift our attention to even-ordered Latin squares, where the results are based on the global relationship between the different R_i values as contrasted with the local nature of Corollary 4.27 and Corollary 4.28 for odd-ordered Latin squares.

Lemma 4.30. *Let $n \equiv 0 \pmod 4$ and L be a Latin square of order n. Then*

$$E_1 + E_3 + \cdots + E_{n-1} \equiv E_2 + E_4 + \cdots + E_n \equiv 0 \pmod 4.$$

Proof. We have,

$$\begin{aligned}\sum_{m=1}^{n/2} E_{2m-1} &= \sum_{m=1}^{n/2}\sum_{r=0}^{2m-1}(-1)^{(2m-1)-r}\binom{n-r}{n-(2m-1)}R_r = \sum_{r=0}^{n-1}(-1)^{n-r-1}\sum_{s=1}^{\lfloor (n-r)/2\rfloor}\binom{n-r}{2s-1}R_r \\ &= \sum_{r=0}^{n-1}(-2)^{n-r-1}R_r \equiv R_{n-1} - 2R_{n-2} \equiv nd_n - 0 \equiv 0 \pmod 4,\end{aligned}$$

where d_n is the number of derangements of order n. Since $\sum_{i=1}^{n} E_i = n! \equiv 0 \pmod 4$, the second congruence follows. □

Lemma 4.31. *Let $n \equiv 0 \pmod 4$ and L be a Latin square of order n. Then*

$$E_{2i-1} \equiv E_{2i} \pmod 2.$$

Proof. We have,

$$\begin{aligned}E_{2i} + E_{2i-1} &= R_{2i} + \sum_{r=1}^{2i-1}\left[\binom{n-r}{n-2i} - \binom{n-r}{n-(2i-1)}\right](-1)^r R_r \\ &= R_{2i} + \sum_{r=1}^{2i-1}\left[\binom{n-r}{n-2i} - \binom{n-r}{n-2i}\left(\frac{2i-r}{n-2i+1}\right)\right](-1)^r R_r \\ &= R_{2i} + \sum_{r=1}^{2i-1}\binom{n-r}{n-2i}\left[\frac{n-4i+r+1}{n-2i+1}\right](-1)^r R_r.\end{aligned}$$

If r is even, then R_r is even. If r is odd, then $n-4i+r+1$ is even, while $n-2i+1$ is odd, so $\binom{n-r}{n-2i}(n-4i+r+1)/(n-2i+1)$ must be even (it is an integer, since our proof shows that it is the difference of two integers). The result follows. □

In general, E_{2i} and E_{2i+1} seem unrelated except in the following situation:

Conjecture 4.32. Let $n \equiv 0 \pmod 4$ and L be a Latin square of order n. Then

$$E_{n-1} \equiv 2E_{n-2} \pmod 4.$$

Note that this conjecture is equivalent to showing that $R_1+R_3+\cdots+R_{n-1} \equiv 0 \pmod 4$ and $R_2 + R_4 + \cdots + R_n \equiv E_n \pmod 4$ by just applying the definition of E_i. When $n \equiv 0 \pmod 4$, we have that $R_1 + R_2 + \cdots + R_n \equiv E_n \pmod 4$.

We conclude with a few additional conjectures for even-ordered Latin squares. These conjectures relate t_{ij} to E_k. It is unclear if these are the only restrictions for even-ordered Latin squares.

Conjecture 4.33. Let $n \equiv 0 \pmod 2$ and L be a Latin square of order n. Then for all i, j,

$$2t_{ij} \equiv E_{n-1} \equiv N_r \pmod 4.$$

Conjecture 4.34. Let $n \equiv 0 \pmod 4$ and L be a Latin square of order n. Then for all i, j,

$$E_n \equiv E_{n-1} \equiv 2E_{n-2} \equiv 2t_{ij} \equiv N_r \pmod 4.$$

Chapter 5

Transversals in Latin Arrays

Now I will have less distraction.
– L. Euler
(Upon losing the use of his right eye)

For $n \times n$ Latin arrays, as the number of distinct symbols increases, there must come a point beyond which it becomes impossible to avoid transversals. This chapter is motivated by the question of when this threshold occurs. Let $\ell(n)$ be the least positive integer such that $\ell(n) \geqslant n$ and every Latin array of order n with at least $\ell(n)$ distinct symbols contains a transversal. This function was introduced by Akbari and Alipour [3], who calculated $\ell(n)$ for $n \leqslant 4$ and showed that $\ell(5) \geqslant 7$ and $\ell(2^k - 2) > 2^k$ for every integer $k > 2$. Counter-intuitively, every Latin square of order 5 contains a transversal, but there is a Latin array of order 5 with six symbols and no transversal. Hence, it is not always true that increasing the number of symbols increases the number of transversals. Nevertheless, $\ell(n)$ is well defined since an $n \times n$ Latin array with n^2 different symbols certainly has a transversal. Akbari and Alipour put forward the following conjectures:

Conjecture 5.1. For every integer $n \geqslant 3$, we have $\ell(n) \leqslant n^2/2$.

Conjecture 5.2. For every integer c, there exists a positive integer n such that $\ell(n) > n+c$.

Up until this point, it was unknown whether there is some constant $c < 1$ such that $\ell(n) \leqslant cn^2$ for every integer $n > 1$. In Section 5.3, we provide a proof of such a result.[1] Later, in Section 7.4, we determine $\ell(n)$ exactly for $n \leqslant 7$.

On first glance, Conjecture 5.1 seems very generous and that maybe $\ell(n)$ even has a linear upper bound. However, the problem is deceptively hard, and the following observation gives some hint as to why.

Proposition 5.3. *Let k be a non-negative integer. If $\ell(n) \leqslant 2kn + n - k^2 - k$ for all n, then every Latin square of order n has a partial transversal of length $n - k$.*

Proof. Let L be any Latin square of order n. Let M be a Latin array of order $n+k$, which has L as the top-left $n \times n$ subarray and all remaining entries are new distinct symbols. The number of symbols in M is $n + 2nk + k^2 \geqslant \ell(n+k)$, so there must be a transversal in M. This transversal hits at most $2k$ cells in the last k rows or columns of M, so it must intersect the copy of L in at least $n-k$ cells, each of which contains a different symbol. □

[1] In [9], we provided two independent proofs of a non-trivial bound: one probabilistic method and one deterministic method. However, the result utilising the probabilistic method will not be shown here as it provides a worse bound and I was not a part of its proof.

Putting $k = 1$, we see that if $\ell(n) \leqslant 3n - 2$ for all n, then every Latin square has a near transversal. This would solve Brualdi's Conjecture. Indeed, any linear upper bound on $\ell(n)$ would imply the existence of a constant c such that every Latin square of order n has a partial transversal of length $n - c$.

There is a broader setting in which quadratically many symbols is known to be best possible, namely row-Latin arrays. For every positive integer n, let $\ell_r(n)$ be the least positive integer such that $\ell_r(n) \geqslant n$ and every $n \times n$ row-Latin array with at least $\ell_r(n)$ distinct symbols contains a transversal. Barát and Wanless [8] showed that $\ell_r(n) > \frac{1}{2}n^2 - O(n)$. In Section 5.3, we prove that $\ell_r(n) \leqslant \left\lceil \frac{1}{4}(5 - \sqrt{5})n^2 \right\rceil$ for every integer $n > 1$.

A related problem is when repeats are allowed within a row or column, but a restriction is placed on how many times a symbol can occur in the entire square. It has been shown in [32, 41] that a transversal must exist if no symbol occurs more than cn times in a square of order n, where c is a small constant. This means that if each symbol occurs roughly the same number of times then linearly many symbols are enough to ensure a transversal.

5.1 Bounds on General Arrays

In this section, we prove a bound on $\ell(n)$ using non-probabilistic methods. We start by proving results about general square arrays, then later use these results to give bounds on the number of symbols in transversal-free row-Latin arrays and transversal-free Latin arrays.

We call a symbol in an array A a *singleton* if it occurs exactly once in A and a *clone* otherwise. We define $R_i(A)$ and $C_j(A)$ to be the set of symbols occurring in row i and column j of A, respectively. As before, let $A(i \mid j)$ denote the array formed from A by deleting row i and column j. Let $\Psi_{ij}(A)$ be the set of symbols that appear in A and not in $A(i \mid j)$.

Lemma 5.4. *Let A be a transversal-free array of order n. If $A(n-1 \mid n-1)$ has a transversal and if $|R_{n-1}(A) \cup C_{n-1}(A)| \geqslant (k+1)n - 1$, then A has at most*

$$\frac{1}{2}(k^2 - 2k + 2)n^2 + \frac{1}{2}(3k - 2)n$$

distinct symbols.

Proof. Assume that T is a near transversal of A that does not meet the last row or column and minimises the number of symbols that it has from $R_{n-1}(A) \cup C_{n-1}(A)$.

We call a symbol *large* if it appears in both T and $R_{n-1}(A) \cup C_{n-1}(A)$ and *small* otherwise. Let λ be the number of large symbols. Permute the first $n-1$ rows and columns of A so that T is located along the main diagonal and all of the large symbols of T appear in the top λ rows. For $0 \leqslant i < n - 1$, note that $A_{i,n-1}$ and $A_{n-1,i}$ cannot be two different small symbols. Otherwise, $(T \setminus \{(i, i, A_{ii})\}) \cup \{(i, n-1, A_{i,n-1}), (n-1, i, A_{n-1,i})\}$ would be a transversal of A. So there are at most $n - 1$ distinct small symbols in the last row and column. Thus,

$$\lambda \geqslant |R_{n-1}(A) \cup C_{n-1}(A)| - (n-1) \geqslant (k+1)n - 1 - (n-1) = kn. \tag{5.1}$$

We now define a subset Γ of the entries of A in which each symbol in A is represented exactly once. We populate Γ in three steps. First, $T \subseteq \Gamma$. Second, for every small symbol s that occurs in the last row or column, select one such entry containing s and add it to Γ. Note that s cannot appear in T, by the definition of "small". Finally, for every symbol

s' in A that does not appear in T or in the last row or column, select one entry with the symbol s' and add it to Γ.

We claim that if (i, j) is in the top λ rows of A with $i < j < n-1$, then at most one of (i, j, A_{ij}) and (j, i, A_{ji}) can be in Γ. Suppose otherwise, and consider

$$\big(T \setminus \{(i, i, A_{ii}), (j, j, A_{jj})\}\big) \cup \{(i, j, A_{ij}), (j, i, A_{ji})\}. \tag{5.2}$$

Note that the symbol A_{ii} is contained in the last row or column of A. By the definition of Γ, we know that (i, j) and (j, i) do not have the same symbol and neither one shares a symbol with any entry in T or in the last row or column. So (5.2) is a near transversal that contains fewer symbols in $R_{n-1}(A) \cup C_{n-1}(A)$ than T, contradicting the choice of T. This implies that within the first λ rows and columns of $A(n-1 \mid n-1)$, there are at least

$$(n-2) + (n-3) + \cdots + (n-\lambda-1) = \lambda n - \frac{\lambda(\lambda+3)}{2}$$

entries not contained in Γ. Within the last row and column of A, there are at most $n-1$ entries in Γ (all containing small symbols), so at least n entries are not in Γ. Thus, the number of distinct symbols in A is

$$|\Gamma| \leqslant n^2 - \left(\lambda n - \frac{\lambda(\lambda+3)}{2}\right) - n = \frac{1}{2}\lambda^2 - \left(n - \frac{3}{2}\right)\lambda + n(n-1). \tag{5.3}$$

This quadratic in λ decreases weakly on the integer points in the interval $kn \leqslant \lambda \leqslant n-1$. Given (5.1), we may substitute $\lambda = kn$ into (5.3) to get the desired result. □

Lemma 5.5. *Let A be an $n \times n$ array with βn^2 distinct symbols. If there are $d \geqslant 1$ clones in row i, then there is some clone A_{ij} such that*

$$|R_i(A) \cup C_j(A)| \geqslant |R_i(A)| + \frac{\beta n^2 - (n-d)(n-1) - |R_i(A)|}{d}.$$

Proof. We will endeavour to find a column j such that $|C_j(A) \setminus R_i(A)|$ is large. Without loss of generality, assume that the rightmost d columns of row i contain clones. First, remove all occurrences of the symbols in $R_i(A)$ from the array. Now, arbitrarily select a representative entry for each of the remaining symbols in the array. Note that there are no representatives in row i and so there are at most $(n-d)(n-1)$ representatives in the first $n-d$ columns. Of the original βn^2 symbols, at least $\beta n^2 - (n-d)(n-1) - |R_i(A)|$ must have their representative in the last d columns. By the pigeon-hole principle, the desired clone A_{ij} occurs in one of the last d columns. □

Let $\mathcal{A}$ be some class of square arrays of symbols that has the following two properties: (i) if any row and column of an array in $\mathcal{A}$ is deleted, the resulting array is in $\mathcal{A}$ and (ii) if in one entry of the array, the symbol is changed to a new symbol that appears nowhere else in the array, then the resulting array is in $\mathcal{A}$. Note that $\mathcal{L}$, the set of all Latin arrays, and $\mathcal{R}$, the set of all row-Latin arrays, both satisfy the requirements listed.

Let $\frac{1}{2} \leqslant \alpha \leqslant 1$. Define $\mathcal{M}_{\mathcal{A}}(\alpha)$ to be the set of transversal-free arrays in $\mathcal{A}$ whose ratio of number of distinct symbols to cells is at least α. Suppose that $\mathcal{M}_{\mathcal{A}}(\alpha)$ is non-empty. Define $\mathcal{M}^*_{\mathcal{A}}(\alpha) \subseteq \mathcal{M}_{\mathcal{A}}(\alpha)$ by the rule that if $A \in \mathcal{M}^*_{\mathcal{A}}(\alpha)$, then no array in $\mathcal{M}_{\mathcal{A}}(\alpha)$ has an order smaller than A and no array in $\mathcal{M}_{\mathcal{A}}(\alpha)$ of the same order as A contains more distinct symbols than A. For example, both $\mathcal{M}^*_{\mathcal{L}}(1/2)$ and $\mathcal{M}^*_{\mathcal{R}}(1/2)$ consist solely of the Latin squares of order 2. For the remainder of the section, we bound the number of symbols in arrays by examining properties of the arrays in $\mathcal{M}^*_{\mathcal{A}}(\alpha)$.

Lemma 5.6. *Let $A \in \mathcal{M}^*_{\mathcal{A}}(\alpha)$ be an array of order n. If A_{ij} is a singleton, then $|\Psi_{ij}(A)| > \alpha(2n-1)$ and $R_i(A)$ (resp., $C_j(A)$) contains more than $(2\alpha-1)n$ symbols that appear only in row i (resp., column j) of A.*

Proof. Any array of order 1 has a transversal, so $n \geqslant 2$. There is no transversal T of $A(i \mid j)$, or else $T \cup \{(i, j, A_{ij})\}$ would be a transversal of A. As $A \in \mathcal{M}^*_{\mathcal{A}}(\alpha)$, we have that $A(i \mid j) \notin \mathcal{M}_{\mathcal{A}}(\alpha)$, so the number of distinct symbols in $A(i \mid j)$ is strictly less than $\alpha(n-1)^2$. Thus,

$$|\Psi_{ij}(A)| > \alpha n^2 - \alpha(n-1)^2 = \alpha(2n-1).$$

At most $n-1$ of the symbols in $\Psi_{ij}(A) \setminus \{A_{ij}\}$ appear in $C_j(A)$, so at least

$$|\Psi_{ij}(A)| - (n-1) > \alpha(2n-1) - (n-1) \geqslant (2\alpha-1)n$$

symbols appear in row i and nowhere else in A. A similar argument applies to $C_j(A)$. □

Lemma 5.7. *Let $A \in \mathcal{M}^*_{\mathcal{A}}(\alpha)$ be an array of order n. If A_{ij} is a clone and $|R_i(A) \cup C_j(A)| \geqslant (k+1)n-1$, then A has at most*

$$\frac{1}{2}(k^2-2k+2)n^2 + \frac{1}{2}(3k-2)n$$

distinct symbols.

Proof. Without loss of generality, $i = j = n-1$. Create A' by changing the symbol in the $(n-1, n-1)$ cell of A to a symbol that did not previously appear in A. Since A_{ij} is a clone in A, we know that A' contains strictly more symbols than A. Since $A \in \mathcal{M}^*_{\mathcal{A}}(\alpha)$, we conclude that A' has a transversal, although A does not. Hence there is a near transversal of A that does not meet row $n-1$ or column $n-1$. By applying Lemma 5.4, the result follows. □

Unfortunately, in the best case, Lemma 5.7 falls just short of proving Conjecture 5.1.

Corollary 5.8. *Let $A \in \mathcal{M}^*_{\mathcal{A}}(\alpha)$ be an array of order n. If A_{ij} is a clone and $|R_i(A) \cup C_j(A)| = 2n-1$, then A has at most $(n^2+n)/2$ distinct symbols.*

Lemmas 5.5, 5.6 and 5.7 form the main framework needed to bound the number of symbols. We utilise Lemmas 5.5 and 5.6 in different ways to find an entry (i, j, k) where k is a clone and row i and column j contain many different symbols. We then apply Lemma 5.7 to bound the number of symbols overall. The following sections concentrate on specific classes for $\mathcal{A}$.

5.2 Bounds on Row-Latin Arrays

In this section, we consider $\mathcal{A} = \mathcal{R}$, the set of row-Latin arrays.

Lemma 5.9. *Let $M \in \mathcal{M}^*_{\mathcal{R}}(\alpha)$ be a row-Latin array of order n. There exists a clone M_{ij} for which $|R_i(M) \cup C_j(M)| \geqslant 2\alpha n - 1$.*

Proof. First suppose that there is a clone M_{ij} that appears in the same column as a singleton. By Lemma 5.6, $C_j(M)$ contains at least $(2\alpha-1)n$ symbols that appear only in $C_j(M)$. One of these symbols may be M_{ij}, but

$$|R_i(M) \cup C_j(M)| = |R_i(M)| + |C_j(M) \setminus R_i(M)| \geqslant n + (2\alpha-1)n - 1 = 2\alpha n - 1,$$

as required.

Hence we may assume that no column contains a singleton and a clone. Let d be the number of columns that contain clones.

If $d \leqslant n/2$, then we can find a transversal in the following way. Let R be the $n \times d$ subarray of M that contains the clones of M. A result of Drisko [25] implies that M has a partial transversal of length d that is wholly inside R. Since this partial transversal covers all columns that contain clones, it can trivially be extended to a transversal using singletons.

So we may assume that $d > n/2$. Since each row contains d clones, we may use Lemma 5.5 with $\beta \geqslant \alpha$ to find some clone M_{ij} such that

$$|R_i(M) \cup C_j(M)| \geqslant \frac{\alpha - 1}{d} n^2 + 2n - 1 > 2(\alpha - 1)n + 2n - 1 = 2\alpha n - 1. \qquad \square$$

We now show one of our main results, that row-Latin arrays with many symbols must have a transversal.

Theorem 5.10. *Let L be a row-Latin array of order n. If L has at least $\frac{1}{4}(5 - \sqrt{5})n^2 \approx 0.6910n^2$ distinct symbols, then L has a transversal.*

Proof. Aiming for a contradiction, suppose that $L \in \mathcal{M}_{\mathcal{R}}(\alpha)$ for $\alpha = (5 - \sqrt{5})/4$. Then there exists $M \in \mathcal{M}^*_{\mathcal{R}}(\alpha)$. Let M have order m. By Lemma 5.9, there is a clone M_{ij} such that $|R_i(M) \cup C_j(M)| \geqslant 2\alpha m - 1$. By Lemma 5.7, the number of distinct symbols in M is at most

$$\frac{1}{2}\left((2\alpha - 1)^2 - 2(2\alpha - 1) + 2\right) m^2 + \frac{1}{2}\left(3(2\alpha - 1) - 2\right) m = \alpha m^2 - \frac{1}{4}(3\sqrt{5} - 5)m.$$

This contradicts the fact that M has at least αm^2 distinct symbols, and we are done. $\square$

5.3 Bounds on Latin Arrays

In this section, we consider $\mathcal{A} = \mathcal{L}$, the set of Latin arrays.

We call a Latin array L of order n *focused* if every singleton in L occurs in a row or a column that contains only singletons and $|\Psi_{ij}(L)| = 2n - 1$ for some (i, j) (that is, row i and column j contain only singletons). We deal with focused and unfocused arrays separately.

For focused arrays, we need the following, which is simply the contrapositive of Theorem 2.29 with $h = 0$.

Corollary 5.11. *Let L be an $n \times n$ Latin array and $0 \leqslant t < n$. If $(n - t)^2 > t$, then L has a partial transversal of length $t + 1$.*

In the following result, recall that we assume $\alpha \geqslant 1/2$.

Lemma 5.12. *Let $M \in \mathcal{M}^*_{\mathcal{L}}(\alpha)$ be a Latin array of order n. If M is focused, then M contains at most $\frac{1}{8}(6 - \sqrt{2})n^2 \approx 0.5732n^2$ distinct symbols.*

Proof. Let $\delta = \lceil (2\alpha - 1)n \rceil$. Suppose M has r rows and c columns that contain singletons. Permute the rows and columns of M so that these singletons occur in the top r rows and leftmost c columns. Since M is focused, $\min(r, c) \geqslant 1$ and the bottom-right $(n-r) \times (n-c)$ subarray does not contain any singletons. Thus, if we consider any singleton in the last row or last column, we get $\min(r, c) \geqslant \delta$ by Lemma 5.6.

If $\alpha \geqslant 3/4$, then $\min(r, c) \geqslant n/2$ and so $\{(i, n - i - 1) : 0 \leqslant i < n\}$ is a set of cells containing only singletons, contradicting the fact that M has no transversal. So $\alpha < 3/4$.

Let M' be the subarray formed by the last $n-\delta$ rows and columns of M. Suppose that M has a partial transversal of length $n-2\delta$ wholly inside M'. Then this partial transversal can easily be extended to a transversal by selecting singletons in the first δ rows and δ columns of M. By assumption M has no transversal, so applying Corollary 5.11 to M' we find that $(\delta+1)^2 \leqslant n-2\delta-1$. Hence

$$0 \geqslant \delta^2+4\delta+2-n \geqslant (2\alpha-1)^2n^2+(8\alpha-5)n+2. \tag{5.4}$$

From the discriminant of this quadratic we learn that $32\alpha^2-48\alpha+17 \geqslant 0$. Since $\alpha < 3/4$ we have $\alpha \leqslant (6-\sqrt{2})/8$. $\square$

For any $\alpha > 1/2$, it is worth noting that (5.4) fails for all large n. So we get an asymptotic version of Conjecture 5.1 holding for focused Latin arrays. We are not able to reach such a strong conclusion for the unfocused case.

Lemma 5.13. *Let $M \in \mathcal{M}^*_{\mathcal{L}}(\alpha)$ be a Latin array of order n. If M is unfocused, then there exists some clone M_{ij} such that $|R_i(M) \cup C_j(M)| \geqslant (\alpha+1)n-1$.*

Proof. Firstly, we consider the case that M has some row or column that contains only clones. Without loss of generality, row i contains only clones. By Lemma 5.5, there is some clone M_{ij} such that $|R_i(M) \cup C_j(M)| \geqslant n+(\alpha n^2-n)/n = (\alpha+1)n-1$.

Secondly, we consider the case that every row and column of M contains a singleton. Since M is unfocused, there is some singleton M_{ik} such that there is a clone in both row i and column k. By Lemma 5.6, we have $|\Psi_{ik}(M)| > \alpha(2n-1)$. Each symbol in $\Psi_{ik}(M)$ appears in either $R_i(M)$ or $C_k(M)$. Also, M_{ik} appears in both $R_i(M)$ and $C_k(M)$, so without loss of generality, $R_i(M)$ contains at least $(|\Psi_{ik}(M)|+1)/2 > \alpha(n-1/2)+1/2 \geqslant \alpha n$ symbols that are in $\Psi_{ik}(M)$. Let M_{ij} be a clone in the same row as M_{ik}. Except possibly for M_{ij}, none of the n symbols in $C_j(M)$ are in $\Psi_{ik}(M)$. Hence, $|R_i(M) \cup C_j(M)| \geqslant \alpha n+n-1$ as required. $\square$

We now show a stronger result than Theorem 5.10 holds for Latin arrays.

Theorem 5.14. *Let L be a Latin array of order n. If L has at least $\left(2-\sqrt{2}\right)n^2 \approx 0.5858n^2$ distinct symbols, then L has a transversal.*

Proof. Aiming for a contradiction, suppose that $L \in \mathcal{M}_{\mathcal{L}}(\alpha)$ for $\alpha = 2-\sqrt{2}$. Then there exists $M \in \mathcal{M}^*_{\mathcal{R}}(\alpha)$. Let M have order m. Note that M cannot be focused, by Lemma 5.12. So, by Lemma 5.13, there is a clone M_{ij} such that $|R_i(M) \cup C_j(M)| \geqslant (\alpha+1)m-1$. By Lemma 5.7, the number of distinct symbols in M is at most

$$\frac{1}{2}\left(\alpha^2-2\alpha+2\right)m^2+\frac{1}{2}\left(3\alpha-2\right)m = \alpha m^2-\frac{1}{2}(3\sqrt{2}-4)m.$$

This contradicts the fact that M has at least αm^2 distinct symbols, and we are done. $\square$

The results of our investigations (both in this chapter and the computational results in Section 7.4) lead us to be sceptical that Conjecture 5.2 is true. However, proving that it is false is likely to be extremely hard, for the reasons explained after Proposition 5.3. Yet, it also seems hard to prove a subquadratic bound on $\ell(n)$, or even to prove Conjecture 5.1. For $\ell_r(n)$ we know more. Thanks to [8] and Theorem 5.10, we know that $\frac{1}{2}n^2-O(n) < \ell_r(n) \leqslant \left\lceil \frac{1}{4}(5-\sqrt{5})n^2 \right\rceil$.

Chapter 6

Covers of Latin Squares

> *Seek simplicity and distrust it.*
> – A. N. Whitehead

We now shift our focus to an object which approaches the search for transversals from a different perspective.

For this chapter, we define $E(L) = \{(i, j, L_{ij}) : i, j \in \mathbb{Z}_n\}$ to be the set of *entries*. The set of all entries in a row, all entries in a column or all entries containing a given symbol in a Latin square L is called a *line*. In particular, a Latin square of order n contains exactly $3n$ lines. We say a line ℓ is *represented* by an entry $\mathbf{e}$ whenever $\mathbf{e} \in \ell$. For a set of entries $\mathscr{C} \subseteq E(L)$, we say ℓ is *represented* by $\mathscr{C}$ whenever $|\mathscr{C} \cap \ell| \geqslant 1$, and we say it is represented $|\mathscr{C} \cap \ell|$ times by $\mathscr{C}$. If $\mathscr{C} \cap \ell = \{\mathbf{e}\}$, we say that ℓ is *uniquely represented* by $\mathbf{e}$.

Definition 6.1. Let L be a Latin square. A *c-cover* of L is a c-subset of $E(L)$ in which every line is represented.

In order for a Latin square of order n to have a c-cover, we must have $c \geqslant n$.

A partial transversal of length $n - d$ is said to have *deficit* d. Since an entry (r, c, s) in a partial transversal uniquely represents three lines (its row, column and symbol), a partial transversal of deficit d represents exactly $3(n - d)$ lines, and so a transversal is an n-cover. Figure 6.1 gives examples of a partial transversal, a transversal and a cover.

In a Latin square L, we say a cover $\mathscr{C}$ of L is *minimal* if, for all $\mathbf{e} \in \mathscr{C}$, the set $\mathscr{C} \setminus \{\mathbf{e}\}$ is not a cover. If $\mathscr{C}$ is not minimal, then it has a *redundant* entry $\mathbf{e} \in \mathscr{C}$ for which $\mathscr{C} \setminus \{\mathbf{e}\}$ is also a cover. We also say $\mathscr{C}$ is *minimum* if every cover of L has size at least $|\mathscr{C}|$. We say a partial transversal T of L is *maximal* if, for all $\mathbf{e} \in E(L) \setminus T$, the set $T \cup \{\mathbf{e}\}$ is not a partial transversal. We stress that the maximality of a partial transversal T is always relative to the whole Latin square L, even when we locate T inside some proper subset of $E(L)$.

0	1	2	3
3	**2**	1	0
2	3	0	**1**
1	0	3	2

0	1	2	3
3	2	**1**	0
2	**3**	0	1
1	0	3	**2**

0	1	2	**3**
3	**2**	1	0
2	3	0	**1**
1	0	**3**	2

Figure 6.1: A Latin square of order $n = 4$ where we highlight a partial transversal of deficit 1 (left), a transversal (middle), and an $(n + 1)$-cover (right).

Pippenger and Spencer [66] showed a very powerful and general result that includes covers of Latin squares as a special case. They showed that as $n \to \infty$, the entries of a Latin square of order n can be decomposed into $n - o(n)$ covers. In particular, this means that all Latin squares have a cover of size $n + o(n)$. A better upper bound on the size of the smallest cover is given in Corollary 6.3.

A Latin square L of order n is equivalent to a tripartite 3-uniform hypergraph with n vertices in each part (corresponding respectively to rows, columns and symbols) and n^2 hyperedges (corresponding to the entries of L). In this framework, a cover of L is precisely an edge cover (a set of hyperedges whose union covers the vertex set) of this hypergraph. Alternatively, L can be considered as an n-uniform hypergraph of order n^2 with edges that are precisely the $3n$ lines of L; a cover of L is precisely a vertex cover (a set of vertices that intersects every edge) of this hypergraph. This relationship with hypergraph covers is one justification for our choice of terminology.

A Latin square L of order n also has a natural representation as an n^2-vertex graph, called a *Latin square graph* which we denote Γ_L, with vertex set $E(L)$ and an edge between two distinct entries whenever they share a row, column or symbol. An example is shown in Figure 6.2. The graph Γ_L is thus the union of $3n$ cliques of size n (one for each line), and a cover in L is equivalent to a selection of vertices in Γ_L in which each of these cliques has at least one representative. A cover of L does not necessarily map to a vertex cover of Γ_L (the example in Figure 6.2 is not a vertex cover of Γ_L).

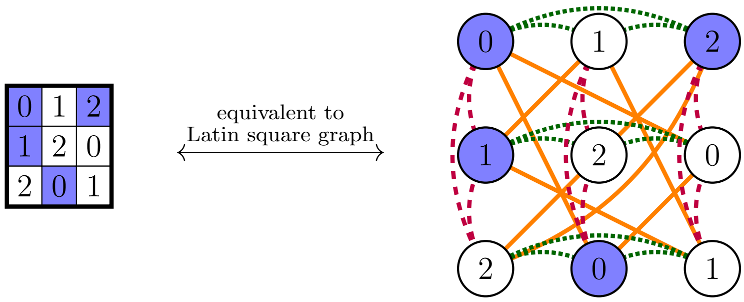


Figure 6.2: Converting between a Latin square L and the equivalent Latin square graph Γ_L, with an $(n+1)$-cover highlighted in both. Edge colours are added to indicate the relationship between neighbouring entries (dotted for the same row, dashed for the same column and solid for the same symbol).

Any cover of L maps to a dominating set of Γ_L. In fact, any cover of L corresponds to a 3-dominating set of Γ_L, *i.e.,* any entry outside the 3-dominating set has 3 or more neighbours inside the 3-dominating set [77, Sec. 7.1] (see also [19]). The converse is not true, *i.e.,* not every 3-dominating set is a cover: a 3-dominating set (actually a 4-dominating set) is formed in Γ_L by the entries with symbols 0 and 1 in any Latin square L of order $n \geqslant 2$. Yet, when $n \geqslant 3$, this 4-dominating set does not cover the symbol 2. A cover therefore corresponds to a special kind of 3-dominating set, where each n-clique (arising from each line in the Latin square) has a representative in the cover.

Let L be a Latin square of order $n \geqslant 3$. The domination number of Γ_L, *i.e.,* the size of the smallest dominating set of Γ_L, denoted $\gamma(\Gamma_L)$, is less than n: to form an $(n-1)$-entry dominating set, select all but one of the entries with symbol 0. In fact, $\gamma(\Gamma_L)$ is likely smaller than $n-1$, since any maximal partial transversal corresponds to a dominating set in Γ_L. However, for a 3-dominating set of cardinality a to exist in Γ_L, we must have

$$a\,3(n-1) \geqslant 3(n^2 - a)$$

since each of the a entries in the 3-dominating set dominates at most $3(n-1)$ vertices, and there are n^2-a entries dominated at least 3 times each. This implies that $a \geqslant n$, implying the 3-domination number of Γ_L, denoted $\gamma_3(\Gamma_L)$, is strictly greater than the domination number, *i.e.,* $\gamma_3(\Gamma_L) > \gamma(\Gamma_L)$. (In fact, $\gamma_3(G) > \gamma(G)$ holds for all graphs G with minimum degree at least 3 [77, Cor. 7.2].)

A theme in our work is to explore a loose kind of duality between covers and partial transversals. In Section 6.1, we demonstrate some relationships between the sizes of maximum partial transversals and minimum covers, and between the numbers of these objects. In Section 6.2, we look at the other end of the spectrum, namely small maximal partial transversals and large minimal covers. Here we find less of a connection. We show that Latin squares of a given size have little variation in the size of their largest minimal covers, but can vary significantly in the size of their smallest maximal partial transversals. In Section 6.3, we summarise our achievements and discuss possible directions for future research.

6.1 Covers and Partial Transversals

In this section, we explore some basic relationships between covers and partial transversals. We first consider how to turn a partial transversal into a cover.

Theorem 6.2. *In a Latin square L of order $n \geqslant 2$, any partial transversal T of deficit d is contained in an $(n+\lceil d/2\rceil)$-cover. Moreover, if T is maximal, then the smallest cover containing T has size $n+\lceil d/2\rceil$.*

Proof. We begin by assuming T is maximal, in which case any entry in $E(L)\setminus T$ covers at most two previously uncovered lines. Let $r_1,\ldots,r_d$, $c_1,\ldots,c_d$ and $s_1,\ldots,s_d$ denote, respectively, the rows, columns and symbols that are unrepresented in T. Start by setting $\mathscr{C}=T$. Then for $i\in\{1,\ldots,\lfloor d/2\rfloor\}$ we add $(r_{2i-1},c_{2i-1},\bullet)$, $(r_{2i},\bullet,s_{2i-1})$ and $(\bullet,c_{2i},s_{2i})$ to $\mathscr{C}$. Finally, if d is odd we add $(r_d,c_d,\bullet)$ and $(\bullet,\bullet,s_d)$ to $\mathscr{C}$. This produces a cover of size $n-d+\lceil 3d/2\rceil = n+\lceil d/2\rceil$. As we covered the maximum possible number of uncovered lines at each step, no smaller cover contains T.

If T is not maximal, then the above approach gives a cover $\mathscr{C}$ of size at most $n+\lceil d/2\rceil$, since there may be duplication among the entries that are added. Assuming $n\geqslant 2$, we can simply add entries from $E(L)\setminus\mathscr{C}$ until we have a cover of size $n+\lceil d/2\rceil$. □

Since Shor and Hatami [46] have shown the existence of a partial transversal with small deficit, we immediately get the following.

Corollary 6.3. *Every Latin square of order n has a cover of size $n+O(\log^2 n)$.*

Proof. Use Theorems 2.31 and 6.2. □

We now consider how to turn a cover into a partial transversal.

Theorem 6.4. *Let L be Latin square of order $n\geqslant 1$. Any $(n+a)$-cover of L contains a partial transversal of deficit $2a$.*

Proof. Let R, C and S respectively be n-subsets of an $(n+a)$-cover $\mathscr{C}$ in which each row, column and symbol is (necessarily uniquely) represented. Note that $T=R\cap C\cap S$ is a partial transversal of L. Since $|\mathscr{C}|=n+a$ and $|\mathscr{C}\setminus R|=|\mathscr{C}\setminus C|=|\mathscr{C}\setminus S|=a$,

$$T=R\cap C\cap S=\mathscr{C}\setminus\big((\mathscr{C}\setminus R)\cup(\mathscr{C}\setminus C)\cup(\mathscr{C}\setminus S)\big)$$

has size at least $n-2a$, so has deficit at most $2a$. Finally, if T has a smaller deficit, we can delete entries to obtain deficit exactly $2a$. □

For any $(n+1)$-cover of a Latin square L, the corresponding $n+1$ vertices of the Latin square graph Γ_L induce a subgraph with 3 edges. (This is an example of a partial Latin square graph [38]. A *partial Latin square* is a matrix in which entries are either empty or contain a single symbol, and no symbol is repeated within any row or column. Alternatively, a partial Latin square can be viewed as a set of triples where no two triples agree in more than one coordinate.) Ignoring isolated vertices and edge colours, there are only 5 such graphs, which we denote $G_1, \dots, G_5$, depicted in Figure 6.3. We refer to these graphs as being the graph *induced* by the cover (specifically, this terminology ignores isolated vertices). Taking a conjugate of L permutes the edge colours in the graph induced by a cover, which does not change the type of graph according to our classification.

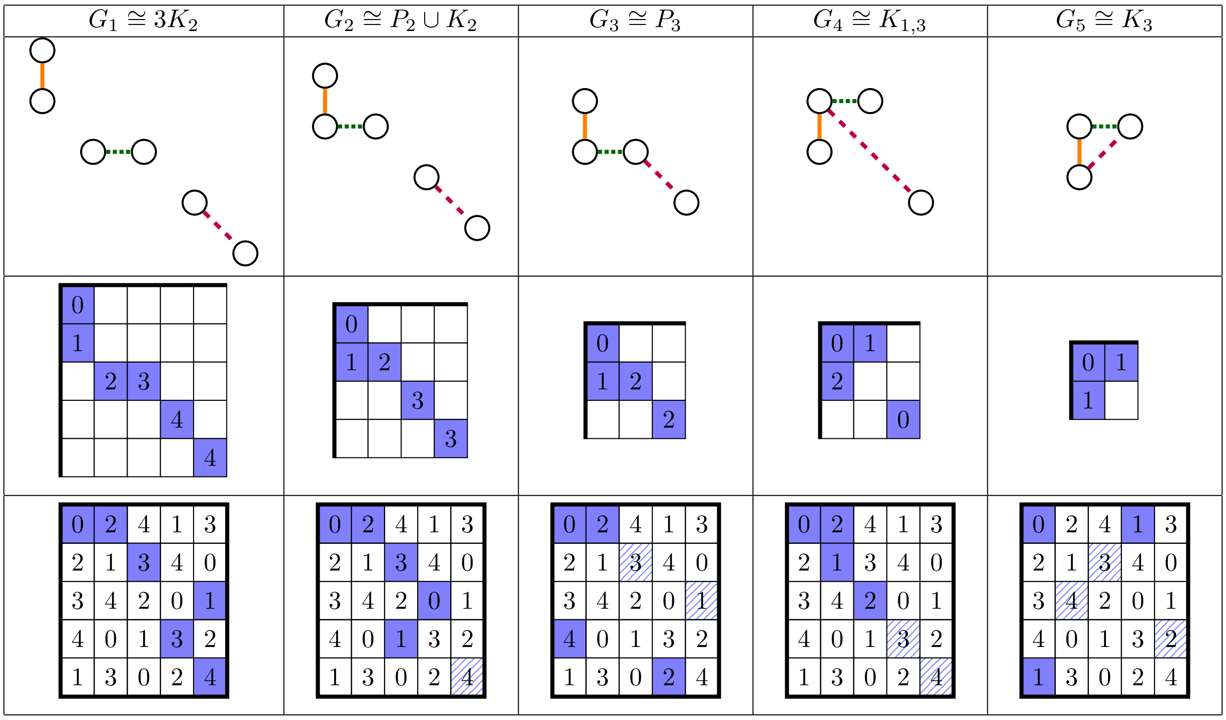


Figure 6.3: *Top row*: The five possible non-isomorphic subgraphs induced by an $(n+1)$-cover of a Latin square graph. *Middle row*: Depicting how the subgraphs $G_1, \dots, G_5$ can arise in a cover. *Bottom row*: An example of a Latin square that simultaneously contains different covers that induce the five graph structures, $G_1, \dots, G_5$.

A consequence of Theorem 6.2 is that any Latin square of order $n \geqslant 2$ with a partial transversal of deficit 1 has an $(n+1)$-cover. Thus, if Brualdi's Conjecture is true, then all Latin squares of order $n \geqslant 2$ have an $(n+1)$-cover. A converse of this statement is not immediate since we cannot always delete 2 entries from an $(n+1)$-cover to give a partial transversal of deficit 1; see Figure 6.3 (under graph G_1) for an example. However, Theorems 6.2 and 6.4 imply that a Latin square L of order $n \geqslant 2$ has a partial transversal of deficit 2 if and only if it has an $(n+1)$-cover. We now extend this observation to minimum covers.

Theorem 6.5. *Let L be a Latin square of order $n \geqslant 2$. The minimum size of a cover of L is $n+a$ if and only if the minimum deficit of a partial transversal of L is either $2a$ or $2a-1$.*

Proof. First suppose that L has an $(n+a)$-cover and no smaller cover. By Theorem 6.4, there is a partial transversal of deficit $2a$. If L has a partial transversal of deficit at most

$2a-2$, then Theorem 6.2 implies there is a cover of size at most $n+a-1$, which we are assuming is not the case. Hence, the minimum deficit of a partial transversal is either $2a$ or $2a-1$.

For the converse, suppose the minimum deficit of a partial transversal is either $2a$ or $2a-1$. By Theorem 6.2, there is an $(n+a)$-cover. If there is a cover of size at most $n+a-1$, then Theorem 6.4 implies there is a partial transversal of deficit at most $2a-2$, which we are assuming is not the case. □

For the $a=0$ case in Theorem 6.5, a transversal of a Latin square of order n is also an n-cover. For the $a=1$ case, cyclic group tables of even order are examples for which the minimum size of a cover is $n+1$ and the minimum deficit of a partial transversal is 1. Brualdi's Conjecture implies the minimum size of a cover of an order n Latin square is n or $n+1$.

Figure 6.3 also includes an example of a Latin square of order 5 in which all five of the possible induced subgraphs are achieved by different $(n+1)$-covers. We make the following observations about deleting vertices from the graphs in Figure 6.3.

- For graph G_4, we can delete one vertex to create an edgeless graph, so deleting the corresponding entry from the $(n+1)$-cover gives a transversal. Thus $(n+1)$-covers that induce G_4 are not minimal, unlike the other four graphs (G_1, G_2, G_3 and G_5).

- For graphs $G_2,\dots,G_5$, we can delete two vertices to create an edgeless graph, and deleting the corresponding entries from the $(n+1)$-cover gives a near-transversal.

- For graph G_1, we must delete at least 3 vertices to create an edgeless graph.

- For any vertex v of any of the five graphs $G_1,\dots,G_5$, it is possible to delete 3 or fewer vertices to create an edgeless graph without deleting v. Thus when $n \geqslant 2$, every entry in an $(n+1)$-cover belongs to a partial transversal of deficit 2.

We define $q_i = q_i(L)$ to be the number of $(n+1)$-covers that induce G_i in a Latin square L. Across all isotopism classes of order $n \leqslant 8$, we found all $(n+1)$-covers. Table 6.1 lists the average number of $(n+1)$-covers that induce each graph across these isotopism classes. Table 6.1 also shows the fewest number of $(n+1)$-covers found of each of the 5 types. It is interesting to note that for each n, the number of all $(n+1)$-covers is fairly consistent across the Latin squares of order n (in the sense that the range is small compared to the average). This is not true, for example, for the number of transversals.

Theorems 6.4 and 6.5 leave open some possibilities, *e.g.,* a Latin square might have two minimum covers that differ in terms of the smallest deficit of the partial transversals that they contain. The following theorem gives some restrictions in this context.

Theorem 6.6. *Let L be a Latin square of order $n \geqslant 2$ in which the minimum deficit of a partial transversal is d. Then:*

1. *the minimum size of a cover of L is $n+\lceil d/2\rceil$,*
2. *any partial transversal T of deficit d is contained in a minimum cover of L,*
3. *any minimum cover contains a partial transversal of deficit d if d is even, or deficit $d+1$ if d is odd, and*
4. *if d is even, any minimum cover that contains an entry $\mathbf{e}$ contains a minimum-deficit partial transversal that contains $\mathbf{e}$.*

	Average number of covers					
	G_1	G_2	G_3	G_4	G_5	All
$n = 5$	62	90	54	180	14	400
$n = 6$	165	889	526	229	60	1 871
$n = 7$	1 137	4 615	2 413	900	132	9 199
$n = 8$	8 067	24 675	10 163	3 419	483	46 808

	Minimum number of covers					
	G_1	G_2	G_3	G_4	G_5	All
$n = 5$	24	0	0	60	0	400
$n = 6$	0	288	0	0	0	1 728
$n = 7$	888	0	0	126	0	8 970
$n = 8$	4 672	0	0	0	0	42 240

	Maximum number of covers					
	G_1	G_2	G_3	G_4	G_5	All
$n = 5$	100	180	108	300	28	400
$n = 6$	384	1 296	972	960	216	1 944
$n = 7$	3 528	5 220	2 700	5 586	195	9 354
$n = 8$	22 016	29 376	12 288	21 504	1 536	48 832

Table 6.1: The number of $(n+1)$-covers that induce G_i, averaged over isotopism classes of Latin square of order n. We also give the minimum and maximum numbers of $(n+1)$-covers found in a Latin square. The columns headed “All” refer to the count of all $(n+1)$-covers irrespective of which G_i they induce.

0	1	2	3	4	5	6	7
1	2	3	4	5	6	7	0
2	3	4	5	6	7	0	1
3	4	5	6	7	0	1	2
4	5	6	7	0	1	2	3
5	6	7	0	1	2	3	4
6	7	0	1	2	3	4	5
7	0	1	2	3	4	5	6

G_1

0	1	2	3	4	5	6	7
1	2	3	4	5	6	7	0
2	3	4	5	6	7	0	1
3	4	5	6	7	0	1	2
4	5	6	7	0	1	2	3
5	6	7	0	1	2	3	4
6	7	0	1	2	3	4	5
7	0	1	2	3	4	5	6

G_2

0	1	2	3	4	5	6	7
1	2	3	4	5	6	7	0
2	3	4	5	6	7	0	1
3	4	5	6	7	0	1	2
4	5	6	7	0	1	2	3
5	6	7	0	1	2	3	4
6	7	0	1	2	3	4	5
7	0	1	2	3	4	5	6

G_3

0	1	2	3	4	5	6	7
1	2	3	4	5	6	7	0
2	3	4	5	6	7	0	1
3	4	5	6	7	0	1	2
4	5	6	7	0	1	2	3
5	6	7	0	1	2	3	4
6	7	0	1	2	3	4	5
7	0	1	2	3	4	5	6

G_5

Figure 6.4: Minimum covers of the Cayley table of $\mathbb{Z}_8$ that induce graphs isomorphic to G_1, G_2, G_3 and G_5, respectively. For the three covers from G_2, G_3 and G_5, we can delete 2 entries from the highlighted cover to give a partial transversal of deficit 1, but we must delete at least 3 entries from the left-most cover to obtain a partial transversal, which will have deficit at least 2.

0	1	2	3	4	5	6	7	8	9
1	2	3	4	5	6	7	8	9	0
2	3	4	5	6	7	8	9	0	1
3	4	5	6	7	8	9	0	1	2
4	5	6	7	8	9	0	1	2	3
5	6	7	8	9	0	1	2	3	4
6	7	8	9	0	1	2	3	4	5
7	8	9	0	1	2	3	4	5	6
8	9	0	1	2	3	4	5	6	7
9	0	1	2	3	4	5	6	7	8

Figure 6.5: A minimum cover $\mathscr{C}$ of the Cayley table of $\mathbb{Z}_{10}$, where $\mathscr{C}$ does not contain a partial transversal of deficit 1, and each partial transversal of deficit 2 in $\mathscr{C}$ is maximal.

Proof. If L has a cover of size less than $n+\lceil d/2\rceil$, then Theorem 6.4 implies it has a partial transversal of deficit less than d, contradicting the assumption that d is the minimum deficit. So any cover has size at least $n+\lceil d/2\rceil$, and Theorem 6.2 implies T is contained in some cover of size $n+\lceil d/2\rceil$.

Theorem 6.5 implies that any $(n+\lceil d/2\rceil)$-cover contains a partial transversal of deficit $2\lceil d/2\rceil$ or $2\lceil d/2\rceil-1$. When d is odd, these equal $d+1$ and d, respectively, and we can delete an entry from a partial transversal of deficit d to obtain one of deficit $d+1$. When d is even, Theorem 6.4 implies the cover contains a partial transversal of deficit d, and we can ensure $\mathbf{e}$ belongs to this partial transversal by choosing $\mathbf{e}\in R\cap C\cap S$ in the proof of Theorem 6.4. □

Theorem 6.6 implies nothing consequential when $d=0$. Cyclic groups of even order have minimum deficit $d=1$, and are thus a convenient example to verify that the conditions of Theorem 6.6 cannot be tightened. To date, we only have examples of Latin squares where the minimum deficit of a partial transversal is $d\in\{0,1\}$, with Brualdi's Conjecture implying that this is always the case, so we cannot inspect $d\geqslant 2$ cases. By inspecting cyclic group tables of even order, we make the following observations:

- A minimum cover might not contain a partial transversal of minimum deficit d, but instead have one of deficit $d+1$. Figure 6.4 gives an example of this; we give four covers of the Cayley table of $\mathbb{Z}_8$, one of which contains no partial transversal of deficit 1.
- A minimum cover might contain two maximal partial transversals, one of deficit d and one of deficit $d+1$. The cover that induces G_2 in Figure 6.4 has this property.
- A minimum cover might contain no partial transversals of minimum deficit d, with the partial transversals of deficit $d+1$ it contains all being maximal. For the the Cayley table of $\mathbb{Z}_{10}$, Figure 6.5 shows an 11-cover that contains no partial transversals of deficit 1, and the 8 partial transversals of deficit 2 it contains are maximal.

Given the five possible graph structures of $(n+1)$-covers in Figure 6.3, we can enumerate the number of partial transversals of deficit d they contain, which we do in Table 6.2. The terms in Table 6.2 are derived as follows: For each way we can delete $b+1$ vertices from the graph G_i to form an edgeless graph, we can form a partial transversal of deficit d by deleting them along with a further $d-b$ entries that are not involved in G_i. Each partial transversal of deficit d generated this way is distinct. Generally, these are not maximal partial transversals, but they are maximal partial transversals when $d=0$, or when $d=1$ for graphs other than G_4.

	Number of partial transversals of deficit d contained in an $(n+1)$-cover that induces G_i	$d=0$	$d=1$	$d=2$
G_1	$8\binom{n-5}{d-2}+12\binom{n-5}{d-3}+6\binom{n-5}{d-4}+\binom{n-5}{d-5}$	–	–	8
G_2	$2\binom{n-4}{d-1}+7\binom{n-4}{d-2}+5\binom{n-4}{d-3}+\binom{n-4}{d-4}$	–	2	$2n-1$
G_3	$3\binom{n-3}{d-1}+4\binom{n-3}{d-2}+\binom{n-3}{d-3}$	–	3	$3n-5$
G_4	$\binom{n-3}{d}+3\binom{n-3}{d-1}+4\binom{n-3}{d-2}+\binom{n-3}{d-3}$	1	$n+3$	$\frac{1}{2}(n^2-n+2)$
G_5	$3\binom{n-2}{d-1}+\binom{n-2}{d-2}$	–	3	$3n-5$

Table 6.2: The number of distinct partial transversals of deficit d within an $(n+1)$-cover that induces the subgraph G_i.

Table 6.2 shows that the number of partial transversals of deficit d in an $(n+1)$-cover depends significantly on its structure, *e.g.,* when $d=2$, the number varies from $\Theta(1)$ to $\Theta(n^2)$. We therefore do not anticipate a simple relationship between the number of partial transversals and the number of $(n+1)$-covers in general. However, we have the following results for $(n+1)$-covers.

Theorem 6.7. *Let T be a maximal partial transversal of deficit d in a Latin square L of order $n \geqslant 3$.*

1. *If $d=0$, then T belongs to exactly n^2-n distinct $(n+1)$-covers, none of which are minimal.*
2. *If $d=1$, then T belongs to exactly $3(n-1)$ distinct $(n+1)$-covers, each of which is minimal.*
3. *If $d=2$, then T belongs to exactly 8 distinct $(n+1)$-covers.*
4. *If $d \geqslant 3$, then T does not belong to any $(n+1)$-cover.*

Proof. The $d=0$ case is trivial, so we begin with the case $d=1$. Assume row i, column j and symbol k are unrepresented by T. Since T is maximal, $(i,j,k) \notin E(L)$, so to extend it to an $(n+1)$-cover, we must add entries of the form

- $(i,j,\bullet)$ and $(\bullet,\bullet,k)$,
- $(i,\bullet,k)$ and $(\bullet,j,\bullet)$, excluding $(i,j,\bullet)$, or
- $(\bullet,j,k)$ and $(i,\bullet,\bullet)$, excluding $(i,j,\bullet)$ and $(i,\bullet,k)$.

This gives $3n-3$ distinct ways to extend T to an $(n+1)$-cover. Each cover induces a graph of type G_2, G_3 or G_5 (*cf.* Figure 6.3). In particular, as G_4 does not arise, the $(n+1)$-covers are minimal.

Now assume $d=2$. Assume rows i and i', columns j and j' and symbols k and k' are unrepresented by T. Since T is maximal, there are no entries of the form (x,y,z) with $x \in \{i,i'\}$, $y \in \{j,j'\}$ and $z \in \{k,k'\}$. One $(n+1)$-cover has the form $T \cup \{(i,j,\bullet),(i',\bullet,k),(\bullet,j',k')\}$, and we obtain all others by some combination of swapping i and i', swapping j and j', and/or swapping k and k'.

When $d \geqslant 3$, Theorem 6.2 implies that T does not extend to an $(n+1)$-cover. □

In the $d=2$ case of Theorem 6.7, the 8 distinct $(n+1)$-covers may or may not be minimal depending on the structure of L. For example, when $n=3$, they are all non-minimal (since Latin squares of order 3 have no minimal 4-covers).

Theorem 6.8. *Let L be a Latin square of order $n \geqslant 3$. Let $p_{\max}$ be the number of maximal partial transversals of deficit 1 in L. Let $q_{\min}$ be the number of minimal $(n+1)$-covers in L. Then $q_{\min} = q_1+q_2+q_3+q_5$ and*

$$0 \leqslant \frac{2(q_{\min}-q_1)}{3n-4} \leqslant p_{\max} \leqslant \frac{2(q_{\min}-q_1)}{3n-6} \leqslant \frac{2q_{\min}}{3n-6}. \tag{6.1}$$

If t is the number of transversals in L, then the number p of (not necessarily maximal) partial transversals of deficit 1 of L and the number q of (not necessarily minimal) $(n+1)$-covers of L satisfy

$$p \leqslant \frac{2q+n(n-4)t}{3n-6}. \tag{6.2}$$

Proof. The theorem is easily checked when $n = 3$ since $p_{\max} = q_{\min} = q_1 = 0$ in this case, so assume $n \geqslant 4$. Theorem 6.7 implies that each maximal partial transversal T of deficit 1 embeds in exactly $3(n-1)$ distinct minimal $(n+1)$-covers. Moreover, in the proof of Theorem 6.7, we observed that these $(n+1)$-covers are of type G_2, G_3 or G_5, which contain exactly 2, 3 and 3 maximal partial transversals of deficit 1, respectively. Thus,

$$3(n-1)p_{\max} = 2q_2 + 3q_3 + 3q_5. \tag{6.3}$$

We also know

$$q_{\min} = q_1 + q_2 + q_3 + q_5$$

since $(n+1)$-covers are minimal if and only if they do not induce G_4. Hence

$$3(n-1)p_{\max} = 2q_{\min} - 2q_1 + q_3 + q_5. \tag{6.4}$$

Let T be a maximal partial transversal of L of deficit 1. Up to isotopism of L, we may assume that $T = \{(i,i,i) : i \in \mathbb{Z}_n \setminus \{z\}\}$, where $z = n-1$, and $L_{zz} = 0$. Define r such that $L_{rz} = z$ and define c such that $L_{zc} = z$. Among the $3(n-1)$ distinct minimal $(n+1)$-covers containing T, we have the following three families:

$$\begin{aligned}
&\big\{T \cup \{(z,z,0),(i,j,z)\} : i,j \in \mathbb{Z}_n \setminus \{0,z\}\big\},\\
&\big\{T \cup \{(r,z,z),(z,j,k)\} : j,k \in \mathbb{Z}_n \setminus \{r,z\}\big\}, \text{ and}\\
&\big\{T \cup \{(z,c,z),(i,z,k)\} : i,k \in \mathbb{Z}_n \setminus \{c,z\}\big\}.
\end{aligned}$$

Each family accounts for at least $n-4$ distinct minimal $(n+1)$-covers containing T and inducing G_2. Since there are $3(n-1)$ minimal $(n+1)$-covers containing T, there can be at most 9 that do not induce G_2, and hence either induce G_3 or G_5. We note that T is contained in at least 3 distinct minimal $(n+1)$-covers that do not induce G_2, corresponding to the three choices of two entries from $\{(r,z,z),(z,c,z),(z,z,0)\}$. This means there are between 3 and 9 distinct $(n+1)$-covers that induce G_3 or G_5 and contain T. Also, recall that each $(n+1)$-cover that induces G_3 or G_5 contains exactly 3 maximal partial transversals of deficit 1. This gives $3p_{\max} \leqslant 3q_3 + 3q_5 \leqslant 9p_{\max}$ or simply $p_{\max} \leqslant q_3 + q_5 \leqslant 3p_{\max}$, which we substitute into (6.4) to obtain (6.1).

The number p of (not necessarily maximal) partial transversals of deficit 1 of L is $p = p_{\max} + nt$ and the number q of (not necessarily minimal) $(n+1)$-covers of L is $q = q_{\min} + q_4 = q_{\min} + n(n-1)t$. Combining this with (6.1), we get (6.2). □

Our next result is motivated by the work of Belyavskaya and Russu (see [22, p. 179]) who showed that Cayley tables of certain groups do not have maximal partial transversals of deficit 1, in which case $p_{\max} = q_2 = q_3 = q_5 = 0$. This is an obstacle to finding a non-trivial lower bound on $p_{\max}$ that is only a function of $q_{\min}$ and n.

Lemma 6.9. *Let L be the Cayley table of an abelian group $\mathscr{G}$ of order n. If the Sylow 2-subgroups of $\mathscr{G}$ are trivial or non-cyclic then L has no maximal partial transversal of deficit 1 (and hence has no $(n+1)$-cover inducing G_2, G_3 or G_5). On the other hand, if the Sylow 2-subgroups of $\mathscr{G}$ are non-trivial and cyclic then L has no transversal (and hence has no $(n+1)$-cover inducing G_4).*

Proof. Let $X_{\mathscr{G}}$ denote the sum of the elements of $\mathscr{G}$. It is well-known (see, for example, [22, p. 9]) that $X_{\mathscr{G}}$ is the identity if the Sylow 2-subgroups of $\mathscr{G}$ are trivial or non-cyclic and is otherwise equal to the unique element of order 2 in $\mathscr{G}$. In the latter case there are no transversals in L ([22, p. 8]) as claimed, so we concentrate on the former case. Suppose T

is a partial transversal of deficit 1 in L and that r, c and s are respectively the row, column and symbol that are not represented in T. Then $-s = X_{\mathscr{G}} - s = (X_{\mathscr{G}} - r) + (X_{\mathscr{G}} - c) = -r - c$ because L is the Cayley table of $\mathscr{G}$. As $s = r + c$, we see immediately that T is not maximal, from which the result follows. $\square$

Table 6.3 gives the value of q_i for the Cayley table of $\mathbb{Z}_n$. The zeroes in Table 6.3 are all explained by Lemma 6.9, except that $q_5 = 0$ in $\mathbb{Z}_6$, which may just be a small order quirk.

	q_1	q_2	q_3	q_4	q_5	All
$\mathbb{Z}_5$	100	0	0	300	0	400
$\mathbb{Z}_6$	144	864	864	0	0	1 872
$\mathbb{Z}_7$	3528	0	0	5 586	0	9 114
$\mathbb{Z}_8$	7424	27 648	9 216	0	1 024	45 312
$\mathbb{Z}_9$	115 668	0	0	145 800	0	261 468
$\mathbb{Z}_{10}$	326 400	864 000	249 600	0	9 600	1 449 600
$\mathbb{Z}_{11}$	4 692 380	0	0	4 163 610	0	8 855 990

Table 6.3: The number q_i of $(n+1)$-covers of $\mathbb{Z}_n$ that induce G_i.

The maximal partial transversal highlighted in the Latin square

0	3	4	5	2	1
3	1	0	4	5	2
5	4	2	0	1	3
4	2	1	3	0	5
2	5	3	1	4	0
1	0	5	2	3	4

is contained in exactly 9 (necessarily minimal) $(n+1)$-covers that do not induce G_2. We also saw during the proof of Theorem 6.8 that all maximal partial transversals of deficit 1 are contained in at least 3 (necessarily minimal) $(n+1)$-covers that do not induce G_2. These observations present some obstacles to improving the bounds given in (6.1).

We also observe that in a general Latin square, (6.3) implies that $q_2 \equiv 0 \pmod 3$. Akbari and Alipour [3] showed that $p_{\max} \equiv 0 \pmod 4$. Thus, $2q_2 \equiv q_3 + q_5 \pmod 4$ by (6.3). We also note $q_4 = n(n-1)t$, where t is the number of transversals, and t is even when the order n is even [7]. Also, we can delete an entry from any $(n+1)$-cover of type G_3 (when $n \geqslant 5$) or type G_5 (when $n \geqslant 4$) and add another entry to obtain an $(n+1)$-cover of type G_2, implying that if $q_2 = 0$, then $q_3 = q_5 = 0$ (which occurs for the odd-ordered cyclic group tables).

The question of which entries within Latin squares belong to transversals has also been studied. The parallel topic for covers plays a role throughout this paper, so we mention the following theorem, which can be derived from [29] and [83].

Theorem 6.10. *For every $n \geqslant 5$, there exists a Latin square of order n that has transversals, but also has an entry that is not in any transversal. Consequently, for every $n \geqslant 5$, there exists a Latin square of order n that contains an entry that is not in any minimum cover nor in any partial transversal of minimum deficit.*

Theorem 6.10 does not extend to any order $n \leqslant 4$ since all Latin squares of those orders are isotopic to the Cayley table of a group. Such Latin squares have autotopism groups that act transitively on entries, and hence every entry is in a partial transversal of minimum deficit and also every entry is in a minimum cover.

Consider a Latin square of L in which the minimum deficit of a partial transversal is d. By Theorem 6.6, every entry of L that is in a partial transversal of deficit d is also in a minimum cover. It is not clear if the converse holds when d is odd (although Theorem 6.6 shows the converse holds when d is even). There is no known Latin square of order $n \geqslant 2$ that has an entry that is not in a partial transversal of deficit 1. If this property holds in general, then every entry that is in a minimum cover is also in a partial transversal of minimum deficit.

To finish this section, we observe that if a Latin square L of order $n \geqslant 5$ has a transversal, then any entry in L belongs to a minimal $(n+1)$-cover. Therefore, the entries in Theorem 6.10 that are not in minimum covers do belong to minimal covers of size 1 larger than minimum.

Theorem 6.11. *If a Latin square L of order $n \geqslant 5$ has a transversal T, then each entry of L belongs to some minimal $(n+1)$-cover.*

Proof. A computer search reveals that any entry in any Latin square of order $n \in \{5, 6\}$ belongs to a minimal $(n+1)$-cover. Now assume $n \geqslant 7$. By applying an isotopism, we may assume that $T = \{(i, i, i) : i \in \mathbb{Z}_n\}$. Let (a, b, c) be an arbitrary entry of L.

First, we consider the case when no transversal contains (a, b, c) (implying that $a \neq b$). Consider

$$\mathscr{C} = (T \setminus \{(a, a, a), (b, b, b)\}) \cup \{(a, b, c), (b, c', a), (c'', a, b)\}.$$

Now, $\mathscr{C}$ is a clearly a cover of L, and is minimal unless $c = c' = c''$ leaving (c, c, c) as a redundant entry. However, if $c = c' = c''$, then $\mathscr{C} \setminus \{(c, c, c)\}$ would be a transversal containing (a, b, c), which we assumed did not exist, so $\mathscr{C}$ must be a minimal $(n+1)$-cover containing (a, b, c).

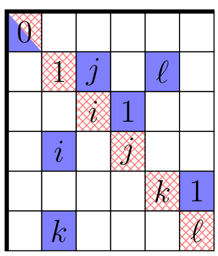
Now we may assume that $(a, b, c) \in T$ and that $a = b = c = 0$. Let i be such that $i \notin \{0, 1\}$ and $L_{1i} \neq 0$. Consider

$$\mathscr{C}_1 = (T \setminus \{(1, 1, 1), (i, i, i)\}) \cup \{(1, i, j), (i, j', 1), (j'', 1, i)\}.$$

By a similar argument as before, if j, j' and j'' are not all the same, then $\mathscr{C}_1$ is a minimal $(n+1)$-cover containing (a, b, c). If $j = j' = j''$, then note that $j \notin \{0, 1, i\}$, and then let k be such that $k \notin \{0, 1, i, j\}$ and $L_{1k} \notin \{0, i\}$ (this choice of k is possible since $n \geqslant 7$). Consider

$$\mathscr{C}_2 = (T \setminus \{(1, 1, 1), (k, k, k)\}) \cup \{(1, k, \ell), (k, \ell', 1), (\ell'', 1, k)\}.$$

By a similar argument as before, if ℓ, ℓ' and ℓ'' are not all the same, then $\mathscr{C}_2$ is a minimal $(n+1)$-cover containing (a, b, c). If $\ell = \ell' = \ell''$, then note that $\ell \notin \{0, 1, j, k\}$ and L must have the following structure:



.

By removing the entries containing the symbols $1, i, j, k$ and ℓ from T and adding the shaded entries, we have a minimal $(n+1)$-cover containing (a, b, c). ∎

Theorem 6.11 does not hold for orders $n \in \{1, 3, 4\}$ as the Latin squares of those orders that have transversals do not have minimal $(n+1)$-covers (and Theorem 6.11 is vacuously true when $n = 2$).

6.2 Large Minimal Covers

In this section, we consider the question of how large a minimal cover in a Latin square of order n can be.

When $n \geqslant 3$, a transversal (which has size n) is the smallest minimal cover possible in a Latin square of order n. The size of the largest minimal cover is harder to establish. It is clear that it cannot be larger than size $3n$, since there are only $3n$ lines and each entry in a minimal cover uniquely represents at least one line. Perhaps surprisingly, this is close to the true answer. We show that every Latin square of order n has a minimal cover with size asymptotically equal to $3n$ as $n \to \infty$.

To work towards finding the size of the largest minimal covers, we begin with a simple observation.

Lemma 6.12. *Every Latin square L of order $n \geqslant 1$ contains a minimal cover of size $2n-1$. Furthermore, any entry of L belongs to a minimal cover of size $2n-1$.*

Proof. Take all entries that are in the r-th row and/or in the c-th column. This gives a set of $2n-1$ entries in which every line is represented. The entry (r, c, L_{rc}) uniquely represents its symbol. The other entries in row r uniquely represent their respective columns, and the other entries in column c uniquely represent their respective rows. Hence the cover is minimal. $\square$

We consider a more general problem that is easier to deal with. If an $n \times n$ partial Latin square on the symbol set $\mathbb{Z}_n$ has each row, column and symbol represented at least once, we call it a *potential cover* of order n. By definition, a cover admits a completion to a Latin square, whereas not all potential covers admit a completion. Figure 6.6 gives an example of two potential covers, one of which is a cover. A potential cover $\mathscr{C}$ of order n is *minimal* if, for all $\mathbf{e} \in \mathscr{C}$, the set $\mathscr{C} \setminus \{\mathbf{e}\}$ is not a potential cover of order n. We bound the maximum size of minimal potential covers, thereby giving an upper bound on the cardinality of minimal covers.

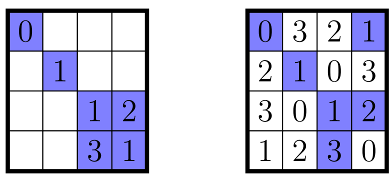


Figure 6.6: Two potential covers of Latin squares of order 4. Only the right potential cover admits a completion to a Latin square of order 4 (as indicated) and is therefore a cover.

Given a potential cover $\mathscr{C}$, define $\mathcal{U}_{\mathrm{R}} = \mathcal{U}_{\mathrm{R}}(\mathscr{C})$ to be the set of all entries that uniquely represent a row but no other line, $\mathcal{U}_{\mathrm{RC}} = \mathcal{U}_{\mathrm{RC}}(\mathscr{C})$ to be the set of all entries that uniquely represent a row and a column but no other line, $\mathcal{U}_{\mathrm{RCS}} = \mathcal{U}_{\mathrm{RCS}}(\mathscr{C})$ to be the set of all entries that uniquely represent a row, column and symbol, and define $\mathcal{U}_{\mathrm{C}}$, $\mathcal{U}_{\mathrm{S}}$, $\mathcal{U}_{\mathrm{RS}}$ and $\mathcal{U}_{\mathrm{CS}}$ accordingly. An example is given in Figure 6.7.

If an entry does not uniquely represent a row, column or symbol, then it can be deleted to give a smaller potential cover, *i.e.,* the potential cover is not minimal. If a potential cover $\mathscr{C}$ is minimal, then $\{\mathcal{U}_{\mathrm{R}}, \mathcal{U}_{\mathrm{C}}, \mathcal{U}_{\mathrm{S}}, \mathcal{U}_{\mathrm{RC}}, \mathcal{U}_{\mathrm{RS}}, \mathcal{U}_{\mathrm{CS}}, \mathcal{U}_{\mathrm{RCS}}\}$ is a partition of $\mathscr{C}$.

2	6	0	5	3	4	1
6	4	2	1	0	3	5
1	2	3	0	6	5	4
0	3	4	6	5	1	2
3	5	1	2	4	6	0
5	0	6	4	1	2	3
4	1	5	3	2	0	6

$$\begin{aligned}
\mathcal{U}_{\mathrm{R}} &= \{(1,1,4),(5,5,2)\}\\
\mathcal{U}_{\mathrm{C}} &= \{(2,6,4)\}\\
\mathcal{U}_{\mathrm{S}} &= \{(6,1,1),(6,5,0)\}\\
\mathcal{U}_{\mathrm{RC}} &= \{(0,0,2)\}\\
\mathcal{U}_{\mathrm{RS}} &= \{(4,1,5)\}\\
\mathcal{U}_{\mathrm{CS}} &= \{(2,2,3)\}\\
\mathcal{U}_{\mathrm{RCS}} &= \{(3,3,6)\}
\end{aligned}$$

Figure 6.7: Illustrating $\mathcal{U}_{\mathrm{R}}$, $\mathcal{U}_{\mathrm{RC}}$, etc. for a minimal cover of a Latin square of order 7.

Throughout the next proof, we edit a minimal potential cover $\mathscr{C}$ by deleting a few entries from it, and adding others, which creates a modified minimal potential cover. After such edits, to verify the result is indeed a minimal potential cover, we need to check the following three properties:

1. *Partial Latin square.* When adding entries, we must ensure we do not violate the partial Latin square property by adding an entry to an already filled cell, or by adding a symbol to a row or column that already contains that symbol.

2. *Potential cover.* After deleting entries from a minimal potential cover, we necessarily end up with some rows, columns and/or symbols unrepresented. These rows, columns and/or symbols must be represented by newly added entries.

3. *Minimality.* We need to verify that each entry uniquely represents some row, column or symbol. We need only check this for the newly added entries and any entries that share a row, column or symbol with a newly added entry. This last point is the most subtle: it is easy to overlook that adding an entry might make another entry redundant.

We omit details of such routine checks without further comment.

Lemma 6.13. *Let $n \geqslant 2$. There exists a minimal potential cover M of order n, which is at least as large as all other minimal potential covers of order n, and has the following additional properties*

$$\begin{aligned}
\mathcal{U}_{\mathrm{RC}} = \mathcal{U}_{\mathrm{RS}} = \mathcal{U}_{\mathrm{CS}} = \mathcal{U}_{\mathrm{RCS}} &= \emptyset,\\
|M| &= |\mathcal{U}_{\mathrm{R}}| + |\mathcal{U}_{\mathrm{C}}| + |\mathcal{U}_{\mathrm{S}}|,\\
0 < |\mathcal{U}_{\mathrm{R}}| &\leqslant (n-|\mathcal{U}_{\mathrm{C}}|)(n-|\mathcal{U}_{\mathrm{S}}|),\\
0 < |\mathcal{U}_{\mathrm{C}}| &\leqslant (n-|\mathcal{U}_{\mathrm{R}}|)(n-|\mathcal{U}_{\mathrm{S}}|), \text{ and}\\
0 < |\mathcal{U}_{\mathrm{S}}| &\leqslant (n-|\mathcal{U}_{\mathrm{R}}|)(n-|\mathcal{U}_{\mathrm{C}}|).
\end{aligned}$$

Proof. We assume that $\mathscr{C}$ is some minimal potential cover of order n of the largest possible size. If $|\mathscr{C}| \leqslant 2n-1$, then the cover described in the proof of Lemma 6.12 satisfies the required conditions. So we may assume that $|\mathscr{C}| \geqslant 2n$ (which implies that $n \geqslant 3$).

We first argue that $\mathcal{U}_{\mathrm{RCS}} = \emptyset$. If $(r,c,s) \in \mathcal{U}_{\mathrm{RCS}}$, then since $|\mathscr{C}| \geqslant 2n$ there is a row $r' \neq r$ that contains at least two entries in $\mathscr{C}$ and, similarly, there is some column $c' \neq c$ that contains at least two entries in $\mathscr{C}$. But then

$$\big(\mathscr{C} \setminus \{(r,c,s)\}\big) \cup \{(r',c,s),(r,c',s)\}$$

is a larger potential cover, contradicting the choice of $\mathscr{C}$. So $\mathcal{U}_{\mathrm{RCS}} = \emptyset$.

Next we explain how we can modify $\mathscr{C}$ in such a way that $\mathcal{U}_{\mathrm{S}}$ and/or $\mathcal{U}_{\mathrm{RC}}$ becomes empty, without decreasing the size of $\mathscr{C}$ nor violating the minimal potential cover property.

Suppose that there exists $(r_0, c_0, s_0) \in \mathcal{U}_{\mathrm{S}}$ and $(r_1, c_1, s_1) \in \mathcal{U}_{\mathrm{RC}}$. Note that these entries cannot agree in any coordinate. We now split into three cases.

Case I: Symbol s_1 does not appear in row r_0 nor in column c_0.

In this case,

$$\big(\mathscr{C} \setminus \{(r_1, c_1, s_1)\}\big) \cup \big\{(r_0, c_1, s_1), (r_1, c_0, s_1)\big\}$$

is a larger minimal potential cover than $\mathscr{C}$, contradicting the choice of $\mathscr{C}$.

Case II: Symbol s_1 is represented at most twice in $\mathscr{C}$.

Since $(r_1, c_1, s_1) \in \mathcal{U}_{\mathrm{RC}}$, we know that s_1 must be represented exactly twice in $\mathscr{C}$. It follows that s_1 cannot occur in both row r_0 and column c_0. Suppose s_1 does not occur in row r_0 (the case when s_1 does not occur in column c_0 can be resolved symmetrically).

Since $|\mathscr{C}| \geqslant 2n$, there exists a column $c' \neq c_0$ that is not uniquely represented in $\mathscr{C}$. Case I implies s_1 occurs in column c_0 and hence does not occur in column c'. Thus,

$$\big(\mathscr{C} \setminus \{(r_1, c_1, s_1)\}\big) \cup \big\{(r_0, c_1, s_1), (r_1, c', s_1)\big\}$$

is a larger potential cover than $\mathscr{C}$, contradicting the choice of $\mathscr{C}$.

Case III: Symbol s_1 is represented at least three times in $\mathscr{C}$.

In this case,

$$\big(\mathscr{C} \setminus \{(r_0, c_0, s_0), (r_1, c_1, s_1)\}\big) \cup \big\{(r_0, c_1, s_0), (r_1, c_0, s_0)\big\} \tag{6.5}$$

is another minimal potential cover, with the same cardinality as $\mathscr{C}$. The switching (6.5) removes one entry from each of $\mathcal{U}_{\mathrm{S}}$ and $\mathcal{U}_{\mathrm{RC}}$, and replaces them with new entries in $\mathcal{U}_{\mathrm{C}}$ and $\mathcal{U}_{\mathrm{R}}$ respectively.

By iteration, we can reach a point where at least one of $\mathcal{U}_{\mathrm{S}}$ and $\mathcal{U}_{\mathrm{RC}}$ is empty. A similar process of switchings allows us to reach a point where one of $\mathcal{U}_{\mathrm{R}}$ and $\mathcal{U}_{\mathrm{CS}}$ is empty, and also one of $\mathcal{U}_{\mathrm{C}}$ and $\mathcal{U}_{\mathrm{RS}}$ is empty. We continue this process until at least one set from each pair is empty. Note that while making switch (6.5), we increase the size of two sets in question. However, no matter which switching we perform, the number of entries in $\mathcal{U}_{\mathrm{RC}} \cup \mathcal{U}_{\mathrm{RS}} \cup \mathcal{U}_{\mathrm{CS}}$ decreases, so the process terminates. Call the resulting minimal potential cover M.

Note that M satisfies $|\mathcal{U}_{\mathrm{R}}| + |\mathcal{U}_{\mathrm{RC}}| + |\mathcal{U}_{\mathrm{RS}}| < n$, since there are only n rows and not all of them are uniquely represented. Similarly, $|\mathcal{U}_{\mathrm{C}}| + |\mathcal{U}_{\mathrm{RC}}| + |\mathcal{U}_{\mathrm{CS}}| < n$. If $\mathcal{U}_{\mathrm{S}} = \emptyset$, then

$$\begin{aligned}|\mathscr{C}| &= |\mathcal{U}_{\mathrm{R}}| + |\mathcal{U}_{\mathrm{C}}| + |\mathcal{U}_{\mathrm{RC}}| + |\mathcal{U}_{\mathrm{RS}}| + |\mathcal{U}_{\mathrm{CS}}| \\ &\leqslant (|\mathcal{U}_{\mathrm{R}}| + |\mathcal{U}_{\mathrm{RC}}| + |\mathcal{U}_{\mathrm{RS}}|) + (|\mathcal{U}_{\mathrm{C}}| + |\mathcal{U}_{\mathrm{RC}}| + |\mathcal{U}_{\mathrm{CS}}|) < 2n,\end{aligned}$$

which contradicts the assumption that $|\mathscr{C}| \geqslant 2n$. Therefore $\mathcal{U}_{\mathrm{S}} \neq \emptyset$. By similar arguments, $\mathcal{U}_{\mathrm{R}} \neq \emptyset$ and $\mathcal{U}_{\mathrm{C}} \neq \emptyset$. By the deductions above, we have $\mathcal{U}_{\mathrm{CS}} = \mathcal{U}_{\mathrm{RS}} = \mathcal{U}_{\mathrm{RC}} = \emptyset$, implying that $|M| = |\mathcal{U}_{\mathrm{R}}| + |\mathcal{U}_{\mathrm{C}}| + |\mathcal{U}_{\mathrm{S}}|$.

The entries in $\mathcal{U}_{\mathrm{S}}$ cannot share a row with any entry in $\mathcal{U}_{\mathrm{R}}$, nor share a column with any entry in $\mathcal{U}_{\mathrm{C}}$, so they lie in an $(n - |\mathcal{U}_{\mathrm{R}}|) \times (n - |\mathcal{U}_{\mathrm{C}}|)$ submatrix, implying that $|\mathcal{U}_{\mathrm{S}}| \leqslant (n - |\mathcal{U}_{\mathrm{R}}|)(n - |\mathcal{U}_{\mathrm{C}}|)$. Symmetric results hold for $\mathcal{U}_{\mathrm{R}}$ and $\mathcal{U}_{\mathrm{C}}$, which completes the proof. □

Theorem 6.14. *Every minimal cover of a Latin square of order n has size at most $\lfloor 3(n + 1/2 - \sqrt{n + 1/4}) \rfloor$.*

Proof. Let x, y, z be real numbers from the interval $[0, n]$, and let $\alpha = (x+y+z)/3 \in [0, n]$. If (x, y, z) satisfies

$$x + yz \geqslant n, \tag{6.6}$$
$$y + xz \geqslant n \text{ and} \tag{6.7}$$
$$z + xy \geqslant n, \tag{6.8}$$

then (α, α, α) also satisfies (6.6)–(6.8) because

$$\begin{aligned} \tfrac{1}{3}(x+y+z) + \left(\tfrac{1}{3}(x+y+z)\right)^2 &= \tfrac{1}{9}\left(3x + 3y + 3z + x^2 + y^2 + z^2 + 2(xy + xz + yz)\right) \\ &\geqslant \tfrac{1}{9}\left(3x + 3y + 3z + 3xy + 3xz + 3yz\right) \\ &\geqslant n, \end{aligned}$$

where the first inequality holds because $x^2 + y^2 + z^2 \geqslant xy + xz + yz$ and the second follows from (6.6)–(6.8). Since $\alpha \geqslant 0$ and $\alpha + \alpha^2 \geqslant n$, it follows that

$$3n - (x + y + z) = 3n - 3\alpha \leqslant 3\big(n + 1/2 - \sqrt{n + 1/4}\big). \tag{6.9}$$

Let M_0 be an arbitrary minimal cover of a Latin square of order n. By Lemma 6.13, there is a minimal potential cover M such that $(n - |\mathcal{U}_{\mathrm{R}}|, n - |\mathcal{U}_{\mathrm{C}}|, n - |\mathcal{U}_{\mathrm{S}}|) \in [0, n]^3$ satisfies (6.6)–(6.8) and $|M| \geqslant |M_0|$. Thus, by (6.9),

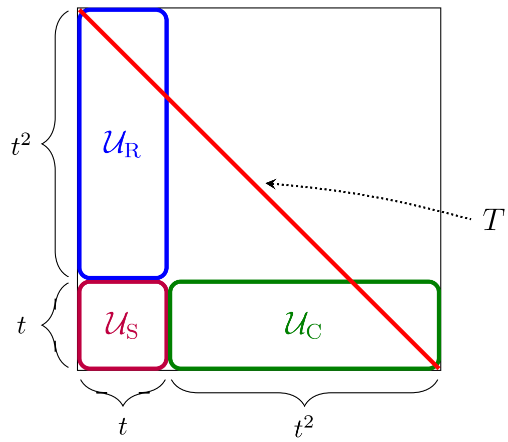
$$|M_0| \leqslant |M| = 3n - \big(n - |\mathcal{U}_{\mathrm{R}}| + n - |\mathcal{U}_{\mathrm{C}}| + n - |\mathcal{U}_{\mathrm{S}}|\big) \leqslant 3\big(n + 1/2 - \sqrt{n + 1/4}\big),$$

from which the result follows. □

We note that $-1/2 + \sqrt{n + 1/4}$ is a positive integer t when $n = t^2 + t$. We next show that the bound in Theorem 6.14 is achieved for orders n of this form, and therefore, by infinitely many covers. Moreover, we show that all theoretically possible minimal cover sizes are simultaneously achieved by different covers in a single Latin square of order $t^2 + t$.

Lemma 6.15. *Let $t \geqslant 2$ and let L be a Latin square of order $n = t^2 + t$ with a transversal T and a minimal cover $\mathscr{C}$ of size $3t^2$ such that $|\mathcal{U}_{\mathrm{R}}| = |\mathcal{U}_{\mathrm{C}}| = |\mathcal{U}_{\mathrm{S}}| = t^2$ and $|\mathcal{U}_{\mathrm{R}} \cap T| = t$. Then L contains a minimal c-cover for all $c \in \{t^2 + t, \ldots, 3t^2\}$.*

Proof. Since $|\mathcal{U}_{\mathrm{R}}| = t^2$, all elements in $\mathcal{U}_{\mathrm{C}} \cup \mathcal{U}_{\mathrm{S}}$ must be contained in t rows. Similarly, $\mathcal{U}_{\mathrm{R}} \cup \mathcal{U}_{\mathrm{S}}$ must be contained in t columns, and thus, $\mathcal{U}_{\mathrm{S}}$ is a $t \times t$ submatrix. We now argue that $\mathscr{C} \cap T = \mathcal{U}_{\mathrm{R}} \cap T$. Note that $\mathcal{U}_{\mathrm{S}} \cap T = \emptyset$ since $\mathcal{U}_{\mathrm{R}} \cup \mathcal{U}_{\mathrm{S}}$ is contained in t columns and $|\mathcal{U}_{\mathrm{R}} \cap T| = t$. Similarly, $\mathcal{U}_{\mathrm{C}} \cap T = \emptyset$ since at most $n - |\mathcal{U}_{\mathrm{S}}| = t$ distinct symbols occur in $\mathcal{U}_{\mathrm{R}} \cup \mathcal{U}_{\mathrm{C}}$ and $|\mathcal{U}_{\mathrm{R}} \cap T| = t$. Permute the rows, columns and symbols of L in such a way that (a) $T = \{(i, i, i) : 0 \leqslant i < t^2 + t\}$, (b) the entries in $\mathcal{U}_{\mathrm{S}}$ comprise the bottom-left $t \times t$ submatrix, and (c) the symbol in the bottom-left entry is $t^2 - 1$ (this simplifies Case III below). Thus, L has the following structure:



Clearly, T itself provides a minimal (t^2+t)-cover, and we also know that L has a minimal (t^2+t+1)-cover by Theorem 6.11. For $c \in \{t^2+t+2, \dots, 3t^2\}$, we break into 3 cases. In each of these cases, a set of entries from T is added to $\mathscr{C}$ and then entries that have become redundant are removed. For each entry added that is not in the first t columns nor last t rows, three redundant entries are removed (one from each of $\mathcal{U}_{\mathrm{R}}$, $\mathcal{U}_{\mathrm{C}}$ and $\mathcal{U}_{\mathrm{S}}$). These entries correspond to the set Y below. For each entry added in the last t rows, two redundant entries are removed (one from each of $\mathcal{U}_{\mathrm{C}}$ and $\mathcal{U}_{\mathrm{S}}$). These entries correspond to the set X below.

The $3t$ lines that are not uniquely represented by $\mathscr{C}$ are (a) the first t columns, (b) the last t rows and (c) the symbols in $\mathcal{U}_{\mathrm{R}} \cap T$. In all cases the modifications that we make leave $\mathcal{U}_{\mathrm{R}} \cap T$ in the resulting cover, so the lines in (a) and (c) are still be represented. The representatives of the last t rows will be addressed in each case. The other checks required to show that the resulting set of entries is a minimal c-cover are straightforward and is omitted. If $Z \subseteq \mathbb{Z}_n$, we define $\mathcal{V}_{\mathrm{R}}(Z) = \{(i,\bullet,\bullet) \in \mathcal{U}_{\mathrm{R}} : i \in Z\}$, and we define $\mathcal{V}_{\mathrm{C}}$ and $\mathcal{V}_{\mathrm{S}}$ similarly. Whenever we use this notation, the elements in Z will be in one-to-one correspondence with elements of $\mathcal{V}_{\mathrm{R}}$ (similarly for $\mathcal{V}_{\mathrm{C}}$ or $\mathcal{V}_{\mathrm{S}}$).

In each case,

$$\mathscr{C}(X,Y) = \big(\mathscr{C} \cup \{(i,i,i) : i \in X \cup Y\}\big) \setminus \big(\mathcal{V}_{\mathrm{C}}(X) \cup \mathcal{V}_{\mathrm{S}}(X) \cup \mathcal{V}_{\mathrm{R}}(Y) \cup \mathcal{V}_{\mathrm{C}}(Y) \cup \mathcal{V}_{\mathrm{S}}(Y)\big).$$

will be a minimal cover of the appropriate size. Note that in each case, $|\mathcal{V}_{\mathrm{C}}(X) \cup \mathcal{V}_{\mathrm{S}}(X) \cup \mathcal{V}_{\mathrm{R}}(Y) \cup \mathcal{V}_{\mathrm{C}}(Y) \cup \mathcal{V}_{\mathrm{S}}(Y)| = 2|X| + 3|Y|$ and $|\mathscr{C}(X,Y)| = 3t^2 - |X| - 2|Y|$.

Case I: $c \in \{3t^2-t+1, \dots, 3t^2\}$. Define $X = \{t^2, \dots, t^2+(3t^2-c)-1\}$ (with $X = \emptyset$ if $c = 3t^2$) and $Y = \emptyset$. Note that since $|X| + 2|Y| < t$, the elements of $\mathscr{C}(X,Y)$ in the bottom-left $t \times t$ submatrix cover the last t rows of L. Thus, $\mathscr{C}(X,Y)$ is a minimal c-cover.

Case II: $c \in \{t^2+t+2, \dots, 3t^2-t\}$ and c is even. Define $X = \{t^2, \dots, t^2+t-1\}$ and $Y = \{t, \dots, t+(3t^2-t-c)/2-1\}$ (with $Y = \emptyset$ if $c = 3t^2-t$). Note that the bottom t rows are covered by $\{(i,i,i) : i \in X\}$. Thus, $\mathscr{C}(X,Y)$ is a minimal c-cover.

Case III: $c \in \{t^2+t+3, \dots, 3t^2-t-1\}$ and c is odd. Define $X = \{t^2, \dots, t^2+t-2\}$ and $Y = \{t, \dots, t+(3t^2-t-c+1)/2-1\}$. Note that $t^2-1 \notin Y$, so $(t^2+t-1, 0, t^2-1) \in \mathscr{C}(X,Y)$, and so the bottom t rows are covered by $\{(i,i,i) : i \in X\} \cup \{(t^2+t-1, 0, t^2-1)\}$. Thus, $\mathscr{C}(X,Y)$ is a minimal c-cover. $\square$

Theorem 6.16. *For all $t \geqslant 2$, there exists a Latin square of order $n = t^2+t$ that contains a minimal c-cover for all $c \in \{t^2+t, \dots, 3t^2\}$.*

Proof. For each order, we give an example of a square that satisfies the properties required in Lemma 6.15. When $t = 2$, the following Latin square satisfies the requirements:

5	2	3	0	4	1
1	4	0	5	2	3
4	0	2	3	1	5
3	5	4	1	0	2
0	1	5	2	3	4
2	3	1	4	5	0

and when $t = 6$, the Latin square given in Figure 6.8 satisfies the requirements.

We may now assume that $t \notin \{2, 6\}$, so there exists a pair (A, B) of orthogonal Latin squares of order t. Define a $(t^2+t) \times (t^2+t)$ matrix D by filling cell $(\alpha t + r, \beta t + c)$, for

$\alpha, \beta \in \{0, \ldots, t\}$ and $r, c \in \{0, \ldots, t-1\}$, with the symbol

$$(A_{rc} - (\alpha + \beta + 1), B_{rc}) \in \mathbb{Z}_{t+1} \times \mathbb{Z}_t.$$

This means, for example, that row 0 has cell $(0, \beta t + c)$ filled with symbol $(A_{0c} - (\beta+1), B_{0c})$ whenever $0 \leqslant \beta \leqslant t$ and $0 \leqslant c \leqslant t-1$. Thus each symbol in $\mathbb{Z}_{t+1} \times \mathbb{Z}_t$ occurs exactly once as we iterate over β and c, so the first row is Latin. A similar argument holds for each row and each column, so D is a Latin square. An example of this construction when $t = 4$ is given in Figure 6.9.

Consider the set of entries in the bottom-left $t \times t$ submatrix of D:

$$\mathcal{D}_S = \big\{ \big(t^2 + r, c, (A_{rc}, B_{rc})\big) \in E(D) : r, c \in \{0, \ldots, t-1\} \big\}.$$

Since A and B are orthogonal, each symbol that occurs in $\mathcal{D}_S$ occurs exactly once. The symbols in $\mathbb{Z}_{t+1} \times \mathbb{Z}_t$ that do not occur in $\mathcal{D}_S$ are thus $X = \{(t, 0), \ldots, (t, t-1)\}$. Define

$$\begin{aligned} \mathcal{D}_R &= \big\{ (r, c, s) \in E(D) : c \in \{0, \ldots, t-1\} \text{ and } s \in X \big\} \text{ and} \\ \mathcal{D}_C &= \big\{ (r, c, s) \in E(D) : r \in \{t^2, \ldots, t^2 + t - 1\} \text{ and } s \in X \big\}. \end{aligned}$$

We next argue that $\mathscr{C} = \mathcal{D}_R \cup \mathcal{D}_C \cup \mathcal{D}_S$ is a minimal cover of D, where $\mathcal{U}_{\mathrm{R}} = \mathcal{D}_R$, $\mathcal{U}_{\mathrm{C}} = \mathcal{D}_C$, and $\mathcal{U}_{\mathrm{S}} = \mathcal{D}_S$. Each symbol is covered by $\mathscr{C}$, as described above. The first t columns are covered by $\mathcal{D}_S$. For any other column $\beta t + c$ (with $\beta \in \{1, \ldots, t\}$ and $c \in \{0, \ldots, t-1\}$), let r be such that $A_{rc} = \beta - 1$. The entry $(t^2 + r, \beta t + c, \bullet) \in \mathcal{D}_C$ covers column $\beta t + c$. Since there were t^2 such columns to cover and $|\mathcal{D}_C| = t^2$, no entries in $\mathcal{D}_C$ are redundant (all entries in $\mathcal{D}_R$ and $\mathcal{D}_S$ are contained in the first t columns). A similar argument holds for covering the rows. Thus, $\mathscr{C}$ is a minimal cover of size $3t^2$ with $|\mathcal{U}_{\mathrm{R}}| = |\mathcal{U}_{\mathrm{C}}| = |\mathcal{U}_{\mathrm{S}}| = t^2$. Before we can apply Lemma 6.15, we must now find a transversal T in D such that $|\mathcal{U}_{\mathrm{R}} \cap T| = t$.

Case I: t is even.

We may assume without loss of generality that $A_{rr} = 0$ and $B_{rr} = r$ for $r \in \{0, \ldots, t-1\}$. We construct the Latin square D as described above. The symbols on the main diagonal of D are

$$\{(-2\alpha - 1, r) : 0 \leqslant \alpha < t + 1 \text{ and } 0 \leqslant r < t\}.$$

Since $t+1$ is odd, this set is $\mathbb{Z}_{t+1} \times \mathbb{Z}_t$, implying that the main diagonal is a transversal. Note that $\mathcal{U}_{\mathrm{R}}$ intersects the first t entries of the main diagonal. Thus, we may apply Lemma 6.15 to D.

Case II: t is odd.

We set $A_{rc} = c - r \pmod t$ and $B_{rc} = 2c - r \pmod t$. Let D be the Latin square from the construction above. Note that

$$\begin{gathered} \big\{ (\alpha t + r, \alpha t + r, (-2\alpha - 1, r)) : 0 \leqslant \alpha < (t+1)/2 \text{ and } 0 \leqslant r < t \big\} \\ \cup \big\{ (\alpha t + r, \alpha t + (r+1), (-2\alpha, r+2)) : (t+1)/2 \leqslant \alpha < t + 1 \text{ and } 0 \leqslant r < t - 1 \big\} \\ \cup \big\{ (\alpha t + r, \alpha t, (-2\alpha, r+2)) : (t+1)/2 \leqslant \alpha < t + 1 \text{ and } r = t - 1 \big\} \end{gathered}$$

is a transversal of D and that $\mathcal{U}_{\mathrm{R}}$ intersects the first t entries of this transversal. Thus, we may apply Lemma 6.15 to D. $\square$

Our next goal is to show that all Latin squares have a minimal cover that is asymptotically equal to the bound in Theorem 6.14. To do so, we introduce the notion of a partial minimal cover. If L is a partial Latin square and $\mathcal{P} \subseteq E(L)$ such that, for some $\mathbf{e} \in \mathcal{P}$, both

36	28	18	35	21	14	9	20	41	39	7	37	22	13	16	25	23	10	11	38	8	15	19	27	24	26	17	12	33	29	34	30	32	31	40	6	5	2	3	1	0	4
17	37	27	31	15	32	34	19	35	3	24	14	28	22	20	6	40	1	38	10	36	12	25	30	33	8	41	18	29	39	26	4	23	13	21	7	11	16	5	9	2	0
10	30	38	1	2	3	37	5	40	36	25	16	26	35	24	17	21	18	9	32	22	28	29	14	34	11	39	0	41	27	13	31	4	23	33	20	6	7	15	8	12	19
35	4	28	39	0	2	22	7	20	34	27	9	14	37	29	41	30	38	18	21	11	16	32	24	8	10	12	25	36	26	31	33	40	1	5	3	23	13	19	6	17	15
2	3	22	5	40	16	11	28	9	23	0	8	6	17	12	7	19	32	31	14	10	13	36	4	1	35	15	37	20	21	30	38	34	33	39	41	29	26	27	25	24	18
1	2	3	4	5	41	10	38	8	22	15	40	23	12	17	26	18	11	6	29	9	39	20	37	25	27	13	16	19	24	36	0	28	21	7	14	35	34	33	31	32	30
25	10	11	2	39	8	12	0	38	14	23	15	21	26	19	34	31	13	20	30	17	22	37	5	6	41	18	33	7	32	16	36	27	29	28	24	1	4	9	3	35	40
34	14	30	36	32	7	8	23	6	35	5	24	19	40	13	39	28	41	10	25	26	31	0	2	15	17	33	3	21	4	9	11	38	22	20	18	16	29	37	12	27	1
41	35	9	10	11	6	16	27	7	38	39	13	29	18	23	32	24	17	12	37	15	20	26	34	31	33	22	19	25	30	40	1	3	2	0	4	14	5	8	28	36	21
26	27	40	29	24	25	7	3	33	4	12	34	5	15	14	21	38	30	39	11	32	17	2	23	37	9	16	13	10	6	28	35	0	36	31	1	18	41	22	20	19	8
19	0	41	20	1	18	6	40	30	8	17	35	25	4	39	12	10	33	34	15	21	32	28	3	27	13	14	5	31	16	38	23	2	37	11	9	22	36	26	24	29	7
40	11	23	17	8	13	24	14	16	20	37	5	1	36	31	3	22	25	35	0	29	7	4	9	2	34	30	38	39	28	33	19	21	26	18	32	27	6	10	15	41	12
32	17	12	13	38	15	19	30	23	31	14	29	11	7	5	37	33	27	22	24	18	0	41	16	26	28	21	36	34	25	1	3	20	4	2	40	8	35	6	10	9	39
14	33	34	6	36	0	20	21	31	25	18	39	3	27	35	24	4	12	41	9	16	1	40	8	11	15	2	28	38	10	22	17	5	32	30	13	37	19	7	26	23	29
3	41	5	16	14	12	32	33	10	2	31	19	38	24	28	9	0	23	8	13	40	26	35	1	17	29	25	15	11	7	4	37	6	34	22	30	21	20	39	36	18	27
21	26	36	0	27	28	41	32	19	33	13	18	34	29	40	15	39	22	14	12	20	25	5	17	10	6	31	24	30	11	3	16	1	7	4	2	38	37	35	23	8	9
9	36	0	7	35	21	15	26	13	37	20	12	39	2	22	30	29	19	17	34	41	27	1	33	16	32	40	23	14	38	11	25	31	8	6	10	4	28	24	18	5	3
31	16	17	12	13	39	26	34	22	27	2	38	0	20	25	33	5	9	30	3	23	29	14	15	32	4	24	7	28	1	41	18	35	19	36	21	40	8	11	37	6	10
8	21	35	23	6	40	33	22	12	28	34	30	18	19	26	11	37	24	32	17	39	10	9	29	13	7	38	2	5	0	15	20	14	41	16	31	25	3	4	27	1	36
23	18	21	32	37	22	35	29	14	24	1	20	13	25	33	5	26	7	36	2	31	30	6	39	4	40	28	41	27	3	8	10	19	11	9	34	15	12	0	17	16	38
22	23	7	19	20	38	25	35	29	30	40	6	31	28	21	4	32	26	27	1	24	36	34	41	3	5	0	14	18	8	2	9	11	10	37	33	39	17	12	16	15	13
11	6	24	37	9	27	39	31	21	32	36	28	33	41	2	8	34	0	7	40	35	18	30	10	20	22	26	29	1	19	17	13	15	14	12	16	3	38	25	5	4	23
38	15	31	25	12	10	2	18	37	7	16	32	27	0	36	29	41	4	19	26	34	24	21	6	40	30	35	8	22	33	23	28	17	39	1	5	20	9	14	13	3	11
27	29	13	38	26	34	31	4	11	19	21	10	20	3	41	40	7	6	0	33	28	37	23	25	9	24	1	32	2	36	12	15	30	5	8	35	17	14	18	22	39	16
4	39	29	24	18	26	30	36	34	5	6	33	37	9	8	14	1	31	25	28	38	11	10	21	0	3	7	27	35	2	32	41	16	15	13	12	19	22	23	40	20	17
5	24	25	26	7	37	0	41	39	11	30	36	2	31	1	28	8	40	15	27	33	23	3	35	29	16	19	17	32	9	14	12	13	20	34	38	10	18	21	4	22	6
28	12	37	8	29	4	36	25	32	26	19	21	40	16	27	10	35	14	33	41	5	3	24	0	39	31	20	22	23	34	6	7	18	9	17	15	13	1	30	11	38	2
39	22	1	18	19	20	23	6	28	0	38	27	4	32	15	35	2	37	3	36	12	34	33	13	7	21	29	26	16	31	5	8	10	30	41	11	9	24	17	14	40	25
33	40	4	11	31	19	29	9	18	10	22	3	24	1	30	0	25	34	37	5	6	2	27	32	36	39	23	20	8	35	21	14	12	38	26	17	41	15	16	7	13	28
29	8	16	41	3	24	21	2	17	1	26	31	30	33	32	38	9	28	23	6	0	35	18	40	14	25	27	34	4	13	39	5	37	12	15	22	7	10	36	19	11	20
16	9	10	40	30	31	23	37	15	29	3	22	32	39	18	27	6	20	13	7	19	38	8	36	21	14	4	1	26	12	35	34	33	28	41	0	24	11	17	2	25	5
7	20	19	22	23	36	40	17	26	12	33	0	9	30	3	1	14	29	24	4	27	41	16	11	38	2	34	31	15	5	10	6	8	18	32	39	28	25	13	35	21	37
20	34	39	30	25	1	14	11	27	6	4	17	7	21	38	2	3	15	16	8	13	33	31	28	35	36	5	10	9	37	18	32	22	0	19	23	12	40	29	41	26	24
15	32	33	34	41	30	18	1	24	9	29	23	10	5	4	13	11	16	28	31	25	19	39	26	12	38	37	40	6	17	7	2	36	3	35	8	0	27	20	21	14	22
13	38	15	14	17	33	27	8	25	18	32	11	36	23	34	31	20	21	29	16	7	4	22	12	5	1	3	30	24	41	19	40	9	6	10	37	26	0	2	39	28	35
37	5	6	28	33	9	13	24	0	41	35	25	8	34	7	16	27	36	4	18	1	14	11	31	30	12	32	21	40	15	20	22	39	17	3	19	2	23	38	29	10	26
0	1	2	3	4	5	28	15	36	40	41	7	17	10	9	18	16	35	21	39	14	6	12	20	23	19	8	11	13	22	27	29	25	24	38	26	34	31	32	30	37	33
6	7	8	9	10	11	1	12	5	13	28	4	41	14	0	23	15	2	26	20	37	21	17	19	22	18	36	35	3	40	29	39	24	16	27	25	33	30	31	38	34	32
12	13	14	15	16	17	38	10	4	21	11	26	35	8	37	22	36	3	2	19	30	5	7	18	41	23	6	9	0	20	25	27	29	40	24	28	32	39	1	34	33	31
18	19	20	21	22	23	3	16	1	17	8	2	12	38	10	36	13	39	5	35	4	9	15	7	28	0	11	6	37	14	24	26	41	27	25	29	31	32	40	33	30	34
24	25	26	27	28	29	4	13	2	16	9	1	15	6	11	20	12	5	40	23	3	8	38	22	19	37	10	39	17	18	0	21	7	35	14	36	30	33	34	32	31	41
30	31	32	33	34	35	5	39	3	15	10	41	16	11	6	19	17	8	1	22	2	40	13	38	18	20	9	4	12	23	37	24	26	25	29	27	36	21	28	0	7	14

Figure 6.8: A Latin square of order 42 and a minimal 108-cover generated by a semi-random computer search. The main diagonal is a transversal.

$$A = \begin{array}{|c|c|c|c|}\hline 0&2&3&1\\\hline 2&0&1&3\\\hline 3&1&0&2\\\hline 1&3&2&0\\\hline\end{array}$$

$$B = \begin{array}{|c|c|c|c|}\hline 0&3&1&2\\\hline 2&1&3&0\\\hline 3&0&2&1\\\hline 1&2&0&3\\\hline\end{array}$$

$$D = \begin{array}{|cccc|cccc|cccc|cccc|cccc|}\hline
16&7&9&2&12&3&5&18&8&19&1&14&4&15&17&10&0&11&13&6\\
6&17&3&8&2&13&19&4&18&9&15&0&14&5&11&16&10&1&7&12\\
11&0&18&5&7&16&14&1&3&12&10&17&19&8&6&13&15&4&2&9\\
1&10&4&19&17&6&0&15&13&2&16&11&9&18&12&7&5&14&8&3\\\hline
12&3&5&18&8&19&1&14&4&15&17&10&0&11&13&6&16&7&9&2\\
2&13&19&4&18&9&15&0&14&5&11&16&10&1&7&12&6&17&3&8\\
7&16&14&1&3&12&10&17&19&8&6&13&15&4&2&9&11&0&18&5\\
17&6&0&15&13&2&16&11&9&18&12&7&5&14&8&3&1&10&4&19\\\hline
8&19&1&14&4&15&17&10&0&11&13&6&16&7&9&2&12&3&5&18\\
18&9&15&0&14&5&11&16&10&1&7&12&6&17&3&8&2&13&19&4\\
3&12&10&17&19&8&6&13&15&4&2&9&11&0&18&5&7&16&14&1\\
13&2&16&11&9&18&12&7&5&14&8&3&1&10&4&19&17&6&0&15\\\hline
4&15&17&10&0&11&13&6&16&7&9&2&12&3&5&18&8&19&1&14\\
14&5&11&16&10&1&7&12&6&17&3&8&2&13&19&4&18&9&15&0\\
19&8&6&13&15&4&2&9&11&0&18&5&7&16&14&1&3&12&10&17\\
9&18&12&7&5&14&8&3&1&10&4&19&17&6&0&15&13&2&16&11\\\hline
0&11&13&6&16&7&9&2&12&3&5&18&8&19&1&14&4&15&17&10\\
10&1&7&12&6&17&3&8&2&13&19&4&18&9&15&0&14&5&11&16\\
15&4&2&9&11&0&18&5&7&16&14&1&3&12&10&17&19&8&6&13\\
5&14&8&3&1&10&4&19&17&6&0&15&13&2&16&11&9&18&12&7\\\hline
\end{array}$$

Figure 6.9: Example of the construction in the proof of Theorem 6.16 after the symbols are relabelled to belong to $\mathbb{Z}_{20}$. Here we have $t = 4$, and we highlight a $3t^2$-cover. We also highlight the main diagonal, which is a transversal.

$\mathcal{P}$ and $\mathcal{P} \setminus \{\mathbf{e}\}$ represent the same lines, then we call $\mathbf{e}$ *redundant*. An entry $(r, c, s) \in \mathcal{P}$ is redundant if and only if there exists three other entries of the form $(r, \bullet, \bullet)$, $(\bullet, c, \bullet)$ and $(\bullet, \bullet, s)$ in $\mathcal{P}$. We define a *partial minimal cover* as any $\mathcal{P} \subseteq E(L)$ that has no redundant entries. We can iteratively delete redundant entries from any $\mathcal{P} \subseteq E(L)$ to obtain a partial minimal cover of size no more than $|\mathcal{P}|$ in which the same lines are represented.

It is important to note that not every partial minimal cover can be extended to a minimal cover, and Figure 6.10 gives two examples of partial minimal covers that cannot be extended to a minimal cover (nor even a larger partial minimal cover).

0	1	**2**	3
1	0	**3**	2
2	3	0	1
3	2	1	0

0	**1**	2	**3**	**4**
1	0	3	4	2
2	3	**4**	**0**	1
3	4	1	2	0
4	2	0	1	3

Figure 6.10: Two Latin squares with partial minimal covers that are not subsets of any minimal cover.

Even though a partial minimal cover does not necessarily extend to a minimal cover, we can find a minimal cover that is at least as large as any partial minimal cover.

Lemma 6.17. *Let L be a Latin square of order n and $\mathcal{P}$ be a partial minimal cover of L. Then L contains a minimal cover of size at least $|\mathcal{P}|$.*

Proof. If $|\mathcal{P}| \leqslant 2n - 1$, then the cover described in the proof of Lemma 6.12 satisfies the constraints, so assume $|\mathcal{P}| \geqslant 2n$. If $\mathcal{P}$ is a minimal cover, then the statement is trivial so suppose there is some line, say row r, that is not covered by $\mathcal{P}$.

Since $|\mathcal{P}| \geqslant 2n$, there exists a column c that is represented at least twice in $\mathcal{P}$. Define $\mathcal{P}' = \mathcal{P} \cup \{(r, c, s)\}$ where $s = L_{rc}$. If $\mathcal{P}'$ is not a partial minimal cover, then there must be some entry in row r, in column c or with symbol s that is redundant. Since (r, c, s) is the only entry in row r, it is not redundant. Since there are at least three entries in column c in $\mathcal{P}'$, no entry in column c is redundant in $\mathcal{P}'$ (otherwise we contradict the minimality of $\mathcal{P}$). However, if s is represented exactly once in $\mathcal{P}$, by (r_0, c_0, s) say, then that entry is redundant in $\mathcal{P}'$ if and only if there are other entries covering row r_0 and column c_0. In this case, we define $\mathcal{P}'' = \mathcal{P}' \setminus \{(r_0, c_0, s)\}$, otherwise, we define $\mathcal{P}'' = \mathcal{P}'$. Note that in either case, $\mathcal{P}''$ is a partial minimal cover that covers strictly more lines than $\mathcal{P}$ and is at least as big as $\mathcal{P}$.

We repeat the above process until all lines are covered. $\square$

Next, we need a technical lemma.

Lemma 6.18. *Fix $\epsilon > 0$. Let G be a bipartite graph with bipartition $A \cup B$ and maximum degree at most $n^{1/2+\epsilon}$. Suppose that G has $n^{3/2+\epsilon} - O(n^{1+2\epsilon})$ edges and that $|A| = n - O(n^{1/2+\epsilon})$ and $|B| = n - O(n^{1/2+\epsilon})$. Then we can find a set $U \subset A$ of vertices such that $|U| = O(n^{1/2+\epsilon})$ and there are at most $O(n^{1/2+2\epsilon})$ vertices in B that do not have a neighbour in U.*

Proof. Let $B' \subseteq B$ be the set of vertices in B of degree at least $\frac{1}{2}n^{1/2+\epsilon}$. Counting edges, we have that

$$n^{3/2+\epsilon} - O(n^{1+2\epsilon}) \leqslant n^{1/2+\epsilon}|B'| + \tfrac{1}{2}n^{1/2+\epsilon}\big(n - O(n^{1/2+\epsilon}) - |B'|\big),$$

which implies that $|B'| \geqslant n - O(n^{1/2+2\epsilon})$. Consider choosing a set $U \subseteq A$ of size $|U| = \lceil n^{1/2+\epsilon} \rceil$ uniformly at random. For any $v \in B'$ the probability that v has no neighbour in U is

$$\frac{\binom{|A|-\deg(v)}{|U|}}{\binom{|A|}{|U|}} \leqslant \frac{\binom{|A|-\frac{1}{2}n^{1/2+\epsilon}}{|U|}}{\binom{|A|}{|U|}} = \prod_{0 \leqslant i < |U|} \frac{|A| - \frac{1}{2}n^{1/2+\epsilon} - i}{|A| - i}$$

$$\leqslant \left(\frac{|A| - \frac{1}{2}n^{1/2+\epsilon}}{|A|} \right)^{|U|} = O\big(\exp(-\tfrac{1}{2}n^{2\epsilon})\big)$$

using the identity $1 - 1/x \leqslant e^{-1/x}$ when $x \neq 0$. So the expected number of vertices in B' with no neighbour in U is $O\big(n\exp(-\frac{1}{2}n^{2\epsilon})\big) = o(1)$. It follows that for large n there is some choice of U whose neighbourhood includes B', and we are done. □

Theorem 6.19. *Fix $\epsilon > 0$. Every Latin square of order n has a minimal cover of size $3n - O(n^{1/2+\epsilon})$.*

Proof. If $\epsilon \geqslant 1/2$, then the theorem follows from Lemma 6.12, so assume that $\epsilon < 1/2$. Suppose L is a Latin square of order n and let $\psi = \lfloor n^{1/2+\epsilon} \rfloor$. We gradually build a large partial minimal cover $\mathscr{C}$ for L.

Define B_1 to be the ψ-regular bipartite graph with vertices $\{c_0, \dots, c_{n-1}\} \cup \{s_0, \dots, s_{n-1}\}$ with an edge $c_i s_j$ if and only if $L_{ki} = j$ for some $k \in \{0, \dots, \psi - 1\}$. Applying Lemma 6.18 to B_1, we find a set $U_1 \subseteq \{c_0, \dots, c_{n-1}\}$ with $|U_1| = O(n^{1/2+\epsilon})$ such that $n - O(n^{1/2+2\epsilon})$ vertices in $\{s_0, \dots, s_{n-1}\}$ have a neighbour in U_1. In other words, the submatrix S formed by the rows indexed $\{0, \dots, \psi - 1\}$ and the columns indexed $\{i : c_i \in U_1\}$ contains a set of $n - O(n^{1/2+2\epsilon})$ entries with distinct symbols, and we (provisionally) initialise $\mathscr{C}$ to be this set of entries. By removing at most ψ entries from $\mathscr{C}$ if necessary, we identify a set Ψ of ψ symbols that are not yet represented in $\mathscr{C}$.

Next we form a bipartite graph B_2. The vertices of B_2 correspond to the rows and columns of L that do not intersect S. We place an edge from row vertex r to column vertex c if and only if $L_{rc} \in \Psi$. Since S has $O(n^{1/2+\epsilon})$ rows and $O(n^{1/2+\epsilon})$ columns, B_2 has $n\psi - O(n^{1/2+\epsilon}\psi) = n^{3/2+\epsilon} - O(n^{1+2\epsilon})$ edges and maximum degree at most ψ. Hence we can apply Lemma 6.18 twice to find a set U_2 of rows and a set U_3 of columns with desired properties that we now describe. First, they do not intersect S. Second, they are small enough that $|U_2| = O(n^{1/2+\epsilon})$ and $|U_3| = O(n^{1/2+\epsilon})$. We (provisionally) include in $\mathscr{C}$ any entry containing a symbol in Ψ in the rows in U_2 and/or the columns of U_3. Lemma 6.18 implies that these entries cover a set U_4 of $n - O(n^{1/2+2\epsilon})$ rows and a set U_5 of $n - O(n^{1/2+2\epsilon})$ columns.

At this point, $\mathscr{C}$ may not be a partial minimal cover, so we iteratively remove redundant entries from $\mathscr{C}$. Afterwards, the following three sets, each comprising of $n - O(n^{1/2+2\epsilon})$ lines, are covered and no entry in $\mathscr{C}$ can cover more than one of the following lines:

- The rows in U_4 that are not in U_2.
- The columns in U_5 that are not in U_3.
- The symbols other than those in Ψ.

Thus $\mathscr{C}$ is a partial minimal cover of size at least $3n - O(n^{1/2+2\epsilon})$. By Lemma 6.17, there is a minimal cover of L of size $3n - O(n^{1/2+2\epsilon})$. We replace ϵ by $\epsilon/2$ to complete the proof. □

Next, we report on some computations of sizes of minimal covers for small Latin squares.

The Cayley tables of the groups $\mathbb{Z}_3$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$ have transversals and $(n+2)$-covers, but do not have any minimal $(n+1)$-covers. Thus the spectrum of sizes of minimal covers is not continuous in these two cases. However, these two Latin squares may be just small anomalies, since we found no other Latin squares of order up to 8 with a gap in their spectrum.

For orders $n \leqslant 5$, the minimal cover constructed in Lemma 6.12 meets the bound in Theorem 6.14 and hence has maximum possible size. For each order in the range $6 \leqslant n \leqslant 9$, we found a Latin square that has no minimal cover meeting the bound in Theorem 6.14. Our computations were exhaustive for $6 \leqslant n \leqslant 8$, where there is a gap of only 1 between the size of the cover in Lemma 6.12 and the bound in Theorem 6.14. For $n = 6$, there are 6 species that meet the bound and 6 that do not; neither group table meets the bound. For $n = 7$ there are 145 species that meet the bound. The 2 species that do not meet the bound contain the group $\mathbb{Z}_7$ and the Steiner quasigroup. For $n = 8$ there are 283654 species that meet the bound. The 3 species that do not meet the bound contain the dihedral group, the elementary abelian group, and the Latin square obtained by turning an intercalate in the elementary abelian group (that is, by replacing a 2×2 Latin subsquare with the other possible subsquare on the same two symbols). Note that the autotopism group of the elementary abelian group acts transitively on the intercalates, so it does not matter which intercalate gets turned.

We could not do exhaustive computations for all Latin squares of order 9, but we confirmed that $\mathbb{Z}_3 \times \mathbb{Z}_3$ meets the bound in Theorem 6.14, whilst $\mathbb{Z}_9$ does not. The largest minimal cover in $\mathbb{Z}_9$ has size 18, which is one more than the size of the example in Lemma 6.12 but one less than the bound in Theorem 6.14.

In Section 6.1, we showed a kind of duality between minimal covers and maximal partial transversals. However, we next reveal a distinction between the behaviours of these objects. We begin with the following theorem, which gives the values of k and n for which there exists a Latin square of order $n \geqslant 5$ with a maximal partial transversals of deficit $d = n - k$.

Theorem 6.20. *For all integers $n \geqslant 5$ and $k \geqslant 1$ satisfying $n \geqslant 2k$, there exists a Latin square $L = [L_{ij}]$ of order n where $L_{ii} = i$ for all $i \in \{0, \ldots, n-k-1\}$ and the intersection of the k rows and columns indexed by $\{n-k, \ldots, n-1\}$ is a subsquare (*i.e., *a submatrix that is a Latin square) on the symbols $\{0, \ldots, k-1\}$. Consequently, L has a maximal partial transversal of length $n - k$.*

Proof. A Latin square $M = [M_{ij}]$ of order m is idempotent if $M_{ii} = i$ for all $i \in \{0, \ldots, m-1\}$. Any Latin square with a transversal can be made idempotent by applying an isotopism.

The $n = 2k$ case of the theorem is immediate, by simply taking a direct product of an idempotent Latin square of order k with a Latin square of order 2. So we may assume that $k \leqslant n - k - 1$. Also, note that $n - k \geqslant \lceil n/2 \rceil \geqslant 3$.

$$\overbrace{\begin{array}{|c|c|c|c|c|c|}\hline 0 & 2 & 9 & 8 & 3 & 1 \\ \hline 3 & 1 & 0 & 7 & 2 & 4 \\ \hline 9 & 0 & 2 & 1 & 6 & 3 \\ \hline 8 & 7 & 1 & 3 & 0 & 2 \\ \hline 1 & 3 & 5 & 2 & 4 & 0 \\ \hline 2 & 6 & 3 & 0 & 1 & 5 \\ \hline \end{array}}^{\substack{M' \text{ when} \\ n-k=6 \text{ and } k=4}} \qquad \overbrace{\begin{array}{|c|c|c|c|c|c|}\hline 0 & 4 & 1 & 6 & 2 & 3 \\ \hline 3 & 1 & 4 & 2 & 7 & 0 \\ \hline 4 & 8 & 2 & 0 & 3 & 1 \\ \hline 1 & 2 & 9 & 3 & 0 & 4 \\ \hline 10 & 0 & 3 & 1 & 4 & 2 \\ \hline 2 & 3 & 0 & 4 & 1 & 5 \\ \hline \end{array}}^{\substack{M' \text{ when} \\ n-k=6 \text{ and } k=5}}$$

Figure 6.11: Matrices used in the $n - k = 6$ case of the proof of Theorem 6.20.

If $n-k=6$ and $k \in \{4,5\}$, we define M' as given in Figure 6.11. In all other relevant cases, we can find a Latin square of order $n-k$ with $k+1$ disjoint transversals [82]. Applying an isotopism, we get an idempotent Latin square $M=[M_{ij}]$ of order $n-k$. It has k disjoint transversals, denoted d_σ for $\sigma \in \{n-k,\dots,n-1\}$, which do not intersect the main diagonal. We replace the symbols in $\{k,\dots,n-k-1\}$ in each d_σ by the symbol σ, and call the result M'. We give an example of this construction in Figure 6.12.

Thus, M' is idempotent and contains $n-k$ copies of each symbol in $\{0,\dots,k-1\}$ and $n-2k$ copies of each symbol in $\{k,\dots,n-1\}$. Ryser's Theorem [69] implies that M' embeds in a Latin square L of order n; this is illustrated for the example in Figure 6.12. Moreover, since M' contains each symbol in $\{0,\dots,k-1\}$ exactly $n-k$ times, the intersection of the k rows and columns indexed by $\{n-k,\dots,n-1\}$ in L must be a subsquare on the symbols $\{0,\dots,k-1\}$. $\square$

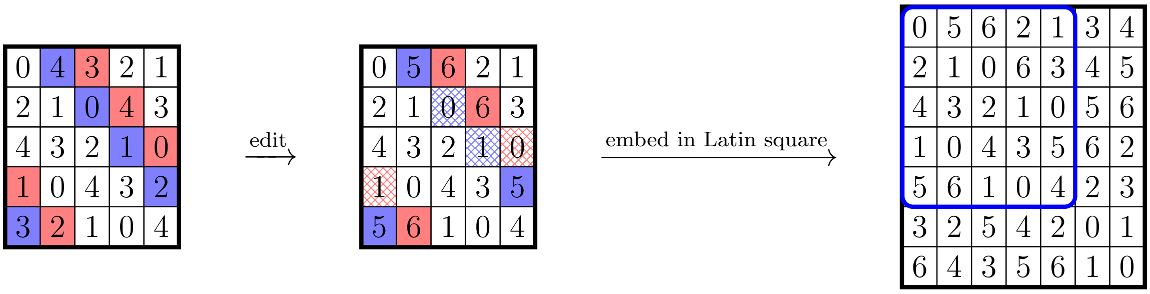


Figure 6.12: Example of the construction in the proof of Theorem 6.20 when $n=7$ and $k=2$.

Any partial transversal of length less than $\lceil n/2 \rceil$ can be extended. Thus, a consequence of Theorem 6.20 is that among all Latin squares of order $n \geqslant 5$, the shortest maximal partial transversal has length $\lceil n/2 \rceil$. Theorem 6.19 shows that the upper bound on minimal covers described in Theorem 6.14 is achieved asymptotically for all Latin squares of order n. However, as we established in Theorem 2.32, most Latin squares do not come close to achieving a maximal partial transversal of length $\lceil n/2 \rceil$. While minimum covers directly relate to maximum partial transversals (see Theorems 6.2 and 6.4), maximum minimal covers seem not to have a direct relationship with minimum maximal partial transversals.

6.3 Concluding Remarks on Covers

We have introduced covers of Latin squares with the aim of using them to better understand partial transversals, focusing primarily on topics relating to extremal sizes.

We found that some properties of covers have analogous properties for partial transversals, while others do not. For example, the maximum size of partial transversals is closely related to the minimum size of covers. However, the minimum size of a maximal partial transversal is $\lceil n/2 \rceil$, which most Latin squares do not come close to achieving (see Theorem 2.32). In contrast, the maximum size of a minimal cover is $3n - O(n^{1/2})$, which is asymptotically achieved by all Latin squares (see Theorem 6.19).

There are $(n+1)$-covers that contain no partial transversals of deficit 0 or 1. The error on the upper bound on the number of partial transversals in Theorem 6.7 grows with the number of such $(n+1)$-covers. Also, while Brualdi's Conjecture implies the existence of $(n+1)$-covers in all Latin squares of order n, we have not established the converse.

Instead, a weaker form of the converse is true: if every Latin square of order $n \geqslant 2$ has an $(n+1)$-cover, then every Latin square of order $n \geqslant 2$ has a partial transversal of deficit 2.

Relating the enumeration of partial transversals with small deficit ($d \in \{1, 2\}$) to the enumeration of $(n+1)$-covers is also difficult because the number of embeddings of a maximal partial transversal of deficit d within an $(n+1)$-cover depends on the structure of the Latin square.

There are switches that can be performed among $(n+1)$-covers, such as

3	**1**	5	2	0	4
1	0	**3**	5	4	2
4	2	1	3	5	**0**
2	5	0	**4**	1	3
0	4	2	1	3	**5**
5	3	4	0	**2**	1

$\longleftrightarrow$

3	**1**	5	2	0	4
1	0	**3**	5	4	2
4	2	1	3	5	0
2	5	0	**4**	1	3
0	4	2	1	3	**5**
5	3	4	0	**2**	1

which converts an $(n+1)$-cover inducing G_5 into an $(n+1)$-cover inducing G_3. However, we did not succeed in making switchings work for converting $(n+1)$-covers inducing G_1 into the other structures, which would yield a partial transversal of deficit 1. It is possible that more complicated switching patterns might succeed in changing the graph structure in $(n+1)$-covers inducing G_1, but it is also possible that identifying such switchings would not be possible without, say, proving Brualdi's Conjecture.

In the case of minimal covers of maximum size, the results in Section 6.2 make significant progress, finding an explicit upper bound that is achieved infinitely often, and that is achieved asymptotically by all Latin squares.

In the proof of Theorem 6.19, we find an $O(n^{1/2+\epsilon}) \times O(n^{1/2+\epsilon})$ submatrix S containing all but $O(n^{1/2+\epsilon})$ symbols. This raises the question as to whether stronger results in this direction hold. Does every $n^2 \times n^2$ Latin square contain an $n \times n$ submatrix that contains every symbol? The 4×4 Latin squares each have 2×2 submatrices containing all four symbols, but the 9×9 Latin square

0	1	2	3	4	5	6	7	8
1	0	3	2	5	6	7	8	4
2	3	1	0	7	8	4	5	6
3	2	0	1	6	7	8	4	5
4	5	8	7	1	2	3	6	0
5	6	4	8	0	1	2	3	7
6	7	5	4	8	0	1	2	3
7	8	6	5	3	4	0	1	2
8	4	7	6	2	3	5	0	1

found by White [85], has the property that no 3×3 submatrix contains all nine symbols. It would be of some interest to find more precise results for general Latin squares as to how small a submatrix contains every symbol, and/or how many distinct symbols we can be sure to find in at least one submatrix of given dimensions.

There are multiple directions in which the study of covers could be extended; we describe some below.

Some of the results here could be extended to Latin rectangles or even special kinds of partial Latin rectangles such as plexes [82]. It would also be interesting to extend the investigation to Latin hypercubes, sets of mutually orthogonal Latin squares, or to MDS codes more generally.

The Cayley tables of groups are of particular interest, since transversals in them are equivalent to orthomorphisms, and problems such as enumeration of orthomorphisms (particularly for cyclic groups) have been studied [59]. Moreover, cyclic group tables have a lot of structure (see, *e.g.,* Lemma 6.9) that may permit a more successful study of switchings than in general Latin squares.

Each of the five structurally distinct $(n+1)$-covers can be embedded in a Latin square of order 5, as shown in Figure 6.3, so by replacing the 5×5 subsquares in the $k = 5$ case of Theorem 6.20, we find that every potential $(n+1)$-cover embeds in a Latin square of order n, for all $n \geqslant 10$. In fact, the same is easily found to be true for orders in $\{5, \ldots, 9\}$ (by searching random Latin squares of these orders). It would be interesting to resolve the general case of this embedding problem, *i.e.,* for which orders n does every potential $(n+a)$-cover complete to a Latin square? A famous problem along these lines is Evan's Conjecture [37], which has since been proved [5, 45, 75], which states that a partial Latin square of order n with at most $n-1$ entries can be completed.

Exhaustive computations for orders $n \leqslant 8$ suggest the following:

Conjecture 6.21. Let L be a Latin square of even order n, with t transversals and $q_{\min}$ minimal $(n+1)$-covers. Then $t \equiv 2q_{\min} \pmod 4$.

Since we know that every Latin square of order $n \equiv 2 \pmod 4$ has $t \equiv 0 \pmod 4$, Conjecture 6.21 would imply that $q_{\min} \equiv 0 \pmod 2$ when $n \equiv 2 \pmod 4$. Another curious observation is that the number of $(n+1)$-covers in every Latin square of order 7 is divisible by 3.

Finally, we mention that the data in Table 6.1 shows approximate consistency in the number of $(n+1)$-covers that Latin squares of order n have. If this is a pattern, it might be worth investigating as a means to prove a weakened form of Brualdi's Conjecture (via Theorem 6.4).

Chapter 7

Computational Results

The real danger is not that computers will begin to think like men, but that men will begin to think like computers.
– S. J. Harris

Computation is an increasingly important part of research in combinatorics. Many problems, such as counting objects, have become much too cumbersome for a person to accomplish without the aid of a computer. We start with a discussion of some computational tips and tricks that can dramatically improve the runtime of the programs that are written for combinatorics. We have kept this section completely separate so that the important topics in each of the subsequent chapters are not bogged down by small implementation details. The reader is safe to skip Section 7.1 if computation is not their forte.

The chapter then continues to describe three separate computational results. First, we show that a generalisation of Brualdi's Conjecture is true for $n \leqslant 11$. We then introduce an idea which allows us to search for projective planes of order 11 and finish by finding all Latin arrays of order $n \leqslant 7$ that do not contain a transversal.

7.1 Computational Tips and Tricks

(The reader may jump to Section 7.2 if computational implementation is not of interest.)

A good portion of computation in combinatorics involves problems that are NP-complete, and thus, we (likely) have no hope of finding an efficient algorithm for our problems. The combinatorial objects are often quite structured and do not always perform well by using heuristics that work well in the *general* case. It is not uncommon for programs to run for many CPU-years, so small improvements can often lead to dramatically smaller runtimes. For an excellent resource on common algorithms, we refer the reader to [21].

Before we get started, we would like to put a global disclaimer: it is extremely rare for optimisations to improve efficiency in every situation. When a program is run for a long period of time, you should test your code on a few different heuristics to determine which works best for your problem/hardware/compiler. It is often the case that the best strategy to just write simple code that the compiler can optimise. Our first (and potentially most important) tip is to ensure that your compiler's optimisations are turned on. For example, `gcc` has the flags `-Ox` for turning on optimisations. We have found that `-O2` is often the fastest for the problems listed here, but you should experiment for your needs.

A bitmask makes use of the binary representation of numbers on modern-day computer architectures. Rather than caring about the integer value of the number, we care about which bits are set in the number. The first use of a bitmask is simply a compressed boolean array, where each bit represents true or false. However, more complicated tasks can be accomplished. For example, if we wish to remove the first non-zero bit in a number, we may simply compute `x & (x-1)`. We call this *extracting a bit* from x. We can also isolate that bit if we wish rather than removing it with `x & (-x)`. An example where this is useful: imagine we are constructing a Latin square one cell at a time. To do this, we simply try placing the symbol 0 into the first cell and recursively ask if we can complete this square. If we can, then we are done. If not, then try placing the symbol 1, etc. But before placing the symbol into the cell, we must first check if that symbol already appears in the current row or column. Instead of just going through each symbol and checking, we can use the above idea. Each row and column must have a bitmask indicating which symbols have been used in it. The bitwise-`and` of the bitmask for the row and column gives a bitmask of all symbols which are allowed in this cell. Using the extraction method above, we can remove candidates one at a time from our bitmask and try each one of those. Note that this does not reduce our overall complexity since we still (on average) look at $O(n)$ symbols for each cell, but the constant hidden by the big-O is improved since we only look at valid symbols. One extremely important topic in bitmasks is determining the number of bits that are set (often called the *popcount* of the number). The first naive implementation is to visit all n bits, checking each one if it is set or not. However, this can immediately be improved by repeatedly extracting a bit (as above) from your number one at a time until you reach 0. However, in practice, a lookup table is often significantly faster. If n is small enough and popcounts are needed continually, we recommend that you store all 2^n popcounts in memory. If n is too large (or memory is a constraint), then we recommend storing the popcounts for the first 2^{16} integers in memory. You may then just do $\lceil n/16 \rceil$ lookups. Note that 16 being a multiple of 8 is important for efficiency on many machines. There are many more uses for bitmasks which we do not include here, but we would recommend visiting [6] for an overview of neat tricks. On many architectures, the idea of DeBruijn sequences can dramatically improve simple operations assuming that multiplication can be done quickly (again, see [6]). We should also note that certain architectures have built-in assembly code for these operations, and if so, they will likely be faster than anything you can code on your own.

As mentioned above, sometimes the structure in our problems do not respond well to general heuristics. As a quick example: one of the best programs for finding cliques in general graphs is `Cliquer` [64]. Unfortunately, once you pass a graph to `Cliquer` of even moderate size, it starts to struggle. However, we can make use of the underlying structure of the graph to perform better than general programs since this information is lost when passed to `Cliquer`. Consider the problem of trying to find n disjoint transversals in a Latin square. To do this, we first find all transversals (see Section 7.1.1 for details on this) and then build a graph G with one vertex for each transversal. An edge is placed between each pair of disjoint transversals. A clique of size n corresponds directly to n disjoint transversals. However, by just focusing on G, we lose a lot of information about our problem. For example, we can easily n-colour G based on which column the transversal intersects the top row. If we wish to find an n-clique, we must take exactly one vertex from each of these colour classes. This observation is crucial in our computations. Moreover, it is often better to not construct the graph explicitly, but rather work on an implicit graph. In the problem above, a naive approach to find every neighbour of a transversal is to iterate through each transversal in the Latin square and check the number of intersections. However, consider the prefix tree of transversals (where each path from the root to a leaf corresponds to a transversal). We simply perform a depth-first search on this tree. The vertex at depth i corresponds to the entry in row i of our transversal. Thus, if there is a clash, we may simply break out of our current recursive call and continue one level up our search tree. This idea was described in [30] by not first compressing it into a prefix tree.

And as a closing remark, we would be remiss to not mention `nauty` [61], an extremely useful tool used to find automorphisms in graphs. This program is kept up to date and is integral to many tasks such as determining equivalence.

7.1.1 Finding Transversals

One very common computational task needed in the study of Latin squares is counting the number of transversals. Unfortunately, to date, there is no known way of counting transversals in a Latin square of even modest sizes. In this section, we describe a few different algorithms to compute this as well as give some C++ code in Appendix A.

The first naive approach of finding a transversal is to iterate through all $n!$ diagonals and check each diagonal exhaustively, but this quickly becomes infeasible with a complexity of $O(n!n)$. This can be sped up by recursively searching row-by-row and attempting to extend the current partial transversal. In this recursive algorithm, you would check each of the entries in the current row and recursively add each entry that does not clash with a column or symbol that has already been selected (note that the row cannot clash since our search is row-by-row). A C++ code snippet of this approach can be found in Algorithm 3. This code is quickly improved in Algorithm 4 by utilising a linked-list to represent which columns are still unused (see Algorithm 4).

Although this approach is still $O(n!n)$, it performs substantially better in practice since we are able to prune certain diagonals early. This algorithm on its own is not likely to be much faster asymptotically than $O(n!n)$ due to the fact that there are diagonals with large weight even if there are no transversals. In fact, on average, a diagonal contains $(1-1/e)n$ symbols (this was first proven by Lindner and Perry, but first published by Stein in [76]). If our goal is a better asymptotic result, we can easily improve our complexity by using dynamic programming. At each step in the recursion, a certain subset of columns and symbols have been used, and if we ever recurse to a point where the same subset of columns and symbols have been used, then we can use the previously computed result rather than computing it again. This reduces the algorithm to $O(4^n n)$. However, due to the use of such a large amount of memory, the addition of dynamic programming often slows the process of counting in practice and becomes nearly impossible for larger n.

The addition of the linked list speeds up the search and allows us to push a few orders further. If we wish to push even further, we can do so by utilising a meet-in-the-middle approach. Instead of building an entire transversal one row at a time, we can create partial transversals in the top half of the square, then create partial transversals in the bottom half of the square and sew these partial transversals together into transversals. To do this efficiently, let S be a k-subset of the columns (any k will work, but choosing $k = \lfloor n/2 \rfloor$ seems to perform the best in practice). We create all partial transversals of length k that are in the $k \times k$ subsquare formed by the top k rows and the k columns that are in S. Then we create all partial transversals of length $n-k$ that are in the $(n-k) \times (n-k)$ subsquare in the bottom $n-k$ rows and the $n-k$ columns that are not in S. We then count the number of pairs of partial transversals that do not share any symbols. For simplicity in the code comments in Algorithms 5 and 6, the completed partial transversals (of length k in the top-half and length $n-k$ in the bottom half) are called "(half-)partial transversals". In our main function, we must set up our linked-lists appropriately so that only the relevant columns are visible to each portion of the search. Finally, one additional idea can improve the runtime of the code. Note that in Algorithm 5, we reset the entire `counter` array at each step, which requires us to touch each of the 2^n memory locations. However, by simply having an auxiliary array which keeps track of the last time that we saw each of the elements, we do not have to necessarily touch each of the 2^n memory locations at each step. This optimisation is implemented in Algorithm 6.

Each of the four code snippets can be modified to actually find each of the transversals (not just count them). Algorithms 3 and 4 are easily modified, but a little care is needed in Algorithms 5 and 6. The naive way to modify Algorithms 5 and 6 is to have a list of (half-)partial transversals for each possible bitmask. However, this slows down the code quite substantially since a certain bitmask may be visited by the top half and not have a match in the bottom half. This requires a substantial amount of (potentially) wasted memory. This can be fixed by doing three searches for each set S (instead of two as above). The first pass is simply to mark which of the bitmasks are actually present in the bottom half. Then when we are doing the top-half, we only need to store the (half-)partial transversals that we know will be completable. The third pass builds the (half-)transversals in the bottom half and connects to the appropriate (half-)transversals in the top half. This implementation has the advantage that it does not require extra memory on top of what is needed for Algorithm 6 and the memory to store the transversals. We have found that this three-pass implementation works better in practice than the naive two pass approach.

Small modifications to the code included in Appendix A can improve the runtimes, but those modifications are quite minimal in comparison to Algorithm 6 and are not included for ease of reading the code.

7.2 Brualdi's Conjecture

When transversals are not present in Latin squares, we shift our focus to finding long partial transversals. If Brualdi's Conjecture is correct, then we know that every Latin square has a near transversal. However, this property may be true more generally. In this section, we extend previously known results about long partial transversals from Latin squares to more general objects.

The key idea needed to prove the best known bound on the length of a partial transversal (Theorem 2.31) is the idea of #-swapping. Say you have a diagonal, T, of weight w. Choose two entries from T, say (i_0, j_0, k_0) and (i_1, j_1, k_1). If $T \setminus \{(i_0, j_0, k_0), (i_1, j_1, k_1)\}$ still covers w symbols, then we consider

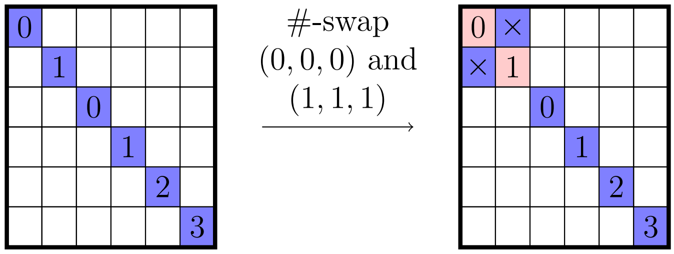
$$(T \setminus \{(i_0, j_0, k_0), (i_1, j_1, k_1)\}) \cup \{(i_0, j_1, \bullet), (i_1, j_0, \bullet)\}.$$

This diagonal is guaranteed to have a weight of either w, $w+1$ or $w+2$. The act of swapping $\{(i_0, j_0, k_0), (i_1, j_1, k_1)\}$ for $\{(i_0, j_1, \bullet), (i_1, j_0, \bullet)\}$ to obtain a new diagonal is called a #-swap. Note that by repeated use of #-swaps, the weight of the diagonal can never decrease. Thus, if we start with a diagonal of maximum weight, it is impossible to #-swap to a diagonal of larger weight and the set of symbols on each of these diagonals are the same.

Throughout the remainder of the section, we use $\times$ to denote that that cell of the square must contain a symbol that appears on the original diagonal. For example, if that cell is reachable via a sequence of #-swaps and the original diagonal has maximum weight, then the symbol in the cell must appear somewhere on the original diagonal. Moreover, if a cell of a square is empty, it means that we do not know anything about it.

Example 7.1. Here is an example of #-swapping on a diagonal of weight $4 = 6 - 2$. If we remove the top left 0 and 1 from the transversal, we still have 4 symbols left, so we may

#-swap on these entries and instead consider the diagonal which contains the two $\times$'s and the bottom four rows unchanged.



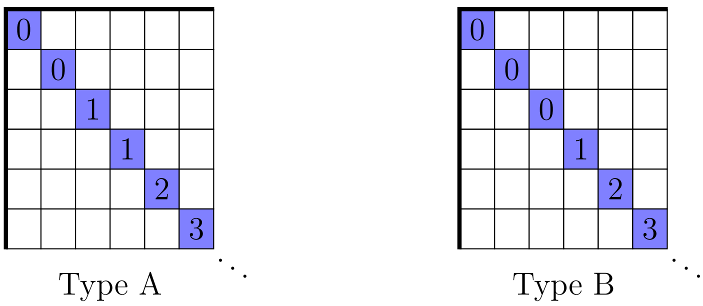
Note that when performing a #-swap, the two cells that were swapped are lightly shaded for further clarity.

For the remainder of this section, we shift our focus to proving a generalisation of Brualdi's Conjecture. We try to find near transversals in all Latin arrays rather than just Latin squares. We focus on diagonals of weight $n-2$ and attempt to uncover a new symbol, which would locate a near transversal. The following simple observation is needed throughout.

Proposition 7.2. *If every Latin array of order n contains a partial transversal of length k, then every Latin array of order $n+1$ contains a partial transversal of length k.*

Proof. Let L be a Latin array of order $n+1$. Let L' be the Latin array corresponding to the first n rows and n columns of L. By our assumption, L' has a partial transversal of length k, which corresponds to a partial transversal of length k in L. $\square$

Throughout the section, we utilise Proposition 7.2 iteratively. We use the fact that all Latin arrays of order n contain a near transversal, and thus, all Latin arrays of order $n+1$ contain a diagonal of weight at least $n-2$. If you have a diagonal of weight $n-2$, there are two different possible configurations for the duplicated symbols:



Type A Type B

We say that the square on the left has type A and the square on the right has type B. We first start by showing that the existence of type B implies the existence of type A in maximal cases.

Lemma 7.3. *Let L be a Latin array of order n with a diagonal of weight $n-2$ and no diagonal of weight greater than $n-2$. If L has a diagonal of type B, then there exists a diagonal of type A that you can get to by using only #-swaps.*

Proof. Assume, on the contrary, that you cannot reach a diagonal of type A. Without loss of generality, the diagonal is the main diagonal and the three repeated symbols are in the top 3 rows.

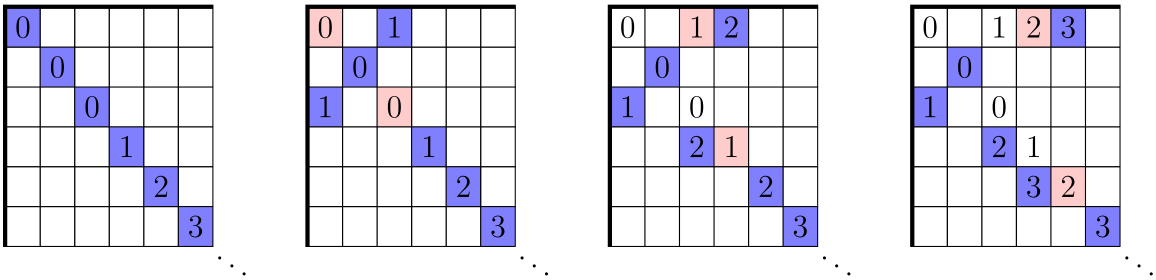
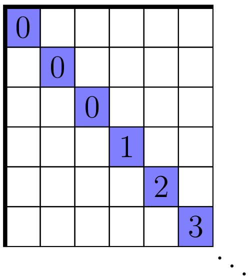


Figure 7.1: The first three #-swaps in Lemma 7.3.



We need to perform $n-2$ #-swaps to arrive at a contradiction. At first, we #-swap $(0,0,0)$ and $(2,2,0)$. The symbols in the cells $(0,2)$ and $(2,0)$ must be the same, otherwise this new diagonal would be of type A. Without loss of generality, these cells contain the symbol 1. We now #-swap $(2,0,1)$ and $(3,3,1)$. By a similar argument, the two uncovered cells must contain the same symbols (which is, without loss of generality, 2). We repeat this same argument $n-2$ times in total. On all steps i (except the first one), we #-swap the entries $(0,i,i-1)$ and $(i+1,i+1,i-1)$ and expose the entries $(0,i+1,i)$ and $(i+1,i,i)$. The first three steps are shown in Figure 7.1.

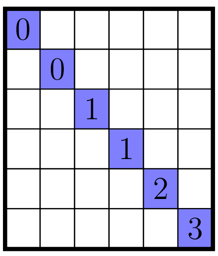
However, at step $n-2$, the uncovered symbol must be some symbol that did not appear on the original diagonal. Thus, we have found a heavier diagonal, a contradiction. $\square$

Note that Pula [68] proved that every Latin square with a partial transversal of length t contains a diagonal of weight t such that each symbol appears at most twice. However, Lemma 7.3 is slightly stronger when the weight is $n-2$ and is needed for the analysis below.

Next, we give a simple example of how using #-swaps is useful.

Lemma 7.4. *In any Latin array of order* 6*, there exists a diagonal of weight at least* 5*.*

Proof. First, it is quite easy to show that the heaviest diagonal must be at least of weight 4 (for example, use Theorem 2.29). We now assume, on the contrary, that there exists a Latin array that contains a diagonal of weight 4, but none of weight 5 or 6. Without loss of generality, the original diagonal of length 4 is along the main diagonal. By Lemma 7.3, we know that it must take the form given here.



At this point, we are presented with four options for which pair of entries to #-swap (choose either 0 and either 1 independently). From this, we can see we have the following.

0		$\times$	$\times$		
	0	$\times$	$\times$		
$\times$	$\times$	1			
$\times$	$\times$		1		
				2	
					3

As explained above, each of the $\times$ s that are uncovered must be one of $0, 1, 2, 3$, otherwise, we would have a heavier diagonal. Let's explore #-swapping the entries in the first row and the third row. The symbol in the $(2, 0)$ cell must be either 2 or 3. Without loss of generality, we assume that it is a 2.

0		$\times$	$\times$		
	0	$\times$	$\times$		
2	$\times$	1			
$\times$	$\times$		1		
				2	
					3

Note that we do not know what symbol is in the $(0, 2)$ cell, but we do know that it is a duplicate symbol (*i.e.,* it appears at least one more time on the diagonal or at least two more times if it is a 2). Thus, we are free to #-swap on that entry now. We #-swap that entry and $(4, 4, 2)$.

0		$\times$	$\times$	$\times$	
	0	$\times$	$\times$		
2	$\times$	1			
$\times$	$\times$		1		
		$\times$		2	
					3

The symbol in the $(4, 2)$ cell must be either 0 or 3 and the symbol in the $(0, 4)$ cell is a duplicate (as described above), and so may be used immediately. At this point, we consider both cases for the $(4, 2)$ cell separately. In either case, we #-swap the entry in the top row with the appropriate duplicated symbol.

0	$\times$	$\times$	$\times$	$\times$	
	0	$\times$	$\times$	$\times$	
2	$\times$	1			
$\times$	$\times$		1		
		0		2	
					3

0		$\times$	$\times$	$\times$	$\times$
	0	$\times$	$\times$		
2	$\times$	1			
$\times$	$\times$		1		
		3		2	
				$\times$	3

In either case, the top row now has five entries whose symbol must come uniquely from the set $\{0, 1, 2, 3\}$, and thus, we have a contradiction and the result follows. $\square$

0	×	×	×	×	×		
	0			1	2		
2		1	3				
3	2		1	0			
	3	0		2			
1			0		3		
						4	
							5

Figure 7.2: One of 14 squares that fail Algorithm 1 for $n = 8$.

Hatami and Shor [46] used this same idea to show the same result as Lemma 7.4. However, in their description, they did not leave all of the symbols in the top row as unknown (×). Instead, they did extra case analysis to determine what those symbols could be. By leaving the top row as unknown symbols, there is the potential for less branching in the algorithm. Moreover, by continually using the top row to #-swap on, there are only two choices of pairs of entries to #-swap (but one of these choices brings you back to the previous diagonal).

We describe two algorithms whose goal it is to show that there is a near transversal in all Latin arrays of order n. Algorithm 1 describes the basic algorithm to show that all Latin arrays of order n contain a near transversal. This algorithm formalises the method used in the proof of Lemma 7.4. This algorithm is refined below in Algorithm 2 to work for larger orders.

It is important to note that in both algorithms below, all variables are considered local variables, so changing the value of a parameter does not affect its value outside of that specific instance.

Algorithm 1 is sufficient to show that all Latin arrays of order $n \leqslant 7$ contain a near transversal. However, for $n = 8$, Algorithm 1 fails to show the desired result as it returns False for Figure 7.2.

A total of 14 squares fail Algorithm 1 for $n = 8$. For $n = 9$, one may expect more squares to fail Algorithm 1, but interestingly, those 14 squares (with one extra row and column added) are the only squares to fail Algorithm 1. For $n = 10$, a total of 82 140 squares fail Algorithm 1.

Thus, a more refined approach is needed to find near transversals in larger orders. The first observation is that after we have cycled back on ourselves and returned False on line 5 of Algorithm 1, we may now choose another row to #-swap on rather than the first one. Recall that by only using #-swaps on the top row, we are only utilising two of the possible #-swaps available (there are 4 possible if the diagonal is of type A and 3 if it is of type B). In fact, one need not use the main diagonal at all as our starting point (though, in our searches, we always centre around the main diagonal). In Algorithm 2, we first #-swap along the top row. Once we cycle around, we then #-swap along the second row, then the third, then the fourth. In Algorithm 1, we arbitrarily selected row 3 to be the initial value for r. In Algorithm 2, when we are #-swapping on rows 0,1,2 and 3, we use the rows 3,2,1 and 0, respectively for the initial value of the "row we just #-swapped on". This is so that we may use the fact that $r_0 + r_1 = 3$ to save space in the algorithm.

The good news is that there are two heuristics that can be added to the search that improve its performance in practice immensely when utilised together. (However, there is a minor drawback to using them, which we will discuss shortly.) The first heuristic is to search

Algorithm 1 Basic algorithm to show that all Latin arrays of order n contain a near transversal. $\textsc{NaiveHash}(L, \epsilon, 0, 3)$ should be called initially, where L is an $n \times n$ ($n \geqslant 4$) array with all cells empty except the main diagonal, which contains $(0, 0, 1, 1, 2, 3, \ldots, n-3)$ and ϵ is the identity permutation. Note that 3 is an arbitrary choice—we could have selected 2 or 3 (the rows of the duplicated symbol 1).

```
Input L is a partial Latin array
Input σ is a permutation defining a diagonal of weight n − 2 in L
Input d is the depth of the search
Input r is the row we just hashed on
Output TRUE if a near transversal is guaranteed in all Latin arrays of order n.
Output FALSE if inconclusive.

procedure NAIVEHASH(L, σ, d, r)
    if Some row or column of L contains at least n − 1 filled cells then
        return TRUE                                   ▷ Near transversal guaranteed
    if d ≠ 0 and σ is the identity and r = 3 then
        return FALSE                        ▷ We have cycled back to where we started

    S ← L(r, σ_r)                                              ▷ Symbol to hash on
    R ← row such that σ_R = S, R > 0 and R ≠ r    ▷ Other row that contains S on σ
    swap(σ_0, σ_R)
    Fill cell (0, σ_0) and (R, σ_R)     ▷ Set to × if it does not already contain a symbol.

    if L(R, σ_R) ≠ × then                      ▷ If we already know what symbol this is
        return NAIVEHASH(L, σ, d + 1, R)
    else                      ▷ If we do not know what symbol is here, then try all valid ones.
        k ← largest symbol in L that appears multiple times
                          ▷ The symbols k + 1, k + 2, ..., n − 3 are all symmetric up to
                                this point, so we only need to consider one of them
                                       (without loss of generality, we use k + 1).
        for s ← 0 to min(k + 1, n − 3) do
            if s is not in row R nor column σ_R then
                L(R, σ_R) ← s
                if NAIVEHASH(L, σ, d + 1, R) = FALSE then
                    return FALSE
        return TRUE       ▷ No matter which symbol we use, there is a near transversal
```

0	6	$\times$	5	3	$\times$	2	4	1		
$\times$	0	3	$\times$	1	$\times$	$\times$	$\times$	$\times$		
2	$\times$	1	4	$\times$	$\times$	$\times$	0	3		
6	2	$\times$	1	$\times$	$\times$	$\times$	$\times$	5		
$\times$	$\times$	0	3	2	$\times$	$\times$	6	4		
$\times$	1	5	$\times$	$\times$	3	6	$\times$	$\times$		
3	$\times$	$\times$	$\times$	5	$\times$	4	$\times$	$\times$		
$\times$	$\times$	$\times$	$\times$	$\times$	2	1	5	0		
$\times$	$\times$	$\times$	2	$\times$	5	0	$\times$	6		
									7	
										8

Figure 7.3: A partial Latin array that loops back in Line 4 when $r_0 = 0$.

for diagonals of weight $n-2$ that may not be reachable via #-swaps. If such a diagonal covers all rows except r_0 and r_1 and all columns except c_0 and c_1, then we know that each of $(r_0, c_0), (r_0, c_1), (r_1, c_0)$ and (r_1, c_1) must also contain symbols from $\{0, 1, \ldots, n-3\}$, or else we would have a near transversal. Thus, if those cells are empty, we may fill them with an $\times$. In practice, every time that we fill in a cell with a specific symbol (not an $\times$), we only search for diagonals that go through that cell. The second heuristic is to choose some $\times$ in the square and decide what that symbol should be by exhaustively trying each one. We define the *liberties* of a cell to be the number of symbols that could be placed into the cell without violating the Latin property. In practice, we choose an $\times$ that has the fewest liberties so that our search does not branch too much. All of these are combined into Algorithm 2.

Algorithm 2 is good enough to show the following.

Theorem 7.5. *Every Latin array of order $n \leqslant 11$ contains a near transversal.*

Proof. The cases where $n < 4$ are easy to see (or you can use Theorem 2.29). We iteratively use Proposition 7.2 and Algorithm 2 for $n = 4, \ldots, 11$. $\square$

In the search of $n = 11$, no Latin arrays needed more than just the top two rows to #-swap. The search for $n \leqslant 10$ can be completed in a matter of minutes, while a few hours is needed for $n = 11$. Based on this progression, we believed $n = 12$ to be possible. However, after running our program for several months with several different pruning heuristics on a small grid, we did not think that the program would finish in a reasonable amount of time. Due to the recursive nature of the algorithm, it is difficult to accurately determine what percentage of the search space was covered over those months. Needless to say, all cases that we searched did not provide a counterexample to Brualdi's Conjecture. Note that if one wishes to use Algorithm 2 to find near transversals in just Latin squares (not all Latin arrays), then we would recommend extra heuristics be added. For example, the partial Latin array in Figure 7.3 is one of many squares that loops back on Line 4 of Algorithm 2 with $r_0 = 0$. The search continues with $r_0 = 1$, however, no further search is needed if we are only concerned with Latin squares. There are only two symbols which are not on the original diagonal (9 and 10), but there are not enough empty cells left to place them into. In particular, at least $2(n-2)-4$ cells in the top $(n-2) \times (n-2)$ submatrix must be empty in order to fit in the two missing symbols.

Algorithm 2 More advanced algorithm to determine if all Latin arrays of order n contain a near transversal. $\textsc{Hash}(L, \epsilon, 0, 0, 3)$ should be called initially, where L is an $n \times n$ $(n \geqslant 4)$ array with all cells empty except the main diagonal, which contains $(0, 0, 1, 1, 2, 3, \ldots, n-3)$ and ϵ is the identity permutation. $\textsc{FillCell}$ simply tries all valid symbols to place in the cell (r, c) and calls $\textsc{Hash}$ with the same parameters, but with depth $d + 1$.

Input L is a partial Latin array
Input σ is a permutation defining a diagonal of weight $n - 2$
Input d is the depth of the search with the current r_0
Input r_0 is the row we are mainly #-swapping on (this was the top row in Algorithm 1)
Input r_1 is the other row that we just #-swapped on
Output $\textsc{True}$ if a near transversal is guaranteed in all Latin arrays of order n.
Output $\textsc{False}$ if inconclusive.

procedure $\textsc{Hash}(L, \sigma, d, r_0, r_1)$
    if Some row or column of L contains at least $n - 1$ filled cells **then**
        return $\textsc{True}$ ▷ Near transversal guaranteed
    if $d \neq 0$ and σ is the identity and $r_0 + r_1 = 3$ **then** ▷ We have cycled back
        if $r_0 \geqslant 3$ **then return** $\textsc{False}$ ▷ We need to try something different.
        else return $\textsc{Hash}(L, \sigma, 0, r_0 + 1, r_1 - 1)$ ▷ Try #-swapping along the next row.

    if $d \equiv 3 \pmod 4$ and there is at least one $\times$ in L **then**
        $(r, c) \leftarrow$ cell such that $L(r, c) = \times$. ▷ If there are multiple $\times$, select one with the fewest liberties, breaking ties by selecting the first one in row-major order.
        return $\textsc{FillCell}(L, r, c, \sigma, d, r_0, r_1)$

    $S \leftarrow L(r_1, \sigma_{r_1})$ ▷ Symbol to #-swap on
    $R \leftarrow$ row where $\sigma_R = S$, $R \neq r_0$ and $R \neq r_1$ ▷ Other row that contains S on σ
    $\text{swap}(\sigma_{r_0}, \sigma_R)$
    return $\textsc{FillCell}(L, R, \sigma_R, \sigma, d, r_0, R)$

procedure $\textsc{FillCell}(L, r, c, \sigma, d, r_0, r_1)$
    if $L(r, c) \in \{0, \ldots, n-3\}$ **then return** $\textsc{Hash}(L, \sigma, d + 1, r_0, r_1)$

    $k \leftarrow$ largest symbol in L that appears multiple times
    for $s \leftarrow 0$ to $\min(k + 1, n - 3)$ **do**
        if s is not in row r nor column c **then**
            $L' \leftarrow L$ ▷ Store a copy of L
            $L(r, c) \leftarrow s$
            for each Partial transversal, T, of length $n - 2$ **do**
                $\{R_1, R_2, C_1, C_2\} \leftarrow$ the two rows and columns missing from T
                Fill in cells $(R_1, C_1), (R_1, C_2), (R_2, C_1), (R_2, C_2)$ with $\times$ if empty
            if $\textsc{Hash}(L, \sigma, d + 1, r_0, r_1) = \textsc{False}$ **then**
                return $\textsc{False}$
            $L \leftarrow L'$ ▷ Restore L to its previous configuration
    return $\textsc{True}$ ▷ No matter which symbol we place here, there is a near transversal

The fact that all Latin arrays, and not just Latin squares, have near transversals is an encouraging sign for Brualdi's Conjecture. In fact, we think a stronger result holds:

Conjecture 7.6. Every Latin array contains a near transversal.

We would like to point out that a similar idea to the above algorithms was employed by Pula [68], and that he effectively proved Theorem 7.5 for $n \leqslant 10$. However, all of his work was focused on Latin squares, not Latin arrays, so some minor details such as his version of Proposition 7.2 were only applicable to Latin squares.

In the proof of Theorem 2.31 by Shor and Hatami [46], one of the key ingredients was sets of diagonals with the same weight that were connected by a sequence of #-swaps. A sequence of integers n_k was discussed in detail. To connect those to our results here, n_2 is defined as the smallest order such that a diagonal of weight $n-2$ cannot be #-swapped to uncover a new symbol. Note that the heuristics employed in Algorithm 2 mean that we cannot use the results from Theorem 7.5 to show that $n_2 \geqslant 12$. However, Pula's search [68] may be used to show that $n_2 \geqslant 11$ and we have verified this utilising a similar idea to Algorithms 1 and 2.

Of course, the fact that these squares are Latin seems to be a very important factor in the potential truth of this conjecture (see Theorem 2.27). The idea used in Theorem 2.27 relies heavily on clumping all n of each symbol into a small space. The idea cannot be easily changed to accommodate only one of each symbol per row. Neither the bound of 10^{60} nor the $\log n$ was optimised in [67]. It would be interesting to know the smallest value of n where an equi-n-square exists that does not contain a near transversal.

Upon first glance, the bound shown by Shor and Hatami [46] ($n - 11.053 \log^2 n$) is weaker than the $n - \sqrt{n}$ bound for small values of n. In fact, for $n \leqslant 7\,731\,462$, it is better to use the $n - \sqrt{n}$ bound. However, the groundwork laid out in the asymptotic proof in [46] can be used in a concrete way to show significantly better bounds for lower orders. The key sequence, n_k, is a bound on the size that a square must have before being able to #-swap from a diagonal of weight $n-k$ to a heavier one. In particular, if you have a Latin array of order $n < n_k$, then it contains a diagonal with weight greater than $n-k$.

The following lemma is taken from [46], except the first inequality is the strengthened version from [68] as explained above.

Lemma 7.7 ([46])**.**

$$n_2 \geqslant 11, \tag{7.1}$$

$$n_k \geqslant n_{k-1} + 2k \quad \textit{for } k > 2 \textit{ and} \tag{7.2}$$

$$(n_k - n_j)(2n_j + n_{k-1} - 2n_k + 2k - j) \leqslant n_j(n_j - n_{j-1} - 2j) \quad \textit{for } 3 \leqslant j < k. \tag{7.3}$$

Shor and Hatami used (7.3) to show that $k \leqslant 11.053 \log^2 n_k$. While this seems worse than Theorem 2.29 for small values, simple induction using (7.1) and (7.2) shows that $n_k > k^2$, giving a better bound than Theorem 2.29 for all n. For small values, the deficiency of $11.053 \log^2 n$ is far from the truth. Table 7.1 shows the smallest values that n_k can take and satisfy Lemma 7.7. It is important to note that in order to minimise n_k, we may need to use non-optimal values for $n_2, \ldots, n_{k-1}$ (for example, $n_4 = 28$ is attainable. However, to achieve $n_5 = 41$, we must use $n_4 = 31$.).

7.3 Projective Planes of Order 11

One of the most interesting applications of Latin squares is their relationship with projective planes. We know that complete sets of mutually orthogonal Latin squares are equivalent

k	One sequence that minimises n_k $[n_2, \ldots, n_k]$
2	$[11]$
3	$[11, 17]$
4	$[11, 17, 28]$
5	$[11, 17, 31, 41]$
6	$[11, 17, 28, 46, 58]$
7	$[11, 17, 28, 42, 64, 78]$
8	$[11, 17, 28, 42, 63, 90, 107]$
9	$[11, 17, 28, 46, 58, 91, 122, 140]$
10	$[11, 17, 28, 42, 64, 78, 122, 157, 177]$
11	$[11, 17, 28, 42, 63, 90, 107, 165, 204, 226]$
12	$[11, 17, 28, 46, 58, 91, 122, 140, 216, 259, 283]$
13	$[11, 17, 28, 42, 64, 78, 122, 157, 177, 272, 320, 346]$
14	$[11, 17, 28, 42, 64, 78, 122, 157, 177, 272, 356, 408, 436]$
15	$[11, 17, 28, 42, 63, 90, 107, 165, 204, 226, 346, 439, 495, 525]$
16	$[11, 17, 28, 46, 58, 91, 122, 140, 216, 259, 283, 432, 534, 594, 626]$
17	$[11, 17, 28, 42, 64, 78, 122, 157, 177, 272, 320, 346, 527, 638, 702, 736]$
18	$[11, 17, 28, 42, 64, 78, 122, 157, 177, 272, 356, 408, 436, 662, 783, 851, 887]$
19	$[11, 17, 28, 42, 63, 90, 107, 165, 204, 226, 346, 439, 495, 525, 796, 933, 1005, 1043]$
20	$[11, 17, 28, 46, 58, 91, 122, 140, 216, 259, 283, 432, 534, 594, 626, 948, 1110, 1192, 1234]$

Table 7.1: Smallest values of n_k that satisfy Lemma 7.7 for $k \leqslant 20$ and one possible sequence of $[n_2, \ldots, n_k]$ to achieve that value.

to finite projective planes. However, searching for projective planes is quite a large computational task. To date, we have the complete story for $n \leqslant 10$, with each order except 6 and 10 having at least one projective plane. The projective planes of each order up to and including 8 are unique. However, for $n = 9$, there are several (depending on how equivalence is considered, see [20] for details). For $n = 10$, Lam [56] performed an extensive computer search and determined that no projective plane of order 10 exists. We know that when no projective plane exists, no *near-complete* set of mutually orthogonal squares can exist [16], which implies that there cannot be a set of seven or more mutually orthogonal Latin squares of order 10. However, as of today, we are nowhere near this bound. Many pairs of orthogonal Latin squares have been found, but no triple has been located. By an exhaustive computer search, McKay *et al.* [60] showed that if such a triple exists, each of the three squares must have no symmetry. This result, while very interesting, barely scratches the surface of the entire search space. To date, the closest example we have is due to Egan and Wanless [30], who found a pair of orthogonal Latin squares that contains seven disjoint transversals.

The idea used in [60] was to search through each Latin square of order 10 that contained symmetry, find all of its orthogonal mates and then see if they could extend that pair to a triple by finding the pair's common transversals. In this section, we explore a similar idea by searching through all Latin squares of order 11 with a "high level of symmetry" (this is defined explicitly below) and check if each of them can be extended to a complete set of mutually orthogonal Latin squares. Conjecture 2.11 would imply that there should only be one such set (the Desarguesian set). However, following the same idea as [60] by first finding all mates is computationally infeasible—even determining if a single square has at least one mate takes around one second, and finding all mates is a huge task. Thus, a different idea is needed to search for complete sets of mutually orthogonal Latin squares.

The key observation needed to make the search computationally feasible is to search for the locations of a specific symbol in the $n-1$ squares of the complete set rather than trying to build each of the individual squares one-by-one. Say we have a complete set of mutually orthogonal Latin squares. Without loss of generality, we assume that the top row of each Latin square in the set is in increasing order $(0, 1, 2, \dots)$. The location of each symbol is quite structured. Consider the location of the symbol 0 in the $n - 1$ squares. The $(0, 0)$ cell in each square contains a 0, which means that no other cell in the top row or leftmost column contains a 0. Furthermore, if some entry not in the top row or leftmost column contains a 0 in any of the squares, say $(r, c, 0)$, then no other square may have the entry $(r, c, 0)$ since the ordered pair $(0, 0)$ in the superimposition occurs in the top-left cell. Thus, since we have $n-1$ Latin squares, for any pair (r, c) not in the top row or leftmost column, exactly one of the squares contains the entry $(r, c, 0)$. We now colour the n^2 cells based on which of the $n - 1$ squares contains the symbol 0 in the corresponding cell (or uncoloured if no square contains a 0). For example, consider the complete set of mutually orthogonal Latin squares of order 5 in Figure 7.4.

Note that the $(n - 1) \times (n - 1)$ submatrix formed by deleting the interesting row and column is a Latin square (with colours as its symbols). This will always be the case based on our observations above. Due to the shape, we call this structure a *fan*. If we wish to emphasise the cell that we are basing the fan on, we call it an (r, c)-fan. Note that the symbols do not need to be the same in each of the $n - 1$ squares. For example, in the $(2, 3)$-fan in Figure 7.4, the symbol in each of the four squares is different. However, we are only concerned about the locations of the symbol that passes through the cell in question in each of the squares (the symbols of each individual square can be permuted independently without affecting orthogonality). The idea of a fan is such a simple substructure in a complete set of MOLS that it has been independently rediscovered several times. For

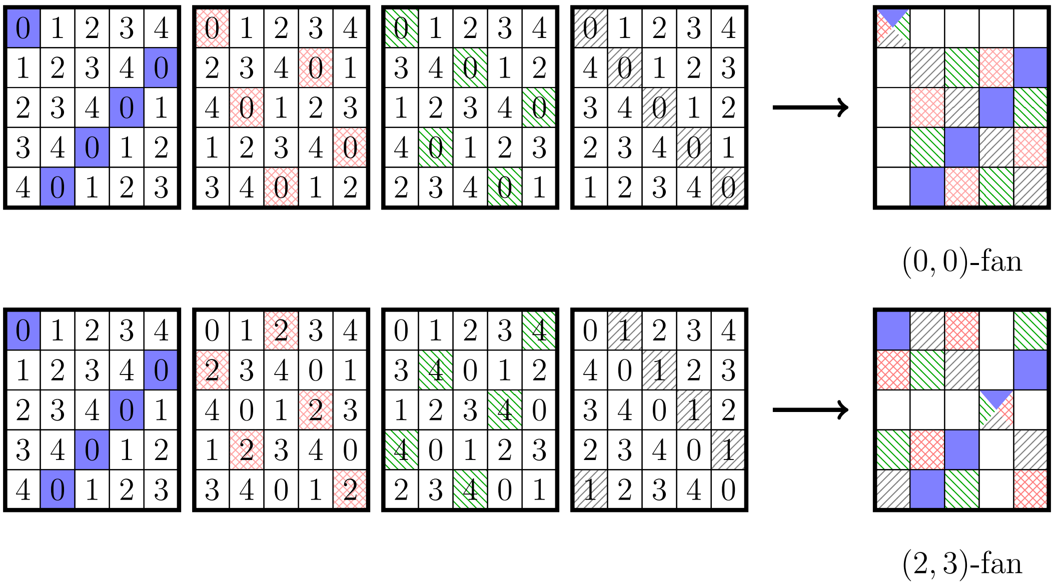


Figure 7.4: A complete set of mutually orthogonal Latin squares of order 5 and the corresponding $(0, 0)$-fan and $(2, 3)$-fan.

0	1	2	3	4	5	6	7	8	9	10
1	9	10	4	2	0	5	8	6	3	7
2	10	1	7	0	4	9	5	3	8	6
3	4	7	10	6	9	0	1	5	2	8
4	2	0	6	7	8	1	9	10	5	3
5	0	4	9	8	6	3	10	1	7	2
6	5	9	0	1	3	8	2	7	10	4
7	8	5	1	9	10	2	3	4	6	0
8	6	3	5	10	1	7	4	2	0	9
9	3	8	2	5	7	10	6	0	4	1
10	7	6	8	3	2	4	0	9	1	5

Figure 7.5: The non-cyclic atomic square of order 11 that possess a fan through every cell.

example, the Latin squares that are embedded into a projective plane's incidence matrix described by Lam, Kolesova and Thiel [55] are equivalent to our fans.

Utilising these fans computationally is quite straightforward. If we fix a square (say L), the colours in any given (r, c)-fan correspond to transversals of L going through (r, c) as well as the unique diagonal of weight 1 that passes through (r, c), so if we can find any cell that does not contain $n - 2$ transversals that can form an (r, c)-fan, then L cannot be part of a complete set of mutually orthogonal Latin squares.

We ran the above search on a library of Latin squares of order 11 whose autoparatopy group is of size 5 or higher (there are a total of 1 010 654 such squares). In almost all cases (approximately 95% of the squares searched), the first cell that was checked did not have an (r, c)-fan. Very few squares required more than two cells to be checked. The only two squares that have a fan through every entry are the cyclic square (which is to be expected) and an atomic Latin square which is equivalent to the one found by Owens and Preece [52] (given in Figure 7.5). An atomic Latin square is a Latin square so that the permutation between any two rows, two columns or two symbols is a single cycle (see [58]).

With these two remaining squares, we wish to determine if they can be part of a complete set, and if so, how many different sets they are a part of (we obviously know that the cyclic square is part of one complete set, possibly more). To accomplish this, consider a graph G where each vertex corresponds to a transversal of the Latin square. Put an edge between two vertices if the corresponding transversals intersect in either 0 or 1 place. In this graph, we search for cliques of size $n(n-2)$. Any such clique then corresponds to $n(n-1)$ diagonals (including the n diagonals from the original Latin square) that meet pairwise at either 0 or 1 location and we may use the following proposition to show that such a clique is sufficient.

Proposition 7.8. *Let D be a set of $n(n-1)$ diagonals in an $n\times n$ array that meet pairwise in at most one location. Then D can be decomposed into a complete set of MOLS.*

Proof. Each cell must have at most $n-1$ diagonals passing through it. If there were more, then the $(n-1)\times(n-1)$ subsquare not including this cell must have another intersection of some pair of diagonals (recall that they already intersect at the cell in question). Since each row has exactly $n(n-1)$ diagonals through it, each cell must have exactly $n-1$ diagonals through it, and each of the cells induces an (r, c)-fan.

Now consider a specific diagonal d_0. Each cell of d_0 intersects exactly $n-2$ other diagonals (and these diagonals must be distinct, otherwise d_0 and that diagonal meet at least two times). Thus, there are $n-1$ diagonals that do not intersect d_0. We will show that these $n-1$ diagonals (plus d_0) form a Latin square. Assume, on the contrary, that they do not form a Latin square. Then some cell is not covered by the diagonals. We know that there are $n-1$ diagonals through this cell and that they induce a Latin square of order $n-1$ (as described above). However, there are only $n-2$ cells from d_0 in this $(n-1)\times(n-1)$ subarray, so there must be some diagonal that does not intersect d_0, which we assumed was not the case. Thus, we have a contradiction and so the diagonals must form a Latin square. Each of the Latin squares constructed in this manner are orthogonal to one another since each diagonal from different squares meet pairwise in exactly one place. □

Witt [86] showed that a complete set of MOLS is equivalent to a set of sharply 2-transitive permutations. A simple counting argument shows that the permutations corresponding to the $n(n-1)$ diagonals form a sharply 2-transitive set, which would provide an alternative proof of Proposition 7.8.

The atomic square produces no such cliques of size $n(n-2)$ and the cyclic square produces exactly one such clique, which corresponds to the Desarguesian plane. We should note that Maenhaut and Wanless [58] performed a computation which showed that the atomic Latin square is not part of any set of 5 or more mutually orthogonal Latin squares, which is a stronger result than we found here. Thus, we have the following interesting result:

Theorem 7.9. *Let $P = \{L_1, L_2, \ldots, L_{10}\}$ be a set of MOLS of order 11. If any L_i has an autoparatopy group of order at least 5, then P is equivalent to the Desarguesian set (constructed in Theorem 2.9).*

Both steps of this search provide a dramatic improvement over the runtimes achieved by traditional methods of finding sets of MOLS. In the first step, we only needed to concern ourselves with transversals through a single cell. We are then effectively searching for a Latin square of order $n-1$ rather than order n. Due to the large combinatorial explosion that occurs around $n = 10$, this reduction in the search space dramatically reduces the runtime. In the second step where we are finding the cliques, the order in which the transversals are searched is extremely important. The *traditional* way to search for the set of MOLS is also done in two steps. First, you find n disjoint transversals, then search for the diagonals that are transversals of both the original square and the new orthogonal square. However, in our search here, we ensure that we select the $n-2$ transversals through the $(0, 0)$ cell first, then the $n-2$ transversals through the $(0, 1)$ cell next, and so on. This ordering seems to prune the search very quickly. The entire search took a few hours to complete, with the majority of the time taken by the cyclic square. The program was also tested on Latin squares of order 9 that are known to be a part of a complete set (some multiple complete sets) and each of these returned the appropriate response.

To give an idea of the difference between the two search styles, note that the cyclic square of order 11 contains 7 372 235 460 687 orthogonal mates [58], while it only contains 2 087 488 fans through each cell. Moreover, as mentioned before, most squares seem to not possess an (r, c)-fan through a randomly selected entry. However, of the squares that we searched, only 1533 of them did not have an orthogonal mate (these squares are called *bachelor squares*), which means that by using the traditional method, we would need the second step of the search much more often.

While performing this search, we also computed the maximum number of disjoint transversals that each Latin square has. For all but the 1533 bachelor squares, there are n disjoint transversals (since they have an orthogonal mate). Of the 1533 bachelor squares, there are 11, 10, 26, 147 and 1339 Latin squares with, respectively, 3, 6, 7, 8 and 9 disjoint transversals (but no more). To date, we know of no Latin square of order 11 with fewer than 3 disjoint transversals. Here is one such example of a Latin square of order 11 with three disjoint transversals highlighted.

10	2	7	4	5	6	3	1	9	8	0
7	1	0	3	9	5	10	8	2	6	4
1	8	4	10	2	9	6	0	7	3	5
6	3	9	1	0	7	8	5	4	10	2
3	10	2	8	6	0	4	9	5	1	7
4	5	10	7	8	1	0	3	6	2	9
5	9	6	0	4	8	2	10	3	7	1
2	0	8	5	10	3	9	7	1	4	6
9	7	1	6	3	4	5	2	10	0	8
8	4	5	2	7	10	1	6	0	9	3
0	6	3	9	1	2	7	4	8	5	10

Only two of the 1533 bachelor squares are confirmed bachelor squares (a Latin square where there exists a cell that has no transversals going through it). The two squares are given here with the highlighted cells indicating that no transversal goes through that cell.

10	2	3	4	1	5	7	6	9	8	0
4	3	0	1	5	2	8	7	6	10	9
1	5	9	0	8	6	10	3	7	2	4
2	1	7	6	0	9	3	4	10	5	8
3	0	10	7	9	1	5	8	2	4	6
5	4	1	9	6	0	2	10	8	3	7
7	8	5	3	10	4	9	0	1	6	2
6	7	8	2	3	10	0	1	4	9	5
9	6	4	10	7	8	1	2	5	0	3
8	10	2	5	4	3	6	9	0	7	1
0	9	6	8	2	7	4	5	3	1	10

0	2	4	3	1	6	5	8	7	10	9
4	1	0	5	8	10	3	2	9	6	7
1	7	9	10	0	8	6	4	5	2	3
3	6	10	0	5	1	4	9	8	7	2
2	0	7	6	4	3	9	10	1	8	5
6	3	8	4	10	7	0	5	2	9	1
5	9	6	1	3	0	2	7	10	4	8
8	10	1	9	2	5	7	6	0	3	4
7	4	5	8	9	2	10	0	3	1	6
10	8	2	7	6	9	1	3	4	5	0
9	5	3	2	7	4	8	1	6	0	10

The style of search described in this section may be able to rule out all Latin squares with an autoparatopy group of order at least 3, but even then, this search is a very small percentage of the total search space. There are more than 10^{24} species of Latin squares of order 11, and even just listing those squares is well outside of our reach.

7.4 Transversal-Free Latin Arrays

In Chapter 5, we found a bound on the number of symbols needed to guarantee a transversal in a Latin array.

We now shift our attention to small values of n where we can compute $\ell(n)$ exactly. Akbari and Alipour [3] determined $\ell(n)$ for $n \leqslant 4$. We extend this search to $n \leqslant 7$ and catalogue all Latin arrays of small orders with no transversals. For $n \geqslant 8$, computing $\ell(n)$ seems challenging. We will mention a couple of unsuccessful attempts to find examples that would provide some insight.

Following [30], we say that two Latin arrays are *trisotopic* if one can be changed into the other by permuting rows, permuting columns, permuting symbols and/or transposing. The set of all Latin arrays trisotopic to a given array is a *trisotopy class*. The number of transversals is a trisotopy class invariant, so to find all transversal-free Latin arrays of a given order it suffices to consider trisotopy class representatives. However, for orders $n > 5$ it becomes difficult to construct a representative of every trisotopy class. The following method allows us to push our results a couple of orders further.

Let L be a transversal-free Latin array. In the first two rows of L, select two entries that do not share a column or symbol (this can always be done for $n \geqslant 3$). Without loss of generality, we may assume that these two entries are $(1, 1, x)$ and $(2, 2, y)$. Let L' be the bottom-right $(n-2) \times (n-2)$ subarray of L where all occurrences of x and y are replaced with a hole (that is, a cell with no symbol; we forbid holes from being chosen in a transversal or partial transversal). There cannot be a partial transversal of length $n-2$ in L', otherwise the corresponding entries in L, together with $(1, 1, x)$ and $(2, 2, y)$, would form a transversal of L.

Thus, to search for transversal-free Latin arrays of order n, we first build a catalogue $\mathcal{C}_{n-2}$ of trisotopy class representatives of transversal-free partial Latin arrays of order $n-2$ with at most two holes in each row and each column. Starting with this catalogue, we can reverse the argument above. At least one representative of each trisotopy class of transversal-free Latin array of order n can be obtained by taking an element of $\mathcal{C}_{n-2}$, filling its holes with x and y, then extending it to a Latin array of order n.

By the above technique, we are able to give a complete catalogue of the transversal-free trisotopy classes for orders $n \leqslant 7$. Table 7.2 gives the value of $\ell(n)$ and the number of trisotopy classes with a specific number of symbols.

Representatives of the trisotopy classes of transversal-free Latin arrays of orders 4 and 5 are:

0	1	2	3
1	2	3	0
2	3	0	1
3	0	1	2

0	1	2	3
1	2	0	4
2	0	3	1
4	3	1	0

0	1	2	3	4
1	2	0	4	5
2	0	1	5	3
4	3	5	2	0
3	5	4	0	1

5	1	2	3	4
1	2	0	4	5
2	0	1	5	3
4	3	5	2	0
3	5	4	0	1

Note that our two representatives of order 5 differ only in their first entry. Both can be completed to Latin squares of order 6; in the first case this Latin square has no transversals, but in the second case it has eight transversals.

		Trisotopy Classes			
n	$\ell(n)$	n symbols	$n+1$ symbols	$n+2$ symbols	Total
2	3	1	-	-	1
3	3	-	-	-	0
4	6	1	1	-	2
5	7	-	2	-	2
6	9	8	19	1	28
7	7	-	-	-	0

Table 7.2: Values of $\ell(n)$ and the number of trisotopy classes of transversal-free Latin arrays.

Many of the transversal-free Latin arrays for order 6 also turn out to be quite similar to one another. There are exactly 28 trisotopy classes for $n = 6$. Previously, nine of these classes were known: eight Latin squares and the array constructed by Akbari and Alipour [3] by removing two rows and columns from the elementary abelian Cayley table of order 8. We now describe the 19 transversal-free trisotopy classes of order 6 with seven symbols. We denote their representative arrays by $L_1, L_2, \ldots, L_{19}$. Let

$$L_1 = \begin{array}{|c|c|c|c|c|c|}\hline 0&1&2&3&4&5\\\hline 1&2&0&5&3&4\\\hline 2&0&1&4&5&3\\\hline 3&4&5&6&1&2\\\hline 5&3&4&1&6&0\\\hline 4&5&3&2&0&6\\\hline\end{array} \quad \text{and } L' = \begin{array}{|c|c|c|c|c|c|}\hline 0&1&2&3&4&5\\\hline 2&5&1&4&3&0\\\hline 1&2&4&5&0&3\\\hline 3&4&5&0&1&2\\\hline 4&3&0&2&5&1\\\hline 5&0&3&1&2&4\\\hline\end{array}.$$

From L', we define $L_2, \ldots, L_8$ by changing some entries on the main diagonal to a new symbol, 6, in the following way. Let

$$R' \in \big\{\{0,1,2,3,4,5\}, \{0,1,3,4,5\}, \{0,2,3,4\}, \{0,2,5\}, \{0,3\}, \{1,2,4,5\}, \{2,3,4,5\}\big\}.$$

For all $r \in R'$, change the symbol on the main diagonal in row r of L' to 6. It turns out that changing the shaded entries in L_1 to 6 results in an array that is trisotopic to L_2. Next, L_9 is obtained by changing the symbol of the shaded entries in L' to a new symbol, 6. Let

$$L_{10} = \begin{array}{|c|c|c|c|c|c|}\hline 0&1&2&3&4&5\\\hline 1&2&6&0&5&4\\\hline 2&5&3&6&0&1\\\hline 3&0&5&4&6&2\\\hline 4&6&0&5&2&3\\\hline 6&4&1&2&3&0\\\hline\end{array} \quad \text{and } L'' = \begin{array}{|c|c|c|c|c|c|}\hline 0&1&2&3&4&5\\\hline 1&2&0&4&5&3\\\hline 2&0&1&5&3&4\\\hline 3&4&5&0&2&1\\\hline 4&5&3&2&1&0\\\hline 5&3&4&1&0&2\\\hline\end{array}.$$

From L_{10}, we can either change the symbol in the $(2,2)$ cell to 4, giving L_{11}, or change the symbol in the $(3,3)$ cell to 1, giving L_{12}.

The array L_{13} is obtained by changing the 3 in rows 2 and 3 of L'' to 6, as well as changing the 5 in row 2 to 3. Next, L_{14} is obtained by changing the 4 in row 3 of L_{13} to 3.

From the Latin square L'', any subset of entries that contain 3 may be changed to a new symbol, 6. This gives rise to 5 trisotopy classes. In particular, we define $L_{15},\dots,L_{19}$ by changing some occurrences of 3 to a new symbol, 6, in the following way. Let

$$R'' \in \big\{\{0\},\{0,1\},\{0,1,2\},\{0,2,4\},\{0,3\}\big\}.$$

For all $r \in R''$, change the 3 in row r of L'' to 6.

One can check that $L_{15},\dots,L_{19}$ are transversal-free by exhaustive computation, but next we give a reason why they have no transversals. The argument is in the style of the highly successful Δ-lemma (see Lemma 2.15). Let L be any Latin array obtained by replacing any subset of the occurrences of 3 in L'' by 6. Define functions ρ,ν to $\mathbb{Z}_3$ by:

$$\begin{aligned}
&\rho(1)=\rho(4)=0,\ \rho(2)=\rho(5)=1,\ \rho(3)=\rho(6)=2,\\
&\nu(0)=\nu(3)=\nu(6)=0,\ \nu(1)=\nu(4)=1,\ \nu(2)=\nu(5)=2.
\end{aligned}$$

Define a function Δ from the entries of L to $\mathbb{Z}_3$ by $\Delta(r,c,s)=\rho(r)+\rho(c)-\nu(s)$. Let D denote the bottom-right 3×3 subsquare of L. Suppose that T is a transversal of L and that $\overline{s}$ is the only symbol in $\{0,1,\dots,6\}$ that does not appear in T. Then

$$\sum_{(r,c,s)\in T}\Delta(r,c,s) = 2\sum_{i=1}^{6}\rho(i) - \sum_{(r,c,s)\in T}\nu(s) = \nu(\overline{s}). \tag{7.4}$$

Also, if T includes x entries in D then overall it has $2x$ entries with symbols in $\{0,1,2\}$, which means that $x=1$ and $\overline{s}\in\{0,1,2\}$. However, $\Delta(r,c,s)=0$ for all entries of L, except those in D, where $\Delta(r,c,s)=\nu(s)$. Hence to satisfy (7.4), the symbol in the only entry of T in D has to be $\overline{s}$, contradicting the fact that this symbol does not appear in T.

The argument we have just presented is specific to order $n=6$ and does not seem to easily generalise to arrays of larger orders.

When performing the search for transversal-free Latin arrays of order 7, we found 15 611 437 trisotopy classes of transversal-free partial Latin arrays of order 5 and at most two holes in each row and column. Table 7.3 provides counts of the trisotopy classes based on number of holes and number of symbols. Since none of these arrays can extend to a Latin array of order 7 with no transversals, we have the following result.

Theorem 7.10. *Every Latin array of order 7 has a transversal.*

The approach that we used to prove Theorem 7.10 is infeasible for $n\geqslant 8$, although we did examine certain interesting sets of Latin arrays of order 8. There are 68 different transversal-free Latin squares of order 8, up to trisotopy. We also considered all Latin arrays which are obtained by removing one row and one column from a Latin square of order 9. We could immediately eliminate any square of order 9 that contains a transversal through every entry. Latin squares that do not contain a transversal through every entry are called *confirmed bachelor squares.* The confirmed bachelor squares of order 9 were generated for [29], providing us with a set of trisotopy class representatives. None of these squares has an order 8 transversal-free subarray. Lastly, we searched all Latin arrays of order 8 with exactly 9 symbols where one of the symbols appears at most 4 times. None of these were transversal-free. The arrays that we have checked are a tiny subset of all Latin arrays of order 8. Without theoretical insight, it seems hopeless to check them all. So all that we can conclude at this stage is that $\ell(8)\geqslant 9$.

It is known that all Latin squares of order 9 have transversals (see, *e.g.,* [29]). We tried, unsuccessfully, to build a transversal-free Latin array of order 9. We did this by

	Number of Symbols										
Number of Holes	3	4	5	6	7	8	9	10	11	12	13
0	-	-	-	2	-	-	-	-	-	-	-
1	-	-	1	17	-	-	-	-	-	-	-
2	-	-	9	271	13	-	-	-	-	-	-
3	-	-	137	4893	1179	61	5	-	-	-	-
4	-	-	1484	54911	31342	5539	1906	462	62	4	-
5	-	3	10686	341251	319750	58257	9823	1175	86	4	-
6	-	19	48436	1155690	1420192	299951	33366	1953	56	-	-
7	-	151	124275	2045859	2754143	670137	63480	2676	30	-	-
8	-	632	159295	1720463	2198260	549316	43912	1710	78	8	1
9	-	916	80609	557285	603320	134056	7120	148	7	1	-
10	3	320	9420	40418	34218	6014	159	1	-	-	-

Table 7.3: The number of trisotopy classes of transversal-free 5×5 partial Latin arrays, which are categorised by number of symbols and number of holes.

removing a row and column from Latin squares of order 10. The squares that we used were representatives of all trisotopy classes for which the autoparatopy group has order 3 or higher, as generated for [60].

Chapter 8

Future Work

Take chances,
make mistakes,
get messy!
– V. F. Frizzle

There are many tantalising problems still open in the study of transversals. The two most famous conjectures, as mentioned above, are Ryser's Conjecture (that every odd-ordered Latin square contains a transversal) and Brualdi's Conjecture (that every Latin square contains a near transversal). Of these two conjectures, Brualdi's Conjecture seems to be more sturdy, as it does not seem to rely on the number of symbols in the square (Section 7.2). However, by allowing extra symbols in the square, the natural generalisation of Ryser's Conjecture is false for $n = 5$ (Section 7.4), so some property of the number of symbols is needed to prove Ryser's Conjecture. We should note that it is possible that $n = 5$ is the sole case where this extended Ryser's Conjecture is false, as we know it is true for $n = 1, 3, 7$. The truth of Ryser's Conjecture is an interesting question because there are not many tangible differences between even-ordered and odd-ordered Latin squares. What about even-ordered squares allows us to not have any transversals? One of the few differences which may lead to a proof is Lemma 2.15.

As of today, the best bound on the length of a partial transversal in a Latin square is $n - O(\log^2 n)$. This bound *feels* too weak to be the best possible bound and we would be surprised if this were the correct asymptotic bound. A bound of $n - O(\log n)$ would be much more believable if Brualdi's Conjecture is indeed false.

The bound of $\ell(n) \leqslant (2 - \sqrt{2})n^2$ shown in Theorem 5.14 leaves a lot to be desired. However, this problem has proven to be stubbornly difficult to gain any amount of traction on. The fact that we cannot show $\ell(n) \leqslant \frac{1}{2}n^2$ is quite distressing as each symbol only appears, on average, twice! We imagine that a substantial leap in this problem can be made based on some small observation; making it both an interesting problem to study, while also being quite an infuriating one. It would be very interesting to know if $\ell(n) > n$ for infinitely many odd n.

In Chapter 4, the type of a transversal was introduced based on the parity of some underlying permutations. Conjecture 4.13 seems true based on small computation and we would be surprised if a counterexample were found. Conjectures 4.32 to 4.34 seem like the next step to take in regards to E_i and R_i and it would be interesting to know if there are deeper patterns modulo 2^k for $k > 2$.

One interesting topic which was not covered in this thesis is the minimum number of transversals across all Latin square of order n. When n is even, the answer to this question is obviously 0, however, it would be interesting to know the smallest non-zero number of transversals that can exist. In particular, is there some Latin square that has exactly one transversal? Other than the $n = 1$ Latin square, no known Latin square contains a single transversal. In a similar vein, it would be interesting to find more odd-ordered Latin squares that do not contain large sets of disjoint transversals. When $n \equiv 3 \pmod 6$, Wanless and Zhang [84] found an infinite family of Latin squares of order n with at most $n/3 + O(1)$ disjoint transversals. However, when $n \not\equiv 3 \pmod 6$, this question is still wide open. In orders 5, 7 and 9, there are examples of Latin squares that possess a transversal that intersects every other transversal of that Latin square [29]. It would be interesting to know if such a square exists for every odd $n \geqslant 5$.

And finally, one interesting application of the fans introduced in Section 7.3 is the ability to provide an alternative proof for the non-existence of projective planes of order 10. The non-existence proof shown by Lam *et al.* [56] relies on studying the codes that can be generated from a projective plane by systematically determining the number of 1s in those underlying codes and then exhaustively searching through all possible codes with that specific number of 1s. However, here we provide a more straightforward approach. Any complete set of MOLS of order 10 must contain a $(0, 0)$-fan. This fan will induce a Latin square of order 9. Consider one such fan. To this fan, we can find all possible $(0, k)$-fans that are *suitable* with our original fan. This can be accomplished by finding *pseudo-transversals* that intersect each diagonal in the $(0, 0)$-fan at most one time. Latin squares of order 9 can be exhaustively generated [60], so if a similar search style is used as in Section 7.3, we believe that exhaustively searching all isotopy classes of Latin squares of order 9 will be possible, and might provide an alternative proof of the non-existence of projective planes of order 10. A similar idea of looking through all Latin squares of order $n - 1$ was utilised by Lam *et al.* [55] when they verified that there were exactly four projective planes of order 9. Their use of the Latin squares of order $n - 1$ turns out to be equivalent to our notion of a fan here, though the search there is conducted in a different way.

We are just starting to scratch the surface on the study of these fascinating objects!

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Appendix A

Useful Code Snippets

We include a few C++ code snippets here. The code is commented, however, some code has been compressed and had comments removed when those pieces of code are the same as has appeared in previous code snippets. Note that for Algorithms 5 and 6, you will need to implement a popcount function. Otherwise, with the inclusion of the appropriate libraries and main(), each piece of code should compile.

Note that we will assume that $n \leqslant 31$ and that the number of transversals is less than 2^{31} so that we may use the `int` data type for simplicity. This can easily be modified for larger values. For simplicity, we utilise some global variables. This improves readability and (on some hardware/compilers) efficiency of the code. We have removed some optimisations from the code in order to maintain readability—these items appear in our normal code but would be too tedious to describe each one.

Algorithm 3 A simple recursive backtracking algorithm to count transversals in a Latin square.

```
// Global Variables:
const int n; // The order of the Latin square
int L[n][n]; // The Latin square in question

// f counts the number of extensions of a partial transversal there are
// -           row : The current row we are working on (all rows above me have been used)
// -        col_BM : A bitmask of which columns have been used so far
// -        sym_BM : A bitmask of which symbols have been used so far
int f(int row, int col_BM, int sym_BM){
  if(row == n) // We found a transversal!
    return 1;

  int num_transversals = 0; // The number of extensions to the current transversal

  for(int col=0; col<n; col++){
    int sym = L[row][col];  // Can we extend our partial transversal with (row,col,sym)?

    if((col_BM >> col) & 1)
      continue;             // We have already used this column

    if((sym_BM >> sym) & 1)
      continue;             // We have already used this symbol

    // If we are here, then we can extend our partial transversal--add (row,col,sym) and
        recurse
    num_transversals += f(row+1, col_BM | (1 << col), sym_BM | (1 << sym));
  }

  return num_transversals;
}

int count_transversals(){
  return f(0, 0, 0); // Count all extensions of an empty partial transversal
}
```

Algorithm 4 A slight modification to Algorithm 3: using a linked-list to determine which columns have been used.

```
// Global Variables:
const int n; // The order of the Latin square
int L[n][n]; // The Latin square in question

// f counts the number of extensions of a partial transversal there are
// -         row : The current row we are working on (all rows above me have been used)
// -      sym_BM : A bitmask of which symbols have been used so far
// -        next : A circularly linked list of columns that have not been used
int f(int row, int sym_BM, int next[n+1]){
  if(row == n) // We found a transversal!
    return 1;

  int num_transversals = 0; // The number of extensions to the current transversal

  int prev = n;              // The index such that next[prev] = col
  for(int col=next[n]; col<n; prev=col, col=next[col]){
    int sym = L[row][col];  // Can we extend our transversal with (row,col,sym)?

    if((sym_BM >> sym) & 1)
      continue;              // We have already used this symbol

    // Take "col" out of our linked list
    next[prev] = next[col];

    // We can extend our partial transversal--add (row,col,sym) and recurse
    num_transversals += f(row+1, sym_BM | (1 << sym), next);

    // Put "col" back into our linked list
    next[prev] = col;
  }

  return num_transversals;
}

int count_transversals(){
  // next[i] points to the next unused column (next[n] is the first unused column)
  int next[n+1];
  for(int i=0; i<n+1; i++)
    next[i] = (i+1) % (n+1);
  return f(0, 0, next);
}
```

Algorithm 5 Counts transversals by searching for (half-)partial transversals then sewing them together.

```
// Global Variables:
const int n; // The order of the Latin square
int L[n][n]; // The Latin square in question

// h counts the number of extensions of a (half-)partial transversal there are
// -         row : The current row we are working on
// -      sym_BM : A bitmask of which symbols have been used so far
// -        next : A circularly linked list of columns that may still be used
// -  counter[i] : The number of (half-)partial transversals in the top half of
//                 the square that use the symbol set 'i'
int h(int row, int sym_BM, int next[n+1], int counter[1 << n]){
  if(next[n] == n){      // This means there are no more columns to use, so we are at the
      end
    if(row != n){
      counter[sym_BM]++; // We are in the top-half. We found a (half-)partial transversal.
      return 0;
    } else {
      // We are in the bottom-half, this can be extended by every (half-)partial
          transversal
      //   whose set of symbols is disjoint from ours
      int missing_syms = ((1 << n)-1) ^ sym_BM;
      return counter[missing_syms];
    }
  }

  int num_transversals = 0;
  int prev = n;
  for(int col=next[n]; col<n; prev=col, col=next[col]){
    int sym = L[row][col];
    if((sym_BM >> sym) & 1) continue;
    next[prev] = next[col];
    num_transversals += h(row+1, sym_BM | (1 << sym), next, counter);
    next[prev] = col;
  }

  return num_transversals;
}

// Makes a circularly linked-list 'next' containing the set bits in BM
void fill_next(int BM, int next[n+1]){
  int prev = n;
  for(int i=0; i<n; i++)
    if((BM >> i) & 1){
      next[prev] = i;
      prev = i;
    }
  next[prev] = n;
}

int count_transversals(){
  int counter[1 << n];

  int num_transversals = 0;
  for(int BM=0; BM<(1 << n); BM++){
    if(popcount(BM) != n/2)
      continue;                        // This bitmask is not the right size

    fill(counter,counter+(1 << n), 0); // Reset the entire counter array to 0

    // Top-half
    fill_next( BM, next);
    h(0, 0, next, counter);

    // Bottom-half
    fill_next(~BM, next);
    num_transversals += h(n/2, 0, next, counter);
  }

  return num_transversals;
}
```

Algorithm 6 A slight modification to Algorithm 5: do not reset 'counter' at each step, but rather keep a separate 'visited' array.

```
// Global Variables:
const int n; // The order of the Latin square
int L[n][n]; // The Latin square in question

// h counts the number of extensions of a (half-)partial transversal there are
// -         row : The current row we are working on
// -      sym_BM : A bitmask of which symbols have been used so far
// -        next : A circularly linked list of columns that may still be used
// -  counter[i] : The number of (half-)partial transversals in the top half of
//                 the square that use the symbol set 'i'
// -  visited[i] : When was the last time that we found a (half-)partial transversal with
//                 the symbol set 'i'?
// -        step : The current step of the search, counter[i] is only relevant if
//                 visited[i] == step
int h(int row, int sym_BM, int next[n+1], int counter[1 << n], int visited[1 << n], int
    step){
  if(next[n] == n){      // This means there are no more columns to use, so we are at the
      end
    if(row != n){
      if(visited[sym_BM] != step){ // We must reset counter
        counter[sym_BM] = 0;
        visited[sym_BM] = step;
      }
      counter[sym_BM]++;
      return 0;
    } else {
      int missing_syms = ((1 << n)-1) ^ sym_BM;
      return (visited[missing_syms] == step ? counter[missing_syms] : 0);
    }
  }

  int num_transversals = 0;
  for(int prev=n, col=next[n]; col<n; prev=col, col=next[col]){
    int sym = L[row][col];
    if((sym_BM >> sym) & 1) continue;
    next[prev] = next[col];
    num_transversals += h(row+1, sym_BM | (1 << sym), next, counter, visited, step);
    next[prev] = col;
  }

  return num_transversals;
}

void fill_next(int BM, int next[n+1]){
  int prev = n;
  for(int i=0; i<n; i++)
    if((BM >> i) & 1){
      next[prev] = i;
      prev = i;
    }
  next[prev] = n;
}

int count_transversals(){
  int counter[1 << n], visited[1 << n];
  fill(visited, visited+n, -1);

  int num_transversals = 0;
  for(int BM=0; BM<(1 << n); BM++){
    if(popcount(BM) != n/2) continue;

    // Top-half
    fill_next( BM, next);
    h(0, 0, next, counter, visited, BM);

    // Bottom-half
    fill_next(~BM, next);
    num_transversals += h(n/2, 0, next, counter, visited, BM);
  }

  return num_transversals;
}
```