Cosmological Newtonian Limits on Large Scales

A thesis submitted for the degree of

Doctor of Philosophy

by

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Abstract

This thesis is about the rigid cosmological Newtonian limits on large time and space scales and answers the following fundamental question:

On what space and time scales can Newtonian cosmological simulations be trusted to approximate relativistic cosmologies?

We will resolve this question under small initial data and the positive cosmological constant condition. Specifically, we focus on solutions which are initially a small perturbation of the Friedmann-Lemaître-Robertson-Walker (FLRW) solution. Here, the FLRW solution is an exact solution to the Einstein field equation that represents a homogeneous, isotropic, fluid-filled universe undergoing accelerated expansion once the cosmological constant is positive.

Our work is planned in the following two steps.

In the first step (see Chapter 2), we only focus on the long time scales by restricting our attentions on a manifold $(0,1]\times\mathbb{T}^3$ (cosmological versions of isolated systems) covered by Newtonian coordinates and conformal compactified temporal coordinates. This is a simplification of our final task which will be investigated in the next step. We establish the existence of 1-parameter families of ϵ-dependent solutions to the Einstein-Euler equations with a positive cosmological constant $\Lambda > 0$ and a linear equation of state $p = \epsilon^2 K\rho$, $0 < K \leq 1/3$, for the parameter values $0 < \epsilon < \epsilon_0$. These solutions exist globally to the future, converge as $\epsilon \searrow 0$ to solutions of the cosmological Poisson-Euler equations of Newtonian gravity, and are inhomogeneous non-linear perturbations of FLRW fluid solutions. The basic idea to achieve this is to rephrase the Einstein-Euler system with $\Lambda > 0$ to a quasilinear symmetric hyperbolic system with jointly singular terms in ϵ and time t by judiciously choosing the conformal factor of a conformal transform and the source term in the wave gauge. Our main aim is to analyze such a singular system, which leads to the long time existence of the solutions to the Einstein-Euler equations, Poisson-Euler equations, and the expected Newtonian error estimates. Initialization must be carefully treated in general relativity due to the constraint equations on the initial hypersurface. Moreover, initial data must be regular in ϵ in Newtonian limits problem to prevent it from blowing up in the Newtonian coordinates. A standard method developed by Lottermoser will be adopted in this chapter to set up the initial data by solving the constraint equations.

In the second step (see Chapter 3), we generalize the above methods and results for long time scales to those for large cosmological spacetime scales by focusing on a manifold $(0,1]\times\mathbb{R}^3$ covered by Newtonian coordinates and compactified temporal coordinates. We establish the similar conclusions as above: the existence of 1-parameter families of ϵ-dependent solutions to the Einstein-Euler equations with a positive cosmological constant $\Lambda > 0$ and a linear equation of state $p = \epsilon^2 K\rho$, $0 < K \leq 1/3$, for the parameter values $0 < \epsilon < \epsilon_0$. These solutions exist globally on the manifold $M = (0,1]\times\mathbb{R}^3$, are future geodesically complete for every $\epsilon > 0$, and converge as $\epsilon \searrow 0$ to solutions of the cosmological Poison-Euler equations of Newtonian gravity. Furthermore, they can be interpreted a representing inhomogeneous nonlinear perturbations of a FLRW fluid solution where the inhomogeneities are driven by localized matter fluctuations that evolve to good approximation according to Newtonian gravity. The key idea and starting point follows the techniques of the above long time results to conformally transform the Einstein-Euler system to the singular symmetric hyperbolic one. However, this system on $\mathbb{R}^3$ becomes more complicated than the previous one on $\mathbb{T}^3$, which involves certain complex function spaces and more delicate analysis. The initialization of this situation becomes more complicated and, in order to solve constraint equations, we introduce Yukawa potential operators which provide better mapping properties than Riesz potential operators. With the help of such operators and Banach's fixed point theorem, we conclude the existence of suitable initial data in certain function spaces.

In summary, we answer the above fundamental question on large spacetime scales and establish, under suitable assumptions, the existence of realistic inhomogeneous cosmological solutions that **(i)** admit a foliation by spacelike (i.e. constant time) hypersurfaces diffeomorphic to $\mathbb{R}^3$, **(ii)** exist globally to the future, **(iii)** can be approximated to arbitrary precision uniformly to the future by a Newtonian solution, and **(iv)** represent non-linear perturbations of a Friedmann-Lemaître-Robertson-Walker fluid solution.

Declaration

In accordance with Monash University Doctorate Regulation 17.2 Doctor of Philosophy and Research Masters regulations the following declarations are made:

I hereby declare that this thesis contains no material which has been accepted for the award of any other degree or diploma at any university or equivalent institution and that, to the best of my knowledge and belief, this thesis contains no material previously published or written by another person, except where due reference is made in the text of the thesis.

This thesis includes 2 submitted publications. The core theme of the thesis is cosmological Newtonian limits on large spacetime scales. The ideas, development and writing up of all the papers in the thesis were the principal responsibility of myself, the student, working within the School of Mathematical Sciences under the supervision of Todd A. Oliynyk.

(The inclusion of co-authors reflects the fact that the work came from active collaboration between researchers and acknowledges input into team-based research.)

In the case of Chapter 2–3 my contribution to the work involved the following:

Thesis Chapter	Publication Title	Status	Nature and % of student contribution	Co-author name(s) Nature and % of Co-authors contribution	Co-author(s), Monash student Y/N
2	Newtonian limits of isolated cosmological systems on long time scales	Accepted	Conducting this research under supervisor's guidance and writing the draft (60%)	Todd A. Oliynyk (PhD supervisor). Inspiring discussions and guidance, significant revising the draft (40%)	N
3	Cosmological Newtonian limits on large spacetime scales	Submitted	Conducting this research under supervisor's guidance and writing the draft (60%)	Todd A. Oliynyk (PhD supervisor). Inspiring discussions and guidance, significant revising the draft (40%)	N

I have not renumbered sections of submitted or published papers in order to generate a consistent presentation within the thesis.

Student signature: [redacted] Date: 28/11/2017

The undersigned hereby certify that the above declaration correctly reflects the nature and extent of the students and co-authors contributions to this work. In instances where I am not the responsible author I have consulted with the responsible author to agree on the respective contributions of the authors.

Main Supervisor signature: [redacted] Date: 5/12/2017

List of publications

1. Chao Liu and Todd A. Oliynyk, Newtonian limits of isolated cosmological systems on long time scales. Accepted by Annales Henri Poincaré;
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Chapter 1
Introduction

Chapter 1

Introduction

Shall I refuse my dinner because I do not fully understand the process of digestion?

Oliver Heaviside

1.1 Background

Until the beginning of the last century, people believed that the universe is flat and described by Euclidean geometry. They were confident in the fact space and time are distinct and nobody could even imagine they are conceptually related. As a consequence, by modeling gravity as a type of force, the Newtonian gravity theory worked very well in describing the universe during that time. Nevertheless, things started to change when Maxwell's electromagnetism theory achieved great success and experiments evidenced that the speed of light is a constant. In 1905, Albert Einstein proposed his famous theory of special relativity which entirely subverted the understanding of space and time. However, there is no room for gravity in such a theory. After ten years of struggle, Einstein finally unveiled the most beautiful gravitational theory up to now which is known as general relativity. In this theory, the core idea is that gravity is not simply a force or a certain phenomenon in spacetime, rather, gravity is the spacetime, warped by matters.

At the level of equations, general relativity is governed by the Einstein field equation which is an equation bridging spacetime geometry and matters. Specifically, the Einstein field equation builds a relationship between the curvature of spacetime and the stress-energy tensor (expressing the mass-energy distribution) of the matter (see (1.3.1) for the exact form). Unlike general relativity, Newtonian gravity presents a much simpler formulation, which encodes the gravitational field by a scalar function known as Newtonian potential and claims this Newtonian potential is determined by a mass distribution (see, for instance, (2.1.51) and (3.1.61)).

Although general relativity has achieved great success both aesthetically and experimentally, Newtonian gravity is still an efficient gravitational theory which also gives excellent predictions under a wide range of conditions (such as slow-moving) due to centuries' worth of experiments. As many authors have pointed out (for examples, see [81, 85]) this implies that the predictions of general relativity should reduce to those of Newtonian gravity under the situations when Newtonian gravity is well-evidenced, for example, when the matter is slow-moving. For convenience, we introduce a dimensionless parameter, ϵ, which expresses the "slowness" of the system and is defined by

$$\epsilon = \frac{v_T}{c},$$

where c is the speed of light, and v_T is a characteristic speed associated with the matter. Using this ϵ, the above statement can be expressed as the predictions of general relativity should reduce to those of Newtonian gravity as $\epsilon \searrow 0$. This seems to be obvious and clear but most references do not give an accurate reason why this is true. In fact, most references formally answered this only at the level of the field equations. That is, the Einstein field equation recovers the Poisson equation under the limit $\epsilon \searrow 0$ or, generally, when the gravitational field equation couples with some matter field, the matter field equation also converges to those in Newtonian gravity as $\epsilon \searrow 0$. It seems these have already been proven, but it doesn't answer the physical phenomenon of general relativity converging to those of Newtonian gravity as $\epsilon \searrow 0$ because the predictions (i.e. physical phenomenon) of general relativity and Newtonian gravity are not described directly by their equations but by their solutions. There is no obvious reason to believe if equations have such similarity, then solutions have such similarity too. Therefore, it is vital to examine the behavior of solutions as $\epsilon \searrow 0$ and this is much more complicated than simply letting $\epsilon \searrow 0$ in the field equation. In this thesis, we attempt to understand the behavior of solutions to Einstein-matter equations in the limit $\epsilon \searrow 0$ and see if such behavior of these solutions tends to the solutions in Newtonian gravity. We refer to this aim as the *(rigorous) Newtonian limit*.

At the level of the field equation, the relationship between Newtonian gravity and general relativity such as Newtonian limits or higher order post-Newtonian expansions have been broadly investigated by many authors (for example, see [8, 9, 16, 19, 21, 22, 30, 46, 47, 48] and references therein) and most works are in the setting of isolated bodies and involve formal calculations. The basic idea of such formal strategies of approximation methods are expanding the metric to certain orders of ϵ assuming this expansion is existent and convergent, then using these expansions to expand the field equation correspondingly, partially solving certain finite order field equations to derive certain order coefficients of the expanding series of the formal solution. The hypothesis of the existence and convergence of this expansion, underlying this method, is very strong. There is no obvious reason to believe it is always true. Our question is to answer whether this assumption is true in certain circumstances. In other words, our goal is to answer Newtonian limits in the level of the solution, by solving the fully nonlinear Einstein field equation and the corresponding equations in Newtonian gravity rigorously and prove that the error between their solutions are dominated by order of ϵ in some situations.

This question was avoided for a long time until Alan Rendall [70] tried to answer it in the case of asymptotically flat solutions of the Vlasov-Einstein system. The main difficulty in understanding this question is, in order to prove this rigidly, one needs a better understanding of the existence of the Cauchy problem for the Einstein equations which was not an easy task for a long time. It was not until a decade ago that Todd A. Oliynyk started his systematic work on this topic and established a firm foundation of rigorous Newtonian limits from the solution point of view (please refer to [59]–[65]). The key analysis tool for rigorous Newtonian limits is from the techniques of singular limits of symmetric hyperbolic equations involving a singular term with respect to ϵ. By rewriting the Einstein-matter equation in the Newtonian coordinate, this system becomes a form that

$$A^0\partial_0 U + A^i\partial_i U + \frac{1}{\epsilon}C^i\partial_i U = H. \tag{1.1.1}$$

Such a system has been investigated by, for example, G. Browning and H.O. Kreiss [12], S. Klainerman and A. Majda [41] and S. Schochet [76, 75]. The rigorous Newtonian limits theory are rooted in such analysis.

Historically, research was focused on the Newtonian limits and post-Newtonian expansions of isolated bodies which are important objects in astrophysics. Physicists use these important techniques to calculate physical quantities for the purpose of comparing general relativity with experiments. Most works, for example, [8, 9, 16, 19, 21, 22, 30, 46, 47, 48] and references therein, only involve formal calculations without answering the underlying questions about the convergence of Newtonian limits or post-Newtonian expansions, with only a few exceptions [70, 63, 60, 59] where rigorous results were established.

Recently, the attention about the Newtonian limits and the post-Newtonian expansions has shifted to cosmological settings from the isolated one, because of questions surrounding the physical interpretation of large scale cosmological simulations using Newtonian gravity and the role of Newtonian gravity in cosmological averaging. From a cosmological perspective, the most important family of solutions to the Einstein-Euler system are the Friedmann-Lemaître-Robertson-Walker (FLRW) solutions that represent a homogeneous, fluid filled universe. Thus, for the Newtonian limit problem in the cosmological setting, we must first verify that the existence of a perturbation of FLRW solutions which are 1-parameter families of ϵ-dependent solutions to the Einstein-Euler system and then try to show that such ϵ-dependent solutions converge to solutions of the cosmological Poison-Euler equations of Newtonian gravity, as $\epsilon \searrow 0$. After generalizing Newtonian gravity to the cosmological setting [74], Newtonian theory can describe gravity on all scales except in regions near compact neutron stars or black holes [15, 39]. Here too, the majority of results of Newtonian limits and post-Newtonian expansions on cosmological scales are based on formal calculations [13, 14, 15, 17, 23, 33, 34, 38, 43, 44, 55, 56, 57, 68, 87] with the articles [61, 62, 64, 65, 51, 50] being the only exceptions where rigorous results have been obtained. We remark that in [61], the author gave a somewhat surprising result that there are cosmological post-Newtonian expansions to any specified order in certain circumstances.

When considering the Newtonian limits on cosmological scales, there are some outstanding differences compared to the isolated situations, as pointed out in [65]. In order for solutions of the cosmological Poisson-Euler system to be cosmologically relevant, the initial data must be chosen wisely. The key requirement for the initial data is that the inhomogeneous component of the fluid density should be composed of localized fluctuations that represent local, near Newtonian subsystems for which the light travel time between the localized fluctuations remains bounded away from zero in the limit $\epsilon \searrow 0$. In other words, the inhomogeneous part of the 1-parameter families of ϵ-independent families of initial data, in the relativistic coordinates, which are defined by (1.3.9), consists of a finite number of spikes with characteristic width $\sim \epsilon$, which is centered at arbitrary ϵ-independent points. Once we have transformed the relativistic coordinates to the Newtonian ones, the initial data becomes a finite number of spikes with characteristic width ~ 1, which is centered at arbitrary ϵ-dependent points, and the distance between centers of these spikes are of order $1/\epsilon$, which verifies that the light travel time between the localized fluctuations remains bounded away from zero in the limit $\epsilon \searrow 0$. We will come back to this problem with more detailed discussions in §1.3.3 and Chapter 3 step by step.

This thesis will contribute to the rigorous Newtonian limits on the cosmological setting with the positive cosmological constant $\Lambda > 0$, and mainly focus on the cosmological Newtonian limits on the long time scale and the large spatial scales, which has already implied all the local results of cosmological Newtonian limits in certain assumptions included in the previous works [61, 62, 64, 65].

As we have mentioned above, [61, 62, 64, 65] are the only results on the rigorous Newtonian limits for the cosmological setting. First, [61, 62] established the local-in-time existence of a large class of one-parameter families of cosmological solutions to the Einstein-Euler equations on a spacetime region $[0, T) \times \mathbb{T}^3$ for some $T > 0$ that have a Newtonian limit. However, one should note that in these short time results, the authors do not require that the cosmological constant be positive. Due to the limitations on the local spacetime region of the above works, there are two further main directions to explore. On one hand, as pointed out in [34], the class of solutions that [61, 62] constructed were not valid on cosmological scales, and therefore did not distinguish the relationship between Newtonian gravity and general relativity on cosmological scales. This is because these solutions have a characteristic size $\sim \epsilon$ and should be interpreted as cosmological versions of isolated systems. To solve this difficulty, [64, 65] introduced a new type of initial data which we have briefly stated in the previous two paragraphs and is interpreted as cosmological relevant initial data. Under these data, the solutions are cosmological relevant as well. However, we point out these papers [64, 65] are still local-in-time results on cosmological Newtonian limits although they bring cosmological versions of isolated systems to authentic cosmological scales. On the other hand, on temporal direction, a key question is to understand how long such solutions with cosmological Newtonian limits can survive. The current thesis intends to answer such questions. There are two steps to proceed in the following two chapters (Chapters 2 and 3). In Chapter 2, we will focus our spatial manifold on $\mathbb{T}^3$ which are cosmological versions of isolated systems as in [61, 62]. The reason for such a simplification is to concentrate on developing a technique to prove the existence of the long time Newtonian limit of solutions around the FLRW background by ignoring the effects of the spatially cosmological relevance. After this, with the successful technique for the long time Newtonian limits in hand, in Chapter 3, we attempt to generalize the method for long time scheme to long time Newtonian limits on cosmological scales which means the spatial manifold is $\mathbb{R}^3$ with carefully chosen cosmological relevant data. Up to this stage, we have provided the cosmological Newtonian limits on the complete spacetime region

$[0, \infty) \times \mathbb{R}^3$. One significant requirement in this thesis comparing with [61, 62] is that the cosmological constant must be positive $\Lambda > 0$, which is a crucial condition for the structure of the analysis for long time results.

In summary, there are four stages of the investigations of rigorous cosmological Newtonian limits. We list them as follows:

1. [61, 62] gave the short time existence of Newtonian limits on cosmological versions of isolated systems;
2. [64, 65] gave the short time existence of Newtonian limits on cosmological scales;
3. Chapter 2 provides the long time existence of Newtonian limits on cosmological versions of isolated systems;
4. Chapter 3 provides the long time existence of Newtonian limits on cosmological scales.

This work can be carried out due to recent developments of the fully nonlinear future stability of Friedmann-Lemaître-Robertson-Walker (FLRW) solutions with the positive cosmological constant. H. Ringström [71] was the first to investigate the future global non-linear stability in the case of Einstein's equations coupled to a non-linear scalar field. Inspired by H. Ringström's work, I. Rodnianski and J. Speck [73] established the future non-linear stability of these FLRW solutions under the condition $0 < K < 1/3$ and the assumption of zero fluid vorticity. Then, M. Hadi and J. Speck [35] and J. Speck [77] answered that this future non-linear stability result remains true for fluids with non-zero vorticity and also for the equation of state parameter values $K = 0$. By employing the conformal method developed by H. Friedrich [26, 27], C. Lübble and J. A. V. Kroon [53] proved the above question for the equation of state parameter values $K = 1/3$ that is the pure radiation universe. After these, T. Oliynyk [66] gave an alternative proof for such non-linear future stability problems of FLRW solutions based on a completely different method which is the basic tool for us, in the current thesis, to extend the short time existence of cosmological Newtonian limits to the long time one. This method transforms the Einstein-Euler system to a singular in time, symmetric hyperbolic system by choosing the conformal factor and the source term of the wave gauge and variables judiciously. Although this system is singular in time, due to the right sign of the singular term, it still behaves well for the analysis, which we will explain in more detail in §1.3. It is worth noting that in 1994, U. Brauer, A. Rendall and O. Reula [11] proved a fully non-linear future stability problem on the Newtonian cosmological model, which laid the foundation on the corresponding stability problem in the Newtonian setting.

1.2 Fundamental question

The fundamental question posed in this thesis is: on what space and time scale can Newtonian cosmological simulations be trusted to approximate relativistic cosmologies?

We will resolve this question under small initial data and the positive cosmological constant condition. Specifically, we focus on solutions which are initially a small perturbation of the Friedmann-Lemaître-Robertson-Walker (FLRW) solution. Here, the FLRW solution is an exact solution to the Einstein field equation that represents a homogeneous, isotropic, fluid-filled universe undergoing accelerated expansion once the cosmological constant is positive.

This question was first explored in [61, 62] which implied the existence of cosmological Newtonian limits on *small space-time scales*. Next, [64, 65] developed a method to answer it on *cosmological scales* but still on small-time regions. Although the above works answered the short time existence of cosmological Newtonian limits completely, there is no result on the rigorous long time existence of Newtonian limits. The motivation of the current thesis is to push previous short time results to the *complete space and time scales*, that is, we attempt to answer this question on very *large space-time scales*. In order to achieve this purpose, however, we have to impose more fundamental assumptions which are small initial data and the positive cosmological constant condition. The reason these assumptions are required is, because the Einstein-Euler system is a fully non-linear system, we have to use these additional conditions to control the long time behavior for the purpose of preventing the nonlinearity from becoming uncontrollable.

In this thesis, we will resolve this question under a small initial data condition. Informally, we establish the initial data set in the cosmological scale which solves the constraint equations and construct the existence of 1-parameter families of ϵ-dependent solutions to the Einstein-Euler system with the positive cosmological constant $\Lambda > 0$ that: *(i)* are defined for $\epsilon \in (0, \epsilon_0)$ for some fixed constant $\epsilon_0 > 0$, *(ii)* exist globally on $(t, x^i) \in [0, +\infty) \times \mathbb{T}^3$ (Chapter 2) and $(t, x^i) \in [0, +\infty) \times \mathbb{R}^3$ with cosmological relevant data (Chapter 3), respectively, *(iii)* converge, in a suitable sense, as $\epsilon \searrow 0$ to solutions of the cosmological Poisson-Euler equations of Newtonian gravity, and *(iv)* are small, non-linear perturbations of the FLRW fluid solutions.

1.3 Methodology

This thesis contributes to the existence of rigorous Newtonian limits of solutions to the Einstein-Euler equations with the positive cosmological constant $\Lambda > 0$. We start with the dimensionless version of the Einstein-Euler equations with positive cosmological constant $\Lambda > 0$ (see Appendix A for a detailed derivation of the *dimensionless* Einstein-Euler system):

$$\tilde{G}^{\mu\nu} + \Lambda \tilde{g}^{\mu\nu} = \tilde{T}^{\mu\nu}, \tag{1.3.1}$$

$$\tilde{\nabla}_\mu \tilde{T}^{\mu\nu} = 0. \tag{1.3.2}$$

where $\tilde{G}^{\mu\nu} = \tilde{R}^{\mu\nu} - \frac{1}{2}\tilde{R}\tilde{g}^{\mu\nu}$ is the Einstein tensor constructed by Ricci tensor $\tilde{R}^{\mu\nu}$ and scalar tensor $\tilde{R}$ of the metric $\tilde{g} = \tilde{g}_{\mu\nu} d\bar{x}^\mu d\bar{x}^\nu$, and

$$\tilde{T}^{\mu\nu} = (\bar{\rho} + \bar{p})\tilde{v}^\mu \tilde{v}^\nu + \bar{p}\tilde{g}^{\mu\nu}$$

is the perfect fluid stress-energy tensor. Here, $\bar{\rho}$ and $\bar{p}$ denote the fluid's proper energy density and pressure, respectively, while $\tilde{v}^\nu$ is the fluid four-velocity, which we assume is normalized by

$$\tilde{v}^\mu \tilde{v}_\mu = -1. \tag{1.3.3}$$

We focus on barotropic fluids with a linear equation of state of the form

$$\bar{p} = \epsilon^2 K \bar{\rho}, \qquad 0 < K \leq \frac{1}{3}. \tag{1.3.4}$$

The dimensionless parameter ϵ can be identified with the ratio

$$\epsilon = \frac{v_T}{c},$$

where c is the speed of light and v_T is a characteristic speed associated with the fluid.

Due to the nonlinearity of above Einstein-Euler system, we must use some exact solution as the reference solutions, by carefully controlling the nonlinearities of Einstein-Euler equations, to find new solutions perturbed around this exact solution. The Friedmann-Lemaître-Robertson-Walker (FLRW) solution, which represents a homogeneous, isotropic fluid-filled universe undergoing accelerated expansion, is an exact solution to above (1.3.1)–(1.3.2). From a cosmological perspective, they are the most important family of solutions to the Einstein-Euler system. In this thesis, we will concentrate on the inhomogeneous non-linear perturbations of these FLRW solutions, that is, we view the FLRW metric as our reference solution. Letting $(\bar{x}^i)$, $i = 1, 2, 3$, denote the standard coordinates on the 3 dimensional manifold $\mathbb{B}$ which in this thesis, we take to be $\mathbb{T}^3$ and $\mathbb{R}^3$ in Chapter 2 and 3, respectively, and $t = \bar{x}^0$ a time coordinate on the interval $(0, 1]$, the FLRW solutions which are composed of the metric $\tilde{h}(t)$, the velocity of homogeneous perfect fluid $\tilde{v}_H(t)$ and the density of the fluid $\mu(t)$ on the manifold covered by $(\bar{x}^\mu)$

$$M := (0,1] \times \mathbb{B}$$

are defined by

$$\tilde{h}(t) = -\frac{3}{\Lambda t^2} dtdt + a(t)^2 \delta_{ij} d\bar{x}^i d\bar{x}^j, \tag{1.3.5}$$

$$\tilde{v}_H(t) = -t\sqrt{\frac{\Lambda}{3}}\partial_t, \tag{1.3.6}$$

$$\mu(t) = \frac{\mu(1)}{a(t)^{3(1+\epsilon^2 K)}}, \tag{1.3.7}$$

where $\mu(1)$ is the initial density (freely specifiable) and $a(t)$ satisfies

$$-ta'(t) = a(t)\sqrt{\frac{3}{\Lambda}}\sqrt{\frac{\Lambda}{3} + \frac{\mu(t)}{3}}, \qquad a(1) = 1. \tag{1.3.8}$$

1.3.1 Analysis of Newtonian limits

In this section, the key idea of the rigorous analysis scheme of Newtonian limits, which were established by [59, 60], will be introduced. The fundamental tools for this scheme are the techniques of the singular in ϵ symmetric hyperbolic equations investigated in [12, 41, 75, 76].

We will refer to the global coordinates $(\bar{x}^\mu)$ on manifold M defined above as *relativistic coordinates*. In order to discuss the Newtonian limit and the sense in which solutions converge as $\epsilon \searrow 0$, we need to introduce the spatially rescaled coordinates (x^μ) defined by

$$t = \bar{x}^0 = x^0 \quad \text{and} \quad \bar{x}^i = \epsilon x^i, \qquad \epsilon > 0, \tag{1.3.9}$$

which we refer to as *Newtonian coordinates*.

The first key step for Newtonian limits is to identify the "right" variables which contain the correct information of the orders of ϵ. Then by choosing a suitable gauge, one can write the Einstein-Euler equations, in terms of Newtonian coordinates, to the following singular symmetric hyperbolic equation

$$A^0(\epsilon, t, x, u)\partial_0 u + A^i(\epsilon, t, x, u)\partial_i u + \frac{1}{\epsilon}C^i\partial_i u = F(\epsilon, t, x, u) \tag{1.3.10}$$

where C^i are constant matrix. The corresponding limiting equations of the above singular hyperbolic equation is defined by

$$\mathring{A}^0(t, x, \mathring{u})\partial_0\mathring{u} + \mathring{A}^i(t, x, \mathring{u})\partial_i\mathring{u} + C^i\partial_i v = \mathring{F}(t, x, \mathring{u}) \tag{1.3.11}$$

$$C^i\partial_i\mathring{u} = 0 \tag{1.3.12}$$

where $\mathring{A}^\mu := \lim_{\epsilon\searrow 0} A^\mu$ and $\mathring{F} := \lim_{\epsilon\searrow 0} F$. In fact, the Poisson-Euler equation in Newtonian gravity can be written in the form of the above limiting equation (1.3.11)-(1.3.12). Therefore, one can regard u is the solution to Einstein-Euler equations and $\mathring{u}$ is the solution to Poisson-Euler equations. By Newtonian limits, we mean under suitable assumptions, we try to prove $\|u - \mathring{u}\|_{\text{some norm}} \leq C\epsilon$. This has a positive answer for the *short time* region investigated in [12, 41, 75, 76] based on some conditions. To handle this singular system in ϵ is to observe that in the energy estimate, $\frac{1}{\epsilon}\langle w, C^i\partial_i w\rangle \equiv 0$ due to C^i being a constant matrix. This will eliminate the worst singular term. However, one difficulty we point out here is, usually, Einstein-Euler equations can not be written in the form of (1.3.10) directly, there will be a $1/\epsilon$ singular term appearing in the error term. To conquer this difficulty, we shift the unknown variables by some quantity ξ, then the $1/\epsilon$ singular term in the errors will be absorbed into $\frac{1}{\epsilon}C^i\partial_i w$ where $w = u - \xi$ and ξ is some quantity related to the Newtonian potential. However, this shift will introduce the nonlocal term into the errors. This shifted component ξ is essentially related to the Newtonian potential and hence the nonlocal term is related to the Poisson equations.

1.3.2 Conformal singularized symmetric hyperbolic PDEs

The main goal of this thesis is to answer the cosmological Newtonian limit question on long time and cosmological spacetime scales. The above scheme of the analysis of Newtonian limits provided in §1.3.1 is not sufficient for this purpose now. We have to make additional assumptions to ensure the short time solutions to Einstein-Euler equations, Poisson-Euler equations and the estimates between these two sets of solutions can be extended to the long time region. The key assumptions are the positive cosmological constant $\Lambda > 0$ and the smallness conditions on the initial data. Here, we will give a brief demonstration on solving the long time scheme by a naive linear model. The detailed nonlinear proof is described throughout Chapter 2 and 3.

In order to proceed the long time Newtonian limit we transform the long time problem on $[0, \infty)$ to a short time one on $(0, 1]$ by a time coordinate transform and conformal transform (see (2.1.9), (2.1.12)–(2.1.13) or (3.1.8), (3.1.15)–(3.1.16) for the details), where the future lies in the direction of decreasing t and the timelike infinity is located at t = 0. To achieve this, we apply the scheme established by Todd A. Oliynyk in [66]. By using suitable variables U (considering suitable variables for long time scheme and suitable orders of ϵ), the conformal factor and the wave gauge (carefully choosing the source term of the wave gauge to eliminate some bad error terms in Einstein-Euler equations), with the help of rescaling spatial coordinates $\bar{x}^i = \epsilon x^i$ $(i = 1, 2, 3)$, the Einstein-Euler system can be re-expressed as the following model

$$A^0(\epsilon, t, x, U)\partial_0 U + A^i(\epsilon, t, x, U)\partial_i U + \frac{1}{\epsilon}C^i\partial_i U = \frac{1}{t}\mathfrak{A}(\epsilon, t, x, U)\mathbb{P}U + H(\epsilon, t, x, U) \tag{1.3.13}$$

for $t \in (0, 1]$, where C^i are constants and $\mathbb{P}$ is a constant projection. The outstanding difference of this model (1.3.13) compared with (1.3.10) is the presence of a singular in time term $\frac{1}{t}\mathfrak{A}(\epsilon, t, x, U)\mathbb{P}U$. If this singular term takes "right" sign, then, with other assumptions which make sure the nonlinearity of it is under control, the solution exists on $t \in (0, 1]$ which implies the global existence on $[0, \infty)$ in terms of the standard time coordinate for FLRW metric. In fact, later on, we will see the rescaled difference $\frac{1}{\epsilon}(U - \mathring{U})$ between solutions U to Einstein-Euler equations and the ones $\mathring{U}$ to the Poisson-Euler equations also satisfies the similar equations to (1.3.13), but the function space for this difference must be larger than the one for solutions U and $\mathring{U}$. Therefore, for the sake of the analysis of the long time scheme, it is necessary to analyze (1.3.13) in more detail. In Chapter 2 and 3, we will research such systems on different initial data sets and background manifolds, respectively.

Because the *local* existence of the above model is back to (1.3.10), standard methods can be applied. By the continuation principle, the key step is to obtain an a priori estimate of U to this equation. The importance of this a priori estimate for such a system is twofold: First, it gives $W^{1,\infty}$ estimates of U, which allows us to extend our solution to $t \in (0, 1]$ eventually; Second, once we write the rescaled difference $\frac{1}{\epsilon}(U - \mathring{U})$ into a equation of the similar form of (1.3.13), this a priori estimate yields the estimate of Newtonian limits directly, that is $\|U - \mathring{U}\|_{\text{some norm}} \leq C\epsilon$. It seems difficulties arise when we analyze this equation (1.3.13) as $\epsilon \searrow 0$ and $t \searrow 0$ due to the singularities in ϵ and time t. In order to illustrate how to eliminate the singularities of ϵ and t in the energy estimates as $\epsilon \searrow 0$ and $t \searrow 0$ and convey the basic spirits of the analysis of such a model equation, we look at the following extremely simplified linear model first,

$$\partial_t U + \frac{1}{\epsilon}a^i\partial_i U = \frac{1}{t}U + aU \tag{1.3.14}$$

for $t \in [-1, 0)$, where a^i, a is a constant. We emphasize that, as we will carry out in the model equations in Chapter 2 and 3, we switch $t \in (0, 1]$ to the standard time orientation $t \in [-1, 0)$, where the future is located in the direction of increasing time, while keeping the singularity located at $t = 0$. We do this in order to make the derivation of the energy estimates as similar as possible to those for non-singular symmetric hyperbolic systems, which we expect will make it easier for readers familiar with such estimates to follow the arguments below. To get back to the time orientation used to formulate the conformal Einstein-Euler equations, we need only apply the trivial time transformation $t \mapsto -t$.

Let us focus on the energy defined by

$$\|U\| = \langle U, U\rangle^{\frac{1}{2}}. \tag{1.3.15}$$

Then this model equation yields, by acting the inner product of U on the both sides of (1.3.14),

$$\partial_t \|U\|^2 = \frac{2}{t}\|U\|^2 + 2a\|U\|^2 \tag{1.3.16}$$

provided

$$\frac{1}{\epsilon}\langle U, a^i \partial_i U\rangle = 0. \tag{1.3.17}$$

This condition (1.3.17), in fact, holds for all of our proof in Chapter 2 and 3 due to the manifolds of integration are $\mathbb{T}^3$ and $\mathbb{R}^3$ respectively. Then we conclude that

$$\partial_t\left(\|U\|^2 + \int_{T_0}^t -\frac{2}{s}\|U\|^2 ds\right) \lesssim \|U\|^2 + \int_{T_0}^t -\frac{2}{s}\|U\|^2 ds. \tag{1.3.18}$$

Then applying Grönwall's inequality leads to the boundness of a new energy

$$\|U\|^2 + \int_{T_0}^t -\frac{2}{s}\|U\|^2 ds \quad \text{(Note that } s < 0). \tag{1.3.19}$$

This is the core analysis for the long time scheme. Of course, our system is far more complicated than this model due to the full nonlinearity. That is why our proof is only valid for small initial data and requires plenty of structural assumptions on the coefficients and remainder terms. In other words, all of those assumptions and smallness of initial data ensure these nonlinearities are under control and make sure this nonlinear system is eventually dominated by such a simple linear model. In order to close such type of estimates by Grönwall's inequality, we have to select appropriate function spaces wisely as our working platform. One requirement for these function spaces is they must be subspaces of $W^{1,\infty}$ in order to apply the continuation principle later.

Once we get the a priori estimates in the suitable norms using above basic and simplified ideas, we are able to use the continuation principle[1] to extend our local solutions to the long time ones because above a priori estimates have already implied that $\|U\|_{W^{1,\infty}} \leq C\sigma$ is bounded for the known local solutions U on $t \in [T_0, T)$. The fact that $\|U\|_{W^{1,\infty}} \leq C\sigma$ will rule out all the other alternatives of Majda's criterion in Corollary D.4.3 (that is, the second and the third alternatives in this Corollary can not occur). This is the spirit and mechanism that we will follow in our main analysis of long time issues in Chapter 2 and 3.

1.3.3 Initialization and function spaces of the evolution equations

Suitably selecting initial data is crucial for the Newtonian limit questions because inappropriate data may lead to a blow-up of Newtonian data on the initial hypersurface as $\epsilon \searrow 0$. The initial data are governed by the constraint equations which are essentially an elliptic system. We have to specify some part of the data, then the other data will be derived from these free ones via constraints. One key issue when specifying the free data for the Newtonian limit problem is to make sure the free data have the suitable order of ϵ. In this thesis, we will use a formulation for constraints developed by M. Lottermoser [52].

[1]The continuation principle can be found in Appendix D.4. In fact, we also give a brief introduction of bootstrap arguments in this appendix because in order to get the a priori estimate above, one may adopt bootstrap to obtain it, but we directly derive this estimate in Chapter 2 and 3 instead. However, we assure that one, of course, could get it by bootstrap arguments via the same key steps.

In Chapter 2, when we investigate the Newtonian limits on the cosmological version of the isolated system, the initialization is standard and we can directly apply the technique of [52] to our system, then using the implicit function theorem to conclude the existence of the complete initial data set.

However, in Chapter 3, when considering the Newtonian limits on cosmological scales, the initialization becomes more complicated. In order to endow this system with cosmological relevant data rather than isolated data, we will adopt the basic idea from [65]. Before stating this idea, let us first illustrate the reason why the results in Chapter 2 fail on the cosmological scales. In Chapter 2, we have established the existence of 1-parameter families of ϵ-dependent solutions to the Einstein-Euler equation on $\mathbb{T}^3$ for the long time scale, that converges to solutions of the cosmological Poisson-Euler equations as $\epsilon \searrow 0$. If we lift these solutions to the covering space, they become periodic solutions on $\mathbb{R}^3$ but with the period $\sim \epsilon$ in relativistic coordinates. When $\epsilon \searrow 0$, all the solutions collapse, and is equivalent to the behavior of the isolated system, rather than the system on cosmological spatial scales in astrophysics. Then in order to prevent such a collapse to the isolated system picture, as [64] and [65] proposed, we have to, in relativistic coordinates, initially fix the inhomogeneous component of fluid density around several fixed points which are independent of ϵ, with the characteristic width $\sim \epsilon$. Then in relativistic coordinates, as $\epsilon \searrow 0$, such inhomogeneous components of density collapse into bumps separately around those fixed points, which gives the exact picture of our universe on cosmological scales. Next, in order to convey the key idea and explain such data in Newtonian coordinates more quantitatively, we give the initial density by

$$\delta\breve{\rho}_{\epsilon,\vec{\mathbf{y}}}(\mathbf{x}) = \sum_{\lambda=1}^{N} \delta\breve{\rho}_{\lambda}\Big(\mathbf{x} - \frac{\mathbf{y}_\lambda}{\epsilon}\Big) \tag{1.3.20}$$

where $\mathbf{x} = (x^i)$, $\delta\breve{\rho}_\lambda$ is in a certain function space and $\vec{\mathbf{y}} = (\mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_N) \in \mathbb{R}^{3N}$ are fixed spatial points in relativistic coordinates. However, we illustrate it here in a very rough fashion and the exact statements will be presented step by step later. Due to (1.3.9), we know every bump given by (1.3.20) is of characteristic width $\sim \epsilon$ in relativistic coordinates and centered at $\mathbf{y}_\lambda \in \mathbb{R}^3$ for $\lambda = 1, \cdots, N$. Therefore, this seems consistent with our aim in relativistic coordinates. Nevertheless, readers may immediately notice that this initial datum, in Newtonian coordinates, is singular in ϵ, which seems unexpected for the Newtonian setting. However, because the light travel time between the localized fluctuations remains bounded away from 0 in the limit $\epsilon \searrow 0$ on the cosmological scales of astrophysics (see [65] and [64] for more details), the above setting precisely gives us the "right" initialization even though this data seems singular. In other words, if there is no such singular behavior in ϵ in Newtonian coordinates, due to the flattened light cone, the light travel time between the localized fluctuations is identically 0, which contradicts the observations in astrophysics. Therefore, such types of initial data do well to depict the universe both in the Newtonian picture and the relativistic one.

In addition, the previous approach of initialization in Chapter 2 fails because the operator Δ^{-1} for functions on $\mathbb{R}^3$ is not as good as the one for functions on $\mathbb{T}^3$. Note that for functions defined on $\mathbb{R}^3$, $(-\Delta)^{-\frac{s}{2}}$ is a bounded operator from L^p to L^q where p, q are described in Theorem B.0.1. [65] gave a method to solve this difficulty by introducing an inhomogeneous conformal factor. Using the inhomogeneous components of this conformal factor ensures that the remainder terms of the constraints possess certain structures. That is, all the remainder terms are in $L^{6/5}$. From a special case of Theorem B.0.1, $\|(-\Delta)^{-1}f\|_{L^6} \lesssim \|f\|_{L^{6/5}}$, one can derive that $u^{0\mu} \in L^6$. Furthermore, one is able to establish a suitable contraction mapping, by which the author concludes the existence of a complete initial data set. However, in our current case, in view of the long time scheme, we use a wave gauge with a special source term which directly leads to the presence of the linear term of $u^{0\mu}$ in the remainders. This causes the failure of the method in [65] because the specified structure of the remainders can not be achieved due to this linear term of $u^{0\mu}$. We do not intend to improve the method of [65], instead, we will propose another way to solve the constraints in this thesis by introducing some fractional Yukawa potential operators $(\kappa^2 - \Delta)^{-\frac{s}{2}}$ ($\kappa > 0$) and investigating some important properties of such operators. The basic idea is to use $(\kappa^2 - \Delta)^{-1}$ to solve the elliptic equation of the constraints instead of using Δ^{-1}. The significant benefits from Yukawa potential operator $(\kappa^2 - \Delta)^{-s/2}$ are the rescaling operators $\epsilon(\kappa^2 - \Delta)^{-1/2}$ and $(\kappa^2 - \Delta)^{-1/2}\partial_j$ are bounded operators from $W^{s,p}$ to $W^{s,p}$. By choosing suitable variables, with the help of the gauge constraints, we write the gravitational constraint

$$(\bar{G}^{0\mu}-\bar{T}^{0\mu})|_{\text{Initial hypersurface}}=0$$

to the following elliptic system,

$$A\begin{pmatrix}\phi\\ \psi^j\end{pmatrix}=\begin{pmatrix}f(\epsilon,\phi,\psi^j,\breve{\xi})\\ g^j(\epsilon,\phi,\psi^k,\breve{\xi})\end{pmatrix} \tag{1.3.21}$$

where ϕ and ψ^j are the constraint data, $\breve{\xi}$ is the free data, $\mathtt{a}>0$, $\mathtt{b}<0$, $\mathtt{c}>0$ and $\mathtt{d}>0$, and

$$A=\begin{pmatrix}\Delta-\epsilon^2\mathtt{a} & -\epsilon\mathtt{b}\partial_j\\ -\epsilon\mathtt{d}\partial^j & \Delta-\epsilon^2\mathtt{c}\end{pmatrix}.$$

Then acting A^{-1} to both sides of (1.3.21) (note that A is not always invertible, so we have to do more work, we just ignore this here to convey the basic idea), we can construct a contraction mapping. It turns out that Yukawa potential operators along with some specific structures of the remainder terms f and g^j make sure such contraction exists. Subsequently, we conclude the existence of complete initial data by the Banach's fixed point theorem. We briefly state this result as follows. Suppose $s\in\mathbb{Z}_{\geq 3}$, $r>0$ and $\vec{\mathbf{y}}=(\mathbf{y}_1,\cdots,\mathbf{y}_N)\in\mathbb{R}^{3N}$. Given free data $\breve{\xi}=\big(\breve{\mathfrak{u}}^{ij}_\epsilon,\breve{\mathfrak{u}}^{ij}_{0,\epsilon},\delta\breve{\rho}_\lambda,\breve{z}^j_\lambda\big)\in R^{s+1}(\mathbb{R}^3,\mathbb{S}_3\times H^s(\mathbb{R}^3,\mathbb{S}_3))\times(L^{\frac{6}{5}}\cap K^s(\mathbb{R}^3,\mathbb{R}))\times(L^{\frac{6}{5}}\cap K^s(\mathbb{R}^3,\mathbb{R}^3))$, for $\lambda=1,\cdots,N$. $\delta\breve{\rho}_{\epsilon,\vec{\mathbf{y}}}$ and $\breve{z}^j_{\epsilon,\vec{\mathbf{y}}}$ are defined by

$$\delta\breve{\rho}_{\epsilon,\vec{\mathbf{y}}}(\mathbf{x})=\sum_{\lambda=1}^N\delta\breve{\rho}_\lambda\Big(\mathbf{x}-\frac{\mathbf{y}_\lambda}{\epsilon}\Big)\quad\text{and}\quad\breve{z}^j_{\epsilon,\vec{\mathbf{y}}}(\mathbf{x})=\sum_{\lambda=1}^N\breve{z}^j_\lambda\Big(\mathbf{x}-\frac{\mathbf{y}_\lambda}{\epsilon}\Big),$$

where $\big(\breve{\mathfrak{u}}^{ij}_\epsilon,\breve{\mathfrak{u}}^{ij}_{0,\epsilon},\delta\breve{\rho}_{\epsilon,\vec{\mathbf{y}}},\breve{z}^j_{\epsilon,\vec{\mathbf{y}}}\big)$ are free data which represent some components of the metric, the inhomogeneous component of the density, and the spatial velocity of the perfect fluid. Then for $r>0$ chosen small enough such that

$$\|\breve{\mathfrak{u}}^{ij}_\epsilon\|_{R^{s+1}}+\|\breve{\mathfrak{u}}^{ij}_{0,\epsilon}\|_{H^s}+\|\delta\breve{\rho}_\lambda\|_{L^{\frac{6}{5}}\cap K^s}+\|\breve{z}^j_\lambda\|_{L^{\frac{6}{5}}\cap K^s}\leq r.$$

Then there exists an small constant $\epsilon_0=\epsilon_0(r)>0$ and a family of one parameter maps

$$\{u^{\mu\nu}_{\epsilon,\vec{\mathbf{y}}},u_{\epsilon,\vec{\mathbf{y}}},u^{ij}_{\gamma,\epsilon,\vec{\mathbf{y}}},u^{0\mu}_{i,\epsilon,\vec{\mathbf{y}}},u^{0\mu}_{0,\epsilon,\vec{\mathbf{y}}},u_{\gamma,\epsilon,\vec{\mathbf{y}}},z_{j,\epsilon,\vec{\mathbf{y}}},\delta\zeta_{\epsilon,\vec{\mathbf{y}}}\}|_{\text{Initial hypersurface}}\in X^s(\mathbb{R}^3) \tag{1.3.22}$$

where

$$X^s(\mathbb{R}^3):=R^{s+1}(\mathbb{R}^3,\mathbb{S}_4)\times R^{s+1}(\mathbb{R}^3,\mathbb{R})\times R^s(\mathbb{R}^3,\mathbb{S}_3)\times\big(R^s(\mathbb{R}^3,\mathbb{R}^3)\big)^2\times R^s(\mathbb{R}^3,\mathbb{R})\times R^s(\mathbb{R}^3,\mathbb{R}^3)\times R^s(\mathbb{R}^3,\mathbb{R}),$$

$$R^s(\mathbb{R}^3,V)=\big\{u\in L^6(\mathbb{R}^3,V)\mid Du\in H^{s-1}(\mathbb{R}^3,V)\big\}$$

and $\{u^{\mu\nu}_{\epsilon,\vec{\mathbf{y}}},u_{\epsilon,\vec{\mathbf{y}}},u^{ij}_{\gamma,\epsilon,\vec{\mathbf{y}}},u^{0\mu}_{i,\epsilon,\vec{\mathbf{y}}},u^{0\mu}_{0,\epsilon,\vec{\mathbf{y}}},u_{\gamma,\epsilon,\vec{\mathbf{y}}},z_{j,\epsilon,\vec{\mathbf{y}}},\delta\zeta_{\epsilon,\vec{\mathbf{y}}}\}$ are the variables we will choose for the analysis of the evolution of the Einstein-Euler system.

From the above initialization, we see that the initial data is in some function space $X^s(\mathbb{R}^3)$ which is composed of function spaces R^s and R^{s+1}. Because of this, in Chapter 3, we analyze the model equation (1.3.13) in function space R^s instead of H^s.

1.4 Notation

In this thesis, the main results and the analysis are located in Chapter 2 and 3. We will adopt slightly different systems of notation in these two chapters for their own benefits. We will introduce the corresponding conventions of notations in §2.1.1 and §3.1.1, respectively. For the readers' convenience, a detailed index of notation can be found in Appendix E which lists all the frequently used definitions and non-standard notations in Chapter 2 and 3, respectively.

1.5 Thesis outline

Chapter 2 is based on the accepted journal paper "Newtonian limits of isolated cosmological systems on long time scales" (Chao Liu and Todd A. Oliynyk). This chapter contributes to the long time existence of cosmological Newtonian limits of solutions around the FLRW metric on $\mathbb{T}^3$, which is a cosmological version of the isolated system. We establish the existence of 1-parameter families of ϵ-dependent solutions to the Einstein-Euler equations with a positive cosmological constant $\Lambda > 0$ and a linear equation of state $p = \epsilon^2 K\rho$, $0 < K \leq 1/3$, for the parameter values $0 < \epsilon < \epsilon_0$. These solutions exist globally to the future, converge as $\epsilon \searrow 0$ to solutions of the cosmological Poisson-Euler equations of Newtonian gravity, and are inhomogeneous non-linear perturbations of FLRW fluid solutions. In this Chapter, we first select suitable variables, the conformal factor and the wave gauge to write the Einstein-Euler system to the model equation (1.3.13). Next, we discuss a local-in-time existence and uniqueness result along with a continuation principle for solutions of the reduced conformal Einstein-Euler equations and the conformal cosmological Poisson-Euler equations, respectively. Then by analyzing the model equation (1.3.13), we establish uniform a priori estimates for solutions to the model which characterizes both the conformal Einstein-Euler equations and the $\epsilon \searrow 0$ limit of these equations. We also establish error estimates by direct applying the model equations as well, that is, a priori estimates for the difference between solutions of the singular hyperbolic equation and the corresponding $\epsilon \searrow 0$ limit equation. After that, we construct ϵ-dependent 1-parameter families of initial data for the reduced conformal Einstein-Euler equations that satisfy the constraint equations on the initial hypersurface. In the end of this Chapter, using the results of the model equation and the constructed initial data, we conclude the Newtonian limits we expect.

Chapter 3 is based on the submitted journal paper "Cosmological Newtonian limits on large spacetime scales" (Chao Liu and Todd A. Oliynyk) which generalizes the results in Chapter 2 to the results on cosmological scales, that is, by judicious cosmological relevant initial data selections, we achieve the long time Newtonian limit approximation of solutions of Einstein-Euler equations on authentic cosmological scales. Specifically, we establish the existence of 1-parameter families of ϵ-dependent solutions on $\mathbb{R}^3$ to the Einstein-Euler equations with a positive cosmological constant $\Lambda > 0$ and a linear equation of state $p = \epsilon^2 K\rho$, $0 < K \leq 1/3$, for the parameter values $0 < \epsilon < \epsilon_0$. These solutions exist globally to the future, converge as $\epsilon \searrow 0$ to solutions of the cosmological Poisson-Euler equations of Newtonian gravity on cosmological scales with cosmological relevant data, and are inhomogeneous non-linear perturbations of FLRW fluid solutions. In this chapter, we first select suitable variables, the conformal factor and the wave gauge to write the Einstein-Euler system to the model equation (1.3.13) as in Chapter 2. However, the difference is in this stage, the remainder term of the model equation (1.3.13) includes the singular term of ϵ, which is localizable. We then discuss the initialization as we briefly discussed in §1.3.3. Next, we build the local-in-time existence and uniqueness result along with a continuation principle for solutions of the reduced conformal Einstein-Euler equations; with the help of uniformly local Sobolev spaces, (we can use uniform local Sobolev spaces due to the localizable equations) and improve the solutions to better function spaces. Similarly, we also build the local-in-time existence and uniqueness result along with a continuation principle for solutions of the conformal cosmological Poisson-Euler equations. After these, in order to get the model equation (1.3.13), we have to shift the variables to new ones. The shifted quantity would also be defined via the Yukawa potential, which is more complicated than the method used in Chapter 2. This shifted variable eliminates the ϵ-singularity in the remainders, but note that the equations become nonlocalizable. Next, we analyze the model equation (1.3.13) with some requirements and use the model equation as a tool to derive a long time Newtonian limit results on cosmological scales. Due to the difficulties of this chapter, we give the following flow chart (Figure 1.1) to help readers track the proof.

In Chapter 4, we review the thesis, provide a general discussion of the cosmological Newtonian limits approximation to general relativity, and pose some future questions.

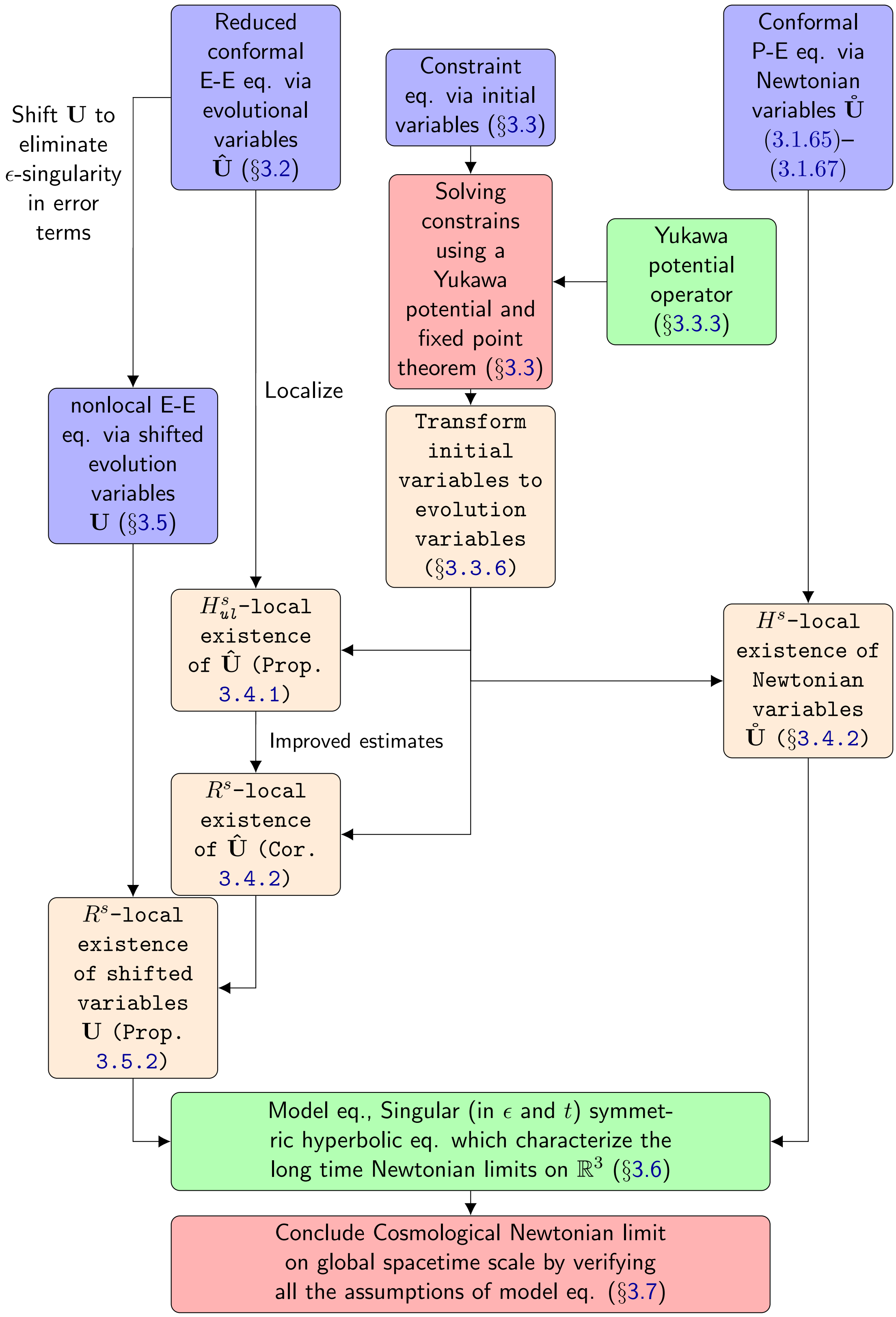

Figure 1.1: *The flow chart of Chapter 3.*

Chapter 2
Newtonian limits of isolated cosmological systems on long time scales

Chapter 2 is based on the accepted journal article: Chao Liu and Todd A. Oliynyk, Newtonian limits of isolated cosmological systems on long time scales, accepted by Annales Henri Poincaré

Abstract. We establish the existence of 1-parameter families of ϵ-dependent solutions to the Einstein-Euler equations with a positive cosmological constant $\Lambda > 0$ and a linear equation of state $p = \epsilon^2 K\rho$, $0 < K \leq 1/3$, for the parameter values $0 < \epsilon < \epsilon_0$. These solutions exist globally to the future, converge as $\epsilon \searrow 0$ to solutions of the cosmological Poisson-Euler equations of Newtonian gravity, and are inhomogeneous non-linear perturbations of FLRW fluid solutions.

References are considered at the end of the thesis.

Chapter 2

Newtonian limits of isolated cosmological systems on long time scales

> When I look for their origin, it goes back into infinity; when I look for their end, it proceeds without termination. Infinite, unceasing, there is no room for words about (the Dao). To regard it as in the category of things is the origin of the language that it is caused or that it is the result of doing nothing; but it would end as it began with things.
>
> Zhuangzi

2.1 Introduction

Gravitating relativistic perfect fluids are governed by the Einstein-Euler equations. The dimensionless version of these equations with a cosmological constant Λ is given by

$$\tilde{G}^{\mu\nu} + \Lambda\tilde{g}^{\mu\nu} = \tilde{T}^{\mu\nu}, \tag{2.1.1}$$

$$\tilde{\nabla}_{\mu}\tilde{T}^{\mu\nu} = 0, \tag{2.1.2}$$

where $\tilde{G}^{\mu\nu}$ is the Einstein tensor of the metric

$$\tilde{g} = \tilde{g}_{\mu\nu}d\bar{x}^{\mu}d\bar{x}^{\nu},$$

and

$$\tilde{T}^{\mu\nu} = (\bar{\rho} + \bar{p})\tilde{v}^{\mu}\tilde{v}^{\nu} + \bar{p}\tilde{g}^{\mu\nu}$$

is the perfect fluid stress-energy tensor. Here, $\bar{\rho}$ and $\bar{p}$ denote the fluid's proper energy density and pressure, respectively, while $\tilde{v}^{\nu}$ is the fluid four-velocity, which we assume is normalized by

$$\tilde{v}^{\mu}\tilde{v}_{\mu} = -1. \tag{2.1.3}$$

In this article, we assume a positive cosmological constant $\Lambda > 0$ and restrict our attention to barotropic fluids with a linear equation of state of the form

$$\bar{p} = \epsilon^2 K \bar{\rho}, \qquad 0 < K \leq \frac{1}{3}.$$

The dimensionless parameter ϵ can be identified with the ratio

$$\epsilon = \frac{v_T}{c},$$

where c is the speed of light and v_T is a characteristic speed associated to the fluid. Understanding the behavior of solutions to (2.1.1)-(2.1.2) in the limit $\epsilon \searrow 0$ is known as the *Newtonian limit*, which has been the subject of many investigations. Most work in this subject has been carried out in the setting of isolated systems and has almost exclusively involved formal calculations, see [8, 9, 16, 19, 21, 22, 30, 46, 47, 48] and references therein, with a few exceptions [59, 60, 70] where rigorous results were established. Due to questions surrounding the physical interpretation of large scale cosmological simulations using Newtonian gravity and the role of Newtonian gravity in cosmological averaging, interest in the Newtonian limit and the related Post-Newtonian expansions has shifted from the isolated systems setting to the cosmological one. Here too, the majority of results are based on formal calculations [13, 14, 15, 17, 23, 33, 34, 38, 43, 44, 55, 56, 57, 68, 87] with the articles [61, 62, 64, 65] being the only exceptions where rigorous results have been obtained.

From a cosmological perspective, the most important family of solutions to the system (2.1.1)-(2.1.2) are the Friedmann-Lemaître-Robertson-Walker (FLRW) solutions that represent a homogeneous, fluid filled universe undergoing accelerated expansion. Letting $(\bar{x}^i)$, $i = 1, 2, 3$, denote the standard periodic coordinates on the 3-torus[1] $\mathbb{T}^3_\epsilon$ and $t = \bar{x}^0$ a time coordinate on the interval $(0, 1]$, the FLRW solutions on the manifold

$$M_\epsilon = (0, 1] \times \mathbb{T}^3_\epsilon$$

are defined by

$$\tilde{h}(t) = -\frac{3}{\Lambda t^2} dt dt + a(t)^2 \delta_{ij} d\bar{x}^i d\bar{x}^j, \tag{2.1.4}$$

$$\tilde{v}_H(t) = -t\sqrt{\frac{\Lambda}{3}}\partial_t, \tag{2.1.5}$$

$$\rho_H(t) = \frac{\rho_H(1)}{a(t)^{3(1+\epsilon^2 K)}}, \tag{2.1.6}$$

where $\rho_H(1)$ is the initial density (freely specifiable) and $a(t)$ satisfies

$$-ta'(t) = a(t)\sqrt{\frac{3}{\Lambda}}\sqrt{\frac{\Lambda}{3} + \frac{\rho_H(t)}{3}}, \qquad a(1) = 1. \tag{2.1.7}$$

Throughout this article, we will refer to the global coordinates $(\bar{x}^\mu)$ on M_ϵ as *relativistic coordinates*. In order to discuss the Newtonian limit and the sense in which solutions converge as $\epsilon \searrow 0$, we need to introduce the spatially rescaled coordinates (x^μ) defined by

$$t = \bar{x}^0 = x^0 \quad \text{and} \quad \bar{x}^i = \epsilon x^i, \qquad \epsilon > 0, \tag{2.1.8}$$

which we refer to as *Newtonian coordinates*. We note that the Newtonian coordinates define a global coordinate system on the ϵ-independent manifold

$$M := M_1 = (0, 1] \times \mathbb{T}^3.$$

Remark 2.1.1. Due to our choice of time coordinate t on $(0, 1]$, the future lies in the direction of *decreasing* t and timelike infinity is located at $t = 0$.

[1] Here, $\mathbb{T}^n_\epsilon = [0, \epsilon]^n / \sim$ where $\sim$ is equivalence relation that follows from the identification of the sides of the box $[0, \epsilon]^n$. When $\epsilon = 1$, we will simply write $\mathbb{T}^n$.

Remark 2.1.2. The non-standard form of the FLRW solution and the ϵ-dependence in the manifold M_ϵ is a consequence of our starting point for the Newtonian limit, which differs from the standard formulation in that the time interval has been compactified from $[0,\infty)$ to $(0,1]$ and the light cones of the metric (2.1.4) do not flatten as $\epsilon \searrow 0$. For comparison, we observe that the standard formulation can be obtained by first switching to Newtonian coordinates, which removes the ϵ-dependence from the spacetime manifold, followed by the introduction of a new time coordinate according to

$$t = e^{-\sqrt{\frac{\Lambda}{3}}\tau}, \tag{2.1.9}$$

which undoes the compactification of the time interval. These new coordinates define a global coordinate system on the ϵ-independent manifold $[0,\infty)\times\mathbb{T}^3$ on which the FLRW metric can be expressed as

$$\hat{h} = -d\tau d\tau + \epsilon^2 \hat{a}(\tau)\delta_{ij}dx^i dx^j$$

where $\hat{a}(\tau) = a(e^{-\sqrt{\frac{\Lambda}{3}}\tau})$. Dividing through by ϵ^2 yields the metric

$$\hat{h}_\epsilon = -\frac{1}{\epsilon^2}d\tau d\tau + \hat{a}(\tau)\delta_{ij}dx^i dx^j,$$

which is now in the standard form for taking the Newtonian limit. In particular, we observe that the light cones of this metric flatten out as $\epsilon \searrow 0$.

Remark 2.1.3. Throughout this article, we take the homogeneous initial density $\rho_H(1)$ to be independent of ϵ. All of the results established in this article remain true if $\rho_H(1)$ is allowed to depend on ϵ in a C^1 manner, that is the map $[0,\epsilon_0) \ni \epsilon \longmapsto \rho_H^\epsilon(1) \in \mathbb{R}_{>0}$ is C^1 for some $\epsilon_0 > 0$.

Remark 2.1.4. As we show in §2.2.1, FLRW solutions $\{a, \rho_H\}$ depend regularly on ϵ and have well defined Newtonian limits. Letting

$$\mathring{a} = \lim_{\epsilon\searrow 0} a \quad \text{and} \quad \mathring{\rho}_H = \lim_{\epsilon\searrow 0} \rho_H \tag{2.1.10}$$

denote the Newtonian limit of a and ρ_H, respectively, it then follows from (2.1.6) and (2.1.7) that $\{\mathring{a}, \mathring{\rho}_H\}$ satisfy

$$\mathring{\rho}_H = \frac{\mathring{\rho}_H(1)}{\mathring{a}(t)^3}$$

and

$$-t\mathring{a}'(t) = \mathring{a}(t)\sqrt{\frac{3}{\Lambda}}\sqrt{\frac{\Lambda}{3} + \frac{\mathring{\rho}_H(t)}{3}}, \qquad \mathring{a}(1) = 1,$$

which define the Newtonian limit of the FLRW equations.

In the articles [61, 62], the second author established the existence of 1-parameter families of solutions[2] $\{\tilde{g}^{\mu\nu}_\epsilon, \bar{\rho}_\epsilon, \tilde{v}^\mu_\epsilon\}$, $0 < \epsilon < \epsilon_0$, to (2.1.1)-(2.1.2), which include the above family of FLRW solutions, on spacetime regions of the form

$$(T_1, 1] \times \mathbb{T}^3_\epsilon \subset M_\epsilon,$$

for some $T_1 \in (0,1]$, that converge, in a suitable sense, as $\epsilon \searrow 0$ to solutions of the cosmological Poisson-Euler equations of Newtonian gravity. Although this result rigorously established the existence of a wide class of solutions to the Einstein-Euler equations that admit a (cosmological) Newtonian limit, the local-in-time nature of the result left open the question of what happens on long time scales. In particular, the question of the existence of 1-parameter families of solutions that converge globally to the future as $\epsilon \searrow 0$ was not addressed. In light of the importance of Newtonian gravity in cosmological simulations [18, 24, 78, 79], this question needs to be resolved in order to understand on what time

[2]To convert the 1-parameter solutions to the Einstein-Euler equations from [61, 62] to solutions of (2.1.1)-(2.1.2), the metric, four-velocity, time coordinate and spatial coordinates must each be rescaled by an appropriate powers of ϵ, after which the rescaled time coordinate must be transformed according to the formula (2.1.9).

scales Newtonian cosmological simulations can be trusted to approximate relativistic cosmologies. In this article, we resolve this question under a small initial data condition. Informally, we establish the existence of 1-parameter families of ϵ-dependent solutions to (2.1.1)-(2.1.2) that: *(i)* are defined for $\epsilon \in (0, \epsilon_0)$ for some fixed constant $\epsilon_0 > 0$, *(ii)* exist globally on M_ϵ, *(iii)* converge, in a suitable sense, as $\epsilon \searrow 0$ to solutions of the cosmological Poison-Euler equations of Newtonian gravity, and *(iv)* are small, non-linear perturbations of the FLRW fluid solutions (2.1.4)-(2.1.7). The precise statement of our results can be found in Theorem 2.1.7.

Before proceeding with the statement of Theorem 2.1.7, we first fix our notation and conventions, and define a number of new variables and equations.

2.1.1 Notation

Index of notation

An index containing frequently used definitions and non-standard notation can be found in Appendix E.1.

Indices and coordinates

Unless stated otherwise, our indexing convention will be as follows: we use lower case Latin letters, e.g. i, j, k, for spatial indices that run from 1 to n, and lower case Greek letters, e.g. α, β, γ, for spacetime indices that run from 0 to n. When considering the Einstein-Euler equations, we will focus on the physical case where $n = 3$, while all of the PDE results established in this article hold in arbitrary dimensions.

For scalar functions $f(t, \bar{x}^i)$ of the relativistic coordinates, we let

$$\underline{f}(t, x^i) := f(t, \epsilon x^i) \tag{2.1.11}$$

denote the representation of f in Newtonian coordinates.

Derivatives

Partial derivatives with respect to the Newtonian coordinates $(x^\mu) = (t, x^i)$ and the relativistic coordinates $(\bar{x}^\mu) = (t, \bar{x}^i)$ will be denoted by $\partial_\mu = \partial/\partial x^\mu$ and $\bar{\partial}_\mu = \partial/\partial \bar{x}^\mu$, respectively, and we use $Du = (\partial_j u)$ and $\partial u = (\partial_\mu u)$ to denote the spatial and spacetime gradients, respectively, with respect to the Newtonian coordinates, and similarly $\bar{\partial} u = (\bar{\partial}_\mu u)$ to denote the spacetime gradient with respect to the relativistic coordinates. We also use Greek letters to denote multi-indices, e.g. $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$, and employ the standard notation $D^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \cdots \partial_n^{\alpha_n}$ for spatial partial derivatives. It will be clear from context whether a Greek letter stands for a spacetime coordinate index or a multi-index.

Given a vector-valued map $f(u)$, where u is a vector, we use Df and $D_u f$ interchangeably to denote the derivative with respect to the vector u, and use the standard notation

$$Df(u) \cdot \delta u := \frac{d}{dt}\bigg|_{t=0} f(u + t\delta u)$$

for the action of the linear operator Df on the vector δu. For vector-valued maps $f(u, v)$ of two (or more) variables, we use the notation $D_1 f$ and $D_u f$ interchangeably for the partial derivative with respect to the first variable, i.e.

$$D_u f(u, v) \cdot \delta u := \frac{d}{dt}\bigg|_{t=0} f(u + t\delta u, v),$$

and a similar notation for the partial derivative with respect to the other variable.

Function spaces

Given a finite dimensional vector space V, we let $H^s(\mathbb{T}^n, V)$, $s \in \mathbb{Z}_{\geq 0}$, denote the space of maps from $\mathbb{T}^n$ to V with s derivatives in $L^2(\mathbb{T}^n)$. When the vector space V is clear from context, we write $H^s(\mathbb{T}^n)$ instead of $H^s(\mathbb{T}, V)$. Letting

$$\langle u, v \rangle = \int_{\mathbb{T}^n} (u(x), v(x))\, d^n x,$$

where $(\cdot, \cdot)$ is a fixed inner product on V, denote the standard L^2 inner product, the H^s norm is defined by

$$\|u\|_{H^s}^2 = \sum_{0 \leq |\alpha| \leq s} \langle D^\alpha u, D^\alpha u \rangle.$$

For any fixed basis $\{\mathbf{e}_I\}_{I=1}^N$ of V, we follow [61] and define a subspace of $H^s(\mathbb{T}^n, V)$ by

$$\bar{H}^s(\mathbb{T}^n, V) = \left\{ u(x) = \sum_{I=1}^N u^I(x)\mathbf{e}_I \in H^s(\mathbb{T}^n, V) \middle| \langle 1, u^I \rangle = 0 \text{ for } I = 1, 2, \cdots, N \right\}.$$

Specializing to $n = 3$, we define, for fixed $\epsilon_0 > 0$ and $r > 0$, the spaces

$$X^s_{\epsilon_0, r}(\mathbb{T}^3) = (-\epsilon_0, \epsilon_0) \times B_r(H^{s+1}(\mathbb{T}^3, \mathbb{S}_3)) \times H^s(\mathbb{T}^3, \mathbb{S}_3) \times B_r(\bar{H}^s(\mathbb{T}^3)) \times \bar{H}^s(\mathbb{T}^3, \mathbb{R}^3),$$

where $\mathbb{S}_N$ denotes the space of symmetric $N \times N$ matrices, and here and throughout this article, we use, for any Banach space Y, $B_r(Y) = \{\, y \in Y \mid \|y\|_Y < r \,\}$ to denote the open ball of radius r.

To handle the smoothness of coefficients that appear in various equations, we introduce the spaces

$$E^p((0, \epsilon_0) \times (T_1, T_2) \times U, V), \quad p \in \mathbb{Z}_{\geq 0},$$

which are defined to be the set of V-valued maps $f(\epsilon, t, \xi)$ that are smooth on the open set $(0, \epsilon_0) \times (T_1, T_2) \times U$, where $U \subset \mathbb{T}^n \times \mathbb{R}^N$ is open, and for which there exist constants $C_{k,\ell} > 0$, $(k, \ell) \in \{0, 1, \ldots, p\} \times \mathbb{Z}_{\geq 0}$, such that

$$|\partial_t{}^k D_\xi^\ell f(\epsilon, t, \xi)| \leq C_{k,\ell}, \quad \forall\, (\epsilon, t, \xi) \in (0, \epsilon_0) \times (T_1, T_2) \times U.$$

If $V = \mathbb{R}$ or V is clear from context, we will drop the V and simply write $E^p((0, \epsilon_0) \times (T_1, T_2) \times U)$. Moreover, we will use the notation $E^p((T_1, T_2) \times U, V)$ to denote the subspace of ϵ-independent maps. Given $f \in E^p((0, \epsilon_0) \times (T_1, T_2) \times U, V)$, we note, by uniform continuity, that the limit $f_0(t, \xi) := \lim_{\epsilon \searrow 0} f(\epsilon, t, \xi)$ exists and defines an element of $E^p((T_1, T_2) \times U, V)$.

Constants

We employ that standard notation

$$a \lesssim b$$

for inequalities of the form

$$a \leq Cb$$

in situations where the precise value or dependence on other quantities of the constant C is not required. On the other hand, when the dependence of the constant on other inequalities needs to be specified, for example if the constant depends on the norms $\|u\|_{L^\infty}$ and $\|v\|_{L^\infty}$, we use the notation

$$C = C(\|u\|_{L^\infty}, \|v\|_{L^\infty}).$$

Constants of this type will always be non-negative, non-decreasing, continuous functions of their arguments, and in general, C will be used to denote constants that may change from line to line. However, when we want to isolate a particular constant for use later on, we will label the constant with a subscript, e.g. C_1, C_2, C_3, etc.

Remainder terms

In order to simplify the handling of remainder terms whose exact form is not important, we will use, unless otherwise stated, upper case calligraphic letters, e.g. $\mathcal{S}(\epsilon,t,x,\xi)$, $\mathcal{T}(\epsilon,t,x,\xi)$ and $\mathcal{U}(\epsilon,t,x,\xi)$, to denote vector valued maps that are elements of the space $E^0\big((0,\epsilon_0)\times(0,2)\times\mathbb{T}^n\times B_R(\mathbb{R}^N)\big)$, and upper case letters in typewriter font, e.g. $\mathtt{S}(\epsilon,t,x,\xi)$, $\mathtt{T}(\epsilon,t,x,\xi)$ and $\mathtt{U}(\epsilon,t,x,\xi)$, to denote vector valued maps that are elements of the space $E^1\big((0,\epsilon_0)\times(0,2)\times\mathbb{T}^n\times B_R(\mathbb{R}^N)\big)$.

2.1.2 Conformal Einstein-Euler equations

The method we use to establish the existence of ϵ-dependent families of solutions to the Einstein-Euler equations that exist globally to the future is based on the one developed in [65]. Following [65], we introduce the conformal metric

$$\bar{g}^{\mu\nu}=e^{2\Psi}\tilde{g}^{\mu\nu} \tag{2.1.12}$$

and the conformal four velocity

$$\bar{v}^{\mu}=e^{\Psi}\tilde{v}^{\mu}. \tag{2.1.13}$$

Under this change of variables, the Einstein equation (2.1.1) transforms as

$$\bar{G}^{\mu\nu}=\bar{T}^{\mu\nu}:=e^{4\Psi}\tilde{T}^{\mu\nu}-e^{2\Psi}\Lambda\bar{g}^{\mu\nu}+2(\bar{\nabla}^{\mu}\bar{\nabla}^{\nu}\Psi-\bar{\nabla}^{\mu}\Psi\bar{\nabla}^{\nu}\Psi)-(2\bar{\Box}\Psi+|\bar{\nabla}\Psi|^2_{\bar{g}})\bar{g}^{\mu\nu}, \tag{2.1.14}$$

where $\bar{\Box}=\bar{\nabla}^{\mu}\bar{\nabla}_{\mu}$, $|\bar{\nabla}\Psi|^2_{\bar{g}}=\bar{g}^{\mu\nu}\bar{\nabla}_{\mu}\Psi\bar{\nabla}_{\nu}\Psi$, and here and in the following, unless otherwise specified, we raise and lower all coordinate tensor indices using the conformal metric. Contracting the free indices of (2.1.14) gives

$$\bar{R}=4\Lambda-\bar{T},$$

where $\bar{T}=\bar{g}_{\mu\nu}\bar{T}^{\mu\nu}$, which we can use to write (2.1.14) as

$$-2\bar{R}^{\mu\nu}=-4\bar{\nabla}^{\mu}\bar{\nabla}^{\nu}\Psi+4\bar{\nabla}^{\mu}\Psi\bar{\nabla}^{\nu}\Psi-2\left[\bar{\Box}\Psi+2|\bar{\nabla}\Psi|^2+\left(\frac{1-\epsilon^2K}{2}\bar{\rho}+\Lambda\right)e^{2\Psi}\right]\bar{g}^{\mu\nu}-2e^{2\Psi}(1+\epsilon^2K)\bar{\rho}\bar{v}^{\mu}\bar{v}^{\nu}. \tag{2.1.15}$$

We will refer to these equations as the *conformal Einstein equations*.

Letting $\tilde{\Gamma}^{\gamma}_{\mu\nu}$ and $\bar{\Gamma}^{\gamma}_{\mu\nu}$ denote the Christoffel symbols of the metrics $\tilde{g}_{\mu\nu}$ and $\bar{g}_{\mu\nu}$, respectively, the difference $\tilde{\Gamma}^{\gamma}_{\mu\nu}-\bar{\Gamma}^{\gamma}_{\mu\nu}$ is readily calculated to be

$$\tilde{\Gamma}^{\gamma}_{\mu\nu}-\bar{\Gamma}^{\gamma}_{\mu\nu}=\bar{g}^{\gamma\alpha}\big(\bar{g}_{\mu\alpha}\bar{\nabla}_{\nu}\Psi+\bar{g}_{\nu\alpha}\bar{\nabla}_{\mu}\Psi-\bar{g}_{\mu\nu}\bar{\nabla}_{\alpha}\Psi\big).$$

Using this, we can express the Euler equations (2.1.2) as

$$\bar{\nabla}_{\mu}\tilde{T}^{\mu\nu}=-6\tilde{T}^{\mu\nu}\nabla_{\mu}\Psi+\bar{g}_{\alpha\beta}\tilde{T}^{\alpha\beta}\bar{g}^{\mu\nu}\bar{\nabla}_{\mu}\Psi, \tag{2.1.16}$$

which we refer to as the *conformal Euler equations*.

Remark 2.1.5. Due to our choice of time orientation, the conformal fluid four-velocity $\bar{v}^{\mu}$, which we assume is future oriented, satisfies

$$\bar{v}^0<0.$$

We also note that $\bar{v}^{\mu}$ is normalized by

$$\bar{v}^{\mu}\bar{v}_{\mu}=-1, \tag{2.1.17}$$

which is a direct consequence of (2.1.3), (2.1.12) and (2.1.13).

2.1.3 Conformal factor

Following [66], we choose

$$\Psi = -\ln t \tag{2.1.18}$$

for the conformal factor, and for later use, we introduce the background metric

$$\bar{h} = -\frac{3}{\Lambda}dtdt + E^2(t)\delta_{ij}d\bar{x}^i d\bar{x}^j, \tag{2.1.19}$$

where

$$E(t) = a(t)t, \tag{2.1.20}$$

which is conformally related to the FLRW metric (2.1.4). Using (2.1.7), we observe that $E(t)$ satisfies

$$\partial_t E(t) = \frac{1}{t}E(t)\Omega(t), \tag{2.1.21}$$

where $\Omega(t)$ is defined by

$$\Omega(t) = 1 - \sqrt{\frac{3}{\Lambda}}\sqrt{\frac{\Lambda}{3} + \frac{\rho_H(t)}{3}}. \tag{2.1.22}$$

A short calculation then shows that the non-vanishing Christoffel symbols of the background metric (2.1.19) are given by

$$\bar{\gamma}^0_{ij} = \frac{\Lambda}{3t}E^2\Omega\delta_{ij} \quad \text{and} \quad \bar{\gamma}^i_{j0} = \frac{1}{t}\Omega\delta^i_j, \tag{2.1.23}$$

from which we compute

$$\bar{\gamma}^\sigma := \bar{h}^{\mu\nu}\bar{\gamma}^\sigma_{\mu\nu} = \frac{\Lambda}{t}\Omega\delta^\sigma_0. \tag{2.1.24}$$

2.1.4 Wave gauge

For the hyperbolic reduction of the conformal Einstein equations, we use the *wave gauge* from [66] that is defined by

$$\bar{Z}^\mu = 0, \tag{2.1.25}$$

where

$$\bar{Z}^\mu = \bar{X}^\mu + \bar{Y}^\mu \tag{2.1.26}$$

with

$$\bar{X}^\mu := \bar{\Gamma}^\mu - \bar{\gamma}^\mu = -\bar{\partial}_\nu\bar{g}^{\mu\nu} + \frac{1}{2}\bar{g}^{\mu\sigma}\bar{g}_{\alpha\beta}\bar{\partial}_\sigma\bar{g}^{\alpha\beta} - \frac{\Lambda}{t}\Omega\delta^\mu_0 \qquad \left(\bar{\Gamma}^\mu = \bar{g}^{\sigma\nu}\bar{\Gamma}^\mu_{\sigma\nu}\right) \tag{2.1.27}$$

and

$$\bar{Y}^\mu := -2\bar{\nabla}^\mu\Psi + \frac{2\Lambda}{3t}\delta^\mu_0 = -2(\bar{g}^{\mu\nu} - \bar{h}^{\mu\nu})\bar{\nabla}_\nu\Psi. \tag{2.1.28}$$

2.1.5 Variables

To obtain variables that are simultaneously suitable for establishing global existence and taking Newtonian limits, we switch to Newtonian coordinates $(x^\mu) = (t, x^i)$ and employ the following rescaled version of the variables[3] introduced in [66]:

$$u^{0\mu} = \frac{1}{\epsilon}\frac{\underline{\bar{g}^{0\mu}} - \bar{\eta}^{0\mu}}{2t}, \tag{2.1.29}$$

$$u_0^{0\mu} = \frac{1}{\epsilon}\left(\underline{\bar{\partial}_0\bar{g}^{0\mu}} - \frac{3(\underline{\bar{g}^{0\mu}} - \bar{\eta}^{0\mu})}{2t}\right), \tag{2.1.30}$$

$$u_i^{0\mu} = \frac{1}{\epsilon}\underline{\bar{\partial}_i\bar{g}^{0\mu}}, \tag{2.1.31}$$

$$u^{ij}(t,x) = \frac{1}{\epsilon}\big(\underline{\bar{\mathfrak{g}}^{ij}} - \bar{\eta}^{ij}\big), \tag{2.1.32}$$

$$u_\mu^{ij} = \frac{1}{\epsilon}\underline{\bar{\partial}_\mu\bar{\mathfrak{g}}^{ij}}, \tag{2.1.33}$$

$$u = \frac{1}{\epsilon}\underline{\bar{\mathfrak{q}}}, \tag{2.1.34}$$

$$u_\mu = \frac{1}{\epsilon}\underline{\bar{\partial}_\mu\bar{\mathfrak{q}}}, \tag{2.1.35}$$

$$z_i = \frac{1}{\epsilon}\underline{\bar{v}_i}, \tag{2.1.36}$$

$$\zeta = \frac{1}{1+\epsilon^2 K}\ln\big(t^{-3(1+\epsilon^2K)}\underline{\bar{\rho}}\big), \tag{2.1.37}$$

and

$$\delta\zeta = \zeta - \zeta_H, \tag{2.1.38}$$

where

$$\bar{\mathfrak{g}}^{ij} = \alpha^{-1}\bar{g}^{ij}, \qquad \alpha := (\det\check{g}_{ij})^{-\frac{1}{3}} = (\det\bar{g}^{kl})^{\frac{1}{3}}, \qquad \check{g}_{ij} = (\bar{g}^{ij})^{-1}, \tag{2.1.39}$$

$$\bar{\mathfrak{q}} = \bar{g}^{00} - \bar{\eta}^{00} - \frac{\Lambda}{3}\ln\alpha - \frac{2\Lambda}{3}\ln E, \tag{2.1.40}$$

$$\bar{\eta}^{\mu\nu} = -\frac{\Lambda}{3}\delta_0^\mu\delta_0^\nu + \delta_i^\mu\delta_j^\nu\delta^{ij}, \tag{2.1.41}$$

and

$$\zeta_H(t) = \frac{1}{1+\epsilon^2K}\ln\big(t^{-3(1+\epsilon^2K)}\rho_H(t)\big). \tag{2.1.42}$$

As we show below in §2.2.1, ζ_H is given by the explicit formula

$$\zeta_H(t) = \zeta_H(1) - \frac{2}{1+\epsilon^2K}\ln\left(\frac{C_0 - t^{3(1+\epsilon^2K)}}{C_0 - 1}\right), \tag{2.1.43}$$

where the constants C_0 and $\zeta_H(1)$ are defined by

$$C_0 = \frac{\sqrt{\Lambda + \rho_H(1)} + \sqrt{\Lambda}}{\sqrt{\Lambda + \rho_H(1)} - \sqrt{\Lambda}} > 1 \tag{2.1.44}$$

and

$$\zeta_H(1) = \frac{1}{1+\epsilon^2K}\ln\rho_H(1), \tag{2.1.45}$$

respectively. Letting

$$\mathring{\zeta}_H = \lim_{\epsilon\searrow 0}\zeta_H \tag{2.1.46}$$

[3]We emphasize that in the subsequent sections, we will only focus on the case that the density ρ is strictly positive. This will be guaranteed later by choosing the perturbations around the FLRW solutions small enough. In this situation, the Euler system can not become degenerate.

denote the Newtonian limit of ζ_H, it is clear from the formula (2.1.43) that

$$\mathring{\zeta}_H(t) = \ln \rho_H(1) - 2\ln\left(\frac{C_0 - t^3}{C_0 - 1}\right). \tag{2.1.47}$$

For later use, we also define

$$z^i = \frac{1}{\epsilon}\underline{\bar{v}}^i. \tag{2.1.48}$$

Remark 2.1.6. It is important to emphasize that the above variables are defined on the ϵ-independent manifold $M = (0,1] \times \mathbb{T}^3$. Effectively, we are treating components of the geometric quantities with respect to the relativistic coordinates as scalars defined on M_ϵ and we are pulling them back as scalars to M by transforming to Newtonian coordinates. This process is necessary to obtain variables that have a well defined Newtonian limit. We stress that for any fixed $\epsilon > 0$, the gravitational and matter fields $\{\bar{g}^{\mu\nu}, \bar{v}^\mu, \bar{\rho}\}$ on M_ϵ are completely determined by the fields $\{u^{0\mu}, u^{ij}, u, z_i, \zeta\}$ on M.

2.1.6 Conformal Poisson-Euler equations

The $\epsilon \searrow 0$ limit of the conformal Einstein-Euler equations on M are the *conformal cosmological Poisson-Euler equations*, which are defined by

$$\partial_t\mathring{\rho} + \sqrt{\frac{3}{\Lambda}}\partial_j\left(\mathring{\rho}\mathring{z}^j\right) = \frac{3(1-\mathring{\Omega})}{t}\mathring{\rho}, \tag{2.1.49}$$

$$\sqrt{\frac{\Lambda}{3}}\mathring{\rho}\partial_t\mathring{z}^j + K\partial^j\mathring{\rho} + \mathring{\rho}\mathring{z}^i\partial_i\mathring{z}^j = \sqrt{\frac{\Lambda}{3}}\frac{1}{t}\mathring{\rho}\mathring{z}^j - \frac{1}{2}\frac{3}{\Lambda}\mathring{\rho}\partial^j\mathring{\Phi} \quad \left(\partial^j := \frac{\delta^{ji}}{\mathring{E}^2}\partial_i\right), \tag{2.1.50}$$

$$\Delta\mathring{\Phi} = \frac{\Lambda}{3}\frac{\mathring{E}^2}{t^2}\Pi\mathring{\rho} \qquad (\Delta := \delta^{ij}\partial_i\partial_j), \tag{2.1.51}$$

where Π is the projection operator defined by

$$\Pi u = u - \langle 1, u\rangle, \tag{2.1.52}$$

for $u \in L^2(\mathbb{T}^3)$,

$$\mathring{E}(t) = \left(\frac{C_0 - t^3}{C_0 - 1}\right)^{\frac{2}{3}} \tag{2.1.53}$$

and

$$\mathring{\Omega}(t) = \frac{2t^3}{t^3 - C_0}, \tag{2.1.54}$$

with C_0 given by (2.1.44).

It will be important for our analysis to introduce the modified density variable $\mathring{\zeta}$ defined by

$$\mathring{\zeta} = \ln(t^{-3}\mathring{\rho}),$$

which is the non-relativistic version of the variable ζ introduced above, see (2.1.37). A short calculation then shows that the conformal cosmological Poisson-Euler equations can be expressed in terms of this modified density as follows:

$$\partial_t\mathring{\zeta} + \sqrt{\frac{3}{\Lambda}}\left(\mathring{z}^j\partial_j\mathring{\zeta} + \partial_j\mathring{z}^j\right) = -\frac{3\mathring{\Omega}}{t}, \tag{2.1.55}$$

$$\sqrt{\frac{\Lambda}{3}}\partial_t\mathring{z}^j + \mathring{z}^i\partial_i\mathring{z}^j + K\partial^j\mathring{\zeta} = \sqrt{\frac{\Lambda}{3}}\frac{1}{t}\mathring{z}^j - \frac{1}{2}\frac{3}{\Lambda}\partial^j\mathring{\Phi}, \tag{2.1.56}$$

$$\Delta\mathring{\Phi} = \frac{\Lambda}{3}t\mathring{E}^2\Pi e^{\mathring{\zeta}}. \tag{2.1.57}$$

2.1.7 Main Theorem

We are in the position to state the main theorem of the article. The proof is given in §2.7.

Theorem 2.1.7. *Suppose $s \in \mathbb{Z}_{\geq 3}$, $0 < K \leq \frac{1}{3}$, $\Lambda > 0$, $\rho_H(1) > 0$, $r > 0$ and the free initial data $\{\breve{\mathfrak{u}}^{ij}, \breve{\mathfrak{u}}^{ij}_0, \breve{\rho}_0, \breve{\nu}^i\}$ is chosen so that $\breve{\mathfrak{u}}^{ij} \in B_r(H^{s+1}(\mathbb{T}^3, \mathbb{S}_3))$, $\breve{\mathfrak{u}}^{ij}_0 \in H^s(\mathbb{T}^3, \mathbb{S}_3)$, $\breve{\rho}_0 \in B_r(\bar{H}^s(\mathbb{T}^3))$, $\breve{\nu}^i \in \bar{H}^s(\mathbb{T}^3, \mathbb{R}^3)$. Then for $r > 0$ chosen small enough, there exists a constant $\epsilon_0 > 0$ and maps $\breve{u}^{\mu\nu} \in C^\omega\big(X^s_{\epsilon_0,r}(\mathbb{T}^3), H^{s+1}(\mathbb{T}^3, \mathbb{S}_4)\big)$, $\breve{u} \in C^\omega\big(X^s_{\epsilon_0,r}(\mathbb{T}^3), H^{s+1}(\mathbb{T}^3)\big)$, $\breve{u}^{\mu\nu}_0 \in C^\omega\big(X^s_{\epsilon_0,r}(\mathbb{T}^3), H^s(\mathbb{T}^3, \mathbb{S}_4)\big)$, $\breve{u}_0 \in C^\omega\big(X^s_{\epsilon_0,r}(\mathbb{T}^3), H^s(\mathbb{T}^3)\big)$, $\breve{z} = (\breve{z}_i) \in C^\omega\big(X^s_{\epsilon_0,r}(\mathbb{T}^3), H^s(\mathbb{T}^3, \mathbb{R}^3)\big)$, and $\delta\breve{\zeta} \in C^\omega\big(X^s_{\epsilon_0,r}(\mathbb{T}^3), H^s(\mathbb{T}^3)\big)$, such that*[4]

$$
\begin{aligned}
u^{\mu 0}|_{t=1} &:= \breve{u}^{\mu 0}(\epsilon, \breve{\mathfrak{u}}^{kl}, \breve{\mathfrak{u}}^{kl}_0, \breve{\rho}_0, \breve{\nu}^k) = \epsilon\frac{\Lambda}{6}\Delta^{-1}\breve{\rho}_0\delta^\mu_0 + \mathrm{O}(\epsilon^2), \\
u^{ij}|_{t=1} &:= \breve{u}^{ij}(\epsilon, \breve{\mathfrak{u}}^{kl}, \breve{\mathfrak{u}}^{kl}_0, \breve{\rho}_0, \breve{\nu}^k) = \epsilon^2\left(\breve{\mathfrak{u}}^{ij} - \frac{1}{3}\breve{\mathfrak{u}}^{pq}\delta_{pq}\delta^{ij}\right) + \mathrm{O}(\epsilon^3), \\
u|_{t=1} &:= \breve{u}(\epsilon, \breve{\mathfrak{u}}^{kl}, \breve{\mathfrak{u}}^{kl}_0, \breve{\rho}_0, \breve{\nu}^k) = \epsilon^2\frac{2\Lambda}{9}\breve{\mathfrak{u}}^{ij}\delta_{ij} + \mathrm{O}(\epsilon^3), \\
z_i|_{t=1} &:= \breve{z}_i(\epsilon, \breve{\mathfrak{u}}^{kl}, \breve{\mathfrak{u}}^{kl}_0, \breve{\rho}_0, \breve{\nu}^k) = \frac{\breve{\nu}^j\delta_{ij}}{\rho_H(1) + \breve{\rho}_0} + \mathrm{O}(\epsilon), \\
\delta\zeta|_{t=1} &:= \delta\breve{\zeta}(\epsilon, \breve{\mathfrak{u}}^{kl}, \breve{\mathfrak{u}}^{kl}_0, \breve{\rho}_0, \breve{\nu}^k) = \ln\left(1 + \frac{\breve{\rho}_0}{\rho_H(1)}\right) + \mathrm{O}(\epsilon^2), \\
u_0|_{t=1} &:= \breve{u}_0(\epsilon, \breve{\mathfrak{u}}^{ij}, \breve{\mathfrak{u}}^{ij}_0, \breve{\rho}_0, \breve{\nu}^i) = \mathrm{O}(\epsilon)
\end{aligned}
$$

and

$$u^{\mu\nu}_0|_{t=1} := \breve{u}^{\mu\nu}_0(\epsilon, \breve{\mathfrak{u}}^{kl}, \breve{\mathfrak{u}}^{kl}_0, \breve{\rho}_0, \breve{\nu}^k) = \mathrm{O}(\epsilon)$$

determine via the formulas (2.1.29), (2.1.30), (2.1.32), (2.1.34), (2.1.36), (2.1.37), and (2.1.38) a solution of the gravitational and gauge constraint equations, see (2.6.3)-(2.6.4) and Remark 2.6.1. Furthermore, there exists a $\sigma > 0$, such that if

$$\|\breve{\mathfrak{u}}^{ij}\|_{H^{s+1}} + \|\breve{\mathfrak{u}}^{ij}_0\|_{H^s} + \|\breve{\rho}_0\|_{H^s} + \|\breve{\nu}^i\|_{H^s} \leq \sigma,$$

then there exist maps

$$
\begin{aligned}
u^{\mu\nu}_\epsilon &\in C^0((0,1], H^s(\mathbb{T}^3, \mathbb{S}_4)) \cap C^1((0,1], H^{s-1}(\mathbb{T}^3, \mathbb{S}_4)), \\
u^{\mu\nu}_{\gamma,\epsilon} &\in C^0((0,1], H^s(\mathbb{T}^3, \mathbb{S}_4)) \cap C^1((0,1], H^{s-1}(\mathbb{T}^3, \mathbb{S}_4)), \\
u_\epsilon &\in C^0((0,1], H^s(\mathbb{T}^3)) \cap C^1((0,1], H^{s-1}((\mathbb{T}^3)), \\
u_{\gamma,\epsilon} &\in C^0((0,1], H^s(\mathbb{T}^3)) \cap C^1((0,1], H^{s-1}((\mathbb{T}^3)), \\
\delta\zeta_\epsilon &\in C^0((0,1], H^s(\mathbb{T}^3)) \cap C^1((0,1], H^{s-1}(\mathbb{T}^3)), \\
z^\epsilon_i &\in C^0((0,1], H^s(\mathbb{T}^3), \mathbb{R}^3)) \cap C^1((0,1], H^{s-1}(\mathbb{T}^3, \mathbb{R}^3)),
\end{aligned}
$$

for $\epsilon \in (0, \epsilon_0)$, and

$$
\begin{aligned}
\mathring{\Phi} &\in C^0((0,1], H^{s+2}(\mathbb{T}^3)) \cap C^1((0,1], H^{s+1}(\mathbb{T}^3)), \\
\delta\mathring{\zeta} &\in C^0((0,1], H^s(\mathbb{T}^3)) \cap C^1((0,1], H^{s-1}(\mathbb{T}^3)), \\
\mathring{z}_i &\in C^0((0,1], H^s(\mathbb{T}^3, \mathbb{R}^3)) \cap C^1((0,1], H^{s-1}(\mathbb{T}^3, \mathbb{R}^3)),
\end{aligned}
$$

such that

(i) $\{u^{\mu\nu}_\epsilon(t,x), u_\epsilon(t,x), \delta\zeta_\epsilon(t,x), z^\epsilon_i(t,x)\}$ determines, via (2.1.12), (2.1.13), (2.1.17), (2.1.29), (2.1.32),

[4] See Lemma 2.6.7 for details.

(2.1.34), (2.1.36), *and* (2.1.37)-(2.1.41), *a* 1*-parameter family of solutions to the Einstein-Euler equations* (2.1.1)-(2.1.2) *in the wave gauge* (2.1.25) *on* M_ϵ,

(ii) $\{\mathring{\Phi}(t,x), \mathring{\zeta}(t,x) := \delta\mathring{\zeta} + \mathring{\zeta}_H, \mathring{z}^i(t,x) := \mathring{E}(t)^{-2}\delta^{ij}\mathring{z}_j(t,x)\}$, *with* $\mathring{\zeta}_H$ *and* $\mathring{E}$ *given by* (2.1.47) *and* (2.1.53), *respectively, solves the conformal cosmological Poisson-Euler equations* (2.1.55)-(2.1.57) *on* M *with initial data* $\mathring{\zeta}|_{t=1} = \ln(\rho_H(1) + \breve{\rho}_0)$ *and* $\mathring{z}^i|_{t=1} = \breve{\nu}^i/(\rho_H(1) + \breve{\rho}_0)$,

(iii) the uniform bounds

$$\|\delta\mathring{\zeta}\|_{L^\infty((0,1],H^s)} + \|\mathring{\Phi}\|_{L^\infty((0,1],H^{s+2})} + \|\mathring{z}_j\|_{L^\infty((0,1],H^s)} + \|\delta\zeta_\epsilon\|_{L^\infty((0,1],H^s)} + \|\mathring{z}^\epsilon_j\|_{L^\infty((0,1],H^s)} \lesssim 1$$

and

$$\|u^{\mu\nu}_\epsilon\|_{L^\infty((0,1],H^s)} + \|u^{\mu\nu}_{\gamma,\epsilon}\|_{L^\infty((0,1],H^s)} + \|u_\epsilon\|_{L^\infty((0,1],H^s)} + \|u_{\gamma,\epsilon}\|_{L^\infty((0,1],H^s)} \lesssim 1,$$

hold for $\epsilon \in (0, \epsilon_0)$,

(iv) and the uniform error estimates

$$\|\delta\zeta_\epsilon - \delta\mathring{\zeta}\|_{L^\infty((0,1],H^{s-1})} + \|z^\epsilon_j - \mathring{z}_j\|_{L^\infty((0,1]\times H^{s-1})} \lesssim \epsilon,$$
$$\|u^{\mu\nu}_{\epsilon,0}\|_{L^\infty((0,1],H^{s-1})} + \|u^{\mu\nu}_{k,\epsilon} - \delta^\mu_0\delta^\nu_0\partial_k\mathring{\Phi}\|_{L^\infty((0,1],H^{s-1})} + \|u^{\mu\nu}_\epsilon\|_{L^\infty((0,1],H^{s-1})} \lesssim \epsilon$$

and

$$\|u_{\gamma,\epsilon}\|_{L^\infty((0,1],H^{s-1})} + \|u_\epsilon\|_{L^\infty((0,1],H^{s-1})} \lesssim \epsilon$$

hold for $\epsilon \in (0, \epsilon_0)$.

2.1.8 Future directions

Although the 1-parameter families of ϵ-dependent solutions to the Einstein-Euler equations from Theorem 2.1.7 do provide a positive answer to the question of the existence of non-homogeneous relativistic cosmological solutions that are globally approximated to the future by solutions of Newtonian gravity, it does not resolve the question for initial data that is relevant to our Universe. This is because these solutions have a characteristic size $\sim \epsilon$ and should be interpreted as cosmological versions of isolated systems [34, 64, 65]. This defect was remedied on short time scales in [65]. There the local-in-time existence of 1-parameter families of ϵ-dependent solutions to the Einstein-Euler equations that converge to solutions of the cosmological Poisson-Euler equations on *cosmological spatial scales* was established.

In Chapter §3, we combine the techniques developed in [65] with a generalization of the ones developed in this article to extend the local-in-time existence result from [65] to a global-in-time result. This resolves the existence question of non-homogeneous relativistic cosmological solutions that are globally approximated to the future on cosmological scales by solutions of Newtonian gravity, at least for initial data that is a small perturbation of FLRW initial data. However, this is far from the end of the story because there are relativistic effects that are important for precision cosmology that are not captured by the Newtonian solutions. To understand these relativistic effects, higher order post-Newtonian (PN) expansions are required starting with the 1/2-PN expansion, which is, by definition, the ϵ order correction to the Newtonian gravity. In particular, it can be shown [67] that the 1-parameter families of solutions must admit a 1/2-PN expansion in order to view them on large scales as a linear perturbation of FLRW solutions. The importance of this result is that it shows it is possible to have rigorous solutions that fit within the standard cosmological paradigm of linear perturbations of FLRW metrics on large scales while, at the same time, are fully non-linear on small scales of order ϵ. Thus the natural next step is to extend the results of Chapter §3 to include the existence of 1-parameter families of ϵ-dependent solutions to the Einstein-Euler equations that admit 1/2-PN expansions globally to the future on cosmological scales. This is work that is currently in progress.

2.1.9 Prior and related work

The future non-linear stability of the FLRW fluid solutions for a linear equation of state $p = K\rho$ was first established under the condition $0 < K < 1/3$ and the assumption of zero fluid vorticity by Rodnianski and Speck in [73] using a generalization of a wave based method developed by Ringström in [71]. Subsequently, it has been shown [28, 35, 53, 77] that this future non-linear stability result remains true for fluids with non-zero vorticity and also for the equation of state parameter values $K = 0$ and $K = 1/3$, which correspond to dust and pure radiation, respectively. It is worth noting that the stability results established in [53] and [28] for $K = 1/3$ and $K = 0$, respectively, rely on Friedrich's conformal method [26, 27], which is completely different from the techniques used in [35, 73, 77] for the parameter values $0 \leq K < 1/3$.

In the Newtonian setting, the global existence to the future of solutions to the cosmological Poisson-Euler equations was established in [11] under a small initial data assumption and for a class of polytropic equations of state.

A new method was introduced in [66] to prove the future non-linear stability of the FLRW fluid solutions that was based on a wave formulation of a conformal version of the Einstein-Euler equations. The global existence results in this article are established using this approach. We also note that this method was recently used to establish the non-linear stability of the FLRW fluid solutions that satisfy the generalized Chaplygin equation of state [49].

2.1.10 Overview

In §2.2, we employ the variables (2.1.29)-(2.1.38) and the wave gauge (2.1.25) to write the conformal Einstein-Euler system, given by (2.1.15) and (2.1.16), as a non-local symmetric hyperbolic system, see (2.2.103), that is jointly singular in ϵ and t.

In §2.3, we state and prove a local-in-time existence and uniqueness result along with a continuation principle for solutions of the reduced conformal Einstein-Euler equations and discuss how solutions to the reduced conformal Einstein-Euler equations determine solutions to the singular system (2.2.103). Similarly, in §2.4, we state and prove a local-in-time existence and uniqueness result and continuation principle for solutions of the conformal cosmological Poisson-Euler equations (2.1.55)-(2.1.57).

We establish in §2.5 uniform a priori estimates for solutions to a class of symmetric hyperbolic equations that are jointly singular in ϵ and t, and include both the formulation (2.2.103) of the conformal Einstein-Euler equations and the $\epsilon \searrow 0$ limit of these equations. We also establish *error estimates*, that is, a priori estimates for the difference between solutions of the singular hyperbolic equation and the corresponding $\epsilon \searrow 0$ limit equation.

In §2.6, we construct ϵ-dependent 1-parameter families of initial data for the reduced conformal Einstein-Euler equations that satisfy the constraint equations on the initial hypersurface $t = 1$ and limit as $\epsilon \searrow 0$ to solutions of the conformal cosmological Poisson-Euler equations.

Using the results from §2.2 to §2.6, we complete the proof of Theorem 2.1.7 in §2.7.

2.2 A singular symmetric hyperbolic formulation of the conformal Einstein-Euler equations

In this section, we employ the variables (2.1.29)-(2.1.38) and the wave gauge (2.1.25) to cast the conformal Einstein-Euler system, given by (2.1.15) and (2.1.16), into a form that is suitable for analyzing the limit $\epsilon \searrow 0$ globally to the future. More specifically, we show that the Einstein-Euler system can be written as a symmetric hyperbolic system that is jointly singular in ϵ and t, and for which the singular terms have a specific structure. Crucially, the ϵ-dependent singular terms are of a form that has been well-studied beginning with the pioneering work of Browning, Klainerman, Kreiss and Majda [12, 40, 41, 45], while the t-dependent singular terms are of the type analyzed in [66].

2.2.1 Analysis of the FLRW solutions

As a first step in the derivation, we find explicit formulas for the functions $\Omega(t)$, $\rho_H(t)$ and $E(t)$ that will be needed to show that the transformed conformal Einstein-Euler systems is of the form analyzed in §2.5. We begin by differentiating (2.1.22) and observe, with the help of (2.1.6), (2.1.20) and (2.1.21), that it satisfies the differential equation

$$-t\partial_t(1-\Omega)+\frac{3}{2}(1+\epsilon^2K)(1-\Omega)^2=\frac{3}{2}(1+\epsilon^2K). \tag{2.2.1}$$

Integrating gives

$$\Omega(t)=\frac{2t^{3(1+\epsilon^2K)}}{t^{3(1+\epsilon^2K)}-C_0}, \tag{2.2.2}$$

where C_0 is as defined above by (2.1.44). Then by (2.1.22), we find that

$$\rho_H(t)=\frac{4C_0\Lambda t^{3(1+\epsilon^2K)}}{(C_0-t^{3(1+\epsilon^2K)})^2}, \tag{2.2.3}$$

which, in turn, shows that $\zeta_H(t)$, as defined by (2.1.42), is given by the formula (2.1.43).

It is clear from the above formulas that Ω, ρ and ζ_H, as functions of (t,ϵ), are in $C^2([0,1]\times[0,\epsilon_0])\cap W^{3,\infty}([0,1]\times[-\epsilon_0,\epsilon_0])$ for any fixed $\epsilon_0>0$. In particular, we can represent $t^{-1}\Omega$ and $\partial_t\Omega$ as

$$\frac{1}{t}\Omega=E^{-1}\partial_tE=t^{2+3\epsilon^2K}\mathtt{Q}_1(t) \qquad\text{and}\qquad \partial_t\Omega=t^{2+3\epsilon^2K}\mathtt{Q}_2(t),$$

respectively, where we are employing the notation from §2.1.1 for the remainder terms $\mathtt{Q}_1$ and $\mathtt{Q}_2$.

Using (2.2.2), we can integrate (2.1.21) to obtain

$$E(t)=\exp\left(\int_1^t\frac{2s^{2+3\epsilon^2K}}{s^{3(1+\epsilon^2K)}-C_0}ds\right)=\left(\frac{C_0-t^{3(1+\epsilon^2K)}}{C_0-1}\right)^{\frac{2}{3(1+\epsilon^2K)}}\geq 1 \tag{2.2.4}$$

for $t\in[0,1]$. From this formula, it is clear that $E\in C^2([0,1]\times[-\epsilon_0,\epsilon_0])\cap W^{3,\infty}([0,1]\times[-\epsilon_0,\epsilon_0])$, and that the Newtonian limit of E, denoted $\mathring{E}$ and defined by

$$\mathring{E}(t)=\lim_{\epsilon\searrow0}E(t),$$

is given by the formula (2.1.53). Similarly, we denote the Newtonian limit of Ω by

$$\mathring{\Omega}(t)=\lim_{\epsilon\searrow0}\Omega(t),$$

which we see from (2.2.2) is given by the formula (2.1.54).

For later use, we observe that E, Ω, ρ_H and ζ_H satisfy

$$-E^{-1}\partial_t^2E+\frac{1}{t}E^{-1}\partial_tE=\frac{1}{2\Lambda t^2}(1+3\epsilon^2K)\rho_H, \tag{2.2.5}$$

$$E^{-1}\partial_t^2E+2E^{-2}(\partial_tE)^2-\frac{5}{t}E^{-1}\partial_tE=\frac{3}{2\Lambda t^2}(1-\epsilon^2K)\rho_H \tag{2.2.6}$$

and

$$\partial_t\zeta_H=-\frac{3}{t}\Omega=-3E^{-1}\partial_tE=-\bar{\gamma}^i_{i0}=-\bar{\gamma}^i_{0i}=t^{2+3\epsilon^2K}\mathtt{Q}_3(t) \tag{2.2.7}$$

as can be verified by a straightforward calculation using the formulas (2.1.43) and (2.2.2)-(2.2.4). By

(2.1.47) and (2.1.54), it is easy to verify

$$\partial_t \mathring{\zeta}_H = -\frac{3}{t}\mathring{\Omega} = \frac{6t^2}{C_0 - t^3}. \tag{2.2.8}$$

We also record the following useful expansions of $t^{1+3\epsilon^2 K}$, $E(\epsilon,t)$ and $\Omega(\epsilon,t)$:

$$t^{1+3\epsilon^2 K} = t + \epsilon^2 \mathcal{X}(\epsilon,t) \quad \text{where} \quad \mathcal{X}(\epsilon,t) = \frac{6K}{\epsilon^2}\int_0^\epsilon \lambda t^{1+3\lambda^2 K}\ln t d\lambda \tag{2.2.9}$$

and

$$E(\epsilon,t) = \mathring{E}(t) + \epsilon \mathtt{E}(\epsilon,t) \quad \text{and} \quad \Omega(\epsilon,t) = \mathring{\Omega}(t) + \epsilon \mathtt{A}(\epsilon,t) \tag{2.2.10}$$

for $(\epsilon,t)\in(0,\epsilon_0)\times(0,1]$, where $\mathcal{X}$, $\mathtt{E}$ and $\mathtt{A}$ are all remainder terms as defined in §2.1.1.

2.2.2 The reduced conformal Einstein equations

The next step in transforming the conformal Einstein-Euler system is to replace the conformal Einstein equations (2.1.15) with the gauge reduced version given by

$$\begin{aligned} -2\bar{R}^{\mu\nu} + 2\bar{\nabla}^{(\mu}\bar{Z}^{\nu)} + \bar{A}^{\mu\nu}_\sigma \bar{Z}^\sigma &= -4\bar{\nabla}^\mu\bar{\nabla}^\nu\Psi + 4\bar{\nabla}^\mu\Psi\bar{\nabla}^\nu\Psi \\ &- 2\left[\bar{\Box}\Psi + 2|\bar{\nabla}\Psi|^2 + \left(\frac{1-\epsilon^2 K}{2}\bar{\rho} + \Lambda\right)e^{2\Psi}\right]\bar{g}^{\mu\nu} - 2e^{2\Psi}(1+\epsilon^2 K)\bar{\rho}\bar{v}^\mu\bar{v}^\nu, \end{aligned} \tag{2.2.11}$$

where

$$\bar{A}^{\mu\nu}_\sigma := -\bar{X}^{(\mu}\delta^{\nu)}_\sigma + \bar{Y}^{(\mu}\delta^{\nu)}_\sigma.$$

We will refer to these equations as the *reduced conformal Einstein equations.*

Proposition 2.2.1. *If the wave gauge (2.1.25) is satisfied, Ψ is chosen as (2.1.18) and $\bar{\gamma}^\nu$ is given by (2.1.24), then the following relations hold:*

$$\begin{aligned} \bar{\nabla}^{(\mu}\bar{\gamma}^{\nu)} &= \bar{g}^{0(\mu}\delta^{\nu)}_0\frac{\Lambda}{t}\left(\partial_t\Omega - \frac{1}{t}\Omega\right) - \frac{\Lambda}{2t}\Omega\partial_t\bar{g}^{\mu\nu}, \\ \bar{\Box}\Psi &= \frac{1}{t^2}\bar{g}^{00} - \frac{1}{t}\bar{Y}^0 + \frac{1}{t}\bar{\gamma}^0, \qquad |\bar{\nabla}\Psi|^2 = \frac{1}{t^2}\bar{g}^{00}, \\ \bar{Y}^\mu\bar{Y}^\nu &= 4\bar{\nabla}^\mu\Psi\bar{\nabla}^\nu\Psi + \frac{8\Lambda}{3t^2}\delta^{(\mu}_0\bar{g}^{\nu)0} + \frac{4\Lambda^2}{9t^2}\delta^\mu_0\delta^\nu_0 \end{aligned}$$

and

$$\bar{\nabla}^{(\mu}\bar{Y}^{\nu)} = -2\bar{\nabla}^\mu\bar{\nabla}^\nu\Psi - \frac{2\Lambda}{3t^2}\bar{g}^{0(\mu}\delta^{\nu)}_0 - \frac{\Lambda}{3t}\bar{\partial}_t\bar{g}^{\mu\nu}.$$

Proof. The proof follows from the formulas (2.1.18), (2.1.24) and (2.1.26)-(2.1.28) via straightforward computation. □

Remark 2.2.2. For the purposes of proving a priori estimates, we can always assume that the wave gauge (2.1.25) holds since this gauge condition is known to propagate for solutions of the reduced Einstein-Euler equations assuming that the gravitational constraint equations and the gauge constraint $\bar{Z}^\mu = 0$ are satisfied on the initial hypersurface. The implication for our strategy of obtaining global solutions to the future by extending local-in-time solutions via a continuation principle through the use of a priori estimates is that we can assume that the wave gauge $\bar{Z}^\mu = 0$ is satisfied, which, in particular, means that we can freely use the relations[5] from Proposition 2.2.1 in the following.

[5]In fact, the only relation from Proposition 2.2.1 that relies on the gauge condition $\bar{Z}^\mu = 0$ being satisfied is $\bar{\Box}\Psi = \frac{1}{t^2}\bar{g}^{00} - \frac{1}{t}\bar{Y}^0 + \frac{1}{t}\bar{\gamma}^0$.

A short computation using the relations from Proposition 2.2.1 then show that the reduced conformal Einstein Equations (2.2.11) can be written as

$$-2\bar{R}^{\mu\nu}+2\bar{\nabla}^{(\mu}\bar{X}^{\nu)}-\bar{X}^{\mu}\bar{X}^{\nu}+\frac{2\Lambda}{t}\Omega\bar{g}^{\mu\nu}=\frac{2\Lambda}{3t}\partial_t\bar{g}^{\mu\nu}-\frac{4\Lambda}{3t^2}\left(\bar{g}^{00}+\frac{\Lambda}{3}\right)\delta^{\mu}_{0}\delta^{\nu}_{0}-\frac{4\Lambda}{3t^2}\bar{g}^{0k}\delta^{(\mu}_{0}\delta^{\nu)}_{k}$$
$$-\frac{2}{t^2}\bar{g}^{\mu\nu}\left(\bar{g}^{00}+\frac{\Lambda}{3}\right)-(1-\epsilon^2K)\frac{\bar{\rho}}{t^2}\bar{g}^{\mu\nu}-2(1+\epsilon^2K)\frac{\bar{\rho}}{t^2}\bar{v}^{\mu}\bar{v}^{\nu}. \quad (2.2.12)$$

Recalling the formula (e.g. see [29, 72])

$$\bar{R}^{\mu\nu}=\frac{1}{2}\bar{g}^{\lambda\sigma}\bar{\partial}_{\lambda}\bar{\partial}_{\sigma}\bar{g}^{\mu\nu}+\bar{\nabla}^{(\mu}\bar{\Gamma}^{\nu)}+\frac{1}{2}(Q^{\mu\nu}-\bar{X}^{\mu}\bar{X}^{\nu}),$$

where

$$Q^{\mu\nu}=\bar{g}^{\lambda\sigma}\bar{\partial}_{\lambda}(\bar{g}^{\alpha\mu}\bar{g}^{\rho\nu})\bar{\partial}_{\sigma}\bar{g}_{\alpha\rho}+2\bar{g}^{\alpha\mu}\bar{\Gamma}^{\eta}_{\lambda\alpha}\bar{g}_{\eta\delta}\bar{g}^{\lambda\gamma}\bar{g}^{\rho\nu}\bar{\Gamma}^{\delta}_{\rho\gamma}+4\bar{\Gamma}^{\lambda}_{\delta\eta}\bar{g}^{\delta\gamma}\bar{g}_{\lambda(\alpha}\bar{\Gamma}^{\eta}_{\rho)\gamma}\bar{g}^{\alpha\mu}\bar{g}^{\rho\nu}+(\bar{\Gamma}^{\mu}-\bar{\gamma}^{\mu})(\bar{\Gamma}^{\nu}-\bar{\gamma}^{\nu}), \quad (2.2.13)$$

we can express (2.2.12) as

$$-\bar{g}^{\lambda\sigma}\bar{\partial}_{\lambda}\bar{\partial}_{\sigma}\bar{g}^{\mu\nu}-2\bar{\nabla}^{(\mu}\bar{\gamma}^{\nu)}-Q^{\mu\nu}+\frac{2\Lambda}{t^2}\Omega\bar{g}^{\mu\nu}=\frac{2\Lambda}{3t}\partial_t\bar{g}^{\mu\nu}-\frac{4\Lambda}{3t^2}\left(\bar{g}^{00}+\frac{\Lambda}{3}\right)\delta^{\mu}_{0}\delta^{\nu}_{0}$$
$$-\frac{4\Lambda}{3t^2}\bar{g}^{0k}\delta^{(\mu}_{0}\delta^{\nu)}_{k}-\frac{2}{t^2}\bar{g}^{\mu\nu}\left(\bar{g}^{00}+\frac{\Lambda}{3}\right)-(1-\epsilon^2K)\frac{\bar{\rho}}{t^2}\bar{g}^{\mu\nu}-2(1+\epsilon^2K)\frac{\bar{\rho}}{t^2}\bar{v}^{\mu}\bar{v}^{\nu}. \quad (2.2.14)$$

By construction, the quadruple $\{\Psi,\bar{h}_{\mu\nu},\rho_H,\bar{v}^{\mu}_{H}\}$, see (2.1.5), (2.1.6), (2.1.18) and (2.1.19), is the conformal representation of a FLRW solution, and as such, it satisfies the conformal Einstein equations (2.1.14) under the replacement $\{\bar{g}_{\mu\nu},\bar{\rho},\bar{v}\}\mapsto\{\bar{h}_{\mu\nu},\rho_H,e^{\Psi}\tilde{v}^{\mu}_{H}\}$. Since $\bar{X}^{\mu}$ and $\bar{Y}^{\mu}$ vanish when $\bar{g}_{\mu\nu}\longmapsto\bar{h}_{\mu\nu}$, it is clear that the conformal Einstein equations (2.1.14) and the reduced conformal Einstein equations (2.2.14) coincide under the replacement $\{\bar{g}_{\mu\nu},\bar{\rho},\bar{v}\}\mapsto\{\bar{h}_{\mu\nu},\rho_H,e^{\Psi}\tilde{v}^{\mu}_{H}\}$, and thus it follows that $\bar{h}_{\mu\nu}$ satisfies

$$-\bar{h}^{00}\bar{\partial}^2_0\bar{h}^{\mu\nu}-2\bar{\nabla}^{(\mu}_{H}\bar{\gamma}^{\nu)}-Q^{\mu\nu}_{H}+\frac{2\Lambda}{t^2}\Omega\bar{h}^{\mu\nu}=\frac{2\Lambda}{3t}\partial_t\bar{h}^{\mu\nu}-(1-\epsilon^2K)\frac{\rho_H}{t^2}\bar{h}^{\mu\nu}-2(1+\epsilon^2K)\frac{\rho_H}{t^2}\frac{\Lambda}{3}\delta^{\mu}_{0}\delta^{\nu}_{0}, \quad (2.2.15)$$

where $\bar{\nabla}_H$ is the Levi-Civita connection of $\bar{h}_{\mu\nu}$,

$$\bar{\nabla}^{(\mu}_{H}\bar{\gamma}^{\nu)}=\bar{h}^{0(\mu}\delta^{\nu)}_{0}\frac{\Lambda}{t}\left(\partial_t\Omega-\frac{1}{t}\Omega\right)-\frac{2\Lambda}{t}\Omega\partial_t\bar{h}^{\mu\nu}$$

and

$$Q^{\mu\nu}_{H}=\bar{h}^{\lambda\sigma}\bar{\partial}_{\lambda}(\bar{h}^{\alpha\mu}\bar{h}^{\rho\nu})\bar{\partial}_{\sigma}\bar{h}_{\alpha\rho}+2\bar{h}^{\alpha\mu}\bar{\gamma}^{\eta}_{\lambda\alpha}\bar{h}_{\eta\delta}\bar{h}^{\lambda\gamma}\bar{h}^{\rho\nu}\bar{\gamma}^{\delta}_{\rho\gamma}+4\bar{\gamma}^{\lambda}_{\delta\eta}\bar{h}^{\delta\gamma}\bar{h}_{\lambda(\alpha}\bar{\gamma}^{\eta}_{\rho)\gamma}\bar{h}^{\alpha\mu}\bar{h}^{\rho\nu}.$$

Using the formulas (2.1.23) for the Christoffel symbols of $\bar{h}_{\mu\nu}$, it is not difficult to verify via a routine calculation that independent components of the equation (2.2.15) agree up to scaling by a constant with the equations (2.2.5)-(2.2.6).

Setting $\nu=0$ and subtracting (2.2.15) from (2.2.14), we obtain the equation

$$-\bar{g}^{\lambda\sigma}\bar{\partial}_{\lambda}\bar{\partial}_{\sigma}(\bar{g}^{\mu0}-\bar{h}^{\mu0})-2(\bar{\nabla}^{(\mu}\bar{\gamma}^{0)}-\bar{\nabla}^{(\mu}_{H}\bar{\gamma}^{0)})-(Q^{\mu0}-Q^{\mu0}_{H})+\frac{2\Lambda}{t^2}\Omega(\bar{g}^{\mu0}-\bar{h}^{\mu0})$$
$$=\frac{2\Lambda}{3t}\partial_t(\bar{g}^{\mu0}-\bar{h}^{\mu0})-\frac{4\Lambda}{3t^2}\left(\bar{g}^{00}+\frac{\Lambda}{3}\right)\delta^{\mu}_{0}\delta^{0}_{0}-\frac{4\Lambda}{3t^2}\bar{g}^{0k}\delta^{(\mu}_{0}\delta^{0)}_{k}-\frac{2}{t^2}\bar{g}^{\mu0}\left(\bar{g}^{00}+\frac{\Lambda}{3}\right)$$
$$-(1-\epsilon^2K)\frac{1}{t^2}\Big[(\bar{\rho}-\rho_H)\bar{g}^{\mu0}+\rho_H(\bar{g}^{\mu0}-\bar{h}^{\mu0})\Big]-2(1+\epsilon^2K)\frac{1}{t^2}\left[(\bar{\rho}-\rho_H)\bar{v}^{\mu}\bar{v}^{0}+\rho_H\left(\bar{v}^{\mu}\bar{v}^{0}-\frac{\Lambda}{3}\delta^{\mu}_{0}\right)\right] \quad (2.2.16)$$

for the difference $\bar{g}^{\mu 0}-\bar{h}^{\mu 0}$. This equation is close to the form that we are seeking. The final step needed to complete the transformation is to introduce a non-local modification which effectively subtracts out the contribution due to the Newtonian potential.

For the spatial components, a more complicated transformation is required to bring those equations into the desired form. The first step is to contract the $\mu = i, \nu = j$ components of (2.2.14) with $\breve{g}_{ij}$, where we recall that $(\breve{g}_{kl}) = (\bar{g}^{kl})^{-1}$. A straightforward calculation, using the identity $\breve{g}_{kl}\bar{\partial}_\mu \bar{g}^{kl} = -3\alpha\bar{\partial}_\mu \alpha^{-1}$ (recall $\alpha = \det(\bar{g}^{kl})$) and (2.2.16) with $\mu = 0$, shows that $\mathfrak{q}$, defined previously by (2.1.40), satisfies the equation

$$\begin{aligned} &-\bar{g}^{\lambda\sigma}\bar{\partial}_\lambda\bar{\partial}_\sigma\mathfrak{q} - 2\Lambda\bar{g}^{00}\frac{1}{t}\left(\partial_t\Omega - \frac{1}{t}\Omega\right) - 2\bar{g}^{\lambda 0}\bar{\Gamma}^0_{\lambda 0}\frac{\Lambda}{t}\Omega + \frac{2\Lambda^2}{9t}\Omega\bar{\Gamma}^k_{i0}\delta^i_k + \frac{2\Lambda^2}{9t}\Omega\bar{g}^{0(i}\bar{\Gamma}^{j)}_{00}\breve{g}_{ij} \\ &\quad - \frac{2\Lambda}{3}\bar{g}^{00}\left(E^{-1}\partial_t^2 E - E^{-2}(\partial_t E)^2\right) - Q + \frac{2\Lambda}{t^2}\Omega\left(\bar{g}^{00} - \frac{\Lambda}{3}\right) = \frac{2\Lambda}{3t}\bar{\partial}_0\mathfrak{q} + \frac{4\Lambda^2}{9t}E^{-1}\partial_t E \\ &\quad - \frac{2}{t^2}\left(\bar{g}^{00} + \frac{\Lambda}{3}\right)^2 - (1-\epsilon^2 K)\frac{\bar{\rho}}{t^2}\left(\bar{g}^{00} - \frac{\Lambda}{3}\right) - 2(1+\epsilon^2 K)\frac{\bar{\rho}}{t^2}\left(\bar{v}^0\bar{v}^0 - \frac{\Lambda}{9}\breve{g}_{ij}\bar{v}^i\bar{v}^j\right), \quad (2.2.17) \end{aligned}$$

where

$$Q = Q^{00} + \frac{\Lambda}{9}\bar{g}^{\lambda\sigma}\bar{\partial}_\lambda\breve{g}_{ij}\bar{\partial}_\sigma\bar{g}^{ij} - \frac{\Lambda}{9}\breve{g}_{ij}Q^{ij}.$$

Under the replacement $\{\bar{g}_{\mu\nu}, \bar{\rho}, \bar{v}\} \mapsto \{\bar{h}_{\mu\nu}, \rho_H, e^\Psi \tilde{v}^\mu_H\}$, equation (2.2.17) becomes

$$\begin{aligned} &-2\Lambda\bar{h}^{00}\frac{1}{t}\left(\partial_t\Omega - \frac{1}{t}\Omega\right) - 2\bar{h}^{\lambda 0}\bar{\gamma}^0_{\lambda 0}\frac{\Lambda}{t}\Omega + \frac{2\Lambda^2}{9t}\Omega\bar{\gamma}^k_{i0}\delta^i_k + \frac{2\Lambda^2}{9t}\Omega\bar{h}^{0(i}\bar{\gamma}^{j)}_{00}\breve{h}_{ij} - \frac{2\Lambda}{3}\bar{h}^{00}\left(E^{-1}\partial_t^2 E - E^{-2}(\partial_t E)^2\right) \\ &\quad - Q_H + \frac{2\Lambda}{t^2}\Omega\left(\bar{h}^{00} - \frac{\Lambda}{3}\right) = \frac{4\Lambda^2}{9t}E^{-1}\partial_t E - (1-\epsilon^2 K)\frac{\rho_H}{t^2}\left(\bar{h}^{00} - \frac{\Lambda}{3}\right) - 2(1+\epsilon^2 K)\frac{\rho_H}{t^2}\frac{\Lambda}{3}, \quad (2.2.18) \end{aligned}$$

where

$$Q_H = Q^{00}_H + \frac{\Lambda}{9}\bar{h}^{\lambda\sigma}\bar{\partial}_\lambda\breve{h}_{ij}\bar{\partial}_\sigma\bar{h}^{ij} - \frac{\Lambda}{9}\breve{h}_{ij}Q^{ij}_H \qquad \text{and} \qquad \breve{h}_{ij} := (\bar{h}^{kl})^{-1} = E^2\delta_{ij},$$

which, for the reasons discussed above, is satisfied by the conformal FRLW solution $\{\Psi, \bar{h}_{\mu\nu}, \rho_H, \bar{v}^\mu_H\}$. Taking the difference between (2.2.17) and (2.2.18) yields the following equation for $\mathfrak{q}$:

$$\begin{aligned} &-\bar{g}^{\lambda\sigma}\bar{\partial}_\lambda\bar{\partial}_\sigma\mathfrak{q} - 2\Lambda(\bar{g}^{00} - \bar{h}^{00})\frac{1}{t}\left(\partial_t\Omega - \frac{1}{t}\Omega\right) - 2(\bar{g}^{\lambda 0}\bar{\Gamma}^0_{\lambda 0} - \bar{h}^{\lambda 0}\bar{\gamma}^0_{\lambda 0})\frac{\Lambda}{t}\Omega + \frac{2\Lambda^2}{9t}\Omega(\bar{\Gamma}^k_{i0} - \bar{\gamma}^k_{i0})\delta^i_k \\ &\quad + \frac{2\Lambda^2}{9t}\Omega(\bar{g}^{0(i}\bar{\Gamma}^{j)}_{00}\breve{g}_{ij} - \bar{h}^{0(i}\bar{\gamma}^{j)}_{00}\breve{h}_{ij}) - \frac{2\Lambda}{3}(\bar{g}^{00} - \bar{h}^{00})\left(E^{-1}\partial_t^2 E - E^{-2}(\partial_t E)^2\right) + \frac{2\Lambda}{t^2}\Omega\left(\bar{g}^{00} - \bar{h}^{00}\right) \\ &\quad - (Q - Q_H) = \frac{2\Lambda}{3t}\bar{\partial}_0\mathfrak{q} - \frac{2}{t^2}\left(\bar{g}^{00} + \frac{\Lambda}{3}\right)^2 - (1-\epsilon^2 K)\frac{1}{t^2}(\bar{\rho} - \rho_H)\left(\bar{g}^{00} - \frac{\Lambda}{3}\right) - (1-\epsilon^2 K)\frac{1}{t^2}\rho_H\left(\bar{g}^{00} + \frac{\Lambda}{3}\right) \\ &\qquad - 2(1+\epsilon^2 K)\frac{1}{t^2}\left[(\bar{\rho} - \rho_H)\left(\bar{v}^0\bar{v}^0 - \frac{\Lambda}{9}\breve{g}_{ij}\bar{v}^i\bar{v}^j\right) + \rho_H\left(\bar{v}^0\bar{v}^0 - \frac{\Lambda}{3} - \frac{\Lambda}{9}\breve{g}_{ij}\bar{v}^i\bar{v}^j\right)\right]. \quad (2.2.19) \end{aligned}$$

Next, denote

$$\mathcal{L}^{ij}_{kl} = \delta^i_k\delta^j_l - \frac{1}{3}\breve{g}_{kl}\bar{g}^{ij},$$

and apply $\frac{1}{\alpha}\mathcal{L}^{ij}_{lm}$ to (2.2.14) with $\mu = l$, $\nu = m$. A calculation using the identities

$$\alpha^{-1}\mathcal{L}^{ij}_{lm}\bar{\partial}_\sigma\bar{g}^{lm} = \bar{\partial}_\sigma\bar{\mathfrak{g}}^{ij} \qquad \text{and} \qquad \mathcal{L}^{ij}_{lm}\bar{g}^{lm} = 0,$$

where we recall that $\bar{\mathfrak{g}}^{ij}$ is defined by (2.1.39), then shows that $\bar{\mathfrak{g}}^{ij}$ satisfies the equation

$$-\bar{g}^{\lambda\sigma}\bar{\partial}_\lambda\bar{\partial}_\sigma(\bar{\mathfrak{g}}^{ij}-\delta^{ij})-\frac{2}{\alpha}\mathcal{L}^{ij}_{lm}\bar{\nabla}^{(l}\bar{\gamma}^{m)}-\tilde{Q}^{ij}=\frac{2\Lambda}{3t}\partial_t(\bar{\mathfrak{g}}^{ij}-\delta^{ij})-\frac{2(1+\epsilon^2K)}{\alpha}\frac{\bar{\rho}}{t^2}\mathcal{L}^{ij}_{lm}\bar{v}^l\bar{v}^m, \tag{2.2.20}$$

where

$$\tilde{Q}^{ij}=-\bar{g}^{\lambda\sigma}\bar{\partial}_\lambda\left(\frac{1}{\alpha}\mathcal{L}^{ij}_{lm}\right)\bar{\partial}_\sigma\bar{g}^{lm}+\frac{1}{\alpha}\mathcal{L}^{ij}_{lm}Q^{lm}.$$

Making the replacement $\{\bar{g}_{\mu\nu},\bar{\rho},\bar{v}\}\mapsto\{\bar{h}_{\mu\nu},\rho_H,e^{\Psi}\tilde{v}^\mu_H\}$, equation (2.2.20) becomes

$$-\frac{2}{\alpha_H}\mathcal{L}^{ij}_{lm,H}\bar{\nabla}^{(l}_H\bar{\gamma}^{m)}-\tilde{Q}^{ij}_H=0, \tag{2.2.21}$$

where

$$\tilde{Q}^{ij}_H=-\bar{h}^{\lambda\sigma}\bar{\partial}_\lambda\left(\frac{1}{\alpha_H}\mathcal{L}^{ij}_{lm,H}\right)\bar{\partial}_\sigma\bar{h}^{lm}+\frac{1}{\alpha_H}\mathcal{L}^{ij}_{lm,H}Q^{lm}_H,\qquad \alpha_H=(\det\check{h}_{ij})^{-\frac{1}{3}}=E^{-2}$$

and

$$\mathcal{L}^{ij}_{kl,H}=\delta^i_k\delta^j_l-\frac{1}{3}\delta_{kl}\delta^{ij}.$$

Subtracting (2.2.20) by (2.2.21) gives

$$\begin{aligned}-\bar{g}^{\lambda\sigma}\bar{\partial}_\lambda\bar{\partial}_\sigma(\bar{\mathfrak{g}}^{ij}-\delta^{ij})-2\left(\frac{1}{\alpha}\mathcal{L}^{ij}_{lm}\bar{\nabla}^{(l}\bar{\gamma}^{m)}-\frac{1}{\alpha_H}\mathcal{L}^{ij}_{lm,H}\bar{\nabla}^{(l}_H\bar{\gamma}^{m)}\right)-(\tilde{Q}^{ij}-\tilde{Q}^{ij}_H)\\ =\frac{2\Lambda}{3t}\partial_t(\bar{\mathfrak{g}}^{ij}-\delta^{ij})-\frac{2(1+\epsilon^2K)}{\alpha}\frac{\bar{\rho}}{t^2}\mathcal{L}^{ij}_{lm}\bar{v}^l\bar{v}^m.\end{aligned} \tag{2.2.22}$$

2.2.3 ϵ-expansions and remainder terms

The next step in the transformation of the reduced conformal Einstein-Euler equations requires us to understand the lowest order ϵ-expansion for a number of quantities. We compute and collect together these expansions in this section. Throughout this section, we work in Newtonian coordinates, and we frequently employ the notation (2.1.11) for evaluation in Newtonian coordinates, and the notation from §2.1.1 for remainder terms.

First, we observe, using (2.1.29), (2.1.34) and (2.1.40), that we can write α as

$$\underline{\alpha}=E^{-2}\exp\left(\epsilon\frac{3}{\Lambda}\left(2tu^{00}-u\right)\right). \tag{2.2.23}$$

Using this, we can write the conformal metric as

$$\underline{\bar{g}^{ij}}=E^{-2}\delta^{ij}+\epsilon\Theta^{ij}, \tag{2.2.24}$$

where

$$\Theta^{ij}=\Theta^{ij}(\epsilon,t,u,u^{\mu\nu}):=\frac{1}{\epsilon}\left(\underline{\alpha}-E^{-2}\right)(\delta^{ij}+\epsilon u^{ij})+E^{-2}u^{ij}, \tag{2.2.25}$$

and Θ^{ij} satisfies $\Theta^{ij}(\epsilon,t,0,0)=0$ and the E^1-regularity properties of a remainder term, see §2.1.1. By the definition of $u^{0\mu}$, see (2.1.29), we have that

$$\underline{\bar{g}^{0\mu}}=\bar{\eta}^{0\mu}+2\epsilon tu^{0\mu} \tag{2.2.26}$$

and, see (2.1.30) and (2.1.31), for the derivatives

$$\underline{\bar{\partial}_0\bar{g}^{0\mu}}=\epsilon(u^{0\mu}_0+3u^{0\mu}),\quad\text{and}\quad\underline{\bar{\partial}_i\bar{g}^{0\mu}}=2t\partial_iu^{0\mu}=\epsilon u^{0\mu}_i. \tag{2.2.27}$$

Additionally, by (2.2.23), we see, with the help of (2.1.21), (2.1.29)-(2.1.31) and (2.1.34)-(2.1.35), that

$$\partial_t\underline{\alpha} = -2\underline{\alpha}\frac{1}{t}\Omega + \epsilon\underline{\alpha}\frac{3}{\Lambda}(3u^{00} + u^{00}_0 - u_0) \quad \text{and} \quad \partial_i\underline{\alpha} = \epsilon^2\frac{3}{\Lambda}\underline{\alpha}(u^{00}_i - u_i).$$

Then differentiating (2.2.24), we find, using the above expression and (2.1.32)-(2.1.33), that

$$\underline{\bar{\partial}_\sigma\bar{g}^{ij}} = \bar{\partial}_\sigma\bar{h}^{ij} - \epsilon\frac{2}{t}\delta^0_\sigma\Omega\Theta^{ij} + \epsilon\underline{\alpha}\left[u^{ij}_\sigma + \frac{3}{\Lambda}(3u^{00}\delta^0_\sigma + u^{00}_\sigma - u_\sigma)(\delta^{ij} + \epsilon u^{ij})\right]. \tag{2.2.28}$$

Since $\breve{g}_{ij}$ is, by definition, the inverse of $\bar{g}^{ij}$, it follows from (2.2.24) and Lemma D.1.2 that we can express $\breve{g}_{ij}$ as

$$\underline{\breve{g}_{ij}} = E^2\delta_{ij} + \epsilon\mathtt{S}_{ij}(\epsilon, t, u, u^{\mu\nu}), \tag{2.2.29}$$

where $\mathtt{S}_{ij}(\epsilon, t, 0, 0) = 0$. From (2.2.24), (2.2.26) and Lemma D.1.2, we then see that

$$\underline{\bar{g}_{\mu\nu}} = \bar{h}_{\mu\nu} + \epsilon\Xi_{\mu\nu}(\epsilon, t, u^{\sigma\gamma}, u), \tag{2.2.30}$$

where $\Xi_{\mu\nu}$ satisfies $\Xi_{\mu\nu}(\epsilon, t, 0, 0) = 0$ and the E^1-regularity properties of a remainder term. Due to the identity

$$\bar{\partial}_\lambda\bar{g}_{\mu\nu} = -\bar{g}_{\mu\sigma}\bar{\partial}_\lambda\bar{g}^{\sigma\gamma}\bar{g}_{\gamma\nu} \tag{2.2.31}$$

we can easily derive from (2.2.28) and (2.2.30) that

$$\underline{\bar{\partial}_\sigma\bar{g}_{\mu\nu}} = \bar{\partial}_\sigma\bar{h}_{\mu\nu} + \epsilon\mathtt{S}_{\mu\nu\sigma}(\epsilon, t, u^{\alpha\beta}, u, u^{\alpha\beta}_\gamma, u_\gamma), \tag{2.2.32}$$

where $\mathtt{S}_{\mu\nu\sigma}(\epsilon, t, 0, 0, 0, 0) = 0$, which in turn, implies that

$$\underline{\bar{\Gamma}^\sigma_{\mu\nu}} - \bar{\gamma}^\sigma_{\mu\nu} = \epsilon\mathtt{I}^\sigma_{\mu\nu}(\epsilon, t, u^{\alpha\beta}, u, u^{\alpha\beta}_\gamma, u_\gamma), \tag{2.2.33}$$

where $\mathtt{I}^\sigma_{\mu\nu}(\epsilon, t, 0, 0, 0, 0) = 0$. Later, we will also need the explicit form of the next order term in the ϵ-expansion for $\bar{\Gamma}^i_{00}$. To compute this, we first observe that the expansions

$$\underline{\bar{\partial}_0\bar{g}_{k0}} = \epsilon\frac{3}{\Lambda}\delta_{ki}E^2[u^{0i}_0 + (3 + 4\Omega)u^{0i}] + \epsilon^2\mathtt{S}_{k00}(\epsilon, t, u^{\alpha\beta}, u, u^{\alpha\beta}_\gamma, u_\gamma) \tag{2.2.34}$$

and

$$\underline{\bar{\partial}_k\bar{g}_{00}} = -\epsilon\left(\frac{3}{\Lambda}\right)^2 u^{00}_k + \epsilon^2\mathtt{S}_{00k}(\epsilon, t, u^{\alpha\beta}, u, u^{\alpha\beta}_\gamma, u_\gamma), \tag{2.2.35}$$

where $\mathtt{S}_{k00}(\epsilon, t, 0, 0, 0, 0) = \mathtt{S}_{00k}(\epsilon, t, 0, 0, 0, 0) = 0$, follow from (2.2.24), (2.2.26), (2.2.28), (2.2.31) and a straightforward calculation. Using (2.2.34) and (2.2.35), it is then not difficult to verify that

$$\underline{\bar{\Gamma}^i_{00}} = \epsilon\frac{3}{\Lambda}[u^{0i}_0 + (3 + 4\Omega)u^{0i}] + \epsilon\frac{1}{2}\left(\frac{3}{\Lambda}\right)^2 E^{-2}\delta^{ik}u^{00}_k + \epsilon^2\mathtt{I}^i_{00}(\epsilon, t, u^{\alpha\beta}, u, u^{\alpha\beta}_\gamma, u_\gamma), \tag{2.2.36}$$

$$\underline{\bar{\Gamma}^i_{i0}} - \underline{\bar{\gamma}^i_{i0}} = \epsilon\Xi_{kj}E^{-2}\frac{\Omega}{t}\delta^{kj} - \epsilon\frac{1}{2}E^2\delta_{kj}\left[-\frac{2}{t}\Omega\Theta^{ij} + E^{-2}\left(u^{ij}_0 + \frac{3}{\Lambda}(3u^{00} + u^{00}_0 - u_0)\delta^{ij}\right)\right] + \epsilon^2\mathtt{I}^i_{i0}(\epsilon, t, u^{\alpha\beta}, u, u^{\alpha\beta}_\gamma, u_\gamma), \tag{2.2.37}$$

where $\mathtt{I}^i_{00}(\epsilon, t, 0, 0, 0, 0) = 0$ and $\mathtt{I}^i_{i0}(\epsilon, t, 0, 0, 0, 0) = 0$.

Continuing on, we observe from (2.1.37) that we can express the proper energy density in terms of ζ by

$$\rho := \underline{\bar{\rho}} = t^{3(1+\epsilon^2K)}e^{(1+\epsilon^2K)\zeta}, \tag{2.2.38}$$

and correspondingly, by (2.1.42),

$$\rho_H = t^{3(1+\epsilon^2 K)} e^{(1+\epsilon^2 K)\zeta_H} \tag{2.2.39}$$

for the FLRW proper energy density. From (2.1.38), (2.2.38) and (2.2.39), it is then clear that we can express the difference between the proper energy densities ρ and ρ_H in terms of $\delta\zeta$ by

$$\delta\rho := \rho - \rho_H = t^{3(1+\epsilon^2 K)} e^{(1+\epsilon^2 K)\zeta_H} \Big(e^{(1+\epsilon^2 K)\delta\zeta} - 1 \Big). \tag{2.2.40}$$

Due to the normalization $\bar{v}^\mu \bar{v}_\mu = -1$, only three components of $\bar{v}_\mu$ are independent. Solving $\bar{v}^\mu \bar{v}_\mu = -1$ for $\bar{v}_0$ in terms of the components $\bar{v}_i$, we obtain

$$\bar{v}_0 = \frac{-\bar{g}^{0i}\bar{v}_i + \sqrt{(\bar{g}^{0i}\bar{v}_i)^2 - \bar{g}^{00}(\bar{g}^{ij}\bar{v}_i\bar{v}_j + 1)}}{\bar{g}^{00}},$$

which, in turn, using the definitions (2.1.29), (2.1.32), (2.1.34), (2.1.36), we can write as

$$\underline{\bar{v}_0} = -\frac{1}{\sqrt{-\underline{\bar{g}^{00}}}} + \epsilon^2 \mathtt{V}_2(\epsilon, t, u, u^{\mu\nu}, z_j), \tag{2.2.41}$$

where $\mathtt{V}_2(\epsilon, t, u, u^{\mu\nu}, 0) = 0$. From this and the definition $\bar{v}^0 = \bar{g}^{0\mu}\bar{v}_\mu$, we get

$$\underline{\bar{v}^0} = \sqrt{-\underline{\bar{g}^{00}}} + \epsilon^2 \mathtt{W}_2(\epsilon, t, u, u^{\mu\nu}, z_j), \tag{2.2.42}$$

where $\mathtt{W}_2(\epsilon, t, u, u^{\mu\nu}, 0) = 0$. We also observe that

$$\underline{\bar{v}^k} = \epsilon(2tu^{0k}\underline{\bar{v}_0} + \underline{\bar{g}^{ik}} z_i) \quad \text{and} \quad z^k = 2tu^{0k}\underline{\bar{v}_0} + \underline{\bar{g}^{ik}} z_i \tag{2.2.43}$$

follow immediately from the definitions (2.1.36) and (2.1.48). For later use, note that z^k can also be written in terms $(\underline{\bar{g}^{\mu\nu}}, z_j)$ by

$$z^i = \underline{\bar{g}^{ij}} z_j + \frac{\underline{\bar{g}^{i0}}}{\underline{\bar{g}^{00}}} \left[-\underline{\bar{g}^{0j}} z_j + \frac{1}{\epsilon}\sqrt{-\underline{\bar{g}^{00}}}\sqrt{1 - \frac{1}{\underline{\bar{g}^{00}}}\epsilon^2(\underline{\bar{g}^{0j}} z_j)^2 + \epsilon^2 \underline{\bar{g}^{jk}} z_j z_k} \right]. \tag{2.2.44}$$

Using the above expansions, we are able to expand $Q^{\mu\nu}$, Q and $\tilde{Q}^{ij}$.

Proposition 2.2.3. *$Q^{\mu\nu}$, Q and $\tilde{Q}^{ij}$ admit the following expansions:*

$$Q^{\mu\nu} - Q^{\mu\nu}_H = \epsilon \mathcal{Q}^{\mu\nu}, \quad Q - Q_H = \epsilon\mathcal{Q}, \quad \text{and} \quad \tilde{Q}^{ij} - \tilde{Q}^{ij}_H = \epsilon\tilde{\mathcal{Q}}^{ij},$$

where

$$\begin{aligned} \mathcal{Q}^{\mu\nu} &= E^{-2}\frac{\Omega}{t}\mathcal{R}^{\mu\nu\gamma}(t)u^{00}_\gamma + \frac{\Omega}{t}\mathcal{R}^{\mu\nu}(t, \mathbf{u}) + \epsilon\mathcal{S}^{\mu\nu}(\epsilon, t, u^{\alpha\beta}, u, u^{\alpha\beta}_\sigma, u_\sigma), \\ \mathcal{Q} &= E^{-2}\frac{\Omega}{t}\mathcal{R}^{\gamma}(t)u^{00}_\gamma + \frac{\Omega}{t}\mathcal{R}(t, \mathbf{u}) + \epsilon\mathcal{S}(\epsilon, t, u^{\alpha\beta}, u, u^{\alpha\beta}_\sigma, u_\sigma), \\ \tilde{\mathcal{Q}}^{ij} &= E^{-2}\frac{\Omega}{t}\tilde{\mathcal{R}}^{ij\gamma}(t)u^{00}_\gamma + \frac{\Omega}{t}\tilde{\mathcal{R}}^{ij}(t, \mathbf{u}) + \epsilon\tilde{\mathcal{S}}^{ij}(\epsilon, t, u^{\alpha\beta}, u, u^{\alpha\beta}_\sigma, u_\sigma), \end{aligned}$$

with[6] $\mathbf{u} = (u^{\alpha\beta}, u, u^{0i}_\sigma, u^{ij}_\sigma, u_\sigma)$*,* $\{\mathcal{R}^{\mu\nu}, \mathcal{R}, \tilde{\mathcal{R}}^{ij}\}$ *linear in* $\mathbf{u}$*,* $\{\mathcal{R}^{\mu\nu\gamma}, \mathcal{R}^{\gamma}, \tilde{\mathcal{R}}^{ij\gamma}\}$ *satisfy*

$$|\partial_t \mathcal{R}^{\mu\nu k}(t)| + |\partial_t \mathcal{R}^{k}(t)| + |\partial_t \tilde{\mathcal{R}}^{ijk}(t)| \lesssim t^2,$$

and the terms $\{\mathcal{S}^{\mu\nu}, \mathcal{S}, \tilde{\mathcal{S}}^{ij}\}$ *vanish for* $(\epsilon, t, u^{\alpha\beta}, u, u^{\alpha\beta}_\sigma, u_\sigma) = (\epsilon, t, 0, 0, 0, 0)$.

[6] Here, in line with our conventions, see §2.1.1, the quantities written with calligraphic letters, e.g. $\mathcal{S}$ and $\mathcal{R}$, denote remainder terms, and consequently also satisfy the properties of remainder terms.

Proof. First, we observe that we can write $Q^{\mu\nu}$ as $Q^{\mu\nu} = Q^{\mu\nu}(\underline{\bar{g}}, \underline{\bar{\partial}\bar{g}})$, where $Q^{\mu\nu}(\underline{\bar{g}}, \underline{\bar{\partial}\bar{g}})$ is quadratic in $\underline{\bar{\partial}\bar{g}} = (\underline{\bar{\partial}_\gamma \bar{g}^{\mu\nu}})$ and analytic in $\underline{\bar{g}} = (\underline{\bar{g}^{\mu\nu}})$ on the region $\det(\underline{\bar{g}}) < 0$. Since $Q^{\mu\nu}_H = Q^{\mu\nu}(\bar{h}, \bar{\partial}\bar{h})$, we can expand $Q^{\mu\nu}(\bar{h} + \epsilon\mathcal{S}, \bar{\partial}\bar{h} + \epsilon\mathcal{T})$ to get

$$Q^{\mu\nu}(\bar{h} + \epsilon\mathcal{S}, \bar{\partial}\bar{h} + \epsilon\mathcal{T}) - Q^{\mu\nu}_H = \epsilon DQ^{\mu\nu}_1(\bar{h}, \bar{\partial}\bar{h}) \cdot \mathcal{S} + \epsilon DQ^{\mu\nu}_2(\bar{h}, \bar{\partial}\bar{h}) \cdot \mathcal{T} + \epsilon^2 \mathcal{G}^{\mu\nu}\big(\epsilon, \bar{h}, \bar{\partial}\bar{h}, \mathcal{S}, \mathcal{T}\big) \quad (2.2.45)$$

where $\mathcal{G}^{\mu\nu}$ is analytic in all variables and vanishes for $(\mathcal{S}, \mathcal{T}) = (0, 0)$, and $DQ^{\mu\nu}_1$ and $DQ^{\mu\nu}_2$ are linear in their second variable. By (2.2.24), (2.2.26), (2.2.27) and (2.2.28), we can choose

$$\mathcal{S} = (\mathcal{S}^{\mu\nu}(\epsilon, t, u, u^{\alpha\beta})) \quad \text{and} \quad \mathcal{T} = (\mathcal{T}^{\mu\nu}_\gamma(\epsilon, t, u, u^{\alpha\beta}, u_\sigma, u^{\alpha\beta}_\sigma))$$

for appropriate remainder terms $\mathcal{S}^{\mu\nu}$ and $\mathcal{T}^{\mu\nu}$, so that

$$\underline{\bar{g}} = \bar{h} + \epsilon\mathcal{S} \quad \text{and} \quad \underline{\bar{\partial}\bar{g}} = \bar{\partial}\bar{h} + \epsilon\mathcal{T}.$$

Using the fact that

$$\bar{h} = \left(-\frac{\Lambda}{3}\delta^\mu_0\delta^\nu_0 + \frac{1}{E^2}\delta^\mu_i\delta^\nu_j\delta^{ij}\right) \quad \text{and} \quad \bar{\partial}\bar{h} = E^{-2}\frac{\Omega}{t}\mathfrak{h},$$

where $\mathfrak{h} = \big(-2\delta^0_\gamma\delta^\mu_i\delta^\nu_j\big)$, we can, using the linearity on the second variable of the derivatives $DQ^{\mu\nu}_\ell$, $\ell = 1, 2$, write (2.2.45) as

$$Q^{\mu\nu} - Q^{\mu\nu}_H = \epsilon E^{-2}\frac{\Omega}{t} DQ^{\mu\nu}_1(\bar{h}, \mathfrak{h}) \cdot \mathcal{S} + \epsilon E^{-2}\frac{\Omega}{t} DQ^{\mu\nu}_2(\bar{h}, \mathfrak{h}) \cdot \mathcal{T} + \epsilon^2\mathcal{G}^{\mu\nu}\big(\epsilon, \bar{h}, \bar{\partial}\bar{h}, \mathcal{S}, \mathcal{T}\big). \qquad (2.2.46)$$

Expanding Θ^{ij}, recall (2.2.25), as

$$\Theta^{ij} = \frac{1}{E^2}\left(\frac{3}{\Lambda}(2tu^{00} - u)\delta^{ij} + u^{ij}\right) + \epsilon\mathcal{A}^{ij}(\epsilon, t, u, u^{\mu\nu}),$$

where $\mathcal{A}^{ij}(\epsilon, t, 0, 0) = 0$, we see from (2.2.24), (2.2.26), (2.2.27) and (2.2.28) that

$$\mathcal{S}^{\mu\nu} = \frac{1}{\epsilon}\big(\underline{\bar{g}^{\mu\nu}} - \bar{h}^{\mu\nu}\big) = 2t\delta^\mu_0\delta^\nu_0 u^{00} + 4t\delta^{(\mu}_0\delta^{\nu)}_j u^{0j} + \delta^\mu_i\delta^\nu_j\frac{1}{E^2}\left(\frac{3}{\Lambda}(2tu^{00} - u)\delta^{ij} + u^{ij}\right) + \epsilon\mathcal{B}^{\mu\nu}(\epsilon, t, u, u^{\alpha\beta}), \qquad (2.2.47)$$

where $\mathcal{B}^{\mu\nu}(\epsilon, t, 0, 0) = 0$, and

$$\begin{aligned}\mathcal{T}^{\mu\nu}_\gamma = \frac{1}{\epsilon}\big(\underline{\bar{\partial}_\gamma\bar{g}^{\mu\nu}} - \bar{\partial}_\gamma\bar{h}^{\mu\nu}\big) &= \delta^\mu_0\delta^\nu_0\delta^0_\gamma(u^{00}_0 + 3u^{00}) + \delta^\mu_0\delta^\nu_0\delta^k_\gamma u^{00}_k + 2\delta^{(\mu}_0\delta^{\nu)}_j\delta^0_\gamma(u^{0j}_0 + 3u^{0j}) + 2\delta^{(\mu}_0\delta^{\nu)}_j\delta^k_\gamma u^{0j}_k \\ &- \frac{2}{t}\Omega\delta^\mu_i\delta^\nu_j\delta^0_\gamma\frac{1}{E^2}\left[\frac{3}{\Lambda}(2tu^{00} - u)\delta^{ij} + u^{ij}\right] + \frac{1}{E^2}\left[u^{ij}_\sigma + \frac{3}{\Lambda}(3u^{00}\delta^0_\sigma + u^{00}_\sigma - u_\sigma)\delta^{ij}\right] + \epsilon\mathcal{C}^{\mu\nu}_\gamma(\epsilon, t, u, u^{\alpha\beta}, u_\sigma, u^{\alpha\beta}_\sigma),\end{aligned} \qquad (2.2.48)$$

where $\mathcal{C}^{\mu\nu}_\gamma(\epsilon, t, 0, 0, 0, 0) = 0$. The stated expansion for $Q^{\mu\mu}$ is then an immediate consequence of (2.2.46), (2.2.47), (2.2.48) and the boundedness and regularity properties of E and Ω, see §2.2.1 for details. The expansions for Q and $\tilde{Q}^{ij}$ can be established in a similar fashion. □

Finally, we collect the last ϵ-expansions that will be needed in the following proposition. The proof follows from the same arguments used to prove Proposition 2.2.3 above.

Proposition 2.2.4. *The following expansions hold:*

$$2(\underline{\bar{\nabla}^{(\mu}\bar{\gamma}^{0)}} - \bar{\nabla}^{(\mu}_H\bar{\gamma}^{0)}) - \frac{2\Lambda}{t^2}\Omega(\underline{\bar{g}^{\mu0}} - \bar{h}^{\mu0}) = \epsilon\mathcal{E}^{\mu0},$$

$$\begin{aligned}&2(\underline{\bar{g}^{00}} - \bar{h}^{00})\frac{\Lambda}{t}\left(\partial_t\Omega - \frac{1}{t}\Omega\right) + 2(\underline{\bar{g}^{\lambda0}\bar{\Gamma}^0_{\lambda0}} - \bar{h}^{\lambda0}\bar{\gamma}^0_{\lambda0})\frac{\Lambda}{t}\Omega - \frac{2\Lambda^2}{9t}\Omega(\underline{\bar{\Gamma}^k_{i0}} - \bar{\gamma}^k_{i0})\delta^i_k \\ &- \frac{2\Lambda^2}{9t}\Omega(\underline{\bar{g}^{0(i}\bar{\Gamma}^{j)}_{00}\breve{g}_{ij}} - \bar{h}^{0(i}\bar{\gamma}^{j)}_{00}\breve{h}_{ij}) + \frac{2\Lambda}{3}(\underline{\bar{g}^{00}} - \bar{h}^{00})\left(E^{-1}\partial^2_t E - E^{-2}(\partial_t E)^2\right) - \frac{2\Lambda}{t^2}\Omega\left(\underline{\bar{g}^{00}} - \bar{h}^{00}\right) = \epsilon\mathcal{E}\end{aligned}$$

and

$$2\left(\underline{|g|^{\frac{1}{3}}\mathcal{L}^{ij}_{lm}\bar{\nabla}^{(l}\bar{\gamma}^{m)}} - |h|^{\frac{1}{3}}\mathcal{L}^{ij}_{lm,H}\bar{\nabla}^{(l}_{H}\bar{\gamma}^{m)}\right) = \epsilon\tilde{\mathcal{E}}^{ij},$$

where

$$\begin{aligned}\mathcal{E}^{\mu 0} &= E^{-2}\frac{\Omega}{t}\mathcal{F}^{\mu 0\gamma}(t)u^{00}_{\gamma} + \frac{\Omega}{t}\mathcal{F}^{\mu 0}(t,\mathbf{u}) + \epsilon\mathcal{S}^{\mu 0}(\epsilon, t, u^{\alpha\beta}, u, u^{\alpha\beta}_{\sigma}, u_{\sigma})\\ \mathcal{E} &= E^{-2}\frac{\Omega}{t}\mathcal{F}^{\gamma}(t)u^{00}_{\gamma} + \frac{\Omega}{t}\mathcal{F}(t,\mathbf{u}) + \epsilon\mathcal{S}(\epsilon, t, u^{\alpha\beta}, u, u^{\alpha\beta}_{\sigma}, u_{\sigma}),\\ \tilde{\mathcal{E}}^{ij} &= E^{-2}\frac{\Omega}{t}\tilde{\mathcal{F}}^{ij\gamma}(t)u_{\gamma} + \frac{\Omega}{t}\tilde{\mathcal{F}}^{ij}(t,\mathbf{u}) + \epsilon\tilde{\mathcal{S}}^{ij}(\epsilon, t, u^{\alpha\beta}, u, u^{\alpha\beta}_{\sigma}, u_{\sigma}),\end{aligned}$$

with $\mathbf{u} = (u^{\alpha\beta}, u, u^{0i}_{\sigma}, u^{ij}_{\sigma}, u_{\sigma})$, $\{\mathcal{F}^{\mu 0\gamma}, \mathcal{F}^{\gamma}, \tilde{\mathcal{F}}^{ij\gamma}\}$ *satisfy*

$$|\partial_t\mathcal{F}^{\mu\nu\gamma}(t)| + |\partial_t\mathcal{F}^{\gamma}(t)| + |\partial_t\tilde{\mathcal{F}}^{ij\gamma}(t)| \lesssim t^2,$$

$\{\mathcal{F}^{\mu 0}, \mathcal{F}, \tilde{\mathcal{F}}^{ij}\}$ *are linear in* $\mathbf{u}$*, and the remainder terms* $\{\mathcal{S}^{\mu 0}, \mathcal{S}, \tilde{\mathcal{S}}^{ij}\}$ *vanish for* $(\epsilon, t, u^{\alpha\beta}, u, u^{\alpha\beta}_{\sigma}, u_{\sigma})$=$(\epsilon, t, 0, 0, 0, 0)$.

2.2.4 Newtonian potential subtraction

Switching to Newtonian coordinates, a straightforward calculation, with the help of Propositions 2.2.3 and 2.2.4, shows that the reduced conformal Einstein equations given by (2.2.16), (2.2.19) and (2.2.22) can be written in first order form using the variables (2.1.29)-(2.1.38) and (2.1.48) as follows:

$$\tilde{B}^0\partial_0\begin{pmatrix}u^{0\mu}_0\\u^{0\mu}_k\\u^{0\mu}\end{pmatrix} + \tilde{B}^k\partial_k\begin{pmatrix}u^{0\mu}_0\\u^{0\mu}_l\\u^{0\mu}\end{pmatrix} + \frac{1}{\epsilon}\tilde{C}^k\partial_k\begin{pmatrix}u^{0\mu}_0\\u^{0\mu}_l\\u^{0\mu}\end{pmatrix} = \frac{1}{t}\tilde{\mathfrak{B}}\mathbb{P}_2\begin{pmatrix}u^{0\mu}_0\\u^{0\mu}_l\\u^{0\mu}\end{pmatrix} + \hat{S}_1, \tag{2.2.49}$$

$$\tilde{B}^0\partial_0\begin{pmatrix}u^{ij}_0\\u^{ij}_k\\u^{ij}\end{pmatrix} + \tilde{B}^k\partial_k\begin{pmatrix}u^{ij}_0\\u^{ij}_l\\u^{ij}\end{pmatrix} + \frac{1}{\epsilon}\tilde{C}^k\partial_k\begin{pmatrix}u^{ij}_0\\u^{ij}_l\\u^{ij}\end{pmatrix} = -\frac{2E^2\underline{\bar{g}^{00}}}{t}\breve{\mathbb{P}}_2\begin{pmatrix}u^{ij}_0\\u^{ij}_l\\u^{ij}\end{pmatrix} + \hat{S}_2, \tag{2.2.50}$$

$$\tilde{B}^0\partial_0\begin{pmatrix}u_0\\u_k\\u\end{pmatrix} + \tilde{B}^k\partial_k\begin{pmatrix}u_0\\u_l\\u\end{pmatrix} + \frac{1}{\epsilon}\tilde{C}^k\partial_k\begin{pmatrix}u_0\\u_l\\u\end{pmatrix} = -\frac{2E^2\underline{\bar{g}^{00}}}{t}\breve{\mathbb{P}}_2\begin{pmatrix}u_0\\u_l\\u\end{pmatrix} + \hat{S}_3, \tag{2.2.51}$$

where

$$\tilde{B}^0 = E^2\begin{pmatrix}-\underline{\bar{g}^{00}} & 0 & 0\\ 0 & \underline{\bar{g}^{kl}} & 0\\ 0 & 0 & -\underline{\bar{g}^{00}}\end{pmatrix}, \qquad \tilde{B}^k = E^2\begin{pmatrix}-4tu^{0k} & -\Theta^{kl} & 0\\ -\Theta^{kl} & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, \tag{2.2.52}$$

$$\tilde{C}^k = \begin{pmatrix}0 & -\delta^{kl} & 0\\ -\delta^{kl} & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, \qquad \tilde{\mathfrak{B}} = E^2\begin{pmatrix}-\underline{\bar{g}^{00}} & 0 & 0\\ 0 & \frac{3}{2}\underline{\bar{g}^{ki}} & 0\\ 0 & 0 & -\underline{\bar{g}^{00}}\end{pmatrix}, \tag{2.2.53}$$

$$\mathbb{P}_2 = \begin{pmatrix}\frac{1}{2} & 0 & \frac{1}{2}\\ 0 & \delta^l_i & 0\\ \frac{1}{2} & 0 & \frac{1}{2}\end{pmatrix}, \qquad \breve{\mathbb{P}}_2 = \begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, \tag{2.2.54}$$

$$\hat{S}_1 = E^2\begin{pmatrix}\mathcal{Q}^{0\mu} + \mathcal{E}^{\mu 0} + 4\epsilon u^{00}u^{0\mu}_0 - 4\epsilon u^{0\mu}u^{00} + 6\epsilon u^{0k}u^{0\mu}_k - 2u^{0\mu}(1-\epsilon^2 K)t^{2+3\epsilon^2 K}e^{(1+\epsilon^2 K)(\zeta_H+\delta\zeta)} + \hat{f}^{0\mu}\\ 0\\ 0\end{pmatrix}, \tag{2.2.55}$$

$$\hat{f}^{0\mu} = -2(1+\epsilon^2K)t^{1+3\epsilon^2K}e^{(1+\epsilon^2K)(\zeta_H+\delta\zeta)}\left(\frac{1}{\epsilon}\left(\underline{v}^0-\sqrt{\frac{\Lambda}{3}}\right)\left(\underline{v}^0+\sqrt{\frac{\Lambda}{3}}\right)\delta^\mu_0+\underline{v}^0z^k\delta^\mu_k\right)$$
$$-\epsilon K\Lambda t^{1+3\epsilon^2K}e^{(1+\epsilon^2K)\zeta_H}(e^{(1+\epsilon^2K)\delta\zeta}-1)\delta^\mu_0-\frac{1}{\epsilon}\frac{\Lambda}{3}\frac{1}{t^2}\delta^\mu_0\delta\rho, \quad (2.2.56)$$

$$\hat{S}_2 = E^2\begin{pmatrix}\tilde{\mathcal{Q}}^{ij}+\tilde{\mathcal{E}}^{ij}+4\epsilon u^{00}u^{ij}_0+\hat{f}^{ij}\\ 0\\ -\underline{\bar{g}}^{00}u^{ij}_0\end{pmatrix}, \quad (2.2.57)$$

$$\hat{S}_3 = E^2\begin{pmatrix}\mathcal{Q}+\mathcal{E}+4\epsilon u^{00}u_0-8\epsilon(u^{00})^2+\hat{f}\\ 0\\ -\underline{\bar{g}}^{00}u_0\end{pmatrix}, \quad (2.2.58)$$

$$\hat{f}^{ij} = -2\epsilon(1+\epsilon^2K)\underline{\alpha}^{-1}\underline{\mathcal{L}^{ij}_{kl}}t^{1+3\epsilon^2K}e^{(1+\epsilon^2K)\zeta}z^kz^l, \quad (2.2.59)$$

and

$$\hat{f} = -\epsilon K\frac{4\Lambda}{3}t^{1+3\epsilon^2K}e^{(1+\epsilon^2K)\zeta_H}\left(e^{(1+\epsilon^2K)\delta\zeta}-1\right)+2\epsilon(1+\epsilon^2K)\frac{\Lambda}{9}\underline{\breve{g}_{ij}}t^{1+3\epsilon^2K}e^{(1+\epsilon^2K)\zeta}z^iz^j$$
$$-2(1-\epsilon^2K)u^{00}t^{2+3\epsilon^2K}e^{(1+\epsilon^2K)(\zeta_H+\delta\zeta)}-2(1+\epsilon^2K)t^{1+3\epsilon^2K}e^{(1+\epsilon^2K)\zeta}\left(\underline{v}^0+\sqrt{\frac{\Lambda}{3}}\right)\frac{1}{\epsilon}\left(\underline{v}^0-\sqrt{\frac{\Lambda}{3}}\right). \quad (2.2.60)$$

At this point, it is important to stress that the equations (2.2.49)-(2.2.51) are completely equivalent to the reduced conformal Einstein equations for $\epsilon > 0$. Moreover, these equations are almost of the form that we need in order to apply the results of §2.5. Since the term $\frac{1}{\epsilon}\left(\underline{v}^0-\sqrt{\frac{\Lambda}{3}}\right)$ is easily seen to be regular in ϵ from the expansion (2.2.42) , the only ϵ-singular term left is $-\frac{1}{\epsilon}\frac{\Lambda}{3}\frac{1}{t^2}E^2\delta\rho\delta^\mu_0$, which can be found in the quantity $\hat{f}^{0\mu}$. Following the method introduced in [59] and then adapted to the cosmological setting in [61], we can remove the singular part of this term while preserving the required structure via the introduction of the shifted variable

$$w^{0\mu}_k = u^{0\mu}_k - \delta^0_0\delta^\mu_0\partial_k\Phi, \quad (2.2.61)$$

where Φ is the potential defined by solving the Poisson equation

$$\triangle\Phi := \frac{\Lambda}{3}\frac{1}{t^2}E^2\Pi\rho^{\frac{1}{1+\epsilon^2K}} = \frac{\Lambda}{3}E^2te^{\zeta_H}\Pi e^{\delta\zeta} \qquad (\Delta = \delta^{ij}\partial_i\partial_j), \quad (2.2.62)$$

which, as we shall show, reduces to the (cosmological) Newtonian gravitational field equations in the limit $\epsilon \searrow 0$. In this sense, we can view (2.2.61) as the subtraction of the gradient of the Newtonian potential from the gravitational field component u^{00}_k.

Using (2.2.62), we can decompose $-\frac{1}{\epsilon}\frac{\Lambda}{3}\frac{1}{t^2}E^2\delta\rho\delta^\mu_0$ as

$$-\frac{1}{\epsilon}\frac{\Lambda}{3}\frac{1}{t^2}E^2\delta\rho\delta^\mu_0 = -\frac{1}{\epsilon}\delta^\mu_0\Delta\Phi-\frac{\Lambda}{3}\delta^\mu_0E^2t^{1+3\epsilon^2K}\phi+\epsilon E^2\mathcal{S}^\mu(\epsilon,t,\delta\zeta), \quad (2.2.63)$$

where

$$\phi := \frac{1}{\epsilon}\left\langle 1, \frac{1}{t^{3(1+\epsilon^2K)}}\delta\rho\right\rangle = \frac{1}{\epsilon}(\mathbb{1}-\Pi)e^{(1+\epsilon^2K)\zeta_H}\left(e^{(1+\epsilon^2K)\delta\zeta}-1\right) \quad (2.2.64)$$

and

$$\mathcal{S}^\mu(\epsilon,t,\delta\zeta)=\frac{\Lambda}{3}\frac{1}{\epsilon^2}\Pi\big[te^{\zeta_H}(e^{\delta\zeta}-1)-t^{1+3\epsilon^2K}e^{(1+\epsilon^2K)\zeta_H}(e^{(1+\epsilon^2K)\delta\zeta}-1)\big]\delta^\mu_0,$$

which obviously satisfies $\mathcal{S}^\mu(\epsilon,t,0)=0$. Although it is not obvious at the moment, ϕ is regular in ϵ, and consequently, with this knowledge, it is clear from (2.2.63) that $-\frac{1}{\epsilon}\delta^\mu_0\Delta\Phi$ is the only ϵ-singular term in $-\frac{1}{\epsilon}\frac{\Lambda}{3}\frac{1}{t^2}E^2\delta\rho$. A straightforward computation using (2.2.62) and (2.2.63) along with the expansions from Propositions 2.2.3 and 2.2.4 then shows that replacing $u^{0\mu}_k$ in (2.2.49) with the shifted variable (2.2.61) removes the ϵ-singular term $-\frac{1}{\epsilon}\delta^\mu_0\Delta\Phi$ and yields the equation

$$\tilde{B}^0\partial_0\begin{pmatrix}u^{0\mu}_0\\ w^{0\mu}_k\\ u^{0\mu}\end{pmatrix}+\tilde{B}^k\partial_k\begin{pmatrix}u^{0\mu}_0\\ w^{0\mu}_l\\ u^{0\mu}\end{pmatrix}+\frac{1}{\epsilon}\tilde{C}^k\partial_k\begin{pmatrix}u^{0\mu}_0\\ w^{0\mu}_l\\ u^{0\mu}\end{pmatrix}=\frac{1}{t}\tilde{\mathfrak{B}}\mathbb{P}_2\begin{pmatrix}u^{0\mu}_0\\ w^{0\mu}_l\\ u^{0\mu}\end{pmatrix}+\tilde{G}_1+\tilde{S}_1, \tag{2.2.65}$$

where

$$\tilde{G}_1=E^2\begin{pmatrix}-E^{-2}\frac{\Omega}{t}\big[\mathcal{D}^{0\mu0}(t)u^{00}_0+\mathcal{D}^{0\mu k}(t)w^{00}_k\big]-\frac{\Omega}{t}\mathcal{D}^{0\mu}(t,\mathbf{u})+4\epsilon u^{00}u^{0\mu}_0-4\epsilon u^{0\mu}u^{00}+f^{0\mu}\\ 0\\ 0\end{pmatrix}, \tag{2.2.66}$$

$$\tilde{S}_1=\begin{pmatrix}-\frac{\Omega}{t}\mathcal{D}^{0\mu k}\partial_k\Phi+\Theta^{kl}\delta^\mu_0\partial_k\partial_l\Phi\\ \frac{3}{2}\frac{1}{t}\delta^\mu_0\underline{\bar{g}^{kl}}\partial_l\Phi-\underline{\bar{g}^{kl}}\delta^\mu_0\partial_0\partial_l\Phi\\ 0\end{pmatrix}+\epsilon\begin{pmatrix}\mathcal{S}^\mu(\epsilon,t,u^{\alpha\beta},u,u^{\alpha\beta}_\sigma,u_\sigma)\\ 0\\ 0\end{pmatrix}, \tag{2.2.67}$$

$$\mathcal{D}^{0\mu\nu}(t)=-\mathcal{R}^{0\mu\nu}(t)-\mathcal{F}^{0\mu\nu}(t),\quad \mathcal{D}^{0\mu}(t,\mathbf{u})=-\mathcal{R}^{0\mu}(t,\mathbf{u})-\mathcal{F}^{0\mu}(t,\mathbf{u}) \tag{2.2.68}$$

and

$$\begin{aligned}f^{0\mu}=&-2(1+\epsilon^2K)t^{1+3\epsilon^2K}e^{(1+\epsilon^2K)(\zeta_H+\delta\zeta)}\left(\frac{1}{\epsilon}\left(\underline{\bar{v}^0}-\sqrt{\frac{\Lambda}{3}}\right)\left(\underline{\bar{v}^0}+\sqrt{\frac{\Lambda}{3}}\right)\delta^\mu_0+\underline{\bar{v}^0}z^k\delta^\mu_k\right)\\&-\Lambda\epsilon Kt^{1+3\epsilon^2K}e^{(1+\epsilon^2K)\zeta_H}(e^{(1+\epsilon^2K)\delta\zeta}-1)\delta^\mu_0-2u^{0\mu}(1-\epsilon^2K)t^{2+3\epsilon^2K}e^{(1+\epsilon^2K)(\zeta_H+\delta\zeta)}\\&-\frac{\Lambda}{3}t^{1+3\epsilon^2K}\phi\delta^\mu_0+\epsilon\mathcal{S}^\mu(\epsilon,t,\delta\zeta).\end{aligned} \tag{2.2.69}$$

With the help of the expansions from Propositions 2.2.3 and 2.2.4, we further decompose $\hat{S}_2$ and $\hat{S}_3$ into a sum of local and nonlocal terms given by

$$\hat{S}_2=\tilde{G}_2+\tilde{S}_2\quad\text{and}\quad\hat{S}_3=\tilde{G}_3+\tilde{S}_3, \tag{2.2.70}$$

where

$$\tilde{G}_2=E^2\begin{pmatrix}-E^{-2}\frac{\Omega}{t}\big[\tilde{\mathcal{D}}^{ij0}(t)u^{00}_0+\tilde{\mathcal{D}}^{ijk}(t)w^{00}_k\big]-\frac{\Omega}{t}\tilde{\mathcal{D}}^{ij}(t,\mathbf{u})+4\epsilon u^{00}u^{ij}_0+\hat{f}^{ij}\\ 0\\ -\underline{\bar{g}^{00}}u^{ij}_0\end{pmatrix}, \tag{2.2.71}$$

$$\tilde{S}_2=\begin{pmatrix}-\frac{\Omega}{t}\tilde{\mathcal{D}}^{ijk}\partial_k\Phi\\ 0\\ 0\end{pmatrix}+\epsilon\begin{pmatrix}\mathcal{S}^{ij}(\epsilon,t,u^{\alpha\beta},u,u^{\alpha\beta}_\sigma,u_\sigma)\\ 0\\ 0\end{pmatrix}, \tag{2.2.72}$$

$$\tilde{G}_3=E^2\begin{pmatrix}-E^{-2}\frac{\Omega}{t}\big[\mathcal{D}^0(t)u^{00}_0+\mathcal{D}^k(t)w^{00}_k\big]-\frac{\Omega}{t}\mathcal{D}(t,\mathbf{u})+4\epsilon u^{00}u_0-8\epsilon(u^{00})^2+\hat{f}\\ 0\\ -\underline{\bar{g}^{00}}u_0\end{pmatrix}, \tag{2.2.73}$$

$$\tilde{S}_3=\begin{pmatrix}-\frac{\Omega}{t}\mathcal{D}^k\partial_k\Phi\\ 0\\ 0\end{pmatrix}+\epsilon\begin{pmatrix}\mathcal{S}(\epsilon,t,u^{\alpha\beta},u,u^{\alpha\beta}_\sigma,u_\sigma)\\ 0\\ 0\end{pmatrix}, \tag{2.2.74}$$

$$\tilde{\mathcal{D}}^{ij\nu}(t) = -\tilde{\mathcal{R}}^{ij\nu}(t) - \tilde{\mathcal{F}}^{ij\nu}(t), \quad \tilde{\mathcal{D}}^{ij}(t,\mathbf{u}) = -\tilde{\mathcal{R}}^{ij}(t,\mathbf{u}) - \tilde{\mathcal{F}}^{ij}(t,\mathbf{u}), \tag{2.2.75}$$

$$\mathcal{D}^{\mu}(t) = -\mathcal{R}^{\mu}(t) - \mathcal{F}^{\mu}(t) \quad \text{and} \quad \mathcal{D}(t,\mathbf{u}) = -\mathcal{R}(t,\mathbf{u}) - \mathcal{F}(t,\mathbf{u}). \tag{2.2.76}$$

Not only is the system of equations (2.2.50), (2.2.51) and (2.2.65) completely equivalent to the reduced conformal Einstein equations for any $\epsilon > 0$, but it is now of the form required to apply the results from §2.5. This completes our transformation of the reduced conformal Einstein equations.

2.2.5 The conformal Euler equations

With the transformation of the reduced conformal Einstein equations complete, we now turn to the problem of transforming the conformal Euler equations. We begin observing that it follows from the computations from [65, §2.2] that conformal Euler equations (2.1.16) can be written in Newtonian coordinates as

$$\bar{B}^0\partial_0\begin{pmatrix}\zeta\\ z^i\end{pmatrix} + \bar{B}^k\partial_k\begin{pmatrix}\zeta\\ z^i\end{pmatrix} = \frac{1}{t}\bar{\mathcal{B}}\hat{\mathbb{P}}_2\begin{pmatrix}\zeta\\ z^i\end{pmatrix} + \bar{S}, \tag{2.2.77}$$

where

$$\bar{B}^0 = \begin{pmatrix} 1 & \epsilon\frac{L^0_i}{\underline{\bar{v}^0}} \\ \epsilon\frac{L^0_j}{\underline{\bar{v}^0}} & K^{-1}M_{ij}\end{pmatrix},$$

$$\bar{B}^k = \begin{pmatrix} \frac{1}{\epsilon}\frac{\underline{\bar{v}^k}}{\underline{\bar{v}^0}} & \frac{L^k_i}{\underline{\bar{v}^0}} \\ \frac{L^k_j}{\underline{\bar{v}^0}} & K^{-1}M_{ij}\frac{1}{\epsilon}\frac{\underline{\bar{v}^k}}{\underline{\bar{v}^0}}\end{pmatrix} = \begin{pmatrix} \frac{1}{\underline{\bar{v}^0}}z^k & \frac{1}{\underline{\bar{v}^0}}\delta^k_i \\ \frac{1}{\underline{\bar{v}^0}}\delta^k_j & K^{-1}\frac{1}{\underline{\bar{v}^0}}M_{ij}z^k\end{pmatrix},$$

$$\bar{\mathfrak{B}} = \begin{pmatrix} 1 & 0 \\ 0 & -K^{-1}(1-3\epsilon^2 K)\frac{\underline{\bar{g}_{ik}}}{\underline{\bar{v}_0}\,\underline{\bar{v}^0}}\end{pmatrix},$$

$$\hat{\mathbb{P}}_2 = \begin{pmatrix} 0 & 0 \\ 0 & \delta^k_j\end{pmatrix},$$

$$\bar{S} = \begin{pmatrix} -L^{\mu}_i\underline{\bar{\Gamma}^i_{\mu\nu}}\,\underline{\bar{v}^{\nu}}\frac{1}{\underline{\bar{v}^0}} \\ -K^{-1}(1-3\epsilon^2 K)\frac{1}{\underline{\bar{v}_0}}\underline{\bar{g}_{0j}} - K^{-1}M_{ij}\underline{\bar{v}^{\mu}}\frac{1}{\epsilon}\underline{\bar{\Gamma}^i_{\mu\nu}}\,\underline{\bar{v}^{\nu}}\frac{1}{\underline{\bar{v}^0}}\end{pmatrix},$$

$$L^{\mu}_i = \delta^{\mu}_i - \frac{\underline{\bar{v}_i}}{\underline{\bar{v}_0}}\delta^{\mu}_0$$

and

$$M_{ij} = \underline{\bar{g}_{ij}} - \frac{\underline{\bar{v}_i}}{\underline{\bar{v}_0}}\underline{\bar{g}_{0j}} - \frac{\underline{\bar{v}_j}}{\underline{\bar{v}_0}}\underline{\bar{g}_{0i}} + \frac{\underline{\bar{g}_{00}}}{(\underline{\bar{v}_0})^2}\underline{\bar{v}_i}\,\underline{\bar{v}_j}.$$

In order to bring (2.2.77) into the required form, we perform a change of variables from z^i to z_j, which are related via a map of the form $z^i = z^i(z_j, \underline{\bar{g}^{\mu\nu}})$, see (2.2.44). Denoting the Jacobian of the transformation by

$$J^{im} := \frac{\partial z^i}{\partial z_m},$$

we observe that

$$\partial_\sigma z^i = J^{im}\partial_\sigma z_m + \delta^0_\sigma\frac{\partial z^i}{\partial\underline{\bar{g}^{\mu\nu}}}\underline{\bar{\partial}_0\bar{g}^{\mu\nu}} + \epsilon\delta^j_\sigma\frac{\partial z^i}{\partial\underline{\bar{g}^{\mu\nu}}}\underline{\bar{\partial}_j\bar{g}^{\mu\nu}}.$$

Multiplying (2.2.77) by the block matrix $\operatorname{diag}(1, J^{jl})$ and changing variables from (ζ, z^i) to $(\delta\zeta, z_j)$, where we recall from (2.1.38) that $\delta\zeta = \zeta - \zeta_H$, we can write the conformal Euler equations (2.2.77) as

$$B^0\partial_0\begin{pmatrix}\delta\zeta\\ z_m\end{pmatrix} + B^k\partial_k\begin{pmatrix}\delta\zeta\\ z_m\end{pmatrix} = \frac{1}{t}\mathfrak{B}\hat{\mathbb{P}}_2\begin{pmatrix}\delta\zeta\\ z_m\end{pmatrix} + \hat{S}, \tag{2.2.78}$$

where

$$B^0 = \begin{pmatrix} 1 & \epsilon \frac{L^0_i}{\underline{\bar{v}^0}} J^{im} \\ \epsilon \frac{L^0_j}{\underline{\bar{v}^0}} J^{jl} & K^{-1} M_{ij} J^{jl} J^{im} \end{pmatrix}, \tag{2.2.79}$$

$$B^k = \begin{pmatrix} \frac{1}{\underline{\bar{v}^0}} z^k & \frac{1}{\underline{\bar{v}^0}} J^{km} \\ \frac{1}{\underline{\bar{v}^0}} J^{kl} & K^{-1} \frac{1}{\underline{\bar{v}^0}} M_{ij} J^{jl} J^{im} z^k \end{pmatrix}, \tag{2.2.80}$$

$$\mathfrak{B} = \begin{pmatrix} 1 & 0 \\ 0 & -K^{-1}(1-3\epsilon^2 K) \frac{1}{\underline{\bar{v}_0 \bar{v}^0}} J^{ml} \end{pmatrix} \tag{2.2.81}$$

and

$$\hat{S} = \begin{pmatrix} -L^0_i \underline{\bar{\Gamma}^i_{00}} - L^\mu_i \underline{\bar{\Gamma}^i_{\mu j}}\, \underline{\bar{v}^j} \frac{1}{\underline{\bar{v}^0}} + (\bar{\gamma}^i_{i0} - \underline{\bar{\Gamma}^i_{i0}}) \\ -K^{-1} J^{jl} M_{ij} \underline{\bar{v}^\mu} \frac{1}{\epsilon} \underline{\bar{\Gamma}^i_{\mu\nu}}\, \underline{\bar{v}^\nu} \frac{1}{\underline{\bar{v}^0}} + \epsilon \frac{L^0_j}{\underline{\bar{v}^0}} J^{jl} \bar{\gamma}^i_{i0} \end{pmatrix} - \begin{pmatrix} \epsilon \frac{L^0_i}{\underline{\bar{v}^0}} \frac{\partial z^i}{\underline{\partial \bar{g}^{\mu\nu}}} \underline{\bar{\partial}_0 \bar{g}^{\mu\nu}} + \epsilon \frac{\delta^k_i}{\underline{\bar{v}^0}} \frac{\partial z^i}{\underline{\partial \bar{g}^{\mu\nu}}} \underline{\bar{\partial}_k \bar{g}^{\mu\nu}} \\ K^{-1} M_{ij} J^{jl} \frac{\partial z^i}{\underline{\partial \bar{g}^{\mu\nu}}} \underline{\bar{\partial}_0 \bar{g}^{\mu\nu}} + \epsilon K^{-1} \bar{M}_{ij} \frac{z^k}{\underline{\bar{v}^0}} J^{jl} \frac{\partial z^i}{\underline{\partial \bar{g}^{\mu\nu}}} \underline{\bar{\partial}_k \bar{g}^{\mu\nu}} \end{pmatrix}.$$

By direct calculation, we see from (2.2.44) and the expansions (2.2.24) and (2.2.26) that

$$J^{ik} = E^{-2}\delta^{ik} + \epsilon \Theta^{ik} + \epsilon^2 \mathtt{S}^{ik}(\epsilon, t, u, u^{\mu\nu}, z_j), \tag{2.2.82}$$

where $\mathtt{S}^{ik}(\epsilon, t, 0, 0, 0) = 0$. Similarly, it is not difficult to see from (2.2.44) and the expansions (2.2.24) and (2.2.26)-(2.2.28) that

$$\delta^0_\sigma \frac{\partial z^i}{\underline{\partial \bar{g}^{\mu\nu}}} \underline{\bar{\partial}_0 \bar{g}^{\mu\nu}} + \epsilon \delta^j_\sigma \frac{\partial z^i}{\underline{\partial \bar{g}^{\mu\nu}}} \underline{\bar{\partial}_j \bar{g}^{\mu\nu}} = -2\delta^0_\sigma \left(E^{-2} \frac{\Omega}{t} z_j \delta^{ij} + \sqrt{\frac{3}{\Lambda}} (u^{0i}_0 + 3u^{0i}) \right) + \epsilon \mathcal{S}^i(\epsilon, t, u, u^{\alpha\beta}, u_\gamma, u^{\alpha\beta}_\gamma, z_j) \tag{2.2.83}$$

and

$$\epsilon \frac{\delta^k_i}{\underline{\bar{v}^0}} \frac{\partial z^i}{\underline{\partial \bar{g}^{\mu\nu}}} \underline{\bar{\partial}_k \bar{g}^{\mu\nu}} = -\epsilon \frac{6}{\Lambda} u^{0i}_k \delta^k_i + \epsilon^2 \mathcal{S}(\epsilon, t, u, u^{\alpha\beta}, u_\gamma, u^{\alpha\beta}_\gamma, z_j), \tag{2.2.84}$$

where $\mathcal{S}^i(\epsilon, t, 0,0,0,0,0) = 0$ and $\mathcal{S}(\epsilon, t, 0,0,0,0,0) = 0$. We further note that the term $-K^{-1} J^{jl} M_{ij} \underline{\bar{v}^\mu} \frac{1}{\epsilon} \underline{\bar{\Gamma}^j_{\mu\nu}}\, \underline{\bar{v}^\nu} \frac{1}{\underline{\bar{v}^0}}$ found in $\hat{S}$ above is not singular in ϵ. This can be seen from the expansions (2.2.24), (2.2.26), (2.2.30) and (2.2.36), which can be used to calculate

$$\begin{aligned} \frac{1}{\epsilon} \underline{\bar{\Gamma}^j_{\mu\nu}}\, \underline{\bar{v}^\mu \bar{v}^\nu} =& 2\underline{\bar{\Gamma}^j_{0i}}\, \underline{\bar{v}^0} z^i + \epsilon \underline{\bar{\Gamma}^j_{ik}} z^i z^k + \frac{1}{\epsilon} \underline{\bar{\Gamma}^j_{00}}\, \underline{\bar{v}^0 \bar{v}^0} \\ =& \sqrt{\frac{\Lambda}{3}} \frac{2\Omega}{t} E^{-2} z_j \delta^{ij} + \left[u^{0i}_0 + (3+4\Omega) u^{0i} \right] + \frac{1}{2} \frac{3}{\Lambda} E^{-2} \delta^{ik} u^{00}_k + \epsilon \mathcal{S}^j(\epsilon, t, u, u^{\alpha\beta}, u_\gamma, u^{\alpha\beta}_\gamma, z_j), \end{aligned} \tag{2.2.85}$$

where $\mathcal{S}^j(\epsilon, t, 0,0,0,0,0) = 0$.

Using the expansions (2.2.82), (2.2.83) and (2.2.85) in conjunction with (2.2.24), (2.2.26), (2.2.30), (2.2.33), (2.2.41), (2.2.42) and (2.2.43), we can expand the matrices $\{B^0, B^k, \mathfrak{B}\}$ and source term S defined above as follows:

$$B^0 = \begin{pmatrix} 1 & 0 \\ 0 & K^{-1} E^{-2} \delta^{lm} \end{pmatrix} + \epsilon \begin{pmatrix} 0 & 0 \\ 0 & K^{-1} \Theta^{lm} \end{pmatrix} + \epsilon^2 \mathtt{S}^0(\epsilon, t, u, u^{\alpha\beta}, u_\gamma, u^{\alpha\beta}_\gamma, z_j), \tag{2.2.86}$$

$$\begin{aligned} B^k =& \sqrt{\frac{3}{\Lambda}} \begin{pmatrix} z^k & E^{-2}\delta^{km} \\ E^{-2}\delta^{kl} & K^{-1} E^{-2} \delta^{lm} z^k \end{pmatrix} + \epsilon \sqrt{\frac{3}{\Lambda}} \begin{pmatrix} \frac{3}{\Lambda} t u^{00} z^k & \Theta^{km} + \frac{3}{\Lambda} t u^{00} E^{-2} \delta^{km} \\ \Theta^{kl} + \frac{3}{\Lambda} t u^{00} E^{-2} \delta^{kl} & K^{-1}(\Theta^{lm} + \frac{3}{\Lambda} t u^{00} E^{-2} \delta^{lm}) z^k \end{pmatrix} \\ &+ \epsilon^2 \mathtt{S}^k(\epsilon, t, u, u^{\alpha\beta}, u_\gamma, u^{\alpha\beta}_\gamma, z_j), \end{aligned} \tag{2.2.87}$$

$$\mathfrak{B} = \begin{pmatrix} 1 & 0 \\ 0 & K^{-1}(1-3\epsilon^2 K) E^{-2} \delta^{lm} \end{pmatrix} + \epsilon \begin{pmatrix} 0 & 0 \\ 0 & K^{-1} \Theta^{lm} \end{pmatrix} + \epsilon^2 \mathtt{S}(\epsilon, t, u, u^{\alpha\beta}, u_\gamma, u^{\alpha\beta}_\gamma, z_j) \tag{2.2.88}$$

and

$$\hat{S} = \begin{pmatrix} 0 \\ -K^{-1}\left[\sqrt{\frac{3}{\Lambda}}(-u_0^{0l} + (-3+4\Omega)u^{0l}) + \frac{1}{2}\left(\frac{3}{\Lambda}\right)^{\frac{3}{2}} E^{-2}\delta^{lk}u_k^{00}\right] \end{pmatrix} + \epsilon \begin{pmatrix} -\Xi_{kj}E^{-2}\frac{\Omega}{t}\delta^{kj} + \frac{1}{2}E^2\delta_{kj}\left[-\frac{2}{t}\Omega\Theta^{kj} + E^{-2}\big(u_0^{kj} + \frac{3}{\Lambda}(3u^{00} + u_0^{00} - u_0)\delta^{kj}\big)\right] + \frac{6}{\Lambda}u_k^{0i}\delta_i^k \\ \mathcal{S}_1(\epsilon, t, u, u^{\alpha\beta}, u_\gamma, u_\gamma^{\alpha\beta}, z_j) \end{pmatrix} + \epsilon^2\mathcal{S}_2(\epsilon, t, u, u^{\alpha\beta}, u_\gamma, u_\gamma^{\alpha\beta}, z_j), \tag{2.2.89}$$

where the remainder terms $\mathtt{S}^0$, $\mathtt{S}^k$, $\mathtt{S}$, $\mathcal{S}_1$ and $\mathcal{S}_2$ all vanish for $(u, u^{\alpha\beta}, u_\gamma, u_\gamma^{\alpha\beta}, z_j) = (0,0,0,0,0)$. We further decompose $\hat{S}$ into a local and nonlocal part by writing

$$\hat{S} = G + S, \tag{2.2.90}$$

where

$$G = \begin{pmatrix} 0 \\ -K^{-1}\left[\sqrt{\frac{3}{\Lambda}}(-u_0^{0l} + (-3+4\Omega)u^{0l}) + \frac{1}{2}\left(\frac{3}{\Lambda}\right)^{\frac{3}{2}} E^{-2}\delta^{lk}w_k^{00}\right] \end{pmatrix}, \tag{2.2.91}$$

and

$$S = \begin{pmatrix} 0 \\ -K^{-1}\frac{1}{2}\left(\frac{3}{\Lambda}\right)^{\frac{3}{2}} E^{-2}\delta^{lk}\partial_k\Phi \end{pmatrix} + \epsilon\mathcal{S}(\epsilon, t, u, u^{\alpha\beta}, u_\gamma, u_\gamma^{\alpha\beta}, z_j). \tag{2.2.92}$$

2.2.6 The complete evolution system

To complete the transformation of reduced conformal Einstein-Euler equations, we need to treat ϕ, defined by (2.2.64), as an independent variable and derive an evolution equation for it. To do so, we see from (2.2.78) that we can write the time derivative of $\delta\zeta$ as

$$\begin{aligned} \partial_t\delta\zeta =& e_0(B^0)^{-1}\left[-B^k\partial_k\begin{pmatrix}\delta\zeta \\ z_m\end{pmatrix} + \frac{1}{t}\mathfrak{B}\hat{\mathbb{P}}_2\begin{pmatrix}\delta\zeta \\ z_m\end{pmatrix} + \hat{S}\right] \\ =& -\sqrt{\frac{\Lambda}{3}}\big(z^k\partial_k\delta\zeta + E^{-2}\delta^{km}\partial_k z_m\big) + \epsilon\mathcal{S}(\epsilon, t, u, u^{\alpha\beta}, u_\gamma, u_\gamma^{\alpha\beta}, z_j), \end{aligned} \tag{2.2.93}$$

where $e_0 = (1,0,0,0)$ and $\mathcal{S}$ vanishes for $(u, u^{\alpha\beta}, u_\gamma, u_\gamma^{\alpha\beta}, z_j) = (0,0,0,0,0)$. Noting that (2.2.7) is equivalent to

$$\frac{1}{1+\epsilon^2 K}\partial_t\rho_H = \frac{3}{t}\rho_H - \frac{3}{t}\Omega\rho_H,$$

it follows directly from the definition of $\delta\zeta$, ρ and $\delta\rho$, see (2.1.38), (2.2.38) and (2.2.40), that

$$\partial_t(\delta\zeta) = \frac{1}{1+\epsilon^2 K}\frac{1}{\rho}\partial_t(\delta\rho) + \frac{3}{t}(\Omega - 1)\frac{\delta\rho}{\rho} \tag{2.2.94}$$

and

$$\partial_k(\delta\zeta) = \frac{1}{1+\epsilon^2 K}\frac{1}{\rho}\partial_k\rho. \tag{2.2.95}$$

Using (2.2.94), (2.2.95) and (2.2.86)-(2.2.89), we can write (2.2.93) as

$$\begin{aligned} \frac{1}{1+\epsilon^2 K}\partial_t(\delta\rho) + \frac{3}{t}(\Omega-1)\delta\rho + \sqrt{\frac{3}{\Lambda}}\partial_k(\rho z^k) =& -\epsilon\left(\frac{3}{\Lambda}\right)^{\frac{3}{2}} t^{4+3\epsilon^2 K}\partial_k\big(u^{00}e^{(1+\epsilon^2 K)\zeta}z^k\big) \\ &+ \epsilon t^{3(1+\epsilon^2 K)}\check{S} + \epsilon^2 t^{3(1+\epsilon^2 K)}\mathcal{S}(\epsilon, t, u, u^{\alpha\beta}, u_\gamma, u_\gamma^{\alpha\beta}, z_j, \partial_k z_j, \delta\zeta, \partial_k\delta\zeta), \end{aligned} \tag{2.2.96}$$

where

$$\check{S} =\frac{1}{2}E^2\delta_{kj}\Big[-\frac{2}{t}\Omega\Theta^{kj}+E^{-2}\big(u^{kj}_0+\frac{3}{\Lambda}(3u^{00}+u^{00}_0-u_0)\delta^{kj}\big)\Big]e^{(1+\epsilon^2K)(\zeta_H+\delta\zeta)}+\frac{6}{\Lambda}u^{0i}_k\delta^k_i e^{(1+\epsilon^2K)(\zeta_H+\delta\zeta)}$$
$$-\Xi_{kj}E^{-2}\frac{\Omega}{t}\delta^{kj}e^{(1+\epsilon^2K)(\zeta_H+\delta\zeta)}$$

and the remainder term $\mathcal{S}$ satisfies $\mathcal{S}(\epsilon,t,0,0,0,0,0,0,\delta\zeta,0)=0$. Taking the L^2 inner product of (2.2.96) with 1 and then multiplying by $1/(\epsilon t^{3(1+\epsilon^2K)})$, we obtain the desired evolution equation for ϕ given by

$$\partial_t\phi=\acute{G}+\acute{S}, \tag{2.2.97}$$

where

$$\acute{G}=(1+\epsilon^2K)\big\langle 1,\check{S}\big\rangle-\frac{3(1+\epsilon^2K)\Omega}{t}\phi \tag{2.2.98}$$

and

$$\acute{S}=\epsilon(1+\epsilon^2K)\big\langle 1,\mathcal{S}(\epsilon,t,u,u^{\alpha\beta},u_\gamma,u^{\alpha\beta}_\gamma,z_j,\partial_k z_j,\delta\zeta,\partial_k\delta\zeta)\big\rangle. \tag{2.2.99}$$

Next, we incorporate the shifted variable (2.2.61) into our set of gravitational variables by defining the vector quantity

$$\mathbf{U}_1=(u^{0\mu}_0,w^{0\mu}_k,u^{0\mu},u^{ij}_0,u^{ij}_k,u^{ij},u_0,u_k,u)^T, \tag{2.2.100}$$

and then combine this with the fluid variables and ϕ by defining

$$\mathbf{U}=(\mathbf{U}_1,\mathbf{U}_2,\phi)^T, \tag{2.2.101}$$

where

$$\mathbf{U}_2=(\delta\zeta,z_i)^T. \tag{2.2.102}$$

Gathering (2.2.50), (2.2.51), (2.2.65), (2.2.78) and (2.2.97) together, we arrive at the following complete evolution equation for $\mathbf{U}$:

$$\mathbf{B}^0\partial_t\mathbf{U}+\mathbf{B}^i\partial_i\mathbf{U}+\frac{1}{\epsilon}\mathbf{C}^i\partial_i\mathbf{U}=\frac{1}{t}\mathbf{BPU}+\mathbf{H}+\mathbf{F}, \tag{2.2.103}$$

where

$$\mathbf{B}^0=\begin{pmatrix}\tilde{B}^0&0&0&0&0\\0&\tilde{B}^0&0&0&0\\0&0&\tilde{B}^0&0&0\\0&0&0&B^0&0\\0&0&0&0&1\end{pmatrix},\quad \mathbf{B}^i=\begin{pmatrix}\tilde{B}^i&0&0&0&0\\0&\tilde{B}^i&0&0&0\\0&0&\tilde{B}^i&0&0\\0&0&0&B^i&0\\0&0&0&0&0\end{pmatrix},\quad \mathbf{C}^i=\begin{pmatrix}\tilde{C}^i&0&0&0&0\\0&\tilde{C}^i&0&0&0\\0&0&\tilde{C}^i&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}, \tag{2.2.104}$$

$$\mathbf{B}=\begin{pmatrix}\tilde{\mathfrak{B}}&0&0&0&0\\0&-2E^2\underline{\bar{g}^{00}}I&0&0&0\\0&0&-2E^2\underline{\bar{g}^{00}}I&0&0\\0&0&0&\mathfrak{B}&0\\0&0&0&0&1\end{pmatrix},\quad \mathbf{P}=\begin{pmatrix}\mathbb{P}_2&0&0&0&0\\0&\breve{\mathbb{P}}_2&0&0&0\\0&0&\breve{\mathbb{P}}_2&0&0\\0&0&0&\hat{\mathbb{P}}_2&0\\0&0&0&0&0\end{pmatrix}, \tag{2.2.105}$$

$$\mathbf{H}=(\tilde{G}_1,\tilde{G}_2,\tilde{G}_3,G,\acute{G})^T\quad\text{and}\quad\mathbf{F}=(\tilde{S}_1,\tilde{S}_2,\tilde{S}_3,S,\acute{S})^T. \tag{2.2.106}$$

The importance of equation (2.2.103) is threefold. First, solutions of the reduced conformal Einstein-Euler equations determine solutions of (2.2.103) as we shall show in the following section. Second, equation (2.2.103) is of the required form so that the a priori estimates from §2.5 apply to its solutions. Finally, estimates for solutions of (2.2.103) that are determined from solutions of the reduced conformal Einstein-Euler equations imply estimates for solutions of the reduced conformal Einstein-Euler equations. In this way, we are able to use the evolution equation (2.2.103) in conjunction with the a priori estimates from §2.5 to establish, for appropriate small data, the global existence of 1-parameter families of ϵ-dependent solutions to the conformal Einstein-Euler equations that exist globally to the future and converge in the limit $\epsilon \searrow 0$ to solutions of the cosmological conformal Poisson-Euler equations of Newtonian gravity.

2.3 Reduced conformal Einstein-Euler equations: local existence and continuation

In this section, we consider the local-in-time existence and uniqueness of solutions to the reduced Einstein-Euler equations and discuss how these solutions determine solutions of (2.2.103). Furthermore, we establish a continuation principle for the Einstein-Euler equations which is based on bounding the H^s norm of $\mathbf{U}$ for $s \in \mathbb{Z}_{\geq 3}$.

Proposition 2.3.1. *Suppose $s \in \mathbb{Z}_{\geq 3}$, $\epsilon_0 > 0$, $\epsilon \in (0, \epsilon_0)$, $T_0 \in (0,1]$, $(\bar{g}_0^{\mu\nu}) \in H^{s+1}(\mathbb{T}^3_\epsilon, \mathbb{S}_4)$, and $(\bar{g}_1^{\mu\nu}) \in H^s(\mathbb{T}^3_\epsilon, \mathbb{S}_4)$, $(\bar{v}_0^\mu) \in H^s(\mathbb{T}^3_\epsilon, \mathbb{R}^4)$ and $\bar{\rho}_0 \in H^s(\mathbb{T}^3_\epsilon)$, where $\bar{v}_0^\mu$ is normalized by $\bar{g}_{0\mu\nu}\bar{v}_0^\mu \bar{v}_0^\nu = -1$, and $\det(\bar{g}_0^{\mu\nu}) < 0$ and $\bar{\rho}_0 > 0$ on $\mathbb{T}^3_\epsilon$. Then there exists a $T_1 \in (0, T_0]$ and a unique classical solution*

$$(\bar{g}^{\mu\nu}, \bar{v}^\mu, \bar{\rho}) \in \bigcap_{\ell=0}^{2} C^\ell((T_1, T_0], H^{s+1-\ell}(\mathbb{T}^3_\epsilon)) \times \bigcap_{\ell=0}^{1} C^\ell((T_1, T_0], H^{s-\ell}(\mathbb{T}^3_\epsilon)) \times \bigcap_{\ell=0}^{1} C^\ell((T_1, T_0], H^{s-\ell}(\mathbb{T}^3_\epsilon)),$$

of the reduced conformal Einstein-Euler equations, given by (2.1.16) and (2.2.12), on the spacetime region $(T_1, T_0] \times \mathbb{T}^3_\epsilon$ that satisfies

$$(\bar{g}^{\mu\nu}, \bar{\partial}_0 \bar{g}^{\mu\nu}, \bar{v}^\mu, \bar{\rho})|_{t=T_0} = (\bar{g}_0^{\mu\nu}, \bar{g}_1^{\mu\nu}, \bar{v}_0^\mu, \bar{\rho}_0).$$

Moreover,

(i) there exists a unique $\Phi \in \bigcap_{\ell=0}^{1} C^\ell((T_1, T_0], \bar{H}^{s+2-\ell}(\mathbb{T}^3))$ that solves equation (2.2.62),

(ii) the vector $\mathbf{U}$, see (2.2.101), is well-defined, lies in the space

$$\mathbf{U} \in \bigcap_{\ell=0}^{1} C^\ell((T_1, T_0], H^{s-\ell}(\mathbb{T}^3, \mathbb{V})),$$

where

$$\mathbb{V} = \mathbb{R}^4 \times \mathbb{R}^{12} \times \mathbb{R}^4 \times \mathbb{S}_3 \times (\mathbb{S}_3)^3 \times \mathbb{S}_3 \times \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R},$$

and solves (2.2.103) on the spacetime region $(T_1, T_0] \times \mathbb{T}^3$, and

(iii) there exists a constant $\sigma > 0$, independent of $\epsilon \in (0, \epsilon_0)$ and $T_1 \in (0, T_0)$, such that if $\mathbf{U}$ satisfies

$$\|\mathbf{U}\|_{L^\infty((T_1, T_0], H^s(\mathbb{T}^3))} < \sigma,$$

then the solution $(\bar{g}^{\mu\nu}, \bar{v}^\mu, \bar{\rho})$ can be uniquely continued as a classical solution with the same regularity to the larger spacetime region $(T_1^*, T_0] \times \mathbb{T}^3_\epsilon$ for some $T_1^* \in (0, T_1)$.

Proof. We begin by noting that the reduced conformal Einstein-Euler equations are well defined as long as the conformal metric $\bar{g}^{\mu\nu}$ remains non-degenerate and the conformal fluid four-velocity remains future directed, that is,

$$\det(\bar{g}^{\mu\nu}) < 0 \quad \text{and} \quad \bar{v}^0 < 0. \tag{2.3.1}$$

Since it is well known that the reduced Einstein-Euler equations can be written as a symmetric hyperbolic system[7] provided that ρ remains strictly positive, we obtain from standard local existence and continuation results for symmetric hyperbolic systems, e.g. Theorems 2.1 and 2.2 of [54], the existence of a unique local-in-time classical solution

$$(\bar{g}^{\mu\nu}, \bar{v}^{\mu}, \bar{\rho}) \in \bigcap_{\ell=0}^{2} C^{\ell}((T_1, T_0], H^{s+1-\ell}(\mathbb{T}^3_\epsilon)) \times \bigcap_{\ell=0}^{1} C^{\ell}((T_1, T_0], H^{s-\ell}(\mathbb{T}^3_\epsilon)) \times \bigcap_{\ell=0}^{1} C^{\ell}((T_1, T_0], H^{s-\ell}(\mathbb{T}^3_\epsilon)) \tag{2.3.2}$$

of the reduced conformal Einstein-Euler equations, for some time $T_1 \in (0, T_0)$, that satisfies

$$(\bar{g}^{\mu\nu}, \bar{\partial}_0 \bar{g}^{\mu\nu}, \bar{v}^{\mu}, \bar{\rho})|_{t=T_0} = (\bar{g}^{\mu\nu}_0, \bar{g}^{\mu\nu}_1, \bar{v}^{\mu}_0, \bar{\rho}_0)$$

for given initial data

$$(\bar{g}^{\mu\nu}, \bar{\partial}_0 \bar{g}^{\mu\nu}, \bar{v}^{\mu}, \bar{\rho}_0) \in H^{s+1}(\mathbb{T}^3_\epsilon, \mathbb{S}_4) \times H^{s}(\mathbb{T}^3_\epsilon, \mathbb{S}_4) \times H^{s}(\mathbb{T}^3_\epsilon, \mathbb{R}^4) \times H^{s}(\mathbb{T}^3_\epsilon)$$

satisfying (2.3.1) and $\bar{\rho}_0 > 0$ on the initial hypersurface $t = T_0$. Moreover, if the solution satisfies

$$\det(\bar{g}^{\mu\nu}(\bar{x}^{\gamma})) \leq c_1 < 0, \quad \bar{v}^0(\bar{x}^{\gamma}) \leq c_2 < 0 \tag{2.3.3}$$

and

$$\bar{\rho}(\bar{x}^{\gamma}) \geq c_3 > 0 \tag{2.3.4}$$

for all $(\bar{x}^{\gamma}) \in (T_1, T_0] \times \mathbb{T}^3_\epsilon$, for some constants c_i, $i = 1, 2, 3$, and

$$\|\bar{g}^{\mu\nu}\|_{L^\infty((T_1,T_0],W^{1,\infty}(\mathbb{T}^3_\epsilon))} + \|\bar{\partial}\bar{g}^{\mu\nu}\|_{L^\infty((T_1,T_0],W^{1,\infty}(\mathbb{T}^3_\epsilon))} + \|\bar{v}^{\mu}\|_{L^\infty((T_1,T_0],W^{1,\infty}(\mathbb{T}^3_\epsilon))} + \|\bar{\rho}\|_{L^\infty((T_1,T_0],W^{1,\infty}(\mathbb{T}^3_\epsilon))} < \infty,$$

then there exists a time $T_1^* \in (0, T_1)$ such that the solution uniquely extends to the spacetime region $(T_1^*, T_0] \times \mathbb{T}^3_\epsilon$ with the same regularity as given by (2.3.2).

Next, we set

$$\mathbf{u} = (u^{\mu\nu}, u^{\mu\nu}_{\gamma}, u, u_{\gamma}, \delta\zeta, z_i),$$

where $u^{\mu\nu}$, $u^{\mu\nu}_{\gamma}$, u, u_{γ}, $\delta\zeta$ and z_i are computed from the solution (2.3.2) via the definitions from §2.1.5. From the definitions (2.1.37) and (2.1.38), the formulas (2.1.43)-(2.1.45), the expansions (2.2.24)-(2.2.26) and (2.2.42), and Sobolev's inequality, see Theorem C.1.1, there exists a constant $\sigma > 0$, independent of $T_1 \in (0, T_0)$ and $\epsilon \in (0, \epsilon_0)$, such that

$$\|(u^{\mu\nu}, u, \delta\zeta, z_i)\|_{L^\infty((T_1,T_0],H^s(\mathbb{T}^3))} < \sigma \tag{2.3.5}$$

implies that the inequalities (2.3.3) and $t^{-3(1+\epsilon^2 K)}\bar{\rho} \geq c_3 > 0$ hold for some constants c_i, $i = 1, 2, 3$. Moreover, for σ small enough[8], we see from the Moser inequality from Lemma C.1.3 and the expansions (2.2.24)-(2.2.28) and (2.2.42)-(2.2.43) that

$$\begin{aligned} &\|\bar{g}^{\mu\nu}\|_{L^\infty((T_1,T_0],W^{1,\infty}(\mathbb{T}^3_\epsilon))} + \|\bar{\partial}\bar{g}^{\mu\nu}\|_{L^\infty((T_1,T_0],W^{1,\infty}(\mathbb{T}^3_\epsilon))} \\ &\quad + \|\bar{v}^{\mu}\|_{L^\infty((T_1,T_0],W^{1,\infty}(\mathbb{T}^3_\epsilon))} + \|\bar{\rho}\|_{L^\infty((T_1,T_0],W^{1,\infty}(\mathbb{T}^3_\epsilon))} \leq C(\sigma)\big(\|\mathbf{u}\|_{L^\infty((T_1,T_0],H^s(\mathbb{T}^3))} + 1\big). \end{aligned}$$

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[7] This follows from writing the wave equation (2.2.14) in first order form and using one of the various methods for expressing the relativistic conformal Euler equations as a symmetric hyperbolic system. One particular way of writing the conformal Euler equations in symmetric hyperbolic form is given in §2.2.5 which is a variation of the method introduced by Rendall in [69]. For other elegant approaches, see [10, 25, 86].

[8] We emphasize that by choosing σ small enough, with the help of Sobolev embedding theorem (i.e. $H^s \subset C^1 \subset W^{1,\infty}$ since $s > 3/2 + 1$) and (2.3.5), we have $\|(u^{\mu\nu}, u, \delta\zeta, z_i)\|_{L^\infty((T_1,T_0],W^{1,\infty}(\mathbb{T}^3))} < \sigma$, which ensures (2.3.3)-(2.3.4) for $(t, \bar{x}^i) \in (T_1, T_0] \times \mathbb{T}^3_\epsilon$ by the definitions from §2.1.5, that is the conformal fluid four-velocity remains future directed, the conformal metric $\bar{g}^{\mu\nu}$ remains non-degenerate and ρ remains strictly positive. This actually help us rule out the third alternative of Majda's criterion in Corollary D.4.3.

Thus by the continuation principle, there exists a $\sigma > 0$ such that if (2.3.5) holds and

$$\|\mathbf{u}\|_{L^\infty((T_1,T_0],H^s(\mathbb{T}^3))} < \infty, \tag{2.3.6}$$

then the solution (2.3.2) can be uniquely continued as a classical solution with the same regularity to the larger spacetime region $(T_1^*, T_0] \times \mathbb{T}^3_\epsilon$ for some $T_1^* \in (0, T_1)$.

Since $\Delta : \bar{H}^{s+2}(\mathbb{T}^3) \longrightarrow \bar{H}^s(\mathbb{T}^3)$ is an isomorphism, we can solve (2.2.62) to get

$$\frac{1}{t}\Phi = \frac{\Lambda}{3}E^2 e^{\zeta_H}\Delta^{-1}\Pi e^{\delta\zeta} \in \bigcap_{\ell=0}^{1} C^\ell((T_1,T_0],\bar{H}^{s+2-\ell}(\mathbb{T}^3)). \tag{2.3.7}$$

As ζ_H and E are uniformly bounded on $(0,1]$, see (2.1.43) and (2.2.4), it then follows via the Moser inequality from Lemma C.1.3 that the derivative $\partial_k\Phi$ satisfies the bound

$$\|t^{-1}\partial_k\Phi(t)\|_{H^{s+1}(\mathbb{T}^3))} \leq C\big(\|\delta\zeta(t)\|_{H^s(\mathbb{T}^3)}\big)\|\delta\zeta(t)\|_{H^s(\mathbb{T}^3)}$$

uniformly for $(t,\epsilon) \in (T_1,T_0] \times (0,\epsilon_0)$, where C is independent of initial data and the times $\{T_1, T_2\}$. But, this implies via the definition of $\mathbf{U}$, see (2.2.101), that

$$\|\mathbf{u}\|_{L^\infty((T_1,T_0],H^s(\mathbb{T}^3))} \leq C\big(\|\mathbf{U}\|_{L^\infty((T_1,T_0],H^s(\mathbb{T}^3))}\big)\|\mathbf{U}\|_{L^\infty((T_1,T_0],H^s(\mathbb{T}^3))}.$$

Since

$$\|(u^{\mu\nu}, u, \delta\zeta, z_i)\|_{L^\infty((T_1,T_0],H^s(\mathbb{T}^3))} \leq \|\mathbf{U}\|_{L^\infty((T_1,T_0],H^s(\mathbb{T}^3))},$$

we find that

$$\|\mathbf{U}\|_{L^\infty((T_1,T_0],H^s(\mathbb{T}^3))} < \sigma \tag{2.3.8}$$

implies that the inequalities (2.3.5) and (2.3.6) both hold. In particular, this shows that if (2.3.8) holds for $\sigma > 0$ small enough, then the solution (2.3.2) can be uniquely continued as a classical solution with the same regularity to the larger spacetime region[9] $(T_1^*, T_0] \times \mathbb{T}^3_\epsilon$ for some $T_1^* \in (0, T_1)$ eventually. □

Remark 2.3.2. In fact, when using continuation principle (Theorem D.4.2 and Corollary D.4.3), we have to rule out the second and the third alternatives of Majda's criterion (Corollary D.4.3). The third alternative is easily ruled out by letting σ small enough as we have mentioned above. About the second alternative, the above proof clearly implies $\|D\mathbf{U}\|_{L^\infty(\mathbb{T}^3)} \lesssim \|\mathbf{U}\|_{H^s} \leq \sigma \leq \infty$ for $t \in (T_1, T_0]$. We need to estimate $\|\partial_t\mathbf{U}\|_{L^\infty(\mathbb{T}^3)}$. From the equation (2.2.103), $\|\partial_t\mathbf{U}\|_{L^\infty(\mathbb{T}^3)}$ can be estimated by $\|\mathbf{U}\|_{W^{1,\infty}(\mathbb{T}^3)}$ by noting that $\mathbf{B}^\mu$ and $\mathbf{B}$ is smooth in the unknown variable $\mathbf{U}$ (with the help of the Moser inequality from Lemma C.1.3). It is easy to prove that via standard steps. We will present similar proofs in §2.5 under the similar structure of the equations (see (2.5.30)) and in that cases, the coefficient matrices are of much better properties (e.g. E^0 class) to make sure $\|\partial_t\mathbf{U}\|_{L^\infty}$ is well controlled.

2.4 Conformal cosmological Poisson-Euler equations: local existence and continuation

In this section, we consider the local-in-time existence and uniqueness of solutions to the conformal cosmological Poisson-Euler equations, and we establish a continuation principle that is based on bounding the H^s norm of $(\mathring{\zeta}, \mathring{z}^j)$.

Proposition 2.4.1. *Suppose $s \in \mathbb{Z}_{\geq 3}$, $\mathring{\zeta}_0 \in H^s(\mathbb{T}^3)$ and $(\mathring{z}_0^i) \in H^s(\mathbb{T}^3, \mathbb{R}^3)$. Then there exists a*

[9] This is because of the continuation principle Theorem D.4.2 and Corollary D.4.3 from Appendix D.4. It is obviously that $H^s \subset C^1 \subset W^{1,\infty}$ for $s > 3/2 + 1$ due to the Sobolev embedding theorem and it implies $\|\mathbf{U}\|_{W^{1,\infty}} \lesssim \|\mathbf{U}\|_{H^s} \leq \sigma$ for $t \in (T_1, T_0]$.

$T_1 \in (0, T_0]$ and a unique classical solution

$$(\mathring{\zeta}, \mathring{z}^i, \mathring{\Phi}) \in \bigcap_{\ell=0}^{1} C^\ell((T_1, T_0], H^{s-\ell}(\mathbb{T}^3)) \times \bigcap_{\ell=0}^{1} C^\ell((T_1, T_0], H^{s-\ell}(\mathbb{T}^3, \mathbb{R}^3)) \times \bigcap_{\ell=0}^{1} C^\ell((T_1, T_0], H^{s+2-\ell}(\mathbb{T}^3)),$$

of the conformal cosmological Poisson-Euler equations, given by (2.1.55)-(2.1.57), on the spacetime region $(T_1, T_0] \times \mathbb{T}^3$ that satisfies

$$(\mathring{\zeta}, \mathring{z}^i)|_{t=T_0} = (\mathring{\zeta}_0, \mathring{z}_0^i)$$

on the initial hypersurface $t = T_0$. Furthermore, if

$$\|(\mathring{\zeta}, \mathring{z}^i)\|_{L^\infty((T_1,T_0],H^s)} < \infty,$$

then the solution $(\mathring{\zeta}, \mathring{z}^i, \mathring{\Phi})$ can be uniquely continued as a classical solution with the same regularity to the larger spacetime region $(T_1^*, T_0] \times \mathbb{T}^3$ for some $T_1^* \in (0, T_1)$.

Proof. Using the fact that $\Delta : \bar{H}^{s+2} \longrightarrow \bar{H}^s$ is an isomorphism, we can solve the Poisson equation (2.1.51) by setting

$$\mathring{\Phi} = \frac{\Lambda}{3} t \mathring{E}^2 \Delta^{-1} \Pi e^{\mathring{\zeta}}. \tag{2.4.1}$$

We can use this to write (2.1.55)-(2.1.57) as

$$\partial_t \mathring{\zeta} + \sqrt{\frac{3}{\Lambda}} \big(\mathring{z}^j \partial_j \mathring{\zeta} + \partial_j \mathring{z}^j \big) = -\frac{3\mathring{\Omega}}{t}, \tag{2.4.2}$$

$$\sqrt{\frac{\Lambda}{3}} \partial_t \mathring{z}^j + \mathring{z}^i \partial_i \mathring{z}^j + K \partial^j \mathring{\zeta} = \sqrt{\frac{\Lambda}{3}} \frac{1}{t} \mathring{z}^j - \frac{1}{2} t \mathring{E}^2 \partial^j \Delta^{-1} \Pi e^{\mathring{\zeta}}. \tag{2.4.3}$$

It is then easy to see that this system can be cast in symmetric hyperbolic form by multiplying (2.4.3) by $\mathring{E}^2 K^{-1} \sqrt{\frac{3}{\Lambda}}$. Even though the resulting system is non-local due to the last term in (2.4.3), all of the standard local existence and uniqueness results and continuation principles that are valid for local symmetric hyperbolic systems, e.g. Theorems 2.1 and 2.2 of [54], continue to apply. Therefore it follows that there exists a unique local-in-time classical solution

$$(\mathring{\zeta}, \mathring{z}^i) \in \bigcap_{\ell=0}^{1} C^\ell((T_1, T_0], H^{s-\ell}(\mathbb{T}^3)) \times \bigcap_{\ell=0}^{1} C^\ell((T_1, T_0], H^{s-\ell}(\mathbb{T}^3, \mathbb{R}^3)) \tag{2.4.4}$$

of (2.4.2)-(2.4.3) for some time $T_1 \in (0, T_0)$ that satisfies

$$(\mathring{\zeta}, \mathring{z}^i)_{t=T_0} = (\mathring{\zeta}_0, \mathring{z}_0^i)$$

for given initial data $(\mathring{\zeta}_0, \mathring{z}_0^i) \in H^s(\mathbb{T}^3) \times H^s(\mathbb{T}^3, \mathbb{R}^3)$. Moreover, if the solution satisfies

$$\|\mathring{\zeta}\|_{L^\infty((T_1,T_0],W^{1,\infty})} + \|\mathring{z}^i\|_{L^\infty((T_1,T_0],W^{1,\infty})} < \infty,$$

then there exists a time $T_1^* \in (0, T_1)$ such that the solution (2.4.4) uniquely extends to the spacetime region $(T_1^*, T_0] \times \mathbb{T}^3$ with the same regularity. By Sobolev's inequality, see Theorem C.1.1, this is clearly implied by the stronger condition

$$\|(\mathring{\zeta}, \mathring{z}^i)\|_{L^\infty((T_1,T_0],H^s)} < \infty.$$

Finally from (2.4.1), (2.4.4) and the Moser inequality from Lemma C.1.3, it is clear that

$$\mathring{\Phi} \in \bigcap_{\ell=0}^{1} C^\ell((T_1, T_0], H^{s+2-\ell}(\mathbb{T}^3)).$$

□

Corollary 2.4.2. *If the initial modified density $\zeta_0 \in H^s(\mathbb{T}^3)$ from Proposition 2.4.1 is chosen so that*

$$\mathring{\zeta}_0 = \ln\left(\frac{\mathring{\rho}_H(T_0) + \breve{\rho}_0}{T_0^3}\right),$$

where $\mathring{\rho}_H = \frac{4C_0\Lambda t^3}{(C_0-t^3)^2}$, $\breve{\rho}_0 \in \bar{H}^s(\mathbb{T}^3)$, and $\mathring{\rho}_H(T_0) + \breve{\rho}_0 > 0$ in $\mathbb{T}^3$, then the solution $(\mathring{\zeta}, \mathring{z}^i, \mathring{\Phi})$ to the conformal cosmological Poisson-Euler equations from Proposition 2.4.1 satisfies

$$\Pi\mathring{\rho} = \delta\mathring{\rho} := \mathring{\rho} - \mathring{\rho}_H \quad in\ (T_1, T_0] \times \mathbb{T}^3.$$

Proof. Since $\mathring{\rho} = t^3 e^{\mathring{\zeta}}$ satisfies (2.1.49), we see after applying $\langle 1, \cdot\rangle$ to this equations that $\langle 1, \mathring{\rho}\rangle$ satisfies

$$\frac{d}{dt}\langle 1, \mathring{\rho}(t)\rangle = \frac{3(1-\mathring{\Omega}(t))}{t}\langle 1, \mathring{\rho}(t)\rangle, \quad T_1 < t \leq T_0,$$

while from the choice of initial data, we have

$$\langle 1, \mathring{\rho}(T_0)\rangle = \mathring{\rho}_H(T_0).$$

By a direct computation, we observe with the help of (2.1.54) that $\mathring{\rho}_H = \frac{4C_0\Lambda t^3}{(C_0-t^3)^2}$ satisfies the differential equation

$$\frac{d}{dt}\mathring{\rho}_H(t) = \frac{3(1-\mathring{\Omega}(t))}{t}\mathring{\rho}_H(t) \quad 0 < t \leq T_0, \tag{2.4.5}$$

and hence, that

$$\langle 1, \mathring{\rho}(t)\rangle = \mathring{\rho}_H(t), \quad T_1 < t \leq T_0, \tag{2.4.6}$$

by the uniqueness of solutions to the initial value problem for ordinary differential equations. The proof now follows since

$$\Pi\mathring{\rho} \overset{(2.1.52)}{=} \mathring{\rho} - \langle 1, \mathring{\rho}\rangle \overset{(2.4.6)}{=} \mathring{\rho} - \mathring{\rho}_H(t) \quad \text{in } (T_1, T_0] \times \mathbb{T}^3.$$

□

Remark 2.4.3. Letting

$$\delta\mathring{\zeta} = \mathring{\zeta} - \mathring{\zeta}_H, \tag{2.4.7}$$

where, see (2.1.10), (2.1.47) and (2.2.3),

$$\mathring{\zeta}_H = \ln(t^{-3}\mathring{\rho}_H), \tag{2.4.8}$$

it is clear that the initial condition

$$\mathring{\zeta}|_{t=T_0} = \ln\left(\frac{\mathring{\rho}_H(T_0) + \breve{\rho}_0}{T_0^3}\right),$$

from Corollary 2.4.2 is equivalent to the initial condition

$$\delta\mathring{\zeta}|_{t=T_0} = \ln\left(1 + \frac{\breve{\rho}_0}{\mathring{\rho}_H(T_0)}\right)$$

for $\delta\mathring{\zeta}$.

2.5 Singular symmetric hyperbolic systems

In this section, we establish uniform a priori estimates for solutions to a class of symmetric hyperbolic systems that are jointly singular in ϵ and t, and include both the formulation of the reduced conformal

Einstein-Euler equations given by (2.2.103) and the $\epsilon \searrow 0$ limit of these equations. We also establish *error estimates*, that is, a priori estimates for the difference between solutions of the ϵ-dependent singular symmetric hyperbolic systems and their corresponding $\epsilon \searrow 0$ limit equations.

The ϵ-dependent singular terms that appear in the symmetric hyperbolic systems we consider are of a type that have been well studied, see [12, 40, 41, 45], while the t-dependent singular terms are of the type analyzed in [66]. The uniform a priori estimates established here follow from combining the energy estimates from [12, 40, 41, 45] with those from [66].

Remark 2.5.1. In this section, we switch to the standard time orientation, where the future is located in the direction of increasing time, while keeping the singularity located at $t = 0$. We do this in order to make the derivation of the energy estimates in this section as similar as possible to those for non-singular symmetric hyperbolic systems, which we expect will make it easier for readers familiar with such estimates to follow the arguments below. To get back to the time orientation used to formulate the conformal Einstein-Euler equations, see Remark 2.1.1, we need only apply the trivial time transformation $t \mapsto -t$.

2.5.1 Uniform estimates

We will establish uniform a priori estimates for the following class of equations:

$$A^0\partial_0 U + A^i\partial_i U + \frac{1}{\epsilon}C^i\partial_i U = \frac{1}{t}\mathfrak{A}\mathbb{P}U + H \quad \text{in} \quad [T_0, T_1) \times \mathbb{T}^n, \tag{2.5.1}$$

where

$$\begin{aligned}
U &= (w, u)^T,\\
A^0 &= \begin{pmatrix} A^0_1(\epsilon,t,x,w) & 0 \\ 0 & A^0_2(\epsilon,t,x,w) \end{pmatrix},\\
A^i &= \begin{pmatrix} A^i_1(\epsilon,t,x,w) & 0 \\ 0 & A^i_2(\epsilon,t,x,w) \end{pmatrix},\\
C^i &= \begin{pmatrix} C^i_1 & 0 \\ 0 & C^i_2 \end{pmatrix}, \quad \mathbb{P} = \begin{pmatrix} \mathbb{P}_1 & 0 \\ 0 & \mathbb{P}_2 \end{pmatrix},\\
\mathfrak{A} &= \begin{pmatrix} \mathfrak{A}_1(\epsilon,t,x,w) & 0 \\ 0 & \mathfrak{A}_2(\epsilon,t,x,w) \end{pmatrix},\\
H &= \begin{pmatrix} H_1(\epsilon,t,x,w) \\ H_2(\epsilon,t,x,w,u) + R_2 \end{pmatrix} + \begin{pmatrix} F_1(\epsilon,t,x) \\ F_2(\epsilon,t,x) \end{pmatrix},\\
R_2 &= \frac{1}{t}M_2(\epsilon,t,x,w,u)\mathbb{P}_3 U,
\end{aligned}$$

and the following assumptions hold for fixed constants $\epsilon_0, R > 0$, $T_0 < T_1 < 0$ and $s \in \mathbb{Z}_{>n/2+1}$:

Assumptions 2.5.2.

1. *The C^i_a, $i = 1, \dots, n$ and $a = 1, 2$, are constant, symmetric $N_a \times N_a$ matrices.*
2. *The $\mathbb{P}_a$, $a = 1, 2$, are constant, symmetric $N_a \times N_a$ projection matrices, i.e. $\mathbb{P}_a^2 = \mathbb{P}_a$. We use $\mathbb{P}_a^\perp = \mathbb{1} - \mathbb{P}_a$ to denote the complementary projection matrix.*
3. *The source terms $H_a(\epsilon,t,x,w)$, $a = 1, 2$, $F_a(\epsilon,t,x)$, $a = 1, 2$, and $M_2(\epsilon,t,x,w,u)$ satisfy $H_1 \in E^0\big((0,\epsilon_0)\times(2T_0,0)\times\mathbb{T}^n\times B_R(\mathbb{R}^{N_1}),\mathbb{R}^{N_1}\big)$, $H_2 \in E^0\big((0,\epsilon_0)\times(2T_0,0)\times\mathbb{T}^n\times B_R(\mathbb{R}^{N_1})\times B_R(\mathbb{R}^{N_2})\times B_R((\mathbb{R}^{N_1})^n),\mathbb{R}^{N_2}\big)$, $F_a \in C^0\big((0,\epsilon_0)\times[T_0,T_1),H^s(\mathbb{T}^n,\mathbb{R}^{N_a})\big)$, $M_2 \in E^0\big((0,\epsilon_0)\times(2T_0,0)\times\mathbb{T}^n\times B_R(\mathbb{R}^{N_1})\times B_R(\mathbb{R}^{N_2}),\mathbb{M}_{N_2\times N_2}\big)$, and*

$$H_1(\epsilon,t,x,0) = 0, \quad H_2(\epsilon,t,x,0,0) = 0 \quad \textit{and} \quad M_2(\epsilon,t,x,0,0) = 0$$

for all $(\epsilon,t,x) \in (0,\epsilon_0)\times(2T_0,0)\times\mathbb{T}^n$.

4. *The matrix valued maps* $A^i_a(\epsilon,t,x,w)$, $i=0,\dots,n$ *and* $a=1,2$, *satisfy* $A^i_a \in E^0\big((0,\epsilon_0)\times(2T_0,0)\times\mathbb{T}^n\times B_R(\mathbb{R}^{N_a}),\mathbb{S}_{N_a}\big)$.

5. *The matrix valued maps* $A^0_a(\epsilon,t,x,w)$, $a=1,2$, *and* $\mathfrak{A}_a(\epsilon,t,x,w)$, $a=1,2$, *can be decomposed as*

$$A^0_a(\epsilon,t,x,w)=\mathring{A}^0_a(t)+\epsilon\tilde{A}^0_a(\epsilon,t,x,w), \tag{2.5.2}$$
$$\mathfrak{A}_a(\epsilon,t,x,w)=\mathring{\mathfrak{A}}_a(t)+\epsilon\tilde{\mathfrak{A}}_a(\epsilon,t,x,w), \tag{2.5.3}$$

where $\mathring{A}^0_a\in E^1\big((2T_0,0),\mathbb{S}_{N_a}\big)$, $\mathring{\mathfrak{A}}_a\in E^1\big((2T_0,0),\mathbb{M}_{N_a\times N_a}\big)$, $\tilde{A}^0_a\in E^1\big((0,\epsilon_0)\times(2T_0,0)\times\mathbb{T}^n\times B_R(\mathbb{R}^{N_1}),\mathbb{S}_{N_a}\big)$, $\tilde{\mathfrak{A}}_a\in E^0\big((0,\epsilon_0)\times(2T_0,0)\times\mathbb{T}^n\times B_R(\mathbb{R}^{N_1}),\mathbb{M}_{N_a\times N_a}\big)$, *and*[10]

$$D_x\tilde{\mathfrak{A}}_a(\epsilon,t,x,0)=D_x\tilde{A}^0_a(\epsilon,t,x,0)=0 \tag{2.5.4}$$

for all $(\epsilon,t,x)\in(0,\epsilon_0)\times(2T_0,0)\in\mathbb{T}^n$.

6. *For* $a=1,2$, *the matrix* $\mathfrak{A}_a$ *commutes with* $\mathbb{P}_a$, *i.e.*

$$[\mathbb{P}_a,\mathfrak{A}_a(\epsilon,t,x,w)]=0 \tag{2.5.5}$$

for all $(\epsilon,t,x,w)\in(0,\epsilon_0)\times(2T_0,0)\times\mathbb{T}^n\times B(\mathbb{R}^{N_1})$.

7. $\mathbb{P}_3$ *is a symmetric* $(N_1+N_2)\times(N_1+N_2)$ *projection matrix that satisfies*

$$\mathbb{P}\mathbb{P}_3=\mathbb{P}_3\mathbb{P}=\mathbb{P}_3, \tag{2.5.6}$$
$$\mathbb{P}_3A^i(\epsilon,t,x,w)\mathbb{P}_3^\perp=\mathbb{P}_3C^i\mathbb{P}_3^\perp=\mathbb{P}_3\mathfrak{A}(\epsilon,t,x,w)\mathbb{P}_3^\perp=0 \tag{2.5.7}$$

and

$$[\mathbb{P}_3,A^0(\epsilon,t,x,w)]=0 \tag{2.5.8}$$

for all $(\epsilon,t,x,w)\in(0,\epsilon_0)\times(2T_0,0)\times\mathbb{T}^n\times B_R(\mathbb{R}^{N_1})$, *where* $\mathbb{P}_3^\perp=\mathbb{1}-\mathbb{P}_3$ *defines the complementary projection matrix.*

8. *There exists constants* κ, γ_1, $\gamma_2>0$, *such that*

$$\frac{1}{\gamma_1}\mathbb{1}\le A^0_a(\epsilon,t,x,w)\le\frac{1}{\kappa}\mathfrak{A}_a(\epsilon,t,x,w)\le\gamma_2\mathbb{1} \tag{2.5.9}$$

for all $(\epsilon,t,x,w)\in(0,\epsilon_0)\times(2T_0,0)\times\mathbb{T}^n\times B(\mathbb{R}^{N_1})$ *and* $a=1,2$.

9. *For* $a=1,2$, *the matrix* A^0_a *satisfies*

$$\mathbb{P}^\perp_aA^0_a(\epsilon,t,x,\mathbb{P}^\perp_1w)\mathbb{P}_a=\mathbb{P}_aA^0_a(\epsilon,t,x,\mathbb{P}^\perp_1w)\mathbb{P}^\perp_a=0 \tag{2.5.10}$$

for all $(\epsilon,t,x,w)\in(0,\epsilon_0)\times(2T_0,0)\times\mathbb{T}^n\times B(\mathbb{R}^{N_1})$.

10. *For* $a=1,2$, *the matrix* $\mathbb{P}^\perp_a[D_wA^0_a\cdot(A^0_1)^{-1}\mathfrak{A}_1\mathbb{P}_1w]\mathbb{P}^\perp_a$ *can be decomposed as*

$$\mathbb{P}^\perp_a\big[D_wA^0_a(\epsilon,t,x,w)\cdot\big(A^0_1(\epsilon,t,x,w)\big)^{-1}\mathfrak{A}_1(\epsilon,t,x,w)\mathbb{P}_1w\big]\mathbb{P}^\perp_a=t\mathfrak{S}_a(\epsilon,t,x,w)+\mathfrak{T}_a(\epsilon,t,x,w,\mathbb{P}_1w) \tag{2.5.11}$$

for some $\mathfrak{S}_a\in E^0\big((0,\epsilon_0)\times(2T_0,0)\times\mathbb{T}^n\times B_R(\mathbb{R}^{N_1}),\mathbb{M}_{N_a\times N_a}\big)$, $a=1,2$, *and* $\mathfrak{T}_a\in E^0\big((0,\epsilon_0)\times(2T_0,0)\times\mathbb{T}^n\times B_R(\mathbb{R}^{N_1})\times\mathbb{R}^{N_1},\mathbb{M}_{N_a\times N_a}\big)$, $a=1,2$, *where the* $\mathfrak{T}_a(\epsilon,t,x,w,\xi)$ *are quadratic in* ξ.

Before proceeding with the analysis, we take a moment to make a few observations about the structure of the singular system (2.5.1). First, if $\mathfrak{A}=0$, then the singular term $\frac{1}{t}\mathfrak{A}\mathbb{P}U$ disappears

[10] Or in other words, the matrices $\tilde{\mathfrak{A}}_a|_{w=0}$ and $\tilde{A}^0_a|_{w=0}$ depend only on (ϵ,t).

from (2.5.1) and it becomes a regular symmetric hyperbolic system. Uniform ϵ-independent a priori estimates that are valid for $t \in [T_1, 0)$ would then follow, under a suitable small initial data assumption, as a direct consequence of the energy estimates from [12, 40, 41, 45]. When $\mathfrak{A} \neq 0$, the positivity assumption (2.5.9) guarantees that the singular term $\frac{1}{t}\mathfrak{A}\mathbb{P}U$ acts like a friction term. This allows us to generalize the energy estimates from [12, 40, 41, 45] in such a way as to obtain, under a suitable small initial data assumption, uniform ϵ-independent a priori estimates that are valid on the time interval $[T_1, 0)$; see (2.5.40), (2.5.41) and (2.5.42) for the key differential inequalities used to derive these a priori estimates.

Remark 2.5.3. The equation for w decouples from the system (2.5.1) and is given by

$$A_1^0\partial_0 w + A_1^i\partial_i w + \frac{1}{\epsilon}C_1^i\partial_i w = \frac{1}{t}\mathfrak{A}_1\mathbb{P}_1 w + H_1 + F_1 \quad \text{in} \quad [T_0, T_1) \times \mathbb{T}^n. \tag{2.5.12}$$

Remark 2.5.4.

1. By Taylor expanding $A_a^0(\epsilon, t, x, \mathbb{P}_1^\perp w + \mathbb{P}_1 w)$ in the variable $\mathbb{P}_1 w$, it follows from (2.5.10) that there exist matrix valued maps $\hat{A}_a^0, \breve{A}_a^0 \in E^1\big((0, \epsilon_0)\times(2T_0, 0)\times\mathbb{T}^n\times B_R(\mathbb{R}^{N_1}), \mathbb{M}_{N_a\times N_a}\big)$, $a = 1, 2$, such that

$$\mathbb{P}_a^\perp A_a^0(\epsilon, t, x, w)\mathbb{P}_a = \mathbb{P}_a^\perp[\hat{A}_a^0(\epsilon, t, x, w)\cdot\mathbb{P}_1 w]\mathbb{P}_a \tag{2.5.13}$$

   and

$$\mathbb{P}_a A_a^0(\epsilon, t, x, w)\mathbb{P}_a^\perp = \mathbb{P}_a[\breve{A}_a^0(\epsilon, t, x, w)\cdot\mathbb{P}_1 w]\mathbb{P}_a^\perp \tag{2.5.14}$$

   for all $(\epsilon, t, x, w) \in (0, \epsilon_0) \times (2T_0, 0) \times \mathbb{T}^n \times B(\mathbb{R}^{N_1})$.

2. It is not difficult to see that the assumptions (2.5.9) and (2.5.10) imply that

$$\mathbb{P}_a^\perp\big(A_a^0(\epsilon, t, x, \mathbb{P}_1^\perp w)\big)^{-1}\mathbb{P}_a = \mathbb{P}_a\big(A_a^0(\epsilon, t, x, \mathbb{P}_1^\perp w)\big)^{-1}\mathbb{P}_a^\perp = 0$$

   for all $(\epsilon, t, x, w) \in (0, \epsilon_0) \times (2T_0, 0) \times \mathbb{T}^n \times B(\mathbb{R}^{N_1})$. By Taylor expanding $(A_a^0(\epsilon, t, x, \mathbb{P}_1^\perp w + \mathbb{P}_1 w))^{-1}$ in the variable $\mathbb{P}_1 w$, it follows that there exist matrix valued maps $\hat{B}_a^0, \breve{B}_a^0 \in E^1\big((0, \epsilon_0)\times(2T_0, 0)\times\mathbb{T}^n\times B_R(\mathbb{R}^{N_1}), \mathbb{M}_{N_a\times N_a}\big)$, $a = 1, 2$, such that

$$\mathbb{P}_a^\perp\big(A_a^0(\epsilon, t, x, w)\big)^{-1}\mathbb{P}_a = \mathbb{P}_a^\perp[\hat{B}_a^0(\epsilon, t, x, w)\cdot\mathbb{P}_1 w]\mathbb{P}_a \tag{2.5.15}$$

   and

$$\mathbb{P}_a\big(A_a^0(\epsilon, t, x, w)\big)^{-1}\mathbb{P}_a^\perp = \mathbb{P}_a[\breve{B}_a^0(\epsilon, t, x, w)\cdot\mathbb{P}_1 w]\mathbb{P}_a^\perp \tag{2.5.16}$$

   for all $(\epsilon, t, x, w) \in (0, \epsilon_0) \times (2T_0, 0) \times \mathbb{T}^n \times B(\mathbb{R}^{N_1})$.

To facilitate the statement and proof of our a priori estimates for solutions of the system (2.5.1), we introduce the following energy norms:

Definition 2.5.5. Suppose $w \in L^\infty([T_0, T_1) \times \mathbb{T}^n, \mathbb{R}^{N_1})$, $k \in \mathbb{Z}_{\geq 0}$, and $\{\mathbb{P}_a, A_a^0\}$, $a = 1, 2$, are as defined above. Then for maps f_a, $a = 1, 2$, and U from the torus $\mathbb{T}^n$ into R^{N_a} and $R^{N_1} \times R^{N_2}$, respectively, the *energy norms*, denoted $|||f_a|||_{a,H^s}$ and $|||U|||_{H^s}$, of f_a and U are defined by

$$|||f_a|||_{a,H^k}^2 := \sum_{0\leq|\alpha|\leq k}\langle D^\alpha f_a, A_a^0\big(\epsilon, t, \cdot, w(t, \cdot)\big)D^\alpha f_a\rangle$$

and

$$|||U|||_{H^k}^2 := \sum_{0\leq|\alpha|\leq k}\langle D^\alpha U, A^0\big(\epsilon, t, \cdot, w(t, \cdot)\big)D^\alpha U\rangle,$$

respectively. In addition to the energy norms, we also define, for $T_0 < T \leq T_1$, the spacetime norm of maps f_a, $a = 1, 2$, from $[T_0, T) \times \mathbb{T}^n$ to R^{N_a} by

$$\|f_a\|_{M^{\infty}_{\mathbb{P}_a,k}([T_0,T)\times\mathbb{T}^n)} := \|f_a\|_{L^\infty([T_0,T),H^k)} + \left(-\int_{T_0}^{T} \frac{1}{t}\|\mathbb{P}_a f_a(t)\|^2_{H^k} dt\right)^{\frac{1}{2}}.$$

Remark 2.5.6. For $w \in L^\infty([T_0, T_1) \times \mathbb{T}^n, \mathbb{R}^{N_1})$ satisfying $\|w\|_{L^\infty([T_0,T_1)\times\mathbb{T}^n)} < R$, we observe, by (2.5.9), that the standard Sobolev norm $\|\cdot\|_{H^k}$ and the energy norms $|||\cdot|||_{a,H^k}$, $a = 1, 2$, are equivalent since they satisfy

$$\frac{1}{\sqrt{\gamma_1}}\|\cdot\|_{H^k} \leq |||\cdot|||_{a,H^k} \leq \sqrt{\gamma_2}\|\cdot\|_{H^k}.$$

With the preliminaries out of the way, we are now ready to state and prove a priori estimates for solutions of the system (2.5.1) that are uniform in ϵ.

Theorem 2.5.7. *Suppose $R > 0$, $s \in \mathbb{Z}_{\geq n/2+1}$, $T_0 < T_1 < 0$, $\epsilon_0 > 0$, $\epsilon \in (0, \epsilon_0)$, Assumptions 2.5.2 hold, the map*

$$U = (w, u) \in \bigcap_{\ell=0}^{1} C^\ell([T_0, T_1), H^{s-\ell}(\mathbb{T}^n, \mathbb{R}^{N_1})) \times \bigcap_{\ell=0}^{1} C^\ell([T_0, T_1), H^{s-1-\ell}(\mathbb{T}^n, \mathbb{R}^{N_2}))$$

defines a solution of the system (2.5.1), and for $t \in [T_0, T_1)$, the source terms F_a, $a = 1, 2$, satisfy the estimates

$$\|F_1(\epsilon, t)\|_{H^s} \leq C(\|w\|_{L^\infty([T_0,t),H^s)})\|w(t)\|_{H^s} \tag{2.5.17}$$

and

$$\|F_2(\epsilon, t)\|_{H^{s-1}} \leq C\big(\|w\|_{L^\infty([T_0,t),H^s)}, \|u\|_{L^\infty([T_0,t),H^{s-1})}\big)(\|w(t)\|_{H^s} + \|u(t)\|_{H^{s-1}}), \tag{2.5.18}$$

where the constants $C(\|w\|_{L^\infty([T_0,t),H^s)})$ and $C\big(\|w\|_{L^\infty([T_0,t),H^s)}, \|u\|_{L^\infty([T_0,t),H^{s-1})}\big)$ are independent of $\epsilon \in (0, \epsilon_0)$ and $T_1 \in (T_0, 0]$. Then there exists a $\sigma > 0$ independent of $\epsilon \in (0, \epsilon_0)$ and $T_1 \in (T_0, 0)$, such that if initially

$$\|w(T_0)\|_{H^s} \leq \sigma \quad \text{and} \quad \|u(T_0)\|_{H^{s-1}} \leq \sigma,$$

then

$$\|w\|_{L^\infty([T_0,T_1)\times\mathbb{T}^n)} \leq \frac{R}{2}$$

and there exists a constant $C > 0$, independent of $\epsilon \in (0, \epsilon_0)$ and $T_1 \in (T_0, 0)$, such that

$$\|w\|_{M^{\infty}_{\mathbb{P}_1,s}([T_0,t)\times\mathbb{T}^n)} + \|u\|_{M^{\infty}_{\mathbb{P}_2,s-1}([T_0,t)\times\mathbb{T}^n)} - \int_{T_0}^{t} \frac{1}{\tau}\|\mathbb{P}_3 U\|_{H^{s-1}}\, d\tau \leq C\sigma$$

for $T_0 \leq t < T_1$.

Proof. Letting C_{Sob} denote the constant from the Sobolev inequality, we have that

$$\|w(T_0)\|_{L^\infty} \leq C_{\text{Sob}}\|w(T_0)\|_{H^s} \leq C_{\text{Sob}}\sigma.$$

We then choose σ to satisfy

$$\sigma \leq \min\left\{1, \frac{\hat{R}}{4}\right\}, \tag{2.5.19}$$

where $\hat{R} = \frac{R}{2C_{\text{Sob}}}$, so that

$$\|w(T_0)\|_{L^\infty} \leq \frac{R}{8}.$$

Next, we define

$$K_1(t) = \|w\|_{L^\infty([T_0,t),H^s)} \quad \text{and} \quad K_2(t) = \|u\|_{L^\infty([T_0,t),H^{s-1})},$$

and observe that $K_1(T_0) + K_2(T_0) \leq \hat{R}/2$, and hence, by continuity, either $K_1(t) + K_2(t) < \hat{R}$ for all $t \in [T_0, T_1)$, or else there exists a first time $T_* \in (T_0, T_1)$ such that $K_1(T_*) + K_2(T_*) = \hat{R}$. Letting $T_* = T_1$ if the first case holds, we then have that

$$K_1(t) + K_2(t) < \hat{R}, \quad 0 \leq t < T_*, \tag{2.5.20}$$

where $T_* = T_1$ or else T_* is the first time in (T_0, T_1) for which $K_1(T_*) + K_2(T_*) = \hat{R}$.

Before proceeding the proof, we first establish a number of preliminary estimates, which we collect together in the following Lemma.

Lemma 2.5.8. *There exists constants $C(K_1(t))$ and $C(K_1(t), K_2(t))$, both independent of $\epsilon \in (0, \epsilon_0)$ and $T_* \in (T_0, T_1]$, such that the following estimates hold for $T_0 \leq t < T_* < 0$:*

$$-\frac{2}{t}\sum_{|\alpha|\leq s}\langle D^\alpha w, A_1^0[(A_1^0)^{-1}\mathfrak{A}_1, D^\alpha]\mathbb{P}_1 w\rangle \leq -\frac{1}{t}C(K_1)\|w\|_{H^s}\|\mathbb{P}_1 w\|^2_{H^s}, \tag{2.5.21}$$

$$-\frac{2}{t}\sum_{|\alpha|\leq s-1}\langle D^\alpha u, A_2^0[(A_2^0)^{-1}\mathfrak{A}_2, D^\alpha]\mathbb{P}_2 u\rangle \leq -\frac{1}{t}C(K_1)(\|u\|_{H^{s-1}} + \|w\|_{H^s})(\|\mathbb{P}_2 u\|^2_{H^{s-1}} + \|\mathbb{P}_2 w\|^2_{H^s}), \tag{2.5.22}$$

$$-\sum_{|\alpha|\leq s}\langle D^\alpha w, A_1^0[D^\alpha, (A_1^0)^{-1}A_1^i]\partial_i w\rangle \leq C(K_1)\|w\|^2_{H^s}, \tag{2.5.23}$$

$$-\sum_{|\alpha|\leq s-1}\langle D^\alpha u, A_2^0[D^\alpha, (A_2^0)^{-1}A_2^i]\partial_i u\rangle \leq C(K_1)\|u\|^2_{H^{s-1}}, \tag{2.5.24}$$

$$-\sum_{|\alpha|\leq s}\langle D^\alpha w, [\tilde{A}_1^0, D^\alpha](A_1^0)^{-1}C_1^i\partial_i w\rangle \leq C(K_1)\|w\|^2_{H^s}, \tag{2.5.25}$$

$$-\sum_{|\alpha|\leq s-1}\langle D^\alpha u, [\tilde{A}_2^0, D^\alpha](A_2^0)^{-1}C_2^i\partial_i u\rangle \leq C(K_1)\|u\|^2_{H^{s-1}}, \tag{2.5.26}$$

$$\sum_{|\alpha|\leq s}\langle D^\alpha w, (\partial_t A_1^0)D^\alpha w\rangle \leq C(K_1)\|w\|^2_{H^s} - \frac{1}{t}C(K_1)\|w\|_{H^s}\|\mathbb{P}_1 w\|^2_{H^s}, \tag{2.5.27}$$

$$\sum_{|\alpha|\leq s-1}\langle D^\alpha u, (\partial_t A_2^0)D^\alpha u\rangle \leq C(K_1)\|u\|^2_{H^{s-1}} - \frac{1}{t}C(K_1, K_2)(\|u\|_{H^{s-1}} + \|w\|_{H^s})(\|\mathbb{P}_2 u\|^2_{H^{s-1}} + \|\mathbb{P}_1 w\|^2_{H^s}) \tag{2.5.28}$$

and

$$\sum_{|\alpha|\leq s-1}\langle D^\alpha \mathbb{P}_3 U, (\partial_t A^0)D^\alpha \mathbb{P}_3 U\rangle \leq -\frac{1}{t}C(K_1)\|\mathbb{P}_1 w\|_{H^s}\|\mathbb{P}_3 U\|^2_{H^{s-1}} + C(K_1)\|\mathbb{P}_3 U\|^2_{H^{s-1}}. \tag{2.5.29}$$

Proof. Using the properties $\mathbb{P}_1^2 = \mathbb{P}_1$, $\mathbb{P}_1 + \mathbb{P}_1^\perp = \mathbb{1}$, $\mathbb{P}_1^{\mathrm{T}} = \mathbb{P}_1$, and $D\mathbb{P}_1 = 0$ of the projection matrix

$\mathbb{P}_1$ repeatedly, we compute

$$\begin{aligned}
&-\frac{2}{t}\sum_{|\alpha|\leq s}\langle D^\alpha w, A_1^0[(A_1^0)^{-1}\mathfrak{A}_1, D^\alpha]\mathbb{P}_1 w\rangle \\
&\quad= -\frac{2}{t}\sum_{|\alpha|\leq s}\langle D^\alpha \mathbb{P}_1 w, A_1^0[(A_1^0)^{-1}\mathfrak{A}_1, D^\alpha]\mathbb{P}_1 w\rangle - \frac{2}{t}\sum_{|\alpha|\leq s}\langle D^\alpha \mathbb{P}_1^\perp w, \mathbb{P}_1^\perp A_1^0[(A_1^0)^{-1}\mathfrak{A}_1, D^\alpha]\mathbb{P}_1 w\rangle \\
&\overset{\text{by (2.5.5)}}{=} -\frac{2}{t}\sum_{|\alpha|\leq s}\langle D^\alpha \mathbb{P}_1 w, A_1^0[(A_1^0)^{-1}\mathfrak{A}_1, D^\alpha]\mathbb{P}_1 w\rangle - \frac{2}{t}\sum_{|\alpha|\leq s}\langle D^\alpha \mathbb{P}_1^\perp w, \mathbb{P}_1^\perp A_1^0[(A_1^0)^{-1}\mathbb{P}_1\mathfrak{A}_1, D^\alpha]\mathbb{P}_1 w\rangle \\
&\quad= -\frac{2}{t}\sum_{|\alpha|\leq s}\langle D^\alpha \mathbb{P}_1 w, A_1^0[(A_1^0)^{-1}\mathfrak{A}_1, D^\alpha]\mathbb{P}_1 w\rangle - \frac{2}{t}\sum_{|\alpha|\leq s}\langle D^\alpha \mathbb{P}_1^\perp w, \mathbb{P}_1^\perp A_1^0\mathbb{P}_1^\perp[\mathbb{P}_1^\perp(A_1^0)^{-1}\mathbb{P}_1\mathfrak{A}_1, D^\alpha]\mathbb{P}_1 w\rangle \\
&\qquad - \frac{2}{t}\sum_{|\alpha|\leq s}\langle D^\alpha \mathbb{P}_1^\perp w, \mathbb{P}_1^\perp A_1^0\mathbb{P}_1[\mathbb{P}_1(A_1^0)^{-1}\mathbb{P}_1\mathfrak{A}_1, D^\alpha]\mathbb{P}_1 w\rangle.
\end{aligned}$$

From this expression, we obtain, with the help the Cauchy-Schwarz inequality, the calculus inequalities from Appendix C, the expansions (2.5.2)-(2.5.3), the relations (2.5.4), (2.5.13), and (2.5.15), and the inequality (2.5.20), the estimate

$$\begin{aligned}
&-\frac{1}{t}\sum_{|\alpha|\leq s}\langle D^\alpha w, A_1^0[(A_1^0)^{-1}\mathfrak{A}_1, D^\alpha]\mathbb{P}_1 w\rangle \\
&\quad\lesssim -\frac{1}{t}\Big[\|A_1^0\|_{H^s}\|\mathbb{P}_1 w\|_{H^s}\|D\big((A_1^0)^{-1}\mathfrak{A}_1\big)]\|_{H^{s-1}} + \|A_1^0\|_{H^s}\|\mathbb{P}_1^\perp w\|_{H^s}\|D\big(\mathbb{P}_1^\perp(A_1^0)^{-1}\mathbb{P}_1\mathfrak{A}_1\big)\|_{H^{s-1}} \\
&\qquad + \|\mathbb{P}_1^\perp A_1^0\mathbb{P}_1\|_{H^s}\|\mathbb{P}_1^\perp w\|_{H^s}\|D\big(\mathbb{P}_1(A_1^0)^{-1}\mathbb{P}_1\mathfrak{A}_1\big)\|_{H^{s-1}}\Big]\|\mathbb{P}_1 w\|_{H^{s-1}} \leq -C(K_1)\frac{1}{t}\|w\|_{H^s}\|\mathbb{P}_1 w\|^2_{H^s}
\end{aligned}$$

for $T_0 \leq t < T_*$, where the constant $C(K_1)$ is independent of $\epsilon \in (0, \epsilon_0)$ and $T_* \in (T_0, T_1]$. This establishes the estimate (2.5.21). By a similar calculation, we find that

$$\begin{aligned}
&-\frac{2}{t}\sum_{|\alpha|\leq s-1}\langle D^\alpha u, A_2^0[(A_2^0)^{-1}\mathfrak{A}_2, D^\alpha]\mathbb{P}_2 u\rangle = -\frac{2}{t}\sum_{|\alpha|\leq s-1}\langle D^\alpha \mathbb{P}_2 u, A_2^0[(A_2^0)^{-1}\mathfrak{A}_2, D^\alpha]\mathbb{P}_2 u\rangle \\
&-\frac{2}{t}\sum_{|\alpha|\leq s-1}\langle D^\alpha \mathbb{P}_2^\perp u, \mathbb{P}_2^\perp A_2^0\mathbb{P}_2^\perp[\mathbb{P}_2^\perp(A_2^0)^{-1}\mathbb{P}_2\mathfrak{A}_2, D^\alpha]\mathbb{P}_2 u\rangle - \frac{2}{t}\sum_{|\alpha|\leq s-1}\langle D^\alpha \mathbb{P}_2^\perp u, \mathbb{P}_2^\perp A_2^0\mathbb{P}_2[\mathbb{P}_2(A_2^0)^{-1}\mathbb{P}_2\mathfrak{A}_2, D^\alpha]\mathbb{P}_2 u\rangle \\
&\leq -\frac{1}{t}C(K_1)\|w\|_{H^s}\|\mathbb{P}_2 u\|^2_{H^{s-1}} - \frac{1}{t}C(K_1)\|u\|_{H^{s-1}}\|\mathbb{P}_1 w\|_{H^s}\|\mathbb{P}_2 u\|_{H^{s-1}} - \frac{1}{t}C(K_1)\|u\|_{H^{s-1}}\|\mathbb{P}_1 w\|_{H^s}\|\mathbb{P}_2 u\|_{H^{s-1}} \\
&\qquad\qquad \leq -\frac{1}{t}C(K_1)(\|u\|_{H^{s-1}} + \|w\|_{H^s})(\|\mathbb{P}_2 u\|^2_{H^{s-1}} + \|\mathbb{P}_2 w\|^2_{H^s}),
\end{aligned}$$

which establishes the estimate (2.5.22).

Next, using the calculus inequalities from Appendix C, we observe that

$$\sum_{0\leq|\alpha|\leq s-1}\langle D^\alpha u, -A_2^0[D^\alpha, (A_2^0)^{-1}A_2^i]\partial_i u\rangle \lesssim \|A_2^0\|_{L^\infty}\|u\|^2_{H^{s-1}}\|D((A_2^0)^{-1}A_2^i)\|_{H^{s-1}} \leq C(K_1)\|u\|^2_{H^{s-1}},$$

which establishes the estimate (2.5.24). Since the estimates (2.5.23), (2.5.25) and (2.5.26) can be obtained in a similar fashion, we omit the details.

Finally, we consider the estimates (2.5.27)-(2.5.28). We begin establishing these estimates by writing (2.5.12) as

$$\epsilon\partial_0 w = \epsilon\frac{1}{t}(A_1^0)^{-1}\mathfrak{A}_1\mathbb{P}_1 w - \epsilon(A_1^0)^{-1}A_1^i\partial_i w - (A_1^0)^{-1}C_1^i\partial_i w + \epsilon(A_1^0)^{-1}H_1 + \epsilon(A_1^0)^{-1}F_1. \tag{2.5.30}$$

Using this and the expansion (2.5.2), we can express the time derivatives $\partial_t A^0_a$, $a = 1, 2$, as

$$\begin{aligned}\partial_t A^0_a =& D_w A^0_a \cdot \partial_t w + D_t A^0_a \\ =& - D_w A^0_a \cdot (A^0_1)^{-1} A^i_1 \partial_i w - [D_w \tilde{A}^0_a \cdot (A^0_1)^{-1} C^i_1 \partial_i w] \\ &+ [D_w A^0_a \cdot (A^0_1)^{-1} H_1] + D_t A^0_a + [D_w A^0_a \cdot (A^0_1)^{-1} F_1] + \frac{1}{t}[D_w A^0_a \cdot (A^0_1)^{-1} \mathfrak{A}_1 \mathbb{P}_1 w]. \end{aligned} \tag{2.5.31}$$

Using (2.5.31) with $a = 2$, we see, with the help of the calculus inequalities from Appendix C, the Cauchy-Schwarz inequality, the estimate (2.5.17), and the expansion (2.5.11) for $a = 2$, that

$$\begin{aligned}\sum_{|\alpha|\leq s-1} \langle D^\alpha u, (\partial_t A^0_2) D^\alpha u\rangle \leq & \sum_{|\alpha|\leq s-1} \Big[\langle D^\alpha u, \mathbb{P}_2^\perp(\partial_t A^0_2)\mathbb{P}_2^\perp D^\alpha u\rangle + \langle D^\alpha u, \mathbb{P}_2^\perp(\partial_t A^0_2)\mathbb{P}_2 D^\alpha u\rangle \\ & + \langle D^\alpha u, \mathbb{P}_2(\partial_t A^0_2)\mathbb{P}_2^\perp D^\alpha u\rangle + \langle D^\alpha u, \mathbb{P}_2(\partial_t A^0_2)\mathbb{P}_2 D^\alpha u\rangle\Big] \\ \leq & C(K_1)\|u\|^2_{H^{s-1}} - \frac{2}{t}\|u\|_{H^{s-1}}\|(A^0_1)^{-1}\mathfrak{A}_1\|_{L^\infty}\|D_w A^0_2\|_{L^\infty}\|\mathbb{P}_2 u\|_{H^{s-1}}\|\mathbb{P}_1 w\|_{H^{s-1}} \\ & - \frac{1}{t}\|\mathbb{P}_1 w\|_{H^s}\|(A^0)^{-1}\mathfrak{A}\|_{L^\infty}\|D_w A^0_2\|_{L^\infty}\|\mathbb{P}_2 u\|^2_{H^{s-1}} - \frac{1}{t}\|u\|^2_{H^{s-1}} C(K_1)\|\mathbb{P}_1 w\|^2_{H^{s-1}} \\ \leq & C(K_1)\|u\|^2_{H^{s-1}} - \frac{1}{t}C(K_1, K_2)(\|u\|_{H^{s-1}} + \|w\|_{H^s})(\|\mathbb{P}_2 u\|^2_{H^{s-1}} + \|\mathbb{P}_1 w\|^2_{H^s}).\end{aligned}$$

This establishes the estimate (2.5.28). Since the estimate (2.5.27) can be established using similar arguments, we omit the details. The last estimate (2.5.29) can also be established using similar arguments with the help of the identity $\mathbb{P}_3\mathbb{P} = \mathbb{P}\mathbb{P}_3 = \mathbb{P}_3$. We again omit the details. □

Applying $A^0 D^\alpha (A^0)^{-1}$ to both sides of (2.5.1), we find that

$$\begin{aligned}A^0\partial_0 D^\alpha U + A^i\partial_i D^\alpha U + \frac{1}{\epsilon}C^i\partial_i D^\alpha U =& -A^0[D^\alpha, (A^0)^{-1}A^i]\partial_i U - [\tilde{A}^0, D^\alpha](A^0)^{-1}C^i\partial_i U \\ &+ \frac{1}{t}\mathfrak{A}D^\alpha\mathbb{P}U + \frac{1}{t}A^0[D^\alpha, (A^0)^{-1}\mathfrak{A}]\mathbb{P}U + A^0 D^\alpha[(A^0)^{-1}H],\end{aligned} \tag{2.5.32}$$

where in deriving this we have used

$$\frac{1}{\epsilon}[A^0, D^\alpha](A^0)^{-1}C^i\partial_i U \overset{(2.5.2)}{=} \frac{1}{\epsilon}[\mathring{A}^0 + \epsilon\tilde{A}^0, D^\alpha](A^0)^{-1}C^i\partial_i U = [\tilde{A}^0, D^\alpha](A^0)^{-1}C^i\partial_i U$$

and

$$\begin{aligned}A^0[D^\alpha, (A^0)^{-1}]C^i\partial_i U =& A^0 D^\alpha\big((A^0)^{-1}C^i\partial_i U\big) - D^\alpha\big(C^i\partial_i U\big) \\ =& A^0 D^\alpha\big((A^0)^{-1}C^i\partial_i U\big) - D^\alpha\big(A^0(A^0)^{-1}C^i\partial_i U\big) = [A^0, D^\alpha](A^0)^{-1}C^i\partial_i U.\end{aligned}$$

Writing A^0_a, $a = 1, 2$, as $A^0_a = (A^0_a)^{\frac{1}{2}}(A^0_a)^{\frac{1}{2}}$, which we can do since A^0_a is a real symmetric and positive-definite, we see from (2.5.9) that

$$(A^0_a)^{-\frac{1}{2}}\mathfrak{A}_{\mathfrak{a}}(A^0_a)^{-\frac{1}{2}} \geq \kappa\mathbb{1}. \tag{2.5.33}$$

Since, by (2.5.5),

$$\frac{2}{t}\langle D^\alpha f, \mathfrak{A}_a D^\alpha\mathbb{P}_a f\rangle = \frac{2}{t}\langle D^\alpha\mathbb{P}_a f, (A^0)^{\frac{1}{2}}[(A^0_a)^{-\frac{1}{2}}\mathfrak{A}_a(A^0_a)^{-\frac{1}{2}}](A^0_a)^{\frac{1}{2}}D^\alpha\mathbb{P}_a f\rangle, \quad a = 1, 2,$$

it follows immediately from (2.5.33) that

$$\frac{2}{t}\sum_{0\leq|\alpha|\leq s-1}\langle D^\alpha u, \mathfrak{A}_2 D^\alpha\mathbb{P}_2 u\rangle \leq \frac{2\kappa}{t}|||\mathbb{P}_2 u|||^2_{2,H^{s-1}} \quad \text{and} \quad \frac{2}{t}\sum_{0\leq|\alpha|\leq s}\langle D^\alpha w, \mathfrak{A}_1 D^\alpha\mathbb{P}_1 w\rangle \leq \frac{2\kappa}{t}|||\mathbb{P}_1 w|||^2_{1,H^s}. \tag{2.5.34}$$

Then, differentiating $\langle D^\alpha w, A_1^0 D^\alpha w\rangle$ with respect to t, we see, from the identities $\langle D^\alpha w, C_1^i \partial_i D^\alpha w\rangle = 0$ and $2\langle D^\alpha w, A_1^i \partial_i D^\alpha w\rangle = -\langle D^\alpha w, (\partial_i A_1^i) D^\alpha w\rangle$, the block decomposition of (2.5.32), which we can use to determine $D^\alpha \partial_t w$, the estimates (2.5.17) and (2.5.34) together with those from from Lemma 2.5.8 and the calculus inequalities from Appendix C, that

$$\begin{aligned}
\partial_t |\!|\!| w |\!|\!|^2_{1,H^s} = & \sum_{0\leq|\alpha|\leq s} \langle D^\alpha w, (\partial_t A_1^0) D^\alpha w\rangle + 2\sum_{0\leq|\alpha|\leq s} \langle D^\alpha w, A_1^0 D^\alpha \partial_t w\rangle \\
\leq & C(K_1)\|w\|^2_{H^s} - \frac{1}{t}C(K_1)\|w\|_{H^s}\|\mathbb{P}_1 w\|^2_{H^s} + \sum_{0\leq|\alpha|\leq s}\langle D^\alpha w, (\partial_i A_1^i) D^\alpha w\rangle \\
& - \frac{2}{\epsilon}\sum_{0\leq|\alpha|\leq s} \overbrace{\langle D^\alpha w, C_1^i \partial_i D^\alpha w\rangle}^{=0} - 2\sum_{0\leq|\alpha|\leq s}\langle D^\alpha w, A_1^0[D^\alpha, (A_1^0)^{-1}A_1^i]\partial_i w\rangle \\
& - 2\sum_{0\leq|\alpha|\leq s}\langle D^\alpha w, [\tilde{A}_1^0, D^\alpha](A_1^0)^{-1}C_1^i\partial_i w\rangle + \frac{2}{t}\sum_{0\leq|\alpha|\leq s}\langle D^\alpha w, \mathfrak{A}_1 D^\alpha \mathbb{P}_1 w\rangle \\
& + \frac{2}{t}\sum_{0\leq|\alpha|\leq s}\langle D^\alpha w, A_1^0[(A_1^0)^{-1}\mathfrak{A}_1, D^\alpha]\mathbb{P}_1 w\rangle + 2\sum_{0\leq|\alpha|\leq s}\langle D^\alpha w, A_1^0 D^\alpha[(A_1^0)^{-1}(H_1+F_1)]\rangle \\
\leq & C(K_1)|\!|\!| w |\!|\!|^2_{1,H^s} + \frac{1}{t}\big[2\kappa - C_1(K_1)\|w\|_{H^s}\big]|\!|\!|\mathbb{P}_1 w|\!|\!|^2_{1,H^s}
\end{aligned} \tag{2.5.35}$$

for $t\in[T_0,T_*)$. By similar calculation, we obtain from differentiating $\langle D^\alpha u, A_2^0 D^\alpha u\rangle$ with respect to t the estimate

$$\begin{aligned}
\partial_t |\!|\!| u |\!|\!|^2_{2,H^{s-1}} = & \sum_{0\leq|\alpha|\leq s-1}\langle D^\alpha u, (\partial_t A_2^0)D^\alpha u\rangle + 2\sum_{0\leq|\alpha|\leq s-1}\langle D^\alpha u, A_2^0 D^\alpha \partial_t u\rangle \\
\leq & C(K_1)\|u\|^2_{H^{s-1}} - \frac{1}{t}C(K_1,K_2)(\|u\|_{H^{s-1}} + \|w\|_{H^s})(\|\mathbb{P}_2 u\|^2_{H^{s-1}} + \|\mathbb{P}_1 w\|^2_{H^s}) \\
& \sum_{0\leq|\alpha|\leq s-1}\langle D^\alpha u, (\partial_i A_2^i)D^\alpha u\rangle - \frac{2}{\epsilon}\sum_{0\leq|\alpha|\leq s-1}\overbrace{\langle D^\alpha u, C_2^i\partial_i D^\alpha u\rangle}^{=0} \\
& - 2\sum_{0\leq|\alpha|\leq s-1}\langle D^\alpha u, A_2^0[D^\alpha,(A_2^0)^{-1}A_2^i]\partial_i u\rangle - 2\sum_{0\leq|\alpha|\leq s-1}\langle D^\alpha u, [\tilde{A}_2^0, D^\alpha](A_2^0)^{-1}C_2^i\partial_i u\rangle \\
& + \frac{2}{t}\sum_{0\leq|\alpha|\leq s}\langle D^\alpha u, \mathfrak{A}_2 D^\alpha \mathbb{P}_2 u\rangle - \frac{2}{t}\sum_{0\leq|\alpha|\leq s-1}\langle D^\alpha u, A_2^0[(A_2^0)^{-1}\mathfrak{A}_2, D^\alpha]\mathbb{P}_2 u\rangle \\
& + 2\sum_{0\leq|\alpha|\leq s-1}\Big\langle D^\alpha u, A_2^0 D^\alpha[(A_2^0)^{-1}\big(H_2 + \frac{1}{t}M_2\mathbb{P}_3 U + F_2\big)]\Big\rangle \\
\leq & C(K_1,K_2)(|\!|\!| u|\!|\!|^2_{2,H^{s-1}} + |\!|\!| w|\!|\!|^2_{1,H^s}) - \frac{1}{2t}C_2(K_1,K_2)(\|u\|_{H^{s-1}} + \|w\|_{H^s})|\!|\!|\mathbb{P}_1 w|\!|\!|^2_{1,H^s} \\
& + \frac{1}{t}\big[2\kappa - C_2(K_1,K_2)(\|u\|_{H^{s-1}} + \|w\|_{H^s})\big]|\!|\!|\mathbb{P}_2 u|\!|\!|^2_{2,H^{s-1}} \\
& - C(K_1)\frac{1}{t}(|\!|\!| u|\!|\!|^2_{2,H^{s-1}} + |\!|\!| w|\!|\!|^2_{1,H^s})|\!|\!|\mathbb{P}_3 U|\!|\!|_{H^{s-1}}
\end{aligned} \tag{2.5.36}$$

for $t\in[T_0,T_*)$.

Applying the operator $A^0 D^\alpha \mathbb{P}^3 (A^0)^{-1}$ to (2.5.1), we see, with the help of (2.5.6)-(2.5.8), that

$$\begin{aligned}
A^0\partial_0 D^\alpha \mathbb{P}_3 U + \mathbb{P}_3 A^i \mathbb{P}_3 \partial_i D^\alpha \mathbb{P}_3 U + \frac{1}{\epsilon}\mathbb{P}_3 C^i \mathbb{P}_3 \partial_i D^\alpha \mathbb{P}_3 U = & - A^0[D^\alpha, (A^0)^{-1}\mathbb{P}_3 A^i \mathbb{P}_3]\partial_i \mathbb{P}_3 U \\
-[\tilde{A}^0, D^\alpha](A^0)^{-1}\mathbb{P}_3 C^i \mathbb{P}_3 \partial_i \mathbb{P}_3 U + \frac{1}{t}\mathbb{P}_3\mathfrak{A}\mathbb{P}_3 D^\alpha \mathbb{P}_3 U + \frac{1}{t}A^0[D^\alpha, & (A^0)^{-1}\mathbb{P}_3\mathfrak{A}\mathbb{P}_3]\mathbb{P}_3 U + A^0 D^\alpha[(A^0)^{-1}\mathbb{P}_3 H].
\end{aligned} \tag{2.5.37}$$

Then, by similar arguments used to derive (2.5.35) and (2.5.36), we obtain from (2.5.37) the estimate

$$\begin{aligned}\partial_t\|\!|\mathbb{P}_3U\|\!|^2_{H^{s-1}} =& \sum_{0\leq|\alpha|\leq s-1}\langle D^\alpha\mathbb{P}_3U,(\partial_tA^0)D^\alpha\mathbb{P}_3U\rangle+2\sum_{0\leq|\alpha|\leq s-1}\langle D^\alpha\mathbb{P}_3U,\mathbb{P}_3A^0\mathbb{P}_3D^\alpha\partial_t\mathbb{P}_3U\rangle\\
\leq& -\frac{1}{t}C(K_1)\|\mathbb{P}_1w\|_{H^s}\|\mathbb{P}_3U\|^2_{H^{s-1}}+C(K_1)\|\mathbb{P}_3U\|^2_{H^{s-1}}\\
&+\sum_{0\leq|\alpha|\leq s-1}\langle D^\alpha\mathbb{P}_3U,(\partial_iA^i)D^\alpha\mathbb{P}_3U\rangle-\frac{2}{\epsilon}\sum_{0\leq|\alpha|\leq s-1}\overbrace{\langle D^\alpha\mathbb{P}_3U,C^i\partial_iD^\alpha\mathbb{P}_3U\rangle}^{=0}\\
&-2\sum_{0\leq|\alpha|\leq s-1}\langle D^\alpha\mathbb{P}_3U,A^0[D^\alpha,(A^0)^{-1}A^i]\partial_i\mathbb{P}_3U+[\tilde{A}^0,D^\alpha](A^0)^{-1}C^i\partial_i\mathbb{P}_3U\rangle\\
&+\frac{2}{t}\sum_{0\leq|\alpha|\leq s}\langle D^\alpha\mathbb{P}_3U,\mathfrak{A}D^\alpha\mathbb{P}_3U\rangle+\frac{2}{t}\sum_{0\leq|\alpha|\leq s-1}\langle D^\alpha\mathbb{P}_3U,A^0[(A^0)^{-1}\mathfrak{A},D^\alpha]\mathbb{P}_3U\rangle\\
&+2\sum_{0\leq|\alpha|\leq s-1}\big\langle D^\alpha\mathbb{P}_3U,A^0D^\alpha[(A^0)^{-1}\mathbb{P}_3H]\big\rangle\\
\leq& C(K_1)\|\mathbb{P}_3U\|^2_{H^{s-1}}+C(K_1)\|\mathbb{P}_3U\|_{H^{s-1}}\Big(\|H_1\|_{H^{s-1}}+\|H_2\|_{H^{s-1}}+\|F_1\|_{H^{s-1}}\\
&+\|F_2\|_{H^{s-1}}\Big)+\frac{1}{t}\Big(2\kappa-C_2(K_1,K_2)\big(\|w\|_{H^s}+\|u\|_{H^{s-1}}\big)\Big)\|\!|\mathbb{P}_3U\|\!|^2_{H^{s-1}}\\
\leq& C(K_1)\|\!|\mathbb{P}_3U\|\!|^2_{H^{s-1}}+C(K_1,K_2)\big(\|\!|w\|\!|_{1,H^s}+\|\!|u\|\!|_{2,H^{s-1}})\big)\|\!|\mathbb{P}_3U\|\!|_{H^{s-1}}\\
&+\frac{1}{t}\Big(2\kappa-C_2(K_1,K_2)\big(\|w\|_{H^s}+\|u\|_{H^{s-1}}\big)\Big)\|\!|\mathbb{P}_3U\|\!|^2_{H^{s-1}}.\end{aligned}$$

Dividing the above estimate by $\|\!|\mathbb{P}_3U\|\!|_{H^{s-1}}$ gives

$$\begin{aligned}\partial_t\|\!|\mathbb{P}_3U\|\!|_{H^{s-1}}\leq& C(K_1)\|\!|\mathbb{P}_3U\|\!|_{H^{s-1}}+C(K_1,K_2)\big(\|\!|w\|\!|_{1,H^s}+\|\!|u\|\!|_{2,H^{s-1}})\big)\\
&+\frac{1}{t}\Big(\kappa-\frac{C_2(K_1,K_2)}{2}\big(\|w\|_{H^s}+\|u\|_{H^{s-1}}\big)\Big)\|\!|\mathbb{P}_3U\|\!|_{H^{s-1}}.\end{aligned}\tag{2.5.38}$$

Next, we choose $\sigma>0$ small enough so that

$$\Big(C_1(\hat{R})+2C_2(\hat{R},\hat{R})\Big)\sigma<\frac{\kappa}{2}$$

in addition to (2.5.19). Then since

$$2\kappa-\Big(C_1(K_1(T_0))\|w(T_0)\|_{H^s}+C_2(K_1(T_0),K_2(T_0))\big(\|w(T_0)\|_{H^s}+\|u(T_0)\|_{H^{s-1}}\big)\Big)>\kappa,$$

we see by continuity that either

$$2\kappa-\Big(C_1(K_1(t))\|w(t)\|_{H^s}+C_2(K_1(t),K_2(t))\big(\|w(t)\|_{H^s}+\|u(t)\|_{H^{s-1}}\big)\Big)>\kappa,\quad 0\leq t<T_*,$$

or else there exists a first time $T^*\in(0,T_*)$ such that

$$2\kappa-\Big(C_1(K_1(T^*))\|w(T^*)\|_{H^s}+C_2(K_1(T^*),K_2(T^*))(\|w(T^*)\|_{H^s}+\|u(T^*)\|_{H^{s-1}}\Big)=\kappa.$$

Thus if we let $T^*=T_*$ if the first case holds, then we have that

$$2\kappa-\Big(C_1(K_1(t))\|w(t)\|_{H^s}+C_2(K_1(t),K_2(t))\big(\|w(t)\|_{H^s}+\|u(t)\|_{H^{s-1}}\big)\Big)>\kappa,\quad 0\leq t<T^*\leq T_*.\tag{2.5.39}$$

Taken together, the estimates (2.5.20), (2.5.35), (2.5.36), (2.5.38) and (2.5.39) imply that

$$\partial_t\|\!|w\|\!|^2_{1,H^s}\leq C(\hat{R})\|\!|w\|\!|^2_{1,H^s}+\frac{\kappa}{t}\|\!|\mathbb{P}_1w\|\!|^2_{1,H^s},\tag{2.5.40}$$

$$\partial_t |||u|||^2_{2,H^{s-1}} \leq C(\hat{R})\big(|||u|||^2_{2,H^{s-1}} + |||w|||^2_{1,H^s}\big) - \frac{1}{t}C_3(\hat{R})\big(|||u|||^2_{2,H^{s-1}} + |||w|||^2_{1,H^s}\big)|||\mathbb{P}_3 U|||_{H^{s-1}} + \frac{\kappa}{2t}|||\mathbb{P}_1 w|||^2_{1,H^s} + \frac{\kappa}{t}|||\mathbb{P}_2 u|||^2_{2,H^{s-1}} \quad (2.5.41)$$

and

$$\partial_t |||\mathbb{P}_3 U|||_{H^{s-1}} \leq C(\hat{R})\big(|||\mathbb{P}_3 U|||_{H^{s-1}} + |||w|||_{1,H^s} + |||u|||_{2,H^{s-1}}\big) + \frac{\kappa}{2t}|||\mathbb{P}_3 U|||_{H^{s-1}} \quad (2.5.42)$$

for $0 \leq t < T^* \leq T_*$.

Next, we set

$$X = |||w|||^2_{1,H^s} + |||u|||^2_{2,H^{s-1}}, \quad Y = |||\mathbb{P}_1 w|||^2_{1,H^s} + |||\mathbb{P}_2 u|||^2_{2,H^{s-1}}, \quad \text{and} \quad Z = |||\mathbb{P}_3 U|||_{H^{s-1}}.$$

Since $C_3(\hat{R})X(T_0)/\sigma \leq C(\hat{R})\sigma$, we can choose σ small enough so that $C_3(\hat{R})X(T_0)/\sigma < \kappa/4$. Then by continuity, either $C_3(\hat{R})X(t)/\sigma \leq \kappa/4$ for $t \in [T_0, T^*)$, or else there exists a first time $T \in (T_0, T^*)$ such that $C_3(\hat{R})X(T)/\sigma = \kappa/4$. Thus if we set $T = T^*$ if the first case holds, then we have that

$$C_3(\hat{R})\frac{X(t)}{\sigma} < \kappa/4, \quad T_0 \leq t < T \leq T^* \leq T_*. \quad (2.5.43)$$

Adding the inequalities (2.5.40) and (2.5.41) and dividing the results by σ, we obtain, with the help of (2.5.43), the inequality

$$\partial_t\left(\frac{X}{\sigma}\right) \leq C(\hat{R})\frac{X}{\sigma} - \frac{\kappa}{4t}Z + \frac{\kappa}{2t}\frac{Y}{\sigma}, \quad T_0 \leq t < T \leq T^* \leq T_*, \quad (2.5.44)$$

while the inequality

$$\partial_t Z \leq C(\hat{R})\left(Z + \sigma + \frac{X}{\sigma}\right) + \frac{\kappa}{2t}Z, \quad T_0 \leq t < T^* \leq T_* \quad (2.5.45)$$

follows from (2.5.42) and Young's inequality. Adding (2.5.44) and (2.5.45), we find that

$$\partial_t\left(\frac{X}{\sigma} + Z - \frac{\kappa}{4}\int_{T_0}^t \frac{1}{\tau}\left(\frac{Y}{\sigma} + Z\right)d\tau + \sigma\right) \leq C(\hat{R})\left(\frac{X}{\sigma} + Z - \frac{\kappa}{4}\int_{T_0}^t \frac{1}{\tau}\left(\frac{Y}{\sigma} + Z\right)d\tau + \sigma\right) \quad (2.5.46)$$

for $T_0 \leq t < T \leq T^* \leq T_*$. Since $X(T_0) \leq C(\hat{R})\sigma^2$ and $Z(T_0) \lesssim \sigma$, it follows directly from (2.5.46) and Grönwall's inequality that

$$\frac{X}{\sigma} + Z - \frac{\kappa}{4}\int_{T_0}^t \frac{1}{\tau}\left(\frac{Y}{\sigma} + Z\right)d\tau + \sigma \leq e^{C(\hat{R})(t-T_0)}C(\hat{R})\sigma, \quad T_0 \leq t < T \leq T^* \leq T_*,$$

from which it follows that

$$\|w\|_{M^\infty_{\mathbb{P}_1,s}([T_0,t)\times\mathbb{T}^n)} + \|u\|_{M^\infty_{\mathbb{P}_2,s-1}([T_0,t)\times\mathbb{T}^n)} - \int_{T_0}^t \frac{1}{\tau}|||\mathbb{P}_3 U|||_{H^{s-1}}\,d\tau \leq C(\hat{R})\sigma, \quad T_0 \leq t < T \leq T^* \leq T_*, \quad (2.5.47)$$

where we stress that the constant $C(\hat{R})$ is independent of ϵ and the times T, T^*, T_*, and T_1. Choosing σ small enough, it is then clear from the estimate (2.5.47) and the definition of the times T, T^*, and T_1 that $T = T^* = T_* = T_1$, which completes the proof. □

2.5.2 Error estimates

In this section, we consider solutions of the singular initial value problem

$$A^0_1(\epsilon, t, x, w)\partial_0 w + A^i_1(\epsilon, t, x, w)\partial_i w + \frac{1}{\epsilon}C^i_1\partial_i w = \frac{1}{t}\mathfrak{A}_1(\epsilon, t, x, w)\mathbb{P}_1 w + H_1 + F_1 \quad \text{in} \quad [T_0, T_1) \times \mathbb{T}^n, \quad (2.5.48)$$

$$w(x)|_{t=T_0} = \mathring{w}^0(x) + \epsilon s^0(\epsilon, x) \qquad \text{in} \quad \{T_0\} \times \mathbb{T}^n, \tag{2.5.49}$$

where the matrices A_1^0, A_1^i, $i = 1, \dots, n$, and $\mathfrak{A}_1$ and the source terms H_1 and F_1 satisfy the conditions from Assumption 2.5.2. Our aim is to use the uniform a priori estimates from Theorem 2.5.7 to establish uniform a priori estimates for solutions of (2.5.48)-(2.5.49) and to establish an error estimate between solutions of (2.5.48)-(2.5.49) and solutions of the *limit equation*, which is defined by

$$\mathring{A}_1^0 \partial_0 \mathring{w} + \mathring{A}_1^i \partial_i \mathring{w} = \frac{1}{t} \mathring{\mathfrak{A}}_1 \mathbb{P}_1 \mathring{w} - C_1^i \partial_i v + \mathring{H}_1 + \mathring{F}_1 \qquad \text{in} \quad [T_0, T_1) \times \mathbb{T}^n, \tag{2.5.50}$$

$$C_1^i \partial_i \mathring{w} = 0 \qquad \text{in} \quad [T_0, T_1) \times \mathbb{T}^n, \tag{2.5.51}$$

$$\mathring{w}(x)|_{t=T_0} = \mathring{w}^0(x) \qquad \text{in} \quad \{T_0\} \times \mathbb{T}^n. \tag{2.5.52}$$

In this system, $\mathring{A}_1^0$ and $\mathring{\mathfrak{A}}_1$ are defined by (2.5.2) and (2.5.3) with $a = 1$, respectively, $\mathring{A}_1^i$ and $\mathring{H}_1$ are defined by the limits

$$\mathring{A}_1^i(t, x, \mathring{w}) = \lim_{\epsilon \searrow 0} A_1^i(\epsilon, t, x, \mathring{w}) \quad \text{and} \quad \mathring{H}_1(t, x, \mathring{w}) = \lim_{\epsilon \searrow 0} H_1(\epsilon, t, x, \mathring{w}), \tag{2.5.53}$$

respectively, and the following assumptions hold for fixed constants $R > 0$, $T_0 < T_1 < 0$ and $s \in \mathbb{Z}_{>n/2+1}$:

Assumptions 2.5.9.

1. *The source terms*[11] $\mathring{F}_1$ *and* v *satisfy* $\mathring{F}_1 \in C^0\big([T_0, T_1), H^s(\mathbb{T}^n, \mathbb{R}^{N_1})\big)$ *and* $v \in \bigcap_{\ell=0}^1 C^\ell\big([T_0, T_1), H^{s+1-\ell}(\mathbb{T}^n, \mathbb{R}^{N_1})\big)$.
2. *The matrices* $\mathring{A}_1^i$, $i = 1, \dots, n$ *and the source term* $\mathring{H}_1$ *satisfy*[12] $t\mathring{A}_1^i \in E^1\big((2T_0, 0) \times \mathbb{T}^n \times B_R(\mathbb{R}^{N_1}), \mathbb{S}_{N_1}\big)$, $t\mathring{H}_1 \in E^1\big((2T_0, 0) \times \mathbb{T}^n \times B_R(\mathbb{R}^{N_1}), \mathbb{R}^{N_1}\big)$, *and*
$$D_t\big(t\mathring{H}_1(t, x, 0)\big) = 0.$$

We are now ready to state and establish uniform a priori estimates for solutions of the singular initial value problem (2.5.48)-(2.5.49) and the associated limit equation defined by (2.5.50)-(2.5.52).

Theorem 2.5.10. *Suppose* $R > 0$, $s \in \mathbb{Z}_{>n/2+1}$, $T_0 < T_1 \le 0$, $\epsilon_0 > 0$, $\mathring{w}^0 \in H^s(\mathbb{T}^n, \mathbb{R}^M)$, $s^0 \in L^\infty\big((0, \epsilon_0), H^s(\mathbb{T}^n, \mathbb{R}^{N_1})\big)$, *Assumptions 2.5.2 and 2.5.9 hold, the maps*

$$(w, \mathring{w}) \in \bigcap_{\ell=0}^1 C^\ell\big([T_0, T_1), H^{s-\ell}(\mathbb{T}^n, \mathbb{R}^{N_1})\big) \times \bigcap_{\ell=0}^1 C^\ell\big([T_0, T_1), H^{s-\ell}(\mathbb{T}^n, \mathbb{R}^{N_1})\big)$$

define a solution to the initial value problems (2.5.48)-(2.5.49) *and* (2.5.50)-(2.5.52), *and for* $t \in [T_0, T_1)$, *the following estimate holds:*

$$\|v(t)\|_{H^{s+1}} - \frac{1}{t}\|\mathbb{P}_1 v(t)\|_{H^{s+1}} + \|\partial_t v(t)\|_{H^s} \le C\big(\|\mathring{w}\|_{L^\infty([T_0,t),H^s)}\big)\|\mathring{w}(t)\|_{H^s}, \tag{2.5.54}$$

$$\|\mathring{F}_1(t)\|_{H^s} + \|t\partial_t \mathring{F}_1(t)\|_{H^{s-1}} \le C\big(\|\mathring{w}\|_{L^\infty([T_0,t),H^s)}\big)\|\mathring{w}(t)\|_{H^s}, \tag{2.5.55}$$

$$\|F_1(\epsilon, t)\|_{H^s} \le C\big(\|w\|_{L^\infty([T_0,t),H^s)}\big)\|w(t)\|_{H^s}, \tag{2.5.56}$$

$$\|A_1^i(\epsilon, t, \cdot, \mathring{w}(t)) - \mathring{A}_1^i(t, \cdot, \mathring{w}(t))\|_{H^{s-1}} \le \epsilon C\big(\|\mathring{w}(t)\|_{L^\infty([T_0,t),H^s)}\big) \tag{2.5.57}$$

and

$$\|H_1(\epsilon, t, \cdot, \mathring{w}(t)) - \mathring{H}_1(t, \cdot, \mathring{w}(t))\|_{H^{s-1}} + \|F_1(\epsilon, t) - \mathring{F}_1(t)\|_{H^{s-1}}$$

[11] The source term $\mathring{F}_1$ should be thought of as the $\epsilon \searrow 0$ limit of F_1. This is made precise by the hypothesis (2.5.58) of Theorem 2.5.10.

[12] From the assumptions, see Assumption 2.5.2.(3)-(4), on A_1^i and H_1, it follows directly from the (2.5.53) that $\mathring{A}_1^i \in E^0\big((2T_0, 0) \times \mathbb{T}^n \times B_R(\mathbb{R}^{N_1}), \mathbb{S}_{N_1}\big)$ and $\mathring{H}_1 \in E^0\big((2T_0, 0) \times \mathbb{T}^n \times B_R(\mathbb{R}^{N_1}), \mathbb{R}^{N_1}\big)$.

$$\leq \epsilon C\big(\|w\|_{L^\infty([T_0,t),H^s)}, \|\mathring{w}\|_{L^\infty([T_0,t),H^s)}\big)(\|w(t)\|_{H^s} + \|z(t)\|_{H^{s-1}} + \|\mathring{w}(t)\|_{H^s}), \tag{2.5.58}$$

where

$$z = \frac{1}{\epsilon}(w - \mathring{w} - \epsilon v)$$

and the constants $C\big(\|w\|_{L^\infty([T_0,t),H^s)}\big)$, $C\big(\|\mathring{w}\|_{L^\infty([T_0,t),H^s)}\big)$ *and* $C\big(\|w\|_{L^\infty([T_0,T_t),H^s)}, \|\mathring{w}\|_{L^\infty([T_0,t),H^s)}\big)$ *are independent of* $\epsilon \in (0, \epsilon_0)$ *and the time* $T_1 \in (T_0, 0)$.

Then there exists a small constant $\sigma > 0$, *independent of* $\epsilon \in (0, \epsilon_0)$ *and* $T_1 \in (T_0, 0)$, *such that if initially*

$$\|\mathring{w}^0\|_{H^s} + \|s^0\|_{H^s} \leq \sigma \quad \textit{and} \quad C_1^i \partial_i \mathring{w}^0 = 0, \tag{2.5.59}$$

then

$$\max\{\|w\|_{L^\infty([T_0,T_1)\times\mathbb{T}^n)}, \|w\|_{L^\infty([T_0,T_1)\times\mathbb{T}^n)}\} \leq \frac{R}{2} \tag{2.5.60}$$

and there exists a constant $C > 0$, *independent of* $\epsilon \in (0, \epsilon_0)$ *and* $T_1 \in (T_0, 0)$, *such that*

$$\|w\|_{M^\infty_{\mathbb{P}_1,s}([T_0,T_1)\times\mathbb{T}^n)} + \|\mathring{w}\|_{M^\infty_{\mathbb{P}_1,s}([T_0,T_1)\times\mathbb{T}^n)} + \|t\partial_t\mathring{w}\|_{M^\infty_{\mathbb{1},s-1}([T_0,T_1)\times\mathbb{T}^n)} + \int_{T_0}^t \|\partial_t \mathring{w}\|_{H^{s-1}} d\tau - \int_{T_0}^t \frac{1}{\tau}\|\mathbb{P}_1\mathring{w}\|_{H^{s-1}} d\tau \leq C\sigma, \tag{2.5.61}$$

$$\|w - \mathring{w}\|_{L^\infty([T_0,t),H^{s-1})} \leq \epsilon C\sigma \tag{2.5.62}$$

and

$$-\int_{T_0}^t \frac{1}{\tau}\|\mathbb{P}_1(w - \mathring{w})\|^2_{H^{s-1}} d\tau \leq \epsilon^2 C\sigma^2 \tag{2.5.63}$$

for $T_0 \leq t < T_1$.

Proof. First, we observe, by (2.5.2) and (2.5.10), that A_1^0 satisfies

$$\mathbb{P}_1^\perp \mathring{A}_1^0 \mathbb{P}_1 = \mathbb{P}_1 \mathring{A}_1^0 \mathbb{P}_1^\perp. \tag{2.5.64}$$

Using this, we find, after applying $\mathbb{P}_1$ to the limit equation (2.5.50), that

$$b = \mathbb{P}_1 \mathring{w} \tag{2.5.65}$$

satisfies the equation

$$\mathbb{P}_1\mathring{A}_1^0\mathbb{P}_1\partial_t b + \mathbb{P}_1\mathring{A}_1^i\mathbb{P}_1\partial_i b = \frac{1}{t}\mathbb{P}_1\mathring{\mathfrak{A}}_1\mathbb{P}_1 b + \mathbb{P}_1\mathring{H}_1 + \mathbb{P}_1\bar{F}_2, \tag{2.5.66}$$

where

$$\bar{F}_2 = -\mathbb{P}_1\mathring{A}_1^i\mathbb{P}_1^\perp\partial_i\mathring{w} + \mathbb{P}_1\mathring{F}_1 - \mathbb{P}_1 C_1^i\partial_i v.$$

Clearly, $\bar{F}_2$ satisfies

$$\|\bar{F}_2(t)\|_{H^{s-1}} \leq C\big(\|\mathring{w}\|_{L^\infty(T_0,t),H^s)}\big)\|\mathring{w}(t)\|_{H^s} \tag{2.5.67}$$

for $0 \leq t < T_1$ by (2.5.54), (2.5.55) and the calculus inequalities from Appendix C, while

$$\|b(T_0)\|_{H^{s-1}} \leq \|\mathring{w}^0\|_{H^s} \leq \sigma, \tag{2.5.68}$$

by the assumption (2.5.59) on the initial data, and $\mathbb{P}_1\mathring{H}_1(t,x,\mathring{w})$ satisfies

$$\mathbb{P}_1\mathring{H}_1(t,x,0)=0 \tag{2.5.69}$$

by Assumption 2.5.2.(3).

Next, we set

$$y=t\partial_t\mathring{w}.$$

In order to derive an evolution equation for y, we apply $t\partial_t$ to (2.5.50) and use the identity

$$t\partial_t f=tD_tf+[D_{\mathring{w}}f\cdot t\partial_t\mathring{w}]=D_t(tf)-f+[D_{\mathring{w}}f\cdot t\partial_t\mathring{w}],\quad f=f(t,x,\mathring{w}(t,x)),$$

to obtain

$$\mathring{A}^0_1\partial_t y+\mathring{A}^i_1\partial_i y=\frac{1}{t}(\mathbb{P}_1\mathring{\mathfrak{A}}_1\mathbb{P}_1+\mathring{A}^0_1)y-\frac{1}{t}\mathring{\mathfrak{A}}_1 b+\tilde{R}_2+\tilde{H}_2+\tilde{F}_2, \tag{2.5.70}$$

where

$$\tilde{H}_2=D_t(t\mathring{H}_1)-\mathring{H}_1+[D_{\mathring{w}}\mathring{H}_1\cdot y]+(D_t\mathring{\mathfrak{A}}_1)b-(D_t\mathring{A}^0_1)y$$

and

$$\tilde{F}_2=-[D_{\mathring{w}}\mathring{A}^i_1\cdot y]\partial_i\mathring{w}-D_t(t\mathring{A}^i_1)\partial_i\mathring{w}+\mathring{A}^i_1\partial_i\mathring{w}+t\partial_t\mathring{F}_1+tC^i_1\partial_i\partial_t v.$$

Note that in deriving the above equation, we have used the identity

$$\mathring{\mathfrak{A}}_1\mathbb{P}_1=\mathbb{P}_1\mathring{\mathfrak{A}}_1=\mathbb{P}_1\mathring{\mathfrak{A}}_1\mathbb{P}_1, \tag{2.5.71}$$

which follows directly from (2.5.3) and (2.5.5). We further note by (2.5.54), (2.5.55) and Assumption 2.5.2.(4) and Assumption 2.5.9.(2), it is clear that $\tilde{F}_2$ and $\tilde{H}_2=\tilde{H}_2(t,x,\mathring{w},b,y)$ satisfy

$$\|\tilde{F}_2(t)\|_{H^{s-1}}\leq C\big(\|\mathring{w}\|_{H^s}\big)(\|y\|_{H^{s-1}}+\|\mathring{w}\|_{H^s}) \tag{2.5.72}$$

for $T_0\leq t<T_1$ and

$$\tilde{H}_2(t,x,0,0,0)=0, \tag{2.5.73}$$

respectively. Using (2.5.50) and (2.5.59), we see that

$$y|_{t=T_0}=\Big[(\mathring{A}^0_1)^{-1}\mathring{\mathfrak{A}}_1\mathbb{P}_1\mathring{w}-t(\mathring{A}^0_1)^{-1}\mathring{A}^i_1\partial_i\mathring{w}-t(\mathring{A}^0_1)^{-1}C^i_1\partial_i v+t(\mathring{A}^0_1)^{-1}\mathring{H}_1+t(\mathring{A}^0_1)^{-1}\mathring{F}_1\Big]\Big|_{t=T_0},$$

which in turn, implies, via (2.5.59), (2.5.54)-(2.5.55), and the calculus inequalities from Appendix C.1, that

$$\|y\|_{H^{s-1}}(T_0)\leq C(\sigma)\sigma.$$

A short computation using (2.5.48), (2.5.50) and (2.5.51) shows that

$$A^0_1\partial_t z+A^i_1\partial_i z+\frac{1}{\epsilon}C^i_1\partial_i z=\frac{1}{t}\mathfrak{A}_1\mathbb{P}_1 z+\hat{R}_2+\hat{F}_2, \tag{2.5.74}$$

where

$$\hat{F}_2=\frac{1}{\epsilon}(H_1-\mathring{H}_1)+\frac{1}{\epsilon}(F_1-\mathring{F}_1)-\frac{1}{\epsilon}(A^i_1-\mathring{A}^i_1)\partial_i\mathring{w}-A^i_1\partial_i v-A^0_1\partial_t v+\frac{1}{t}\mathbb{P}_1\mathfrak{A}_1\mathbb{P}_1 v$$

and

$$\hat{R}_2=-\frac{1}{t}\tilde{A}^0_1 y+\frac{1}{t}\tilde{\mathfrak{A}}_1 b,$$

and we recall that $\tilde{A}_1^0$ and $\tilde{\mathfrak{A}}_1$ are defined by the expansions (2.5.2)-(2.5.3). Next, we estimate

$$\begin{aligned}
&\frac{1}{\epsilon}\|H_1(\epsilon,t,\cdot,w(t)) - \mathring{H}_1(t,\cdot,\mathring{w}(t))\|_{H^{s-1}} \\
&\qquad \leq \frac{1}{\epsilon}\|H_1(\epsilon,t,\cdot,w(t)) - H_1(\epsilon,t,\cdot,\mathring{w}(t))\|_{H^{s-1}} + \frac{1}{\epsilon}\|H_1(\epsilon,t,\cdot,\mathring{w}(t)) - \mathring{H}_1(t,\cdot,\mathring{w}(t))\|_{H^{s-1}} \\
&\qquad \leq C\big(\|w\|_{L^\infty([T_0,t),H^s)}, \|\mathring{w}\|_{L^\infty([T_0,t),H^s)}\big)(\|w(t)\|_{H^s} + \|z(t)\|_{H^{s-1}} + \|\mathring{w}(t)\|_{H^s}),
\end{aligned} \tag{2.5.75}$$

for $T_0 \leq t < T_1$, where in deriving the second inequality, we used (2.5.58), Taylor's Theorem (in the last variable), and the calculus inequalities. By similar arguments and (2.5.57), we also get that

$$\begin{aligned}
&\frac{1}{\epsilon}\|(A_1^i(\epsilon,t,\cdot,w(t)) - \mathring{A}_1^i(t,\cdot,\mathring{w}(t))\|_{H^{s-1}} \\
&\qquad \leq C\big(\|w\|_{L^\infty([T_0,t),H^s)}, \|\mathring{w}\|_{L^\infty([T_0,t),H^s)}\big)(\|w(t)\|_{H^s} + \|z(t)\|_{H^{s-1}} + \|\mathring{w}(t)\|_{H^s}),
\end{aligned} \tag{2.5.76}$$

again for $T_0 \leq t < T_1$. Taken together, the estimates (2.5.54), (2.5.58), (2.5.75) and (2.5.76) along with the calculus inequalities imply that

$$\|\hat{F}_2(\epsilon,t)\|_{H^{s-1}} \leq C(\|w\|_{L^\infty([T_0,t),H^s)}, \|\mathring{w}\|_{L^\infty([T_0,t),H^s)})(\|w(t)\|_{H^s} + \|z(t)\|_{H^{s-1}} + \|\mathring{w}(t)\|_{H^s}) \tag{2.5.77}$$

for $T_0 \leq t < T_1$. Furthermore, we see from (2.5.54) and (2.5.59) that we can estimate z at $t = T_0$ by

$$\|z\|_{H^{s-1}}(T_0) \leq C(\sigma)\sigma. \tag{2.5.78}$$

We can combine the two equations (2.5.48) and (2.5.50) together into the equation

$$\begin{aligned}
&\begin{pmatrix} A_1^0 & 0 \\ 0 & \mathring{A}_1^0 \end{pmatrix} \partial_t \begin{pmatrix} w \\ \mathring{w} \end{pmatrix} + \begin{pmatrix} A_1^i & 0 \\ 0 & \mathring{A}_1^i \end{pmatrix} \partial_i \begin{pmatrix} w \\ \mathring{w} \end{pmatrix} + \frac{1}{\epsilon} \begin{pmatrix} C_1^i & 0 \\ 0 & 0 \end{pmatrix} \partial_i \begin{pmatrix} w \\ \mathring{w} \end{pmatrix} \\
&\qquad = \frac{1}{t} \begin{pmatrix} \mathfrak{A}_1 & 0 \\ 0 & \mathring{\mathfrak{A}}_1 \end{pmatrix} \begin{pmatrix} \mathbb{P}_1 & 0 \\ 0 & \mathbb{P}_1 \end{pmatrix} \begin{pmatrix} w \\ \mathring{w} \end{pmatrix} + \begin{pmatrix} H_1 \\ \mathring{H}_1 \end{pmatrix} + \begin{pmatrix} F_1 \\ \mathring{F}_1 - C_1^i \partial_i v \end{pmatrix},
\end{aligned} \tag{2.5.79}$$

and collect the three equations (2.5.66), (2.5.70) and (2.5.74) together into the equation

$$A_2^0 \partial_t \begin{pmatrix} b \\ y \\ z \end{pmatrix} + A_2^i \partial_i \begin{pmatrix} b \\ y \\ z \end{pmatrix} + \frac{1}{\epsilon} C_2^i \partial_i \begin{pmatrix} b \\ y \\ z \end{pmatrix} = \frac{1}{t} \mathfrak{A}_2 \mathbb{P}_2 \begin{pmatrix} b \\ y \\ z \end{pmatrix} + H_2 + R_2 + F_2, \tag{2.5.80}$$

where

$$A_2^0 := \begin{pmatrix} \mathbb{P}_1 \mathring{A}_1^0 \mathbb{P}_1 & 0 & 0 \\ 0 & \mathring{A}_1^0 & 0 \\ 0 & 0 & A_1^0 \end{pmatrix}, \quad A_2^i := \begin{pmatrix} \mathbb{P}_1 \mathring{A}_1^i \mathbb{P}_1 & 0 & 0 \\ 0 & \mathring{A}_1^i & 0 \\ 0 & 0 & A_1^i \end{pmatrix}, \tag{2.5.81}$$

$$C_2^i := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & C_1^i \end{pmatrix}, \quad \mathbb{P}_2 := \begin{pmatrix} \mathbb{P}_1 & 0 & 0 \\ 0 & \mathbb{1} & 0 \\ 0 & 0 & \mathbb{P}_1 \end{pmatrix}, \quad \mathfrak{A}_2 = \begin{pmatrix} \mathbb{P}_1 \mathring{\mathfrak{A}}_1 \mathbb{P}_1 & 0 & 0 \\ -\mathbb{P}_1 \mathring{\mathfrak{A}}_1 \mathbb{P}_1 & \mathbb{P}_1 \mathring{\mathfrak{A}}_1 \mathbb{P}_1 + \mathring{A}_1^0 & 0 \\ 0 & 0 & \mathfrak{A}_1 \end{pmatrix}, \tag{2.5.82}$$

$$H_2 := \begin{pmatrix} \mathbb{P}_1 \mathring{H}_1 \\ \tilde{H}_2 \\ 0 \end{pmatrix}, \quad R_2 := \begin{pmatrix} 0 \\ 0 \\ \hat{R}_2 \end{pmatrix} \quad \text{and} \quad F_2 := \begin{pmatrix} \mathbb{P}_1 \bar{F}_2 \\ \tilde{F}_2 \\ \hat{F}_2 \end{pmatrix}. \tag{2.5.83}$$

We remark that due to the projection operator $\mathbb{P}_1$ that appears in the definition (2.5.65) of b and in the top row of (2.5.81), the vector $(b, y, z)^{\mathrm{T}}$ takes values in the vector space $\mathbb{P}_1\mathbb{R}^{N_1} \times \mathbb{R}^{N_1} \times \mathbb{R}^{N_1}$ and (2.5.81) defines a symmetric hyperbolic system, i.e. A_2^0 and A_2^i define symmetric linear operators on $\mathbb{P}_1\mathbb{R}^{N_1} \times \mathbb{R}^{N_1} \times \mathbb{R}^{N_1}$ and A_2^0 is non-degenerate.

Setting

$$\mathbb{P}_3 := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbb{P}_1 & 0 & 0 \\ 0 & 0 & 0 & \mathbb{1} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

it is then not difficult to verify from the estimates (2.5.54), (2.5.56), (2.5.67), (2.5.72) and (2.5.77), the initial bounds (2.5.59), (2.5.68) and (2.5.78), the relations (2.5.64), (2.5.69), (2.5.71) and (2.5.73), and the assumptions on the coefficients $\{A_1^0, A_1^i, \mathring{A}_1^0, \mathring{A}_1^i, \mathfrak{A}_1, \mathring{\mathfrak{A}}_1, H, F\}$, see Assumptions 2.5.2 and 2.5.9, that the system consisting of (2.5.79) and (2.5.80) and the solution $U = (w, \mathring{w}, b, y, z)^{\mathrm{T}}$ satisfy the hypotheses of Theorem 2.5.7, and thus, for $\sigma > 0$ chosen small enough, there exists a constant $C > 0$ independent of $\epsilon \in (0, \epsilon_0)$ and $T_1 \in (T_0, 0)$ such that

$$\|(w, \mathring{w})\|_{L^\infty([T_0,T_1)\times\mathbb{T}^n)} \leq \frac{R}{2} \tag{2.5.84}$$

and

$$\|(w, \mathring{w})\|_{M^\infty_{\mathbb{P}_1,s}([T_0,t)\times\mathbb{T}^n)} + \|(b, y, z)\|_{M^\infty_{\mathbb{P}_2,s-1}([T_0,t)\times\mathbb{T}^n)} - \int_{T_0}^t \frac{1}{\tau}\|\mathbb{P}_3 U\|_{H^{s-1}}\, d\tau \leq C\sigma \tag{2.5.85}$$

for $T_0 \leq t < T_1$. This completes the proof since the estimates (2.5.60)-(2.5.63) follow immediately from (2.5.84) and (2.5.85). □

2.6 Initial data

As is well known, the initial data for the reduced conformal Einstein-Euler equations cannot be chosen freely on the initial hypersurface

$$\Sigma_{T_0} = \{T_0\} \times \mathbb{T}^3 \subset M = (0, T_0] \times \mathbb{T}^3 \qquad (T_0 > 0).$$

Indeed, a number of constraints, which we can separate into gravitational, gauge and velocity normalization, must be satisfied on Σ_{T_0}. There are a number of distinct methods available to solve these constraint equations. Here, we will follow the method used in [61, 62], which is an adaptation of the method introduced by Lottermoser in [52].

The goal of this section is to construct 1-parameter families of ϵ-dependent solutions to the constraint equations that behave appropriately in the limit $\epsilon \searrow 0$. In order to use the method from [61, 62] to solve the constraint equations, we need to introduce new gravitational variables $\hat{\mathfrak{u}}^{\mu\nu}$ and $\hat{\mathfrak{u}}^{\mu\nu}_\sigma$ defined via the formulas

$$\hat{g}^{\mu\nu} := \theta\underline{\bar{g}^{\mu\nu}} = E^3(\bar{h}^{\mu\nu} + \epsilon^2\hat{\mathfrak{u}}^{\mu\nu}) = \hat{h}^{\mu\nu} + \epsilon^2 E^3\hat{\mathfrak{u}}^{\mu\nu} \qquad \text{and} \qquad \hat{\mathfrak{u}}^{\mu\nu}_\sigma := \bar{\partial}_\sigma\hat{\mathfrak{u}}^{\mu\nu}, \tag{2.6.1}$$

respectively, where

$$\theta = \frac{\sqrt{|\underline{\bar{g}}|}}{\sqrt{|\bar{\eta}|}} = \sqrt{\frac{\Lambda}{3}|\underline{\bar{g}}|}, \quad |\bar{g}| = -\det \bar{g}_{\mu\nu}, \quad \hat{h}^{\mu\nu} = E^3\bar{h}^{\mu\nu} \quad \text{and} \quad |\bar{\eta}| = -\det \bar{\eta}_{\mu\nu} = \frac{3}{\Lambda}. \tag{2.6.2}$$

Notation: In the following, we will use upper case script letters, e.g. $\mathscr{Q}(\xi)$, $\mathscr{R}(\xi)$, $\mathscr{S}(\xi)$, to denote analytic maps of the variable ξ whose exact form is not important. The domain of analyticity of these maps will be clear from context. Generally, we will use $\mathscr{S}$ to denote maps that may change line to line, while other letters will be used to denote maps that need to be distinguished for later use. We also introduce the following derivative notation to facilitate the statements,

$$\hat{\partial}_\mu = \frac{1}{\epsilon}\delta^i_\mu\partial_i + \delta^0_\mu\partial_0.$$

The total set of constraints that we need to solve on Σ_{T_0} are:

$$(\underline{\bar{G}^{0\mu}} - \underline{\bar{T}^{0\mu}})|_{t=T_0} = 0 \quad \text{(Gravitational Constraints)}, \tag{2.6.3}$$

$$\left(\hat{\partial}_\nu(E^3\hat{\mathfrak{u}}^{\mu\nu}) - \frac{2}{t}E^3\hat{\mathfrak{u}}^{\mu 0} - \frac{2\Lambda}{3t}\frac{\theta - E^3}{\epsilon^2}\delta^\mu_0 + \frac{\theta - E^3}{\epsilon^2}\frac{\Lambda}{t}\Omega\delta^\mu_0\right)\Bigg|_{t=T_0} = 0 \quad \text{(Gauge constraint)} \tag{2.6.4}$$

and

$$(\underline{\bar{v}^\mu\bar{v}_\mu} + 1)|_{t=T_0} = 0 \quad \text{(Velocity Normalization)}. \tag{2.6.5}$$

Remark 2.6.1. It is not difficult to verify that the constraint (2.6.4) is equivalent to the wave gauge condition $\underline{\bar{Z}^\mu} = 0$ on the initial hypersurface Σ_{T_0}. Indeed, it is enough to notice that $\bar{\partial}_\nu(\hat{h}^{\mu\nu}) = -E^3\frac{\Lambda}{t}\Omega\delta^\mu_0$ and

$$\underline{\bar{X}^\mu} = -\hat{\partial}_\nu\underline{\bar{g}^{\mu\nu}} - \underline{\bar{g}^{\mu\nu}}\frac{1}{\sqrt{|\underline{\bar{g}}|}}\hat{\partial}_\nu\sqrt{|\underline{\bar{g}}|} - \frac{\Lambda}{t}\Omega\delta^\mu_0 = \frac{1}{\theta}(-\theta\hat{\partial}_\nu\underline{\bar{g}^{\mu\nu}} - \underline{\bar{g}^{\mu\nu}}\hat{\partial}_\nu\theta) - \frac{\Lambda}{t}\Omega\delta^\mu_0 = -\frac{1}{\theta}\hat{\partial}_\nu\hat{g}^{\mu\nu} - \frac{\Lambda}{t}\Omega\delta^\mu_0.$$

2.6.1 Reduced conformal Einstein-equations

Before proceeding, we state in the following lemma a result that will be used repeatedly in this section. The proof follows from the definition of θ, see (2.6.2), and a direct calculation. We omit the details.

Lemma 2.6.2.

$$\theta(\epsilon, \hat{\mathfrak{u}}^{\mu\nu}) = E^6\sqrt{-\frac{3}{\Lambda}\det(\bar{h}^{\mu\nu} + \epsilon^2\hat{\mathfrak{u}}^{\mu\nu})} = E^3 + \frac{1}{2}\epsilon^2E^3\left(-\frac{3}{\Lambda}\hat{\mathfrak{u}}^{00} + E^2\hat{\mathfrak{u}}^{ij}\delta_{ij}\right) + \epsilon^4\mathscr{S}(\epsilon, t, E, \hat{\mathfrak{u}}^{\mu\nu}), \tag{2.6.6}$$

where $\mathscr{S}(\epsilon, t, E, 0) = 0$.

Using this lemma, we can express the gauge constraint (2.6.4) as follows:

$$\begin{cases} \partial_t(E^3\hat{\mathfrak{u}}^{00}) = -\frac{1}{\epsilon}\partial_k(E^3\hat{\mathfrak{u}}^{0k}) + \frac{2}{t}E^3\hat{\mathfrak{u}}^{00} + \frac{\Lambda}{3t}E^3\left(-\frac{3}{\Lambda}\hat{\mathfrak{u}}^{00} + E^2\hat{\mathfrak{u}}^{ij}\delta_{ij}\right) \\ \qquad - \frac{1}{2}E^3\frac{\Lambda}{t}\Omega\left(-\frac{3}{\Lambda}\hat{\mathfrak{u}}^{00} + E^2\hat{\mathfrak{u}}^{ij}\delta_{ij}\right) + \epsilon^2\mathscr{S}(\epsilon, t, E, \Omega/t, \hat{\mathfrak{u}}^{\mu\nu}) \\ \partial_t(E^3\hat{\mathfrak{u}}^{j0}) = -\frac{1}{\epsilon}\partial_k(E^3\hat{\mathfrak{u}}^{jk}) + \frac{2}{t}E^3\hat{\mathfrak{u}}^{j0} \end{cases} \tag{2.6.7}$$

where $\mathscr{S}(\epsilon, t, E, \Omega/t, 0) = 0$. The importance of the relations (2.6.7) is that they allow us to determine the time derivatives $\partial_0\hat{\mathfrak{u}}^{\mu 0}$ on the initial hypersurface Σ_{T_0} from the metric variables $\hat{\mathfrak{u}}^{\mu\nu}$ and their spatial derivatives on Σ_{T_0}.

Lemma 2.6.3.

$$\partial_t(\theta - E^3) = \frac{3}{2\Lambda}\epsilon E^3\partial_k\hat{\mathfrak{u}}^{0k} + \epsilon^2\mathscr{A}(\epsilon, t, E, \Omega/t, \hat{\mathfrak{u}}^{\mu\nu}, \partial_l\hat{\mathfrak{u}}^{\mu\nu}, \hat{\mathfrak{u}}^{ij}_0) \tag{2.6.8}$$

and

$$\partial_i\theta = -\frac{3}{2\Lambda}\epsilon^2E^3\partial_i\hat{\mathfrak{u}}^{00} + \frac{1}{2}\epsilon^2E^5\delta_{kl}\partial_i\hat{\mathfrak{u}}^{kl} + \epsilon^4\mathscr{S}_i(\epsilon, t, E, \Omega/t, \hat{\mathfrak{u}}^{\mu\nu}, \partial_l\hat{\mathfrak{u}}^{\mu\nu}, \hat{\mathfrak{u}}^{ij}_0), \tag{2.6.9}$$

where the $\mathscr{A}$ *and* $\mathscr{S}_i$ *are linear in* $(\partial_l\hat{\mathfrak{u}}^{\mu\nu}, \hat{\mathfrak{u}}^{ij}_0)$ *and vanish for* $(\epsilon, t, E, \Omega/t, 0, 0, 0) = 0$.

Proof. The proof of this Lemma follows from straightforward calculations; we only prove (2.6.8). Noticing

$$\theta^{-1}\partial\theta = \frac{1}{2}\underline{\bar{g}^{\mu\nu}}\partial\underline{\bar{g}_{\mu\nu}} = \frac{1}{2}\hat{g}_{\mu\nu}\partial\hat{g}^{\mu\nu} \qquad (\hat{g}_{\mu\nu} = \theta^{-1}\underline{\bar{g}_{\mu\nu}}), \tag{2.6.10}$$

it is not difficult to verify that

$$\partial_t(\theta - E^3) = \frac{1}{2}\theta\hat{g}_{\mu\nu}\partial_0\hat{g}^{\mu\nu} - 3E^3\frac{\Omega}{t} = \frac{3}{2\Lambda}\epsilon E^3\partial_k\hat{\mathfrak{u}}^{0k} + 3E^3\frac{\Omega}{t} - 3E^3\frac{\Omega}{t} + \epsilon^2\mathscr{A}$$

follows from (2.6.7). □

We proceed by differentiating (2.6.7) with respect to time t to obtain, with the help of Lemma 2.6.3, the following:

$$\begin{cases} \partial_t^2(E^3\hat{\mathfrak{u}}^{00}) = \frac{1}{\epsilon^2}\partial_k\partial_i(E^3\hat{\mathfrak{u}}^{ik}) - \frac{1}{\epsilon}\frac{4}{t}\partial_k(E^3\hat{\mathfrak{u}}^{k0}) + \frac{2}{t^2}E^3\hat{\mathfrak{u}}^{00} + \left(\frac{2}{3} - \Omega\right)\frac{\Lambda}{t^2}\frac{\theta - E^3}{\epsilon^2} \\ \qquad - \frac{\theta - E^3}{\epsilon^2}\frac{\Lambda}{t}\partial_t\Omega + \frac{\Lambda}{t}\left(\frac{2}{3} - \Omega\right)\partial_t\frac{\theta - E^3}{\epsilon^2} \\ \qquad = \frac{1}{\epsilon^2}\partial_k\partial_i(E^3\hat{\mathfrak{u}}^{ik}) - \frac{1}{\epsilon}\frac{3}{t}\left(1 + \frac{1}{2}\Omega\right)E^3\partial_k\hat{\mathfrak{u}}^{k0} + \frac{1}{t^2}\mathscr{S}(\epsilon, t, E, \Omega/t, \partial_t\Omega, \hat{\mathfrak{u}}^{\mu\nu}, \partial_l\hat{\mathfrak{u}}^{\mu k}, \hat{\mathfrak{u}}^{ij}_0) \\ \partial_t^2(E^3\hat{\mathfrak{u}}^{j0}) = -\frac{1}{\epsilon}\partial_k\partial_0(E^3\hat{\mathfrak{u}}^{jk}) + \frac{2}{t^2}E^3\hat{\mathfrak{u}}^{j0} - \frac{2}{t}\frac{1}{\epsilon}\partial_k(E^3\hat{\mathfrak{u}}^{jk}) \\ \qquad = -\frac{1}{\epsilon}E^3\partial_k\hat{\mathfrak{u}}^{jk}_0 + \frac{2}{t^2}E^3\hat{\mathfrak{u}}^{j0} - \frac{2 + 3\Omega}{t}\frac{1}{\epsilon}\partial_k(E^3\hat{\mathfrak{u}}^{jk}) \end{cases} \tag{2.6.11}$$

where $\mathscr{S}(\epsilon, t, E, \Omega/t, \partial_t\Omega, 0, 0, 0) = 0$.

Next, we consider the following reduced version of the conformal Einstein equations (2.1.14), which we write using the metric variable $\hat{g}^{\mu\nu}$ defined by (2.6.1):

$$\frac{1}{2\theta^2}\hat{g}^{\lambda\sigma}\hat{\partial}_\lambda\hat{\partial}_\sigma\hat{g}^{\mu\nu} + \underline{\bar{\nabla}^{(\mu}\bar{\Gamma}^{\nu)}} - \frac{1}{2\theta}\hat{g}^{\mu\nu}\underline{\bar{\nabla}_\lambda\bar{\Gamma}^\lambda} + \frac{1}{\theta^2}\hat{Q}^{\mu\nu}(\hat{g}, \bar{\partial}\hat{g}, \theta) - \frac{1}{2}\underline{\bar{X}^\mu\bar{X}^\nu} - \underline{\bar{\nabla}^{(\mu}\bar{Z}^{\nu)}} + \frac{1}{2\theta}\hat{g}^{\mu\nu}\underline{\bar{\nabla}_\lambda\bar{Z}^\lambda}$$
$$- \frac{1}{2}\underline{\bar{A}^{\mu\nu}_\lambda}\underline{\bar{Z}^\lambda} = e^{4\Phi}\underline{\tilde{T}^{\mu\nu}} - \frac{1}{\theta}e^{2\Phi}\Lambda\hat{g}^{\mu\nu} + 2(\bar{\nabla}^\mu\bar{\nabla}^\nu\Psi - \bar{\nabla}^\mu\Psi\bar{\nabla}^\nu\Psi) - (2\bar{\Box}\Psi + |\bar{\nabla}\Psi|^2_{\bar{g}})\frac{1}{\theta}\hat{g}^{\mu\nu}, \tag{2.6.12}$$

where

$$\hat{Q}^{\mu\nu}(\hat{g}, \hat{\partial}\hat{g}, \theta) = \frac{1}{2}\theta^2\underline{Q^{\mu\nu}} - \frac{1}{4}\hat{g}^{\mu\nu}\hat{g}_{\alpha\beta}(\theta^2\underline{Q^{\alpha\beta}} - \theta^2\underline{\bar{X}^\alpha\bar{X}^\beta}) - \frac{1}{2}\hat{g}^{\lambda\sigma}\hat{g}_{\alpha\beta}\hat{\partial}_\sigma\hat{g}^{\mu\nu}\hat{\partial}_\lambda\hat{g}^{\alpha\beta}$$
$$+ \frac{1}{8}\hat{g}^{\lambda\sigma}\hat{g}^{\mu\nu}\hat{g}_{\gamma\rho}\hat{g}_{\alpha\beta}\hat{\partial}_\lambda\hat{g}^{\gamma\rho}\hat{\partial}_\sigma\hat{g}^{\alpha\beta} + \frac{1}{4}\hat{g}^{\lambda\sigma}\hat{g}^{\mu\nu}\hat{\partial}_\lambda\hat{g}_{\alpha\beta}\hat{\partial}_\sigma\hat{g}^{\alpha\beta}$$

with $Q^{\mu\nu}$ as defined previously by (2.2.13). By (2.1.27), (2.2.13), (2.6.1), (2.6.10) and the identity

$$\underline{\bar{\Gamma}^\lambda_{\mu\nu}} = -\hat{g}_{\sigma(\mu}\hat{\partial}_{\nu)}\hat{g}^{\lambda\sigma} + \frac{1}{2}\hat{g}^{\lambda\sigma}\hat{g}_{\alpha\mu}\hat{g}_{\beta\nu}\hat{\partial}_\sigma\hat{g}^{\alpha\beta} + \frac{1}{4}\left(2\hat{g}_{\alpha\beta}\delta^\lambda_{(\mu}\hat{\partial}_{\nu)}\hat{g}^{\alpha\beta} - \hat{g}^{\lambda\sigma}\hat{g}_{\mu\nu}\hat{g}_{\alpha\beta}\hat{\partial}_\sigma\hat{g}^{\alpha\beta}\right), \tag{2.6.13}$$

it is obvious that $\theta^2\underline{Q^{\mu\nu}}$ is analytic in $\hat{g}^{\mu\nu}$, $\hat{\partial}\hat{g}^{\mu\nu}$ and θ. From this and the formula (2.6.13), it is clear that $\hat{Q}^{\mu\nu}$ is analytic in $\hat{g}^{\mu\nu}$, $\hat{\partial}\hat{g}^{\mu\nu}$ and θ. Moreover, using (2.6.7) and (2.6.13), it can be verified by a straightforward calculation that $\hat{Q}^{\mu\nu}$ satisfies

$$\hat{Q}^{\mu\nu}(\hat{g}, \bar{\partial}\hat{g}, \theta) - \hat{Q}^{\mu\nu}_H(\hat{h}, \bar{\partial}\hat{h}, E^3) = \epsilon\mathcal{T}^{\mu\nu k}_{\alpha\beta}(t)\partial_k\hat{\mathfrak{u}}^{\alpha\beta} + \epsilon^2\hat{\mathscr{Q}}^{\mu\nu}(\epsilon, t, E, \Omega/t, x, \hat{\mathfrak{u}}^{\alpha\beta}, \partial_k\hat{\mathfrak{u}}^{\alpha\beta}, \hat{\mathfrak{u}}^{ij}_0)$$

for coefficients $\mathcal{T}^{\mu\nu k}_{\alpha\beta}$ that depend only on t and where $\hat{\mathscr{Q}}^{\mu\nu}(\epsilon, t, E, \Omega/t, x, \hat{\mathfrak{u}}^{\alpha\beta}, 0, 0) = 0$.

Using the easy to verify identities

$$\underline{\bar{\Gamma}^\lambda_{\lambda 0}} = \frac{1}{2}\underline{\bar{g}^{\lambda\sigma}}\hat{\partial}_0\underline{\bar{g}_{\lambda\sigma}} = \frac{1}{2}\hat{g}_{\lambda\sigma}\hat{\partial}_0\hat{g}^{\lambda\sigma} = \frac{1}{\theta}\hat{\partial}_0\theta,$$

$$\bar{\nabla}_\lambda \bar{\gamma}^\lambda = \frac{1}{t}\left(\partial_t \Omega - \frac{1}{t}\Omega + \bar{\Gamma}^\lambda_{\lambda 0}\Omega\right) \quad \text{and} \quad \bar{\nabla}_\lambda \bar{Y}^\lambda = -2\bar{\Box}\Psi - \frac{2\Lambda}{3t^2} + \frac{2\Lambda}{3t}\bar{\Gamma}^\lambda_{\lambda 0},$$

we can write the reduced conformal Einstein equations (2.6.12) as

$$\begin{aligned} &\frac{1}{2\theta^2}\hat{g}^{\lambda\sigma}\hat{\partial}_\lambda\hat{\partial}_\sigma \hat{g}^{\mu\nu} + \bar{\nabla}^{(\mu}\bar{\gamma}^{\nu)} - \frac{1}{2\theta}\hat{g}^{\mu\nu}\frac{1}{t}\left(\partial_t\Omega - \frac{1}{t}\Omega + \underline{\bar{\Gamma}^\lambda_{\lambda 0}}\Omega\right) + \frac{1}{\theta^2}\hat{Q}^{\mu\nu} + \frac{1}{\theta}\hat{g}^{\mu\nu}\frac{1}{t}\bar{\gamma}^0 \\ &= -\frac{\Lambda}{3t}\frac{1}{\theta}\hat{\partial}_0\hat{g}^{\mu\nu} + \frac{2\Lambda}{3t^2}\left[\left(\underline{\bar{g}^{00}} + \frac{\Lambda}{3}\right)\delta^\mu_0\delta^\nu_0 + \underline{\bar{g}^{0k}}\delta^{(\mu}_k\delta^{\nu)}_0\right] + \frac{1}{t^2}(1+\epsilon^2 K)\rho\underline{\bar{v}^\mu\bar{v}^\nu} + \frac{1}{\theta}\frac{1}{t^2}\epsilon^2 K\rho\hat{g}^{\mu\nu}. \end{aligned} \tag{2.6.14}$$

This equation is satisfied for the FLRW solutions (2.1.4)-(2.1.7), i.e. we can substitute $\{\hat{g}^{\mu\nu}, \bar{\rho}, \bar{v}^\mu\} \mapsto \{\hat{h}^{\mu\nu}, \rho_H, e^\Psi \tilde{v}^\mu_H\}$. Dividing the resulting FLRW equation through by θ^2, we get

$$\begin{aligned} &\frac{1}{2\theta^2}\hat{h}^{\lambda\sigma}\hat{\partial}_\lambda\hat{\partial}_\sigma \hat{h}^{\mu\nu} + \frac{E^6}{\theta^2}\bar{\nabla}^{(\mu}_H\bar{\gamma}^{\nu)} - \frac{E^3}{2\theta^2}\hat{h}^{\mu\nu}\frac{1}{t}\left(\partial_t\Omega - \frac{1}{t}\Omega + \bar{\gamma}^\lambda_{\lambda 0}\Omega\right) + \frac{1}{\theta^2}\hat{Q}^{\mu\nu}_H + \frac{E^3}{\theta^2}\hat{h}^{\mu\nu}\frac{1}{t}\bar{\gamma}^0 \\ &= -\frac{\Lambda}{3t}\frac{E^3}{\theta^2}\hat{\partial}_0\hat{h}^{\mu\nu} + \frac{E^6}{\theta^2}\frac{2\Lambda}{3t^2}\left(\bar{h}^{00} + \frac{\Lambda}{3}\right)\delta^\mu_0\delta^\nu_0 + \frac{E^6}{\theta^2}\frac{\Lambda}{3}\frac{1}{t^2}(1+\epsilon^2 K)\rho_H\delta^\mu_0\delta^\nu_0 + \frac{E^3}{\theta^2}\frac{1}{t^2}\epsilon^2 K\rho_H\hat{h}^{\mu\nu}. \end{aligned} \tag{2.6.15}$$

Subtracting (2.6.15) from (2.6.14) yields

$$\begin{aligned} &\hat{g}^{\lambda\sigma}\hat{\partial}_\lambda\hat{\partial}_\sigma(\hat{g}^{\mu\nu} - \hat{h}^{\mu\nu}) + (\hat{g}^{\lambda\sigma} - \hat{h}^{\lambda\sigma})\hat{\partial}_\lambda\hat{\partial}_\sigma\hat{h}^{\mu\nu} + 2\theta^2\left(\bar{\nabla}^{(\mu}\bar{\gamma}^{\nu)} - \frac{E^6}{\theta^2}\bar{\nabla}^{(\mu}_H\bar{\gamma}^{\nu)}\right) \\ &- \theta(\hat{g}^{\mu\nu} - \hat{h}^{\mu\nu})\frac{1}{t}\left(\partial_t\Omega - \frac{1}{t}\Omega\right) + (E^3 - \theta)\hat{h}^{\mu\nu}\frac{1}{t}\left(\partial_t\Omega - \frac{1}{t}\Omega\right) + \theta(\hat{h}^{\mu\nu} - \hat{g}^{\mu\nu})\frac{1}{t}\bar{\gamma}^\lambda_{\lambda 0}\Omega \\ &+ (E^3 - \theta)\hat{h}^{\mu\nu}\frac{1}{t}\bar{\gamma}^\lambda_{\lambda 0}\Omega + \theta\hat{g}^{\mu\nu}\frac{1}{t}(\underline{\bar{\gamma}^\lambda_{\lambda 0}} - \underline{\bar{\Gamma}^\lambda_{\lambda 0}})\Omega + 2(\hat{Q}^{\mu\nu} - \hat{Q}^{\mu\nu}_H) + 2\theta\frac{1}{t}\bar{\gamma}^0\left(\hat{g}^{\mu\nu} - \frac{E^3}{\theta}\hat{h}^{\mu\nu}\right) \\ &= -\frac{2\Lambda}{3t}\theta\hat{\partial}_0(\hat{g}^{\mu\nu} - \hat{h}^{\mu\nu}) - \frac{2\Lambda}{3t}\theta\left(1 - \frac{E^3}{\theta}\right)\hat{\partial}_0\hat{h}^{\mu\nu} + \frac{4\Lambda}{3t^2}\theta^2\left[\left(\left(\underline{\bar{g}^{00}} + \frac{\Lambda}{3}\right) - \frac{E^6}{\theta^2}\left(\bar{h}^{00} + \frac{\Lambda}{3}\right)\right)\delta^\mu_0\delta^\nu_0\right. \\ &\left. + \underline{\bar{g}^{0k}}\delta^{(\mu}_k\delta^{\nu)}_0\right] + 2\theta^2\frac{1}{t^2}(1+\epsilon^2 K)\left(\rho\underline{\bar{v}^\mu\bar{v}^\nu} - \frac{E^6}{\theta^2}\frac{\Lambda}{3}\rho_H\delta^\mu_0\delta^\nu_0\right) + 2\theta\frac{1}{t^2}\epsilon^2 K\left(\rho\hat{g}^{\mu\nu} - \frac{E^3}{\theta}\rho_H\hat{h}^{\mu\nu}\right). \end{aligned} \tag{2.6.16}$$

2.6.2 Transformation formulas

Before proceeding, we collect in the following lemma a set of formulas that can be used to transform from the gravitational variables used in this section to those introduced previously in §2.1.5 for the formulation of the evolution equations.

Lemma 2.6.4. *The evolution variables $u^{0\mu}$, u^{ij} and u can be expressed in terms of the gravitational variables $\hat{\mathfrak{u}}^{\mu\nu}$ by the following expressions:*

$$u^{0\mu} = \frac{\epsilon}{2t}\left(\frac{1}{2}\hat{\mathfrak{u}}^{00}\delta^\mu_0 + \hat{\mathfrak{u}}^{0k}\delta^\mu_k + \frac{\Lambda}{6}E^2\hat{\mathfrak{u}}^{ij}\delta_{ij}\delta^\mu_0\right) + \epsilon^3\mathscr{S}^\mu(\epsilon, t, E, \Omega/t, \hat{\mathfrak{u}}^{\alpha\beta}), \tag{2.6.17}$$

$$u = \epsilon\frac{2\Lambda}{9}E^2\hat{\mathfrak{u}}^{ij}\delta_{ij} + \epsilon^3\mathscr{S}(\epsilon, t, E, \Omega/t, \hat{\mathfrak{u}}^{\alpha\beta}), \tag{2.6.18}$$

$$u^{ij} = \epsilon E^2\left(\hat{\mathfrak{u}}^{ij} - \frac{1}{3}\hat{\mathfrak{u}}^{kl}\delta_{kl}\delta^{ij}\right) + \epsilon^3\mathscr{S}^{ij}(\epsilon, t, E, \Omega/t, \hat{\mathfrak{u}}^{\alpha\beta}), \tag{2.6.19}$$

where all of the remainder terms vanish for $(\epsilon, t, E, \Omega/t, 0) = 0$. Moreover, the 0-component of the conformal fluid four-velocity $\bar{v}^\mu$ can be written as

$$\underline{\bar{v}^0} = \sqrt{\frac{\Lambda}{3}} + \epsilon^2\mathscr{S}(\epsilon, t, E, \Omega/t, \hat{\mathfrak{u}}^{\alpha\beta}, z_j). \tag{2.6.20}$$

Proof. First, we observe that (2.6.17) follows directly from (2.2.26) and Lemma 2.6.2. Next, using

(2.6.1), it is not hard to show that

$$\det(\underline{\bar{g}^{kl}}) = (\theta E^{-3})^{-3}(E^{-6} + \epsilon^2 E^{-4}\hat{\mathfrak{u}}^{ij}\delta_{ij}) + \epsilon^4\mathscr{S} = E^{-6} + \frac{1}{2}\epsilon^2 E^{-6}\left(\frac{9}{\Lambda}\hat{\mathfrak{u}}^{00} - E^2\hat{\mathfrak{u}}^{ij}\delta_{ij}\right) + \epsilon^4\mathscr{S}(\epsilon, t, E, \Omega/t, \hat{\mathfrak{u}}^{\alpha\beta}),$$

from which it follows that

$$\underline{\alpha}E^2 = 1 + \frac{1}{6}\epsilon^2\left(\frac{9}{\Lambda}\hat{\mathfrak{u}}^{00} - E^2\hat{\mathfrak{u}}^{ij}\delta_{ij}\right) + \epsilon^4\mathscr{S}(\epsilon, t, E, \Omega/t, \hat{\mathfrak{u}}^{\alpha\beta}) \tag{2.6.21}$$

by (2.1.39). Then by (2.1.34), (2.1.40) and (2.6.21), we have

$$u = 2tu^{00} - \frac{1}{\epsilon}\frac{\Lambda}{3}\ln[1 + (\underline{\alpha}E^2 - 1)] = \epsilon\frac{2\Lambda}{9}E^2\hat{\mathfrak{u}}^{ij}\delta_{ij} + \epsilon^3\mathscr{S}(\epsilon, t, E, \Omega/t, \hat{\mathfrak{u}}^{\alpha\beta}),$$

while

$$u^{ij} = \frac{1}{\epsilon}\left((\underline{\alpha}\theta)^{-1}\hat{g}^{ij} - E^{-1}\hat{h}^{ij}\right) = \epsilon E^2\left(\hat{\mathfrak{u}}^{ij} - \frac{1}{3}\hat{\mathfrak{u}}^{kl}\delta_{kl}\delta^{ij}\right) + \epsilon^3\mathscr{S}^{ij}(\epsilon, t, E, \Omega/t, \hat{\mathfrak{u}}^{\alpha\beta})$$

follows from (2.2.24), (2.6.1), (2.6.21) and

$$(\underline{\alpha}\theta)^{-1} = E^{-1} - \epsilon^2\frac{1}{3}E\hat{\mathfrak{u}}^{ij}\delta_{ij} + \epsilon^4\mathscr{S}(\epsilon, t, E, \Omega/t, \hat{\mathfrak{u}}^{\mu\nu}).$$

Finally, (2.6.20) follows from (2.2.42), (2.2.26) and (2.6.17)-(2.6.19) □

2.6.3 Solving the constraint equations

We now need to write the constraint equations in a form that is suitable to used the methods from [61, 62]. We begin by defining the rescaled variables

$$\hat{\mathfrak{u}}^{ij}|_{t=T_0} = \epsilon\breve{\mathfrak{u}}^{ij}, \quad \hat{\mathfrak{u}}^{ij}_0|_{t=T_0} = \breve{\mathfrak{u}}^{ij}_0, \quad \hat{\mathfrak{u}}^{0\mu}|_{t=T_0} = \breve{\mathfrak{u}}^{0\mu} \quad \text{and} \quad \hat{\mathfrak{u}}^{0\mu}_0|_{t=T_0} = \breve{\mathfrak{u}}^{0\mu}_0,$$

and noting that

$$\partial_k\hat{\mathfrak{u}}^{ij}|_{t=T_0} = \epsilon\partial_k\breve{\mathfrak{u}}^{ij}. \tag{2.6.22}$$

We then observe that the following terms from (2.6.16) can be represented as

$$\begin{aligned}\epsilon^2 E^3\hat{\mathfrak{u}}^{\lambda\sigma}\hat{\partial}_\lambda\hat{\partial}_\sigma\hat{h}^{\mu0} &+ 2(\theta^2\bar{\nabla}^{(\mu}\bar{\gamma}^{0)} - E^6\bar{\nabla}^{(\mu}_H\bar{\gamma}^{0)}) - \epsilon^2\theta E^3\hat{\mathfrak{u}}^{\mu0}\frac{1}{t}\left(\partial_t\Omega - \frac{1}{t}\Omega\right) + \frac{2}{t}\bar{\gamma}^0(\theta\hat{g}^{0\mu} - E^3\hat{h}^{\mu0})\\ &+ (E^3-\theta)\hat{h}^{\mu\nu}\frac{1}{t}\left(\partial_t\Omega - \frac{1}{t}\Omega\right) + \theta(\hat{h}^{\mu\nu} - \hat{g}^{\mu\nu})\frac{1}{t}\bar{\gamma}^\lambda_{\lambda0}\Omega + (E^3-\theta)\hat{h}^{\mu\nu}\frac{1}{t}\bar{\gamma}^\lambda_{\lambda0}\Omega + \frac{2\Lambda}{3t}(\theta - E^3)\partial_t\hat{h}^{0\mu}\\ &= \epsilon^2\mathscr{S}^\mu(\epsilon, t, E, \Omega/t, \partial_t\Omega, x, \hat{\mathfrak{u}}^{\alpha\beta}, \partial_k\hat{\mathfrak{u}}^{\alpha\beta}, \hat{\mathfrak{u}}^{ij}_0).\end{aligned} \tag{2.6.23}$$

We also note that

$$\underline{\bar{\Gamma}^\lambda_{\lambda0}} - \bar{\gamma}^\lambda_{\lambda0} = \frac{1}{\theta}\partial_t(\theta - E^3) + \frac{E^{-3}}{\theta E^{-3}}\partial_t E^3 - \frac{3}{t}\Omega = \epsilon\frac{3}{2\Lambda}\partial_k\hat{\mathfrak{u}}^{0k} + \epsilon^2\mathscr{S}(\epsilon, t, E, \Omega/t, \hat{\mathfrak{u}}^{\mu\nu}, \partial_k\hat{\mathfrak{u}}^{\mu\nu}, \hat{\mathfrak{u}}^{ij}_0), \tag{2.6.24}$$

Using (2.6.7) and (2.6.11) to replace the first and second time derivatives of $\hat{\mathfrak{u}}^{\mu0}$ by spatial derivatives of $\hat{\mathfrak{u}}^{\mu\nu}$ and the time derivatives $\hat{\mathfrak{u}}^{ij}_0$ in (2.6.16) with $\nu = 0$, we obtain, with the help of (2.6.23)-(2.6.24), the following elliptic equations on Σ_{T_0} for $\hat{\mathfrak{u}}^{\mu0}$:

$$\Delta\breve{\mathfrak{u}}^{00} - \frac{2\Lambda}{3T_0^2}E^2(T_0)\delta\rho + \epsilon\left(\partial_k(\mathscr{T}^{00k}_{\alpha\beta}\breve{\mathfrak{u}}^{\alpha\beta}) - \frac{\Lambda}{3}E^2(T_0)\partial_k\partial_i\breve{\mathfrak{u}}^{ik} + (\frac{\Lambda}{3t} + \frac{\Lambda+1}{2t}\Omega)E^2(T_0)\partial_k\breve{\mathfrak{u}}^{k0}\right)$$

$$+\epsilon^2\mathscr{R}^0(\epsilon,\breve{\mathfrak{u}}^{\mu\nu},\partial_k\breve{\mathfrak{u}}^{\mu\nu},\breve{\mathfrak{u}}_0^{ij},\partial_i\partial_j\breve{\mathfrak{u}}^{0\mu},\delta\rho,z^j)=0, \qquad (2.6.25)$$

$$\Delta\breve{\mathfrak{u}}^{i0}+\epsilon\left(\frac{\Lambda}{3}E^2(T_0)\partial_k\breve{\mathfrak{u}}_0^{ik}-\sqrt{\frac{\Lambda}{3}}\frac{2}{T_0^2}E^2(T_0)\rho z^i+\partial_k(\mathcal{T}^{0ik}_{\alpha\beta}\hat{\mathfrak{u}}^{\alpha\beta})\right)$$
$$+\epsilon^2\mathscr{R}^i(\epsilon,\breve{\mathfrak{u}}^{\mu\nu},\partial_k\breve{\mathfrak{u}}^{\mu\nu},\breve{\mathfrak{u}}_0^{ij},\partial_i\partial_j\breve{\mathfrak{u}}^{0\mu},\delta\rho,z^j)=0, \qquad (2.6.26)$$

where the coefficients $\mathcal{T}^{0\mu k}_{\alpha\beta}=\mathcal{T}^{0\mu k}_{\alpha\beta}(T_0)$ are constant on Σ_{T_0} and the remainder terms satisfy $\mathscr{R}^\mu(\epsilon,0,0,0,0,0,0)=0$.

Remark 2.6.5. From the above calculations, it is not difficult to see that the elliptic equations (2.6.25)-(2.6.26) are equivalent to the gravitational constraint equations (2.6.3) provided that the gauge constraint (2.6.4) is also satisfied. Recalling that (2.6.7) is equivalent to the gauge constraints, it is clear that we can solve the gauge constraints by using (2.6.7) to determine the time derivatives $\partial_t\breve{\mathfrak{u}}^{\mu0}$ from the metric variables $\breve{\mathfrak{u}}^{\mu\nu}$ and their spatial derivatives.

Decomposing $\delta\rho=\rho-\rho_H$ and ρz^i on Σ_{T_0} as

$$\delta\rho|_{t=T_0}=\breve{\rho}_0+\epsilon\breve{\phi} \quad\text{and}\quad (\rho z^i)|_{t=T_0}=\breve{\psi}^i+\breve{\nu}^i,$$

where

$$\breve{\rho}_0:=\Pi\delta\rho|_{t=T_0},\quad \breve{\phi}:=\frac{1}{\epsilon}\langle 1,\delta\rho\rangle|_{t=T_0},\quad \breve{\psi}^i:=\langle 1,\rho z^i\rangle|_{t=T_0}\quad\text{and}\quad \breve{\nu}^i=\Pi(\rho z^i)|_{t=T_0},$$

it is clear that $z^i|_{t=T_0}$ and $\delta\rho|_{t=T_0}$ depend analytically on $(\breve{\nu}^i,\breve{\psi}^i,\breve{\rho}_0,\breve{\phi})$, and in particular,

$$z^i|_{t=T_0}=\frac{\breve{\nu}^i+\breve{\psi}^i}{\rho_H(T_0)+\breve{\rho}_0+\epsilon\breve{\phi}}. \qquad (2.6.27)$$

From this and the fact that the spatial derivatives $\partial_i: H^s(\mathbb{T}^n)\to H^{s-1}(\mathbb{T}^n)$ define bounded linear maps, we can, by Lemmas D.2.1 and D.2.2 from Appendix D.2, view the remainder terms $\mathscr{R}^\mu$ from (2.6.25)-(2.6.26) as defining analytic maps

$$(-\epsilon_0,\epsilon_0)\times B_r(H^{s+1}(\mathbb{T}^3))\times H^s(\mathbb{T}^3)\times B_r(H^s(\mathbb{T}^3))\times B_r(\mathbb{R})\times\mathbb{R}^3\times H^s(\mathbb{T}^3)\ni(\epsilon,\breve{\mathfrak{u}}^{\mu\nu},\breve{\mathfrak{u}}_0^{ik},\breve{\rho}_0,\breve{\phi},\breve{\psi}^i,\breve{\nu}^i)$$
$$\longmapsto\mathscr{R}^\mu(\epsilon,\breve{\mathfrak{u}}^{\mu\nu},\breve{\mathfrak{u}}_0^{ik},\breve{\rho}_0,\breve{\phi},\breve{\psi}^i,\breve{\nu}^i)\in H^{s-1}(\mathbb{T}^3) \qquad (2.6.28)$$

for $r>0$ chosen small enough. Using this observation, we can proceed with the existence proof for solutions to the constraint equations.

Theorem 2.6.6. *Suppose $s\in\mathbb{Z}_{>n/2+1}$ and $r>0$, $\breve{\mathfrak{u}}^{ij}\in B_r(H^{s+1}(\mathbb{T}^3,\mathbb{S}_3))$, $\breve{\mathfrak{u}}_0^{ij}\in H^s(\mathbb{T}^3,\mathbb{S}_3)$, $\breve{\rho}_0\in B_r(\bar{H}^s(\mathbb{T}^3))$, $\breve{\nu}^i\in\bar{H}^s(\mathbb{T}^3,\mathbb{R}^3)$. Then for $r>0$ chosen small enough so that the map (2.6.28) is well-defined and analytic, there exists an $\epsilon_0>0$, and analytic maps $\breve{\phi}\in C^\omega(X^s_{\epsilon_0,r},\mathbb{R})$, $\breve{\psi}^l\in C^\omega(X^s_{\epsilon_0,r},\mathbb{R}^3)$, $\breve{\mathfrak{u}}^{0\mu}\in C^\omega(X^s_{\epsilon_0,r},\bar{H}^{s+1}(\mathbb{T}^3,\mathbb{R}^4))$ and $\breve{\mathfrak{u}}_0^{0\mu}\in C^\omega(X^s_{\epsilon_0,r},H^s(\mathbb{T}^3,\mathbb{R}^4))$ that satisfy*

$$\breve{\phi}(\epsilon,0,0,0,0)=0,\quad \breve{\psi}^l(\epsilon,0,0,0,0)=0,\quad \breve{\mathfrak{u}}^{0\mu}(\epsilon,0,0,0,0)=0\quad\textit{and}\quad \breve{\mathfrak{u}}_0^{0\mu}(\epsilon,0,0,0,0)=0$$

such that

$$\rho|_{t=T_0}=\rho_H(T_0)+\breve{\rho}_0+\epsilon\breve{\phi},$$
$$z^i|_{t=T_0}=\frac{\breve{\nu}^i+\breve{\psi}^i}{\rho_H(T_0)+\breve{\rho}_0+\epsilon\breve{\phi}},$$
$$\mathfrak{u}^{\mu\nu}|_{t=T_0}=\begin{pmatrix}\breve{\mathfrak{u}}^{00} & \breve{\mathfrak{u}}^{0j}\\ \breve{\mathfrak{u}}^{i0} & \epsilon\breve{\mathfrak{u}}^{ij}\end{pmatrix},$$
$$\partial_0\mathfrak{u}^{\mu\nu}|_{t=T_0}=\begin{pmatrix}\breve{\mathfrak{u}}_0^{00} & \breve{\mathfrak{u}}_0^{0j}\\ \breve{\mathfrak{u}}_0^{i0} & \breve{\mathfrak{u}}_0^{ij}\end{pmatrix},$$

where the $\breve{\mathfrak{u}}^{\mu 0}_0$ *are determined by* (2.6.7)*, solve the constraints* (2.6.3)*,* (2.6.4) *and* (2.6.5)*. Moreover, the fields* $\{\breve{\phi}, \breve{\psi}^i, \breve{\mathfrak{u}}^{00}, \breve{\mathfrak{u}}^{0i}\}$ *satisfy the estimate*

$$|\breve{\phi}| + |\breve{\psi}^i| + \|\breve{\mathfrak{u}}^{0\mu}\|_{H^{s+1}} + \|\breve{\mathfrak{u}}^{0\mu}_0\|_{H^s} \lesssim \|\breve{\mathfrak{u}}^{ik}\|_{H^{s+1}} + \|\breve{\mathfrak{u}}^{ik}_0\|_{H^s} + \|\breve{\rho}_0\|_{H^s} + \|\breve{\nu}^i\|_{H^s}$$

uniformly for $\epsilon \in (-\epsilon_0, \epsilon_0)$ *and can be expanded as*

$$\breve{\phi} = \epsilon \mathscr{S}(\epsilon, \breve{\mathfrak{u}}^{jk}, \breve{\mathfrak{u}}^{jk}_0, \breve{\rho}_0, \breve{\nu}^i), \quad \breve{\psi}^i = \epsilon \mathscr{S}^i(\epsilon, \breve{\mathfrak{u}}^{jk}, \breve{\mathfrak{u}}^{jk}_0, \breve{\rho}_0, \breve{\nu}^i), \tag{2.6.29}$$

$$\breve{\mathfrak{u}}^{00} = \frac{2\Lambda}{3T_0^2} E^2(T_0) \Delta^{-1} \breve{\rho}_0 + \epsilon \mathscr{S}(\epsilon, \breve{\mathfrak{u}}^{jk}, \breve{\mathfrak{u}}^{jk}_0, \breve{\rho}_0, \breve{\nu}^i) \quad \textit{and} \quad \breve{\mathfrak{u}}^{0i} = \epsilon \mathscr{S}^i(\epsilon, \breve{\mathfrak{u}}^{jk}, \breve{\mathfrak{u}}^{jk}_0, \breve{\rho}_0, \breve{\nu}^i), \tag{2.6.30}$$

where the maps $\mathscr{S}$ *and* $\mathscr{S}^i$ *that are analytic on* $X^s_{\epsilon_0, r}$ *and vanish for* $(\epsilon, \breve{\mathfrak{u}}^{jk}, \breve{\mathfrak{u}}^{jk}_0, \breve{\rho}_0, \breve{\phi}) = (\epsilon, 0, 0, 0, 0)$.

Proof. Acting on (2.6.25) and (2.6.26) with $\langle 1, \cdot \rangle$ and Π , we obtain, with the help of (2.6.20) and (2.6.28), the equations

$$\breve{\phi} - \epsilon \left\langle 1, \mathscr{R}^0 \left(\epsilon, \breve{\mathfrak{u}}^{\mu\nu}, \breve{\mathfrak{u}}^{jk}_0, \breve{\rho}_0, \breve{\phi}, \breve{\psi}^j, \breve{\nu}^j\right)\right\rangle = 0, \tag{2.6.31}$$

$$\Delta \breve{\mathfrak{u}}^{00} - \frac{2\Lambda}{3T_0^2} E^2(T_0) \breve{\rho}_0 + \epsilon \Pi \mathscr{R}^0 \left(\epsilon, \breve{\mathfrak{u}}^{\mu\nu}, \breve{\mathfrak{u}}^{jk}_0, \breve{\rho}_0, \breve{\phi}, \breve{\psi}^j, \breve{\nu}^j\right) = 0, \tag{2.6.32}$$

$$\breve{\psi}^i + \epsilon \left\langle 1, \mathscr{R}^i \left(\epsilon, \breve{\mathfrak{u}}^{\mu\nu}, \breve{\mathfrak{u}}^{jk}_0, \breve{\rho}_0, \breve{\phi}, \breve{\psi}^j, \breve{\nu}^j\right)\right\rangle = 0 \tag{2.6.33}$$

and

$$\Delta \breve{\mathfrak{u}}^{0i} + \epsilon \Pi \mathscr{R}^i \left(\epsilon, \breve{\mathfrak{u}}^{\mu\nu}, \breve{\mathfrak{u}}^{jk}_0, \breve{\rho}_0, \breve{\phi}, \breve{\psi}^j, \breve{\nu}^j\right) = 0, \tag{2.6.34}$$

which are clearly equivalent to (2.6.25)-(2.6.26). Next, we let

$$\iota := (\breve{\mathfrak{u}}^{ik}, \breve{\mathfrak{u}}^{ik}_0, \breve{\rho}_0, \breve{\nu}^i) \quad \text{and} \quad \beta := \left(\breve{\phi}, \breve{\psi}^i, \breve{\mathfrak{u}}^{0\mu}\right),$$

and write (2.6.31)-(2.6.32) more compactly as

$$F(\epsilon, \iota, \beta) := L(\iota, \beta) + \epsilon M(\epsilon, \iota, \beta) = 0, \tag{2.6.35}$$

where

$$L(\iota, \beta) = \begin{pmatrix} \breve{\phi} \\ \breve{\psi}^i \\ \Delta \breve{\mathfrak{u}}^{0\mu} - \frac{2\Lambda}{3T_0^2} E^2(T_0) \delta^\mu_0 \breve{\rho}_0 \end{pmatrix}.$$

Recalling that the Laplacian Δ defines an isomorphism from $\bar{H}^{s+1}(\mathbb{T}^3)$ to $\bar{H}^{s-1}(\mathbb{T}^3)$, we observe that

$$(0, \iota, \beta) = \left(0, \iota, \begin{pmatrix} 0 \\ 0 \\ \frac{2\Lambda}{3T_0^2} E^2(T_0) \delta^\mu_0 \Delta^{-1} \breve{\rho}_0 \end{pmatrix}\right)$$

solves (2.6.35). Since $D_\beta F(0, \iota, \beta) \cdot \delta\beta = L(0, \delta\beta)$, we can solve (2.6.35) via an analytic version of the Implicit Function Theorem [20, Theorem 15.3], at least for small ϵ, if we can show that

$$\tilde{L}(\delta\beta) = \begin{pmatrix} \delta\breve{\phi} \\ \delta\breve{\psi}^i \\ \Delta \delta \breve{\mathfrak{u}}^{0\mu} \end{pmatrix}$$

defines an isomorphism from $\mathbb{R} \times \mathbb{R}^3 \times \bar{H}^{s+1}(\mathbb{T}^3, \mathbb{R}^4)$ to $\mathbb{R} \times \mathbb{R}^3 \times \bar{H}^{s-1}(\mathbb{T}^3, \mathbb{R}^4)$. But this is clear since $\Delta : \bar{H}^{s+1}(\mathbb{T}^3) \mapsto \bar{H}^{s-1}(\mathbb{T}^3)$ is an isomorphism. Thus, for $r > 0$ chosen small enough and any $R > 0$, there exists an $\epsilon_0 > 0$ and a unique analytic map

$$P : X^s_{\epsilon_0,r} \mapsto \mathbb{R} \times \mathbb{R}^3 \times \bar{H}^{s+1}(\mathbb{T}^3, \mathbb{R}^4)$$

that satisfies

$$F(\iota, P(\epsilon,\iota), \epsilon) = 0$$

for all $(\epsilon, \iota) \in (-\epsilon_0, \epsilon_0) \times B_R\big(H^{s+1}(\mathbb{T}^3, \mathbb{S}_3)\big) \times B_R\big(H^s(\mathbb{T}^3, \mathbb{S}_3)\big) \times B_r\big(\bar{H}^s(\mathbb{T}^3)\big) \times B_r\big(\bar{H}^s(\mathbb{T}^3, \mathbb{R}^3)\big)$ and

$$P(\epsilon,\iota) = \begin{pmatrix} 0 \\ 0 \\ \frac{2\Lambda}{3T_0^2} E^2(T_0)\delta^\mu_0 \Delta^{-1}\breve{\rho}_0 \end{pmatrix} + \mathrm{O}(\epsilon). \tag{2.6.36}$$

Finally, the estimate

$$|\breve{\phi}| + |\breve{\psi}^i| + \|\breve{\mathrm{u}}^{0\mu}\|_{H^{s+1}} + \|\mathrm{u}^{0\mu}_0\|_{H^s} \lesssim \|\breve{\mathrm{u}}^{ik}\|_{H^{s+1}} + \|\breve{\mathrm{u}}^{ik}_0\|_{H^s} + \|\breve{\rho}_0\|_{H^s} + \|\breve{\nu}^i\|_{H^s}$$

follows from analyticity of P, (2.6.36) and (2.6.7). □

2.6.4 Bounding $\mathbf{U}|_{t=T_0}$

For the evolution problem, we need to bound $\mathbf{U}|_{t=T_0}$, see (2.2.101), by the free initial data $\{\breve{\mathrm{u}}^{ik}, \breve{\mathrm{u}}^{ik}_0, \breve{\rho}_0, \breve{\nu}^i\}$ uniformly in ϵ. The required bound is the content of the following lemma.

Lemma 2.6.7. *Suppose that the hypotheses of Theorem 2.6.6 hold, and that $\breve{\phi} \in C^\omega(X^s_{\epsilon_0,r}, \mathbb{R})$, $\breve{\psi}^l \in C^\omega(X^s_{\epsilon_0,r}, \mathbb{R}^3)$, $\breve{\mathrm{u}}^{0\mu} \in C^\omega(X^s_{\epsilon_0,r}, \bar{H}^{s+1}(\mathbb{T}^3, \mathbb{R}^4))$ and $\breve{\mathrm{u}}^{0\mu}_0 \in C^\omega(X^s_{\epsilon_0,r}, H^s(\mathbb{T}^3, \mathbb{R}^4))$ are the analytic maps from that theorem. Then on the initial hypersurface Σ_{T_0}, the gravitational and matter fields*

$$\{u^{\mu\nu}, u^{ij}_\gamma, w^{0\mu}_i, u^{0\mu}_0, u, u_\gamma, z_j, \delta\zeta\}$$

can be expanded as follows:

$$\begin{aligned}
u^{0\mu}|_{t=T_0} &= \epsilon\frac{\Lambda}{6T_0^3}E^2(T_0)\Delta^{-1}\breve{\rho}_0\delta^\mu_0 + \epsilon^2\mathscr{S}^\mu(\epsilon, \breve{\mathrm{u}}^{kl}, \breve{\mathrm{u}}^{kl}_0, \breve{\rho}_0, \breve{\nu}^l), \\
u|_{t=T_0} &= \epsilon^2\frac{2\Lambda}{9}E^2(T_0)\breve{\mathrm{u}}^{ij}\delta_{ij} + \epsilon^3\mathscr{S}(\epsilon, \breve{\mathrm{u}}^{kl}, \breve{\mathrm{u}}^{kl}_0, \breve{\rho}_0, \breve{\nu}^l), \\
u^{ij}|_{t=T_0} &= \epsilon^2E^2(T_0)\left(\breve{\mathrm{u}}^{ij} - \frac{1}{3}\breve{\mathrm{u}}^{kl}\delta_{kl}\delta^{ij}\right) + \epsilon^3\mathscr{S}^{ij}(\epsilon, \breve{\mathrm{u}}^{kl}, \breve{\mathrm{u}}^{kl}_0, \breve{\rho}_0, \breve{\nu}^l), \\
z_j|_{t=T_0} &= E^2(T_0)\frac{\breve{\nu}^i\delta_{ij}}{\rho_H(T_0) + \breve{\rho}_0} + \epsilon\mathscr{S}_j(\epsilon, \breve{\mathrm{u}}^{kl}, \breve{\mathrm{u}}^{kl}_0, \breve{\rho}_0, \breve{\nu}^l), \\
\delta\zeta|_{t=T_0} &= \frac{1}{1+\epsilon^2K}\ln\left(1 + \frac{\breve{\rho}_0 + \epsilon\breve{\phi}}{\rho_H(T_0)}\right) = \ln\left(1 + \frac{\breve{\rho}_0}{\rho_H(T_0)}\right) + \epsilon^2\mathscr{S}(\epsilon, \breve{\mathrm{u}}^{kl}, \breve{\mathrm{u}}^{kl}_0, \breve{\rho}_0, \breve{\nu}^l), \\
w^{0\mu}_i|_{t=T_0} &= \epsilon\mathscr{S}^\mu_i(\epsilon, \breve{\mathrm{u}}^{kl}, \breve{\mathrm{u}}^{kl}_0, \breve{\rho}_0, \breve{\nu}^l), \\
u^{0\mu}_0|_{t=T_0} &= \epsilon\mathscr{S}^\mu(\epsilon, \breve{\mathrm{u}}^{kl}, \breve{\mathrm{u}}^{kl}_0, \breve{\rho}_0, \breve{\nu}^l), \\
u_\gamma|_{t=T_0} &= \epsilon\mathscr{S}_\gamma(\epsilon, \breve{\mathrm{u}}^{kl}, \breve{\mathrm{u}}^{kl}_0, \breve{\rho}_0, \breve{\nu}^l)
\end{aligned}$$

and

$$u^{ij}_\gamma|_{t=T_0} = \epsilon\mathscr{S}^{ij}_\gamma(\epsilon, \breve{\mathrm{u}}^{kl}, \breve{\mathrm{u}}^{kl}_0, \breve{\rho}_0, \breve{\nu}^l),$$

for maps $\mathscr{S}$ that are analytic on $X^s_{\epsilon_0,r}$. Moreover, the estimates

$$\begin{aligned}
&\|u^{\mu\nu}|_{t=T_0}\|_{H^{s+1}} + \|u|_{t=T_0}\|_{H^{s+1}} + \|w^{0\mu}_i|_{t=T_0}\|_{H^s} + \|u^{0\mu}_0|_{t=T_0}\|_{H^s} + \|u_\mu|_{t=T_0}\|_{H^s} \\
&\qquad + \|u^{ij}_\mu|_{t=T_0}\|_{H^s} + |\phi(T_0)| \lesssim \epsilon(\|\breve{\mathrm{u}}^{ij}\|_{H^{s+1}} + \|\breve{\mathrm{u}}^{ij}_0\|_{H^s} + \|\breve{\rho}_0\|_{H^s} + \|\breve{\nu}^i\|_{H^s})
\end{aligned}$$

and

$$\|z_j|_{t=T_0}\|_{H^s} + \|\delta\zeta|_{t=T_0}\|_{H^s} \lesssim \|\breve{\mathfrak{u}}^{ij}\|_{H^{s+1}} + \|\breve{\mathfrak{u}}^{ij}_0\|_{H^s} + \|\breve{\rho}_0\|_{H^s} + \|\breve{\nu}^i\|_{H^s}$$

hold uniformly for $\epsilon \in (-\epsilon_0, \epsilon_0)$*.*

Proof. First, we observe by (2.2.27), (2.2.61), (2.2.62), (2.2.63), (2.6.1), (2.6.22) and Lemma 2.6.2 that

$$w^{0\mu}_i|_{t=T_0} = \frac{1}{2}\partial_i \breve{\mathfrak{u}}^{00}\delta^\mu_0 - \delta^\mu_0 \frac{\Lambda}{3T_0^2}E^2(T_0)\partial_i\Delta^{-1}\breve{\rho}_0 + \partial_i\breve{\mathfrak{u}}^{0k}\delta^\mu_k + \epsilon^2\mathscr{S}^\mu_i(\epsilon, \breve{\mathfrak{u}}^{kl}, \breve{\mathfrak{u}}^{kl}_0, \breve{\rho}_0, \breve{\nu}^l), \tag{2.6.37}$$

where $\mathscr{S}^\mu_i(\epsilon, 0, 0, 0, 0) = 0$, which in turn, implies by (2.6.30) that

$$w^{0\mu}_i|_{t=T_0} = \epsilon\mathscr{S}^\mu_i(\epsilon, \breve{\mathfrak{u}}^{kl}, \breve{\mathfrak{u}}^{kl}_0, \breve{\rho}_0, \breve{\nu}^l), \tag{2.6.38}$$

where again $\mathscr{S}^\mu_i(\epsilon, 0, 0, 0, 0) = 0$. Furthermore, by (2.2.27), (2.6.1), (2.6.17), Lemma 2.6.2 and Theorem 2.6.6, we see that

$$u^{0\mu}_0|_{t=T_0} = \frac{1}{\epsilon}\frac{1}{\theta}\partial_0\hat{g}^{0\mu} - \frac{1}{\epsilon}\hat{g}^{0\mu}\frac{1}{\theta^2}\partial_0\theta - 3u^{0\mu} = \epsilon\mathscr{S}^\mu(\epsilon, \breve{\mathfrak{u}}^{ik}, \breve{\mathfrak{u}}^{ik}_0, \breve{\rho}_0, \breve{\nu}^i), \tag{2.6.39}$$

where $\mathscr{S}^\mu(\epsilon, 0, 0, 0, 0) = 0$.

Next, we see from (2.1.21), (2.1.39), (2.6.1), (2.6.8), (2.6.9) and Theorem 2.6.6, that we can express $\partial_\mu\underline{\alpha}$ as

$$\underline{\alpha}^{-3}\partial_t\underline{\alpha}^3 = 3\underline{\alpha}^{-1}\partial_t\underline{\alpha} = \underline{\breve{g}_{kl}}\partial_t\underline{\bar{g}^{kl}} = -\frac{6\Omega(T_0)}{T_0} + \epsilon^2\mathscr{S}(\epsilon, \breve{\mathfrak{u}}^{ik}, \breve{\mathfrak{u}}^{ik}_0, \breve{\rho}_0, \breve{\nu}^i) \tag{2.6.40}$$

and

$$\begin{aligned}\underline{\alpha}^{-3}\partial_j\underline{\alpha}^3 =& 3\underline{\alpha}^{-1}\partial_j\underline{\alpha} = \epsilon^2\frac{9}{2\Lambda}\partial_j\breve{\mathfrak{u}}^{00} + \epsilon^3\mathscr{S}(\epsilon, \breve{\mathfrak{u}}^{ik}, \breve{\mathfrak{u}}^{ik}_0, \breve{\rho}_0, \breve{\nu}^i)\\ =&\epsilon^2\frac{3}{T_0^2}E^2(T_0)\Delta^{-1}\partial_j\breve{\rho}_0 + \epsilon^3\mathscr{S}(\epsilon, \breve{\mathfrak{u}}^{ik}, \breve{\mathfrak{u}}^{ik}_0, \breve{\rho}_0, \breve{\nu}^i),\end{aligned} \tag{2.6.41}$$

where the error terms $\mathscr{S}$ vanish for $(\epsilon, \breve{\mathfrak{u}}^{ik}, \breve{\mathfrak{u}}^{ik}_0, \breve{\rho}_0, \breve{\nu}^i) = (\epsilon, 0, 0, 0, 0)$. Using (2.6.17) and (2.6.39), we then find with the help of (2.1.35) and (2.6.40) that

$$u_0|_{t=T_0} = 3u^{00} + u^{00}_0 - \frac{1}{\epsilon}\frac{\Lambda}{9}\underline{\alpha}^{-3}\partial_t\underline{\alpha}^3 - \frac{1}{\epsilon}\frac{2\Lambda}{3}\frac{\Omega(T_0)}{T_0} = \epsilon\mathscr{S}_0(\epsilon, \breve{\mathfrak{u}}^{ik}, \breve{\mathfrak{u}}^{ik}_0, \breve{\rho}_0, \breve{\nu}^i), \tag{2.6.42}$$

while we note that

$$u_k|_{t=T_0} = w^{00}_k + \frac{\Lambda}{3T_0^2}E^2(T_0)\partial_k\Delta^{-1}\breve{\rho}_0 - \frac{1}{\epsilon^2}\frac{\Lambda}{9}\underline{\alpha}^{-3}\partial_k\underline{\alpha}^3 = \epsilon\mathscr{S}_k(\epsilon, \breve{\mathfrak{u}}^{ij}, \breve{\mathfrak{u}}^{ij}_0, \breve{\rho}_0, \breve{\nu}^i) \tag{2.6.43}$$

follows from (2.2.62), (2.6.41) and (2.6.37). Again the error terms $\mathscr{S}_\mu$ vanish for $(\epsilon, \breve{\mathfrak{u}}^{ik}, \breve{\mathfrak{u}}^{ik}_0, \breve{\rho}_0, \breve{\nu}^i) = (\epsilon, 0, 0, 0, 0)$. Starting from (2.1.33) and (2.1.39), we see, with the help of (2.6.42), Lemmas 2.6.2 and 2.6.3 along with Theorem 2.6.6, that

$$u^{ij}_0|_{t=T_0} = \frac{1}{\epsilon}\partial_0(\underline{\alpha}^{-1}\theta^{-1}\hat{g}^{ij}) = \epsilon\mathscr{S}^{ij}(\epsilon, \breve{\mathfrak{u}}^{kl}, \breve{\mathfrak{u}}^{kl}_0, \breve{\rho}_0, \breve{\nu}^l), \tag{2.6.44}$$

where $\mathscr{S}^{ij}(\epsilon, 0, 0, 0, 0) = 0$. By a similar calculation, we find with the help of (2.6.38) and (2.6.43) that

$$u^{ij}_k|_{t=T_0} = \frac{1}{\epsilon}\partial_k(\underline{\alpha}^{-1}\theta^{-1}\hat{g}^{ij}) = \epsilon\mathscr{S}^{ij}(\epsilon, \breve{\mathfrak{u}}^{kl}, \breve{\mathfrak{u}}^{kl}_0, \breve{\rho}_0, \breve{\nu}^l), \tag{2.6.45}$$

where $\mathscr{S}^{ij}(\epsilon,0,0,0,0)=0$. Noting that

$$\phi(T_0)=\frac{1}{T_0^{3(1+\epsilon^2K)}}\breve{\phi}=\epsilon\mathscr{S}(\epsilon,\breve{\mathfrak{u}}^{ik},\breve{\mathfrak{u}}^{ik}_0,\breve{\rho}_0,\breve{\nu}^i)$$

by Theorem 2.6.6, the estimate

$$\|u^{\mu\nu}|_{t=T_0}\|_{H^{s+1}}+\|u|_{t=T_0}\|_{H^{s+1}}+\|w_i^{0\mu}|_{t=T_0}\|_{H^s}+\|u_0^{0\mu}|_{t=T_0}\|_{H^s}+\|u_\mu|_{t=T_0}\|_{H^s}$$
$$+\|u_\mu^{ij}|_{t=T_0}\|_{H^s}+|\phi(T_0)|\lesssim\epsilon(\|\breve{\mathfrak{u}}^{ij}\|_{H^{s+1}}+\|\breve{\mathfrak{u}}^{ij}_0\|_{H^s}+\|\breve{\rho}_0\|_{H^s}+\|\breve{\nu}^i\|_{H^s}),$$

which holds uniformly for $\epsilon\in(-\epsilon_0,\epsilon_0)$, follows directly from (2.6.38), (2.6.39), (2.6.42), (2.6.43), (2.6.44), (2.6.45), Lemma 2.6.4 and Theorem 2.6.6.

Next, we observe from $z_j=\frac{1}{\epsilon}\underline{\bar{g}_{j0}\bar{v}^0}+\underline{\bar{g}_{ij}}z^i$, (2.2.42), (2.6.1), (2.6.17)-(2.6.19), (2.6.27) and Theorem 2.6.6 that we can write $z_j|_{t=T_0}$ as

$$z_j|_{t=T_0}=E^2(T_0)\frac{\breve{\nu}^i\delta_{ij}}{\rho_H(T_0)+\breve{\rho}_0}+\epsilon\mathscr{S}(\epsilon,\breve{\mathfrak{u}}^{ik},\breve{\mathfrak{u}}^{ik}_0,\breve{\rho}_0,\breve{\nu}^i),\tag{2.6.46}$$

where $\mathscr{S}(\epsilon,0,0,0,0)=0$. In addition, we note that

$$\delta\zeta|_{t=T_0}=\frac{1}{1+\epsilon^2K}\ln\left(1+\frac{\breve{\rho}_0+\epsilon\breve{\phi}}{\rho_H(T_0)}\right)=\ln\left(1+\frac{\breve{\rho}_0}{\rho_H(T_0)}\right)+\epsilon^2\mathscr{S}(\epsilon,\breve{\mathfrak{u}}^{ik},\breve{\mathfrak{u}}^{ik}_0,\breve{\rho}_0,\breve{\nu}^i)\tag{2.6.47}$$

follows from (2.2.39), (2.2.40) and Theorem 2.6.6, where $\mathscr{S}(\epsilon,0,0,0,0)=0$. Together, (2.6.46) and (2.6.47) imply that the estimate

$$\|z_j|_{t=T_0}\|_{H^s}+\|\delta\zeta|_{t=T_0}\|_{H^s}\lesssim\|\breve{\mathfrak{u}}^{ij}\|_{H^{s+1}}+\|\breve{\mathfrak{u}}^{ij}_0\|_{H^s}+\|\breve{\rho}_0\|_{H^s}+\|\breve{\nu}^i\|_{H^s}$$

holds uniformly for $\epsilon\in(-\epsilon_0,\epsilon_0)$. □

2.7 Proof of Theorem 2.1.7

2.7.1 Transforming the conformal Einstein-Euler equations

The first step of the proof is to observe that the non-local formulation of the conformal Einstein-Euler equations given by (2.2.103) can be transformed into the form (2.5.48) analyzed in §2.5 by making the simple change of time coordinate

$$t\mapsto\hat{t}:=-t\tag{2.7.1}$$

and the substitutions

$$w(\hat{t},x)=\mathbf{U}(-\hat{t},x),\;A_1^0(\epsilon,-\hat{t},w)=\mathbf{B}^0(\epsilon,-\hat{t},\mathbf{U}),\;A_1^i(\epsilon,\hat{t},w)=-\mathbf{B}^i(\epsilon,-\hat{t},\mathbf{U}),\;\mathfrak{A}_1(\epsilon,\hat{t},w)=\mathbf{B}(\epsilon,-\hat{t},\mathbf{U}),\tag{2.7.2}$$

$$C_1^i=-\mathbf{C}^i,\;\mathbb{P}_1=\mathbf{P},\;H_1(\epsilon,\hat{t},w)=-\mathbf{H}(\epsilon,-\hat{t},\mathbf{U})\quad\text{and}\quad F_1(\epsilon,\hat{t},x)=-\mathbf{F}(\epsilon,-\hat{t},x,\mathbf{U},\partial_k\Phi,\partial_t\partial_k\Phi,\partial_k\partial_l\Phi).\tag{2.7.3}$$

With these choices, it is clear that the evolution equations (2.2.103) on the spacetime region $t\in(T_1,1]$, $0<T_1<1$, are equivalent to

$$A_1^0\partial_{\hat{t}}w+A_1^i\partial_iw+\frac{1}{\epsilon}C_1^i\partial_iw=\frac{1}{\hat{t}}\mathfrak{A}_1\mathbb{P}_1w+H_1+F_1\qquad\text{for }(\hat{t},x)\in[-1,-T_1)\times\mathbb{T}^3,$$

which is of the form studied in §2.5.2, see (2.5.48). Furthermore, it is not difficult to verify (see [66, §3] for details) that matrices $\{A_1^\mu,C_1^i,\mathfrak{A}_1,\mathbb{P}_1\}$ and the source term H_1 satisfy the Assumptions 2.5.2.(1)-(9) from §2.5.1 for some positive constants $\kappa,\gamma_1,\gamma_2>0$.

To see that Assumption 2.5.2.(10) is also satisfied is more involved. First, we note that this assumption is equivalent to verifying $\mathbf{P}^{\perp}[D_{\mathbf{U}}\mathbf{B}^0 \cdot (\mathbf{B}^0)^{-1}\mathbf{BPU}]\mathbf{P}^{\perp}$ admits an expansion of the type (2.5.11). To see why this is the case, we recall that $\mathbf{B}^0$ and $\mathbf{P}$ are block matrices, see (2.2.104)-(2.2.105), from which it is clear using (2.2.52)-(2.2.54) that we can expand $\mathbf{P}^{\perp}[D_{\mathbf{U}}\mathbf{B}^0 \cdot (\mathbf{B}^0)^{-1}\mathbf{BPU}]\mathbf{P}^{\perp}$ as

$$\begin{pmatrix} \mathbb{P}_2^{\perp}[D_{\mathbf{U}}\tilde{B}^0 \cdot \mathbf{W}]\mathbb{P}_2^{\perp} & 0 & 0 & 0 & 0 \\ 0 & \breve{\mathbb{P}}_2^{\perp}[D_{\mathbf{U}}\tilde{B}^0 \cdot \mathbf{W}]\breve{\mathbb{P}}_2^{\perp} & 0 & 0 & 0 \\ 0 & 0 & \breve{\mathbb{P}}_2^{\perp}[D_{\mathbf{U}}\tilde{B}^0 \cdot \mathbf{W}]\breve{\mathbb{P}}_2^{\perp} & 0 & 0 \\ 0 & 0 & 0 & \hat{\mathbb{P}}_2^{\perp}[D_{\mathbf{U}}B^0 \cdot \mathbf{W}]\hat{\mathbb{P}}_2^{\perp} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{2.7.4}$$

where

$$\mathbf{W} := (\mathbf{B}^0)^{-1}\mathbf{BPU} = \begin{pmatrix} (\tilde{B}^0)^{-1}\tilde{\mathfrak{B}}\mathbb{P}_2 & 0 & 0 & 0 & 0 \\ 0 & -2E^2\underline{\bar{g}^{00}}(\tilde{B}^0)^{-1}\breve{\mathbb{P}}_2 & 0 & 0 & 0 \\ 0 & 0 & -2E^2\underline{\bar{g}^{00}}(\tilde{B}^0)^{-1}\breve{\mathbb{P}}_2 & 0 & 0 \\ 0 & 0 & 0 & (B^0)^{-1}\mathfrak{B}\hat{\mathbb{P}}_2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}\mathbf{U}$$
$$= \mathbf{P}\begin{pmatrix} \mathbf{Y} & 0 & 0 & 0 & 0 \\ 0 & -2\mathbb{1} & 0 & 0 & 0 \\ 0 & 0 & -2\mathbb{1} & 0 & 0 \\ 0 & 0 & 0 & (B^0)^{-1}\mathfrak{B}\hat{\mathbb{P}}_2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}\mathbf{U}$$

with

$$\mathbf{Y} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{2}\delta^i_j & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Next, by (2.1.32), (2.1.39), (2.2.23), (2.2.26), and (2.2.52), we observe that $\tilde{B}^0$ can be expressed as

$$\tilde{B}^0 = E^2 \begin{pmatrix} \frac{\Lambda}{3} - 2\epsilon t u^{00} & 0 & 0 \\ 0 & (\delta^{ij} + \epsilon u^{ij})E^{-2}\exp\left(\epsilon\frac{3}{\Lambda}(2tu^{00} - u)\right) & 0 \\ 0 & 0 & \frac{\Lambda}{3} - 2\epsilon t u^{00} \end{pmatrix}.$$

Noting from definition (2.2.100) of $\mathbf{U}_1$ that u^{ij} and u are components of the vector $\mathbf{P}_1^{\perp}\mathbf{U}_1$, where

$$\mathbf{P}_1 = \operatorname{diag}(\mathbb{P}_2, \breve{\mathbb{P}}_2, \breve{\mathbb{P}}_2),$$

it is clear that $\tilde{B}^0$, as a map, depends only on the the variables $(\epsilon, t\mathbf{U}_1, \mathbf{P}_1^{\perp}\mathbf{U}_1)$. To make this explicit, we define the map $\hat{B}^0(\epsilon, t\mathbf{U}_1, \mathbf{P}_1^{\perp}\mathbf{U}_1) := \tilde{B}^0(\epsilon, t, \mathbf{U})$. Letting $\mathscr{P}$ denote linear maps that projects out the components $\mathbf{U}_1$ from $\mathbf{U}$, i.e.

$$\mathbf{U}_1 = \mathscr{P}\mathbf{U},$$

we can then differentiate $\tilde{B}^0$ with respect to $\mathbf{U}$ in the direction $\mathbf{W}$ to get

$$\begin{aligned} D_{\mathbf{U}}\tilde{B}^0 \cdot \mathbf{W} = D_{\mathbf{U}}\hat{B}^0(\epsilon, t\mathbf{U}_1, \mathbf{P}_1^{\perp}\mathbf{U}_1) \cdot \mathbf{W} &= \left(D_2\hat{B}^0 D_{\mathbf{U}}(t\mathbf{U}_1) + D_3\hat{B}^0 D_{\mathbf{U}}(\mathbf{P}_1^{\perp}\mathbf{U}_1)\right) \cdot \mathbf{W} \\ &= \left(tD_2\hat{B}^0 D_{\mathbf{U}}(\mathscr{P}\mathbf{U}) + D_3\hat{B}^0 D_{\mathbf{U}}(\mathbf{P}_1^{\perp}\mathscr{P}\mathbf{U})\right) \cdot \mathbf{W} \\ &= tD_2\hat{B}^0\mathscr{P}\mathbf{W} + D_3\hat{B}^0(\mathbf{P}_1^{\perp}\mathscr{P})\mathbf{W} = tD_2\hat{B}^0\mathscr{P}\mathbf{W}, \end{aligned} \tag{2.7.5}$$

where in the above calculations, we employed the identities

$$\mathscr{P}\mathbf{W} = \begin{pmatrix} \mathbb{P}_2 & 0 & 0 & 0 & 0 \\ 0 & \breve{\mathbb{P}}_2 & 0 & 0 & 0 \\ 0 & 0 & \breve{\mathbb{P}}_2 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{Y} & 0 & 0 & 0 & 0 \\ 0 & -2\mathbb{1} & 0 & 0 & 0 \\ 0 & 0 & -2\mathbb{1} & 0 & 0 \\ 0 & 0 & 0 & (B^0)^{-1}\mathfrak{B}\hat{\mathbb{P}}_2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \mathbf{U} = \begin{pmatrix} \mathbf{Y} & 0 & 0 \\ 0 & -2\mathbb{1} & 0 \\ 0 & 0 & -2\mathbb{1} \end{pmatrix} \mathbf{P}_1\mathbf{U}_1$$

and

$$\mathbf{P}_1^\perp \mathscr{P}\mathbf{W} = 0.$$

By (2.2.79), it is not difficult to see that

$$(\hat{\mathbb{P}}_2)^\perp [D_\mathbf{U} B^0 \mathbf{W}] (\hat{\mathbb{P}}_2)^\perp = \begin{pmatrix} D_\mathbf{U} 1 \cdot \mathbf{W} & 0 \\ 0 & 0 \end{pmatrix} = 0, \tag{2.7.6}$$

which in turn, implies via (2.7.4), (2.7.5) and (2.7.6) that

$$\begin{aligned} &\mathbf{P}^\perp [D_\mathbf{U} \mathbf{B}^0 \cdot (\mathbf{B}^0)^{-1} \mathbf{B}\mathbf{P}\mathbf{U}] \mathbf{P}^\perp \\ &\quad = t\,\mathrm{diag}\big(\mathbb{P}_2^\perp D_2 \hat{B}^0 \mathscr{P}\mathbf{W}\mathbb{P}_2^\perp, \breve{\mathbb{P}}_2^\perp D_2 \hat{B}^0 \mathscr{P}\mathbf{W}\breve{\mathbb{P}}_2^\perp, \breve{\mathbb{P}}_2^\perp D_2 \hat{B}^0 \mathscr{P}\mathbf{W}\breve{\mathbb{P}}_2^\perp, 0, 0\big). \end{aligned}$$

From this it is then clear that $\mathbf{P}^\perp [D_\mathbf{U} \mathbf{B}^0 \cdot (\mathbf{B}^0)^{-1} \mathbf{B}\mathbf{P}\mathbf{U}] \mathbf{P}^\perp$ satisfies Assumption 2.5.2.(10).

2.7.2 Limit equations

Setting

$$\mathring{\mathbf{U}} = (\mathring{u}^{0\mu}_0, \mathring{w}^{0\mu}_k, \mathring{u}^{0\mu}, \mathring{u}^{ij}_0, \mathring{u}^{ij}_k, \mathring{u}^{ij}, \mathring{u}_0, \mathring{u}_k, \mathring{u}, \delta\mathring{\zeta}, \mathring{z}_i, \mathring{\phi})^{\mathrm{T}}, \tag{2.7.7}$$

the limit equation, see §2.5.2, associated to (2.2.103) on the spacetime region $(T_2, 1] \times \mathbb{T}^3$, $0 < T_2 < 1$, is given by

$$\mathring{\mathbf{B}}^0 \partial_t \mathring{\mathbf{U}} + \mathring{\mathbf{B}}^i \partial_i \mathring{\mathbf{U}} + \mathbf{C}^i \partial_i \mathbf{V} = \frac{1}{t} \mathring{\mathbf{B}}\mathbf{P}\mathring{\mathbf{U}} + \mathring{\mathbf{H}} + \mathring{\mathbf{F}} \qquad \text{in } (T_2, 1] \times \mathbb{T}^3, \tag{2.7.8}$$

$$\mathbf{C}^i \partial_i \mathring{\mathbf{U}} = 0 \qquad \text{in } (T_2, 1] \times \mathbb{T}^3, \tag{2.7.9}$$

where

$$\mathring{\mathbf{B}}^\mu(t, \mathring{\mathbf{U}}) := \lim_{\epsilon \searrow 0} \mathbf{B}^\mu(\epsilon, t, \mathring{\mathbf{U}}), \quad \mathring{\mathbf{B}}(t, \mathring{\mathbf{U}}) := \lim_{\epsilon \searrow 0} \mathbf{B}(\epsilon, t, \mathring{\mathbf{U}}), \quad \mathring{\mathbf{H}}(t, \mathring{\mathbf{U}}) := \lim_{\epsilon \searrow 0} \mathbf{H}(\epsilon, t, \mathring{\mathbf{U}}), \tag{2.7.10}$$

and

$$\begin{aligned} \mathring{\mathbf{F}} := \bigg(&-\frac{\mathring{\Omega}}{t} \mathcal{D}^{0\mu j} \partial_j \mathring{\Phi}, \frac{3}{2t} \delta^\mu_0 \mathring{E}^{-2} \delta^{kl} \partial_l \mathring{\Phi} - \mathring{E}^{-2} \delta^{kl} \delta^\mu_0 \partial_0 \partial_l \mathring{\Phi}, 0, -\frac{\mathring{\Omega}}{t} \tilde{\mathcal{D}}^{ijr} \partial_r \mathring{\Phi}, \\ &0, 0, -\frac{\mathring{\Omega}}{t} \mathcal{D}^j \partial_j \mathring{\Phi}, 0, 0, 0, -K^{-1} \frac{1}{2} \left(\frac{3}{\Lambda}\right)^{\frac{3}{2}} \mathring{E}^{-2} \delta^{lk} \partial_k \mathring{\Phi}, 0 \bigg)^{\mathrm{T}}. \end{aligned} \tag{2.7.11}$$

In $\mathring{\mathbf{F}}$, the coefficients $\mathcal{D}^{0\mu j}$ and $\tilde{\mathcal{D}}^{ijr}$ are as defined by (2.2.68) and (2.2.75), $\mathring{\Phi}$ is the Newtonian potential, see (2.1.57), and $\mathring{E}$ and $\mathring{\Omega}$ are defined by (2.1.53) and (2.1.54), respectively.

We then observe that under the change of time coordinate (2.7.1) and the substitutions

$$\mathring{w}(\hat{t}, x) = \mathring{\mathbf{U}}(-\hat{t}, x),\ \mathring{A}^0_1(\hat{t}, w) = \mathring{\mathbf{B}}^0(-\hat{t}, \mathring{\mathbf{U}}),\ \mathring{A}^i_1(\hat{t}, w) = -\mathring{\mathbf{B}}^i(-\hat{t}, \mathring{\mathbf{U}}),\ \mathring{\mathfrak{A}}_1(\hat{t}, w) = \mathring{\mathbf{B}}(-\hat{t}, \mathring{\mathbf{U}}),\ C^i_1 = -\mathbf{C}^i, \tag{2.7.12}$$

$$v(\hat{t}, x) = \mathbf{V}(-\hat{t}, x),\ \mathbb{P}_1 = \mathbf{P}, \quad \mathring{H}_1(\hat{t}, w) = -\mathring{\mathbf{H}}(-\hat{t}, \mathring{\mathbf{U}}) \text{ and } \mathring{F}_1(\hat{t}, x) = -\mathring{\mathbf{F}}(-\hat{t}, x), \tag{2.7.13}$$

the limit equation (2.7.8)-(2.7.9) transforms into

$$\mathring{A}_1^0\partial_{\hat{t}}\mathring{w}+\mathring{A}_1^i\partial_i\mathring{w}=\frac{1}{\hat{t}}\mathring{\mathfrak{A}}_1\mathbb{P}_1\mathring{w}-C_1^i\partial_i v+\mathring{H}_1+\mathring{F}_1 \qquad \text{in} \quad [-1,-T_2)\times\mathbb{T}^3,$$
$$C_1^i\partial_i\mathring{w}=0 \qquad \text{in} \quad [-1,-T_2)\times\mathbb{T}^3,$$

which is of the form analyzed in §2.5.2, see (2.5.50)-(2.5.51) and (2.5.53). It is also not difficult to verify the matrices $\mathring{A}_1^i$ and the source term $\mathring{H}_1$ satisfy the Assumptions 2.5.9.(2) from §2.5.2.

2.7.3 Local existence and continuation

For fixed $\epsilon\in(0,\epsilon_0)$, we know from Proposition 2.3.1 that for $T_1\in(0,1)$ chosen close enough to 1 there exists a unique solution

$$\mathbf{U}\in\bigcap_{\ell=0}^{1}C^\ell\big((T_1,1],H^{s-\ell}(\mathbb{T}^3,\mathbb{V})\big)$$

to (2.2.103) satisfying the initial condition

$$\mathbf{U}|_{t=1}=\big(u_0^{0\mu}|_{t=1},w_k^{0\mu}|_{t=1},u^{0\mu}|_{t=1},u_0^{ij}|_{t=1},u_k^{ij}|_{t=1},u^{ij}|_{t=1},u_0|_{t=1},u_k|_{t=1},u|_{t=1},\delta\zeta|_{t=1},z_i|_{t=1},\phi|_{t=1}\big)^{\mathrm{T}},$$

where the initial data, $u_0^{0\mu}|_{t=1}$, $w_k^{0\mu}|_{t=1}$, ..., is determined from Lemma 2.6.7. Moreover, this solution can be continued beyond T_1 provided that

$$\sup_{t\in(T_1,1]}\|\mathbf{U}(t)\|_{H^s}<\infty.$$

Next, by Proposition 2.4.1, there exists, for some $T_2\in(0,1]$, a unique solution

$$(\mathring{\zeta},\mathring{z}^i,\mathring{\Phi})\in\bigcap_{\ell=0}^{1}C^\ell((T_1,T_0],H^{s-\ell}(\mathbb{T}^3))\times\bigcap_{\ell=0}^{1}C^\ell((T_1,T_0],H^{s-\ell}(\mathbb{T}^3,\mathbb{R}^3))\times\bigcap_{\ell=0}^{1}C^\ell((T_1,T_0],H^{s+2-\ell}(\mathbb{T}^3)), \tag{2.7.14}$$

to the conformal cosmological Poisson-Euler equations, given by (2.1.55)-(2.1.57), satisfying the initial condition

$$(\mathring{\zeta},\mathring{z}_i)|_{t=1}=\left(\ln\big(\rho_H(1)+\breve{\rho}_0\big),\frac{\breve{\nu}^i\delta_{ij}}{\rho_H(1)+\breve{\rho}_0}\right). \tag{2.7.15}$$

Setting

$$\mathbf{V}=\left(V_0^{0\mu},V_k^{0\mu},V^{0\mu},0,V_k^{ij},0,0,V_k,0,0,0,0\right), \tag{2.7.16}$$

where

$$V_0^{0\mu}=-\mathring{E}^2\frac{3}{2t}\delta_0^\mu\mathring{\Phi}+\mathring{E}^2\delta_0^\mu\partial_t\mathring{\Phi}=-\frac{1}{2t}\mathring{E}^2\delta_0^\mu\mathring{\Phi}+\delta_0^\mu t\mathring{E}^2\partial_t\left(\frac{\mathring{\Phi}}{t}\right), \tag{2.7.17}$$

$$V_k^{0\mu}=\frac{\mathring{\Omega}}{t}\mathcal{D}^{0\mu j}\Delta^{-1}\partial_k\partial_j\mathring{\Phi}+2\mathring{E}^2\sqrt{\frac{\Lambda}{3}}\frac{1}{t^2}\Delta^{-1}\partial_k(\mathring{\rho}\mathring{z}^j)\delta_j^\mu, \tag{2.7.18}$$

$$V^{0\mu}=\frac{1}{2}\delta_0^\mu\mathring{E}^2\frac{\mathring{\Phi}}{t}+\delta_0^\mu\frac{\Lambda}{3t^3}\mathring{E}^4\mathring{\Omega}\Delta^{-1}\delta\mathring{\rho}, \tag{2.7.19}$$

$$V_k^{ij}=\frac{\mathring{\Omega}}{t}\tilde{\mathcal{D}}^{ijr}\Delta^{-1}\partial_k\partial_r\mathring{\Phi}, \tag{2.7.20}$$

and

$$V_k=\frac{\mathring{\Omega}}{t}\mathcal{D}^j\Delta^{-1}\partial_k\partial_j\mathring{\Phi}, \tag{2.7.21}$$

it follows from Corollary 2.4.2 and (2.7.14) that $\mathbf{V}$ is well-defined and lies in the space

$$\mathbf{V} \in \bigcap_{\ell=0}^{1} C^{\ell}\big((T_2, 1], H^{s-\ell}(\mathbb{T}^3, \mathbb{V})\big).$$

Defining

$$\mathring{\mathbf{U}} = (0, 0, 0, 0, 0, 0, 0, 0, 0, \delta\mathring{\zeta}, \mathring{z}_i, 0), \tag{2.7.22}$$

where we recall, see (2.4.7), (2.4.8) and Theorem 2.1.7.(ii), that

$$\delta\mathring{\zeta} = \mathring{\zeta} - \mathring{\zeta}_H \quad \text{and} \quad \mathring{z}^i = \mathring{E}^{-2}\delta^{ij}\mathring{z}_j, \tag{2.7.23}$$

we see from Remark 2.4.3, (2.7.7) and (2.7.14)-(2.7.15) that $\mathring{\mathbf{U}}$ lies in the space

$$\mathring{\mathbf{U}} \in \bigcap_{\ell=0}^{1} C^{\ell}\big((T_2, 1], H^{s-\ell}(\mathbb{T}^3, \mathbb{V})\big)$$

and satisfies

$$\mathring{\mathbf{U}}|_{t=1} = \left(0, 0, 0, 0, 0, 0, 0, 0, 0, \ln\left(1 + \frac{\breve{\rho}_0}{\rho_H(1)}\right), \frac{\breve{\nu}^i\delta_{ij}}{\rho_H(1) + \breve{\rho}_0}, 0\right)^{\mathrm{T}}.$$

It can be verified by a direct calculation that the pair $(\mathbf{V}, \mathring{\mathbf{U}})$ determines a solution of the limit equation (2.7.8)-(2.7.9). Moreover, by Proposition 2.4.1, it is clear that this solution can be continued past T_2 provided that

$$\sup_{t\in(T_2,1]} \|\mathring{\mathbf{U}}(t)\|_{H^s} < \infty.$$

2.7.4 Global existence and error estimates

For the last step of the proof, we will use the a priori estimates from Theorem 2.5.10 to show that the solutions $\mathbf{U}$ and $(\mathbf{V}, \mathring{\mathbf{U}})$ to the reduced conformal Einstein-Euler equations and the corresponding limit equation, respectively, can be continued all the way to $t = 0$, i.e. $T_1 = T_2 = 0$, with uniform bounds and error estimates. In order to apply Theorem 2.5.10, we need to verify that the estimates (2.5.54)-(2.5.58) hold for the solutions $\mathbf{U}$ and $(\mathbf{V}, \mathring{\mathbf{U}})$. We begin by observing the equation

$$\partial_t \delta\mathring{\rho} = -\sqrt{\frac{3}{\Lambda}}\partial_j\big(\mathring{\rho}\mathring{z}^j\big) + \frac{3(1-\mathring{\Omega})}{t}\delta\mathring{\rho} \tag{2.7.24}$$

holds in $(T_2, 1] \times \mathbb{T}^3$ by (2.1.49), (2.4.5) and the equivalence of the two formulations (2.1.49)- (2.1.51) and (2.1.55)-(2.1.57) of the conformal Poisson-Euler equations. From this equation, (2.1.54) and the calculus inequalities from Appendix C, we obtain the estimate

$$\left\|\partial_t\left(\frac{\delta\mathring{\rho}}{t^3}\right)\right\|_{H^{s-1}} \le C\big(\|\delta\mathring{\zeta}\|_{L^\infty((t,1],H^s)}, \|\mathring{z}_i\|_{L^\infty((t,1],H^s)}\big)\big(\|\delta\mathring{\zeta}(t)\|_{H^s} + \|\mathring{z}_i(t)\|_{H^s}\big), \quad T_2 < t \le 1 \tag{2.7.25}$$

Recalling that we can write the Newtonian potential as

$$\mathring{\Phi} = \frac{\Lambda}{3}\frac{1}{t^2}\mathring{E}^2\Delta^{-1}\delta\mathring{\rho} = \frac{\Lambda}{3}t\mathring{E}^2 e^{\mathring{\zeta}_H}\Delta^{-1}(e^{\delta\mathring{\zeta}} - 1) \quad \text{in } (T_2, 1] \times \mathbb{T}^3 \tag{2.7.26}$$

by (2.1.47), (2.4.1) and Corollary 2.4.2 we see, using the calculus inequalities from Appendix C and invertibility of the Laplacian $\Delta : \bar{H}^{k+1}(\mathbb{T}^3) \to \bar{H}^{k-1}(\mathbb{T}^3)$, $k \in \mathbb{Z}_{\ge 1}$, that we can estimate $\frac{1}{t}\mathring{\Phi}$ by

$$\left\|\frac{1}{t}\mathring{\Phi}(t)\right\|_{H^{s+1}} \le C\big(\|\delta\mathring{\zeta}\|_{L^\infty((t,1],H^{s-1})}\big)\|\delta\mathring{\zeta}(t)\|_{H^{s-1}}, \quad T_2 < t \le 1. \tag{2.7.27}$$

Dividing (2.7.26) by t and then differentiating with respect to t, we find using (2.7.24) that

$$\partial_t\left(\frac{\mathring{\Phi}}{t}\right)+\frac{1}{3}\frac{\Lambda}{t^4}\mathring{E}^2\mathring{\Omega}\Delta^{-1}\delta\mathring{\rho}+\sqrt{\frac{\Lambda}{3}}\frac{1}{t^3}\mathring{E}^2\partial_k\Delta^{-1}\left(\mathring{\rho}\mathring{z}^k\right)=0, \tag{2.7.28}$$

which, by (2.7.26), is also equivalent to

$$\partial_t\mathring{\Phi}=\frac{\Lambda}{3}\frac{1}{t^3}\mathring{E}^2(1-\mathring{\Omega})\Delta^{-1}\delta\mathring{\rho}-\sqrt{\frac{\Lambda}{3}}\frac{1}{t^2}\mathring{E}^2\partial_k\Delta^{-1}\left(\mathring{\rho}\mathring{z}^k\right). \tag{2.7.29}$$

From (2.7.28) and (2.7.29), we then obtain, with the help of the calculus inequalities and the invertibility of the Laplacian, the estimate

$$\|\partial_t\mathring{\Phi}\|_{H^{s+1}}+\left\|t\partial_t\left(\frac{\mathring{\Phi}}{t}\right)\right\|_{H^{s+1}}\leq C(\|\delta\mathring{\zeta}\|_{L^\infty((t,1],H^s)},\|\mathring{z}_i\|_{L^\infty((t,1]),H^s)})(\|\delta\mathring{\zeta}(t)\|_{H^s}+\|\mathring{z}_i(t)\|_{H^s}), \tag{2.7.30}$$

for $T_2<t\leq 1$. Continuing on, we differentiate (2.7.29) with respect to t to get

$$\partial_t^2\mathring{\Phi}=\frac{\Lambda}{3}\frac{1}{t^4}\mathring{\Omega}\left(\frac{5}{2}\mathring{\Omega}-4\right)\mathring{E}^2\Delta^{-1}\delta\mathring{\rho}-\sqrt{\frac{\Lambda}{3}}\frac{1}{t^2}\mathring{E}^2\Delta^{-1}\partial_k\partial_t\left(\mathring{\rho}\mathring{z}^k\right)+\sqrt{\frac{\Lambda}{3}}\frac{1}{t^3}\mathring{E}^2(1-\mathring{\Omega})\Delta^{-1}\partial_k\left(\mathring{\rho}\mathring{z}^k\right), \tag{2.7.31}$$

where in deriving this we have used the fact that $\mathring{\Omega}$ satisfies (2.2.1) with $\epsilon=0$ and that $\Delta^{-1}\delta\mathring{\rho}$ is well defined by Corollary 2.4.2. Adding the conformal cosmological Poisson-Euler equations (2.1.49)-(2.1.50) together, we obtain the following equation for $\mathring{\rho}\mathring{z}^j$:

$$\partial_t\left(\mathring{\rho}\mathring{z}^j\right)+\sqrt{\frac{3}{\Lambda}}K\partial^j\mathring{\rho}+\sqrt{\frac{3}{\Lambda}}\partial_i\left(\mathring{\rho}\mathring{z}^i\mathring{z}^j\right)=\frac{4-3\mathring{\Omega}}{t}\mathring{\rho}\mathring{z}^j-\frac{1}{2}\left(\frac{3}{\Lambda}\right)^{\frac{3}{2}}\mathring{\rho}\partial^j\mathring{\Phi}.$$

Substituting this into (2.7.31) yields the estimate

$$\|\partial_t^2\mathring{\Phi}\|_{H^s}\leq C(\|\delta\mathring{\zeta}\|_{L^\infty((t,1],H^s)},\|\mathring{z}_i\|_{L^\infty((t,1]),H^s)})(\|\delta\mathring{\zeta}(t)\|_{H^s}+\|\mathring{z}_i(t)\|_{H^s}),\quad T_2<t\leq 1, \tag{2.7.32}$$

by (2.7.27), the invertibility of the Laplacian $\Delta:\bar{H}^{k+1}(\mathbb{T}^3)\to\bar{H}^{k-1}(\mathbb{T}^3)$, $k\in\mathbb{Z}_{\geq 1}$, and the calculus inequalities from Appendix C. Next, from the definition of $\mathbf{P}$, see (2.2.105), and (2.7.28), we compute

$$\frac{1}{t}\mathbf{PV}=\left(\frac{1}{2t}(V_0^{0\mu}+V^{0\mu}),\frac{1}{t}V_i^{0\mu},\frac{1}{2t}(V_0^{0\mu}+V^{0\mu}),0,0,0,0,0,0,0,0,0\right)^{\mathrm{T}}, \tag{2.7.33}$$

where the components are given by

$$\frac{1}{2t}(V_0^{0\mu}+V^{0\mu})=\frac{1}{2}\delta_0^\mu\mathring{E}^2\partial_t\left(\frac{\mathring{\Phi}}{t}\right)+\frac{1}{2t}\delta_0^\mu\frac{1}{3}\frac{\Lambda}{t^3}\mathring{E}^4\mathring{\Omega}\Delta^{-1}\delta\mathring{\rho}=-\delta_0^\mu\frac{1}{2}\sqrt{\frac{\Lambda}{3}}\frac{1}{t^3}\mathring{E}^4\partial_k\Delta^{-1}\left(\mathring{\rho}\mathring{z}^k\right), \tag{2.7.34}$$

$$\frac{1}{t}V_k^{0\mu}=\frac{\mathring{\Omega}}{t^2}\mathcal{D}^{0\mu j}\Delta^{-1}\partial_k\partial_j\mathring{\Phi}+2E^2\sqrt{\frac{\Lambda}{3}}\frac{1}{t^3}\Delta^{-1}\partial_k(\mathring{\rho}\mathring{z}^j)\delta_j^\mu. \tag{2.7.35}$$

Routine calculations also show that the components of $\partial_t\mathbf{V}$ are given by

$$\partial_tV_0^{0\mu}=\mathring{E}^2\delta_0^\mu\left(2\mathring{\Omega}-\frac{3}{2}\right)\partial_t\left(\frac{\mathring{\Phi}}{t}\right)+\mathring{E}^2\delta_0^\mu\partial_t^2\mathring{\Phi}-\frac{\mathring{\Omega}}{t^2}\mathring{E}^2\delta_0^\mu\mathring{\Phi}, \tag{2.7.36}$$

$$\partial_tV^{0\mu}=\delta_0^\mu\mathring{E}^2\frac{\mathring{\Omega}}{t}\frac{\mathring{\Phi}}{t}+\frac{1}{2}\delta_0^\mu\mathring{E}^2\partial_t\left(\frac{\mathring{\Phi}}{t}\right)+\delta_0^\mu\frac{\Lambda}{3}\mathring{E}^4\left(4\frac{\mathring{\Omega}^2}{t}+\partial_t\mathring{\Omega}\right)\Delta^{-1}\frac{\delta\mathring{\rho}}{t^3}+\delta_0^\mu\frac{\Lambda}{3}\mathring{E}^4\mathring{\Omega}\Delta^{-1}\partial_t\left(\frac{\delta\mathring{\rho}}{t^3}\right), \tag{2.7.37}$$

$$\partial_tV_k^{ij}=\partial_t\left(\frac{\mathring{\Omega}}{t}\right)\tilde{\mathcal{D}}^{ijr}\Delta^{-1}\partial_k\partial_r\mathring{\Phi}+\frac{\mathring{\Omega}}{t}(\partial_t\tilde{\mathcal{D}}^{ijr})\Delta^{-1}\partial_k\partial_r\mathring{\Phi}+\frac{\mathring{\Omega}}{t}\tilde{\mathcal{D}}^{ijr}\Delta^{-1}\partial_k\partial_r\left(\partial_t\mathring{\Phi}\right), \tag{2.7.38}$$

$$\partial_t V_k = \partial_t\left(\frac{\mathring{\Omega}}{t}\right)\mathcal{D}^j\Delta^{-1}\partial_k\partial_j\mathring{\Phi} + \frac{\mathring{\Omega}}{t}(\partial_t\mathcal{D}^j)\Delta^{-1}\partial_k\partial_j\mathring{\Phi} + \frac{\mathring{\Omega}}{t}\mathcal{D}^j\Delta^{-1}\partial_k\partial_j\left(\partial_t\mathring{\Phi}\right) \tag{2.7.39}$$

and

$$\begin{aligned}\partial_t V_k^{0\mu} =& \partial_t\left(\frac{\mathring{\Omega}}{t}\right)\mathcal{D}^{0\mu j}\Delta^{-1}\partial_k\partial_j\mathring{\Phi} + \frac{\mathring{\Omega}}{t}(\partial_t\mathcal{D}^{0\mu j})\Delta^{-1}\partial_k\partial_j\mathring{\Phi} + \frac{\mathring{\Omega}}{t}\mathcal{D}^{0\mu j}\Delta^{-1}\partial_k\partial_j\left(\partial_t\mathring{\Phi}\right)\\ &+ 2E^2\sqrt{\frac{\Lambda}{3}}\frac{1}{t^2}\delta^\mu_j\left[\frac{2\Omega}{t}\Delta^{-1}\partial_k(\mathring{\rho}\mathring{z}^j) - \frac{2}{t}\Delta^{-1}\partial_k(\mathring{\rho}\mathring{z}^j) + \Delta^{-1}\partial_k\partial_t(\mathring{\rho}\mathring{z}^j)\right].\end{aligned} \tag{2.7.40}$$

Recalling the the coefficients $\mathcal{D}^{0\mu\nu}$, $\tilde{\mathcal{D}}^{ijk}$ and $\mathcal{D}^j$ are remainder terms as defined in §2.1.1, it is then clear that the estimate

$$\begin{aligned}\|\mathbf{V}(t)\|_{H^{s+1}}+&\|t^{-1}\mathbf{P}\mathbf{V}(t)\|_{H^{s+1}} + \|\partial_t\mathbf{V}(t)\|_{H^s}\\ &\leq C(\|\delta\mathring{\zeta}\|_{L^\infty((t,1],H^s)}, \|\mathring{z}_i\|_{L^\infty((t,1),H^s)})(\|\delta\mathring{\zeta}(t)\|_{H^s} + \|\mathring{z}_i(t)\|_{H^s}),\end{aligned} \tag{2.7.41}$$

which holds for $T_2 < t \leq 1$, follows from the formulas (2.1.53), (2.1.54), (2.7.16)-(2.7.21) and (2.7.33)-(2.7.40), the estimates (2.7.25), (2.7.27), (2.7.30), (2.7.32), the calculus inequalities and the invertibility of the Laplacian. By similar reasoning, it is also not difficult to verify that $\mathring{\mathbf{F}}$, defined by (2.7.11), satisfies the estimate

$$\|\mathring{\mathbf{F}}(t)\|_{H^s} + \|t\partial_t\mathring{\mathbf{F}}(t)\|_{H^{s-1}} \leq C(\|\delta\mathring{\zeta}\|_{L^\infty((t,1],H^s)}, \|\mathring{z}_i\|_{L^\infty((t,1],H^s)})(\|\delta\mathring{\zeta}(t)\|_{H^s} + \|\mathring{z}_i(t)\|_{H^s}), \tag{2.7.42}$$

for $T_2 < t \leq 1$.

From the definition of $\mathbf{F}$, see (2.2.106), along with (2.2.67), (2.2.72), (2.2.74), (2.2.92) and (2.2.99), the definitions (2.2.61) and (2.2.100)-(2.2.102), and the calculus inequalities, we see that $\mathbf{F}$ can be estimated as

$$\|\mathbf{F}(t)\|_{H^s} \leq C\big(\|\mathbf{U}\|_{L^\infty((t,1],H^s)}, \|\partial_k\Phi\|_{L^\infty((t,1],H^s)}\big)(\|\mathbf{U}(t)\|_{H^s}+\|\partial_l\partial_k\Phi(t)\|_{H^s}+\|\partial_t\partial_k\Phi(t)\|_{H^s}), \tag{2.7.43}$$

for $T_1 < t < 1$. Appealing again to the invertibility of the map $\Delta : \bar{H}^{k+1}(\mathbb{T}^3) \rightarrow \bar{H}^{k-1}(\mathbb{T}^3)$, $k \in \mathbb{Z}_{\geq 1}$, it follows from (2.2.62) and the calculus inequalities that we can estimate the spatial derivatives of Φ as follows:

$$\|\partial_k\Phi(t)\|_{H^s} + \|\partial_l\partial_k\Phi(t)\|_{H^s} \leq C\big(\|\mathbf{U}\|_{L^\infty((t,1],H^s)}\big)\|\mathbf{U}(t)\|_{H^s} \tag{2.7.44}$$

for $T_1 < t < 1$. Using (2.2.7) and (2.3.7), we see that $\partial_t\partial_k\Phi$ satisfies

$$\partial_t\partial_k\Phi = \frac{\Lambda}{3}E^2e^{\zeta_H}(1-\Omega)\partial_k\Delta^{-1}\Pi e^{\delta\zeta} + \frac{\Lambda}{3}E^2te^{\zeta_H}\partial_k\Delta^{-1}\Big(e^{\delta\zeta}\partial_t\delta\zeta\Big).$$

Replacing $\partial_t\delta\zeta$ in the above equation with the right hand side of (2.2.93), we see, with the help of the calculus properties and the invertibility of the Laplacian that $\partial_t\partial_k\Phi$ can by estimated by

$$\|\partial_t\partial_k\Phi\|_{H^s} \leq C(\|\delta\zeta\|_{L^\infty((t,1],H^s)})(\|\partial_t\delta\zeta\|_{H^{s-1}} + \|\delta\zeta\|_{H^{s-1}}) \leq C(\|\mathbf{U}\|_{L^\infty((t,1],H^s)})\|\mathbf{U}\|_{H^s} \tag{2.7.45}$$

for $T_1 < t < 1$. Combining the estimates (2.7.43)-(2.7.45) gives

$$\|\mathbf{F}(t)\|_{H^s} \leq C\big(\|\mathbf{U}\|_{L^\infty((t,1],H^s)}\big)\|\mathbf{U}(t)\|_{H^s}, \quad T_1 < t < 1. \tag{2.7.46}$$

Together, (2.7.41), (2.7.42) and (2.7.46) show that source terms $\{F_1, \mathring{F}_1, v\}$, as defined by (2.7.3) and (2.7.13), satisfy the estimates (2.5.54)-(2.5.56) from Theorem 2.5.10 for times $-1 \leq \hat{t} < -T_3$, where

$$T_3 = \max\{T_1, T_2\}.$$

This leaves us to verify the Lipschitz estimates (2.5.57)-(2.5.58). We begin by noticing, with the help of (2.2.52), (2.2.87) and (2.7.22), that

$$\tilde{B}^i(\epsilon,t,\mathring{\mathbf{U}})=0,$$
$$B^i(\epsilon,t,\mathring{\mathbf{U}})=\sqrt{\frac{3}{\Lambda}}\begin{pmatrix}\mathring{z}^i & E^{-2}\delta^{im}\\ E^{-2}\delta^{il} & K^{-1}E^{-2}\delta^{lm}\mathring{z}^i\end{pmatrix}+\epsilon^2\mathcal{S}^i(\epsilon,t,\mathring{\mathbf{U}})$$

and

$$B^i(0,t,\mathring{\mathbf{U}})=\sqrt{\frac{3}{\Lambda}}\begin{pmatrix}\mathring{z}^i & \mathring{E}^{-2}\delta^{im}\\ \mathring{E}^{-2}\delta^{il} & K^{-1}\mathring{E}^{-2}\delta^{lm}\mathring{z}^i\end{pmatrix}.$$

From the above expressions, (2.2.10) and the calculus inequalities, we then obtain the estimate

$$\|\mathbf{B}^i(\epsilon,t,\mathring{\mathbf{U}})-\mathring{\mathbf{B}}^i(t,\mathring{\mathbf{U}})\|_{H^{s-1}}\leq\epsilon C(\|\mathring{\mathbf{U}}\|_{L^\infty((t,1],H^s)}),\quad T_3<t\leq 1.$$

Next, using (2.2.59), (2.2.60), (2.2.66), (2.2.69), (2.2.71), (2.2.73) and (2.7.22), we compute the components of $\mathbf{H}(\epsilon,t,\mathring{\mathbf{U}})$, see (2.2.106), as follows:

$$\tilde{G}_1(\epsilon,t,\mathring{\mathbf{U}})=\Big(-2(1+\epsilon^2K)E^2t^{1+3\epsilon^2K}e^{(1+\epsilon^2K)(\zeta_H+\delta\mathring{\zeta})}\sqrt{\frac{\Lambda}{3}}\mathring{z}^k\delta^\mu_k+\epsilon\mathcal{S}^\mu(\epsilon,t,\mathring{\mathbf{U}}),0,0\Big)^{\mathrm{T}},$$
$$\tilde{G}_2(\epsilon,t,\mathring{\mathbf{U}})=\big(\epsilon\mathcal{S}^{ij}(\epsilon,t,\mathring{\mathbf{U}}),0,0\big)^{\mathrm{T}},\quad \tilde{G}_3(\epsilon,t,\mathring{\mathbf{U}})=\big(\epsilon\mathcal{S}(\epsilon,t,\mathring{\mathbf{U}}),0,0\big)^{\mathrm{T}},$$
$$G(\epsilon,t,\mathring{\mathbf{U}})=(0,0)^{\mathrm{T}}\quad\text{and}\quad\acute{G}(\epsilon,t,\mathring{\mathbf{U}})=0,$$

where $\mathcal{S}^\mu$, $\mathcal{S}^{ij}$ and $\mathcal{S}$ all vanish for $\mathring{\mathbf{U}}=0$. It follows immediately from these expressions and the definitions (2.2.106) and (2.7.10) that

$$\mathring{\mathbf{H}}(t,\mathring{\mathbf{U}})=\Big(-2\mathring{E}^2te^{\mathring{\zeta}_H+\delta\mathring{\zeta}}\sqrt{\frac{\Lambda}{3}}\mathring{z}^k\delta^\mu_k,0,0,0,0,0,0,0,0,0,0,0\Big)^T,$$

and, with the help of the calculus inequalities and (2.2.9)-(2.2.10), that

$$\|\mathbf{H}(\epsilon,t,\mathring{\mathbf{U}})-\mathring{\mathbf{H}}(t,\mathring{\mathbf{U}})\|_{H^{s-1}}\leq\epsilon C(\|\mathring{\mathbf{U}}\|_{L^\infty((t,1],H^s)})\|\mathring{\mathbf{U}}\|_{H^{s-1}},\quad T_3<t\leq 1. \tag{2.7.47}$$

To proceed, we define

$$\mathbf{Z}=\frac{1}{\epsilon}(\mathbf{U}-\mathring{\mathbf{U}}-\epsilon\mathbf{V}), \tag{2.7.48}$$

and set

$$z(\hat{t},x)=\mathbf{Z}(-\hat{t},x). \tag{2.7.49}$$

In view of the definitions (2.2.106) and (2.7.11), we see that the estimate

$$\begin{aligned}\|\mathbf{F}(\epsilon,t,\cdot)-\mathring{\mathbf{F}}(t,\cdot)\|_{H^{s-1}}\leq& C\big(\|\mathbf{U}\|_{L^\infty((t,1],H^s)}\big)\Big(\epsilon\|\mathbf{U}\|_{H^{s-1}}+\|\partial_k\Phi\|_{H^{s-1}}+\|\partial_k\partial_l\Phi\|_{H^{s-1}}\\ &+\|\partial_0\partial_l\Phi\|_{H^{s-1}}+\|\partial_k(E^{-2}\Phi-\mathring{E}^{-2}\mathring{\Phi})\|_{H^{s-1}}+\|\partial_0\partial_l(E^{-2}\Phi-\mathring{E}^{-2}\mathring{\Phi})\|_{H^{s-1}}\Big)\\ \leq C\big(\|\mathbf{U}\|_{L^\infty((t,1],H^s)}\big)&\Big(\epsilon\|\mathbf{U}\|_{H^s}+\|\partial_k(E^{-2}\Phi-\mathring{E}^{-2}\mathring{\Phi})\|_{H^{s-1}}+\|\partial_0\partial_l(E^{-2}\Phi-\mathring{E}^{-2}\mathring{\Phi})\|_{H^{s-1}}\Big),\end{aligned} \tag{2.7.50}$$

which holds for $T_3<t\leq 1$, follows from (2.2.9)-(2.2.10), the estimates (2.7.41), (2.7.44) and (2.7.45), the calculus inequalities, and the estimate

$$\|\acute{S}\|_{H^{s-1}}\lesssim\epsilon\langle 1,\mathcal{S}\rangle\lesssim\epsilon\|\mathcal{S}\|_{L^2}\leq\epsilon C(\|\mathbf{U}\|_{L^\infty((t,1],H^s)})\|\mathbf{U}\|_{H^1}\leq\epsilon C(\|\mathbf{U}\|_{L^\infty((t,1],H^s)})\|\mathbf{U}\|_{H^{s-1}}.$$

By (2.1.42), (2.1.47), (2.2.9), (2.2.62), (2.7.26), (2.7.41), (2.7.48), the invertibility of the Laplacian and the calculus inequalities, we see also that

$$\begin{aligned}\|\partial_k(E^{-2}\Phi-\mathring{E}^{-2}\mathring{\Phi})\|_{H^{s-1}}&=\|E^{-2}\Phi-\mathring{E}^{-2}\mathring{\Phi}\|_{H^s}\lesssim\|e^{\zeta_H}\Pi e^{\delta\zeta}-e^{\mathring{\zeta}_H}\Pi e^{\delta\mathring{\zeta}}\|_{H^{s-2}}\\&\lesssim|e^{\zeta_H}-e^{\mathring{\zeta}_H}|\|\Pi(e^{\delta\zeta}-1)\|_{H^{s-2}}+\|e^{\mathring{\zeta}_H}\Pi(e^{\delta\zeta}-e^{\delta\mathring{\zeta}})\|_{H^{s-2}}\\&\leq C\big(\|\delta\zeta\|_{L^\infty((t,1],H^s)},\|\delta\mathring{\zeta}\|_{L^\infty((t,1],H^s)}\big)\big(|\zeta_H-\mathring{\zeta}_H|\|e^{\delta\zeta}-1\|_{H^{s-2}}+\|\delta\zeta-\delta\mathring{\zeta}\|_{H^{s-1}}\big)\\&\leq\epsilon C\big(\|\delta\zeta\|_{L^\infty((t,1],H^s)},\|\delta\mathring{\zeta}\|_{L^\infty((t,1],H^s)}\big)\big(\epsilon\|\delta\zeta\|_{H^{s-1}}+\|\mathbf{Z}\|_{H^{s-1}}+\|\mathbf{V}\|_{H^{s-1}}\big)\\&\leq\epsilon C\big(\|\mathbf{U}\|_{L^\infty((t,1],H^s)},\|\mathring{\mathbf{U}}\|_{L^\infty((t,1],H^s)}\big)\big(\|\mathbf{U}\|_{H^s}+\|\mathbf{Z}\|_{H^{s-1}}+\|\mathring{\mathbf{U}}\|_{H^s}\big)\end{aligned}\tag{2.7.51}$$

for $T_3<t\leq1$, while similar calculations using (2.2.7), (2.2.8) and (2.2.10) show that

$$\begin{aligned}\|\partial_0\partial_l(E^{-2}\Phi-\mathring{E}^{-2}\mathring{\Phi})\|_{H^{s-1}}&=\|\partial_0(E^{-2}\Phi-\mathring{E}^{-2}\mathring{\Phi})\|_{H^s}\lesssim\|\partial_t(e^{\zeta_H}\Pi e^{\delta\zeta}-e^{\mathring{\zeta}_H}\Pi e^{\delta\mathring{\zeta}})\|_{H^{s-2}}\\\leq C\big(\|\delta\zeta\|_{L^\infty((t,1],H^s)},\|\delta\mathring{\zeta}\|_{L^\infty((t,1],H^s)}\big)&\Big(\epsilon\|\delta\zeta\|_{H^{s-1}}+\epsilon^2\|\partial_t\delta\zeta\|_{H^{s-1}}+\|\delta\mathring{\zeta}-\delta\zeta\|_{H^{s-1}}\\&+\|e^{\delta\zeta}\partial_t(\delta\zeta-\delta\mathring{\zeta})\|_{H^{s-2}}+\|\delta\zeta-\delta\mathring{\zeta}\|_{H^{s-1}}\|\partial_t\delta\mathring{\zeta}\|_{H^{s-1}}\Big)\\\leq C\big(\|\delta\zeta\|_{L^\infty((t,1],H^s)},\|\delta\mathring{\zeta}\|_{L^\infty((t,1],H^s)}\big)&\Big(\epsilon\|\delta\zeta\|_{H^{s-1}}+\epsilon^2\|\partial_t\delta\zeta\|_{H^{s-1}}+\epsilon\|\mathbf{Z}\|_{H^{s-1}}+\epsilon\|\mathbf{V}\|_{H^{s-1}}\\&+\|e^{\delta\zeta}\partial_t(\delta\zeta-\delta\mathring{\zeta})\|_{H^{s-2}}+\epsilon(\|\mathbf{Z}\|_{H^{s-1}}+\|\mathbf{V}\|_{H^{s-1}})\|\partial_t\delta\mathring{\zeta}\|_{H^{s-1}}\Big)\end{aligned}\tag{2.7.52}$$

for $T_3<t\leq1$.

Next, by (2.2.8), it is easy to see that (2.1.55) is equivalent to

$$\partial_t\delta\mathring{\zeta}+\sqrt{\frac{3}{\Lambda}}(\mathring{z}^j\partial_j\delta\mathring{\zeta}+\partial_j\mathring{z}^j)=0.$$

Using this, we derive the estimate

$$\|\partial_t\delta\mathring{\zeta}\|_{H^{s-1}}\leq C(\|\mathring{z}_j\|_{L^\infty((t,1],H^s)})\big(\|\delta\mathring{\zeta}\|_{H^s}+\|\mathring{z}_j\|_{H^s}\big),\quad T_3<t\leq1,\tag{2.7.53}$$

while we see from (2.2.93) and (2.7.44) that

$$\begin{aligned}\|\partial_t\delta\zeta\|_{H^{s-1}}&\leq C(\|\mathbf{U}\|_{L^\infty((t,1],H^s)},\|\partial_k\Phi\|_{L^\infty((t,1],H^s)})\big(\|\delta\zeta\|_{H^s}+\|z_j\|_{H^s}+\epsilon(\|\mathbf{U}\|_{H^{s-1}}+\|\partial_k\Phi\|_{H^{s-1}})\big)\\&\leq C(\|\mathbf{U}\|_{L^\infty((t,1],H^s)})\|\mathbf{U}\|_{H^s}\end{aligned}\tag{2.7.54}$$

for $T_3<t\leq1$. We also observe that

$$\begin{aligned}\|e^{\delta\zeta}\mathring{z}^k\partial_k(\delta\zeta-\delta\mathring{\zeta})\|_{H^{s-2}}&\leq\|\partial_k\big[e^{\delta\zeta}\mathring{z}^k(\delta\zeta-\delta\mathring{\zeta})\big]\|_{H^{s-2}}+\|\partial_k\big[e^{\delta\zeta}\mathring{z}^k\big](\delta\zeta-\delta\mathring{\zeta})\|_{H^{s-2}}\\&\leq C(\|\delta\zeta\|_{L^\infty((t,1],H^s)},\|\mathring{z}^k\|_{L^\infty((t,1],H^s)})\|\delta\zeta-\delta\mathring{\zeta}\|_{H^{s-1}}\end{aligned}\tag{2.7.55}$$

and

$$\|e^{\delta\zeta}\partial_k(z_m-\mathring{z}_m)\|_{H^{s-2}}\leq C(\|\delta\zeta\|_{L^\infty((t,1],H^s)})\|z_m-\mathring{z}_m\|_{H^{s-1}}\tag{2.7.56}$$

hold for $T_3<t\leq1$. Furthermore, by (2.2.24), (2.2.26), (2.2.41), (2.2.43), (2.7.23) and (2.7.48), we see that

$$\|z^k-\mathring{z}^k\|_{H^{s-1}}\leq\epsilon C(\|\mathbf{U}\|_{L^\infty((t,1],H^s)})\big(\|\mathbf{Z}\|_{H^{s-1}}+\|\mathbf{V}\|_{H^{s-1}}+\|z_j\|_{H^{s-1}}\big)\tag{2.7.57}$$

and, with the help of (2.2.10), (2.7.41), (2.7.44) and (2.7.55)-(2.7.57), that

$$\begin{aligned}\|e^{\delta\zeta}\partial_t(\delta\zeta-\delta\mathring{\zeta})\|_{H^{s-2}}&\lesssim\|e^{\delta\zeta}(z^k\partial_k\delta\zeta-\mathring{z}^k\partial_k\delta\mathring{\zeta})\|_{H^{s-2}}+\|e^{\delta\zeta}(E^{-2}\partial_kz_m-\mathring{E}^{-2}\partial_k\mathring{z}_m)\|_{H^{s-2}}+\epsilon\|e^{\delta\zeta}\mathcal{S}\|_{H^{s-2}}\\&\lesssim\|e^{\delta\zeta}(z^k-\mathring{z}^k)\partial_k\delta\zeta\|_{H^{s-2}}+\|e^{\delta\zeta}\mathring{z}^k\partial_k(\delta\zeta-\delta\mathring{\zeta})\|_{H^{s-2}}+\|e^{\delta\zeta}(E^{-2}-\mathring{E}^{-2})\partial_kz_m\|_{H^{s-2}}\\&\quad+\|e^{\delta\zeta}\partial_k(z_m-\mathring{z}_m)\|_{H^{s-2}}+\epsilon\|e^{\delta\zeta}\mathcal{S}\|_{H^{s-1}}\end{aligned}$$

$$\leq \epsilon C(\|\mathbf{U}\|_{L^\infty((t,1],H^s)}, \|\mathring{\mathbf{U}}\|_{L^\infty((t,1],H^s)})(\|\mathbf{Z}\|_{H^{s-1}} + \|\mathring{\mathbf{U}}\|_{H^s} + \|\mathbf{U}\|_{H^s}), \tag{2.7.58}$$

where both estimates hold for $T_3 < t \leq 1$. We also observe that (2.7.41) and (2.7.52)-(2.7.58) imply

$$\|\partial_0\partial_l(E^{-2}\Phi - \mathring{E}^{-2}\mathring{\Phi})\|_{H^{s-1}} \leq \epsilon C(\|\mathbf{U}\|_{L^\infty((t,1],H^s)}, \|\mathring{\mathbf{U}}\|_{L^\infty((t,1],H^s)})\big(\|\mathbf{U}\|_{H^s} + \|\mathbf{Z}\|_{H^{s-1}} + \|\mathring{\mathbf{U}}\|_{H^s}\big) \tag{2.7.59}$$

for $T_3 < t \leq 1$. Gathering (2.7.50), (2.7.51) and (2.7.59) together, we obtain the estimate

$$\|\mathbf{F}(\epsilon, t, \cdot) - \mathring{\mathbf{F}}(t, \cdot)\|_{H^{s-1}} \leq \epsilon C(\|\mathbf{U}\|_{L^\infty((t,1],H^s)}, \|\mathring{\mathbf{U}}\|_{L^\infty((t,1],H^s)})(\|\mathbf{U}\|_{H^s} + \|\mathbf{Z}\|_{H^{s-1}} + \|\mathring{\mathbf{U}}\|_{H^s}), \tag{2.7.60}$$

for $T_3 < t \leq 1$. The estimates (2.7.47) and (2.7.60) show that source terms $\{H_1, \mathring{H}_1, F_1, \mathring{F}_1\}$, as defined by (2.7.3) and (2.7.13), and z, defined by (2.7.49), verify the Lipschitz estimate (2.5.58) from Theorem 2.5.10 for times $-1 \leq \hat{t} < -T_3$.

Having verified that all of the hypotheses of Theorem 2.5.10 are satisfied, we conclude, with the help of Lemma 2.6.7, that there exists a constant $\sigma > 0$, independent of $\epsilon \in (0, \epsilon_0)$, such that if the free initial data is chosen so that

$$\|\breve{\mathfrak{u}}^{ij}\|_{H^{s+1}} + \|\breve{\mathfrak{u}}^{ij}_0\|_{H^s} + \|\breve{\rho}_0\|_{H^s} + \|\breve{\nu}^i\|_{H^s} \leq \sigma,$$

then the estimates

$$\|\mathbf{U}\|_{L^\infty((T_3,1],H^s)} \leq C\sigma, \quad \|\mathring{\mathbf{U}}\|_{L^\infty((T_3,1],H^s)} \leq C\sigma \quad \text{and} \quad \|\mathbf{U} - \mathring{\mathbf{U}}\|_{L^\infty((T_3,1],H^{s-1})} \leq \epsilon C\sigma \tag{2.7.61}$$

hold for some constant $C > 0$, independent of $T_3 \in (0, 1)$ and $\epsilon \in (0, \epsilon_0)$. Furthermore, from the continuation criterion discussed in §2.7.3, it is clear that the bounds (2.7.61) imply that the solutions $\mathbf{U}$ and $\mathring{\mathbf{U}}$ exist globally on $M = (0, 1] \times \mathbb{T}^3$ and satisfy the estimates (2.7.61) with $T_3 = 0$ and uniformly for $\epsilon \in (0, \epsilon_0)$. In particular, this implies via the definitions (2.2.101) and (2.7.7) of $\mathbf{U}$ and $\mathring{\mathbf{U}}$ that

$$\|\delta\zeta(t) - \delta\mathring{\zeta}(t)\|_{H^{s-1}} \leq \epsilon C\sigma, \quad \|z_j(t) - \mathring{z}_j(t)\|_{H^{s-1}} \leq \epsilon C\sigma,$$
$$\|u_0^{\mu\nu}(t)\|_{H^{s-1}} \leq \epsilon C\sigma, \quad \|u_k^{\mu\nu}(t) - \delta^\mu_0\delta^\nu_0\partial_k\mathring{\Phi}(t)\|_{H^{s-1}} \leq \epsilon C\sigma, \quad \|u^{\mu\nu}(t)\|_{H^{s-1}} \leq \epsilon C\sigma,$$
$$\|u_0(t)\|_{H^{s-1}} \leq \epsilon C\sigma, \quad \|u_k(t)\|_{H^{s-1}} \leq \epsilon C\sigma \quad \text{and} \quad \|u(t)\|_{H^{s-1}} \leq \epsilon C\sigma$$

for $0 < t \leq 1$, while, from (2.2.42), we see that

$$\left\|\underline{\bar{v}}^0(t) - \sqrt{\frac{\Lambda}{3}}\right\|_{H^{s-1}} \leq C\epsilon\sigma$$

holds for $0 < t \leq 1$. This concludes the proof of Theorem 2.1.7.

Chapter 3
Cosmological Newtonian limits on large spacetime scales

Chapter 3 is based on the submitted journal article: Chao Liu and Todd A. Oliynyk, Cosmological Newtonian limits on large spacetime scales.

Abstract. We establish the existence of 1-parameter families of ϵ-dependent solutions to the Einstein-Euler equations with a positive cosmological constant $\Lambda > 0$ and a linear equation of state $p = \epsilon^2 K\rho$, $0 < K \leq 1/3$, for the parameter values $0 < \epsilon < \epsilon_0$. These solutions exist globally on the manifold $M = (0,1] \times \mathbb{R}^3$, are future geodesically complete for every $\epsilon > 0$, and converge as $\epsilon \searrow 0$ to solutions of the cosmological Poisson-Euler equations. They represent inhomogeneous, nonlinear perturbations of a FLRW fluid solution where the inhomogeneities are driven by localized matter fluctuations that evolve to good approximation according to Newtonian gravity.

References are considered at the end of the thesis.

Chapter 3

Cosmological Newtonian limits on large spacetime scales

Two things are infinite: the universe and human stupidity; and I'm not sure about the universe.

Albert Einstein

3.1 Introduction

Galaxies and clusters of galaxies are prime examples of large scale structures in our universe. Their formation requires non-linear interactions and cannot be analyzed using perturbation theory alone. Currently, cosmological Newtonian N-body simulations [18, 24, 36, 78, 79, 84] are the only well developed tool for studying structure formation. However, the Universe is fundamentally relativistic, and so the use of Newtonian simulations must be carefully justified. This leads naturally to the question: *On what scales can Newtonian cosmological simulations be trusted to approximate realistic relativistic cosmologies?* The main aim of this article is to rigorously answer this question. Informally, we establish, under suitable assumptions, the existence of realistic inhomogeneous cosmological solutions that **(i)** admit a foliation by spacelike (i.e. constant time) hypersurfaces diffeomorphic to $\mathbb{R}^3$, **(ii)** exist globally to the future, **(iii)** can be approximated to arbitrary precision by a Newtonian solution, and **(iv)** represent a non-linear perturbation of a Friedmann-Lemaître-Robertson-Walker (FLRW) fluid solution; see Theorem 3.1.6 for the precise statement.

In this article, we treat all matter in the Universe as a perfect fluid with a linear equation of state, a widely used approximation in cosmological studies, and we assume a positive cosmological constant $\Lambda > 0$ in concordance with observational evidence. The evolution of such fluids are governed by the Einstein-Euler equations given by

$$\tilde{G}^{\mu\nu} + \Lambda \tilde{g}^{\mu\nu} = \tilde{T}^{\mu\nu}, \tag{3.1.1}$$

$$\tilde{\nabla}_\mu \tilde{T}^{\mu\nu} = 0, \tag{3.1.2}$$

where $\tilde{G}^{\mu\nu}$ is the Einstein tensor of the metric $\tilde{g} = \tilde{g}_{\mu\nu} d\bar{x}^\mu d\bar{x}^\nu$,

$$\tilde{T}^{\mu\nu} = (\bar{\rho} + \bar{p})\tilde{v}^\mu \tilde{v}^\nu + \bar{p}\tilde{g}^{\mu\nu}$$

is the perfect fluid stress-energy tensor, the pressure $\bar{p}$ is determined by the proper energy density $\bar{\rho}$ via the linear equation of state

$$\bar{p} = \epsilon^2 K \bar{\rho}, \qquad 0 < K \leq \frac{1}{3},$$

the fluid four-velocity $\tilde{v}^\nu$ is normalized by

$$\tilde{v}^\mu \tilde{v}_\mu = -1, \tag{3.1.3}$$

and the dimensionless parameter ϵ can be identified with the ratio $\epsilon = \frac{v_T}{c}$, where c is the speed of light and v_T is a characteristic speed associated to the fluid.

The proof of our main result, Theorem 3.1.6, is based on a rigorous *Newtonian limit* argument, that is, taking the $\epsilon \searrow 0$ limit of solutions to the Einstein-Euler equations. The starting point for the Newtonian limit argument is the introduction of a suitable 1-parameter family of background solutions to the Einstein-Euler equations (3.1.1)-(3.1.2) that have a well defined Newtonian limit. For our argument, we use a 1-parameter family of FLRW solutions that represent a family of homogeneous, fluid filled universes undergoing accelerated expansion. Letting $(\bar{x}^i)$, $i = 1, 2, 3$, denote the standard coordinates on the $\mathbb{R}^3$ and $t = \bar{x}^0$ a time coordinate on the interval $(0, 1]$, the FLRW family we employ is defined on the manifold

$$M = (0, 1] \times \mathbb{R}^3$$

and the metric, four-velocity, and proper energy density are given by

$$\tilde{h}(t) = -\frac{3}{\Lambda t^2} dt dt + a(t)^2 \delta_{ij} d\bar{x}^i d\bar{x}^j, \tag{3.1.4}$$

$$\tilde{v}_H(t) = -t\sqrt{\frac{\Lambda}{3}}\partial_t, \tag{3.1.5}$$

and

$$\mu(t) = \frac{\mu(1)}{a(t)^{3(1+\epsilon^2 K)}}, \tag{3.1.6}$$

respectively, where the initial proper energy density $\mu(1)$ is freely specifiable and $a(t)$ satisfies

$$-ta'(t) = a(t)\sqrt{\frac{3}{\Lambda}}\sqrt{\frac{\Lambda}{3} + \frac{\mu(t)}{3}}, \qquad a(1) = 1. \tag{3.1.7}$$

Remark 3.1.1. For simplicity, we assume that the homogeneous initial density $\mu(1)$ is independent of ϵ. All of the results established in this article remain true if $\mu(1)$ is allowed to depend on ϵ in a C^1 manner, that is, the map $[0, \epsilon_0] \ni \epsilon \longmapsto \mu^\epsilon(1) \in \mathbb{R}_{>0}$ is C^1 for some $\epsilon_0 > 0$.

Remark 3.1.2. The representation (3.1.4)-(3.1.6) of the FLRW solutions is not the standard one due to the choice of time coordinate that compactifies the time interval from $[0, \infty)$ in the standard presentation to $(0, 1]$ in the coordinates used here. Letting τ denote the standard time coordinate, the relationship between the two time coordinates is

$$t = e^{-\sqrt{\frac{\Lambda}{3}}\tau}. \tag{3.1.8}$$

Due to our choice of time coordinate, the future lies in the direction of *decreasing* t and timelike infinity is located at $t = 0$.

Remark 3.1.3. As we show in §3.2.1, the FLRW solutions $\{a, \mu\}$ depend regularly on ϵ and have well defined Newtonian limits. Letting

$$\mathring{a} = \lim_{\epsilon \searrow 0} a \quad \text{and} \quad \mathring{\mu} = \lim_{\epsilon \searrow 0} \mu \tag{3.1.9}$$

denote the Newtonian limit of a and μ, respectively, it then follows from (3.1.6) and (3.1.7) that $\{\mathring{a}, \mathring{\mu}\}$ satisfy

$$\mathring{\mu} = \frac{\mathring{\mu}(1)}{\mathring{a}(t)^3}$$

and

$$-t\mathring{a}'(t) = \mathring{a}(t)\sqrt{\frac{3}{\Lambda}}\sqrt{\frac{\Lambda}{3}+\frac{\mathring{\mu}(t)}{3}}, \qquad \mathring{a}(1) = 1,$$

which define the Newtonian limit of the FLRW equations. We further note that

$$\mu(1) = \mathring{\mu}(1).$$

Throughout this article, we will refer to the global coordinates $(\bar{x}^\mu)$ on the manifold M, defined above, as *relativistic coordinates*. In addition to the relativistic coordinates, we need to introduce the spatially rescaled coordinates (x^μ) on M defined by

$$t = \bar{x}^0 = x^0 \quad \text{and} \quad \bar{x}^i = \epsilon x^i, \qquad \epsilon > 0, \tag{3.1.10}$$

which we will refer to as *Newtonian coordinates*. These coordinates are necessary for the definition of the Newtonian limit since they are used to define the sense in which solutions converge as $\epsilon \searrow 0$.

Before proceeding with our discussion of the Newtonian limit and the statement of Theorem 3.1.6, we need to first fix our notation and conventions, and introduce a number of new variables that will be needed to state our main result.

3.1.1 Notation

Index of notation

An index containing frequently used definitions and non-standard notation can be found in Appendix E.2.

Indices and coordinates

Unless stated otherwise, our indexing convention will be as follows: we use lower case Latin letters, e.g. i, j, k, for spatial indices that run from 1 to n, and lower case Greek letters, e.g. α, β, γ, for spacetime indices that run from 0 to n. When considering the Einstein-Euler equations, we will restrict our attention to the physical case $n = 3$.

For scalar functions $\bar{f}(\bar{x}^0, \bar{x}^i)$ that are given in terms of the relativistic coordinates, we will use the notation

$$\underline{\bar{f}}(t, x^i) := \bar{f}(t, \epsilon x^i) \tag{3.1.11}$$

to denote the representation of $\bar{f}$ in Newtonian coordinates. More generally, we use this notation for components of tensors. For example, given the representation $\bar{X} = \bar{X}^j(\bar{x}^0, x^i)\bar{\partial}_j$ of the vector field $\bar{X}$ in relativistic coordinates, then $\underline{\bar{X}^j}$ is defined by $\underline{\bar{X}^j}(t, x^i) = \bar{X}^j(t, \epsilon x^i)$.

Derivatives

Partial derivatives with respect to the Newtonian coordinates $(x^\mu) = (t, x^i)$ and the relativistic coordinates $(\bar{x}^\mu) = (t, \bar{x}^i)$ will be denoted by $\partial_\mu = \partial/\partial x^\mu$ and $\bar{\partial}_\mu = \partial/\partial \bar{x}^\mu$, respectively, and we use $Du = (\partial_j u)$ and $\partial u = (\partial_\mu u)$ to denote the spatial and spacetime gradients, respectively, with respect to the Newtonian coordinates, and $\bar{\partial} u = (\bar{\partial}_\mu u)$ to denote the spacetime gradient with respect to the relativistic coordinates.

Greek letters will also be used to denote multi-indices, e.g. $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$, and we will employ the standard notation $D^\alpha = \partial_1^{\alpha_1}\partial_2^{\alpha_2}\cdots\partial_n^{\alpha_n}$ for spatial partial derivatives. It will be clear from context whether a Greek letter stands for a spacetime coordinate index or a multi-index. Furthermore, we will use $D^k u = \{D^\alpha u \,|\, |\alpha| = k\}$ to denote the collection of partial derivatives of order k, and we will have occasion to use the notation $\partial^i = \delta^{ij}\partial_j$ for spatial partial derivatives.

Given a vector-valued map $f(u)$, where u is a vector, we use Df and $D_u f$ interchangeably to denote the derivative with respect to the vector u, and use the standard notation

$$Df(u)\cdot\delta u := \frac{d}{dt}\Big|_{t=0} f(u+t\delta u)$$

for the action of the linear operator Df on the vector δu. For vector-valued maps $f(u,v)$ of two (or more) variables, we use the notation $D_1 f$ and $D_u f$ interchangeably for the partial derivative with respect to the first variable, i.e.

$$D_u f(u,v)\cdot\delta u := \frac{d}{dt}\Big|_{t=0} f(u+t\delta u, v),$$

and a similar notation for the partial derivative with respect to the other variable.

Function spaces

Given a finite dimensional vector space V, we let $H^s(\mathbb{R}^n,V)$, $s\in\mathbb{Z}_{\geq 0}$, denote the space of maps from $\mathbb{R}^n$ to V with s derivatives in $L^2(\mathbb{R}^n)$. When the vector space V is clear from context, we write $H^s(\mathbb{R}^n)$ instead of $H^s(\mathbb{R}^n,V)$. Letting

$$\langle u,v\rangle = \int_{\mathbb{R}^n}(u(x),v(x))\,d^n x,$$

where $(\cdot,\cdot)$ is a fixed inner product on V, denote the standard L^2 inner product, the H^s norm is defined by

$$\|u\|^2_{H^s} = \sum_{0\leq|\alpha|\leq s}\langle D^\alpha u, D^\alpha u\rangle.$$

We let $H^s_{ul}(\mathbb{R}^n,V)$ denote the *uniformly local Sobolev spaces*, which we recall are defined as follows: let $\theta\in C_0^\infty(\mathbb{R}^n)$ be a function such that $\theta>0$ and

$$\theta(x) = \begin{cases} 1, & |x|\leq\frac{1}{2} \\ 0, & |x|>1 \end{cases}$$

and define $\theta_{d,y}(x)$ by $\theta_{d,y}=\theta((x-y)/d)$. Then u belongs to $H^s_{ul}(\mathbb{R}^n,V)$ if there exists a $d>0$ such that

$$\|u\|_{H^s_{ul}} := \sup_{y\in\mathbb{R}^n}\|\theta_{d,y}u\|_{H^s}<\infty.$$

We note that the norms corresponding to different $d>0$ are equivalent, and in addition, they are equivalent to the norm $\sqrt{\sup_{y\in\mathbb{R}^3}\sum_{0\leq|\alpha|\leq s}\int_{\mathbb{R}^n}\theta_{d,y}[D^\alpha u(x)]^2 d^n x}$.

For $s\in\mathbb{Z}_{\geq 1}$, we define the spaces

$$R^s(\mathbb{R}^3,V) = \{u\in L^6(\mathbb{R}^3,V)\mid Du\in H^{s-1}(\mathbb{R}^3,V)\}$$

and

$$K^s(\mathbb{R}^n,V) = \{\, u\in L^\infty(\mathbb{R}^n,V)\mid Du\in H^{s-1}(\mathbb{R}^n,V)\,\}$$

with norms

$$\|u\|_{R^s} = \|Du\|_{H^{s-1}}+\|u\|_{L^6} \quad\text{and}\quad \|u\|_{K^s} = \|u\|_{L^\infty}+\|Du\|_{H^{s-1}}, \tag{3.1.12}$$

respectively. On $\mathbb{R}^3$ and for $s\in\mathbb{Z}_{\geq 2}$, the inequalities

$$\|Du\|_{H^{s-1}}+\|u\|_{W^{s-1,6}}+\|u\|_{W^{s-2,\infty}} \lesssim \|u\|_{R^s} \lesssim \|Du\|_{H^{s-1}}+\|u\|_{W^{s-1,6}}+\|u\|_{W^{s-2,\infty}}, \tag{3.1.13}$$

$$\|u\|_{R^s}\lesssim\|u\|_{H^s}\lesssim\|u\|_{L^{\frac{6}{5}}}+\|u\|_{K^s} \tag{3.1.14}$$

are a direct consequences of the Sobolev and interpolation inequalities, see Theorems C.2.1(2) and C.2.2.

To handle the smoothness of coefficients that appear in various equations, we introduce the spaces

$$E^p((0,\epsilon_0)\times(T_1,T_2)\times U,V), \quad p\in\mathbb{Z}_{\geq 0},$$

which are defined to be the set of V-valued maps $f(\epsilon,t,\xi)$ that are smooth on the open set $(0,\epsilon_0)\times(T_1,T_2)\times U$, where $U\subset\mathbb{R}^n\times\mathbb{R}^N$ is open, and for which there exist constants $C_{k,\ell}>0$, $(k,\ell)\in\{0,1,\dots,p\}\times\mathbb{Z}_{\geq 0}$, such that

$$|\partial_t{}^k D^\ell_\xi f(\epsilon,t,\xi)|\leq C_{k,\ell}, \quad \forall\,(\epsilon,t,\xi)\in(0,\epsilon_0)\times(T_1,T_2)\times U.$$

If $V=\mathbb{R}$ or V clear from context, we will drop the V and simply write $E^p((0,\epsilon_0)\times(T_1,T_2)\times U)$. Moreover, we will use the notation $E^p((T_1,T_2)\times U,V)$ to denote the subspace of ϵ-independent maps. By uniform continuity, the limit $f_0(t,\xi):=\lim_{\epsilon\searrow 0}f(\epsilon,t,\xi)$ exists for each $f\in E^p((0,\epsilon_0)\times(T_1,T_2)\times U,V)$ and defines an element of $E^p((T_1,T_2)\times U,V)$.

We further define, for fixed $\epsilon_0>0$, the spaces

$$X^s_{\epsilon_0}(\mathbb{R}^3)=(0,\epsilon_0)\times R^{s+1}(\mathbb{R}^3,\mathbb{S}_3)\times H^s(\mathbb{R}^3,\mathbb{S}_3)\times\Big(L^{\frac{6}{5}}\cap K^s(\mathbb{R}^3)\Big)\times\Big(L^{\frac{6}{5}}\cap K^s(\mathbb{R}^3,\mathbb{R}^3)\Big)$$

and

$$X^s(\mathbb{R}^3)=R^{s+1}(\mathbb{R}^3,\mathbb{S}_4)\times R^{s+1}(\mathbb{R}^3,\mathbb{R})\times R^s(\mathbb{R}^3,\mathbb{S}_3)\times\big(R^s(\mathbb{R}^3,\mathbb{R}^3)\big)^2\times R^s(\mathbb{R}^3,\mathbb{R})\times R^s(\mathbb{R}^3,\mathbb{R}^3)\times R^s(\mathbb{R}^3,\mathbb{R}),$$

where $\mathbb{S}_N$ denotes the space of symmetric $N\times N$ matrices.

If X and Y are two Banach spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively, then we use

$$\|f\|_{X\cap Y}:=\|f\|_X+\|f\|_Y, \quad f\in X\cap Y,$$

to denote the intersection norm. We will also employ the notation $B_r(X)=\{\,f\in X\,|,\|f\|_X<r\,\}$ to denote the open ball of radius r in X that is centered at 0.

Constants

We employ that standard notation

$$a\lesssim b$$

for inequalities of the form

$$a\leq Cb$$

in situations where the precise value or dependence on other quantities of the constant C is not required. On the other hand, when the dependence of the constant on other inequalities needs to be specified, for example if the constant depends on the norms $\|u\|_{L^\infty}$ and $\|v\|_{L^\infty}$, we use the notation

$$C=C(\|u\|_{L^\infty},\|v\|_{L^\infty}).$$

Constants of this type will always be non-negative, non-decreasing, continuous functions of their arguments, and in general, C will be used to denote constants that may change from line to line. When we want to isolate a particular constant for use later on, we will label the constant with a subscript, e.g. C_1,C_2,C_3, etc.

Remainder terms

In order to simplify the handling of remainder terms whose exact form is not important, we will, unless otherwise stated, use upper case script letters, e.g. $\mathscr{S}(\epsilon,t,x,\xi)$ and $\mathscr{T}(\epsilon,t,x,\xi)$, and upper case script letters with a hat, e.g. $\hat{\mathscr{S}}(\epsilon,t,x,\xi)$ and $\hat{\mathscr{T}}(\epsilon,t,x,\xi)$, to denote vector valued maps that, for some $\epsilon_0, R > 0$ and $N \in \mathbb{Z}_{\geq 1}$, are elements of the spaces $E^0\big((0,\epsilon_0)\times(0,2)\times\mathbb{R}^n\times B_R(\mathbb{R}^N)\big)$ and $E^1\big((0,\epsilon_0)\times(0,2)\times\mathbb{R}^n\times B_R(\mathbb{R}^N)\big)$, respectively. In addition, we will use upper case script letters with a breve, e.g. $\breve{\mathscr{Q}}(\xi)$ and $\breve{\mathscr{R}}(\xi)$, to denote analytic maps of the variable ξ whose exact form is not important; for these maps, the domain of analyticity will be clear from context.

We will say that a function $f(x,y)$ *vanishes to the n^{th} order in y* if it satisfies $f(x,y)\sim \mathrm{O}(y^n)$ as $y\to 0$, that is, there exists a positive constant C such that $|f(x,y)|\leq C|y|^n$ as $y\to 0$.

3.1.2 Conformal Einstein-Euler equations

Returning to the setup of the Newtonian limit, we follow [51] and replace the physical (inverse) metric $\tilde{g}^{\mu\nu}$ and fluid four-velocity $\tilde{v}^\mu$ by the conformally rescaled versions defined by

$$\bar{g}^{\mu\nu} = e^{2\Psi}\tilde{g}^{\mu\nu} \tag{3.1.15}$$

and

$$\bar{v}^{\mu} = e^{\Psi}\tilde{v}^{\mu}, \tag{3.1.16}$$

respectively. Recalling the well known identity

$$\tilde{R}_{\mu\nu} - \bar{R}_{\mu\nu} = -\bar{g}_{\mu\nu}\bar{\Box}\Psi - 2\bar{\nabla}_\mu\bar{\nabla}_\nu\Psi + 2(\bar{\nabla}_\mu\Psi\bar{\nabla}_\nu\Psi - |\bar{\nabla}\Psi|^2_{\bar{g}}\bar{g}_{\mu\nu}),$$

where $\bar{\nabla}_\mu$ and $\bar{R}_{\mu\nu}$ are the covariant derivative and Ricci tensor of $\bar{g}_{\mu\nu}$, respectively, $\bar{\Box} = \bar{\nabla}^\mu\bar{\nabla}_\mu$, and $|\bar{\nabla}\Psi|^2_{\bar{g}} = \bar{g}^{\mu\nu}\bar{\nabla}_\mu\Psi\bar{\nabla}_\nu\Psi$, we find that under the change of variables (3.1.15)-(3.1.16) the Einstein equation (3.1.1) transforms as

$$\bar{G}^{\mu\nu} = \bar{T}^{\mu\nu} := e^{4\Psi}\tilde{T}^{\mu\nu} - e^{2\Psi}\Lambda\bar{g}^{\mu\nu} + 2(\bar{\nabla}^\mu\bar{\nabla}^\nu\Psi - \bar{\nabla}^\mu\Psi\bar{\nabla}^\nu\Psi) - (2\bar{\Box}\Psi + |\bar{\nabla}\Psi|^2_{\bar{g}})\bar{g}^{\mu\nu}, \tag{3.1.17}$$

where here and in the following, unless otherwise specified, we raise and lower all coordinate tensor indices using the conformal metric $\bar{g}_{\mu\nu}$. Contracting the free indices of (3.1.17) gives $\bar{R} = 4\Lambda - \bar{T}$, where $\bar{T} = \bar{g}_{\mu\nu}\bar{T}^{\mu\nu}$ and $\bar{R}$ is the Ricci scalar of the conformal metric. Using this and the definition $\bar{G}^{\mu\nu} = \bar{R}^{\mu\nu} - \frac{1}{2}\bar{R}\bar{g}^{\mu\nu}$ of the Einstein tensor, we can write (3.1.17) as

$$\bar{R}^{\mu\nu} = 2\bar{\nabla}^\mu\bar{\nabla}^\nu\Psi - 2\bar{\nabla}^\mu\Psi\bar{\nabla}^\nu\Psi + \left[\bar{\Box}\Psi + 2|\bar{\nabla}\Psi|^2 + \left(\frac{1-\epsilon^2K}{2}\bar{\rho} + \Lambda\right)e^{2\Psi}\right]\bar{g}^{\mu\nu} + e^{2\Psi}(1+\epsilon^2K)\bar{\rho}\bar{v}^\mu\bar{v}^\nu, \tag{3.1.18}$$

which we will refer to as *the conformal Einstein equations.*

Following [51, 66], we fix the conformal factor by setting

$$\Psi = -\ln t, \tag{3.1.19}$$

and we introduce the background metric

$$\bar{h} = \bar{h}_{\mu\nu}d\bar{x}^\mu d\bar{x}^\nu := -\frac{3}{\Lambda}dtdt + E^2(t)\delta_{ij}d\bar{x}^i d\bar{x}^j, \tag{3.1.20}$$

where

$$E(t) = a(t)t. \tag{3.1.21}$$

We note that the background metric $\bar{h}_{\mu\nu}$ is conformally related to the FLRW metric (3.1.4) according to (3.1.15) under the replacement $\bar{g}^{\mu\nu}\mapsto\bar{h}^{\mu\nu}$ and $\tilde{g}^{\mu\nu}\mapsto\tilde{h}^{\mu\nu}$, where $\bar{h}^{\mu\nu}$ denotes the inverse of $\bar{h}_{\mu\nu}$. By (3.1.7), we observe that $E(t)$ satisfies

$$\partial_t E(t) = \frac{1}{t}E(t)\Omega(t), \tag{3.1.22}$$

where $\Omega(t)$ is defined by

$$\Omega(t) = 1 - \sqrt{1 + \frac{\mu(t)}{\Lambda}} < 0. \tag{3.1.23}$$

For use below, we observe that the relation

$$\mu = \Omega(\Omega - 2)\Lambda \tag{3.1.24}$$

follow directly from the definition of Ω. Straightforward calculations show that the non-vanishing Christoffel symbols $\bar{\gamma}^\sigma_{\mu\nu}$, contracted Christoffel symbols $\bar{\gamma}^\sigma$, Riemannian tensor $\bar{\mathcal{R}}_{\alpha\beta\sigma}{}^\mu$, Ricci tensors $\bar{\mathcal{R}}_{\mu\nu}$ and $\bar{\mathcal{R}}^{\mu\nu} = \bar{h}^{\mu\alpha}\bar{h}^{\mu\beta}\bar{\mathcal{R}}_{\alpha\beta}$, and Ricci scalar $\bar{\mathcal{R}}$ of the background metric (3.1.20) are given by

$$\bar{\gamma}^0_{ij} = \frac{\Lambda}{3t}E^2\Omega\delta_{ij} \qquad \bar{\gamma}^i_{j0} = \frac{1}{t}\Omega\delta^i_j, \qquad \bar{\gamma}^\sigma := \bar{h}^{\mu\nu}\bar{\gamma}^\sigma_{\mu\nu} = \frac{\Lambda}{t}\Omega\delta^\sigma_0, \tag{3.1.25}$$

$$\bar{\mathcal{R}}_{0i0}{}^j = -\bar{\mathcal{R}}_{i00}{}^j = \frac{1}{t^2}(\Omega - \Omega^2 - t\partial_t\Omega)\delta^j_i, \tag{3.1.26}$$

$$\bar{\mathcal{R}}_{0ij}{}^0 = -\bar{\mathcal{R}}_{i0j}{}^0 = \frac{\Lambda}{3t^2}E^2(\Omega - \Omega^2 - t\partial_t\Omega)\delta_{ij}, \qquad \bar{\mathcal{R}}_{ijk}{}^l = \frac{\Lambda}{3t^2}E^2\Omega^2(\delta_{ik}\delta^l_j - \delta_{jk}\delta^l_i), \tag{3.1.27}$$

$$\bar{\mathcal{R}}_{00} = \bar{\mathcal{R}}_{0i0}{}^i = \frac{3}{t^2}(\Omega - \Omega^2 - t\partial_t\Omega), \qquad \bar{\mathcal{R}}_{kl} = -\frac{\Lambda}{3t^2}E^2(\Omega - 3\Omega^2 - t\partial_t\Omega)\delta_{kl}, \tag{3.1.28}$$

$$\bar{\mathcal{R}}^{00} = \frac{\Lambda^2}{3t^2}(\Omega - \Omega^2 - t\partial_t\Omega), \qquad \bar{\mathcal{R}}^{ij} = -\frac{\Lambda}{3t^2}E^{-2}(\Omega - 3\Omega^2 - t\partial_t\Omega)\delta^{ij} \tag{3.1.29}$$

and

$$\bar{\mathcal{R}} = -\frac{2\Lambda}{t^2}(\Omega - 2\Omega^2 - t\partial_t\Omega). \tag{3.1.30}$$

Since the FLRW solution (3.1.4)-(3.1.6) satisfies the Einstein equations, we deduce from (3.1.18) that $\bar{h}^{\mu\nu}$ and μ satisfy

$$\bar{\mathcal{R}}^{\mu\nu} = 2\bar{\nabla}^\nu\bar{\nabla}^\nu\Psi - 2\bar{\nabla}^\mu\Psi\bar{\nabla}^\nu\Psi + \Big[\bar{\Box}\Psi + 2|\bar{\nabla}\Psi|^2_{\bar{h}} + \Big(\frac{1-\epsilon^2K}{2}\bar{\mu} + \Lambda\Big)e^{2\Psi}\Big]\bar{h}^{\mu\nu} + e^{2\Psi}(1+\epsilon^2K)\mu\frac{\Lambda}{3}\delta^\mu_0\delta^\nu_0, \tag{3.1.31}$$

where $\bar{\nabla}_\mu$ is the covariant derivative with respect to the background metric $\bar{h}_{\alpha\beta}$, $\bar{\Box} = \bar{h}^{\mu\nu}\bar{\nabla}_\mu\bar{\nabla}_\nu$ and $|\bar{\nabla}\Psi|^2_{\bar{h}} = \bar{h}^{\mu\nu}\bar{\nabla}_\mu\Psi\bar{\nabla}_\nu\Psi$.

Routine calculations show the Ricci tensor can be expressed as

$$\bar{R}^{\mu\nu} = \frac{1}{2}\bar{g}^{\lambda\sigma}\bar{\nabla}_\lambda\bar{\nabla}_\sigma\bar{g}^{\mu\nu} + \bar{\nabla}^{(\mu}\bar{X}^{\nu)} + \bar{\mathcal{R}}^{\mu\nu} + \bar{P}^{\mu\nu} + \bar{Q}^{\mu\nu} \tag{3.1.32}$$

where

$$\bar{X}^\alpha = \bar{g}^{\beta\gamma}\bar{X}^\alpha{}_{\beta\gamma} = -\bar{\nabla}_\lambda\bar{g}^{\alpha\lambda} + \frac{1}{2}\bar{g}^{\alpha\lambda}\bar{g}_{\sigma\delta}\bar{\nabla}_\lambda\bar{g}^{\sigma\delta} \text{ and } \bar{X}^\alpha{}_{\beta\gamma} = -\frac{1}{2}\big(\bar{g}_{\lambda\gamma}\bar{\nabla}_\beta\bar{g}^{\alpha\lambda} + \bar{g}_{\beta\lambda}\bar{\nabla}_\gamma\bar{g}^{\alpha\lambda} - \bar{g}^{\alpha\lambda}\bar{g}_{\beta\sigma}\bar{g}_{\gamma\delta}\bar{\nabla}_\lambda\bar{g}^{\sigma\delta}\big), \tag{3.1.33}$$

$$\begin{aligned}\bar{P}^{\mu\nu} = &-\frac{1}{2}(\bar{g}^{\mu\lambda} - \bar{h}^{\mu\lambda})\bar{h}^{\alpha\beta}\bar{\mathcal{R}}_{\lambda\alpha\beta}{}^\nu - \frac{1}{2}\bar{h}^{\mu\lambda}(\bar{g}^{\alpha\beta} - \bar{h}^{\alpha\beta})\bar{\mathcal{R}}_{\lambda\alpha\beta}{}^\nu - \frac{1}{2}(\bar{g}^{\mu\lambda} - \bar{h}^{\mu\lambda})(\bar{g}^{\alpha\beta} - \bar{h}^{\alpha\beta})\bar{\mathcal{R}}_{\lambda\alpha\beta}{}^\nu \\ &- \frac{1}{2}(\bar{g}^{\nu\lambda} - \bar{h}^{\nu\lambda})\bar{h}^{\alpha\beta}\bar{\mathcal{R}}_{\lambda\alpha\beta}{}^\mu - \frac{1}{2}h^{\nu\lambda}(\bar{g}^{\alpha\beta} - \bar{h}^{\alpha\beta})\bar{\mathcal{R}}_{\lambda\alpha\beta}{}^\mu - \frac{1}{2}(\bar{g}^{\nu\lambda} - \bar{h}^{\nu\lambda})(\bar{g}^{\alpha\beta} - \bar{h}^{\alpha\beta})\bar{\mathcal{R}}_{\lambda\alpha\beta}{}^\mu,\end{aligned} \tag{3.1.34}$$

and

$$\bar{Q}^{\mu\nu} = -\frac{1}{4}\big(\bar{g}^{\mu\sigma}\bar{g}^{\nu\beta}\bar{\nabla}_\sigma\bar{g}_{\lambda\alpha}\bar{\nabla}_\beta\bar{g}^{\lambda\alpha} + \bar{g}^{\mu\sigma}\bar{g}^{\nu\beta}\bar{\nabla}_\beta\bar{g}_{\lambda\alpha}\bar{\nabla}_\sigma\bar{g}^{\lambda\alpha} + \bar{g}_{\alpha\beta}\bar{g}^{\mu\lambda}\bar{\nabla}_\lambda\bar{g}^{\nu\sigma}\bar{\nabla}_\sigma\bar{g}^{\alpha\beta} + \bar{g}_{\alpha\beta}\bar{g}^{\nu\sigma}\bar{\nabla}_\sigma\bar{g}^{\mu\lambda}\bar{\nabla}_\lambda\bar{g}^{\alpha\beta}\big)$$

$$+\frac{1}{2}\big(\bar{g}^{\mu\alpha}\bar{g}_{\beta\lambda}\bar{\nabla}_\alpha\bar{g}^{\nu\sigma}\bar{\nabla}_\sigma\bar{g}^{\beta\lambda}+\bar{g}^{\mu\alpha}\bar{g}^{\nu\sigma}\bar{\nabla}_\alpha\bar{g}_{\beta\lambda}\bar{\nabla}_\sigma\bar{g}^{\beta\lambda}-\bar{\nabla}_\lambda\bar{g}^{\mu\alpha}\bar{\nabla}_\alpha\bar{g}^{\lambda\nu}-\bar{\nabla}_\alpha\bar{g}^{\nu\lambda}\bar{\nabla}_\lambda\bar{g}^{\alpha\mu}\big)-\bar{g}^{\mu\lambda}\bar{X}^\nu{}_{\lambda\sigma}\bar{X}^\sigma$$
$$+\bar{g}^{\mu\beta}\bar{g}^{\nu\sigma}\bar{X}^\lambda{}_{\beta\sigma}\bar{X}^\alpha{}_{\lambda\alpha}-\bar{g}^{\mu\beta}\bar{g}^{\nu\sigma}\bar{X}^\lambda{}_{\alpha\sigma}\bar{X}^\alpha{}_{\lambda\beta}+\bar{X}^\alpha{}_{\alpha\sigma}\bar{g}^{\mu\beta}\bar{\nabla}_\beta\bar{g}^{\nu\sigma}-\bar{X}^\alpha{}_{\beta\sigma}\bar{g}^{\mu\beta}\bar{\nabla}_\alpha\bar{g}^{\nu\sigma}-\bar{X}^\alpha{}_{\beta\sigma}g^{\nu\sigma}\bar{\nabla}_\alpha\bar{g}^{\mu\beta}. \tag{3.1.35}$$

Employing (3.1.32), we can write the conformal Einstein equations (3.1.18) as

$$-\bar{g}^{\alpha\beta}\bar{\nabla}_\alpha\bar{\nabla}_\beta\bar{g}^{\mu\nu}-2\bar{\nabla}^{(\mu}\bar{X}^{\nu)}-2\bar{\mathcal{R}}^{\mu\nu}-2\bar{P}^{\mu\nu}-2\bar{Q}^{\mu\nu}=-4\bar{\nabla}^\mu\bar{\nabla}^\nu\Psi+4\bar{\nabla}^\mu\Psi\bar{\nabla}^\nu\Psi$$
$$-2\left[\bar{\Box}\Psi+2|\bar{\nabla}\Psi|^2+\left(\frac{1-\epsilon^2K}{2}\bar{\rho}+\Lambda\right)e^{2\Psi}\right]\bar{g}^{\mu\nu}-2e^{2\Psi}(1+\epsilon^2K)\bar{\rho}\bar{v}^\mu\bar{v}^\nu.$$

Letting $\tilde{\Gamma}^\gamma_{\mu\nu}$ and $\bar{\Gamma}^\gamma_{\mu\nu}$ denote the Christoffel symbols of the metrics $\tilde{g}_{\mu\nu}$ and $\bar{g}_{\mu\nu}$, respectively, the difference $\tilde{\Gamma}^\gamma_{\mu\nu}-\bar{\Gamma}^\gamma_{\mu\nu}$ is readily calculated to be $\tilde{\Gamma}^\gamma_{\mu\nu}-\bar{\Gamma}^\gamma_{\mu\nu}=\bar{g}^{\gamma\alpha}\big(\bar{g}_{\mu\alpha}\bar{\nabla}_\nu\Psi+\bar{g}_{\nu\alpha}\bar{\nabla}_\mu\Psi-\bar{g}_{\mu\nu}\bar{\nabla}_\alpha\Psi\big)$. Using this, we can express the Euler equations (3.1.2) as

$$\bar{\nabla}_\mu\tilde{T}^{\mu\nu}=-6\tilde{T}^{\mu\nu}\bar{\nabla}_\mu\Psi+\bar{g}_{\alpha\beta}\tilde{T}^{\alpha\beta}\bar{g}^{\mu\nu}\bar{\nabla}_\mu\Psi, \tag{3.1.36}$$

which we refer to as the *conformal Euler equations*.

Remark 3.1.4. Due to our choice of time orientation, the conformal fluid four-velocity $\bar{v}^\mu$, which we assume is future oriented, satisfies $\bar{v}^0<0$. Furthermore, it follows directly from (3.1.3), (3.1.15) and (3.1.16) that $\bar{v}^\mu$ is normalized, that is,

$$\bar{v}^\mu\bar{v}_\mu=-1. \tag{3.1.37}$$

Wave gauge

In order to obtain a hyperbolic reduction of the conformal Einstein equations that is useful for analyzing the Newtonian limit over long time scales, we need to choose a gauge that is well defined on long time scales and in the limit $\epsilon\searrow 0$. For this, we follow [51] and employ the *wave gauge* defined by

$$\bar{Z}^\mu:=\bar{X}^\mu+\bar{Y}^\mu=0 \tag{3.1.38}$$

where

$$\bar{X}^\mu=-\bar{\nabla}_\nu\bar{g}^{\mu\nu}+\frac{1}{2}\bar{g}^{\mu\sigma}\bar{g}_{\alpha\beta}\bar{\nabla}_\sigma\bar{g}^{\alpha\beta} \tag{3.1.39}$$

and

$$\bar{Y}^\mu=-2(\bar{g}^{\mu\nu}-\bar{h}^{\mu\nu})\bar{\nabla}_\nu\Psi=\frac{2}{t}\left(\bar{g}^{\mu0}+\frac{\Lambda}{3}\delta^\mu_0\right). \tag{3.1.40}$$

Field variables

The gravitational and matter field variables $\{\bar{g}^{\mu\nu}(\bar{x}),\bar{\rho}(\bar{x}),\bar{v}^\mu(\bar{x})\}$ in relativistic coordinates, as they stand, are not suitable for establishing the global existence of solutions or taking the Newtonian limit $\epsilon\searrow 0$. In order to obtain suitable variables, we switch to Newtonian coordinates (x^μ), $t=x^0$, and employ the following field variables, which are closely related to the ones used in [51]:

$$u^{0\mu}=\frac{1}{\epsilon}\frac{\underline{\bar{g}^{0\mu}}-\bar{h}^{0\mu}}{2t}, \tag{3.1.41}$$
$$u^{0\mu}_0=\frac{1}{\epsilon}\left(\delta^0_\nu\underline{\bar{\nabla}_0\bar{g}^{\mu\nu}}-\frac{3(\underline{\bar{g}^{0\mu}}-\bar{h}^{0\mu})}{2t}\right), \tag{3.1.42}$$
$$u^{0\mu}_i=\frac{1}{\epsilon}\delta^0_\nu\underline{\bar{\nabla}_i\bar{g}^{\mu\nu}}, \tag{3.1.43}$$
$$u^{ij}=\frac{1}{\epsilon}\big(\underline{\bar{\mathfrak{g}}^{ij}}-\bar{h}^{ij}\big), \tag{3.1.44}$$

$$u^{ij}_\mu = \frac{1}{\epsilon}\delta^i_\sigma\delta^j_\nu\underline{\bar{\nabla}_\mu(\alpha^{-1}\bar{g}^{\sigma\nu} - \bar{h}^{\sigma\nu})}, \tag{3.1.45}$$
$$u = \frac{1}{\epsilon}\underline{\bar{\mathfrak{q}}}, \tag{3.1.46}$$
$$u_\mu = \frac{1}{\epsilon}\left(\underline{\delta^0_\sigma\delta^0_\nu\bar{\nabla}_\mu(\bar{g}^{\sigma\nu} - \bar{h}^{\sigma\nu})} - \frac{\Lambda}{3}\underline{\bar{\nabla}_\mu \ln\alpha}\right), \tag{3.1.47}$$
$$z_i = \frac{1}{\epsilon}\underline{\bar{v}_i}, \tag{3.1.48}$$
$$\zeta = \frac{1}{1+\epsilon^2 K}\ln\big(t^{-3(1+\epsilon^2 K)}\underline{\bar{\rho}}\big), \tag{3.1.49}$$

and

$$\delta\zeta = \zeta - \zeta_H \tag{3.1.50}$$

where

$$\bar{\mathfrak{g}}^{ij} = \alpha^{-1}\bar{g}^{ij}, \quad \alpha := (\det\bar{g}^{kl})^{\frac{1}{3}}/(\det\bar{h}^{kl})^{\frac{1}{3}} = E^2(\det\breve{g}_{ij})^{-\frac{1}{3}} = E^2(\det\bar{g}^{kl})^{\frac{1}{3}}, \quad \breve{g}_{ij} = (\bar{g}^{ij})^{-1}, \tag{3.1.51}$$
$$\bar{\mathfrak{q}} = \bar{g}^{00} - \bar{h}^{00} - \frac{\Lambda}{3}\ln\alpha, \tag{3.1.52}$$
$$\zeta_H(t) = \frac{1}{1+\epsilon^2 K}\ln\big(t^{-3(1+\epsilon^2 K)}\mu(t)\big) \tag{3.1.53}$$

and we are freely using the notation (3.1.11). As we show below in §3.2.1, ζ_H is given by the explicit formula

$$\zeta_H(t) = \zeta_H(1) - \frac{2}{1+\epsilon^2 K}\ln\left(\frac{C_0 - t^{3(1+\epsilon^2 K)}}{C_0 - 1}\right) \tag{3.1.54}$$

where the constants C_0 and $\zeta_H(1)$ are defined by

$$C_0 = \frac{\sqrt{\Lambda + \mu(1)} + \sqrt{\Lambda}}{\sqrt{\Lambda + \mu(1)} - \sqrt{\Lambda}} > 1 \tag{3.1.55}$$

and

$$\zeta_H(1) = \frac{1}{1+\epsilon^2 K}\ln\mu(1),$$

respectively. Letting

$$\mathring{\zeta}_H = \lim_{\epsilon\searrow 0}\zeta_H \tag{3.1.56}$$

denote the Newtonian limit of ζ_H, it is clear from (3.1.54) that

$$\mathring{\zeta}_H(t) = \ln\mu(1) - 2\ln\left(\frac{C_0 - t^3}{C_0 - 1}\right). \tag{3.1.57}$$

For later use, we also define

$$z^i = \frac{1}{\epsilon}\underline{\bar{v}^i}. \tag{3.1.58}$$

Remark 3.1.5. It is important to emphasize that in the above variables we are treating components of the geometric quantities with respect to the relativistic coordinates as scalars when transforming to Newtonian coordinates. This procedure is necessary in order to obtain variables that have a well defined Newtonian limit. We further emphasize that, for any fixed $\epsilon > 0$, the gravitational and matter fields $\{\bar{g}^{\mu\nu}(\bar{x}), \bar{v}^\mu(\bar{x}), \bar{\rho}(\bar{x})\}$ in relativistic coordinates are completely equivalent to the fields $\{u^{0\mu}(x), u^{ij}(x), u(x), z_i(x), \zeta(x)\}$ defined in the Newtonian coordinates.

3.1.3 Conformal Poisson-Euler equations

The $\epsilon \searrow 0$ limit of the conformal Einstein-Euler equations define the *conformal cosmological Poisson-Euler equations* and are given by

$$\partial_t \mathring{\rho} + \sqrt{\frac{3}{\Lambda}}\partial_j \left(\mathring{\rho}\mathring{z}^j\right) = \frac{3(1-\mathring{\Omega})}{t}\mathring{\rho}, \tag{3.1.59}$$

$$\sqrt{\frac{\Lambda}{3}}\mathring{\rho}\partial_t \mathring{z}^j + K\frac{\delta^{ji}}{\mathring{E}^2}\partial_i\mathring{\rho} + \mathring{\rho}\mathring{z}^i\partial_i\mathring{z}^j = \sqrt{\frac{\Lambda}{3}}\frac{1}{t}\mathring{\rho}\mathring{z}^j - \frac{1}{2}\frac{3}{\Lambda}\frac{t}{\mathring{E}}\mathring{\rho}\frac{\delta^{ji}}{\mathring{E}^2}\partial_i\mathring{\Phi}, \tag{3.1.60}$$

$$\Delta\mathring{\Phi} = \frac{\Lambda}{3}\frac{\mathring{E}^3}{t^3}\delta\mathring{\rho} \tag{3.1.61}$$

where $\Delta := \delta^{ij}\partial_i\partial_j$ is the Euclidean Laplacian on $\mathbb{R}^3$,

$$\mathring{E}(t) = \left(\frac{C_0 - t^3}{C_0 - 1}\right)^{\frac{2}{3}}, \tag{3.1.62}$$

$$\delta\mathring{\rho} = \mathring{\rho} - \mathring{\mu}, \tag{3.1.63}$$

and

$$\mathring{\Omega}(t) = \frac{2t^3}{t^3 - C_0}, \tag{3.1.64}$$

with C_0 as defined above by (3.1.55) and $\mathring{\mu}$ defined by (3.1.6) and (3.1.9). It will be important for our analysis to introduce the modified density variable $\mathring{\zeta}$ defined by

$$\mathring{\zeta} = \ln(t^{-3}\mathring{\rho}),$$

which is the non-relativistic version of the variable ζ defined above by (3.1.49). A short calculation shows that the conformal cosmological Poisson-Euler equations can be expressed in terms of this modified density as follows:

$$\partial_t\mathring{\zeta} + \sqrt{\frac{3}{\Lambda}}(\mathring{z}^j\partial_j\mathring{\zeta} + \partial_j\mathring{z}^j) = -\frac{3\mathring{\Omega}}{t}, \tag{3.1.65}$$

$$\sqrt{\frac{\Lambda}{3}}\partial_t\mathring{z}^j + \mathring{z}^i\partial_i\mathring{z}^j + K\frac{\delta^{ji}}{\mathring{E}^2}\partial_i\mathring{\zeta} = \sqrt{\frac{\Lambda}{3}}\frac{1}{t}\mathring{z}^j - \frac{1}{2}\frac{3}{\Lambda}\frac{t}{\mathring{E}}\frac{\delta^{ji}}{\mathring{E}^2}\partial_i\mathring{\Phi}, \tag{3.1.66}$$

$$\Delta\mathring{\Phi} = \frac{\Lambda}{3}\mathring{E}^3(e^{\mathring{\zeta}} - e^{\mathring{\zeta}_H}). \tag{3.1.67}$$

3.1.4 Initial Data

Thus far, the set up for the Newtonian limit closely mirrors that from [51] with the essential difference being that in this article, we are concerned with a fixed spacetime of the form $M = [0,1)\times\mathbb{R}^3$ as opposed to ϵ-dependent spacetimes of the form[1] $[0,1)\times\mathbb{T}^3_\epsilon$. The change in the spatial hypersurfaces from $\mathbb{T}^3_\epsilon$ to $\mathbb{R}^3$ is important because it will allow us to consider initial data that is physically relevant in the cosmological setting. To understand this improvement, we first recall that the 1-parameter families of ϵ-dependent solutions to the Einstein-Euler equations that were shown in [51] to exist globally to the future on $[0,1)\times\mathbb{T}^3_\epsilon$ and converge as $\epsilon \searrow 0$ to solutions of the cosmological Poisson-Euler equations were interpreted as the cosmological analogues of isolated systems. This interpretation, first discussed in [34], comes from lifting these solutions to the covering space where they become periodic solutions on $[0,1)\times\mathbb{R}^3$ with period $\sim\epsilon$. Since the period determines the spatial size of the universe, the matter in these solutions have a characteristic size $\sim\epsilon$ that shrinks to zero in the Newtonian limit, which is analogous to the behavior of matter in an isolated system under the Newtonian limit.

[1] In [51], unlike the current article, the manifold changes according to the coordinate system used. In relativistic coordinates, the manifold is $[0,1)\times\mathbb{T}^3_\epsilon$, where $\mathbb{T}^3_\epsilon$ is the 3-torus obtain from identifying the sides of the box $[0,\epsilon]^3$, while in Newtonian coordinates, the relevant manifold is $[0,1)\times\mathbb{T}^3_1$.

This leads to the conclusion that the solutions from [51] do not represent gravitating systems on spatial cosmological scales. Instead, they represent gravitating systems on spatial scales comparable to isolated systems. However, they do exist globally to the future which certainly includes time scales that are cosmologically relevant. We further note that the isolated system interpretation also applies to the local-in-time cosmological Newtonian limits from [61, 62], which while only being shown to exist locally in (cosmological) time, do not require a small initial data assumption.

To obtain solutions that are relevant for cosmology, the initial data must be chosen correctly. In particular, the inhomogeneous component of the fluid density should be composed of localized fluctuations, each one of which behaves like an isolated Newtonian system. Furthermore, the fluctuations should be separated from one another by light travel times that remain bounded away from zero as $\epsilon \searrow 0$. Thus, we need to specify, in relativistic coordinates, 1-parameter families of ϵ-dependent families of initial data that can be separated into homogeneous and inhomogeneous components where the homogeneous component has a regular limit as $\epsilon \searrow 0$ while the inhomogeneous component consists of a finite number of spikes with characteristic width $\sim \epsilon$ that can be centered at arbitrarily chosen, ϵ-independent spatial points. Initial data of this type represents cosmological initial data that deviates from homogeneity due to the presence of a finite number of matter fluctuations that remain casually separated and behave as isolated systems in the limit $\epsilon \searrow 0$.

The starting point for constructing this type of initial data is to first select initial data for the matter, which we will specify on the initial hypersurface

$$\Sigma := \{1\} \times \mathbb{R}^3 \cong \mathbb{R}^3.$$

In relativistic coordinates, we choose 1-parameter families of initial data for the proper energy density $\bar{\rho}$ and the spatial components $\bar{v}^I$ of the conformal 3 velocity by setting

$$\bar{\rho}(1, \bar{\mathbf{x}}) = \mu(1) + \delta\breve{\rho}_{\epsilon,\vec{\mathbf{y}}}\Big(\frac{\bar{\mathbf{x}}}{\epsilon}\Big) \quad \text{and} \quad \bar{v}^j(1, \bar{\mathbf{x}}) = \epsilon \breve{z}^j_{\epsilon,\vec{\mathbf{y}}}\Big(\frac{\bar{\mathbf{x}}}{\epsilon}\Big),$$

where

$$\delta\breve{\rho}_{\epsilon,\vec{\mathbf{y}}}(\mathbf{x}) = \sum_{\lambda=1}^{N} \delta\breve{\rho}_\lambda\Big(\mathbf{x} - \frac{\mathbf{y}_\lambda}{\epsilon}\Big), \tag{3.1.68}$$

$$\breve{v}^j_{\epsilon,\vec{\mathbf{y}}}(\mathbf{x}) = \sum_{\lambda=1}^{N} \breve{z}^j_\lambda\Big(\mathbf{x} - \frac{\mathbf{y}_\lambda}{\epsilon}\Big), \tag{3.1.69}$$

and the profile functions $\delta\breve{\rho}_\lambda$ and $\breve{z}^j_\lambda$ are elements of $L^{\frac{6}{5}} \cap K^s$ for some $s \in \mathbb{Z}_{\geq 3}$. It is clear from these formulas that this initial data represents a perturbation of the FLRW initial data $(\bar{\rho}|_\Sigma, \bar{v}^j|_\Sigma) = (\mu(1), 0)$ by N fluctuation of width $\sim \epsilon$ that are centered at the fixed spatial points $\mathbf{y}_\lambda \in \mathbb{R}^3$, $\lambda = 1, 2, \ldots, N$, on the initial hypersurface Σ. As is well known, initial data for the Einstein equations cannot be chosen freely due to presence of constraints that must be satisfied on the initial hypersurface. This has the effect that the description of the initial data for the gravitational field is much more complicated, and consequently, we defer further discussion of the initial data to §3.3.

3.1.5 Main Theorem

With the set up complete, we are now able to state the main result of this article. The proof is given in §3.7.

Theorem 3.1.6. *Suppose* $s \in \mathbb{Z}_{\geq 3}$, $0 < K \leq \frac{1}{3}$, $\Lambda > 0$, $\mu(1) > 0$, $r > 0$, $\vec{\mathbf{y}} = (\mathbf{y}_1, \cdots, \mathbf{y}_N) \in \mathbb{R}^{3N}$, $\Sigma = \{1\} \times \mathbb{R}^3$ *is the initial hypersurface, and the free initial data* $\{\breve{\mathfrak{u}}^{ij}_\epsilon, \breve{\mathfrak{u}}^{ij}_{0,\epsilon}, \delta\breve{\rho}_\lambda, \breve{z}^i_\lambda\}$ *is chosen on* Σ *so that:* $\breve{\mathfrak{u}}^{ij}_\epsilon \in R^{s+1}(\mathbb{R}^3, \mathbb{S}_3)$, $\breve{\mathfrak{u}}^{ij}_{0,\epsilon} \in H^s(\mathbb{R}^3, \mathbb{S}_3)$, $\delta\breve{\rho}_\lambda \in L^{\frac{6}{5}} \cap K^s(\mathbb{R}^3, \mathbb{R})$ *and* $\breve{z}^j_\lambda \in L^{\frac{6}{5}} \cap K^s(\mathbb{R}^3, \mathbb{R}^3)$ *for* $\lambda = 1, \cdots, N$, *and* $\delta\breve{\rho}_{\epsilon,\vec{\mathbf{y}}}$ *and* $\breve{z}^j_{\epsilon,\vec{\mathbf{y}}}$ *are as defined above by* (3.1.68) *and* (3.1.69)*, respectively. Then* *there exists a constant $r > 0$ such that if the free initial data is chosen to satisfy*

$$\|\breve{\xi}_\epsilon\|_s := \|\breve{\mathfrak{u}}^{ij}_\epsilon\|_{R^{s+1}} + \|\breve{\mathfrak{u}}^{ij}_{0,\epsilon}\|_{H^s} + \|\delta\breve{\rho}_\lambda\|_{L^{\frac{6}{5}}\cap K^s} + \|\breve{z}^j_\lambda\|_{L^{\frac{6}{5}}\cap K^s} \leq r,$$

then there exists a constant $\epsilon_0 = \epsilon_0(r) > 0$ and maps $\breve{u}^{\mu\nu}_{\epsilon,\vec{\mathbf{y}}} : X^s_{\epsilon_0}(\mathbb{R}^3) \to R^{s+1}(\mathbb{R}^3, \mathbb{S}_4)$, $\breve{u}_{\epsilon,\vec{\mathbf{y}}} : X^s_{\epsilon_0}(\mathbb{R}^3) \to R^{s+1}(\mathbb{R}^3)$, $\breve{u}^{\mu\nu}_{0,\epsilon,\vec{\mathbf{y}}} : X^s_{\epsilon_0}(\mathbb{R}^3) \to R^s(\mathbb{R}^3, \mathbb{S}_4)$, $\breve{u}_{0,\epsilon,\vec{\mathbf{y}}} : X^s_{\epsilon_0}(\mathbb{R}^3) \to R^s(\mathbb{R}^3)$, $\breve{z}_{i,\epsilon,\vec{\mathbf{y}}} : X^s_{\epsilon_0}(\mathbb{R}^3) \to R^s(\mathbb{R}^3, \mathbb{R}^3)$, and $\delta\breve{\zeta}_{\epsilon,\vec{\mathbf{y}}} : (0, \epsilon_0) \times \big(L^{\frac{6}{5}} \cap K^s(\mathbb{R}^3)\big) \to R^s(\mathbb{R}^3)$, such that

$$\begin{aligned}
u^{0\mu}_{\epsilon,\vec{\mathbf{y}}}|_\Sigma =& \breve{u}^{0\mu}_{\epsilon,\vec{\mathbf{y}}}(\epsilon, \breve{\mathfrak{u}}^{kl}_\epsilon, \breve{\mathfrak{u}}^{kl}_{0,\epsilon}, \delta\breve{\rho}_{\epsilon,\vec{\mathbf{y}}}, \breve{z}^l_{\epsilon,\vec{\mathbf{y}}}) = \mathrm{O}(\epsilon),\\
u_{\epsilon,\vec{\mathbf{y}}}|_\Sigma =& \breve{u}_{\epsilon,\vec{\mathbf{y}}}(\epsilon, \breve{\mathfrak{u}}^{kl}_\epsilon, \breve{\mathfrak{u}}^{kl}_{0,\epsilon}, \delta\breve{\rho}_{\epsilon,\vec{\mathbf{y}}}, \breve{z}^l_{\epsilon,\vec{\mathbf{y}}}) = \epsilon^2 \frac{2\Lambda}{9}\breve{\mathfrak{u}}^{ij}_\epsilon \delta_{ij} + \mathrm{O}(\epsilon^3),\\
u^{ij}_{\epsilon,\vec{\mathbf{y}}}|_\Sigma =& \breve{u}^{ij}_{\epsilon,\vec{\mathbf{y}}}(\epsilon, \breve{\mathfrak{u}}^{kl}_\epsilon, \breve{\mathfrak{u}}^{kl}_{0,\epsilon}, \delta\breve{\rho}_{\epsilon,\vec{\mathbf{y}}}, \breve{z}^l_{\epsilon,\vec{\mathbf{y}}}) = \epsilon^2 \left(\breve{\mathfrak{u}}^{ij}_\epsilon - \frac{1}{3}\breve{\mathfrak{u}}^{kl}_\epsilon \delta_{kl}\delta^{ij}\right) + \mathrm{O}(\epsilon^3),\\
z_{j,\epsilon,\vec{\mathbf{y}}}|_\Sigma =& \breve{z}_{j,\epsilon,\vec{\mathbf{y}}}(\epsilon, \breve{\mathfrak{u}}^{kl}_\epsilon, \breve{\mathfrak{u}}^{kl}_{0,\epsilon}, \delta\breve{\rho}_{\epsilon,\vec{\mathbf{y}}}, \breve{z}^l_{\epsilon,\vec{\mathbf{y}}}) = \delta_{kl}\breve{z}^k_{\epsilon,\vec{\mathbf{y}}} + \mathrm{O}(\epsilon),\\
\delta\zeta_{\epsilon,\vec{\mathbf{y}}}|_\Sigma =& \delta\breve{\zeta}_{\epsilon,\vec{\mathbf{y}}}(\epsilon, \delta\breve{\rho}_{\epsilon,\vec{\mathbf{y}}}) = \frac{1}{1+\epsilon^2 K}\ln\left(1 + \frac{\delta\breve{\rho}_{\epsilon,\vec{\mathbf{y}}}}{\mu(1)}\right),\\
u^{\mu\nu}_{0,\epsilon,\vec{\mathbf{y}}}|_\Sigma =& \breve{u}^{\mu\nu}_{0,\epsilon,\vec{\mathbf{y}}}(\epsilon, \breve{\mathfrak{u}}^{kl}_\epsilon, \breve{\mathfrak{u}}^{kl}_{0,\epsilon}, \delta\breve{\rho}_{\epsilon,\vec{\mathbf{y}}}, \breve{z}^l_{\epsilon,\vec{\mathbf{y}}}) = \mathrm{O}(\epsilon),
\end{aligned}$$

and

$$u_{0,\epsilon,\vec{\mathbf{y}}}|_\Sigma = \breve{u}_{0,\epsilon,\vec{\mathbf{y}}}(\epsilon, \breve{\mathfrak{u}}^{kl}_\epsilon, \breve{\mathfrak{u}}^{kl}_{0,\epsilon}, \delta\breve{\rho}_{\epsilon,\vec{\mathbf{y}}}, \breve{z}^l_{\epsilon,\vec{\mathbf{y}}}) = \mathrm{O}(\epsilon)$$

determine, via the formulas (3.1.41), (3.1.42), (3.1.44), (3.1.46), (3.1.48), (3.1.49), and (3.1.50), a solution of the gravitational and gauge constraint equations, see (3.3.3)-(3.3.4). Furthermore, there exists a constant $\sigma \in (0, r]$, such that if the free initial data is chosen to satisfy

$$\|\breve{\mathfrak{u}}^{ij}_\epsilon\|_{R^{s+1}} + \|\breve{\mathfrak{u}}^{ij}_{0,\epsilon}\|_{H^s} + \|\delta\breve{\rho}_\lambda\|_{L^{\frac{6}{5}}\cap K^s} + \|\breve{z}^j_\lambda\|_{L^{\frac{6}{5}}\cap K^s} \leq \sigma,$$

then there exist maps

$$\begin{aligned}
u^{\mu\nu}_{\epsilon,\vec{\mathbf{y}}} &\in C^0((0,1], R^s(\mathbb{R}^3, \mathbb{S}_4)) \cap C^1((0,1], R^{s-1}(\mathbb{R}^3, \mathbb{S}_4)),\\
u^{\mu\nu}_{\gamma,\epsilon,\vec{\mathbf{y}}} &\in C^0((0,1], R^s(\mathbb{R}^3, \mathbb{S}_4)) \cap C^1((0,1], R^{s-1}(\mathbb{R}^3, \mathbb{S}_4)),\\
u_{\epsilon,\vec{\mathbf{y}}} &\in C^0((0,1], R^s(\mathbb{R}^3)) \cap C^1((0,1], R^{s-1}((\mathbb{R}^3)),\\
u_{\gamma,\epsilon,\vec{\mathbf{y}}} &\in C^0((0,1], R^s(\mathbb{R}^3)) \cap C^1((0,1], R^{s-1}((\mathbb{R}^3)),\\
\delta\zeta_{\epsilon,\vec{\mathbf{y}}} &\in C^0((0,1], R^s(\mathbb{R}^3)) \cap C^1((0,1], R^{s-1}(\mathbb{R}^3)),\\
z_{i,\epsilon,\vec{\mathbf{y}}} &\in C^0((0,1], R^s(\mathbb{R}^3, \mathbb{R}^3)) \cap C^1((0,1], R^{s-1}(\mathbb{R}^3, \mathbb{R}^3)),
\end{aligned}$$

for $\epsilon \in (0, \epsilon_0)$, and

$$\begin{aligned}
\mathring{\Phi}_{\epsilon,\vec{\mathbf{y}}} &\in C^0((0,1], R^{s+2}(\mathbb{R}^3)) \cap C^1((0,1], R^{s+1}(\mathbb{R}^3)),\\
\delta\mathring{\zeta}_{\epsilon,\vec{\mathbf{y}}} &\in C^0((0,1], H^s(\mathbb{R}^3)) \cap C^1((0,1], H^{s-1}(\mathbb{R}^3)),\\
\mathring{z}_{i,\epsilon,\vec{\mathbf{y}}} &\in C^0((0,1], H^s(\mathbb{R}^3, \mathbb{R}^3)) \cap C^1((0,1], H^{s-1}(\mathbb{R}^3, \mathbb{R}^3)),
\end{aligned}$$

such that

(i) *$\{u^{\mu\nu}_{\epsilon,\vec{\mathbf{y}}}(t,x), u_{\epsilon,\vec{\mathbf{y}}}(t,x), \delta\zeta_{\epsilon,\vec{\mathbf{y}}}(t,x), z_{i,\epsilon,\vec{\mathbf{y}}}(t,x)\}$ determines, via (3.1.15), (3.1.16), (3.1.37), (3.1.41), (3.1.44), (3.1.46), (3.1.48) and (3.1.49)-(3.1.52), a 1-parameter family of solutions to the Einstein-Euler equations (3.1.1)-(3.1.2) in the wave gauge (3.1.38) that exists globally to the future on $M = (0,1] \times \mathbb{R}^3$,*

(ii) *$\{\mathring{\Phi}_{\epsilon,\vec{\mathbf{y}}}(t,x), \mathring{\zeta}_{\epsilon,\vec{\mathbf{y}}}(t,x) := \delta\mathring{\zeta}_{\epsilon,\vec{\mathbf{y}}} + \mathring{\zeta}_H, \mathring{z}^i_{\epsilon,\vec{\mathbf{y}}}(t,x) := \mathring{E}(t)^{-2}\delta^{ij}\mathring{z}_{j,\epsilon,\vec{\mathbf{y}}}(t,x)\}$, with $\mathring{\zeta}_H$ and $\mathring{E}$ given by*

(3.1.57) *and* (3.1.62)*, respectively, solves the conformal cosmological Poisson-Euler equations* (3.1.65)-(3.1.67) *on that exists globally to the future on* M *and satisfy the initial conditions*

$$\mathring{\zeta}_{\epsilon,\vec{\mathbf{y}}}|_{\Sigma} = \ln\left(\frac{4C_0\Lambda}{(C_0-1)^2} + \delta\breve{\rho}_{\epsilon,\vec{\mathbf{y}}}\right) \quad \textit{and} \quad \mathring{z}^i_{\epsilon,\vec{\mathbf{y}}}|_{\Sigma} = \breve{z}^i_{\epsilon,\vec{\mathbf{y}}},$$

(iii) the uniform bounds

$$\|\delta\mathring{\zeta}_{\epsilon,\vec{\mathbf{y}}}\|_{L^\infty((0,1],H^s)} + \|\mathring{\Phi}_{\epsilon,\vec{\mathbf{y}}}\|_{L^\infty((0,1],H^{s+2})} + \|\mathring{z}_{j,\epsilon,\vec{\mathbf{y}}}\|_{L^\infty((1,0]\times H^s)} + \|\delta\zeta_{\epsilon,\vec{\mathbf{y}}}\|_{L^\infty((0,1],R^s)} + \|z_{j,\epsilon,\vec{\mathbf{y}}}\|_{L^\infty((0,1],R^s)} \lesssim 1$$

and

$$\|u^{\mu\nu}_{\epsilon,\vec{\mathbf{y}}}\|_{L^\infty((1,0],R^s)} + \|u^{\mu\nu}_{\gamma,\epsilon,\vec{\mathbf{y}}}\|_{L^\infty((0,1],R^s)} + \|u_{\epsilon,\vec{\mathbf{y}}}\|_{L^\infty((0,1],R^s)} + \|u_{\gamma,\epsilon,\vec{\mathbf{y}}}\|_{L^\infty((0,1],R^s)} \lesssim 1,$$

hold for $\epsilon \in (0,\epsilon_0)$*,*

(iv) and the uniform error estimates

$$\|\delta\zeta_{\epsilon,\vec{\mathbf{y}}} - \delta\mathring{\zeta}_{\epsilon,\vec{\mathbf{y}}}\|_{L^\infty((0,1],R^{s-1})} + \|z_{j,\epsilon,\vec{\mathbf{y}}} - \mathring{z}_{j,\epsilon,\vec{\mathbf{y}}}\|_{L^\infty((1,0]\times R^{s-1})} \lesssim \epsilon,$$
$$\|u^{\mu\nu}_{0,\epsilon,\vec{\mathbf{y}}}\|_{L^\infty((1,0],R^{s-1})} + \|u^{\mu\nu}_{k,\epsilon,\vec{\mathbf{y}}} - \delta^\mu_0\delta^\nu_0\partial_k\mathring{\Phi}_{\epsilon,\vec{\mathbf{y}}}\|_{L^\infty((0,1],R^{s-1})} + \|u^{\mu\nu}_{\epsilon,\vec{\mathbf{y}}}\|_{L^\infty((0,1],R^{s-1})} \lesssim \epsilon$$

and

$$\|u_{\gamma,\epsilon,\vec{\mathbf{y}}}\|_{L^\infty((0,1],R^{s-1})} + \|u_{\epsilon,\vec{\mathbf{y}}}\|_{L^\infty((0,1],R^{s-1})} \lesssim \epsilon$$

hold for $\epsilon \in (0,\epsilon_0)$*.*

3.1.6 Future directions

While Theorem 3.1.6 establishes the existence of a large class of inhomogeneous cosmological solutions that are approximated on large cosmological scales by solutions of Newtonian gravity and gives a positive answer to the question at the beginning of the introduction, many questions remain to be answered. For example, the small initial data assumption needed to establish Theorem 3.1.6 suggests the problem of understanding when to expect a similar result to hold without a small initial data assumption. Due to phenomena such as black hole formation, such a result will not hold for all choices of initial data. However, it could hold for carefully chosen large initial data.

In a separate direction, there are relativistic effects that are important for precision cosmology that are not captured by the Newtonian solutions. To understand these effects would require that the Theorem 3.1.6 is generalized to account for higher order post-Newtonian (PN) corrections starting with the 1/2-PN expansion, which is, by definition, the ϵ order correction to the Newtonian gravity. Preliminary work in this direction is currently in preparation [67]. An interesting result of this work is that it characterizes the subset of the 1-parameter families of solutions from Theorem 3.1.6 that can be interpreted on large scales as a linear perturbation of an FLRW solution as those that admit a 1/2-PN expansion. Thus a generalization of Theorem 3.1.6 to include the existence of 1-parameter families of solutions to the Einstein-Euler equations that admit a 1/2-PN expansion would provide a mathematically rigorous resolution to the following perplexing question: *How can the Universe be accurately modelled on small scales using Newtonian gravity, yet, at the same time, be accurately modelled on large scales as a fully relativistic perturbation of an FLRW spacetime?*

3.1.7 Prior and related work

The future non-linear stability of the FLRW fluid solutions for a linear equation of state $p = K\rho$ was first established under the condition $0 < K < 1/3$ and the assumption of zero fluid vorticity by Rodnianski and Speck in [73] using a generalization of a wave based method developed by Ringström in [71]. Subsequently, it has been shown [28, 35, 53, 77] that this future non-linear stability result remains true for fluids with non-zero vorticity and also for the equation of state parameter values $K = 0$ and $K = 1/3$, which correspond to dust and pure radiation, respectively. It is worth noting that the stability results established in [53] and [28] for $K = 1/3$ and $K = 0$, respectively, rely on Friedrich's conformal method [26, 27], which is completely different from the techniques used in [35, 73, 77] for the parameter values $0 \leq K < 1/3$.

In the Newtonian setting, the global existence to the future of solutions to the cosmological Poisson-Euler equations was established in [11] under a small initial data assumption and for a class of polytropic equations of state.

A new method was introduced in [66] to prove the future non-linear stability of the FLRW fluid solutions that was based on a wave formulation of a conformal version of the Einstein-Euler equations. The global existence results in this article are established using this approach. We also note that this method was recently used to establish the non-linear stability of the FLRW fluid solutions that satisfy the generalized Chaplygin equation of state [49].

3.1.8 Overview

In §3.2, we employ the variables (3.1.41)-(3.1.50) and the wave gauge (3.1.38) to write the conformal Einstein-Euler system, given by (3.1.18) and (3.1.36), as a symmetric hyperbolic system that is jointly singular in ϵ and t. By the results of [66], this system is suitable for obtaining the existence of solutions that exist globally to the future; however, it not suitable for obtaining such solutions in the limit $\epsilon \searrow 0$ due to the singular dependence of the solutions on the parameter ϵ. This defect is remedied in §3.5, where we introduce a non-local transformation that brings the system into a form, given by (3.5.23), that is suitable for establishing global existence with uniform control as $\epsilon \searrow 0$.

In §3.3, we use a fixed point method to construct ϵ-dependent 1-parameter families of initial data for the reduced conformal Einstein-Euler equations that satisfy the constraint equations on the initial hypersurface Σ. The fixed point method is similar to one employed in [51]. However, due to the non-compact nature of the initial hypersurface and the translation invariance of the the norms, the proof is technically more difficult and relies crucially on potential theory, in particular, the Riesz and Yukawa potential operators.

In §3.4.1, we state and prove a local-in-time existence and uniqueness result for solutions of the reduced conformal Einstein-Euler equations along with a continuation principle. We establish this local-in-time result by first working in uniformly local Sobolev spaces where we can apply standard theorems. We then show that these results continue to hold in the functions spaces used to obtain global existence in §3.7. Similarly, in §3.4.2, we state and prove a local-in-time existence and uniqueness result and continuation principle for solutions of the conformal cosmological Poisson-Euler equations.

We generalize, in §3.6, the uniform a priori and error estimates established in [51] to hold for a closely related class of symmetric hyperbolic equations on $[T_0, T_1) \times \mathbb{R}^3$ that are jointly singular in ϵ and t. This class includes both the formulation (3.5.23) of the conformal Einstein-Euler equations and the $\epsilon \searrow 0$ limit of these equations.

Finally, in §3.7, we complete the proof of Theorem 3.1.6 by using the results from §3.3 to §3.6 to verify that all the assumptions from §3.6 hold for the non-local formulation (3.5.23) of the conformal Einstein-Euler equations. This allows us to apply Theorem 3.6.10 to get the desired conclusion.

3.2 A singular symmetric hyperbolic formulation of the conformal Einstein–Euler system

In this section, we employ the variables (3.1.41)-(3.1.50) and the wave gauge (3.1.38) to transform the conformal Einstein-Euler system, given by (3.1.18) and (3.1.36), into the form of a symmetric hyperbolic system that has a particular singular dependence on ϵ and t; more specifically, the ϵ-dependent singular terms are of a form that has been well-studied beginning with the pioneering work of Browning, Klainerman, Kreiss and Majda [12, 40, 41, 45], while the t-dependent singular terms are of the type analyzed in [66]. This type of system has been investigated on spacetime regions of the form $(0,1]\times\mathbb{T}^3$ in our previous work [51]. In this section, we derive a modified version of this system that is adapted to spacetime regions of the form $(0,1]\times\mathbb{R}^3$.

3.2.1 Analysis of the FLRW solutions

In [51], we derived some explicit formulas for the functions $\Omega(t)$, $\mu(t)$ and $E(t)$ that will be needed again in this article. For reader's convenience, we reproduce them here beginning with

$$\Omega(t)=\frac{2t^{3(1+\epsilon^2K)}}{t^{3(1+\epsilon^2K)}-C_0} \quad\text{and}\quad \mu(t)=\frac{4C_0\Lambda t^{3(1+\epsilon^2K)}}{(C_0-t^{3(1+\epsilon^2K)})^2}, \tag{3.2.1}$$

where C_0 is as defined above by (3.1.55). From the formula for $\mu(t)$, it is then clear that the formula (3.1.54) for $\zeta_H(t)$ follows immediately from the definition (3.1.53). Furthermore, it clear from the above formulas that Ω, μ and ζ_H, as functions of (t,ϵ), lie in $C^2([0,1]\times[0,\epsilon_0])\cap W^{3,\infty}([0,1]\times[-\epsilon_0,\epsilon_0])$ for any fixed $\epsilon_0>0$, from which it follows that we can represent $t^{-1}\Omega$ and $\partial_t\Omega$ as

$$\frac{1}{t}\Omega=E^{-1}\partial_tE=t^{2+3\epsilon^2K}\hat{\mathscr{Q}}_1(t) \qquad\text{and}\qquad \partial_t\Omega=t^{2+3\epsilon^2K}\hat{\mathscr{Q}}_2(t),$$

respectively, where here, we are employing the notation from §3.1.1 for the remainder terms $\hat{\mathscr{Q}}_1$ and $\hat{\mathscr{Q}}_2$.

Using (3.2.1), we can integrate (3.1.22) to obtain

$$E(t)=\exp\left(\int_1^t\frac{2s^{2+3\epsilon^2K}}{s^{3(1+\epsilon^2K)}-C_0}ds\right)=\left(\frac{C_0-t^{3(1+\epsilon^2K)}}{C_0-1}\right)^{\frac{2}{3(1+\epsilon^2K)}}\geq 1 \tag{3.2.2}$$

for $t\in[0,1]$. From this formula, it is clear that $E\in C^2([0,1]\times[-\epsilon_0,\epsilon_0])\cap W^{3,\infty}([0,1]\times[-\epsilon_0,\epsilon_0])$, and the Newtonian limit of E, denoted $\mathring{E}$ and defined by

$$\mathring{E}(t)=\lim_{\epsilon\searrow0}E(t),$$

is given by the formula (3.1.62). Similarly, we denote the Newtonian limit of Ω by

$$\mathring{\Omega}(t)=\lim_{\epsilon\searrow0}\Omega(t),$$

which we see from (3.2.1) is given by the formula (3.1.64). For latter use, we observe that E, Ω and ζ_H satisfy

$$\partial_t\zeta_H=-\frac{3}{t}\Omega=-3E^{-1}\partial_tE=-\bar{\gamma}^i_{i0}=-\bar{\gamma}^i_{0i}=t^{2+3\epsilon^2K}\hat{\mathscr{Q}}_3(t)$$

as can be verified by a straightforward calculation using the formulas (3.1.54) and (3.2.1)-(3.2.2). By (3.1.57) and (3.1.64), it is also easy to verify

$$\partial_t\mathring{\zeta}_H=-\frac{3}{t}\mathring{\Omega}=\frac{6t^2}{C_0-t^3}. \tag{3.2.3}$$

We record the following useful expansions of $t^{1+3\epsilon^2K}$, $E(\epsilon,t)$ and $\Omega(\epsilon,t)$:

$$t^{1+3\epsilon^2K}=t+\epsilon^2\mathscr{X}(\epsilon,t),\quad \mathscr{X}(\epsilon,t)=\frac{6K}{\epsilon^2}\int_0^\epsilon\lambda t^{1+3\lambda^2K}\ln t d\lambda, \tag{3.2.4}$$

$$E(\epsilon,t)=\mathring{E}(t)+\epsilon\hat{\mathscr{E}}(\epsilon,t) \quad\text{and}\quad \Omega(\epsilon,t)=\mathring{\Omega}(t)+\epsilon\hat{\mathscr{A}}(\epsilon,t) \tag{3.2.5}$$

for $(\epsilon, t) \in (0, \epsilon_0) \times (0, 1]$, where $\mathscr{X}$, $\hat{\mathscr{E}}$ and $\hat{\mathscr{A}}$ are all remainder terms as defined in §3.1.1.

3.2.2 ϵ-expansions and remainder terms

In order to transform the reduced conformal Einstein-Euler equations into the desired form, we need to understand the lowest order ϵ-expansion for a number of quantities. We compute and collect together these expansions in this section. Throughout this section, we work in Newtonian coordinates, and we frequently employ the notation (3.1.11) for evaluation in Newtonian coordinates and the notation from §3.1.1 for remainder terms.

First, we observe, using (3.1.41), (3.1.46) and (3.1.52), that we can write α as

$$\underline{\alpha} = \exp\left(\epsilon\frac{3}{\Lambda}\left(2tu^{00} - u\right)\right) = 1 + \epsilon\frac{3}{\Lambda}\left(2tu^{00} - u\right) + \epsilon^2\hat{\mathscr{Z}}(\epsilon, t, u^{\mu\nu}, u),$$

where $\hat{\mathscr{Z}}(\epsilon, t, 0, 0) = 0$. Using this, we can write the conformal metric as

$$\underline{\bar{g}^{ij}} = E^{-2}\delta^{ij} + \epsilon\Theta^{ij}, \tag{3.2.6}$$

where

$$\Theta^{ij} = \Theta^{ij}(\epsilon, t, u, u^{\mu\nu}) := \frac{1}{\epsilon}\left(\underline{\alpha} - 1\right)E^{-2}\delta^{ij} + \underline{\alpha}u^{ij},$$

and Θ^{ij} satisfies $\Theta^{ij}(\epsilon, t, 0, 0) = 0$ and the E^1-regularity properties of a remainder term, see §3.1.1. From the definition of $u^{0\mu}$, see (3.1.41), we have that

$$\underline{\bar{g}^{0\mu}} = \bar{h}^{0\mu} + 2\epsilon t u^{0\mu}, \tag{3.2.7}$$

and by (3.1.42) and (3.1.43),

$$\delta^0_\nu\underline{\bar{\nabla}_0\bar{g}^{\mu\nu}} = \epsilon(u^{0\mu}_0 + 3u^{0\mu}) \quad \text{and} \quad \delta^0_\nu\underline{\bar{\nabla}_i\bar{g}^{\mu\nu}} = \epsilon u^{0\mu}_i \tag{3.2.8}$$

for the derivatives. Additionally, with the help of (3.1.22), (3.1.41)-(3.1.43) and (3.1.46)-(3.1.47), we have also that

$$\underline{\bar{\nabla}_\beta\alpha} = \epsilon\underline{\alpha}\frac{3}{\Lambda}\big(3u^{00}\delta^0_\beta + u^{00}_\beta - u_\beta\big). \tag{3.2.9}$$

Then differentiating (3.2.6), we find, using the above expression and (3.1.44)-(3.1.45), that

$$\delta^i_\mu\delta^j_\nu\underline{\bar{\nabla}_\sigma\bar{g}^{\mu\nu}} = \epsilon\underline{\alpha}u^{ij}_\sigma + \epsilon\frac{3}{\Lambda}\underline{\alpha}(\bar{h}^{ij} + \epsilon u^{ij})\big(3u^{00}\delta^0_\sigma + u^{00}_\sigma - u_\sigma\big). \tag{3.2.10}$$

Since $\breve{g}_{ij}$ is, by definition, the inverse of $\bar{g}^{ij}$, it follows from (3.2.6) and Lemma D.1.2 that we can express $\breve{g}_{ij}$ as

$$\underline{\breve{g}_{ij}} = E^2\delta_{ij} + \epsilon\hat{\mathscr{Z}}_{ij}(\epsilon, t, u, u^{\mu\nu}), \tag{3.2.11}$$

where $\hat{\mathscr{Z}}_{ij}(\epsilon, t, 0, 0) = 0$. From (3.2.6), (3.2.7) and Lemma D.1.2, we then see that

$$\underline{\bar{g}_{\mu\nu}} = \bar{h}_{\mu\nu} + \epsilon\Xi_{\mu\nu}(\epsilon, t, u^{\sigma\gamma}, u), \tag{3.2.12}$$

where $\Xi_{\mu\nu}$ satisfies $\Xi_{\mu\nu}(\epsilon, t, 0, 0) = 0$ and the E^1-regularity properties of a remainder term. Due to the identity $\bar{\nabla}_\lambda\bar{g}_{\mu\nu} = -\bar{g}_{\mu\sigma}\bar{\nabla}_\lambda\bar{g}^{\sigma\gamma}\bar{g}_{\gamma\nu}$, we can easily derive from (3.2.8), (3.2.10) and (3.2.12) that

$$\underline{\bar{\nabla}_\sigma\bar{g}_{\mu\nu}} = \epsilon\hat{\mathscr{Z}}_{\mu\nu\sigma}(\epsilon, t, \mathbf{u}),$$

where

$$\mathbf{u} = (u^{\alpha\beta}, u, u^{\alpha\beta}_\sigma, u_\sigma),$$

and $\hat{\mathscr{Z}}_{\mu\nu\sigma}(\epsilon,t,0)=0$, which in turn, implies that

$$\underline{X^{\sigma}_{\mu\nu}}=\underline{\bar{\Gamma}^{\sigma}_{\mu\nu}}-\bar{\gamma}^{\sigma}_{\mu\nu}=\epsilon\hat{\mathscr{I}}^{\sigma}_{\mu\nu}(\epsilon,t,\mathbf{u}), \tag{3.2.13}$$

where $\hat{\mathscr{I}}^{\sigma}_{\mu\nu}(\epsilon,t,0)=0$. Later, we will also need the explicit form of the next order term in the ϵ-expansion for $\bar{\Gamma}^{i}_{00}$. By a straightforward calculation, it is then not difficult to verify that

$$\underline{\bar{\Gamma}^{i}_{00}}=\epsilon\frac{3}{\Lambda}(u^{0i}_0+3u^{0i})-\epsilon\frac{1}{2}\left(\frac{3}{\Lambda}\right)^2E^{-2}\delta^{ik}u^{00}_k+\epsilon^2\hat{\mathscr{I}}^{i}_{00}(\epsilon,t,\mathbf{u}), \tag{3.2.14}$$

$$\underline{\bar{\Gamma}^{i}_{k0}}-\underline{\bar{\gamma}^{i}_{k0}}=-\frac{3}{2\Lambda}\epsilon(\delta^{ij}\delta_{kl}u^{0l}_j-u^{0i}_k)-\epsilon\frac{1}{2}E^2\delta_{kj}\left[\left(u^{ij}_0+\frac{3}{\Lambda}E^{-2}(3u^{00}+u^{00}_0-u_0)\delta^{ij}\right)\right]+\epsilon^2\hat{\mathscr{I}}^{i}_{i0}(\epsilon,t,\mathbf{u}), \tag{3.2.15}$$

where $\hat{\mathscr{I}}^{i}_{00}(\epsilon,t,0)=0$ and $\hat{\mathscr{I}}^{i}_{i0}(\epsilon,t,0)=0$.

Continuing on, we observe from (3.1.49) that we can express the proper energy density in terms of ζ by

$$\rho:=\underline{\bar{\rho}}=t^{3(1+\epsilon^2K)}e^{(1+\epsilon^2K)\zeta}, \tag{3.2.16}$$

and correspondingly, by (3.1.53),

$$\mu=t^{3(1+\epsilon^2K)}e^{(1+\epsilon^2K)\zeta_H} \tag{3.2.17}$$

for the FLRW proper energy density. From (3.1.50), (3.2.16) and (3.2.17), it is then clear that we can express the difference between ρ and μ in terms of $\delta\zeta$ by

$$\delta\rho:=\rho-\mu=t^{3(1+\epsilon^2K)}e^{(1+\epsilon^2K)\zeta_H}\Big(e^{(1+\epsilon^2K)\delta\zeta}-1\Big). \tag{3.2.18}$$

Due to the normalization $\bar{v}^{\mu}\bar{v}_{\mu}=-1$, only three components of $\bar{v}_{\mu}$ are independent. Solving $\bar{v}^{\mu}\bar{v}_{\mu}=-1$ for $\bar{v}_0$ in terms of the components $\bar{v}_i$, we obtain

$$\bar{v}_0=\frac{-\bar{g}^{0i}\bar{v}_i+\sqrt{(\bar{g}^{0i}\bar{v}_i)^2-\bar{g}^{00}(\bar{g}^{ij}\bar{v}_i\bar{v}_j+1)}}{\bar{g}^{00}}, \tag{3.2.19}$$

which, in turn, using the definitions (3.1.41), (3.1.44), (3.1.46), (3.1.48), we can write as

$$\underline{\bar{v}_0}=-\frac{1}{\sqrt{-\underline{\bar{g}^{00}}}}+\epsilon^2\hat{\mathscr{V}}_2(\epsilon,t,u,u^{\mu\nu},z_j), \tag{3.2.20}$$

where $\hat{\mathscr{V}}_2(\epsilon,t,u,u^{\mu\nu},0)=0$. From this and the definition $\bar{v}^0=\bar{g}^{0\mu}\bar{v}_\mu$, we get

$$\underline{\bar{v}^0}=\sqrt{-\underline{\bar{g}^{00}}}+\epsilon^2\hat{\mathscr{W}}_2(\epsilon,t,u,u^{\mu\nu},z_j), \tag{3.2.21}$$

where $\hat{\mathscr{W}}_2(\epsilon,t,u,u^{\mu\nu},0)=0$. We also observe that

$$\underline{\bar{v}^k}=\epsilon(2tu^{0k}\underline{\bar{v}_0}+\underline{\bar{g}^{ik}}z_i)\quad\text{and}\quad z^k=2tu^{0k}\underline{\bar{v}_0}+\underline{\bar{g}^{ik}}z_i \tag{3.2.22}$$

follow immediately from the definitions (3.1.48) and (3.1.58). For later use, note that z^k can also be written in terms of $(\underline{\bar{g}^{\mu\nu}},z_j)$ as

$$z^i=\underline{\bar{g}^{ij}}z_j+\frac{\underline{\bar{g}^{i0}}}{\underline{\bar{g}^{00}}}\left[-\underline{\bar{g}^{0j}}z_j+\frac{1}{\epsilon}\sqrt{-\underline{\bar{g}^{00}}}\sqrt{1-\frac{1}{\underline{\bar{g}^{00}}}\epsilon^2(\underline{\bar{g}^{0j}}z_j)^2+\epsilon^2\underline{\bar{g}^{jk}}z_jz_k}\right]. \tag{3.2.23}$$

3.2.3 The reduced conformal Einstein equations

The next step in transforming the conformal Einstein-Euler system is to replace the conformal Einstein equations (3.1.18) with the gauge reduced version given by

$$-\bar{g}^{\alpha\beta}\bar{\nabla}_\alpha\bar{\nabla}_\beta\bar{g}^{\mu\nu} - 2\bar{\nabla}^{(\mu}\bar{X}^{\nu)} - 2\bar{\mathcal{R}}^{\mu\nu} - 2\bar{P}^{\mu\nu} - 2\bar{Q}^{\mu\nu} + 2\bar{\nabla}^{(\mu}\bar{Z}^{\nu)} + \bar{A}^{\mu\nu}_\sigma\bar{Z}^\sigma = -4\bar{\nabla}^\mu\bar{\nabla}^\nu\Psi$$
$$+ 4\bar{\nabla}^\mu\Psi\bar{\nabla}^\nu\Psi - 2\left[\bar{\Box}\Psi + 2|\bar{\nabla}\Psi|^2 + \left(\frac{1-\epsilon^2 K}{2}\bar{\rho} + \Lambda\right)e^{2\Psi}\right]\bar{g}^{\mu\nu} - 2e^{2\Psi}(1+\epsilon^2 K)\bar{\rho}\bar{v}^\mu\bar{v}^\nu, \qquad (3.2.24)$$

where

$$\bar{A}^{\mu\nu}_\sigma := -\bar{X}^{(\mu}\delta^{\nu)}_\sigma + \bar{Y}^{(\mu}\delta^{\nu)}_\sigma.$$

We will refer to these equations as the *reduced conformal Einstein equations.* From (3.1.34) and (3.1.35), it is not difficult to verify, using (3.1.41)-(3.1.47), that $\bar{P}^{\mu\nu}$ and $\bar{Q}^{\mu\nu}$, when expressed in Newtonian coordinates, can be expanded as

$$\underline{\bar{P}^{\mu\nu}} = \epsilon\mathscr{L}^{\mu\nu}(\epsilon, t, u^{\alpha\beta}, u) + \epsilon^2\mathscr{K}^{\mu\nu}(\epsilon, t, u^{\alpha\beta}, u),$$
$$\underline{\bar{Q}^{\mu\nu}} = \epsilon^2\mathscr{W}^{\mu\nu}(\epsilon, t, \mathbf{u}),$$

where $\mathscr{K}^{\mu\nu}$ is quadratic in $(u^{\alpha\beta}, u)$, $\mathscr{W}^{\mu\nu}$ vanishes to second order in $\mathbf{u}$, and $\mathscr{L}^{\mu\nu}$ is linear in $(u^{\alpha\beta}, u)$.

Remark 3.2.1. In the above formula and for the rest of this section, the remainder terms satisfy the following properties: $\mathscr{W}^{\mu\nu}(\epsilon, t, \mathbf{u})$ vanishes to second order in $\mathbf{u}$, $\mathscr{L}(\epsilon, t, \mathbf{u})$, $\mathscr{L}^\mu(\epsilon, t, \mathbf{u})$, $\mathscr{L}^\mu_l(\epsilon, t, \mathbf{u})$, $\mathscr{L}^{\mu\nu\lambda}(\epsilon, t, \mathbf{u})$ and $\mathscr{L}^{\mu\nu}(\epsilon, t, \mathbf{u})$ are linear in $\mathbf{u}$, while $\mathscr{J}(\epsilon, t, \mathbf{u})$, $\mathscr{J}^\mu(\epsilon, t, \mathbf{u})$, $\mathscr{J}^{\mu\nu}(\epsilon, t, \mathbf{u})$, and $\mathscr{J}^{\mu\nu\lambda}(\epsilon, t, \mathbf{u})$ vanish for $\mathbf{u} = 0$.

Using (3.1.31) and (3.1.38), we observe that the reduced conformal Einstein equations (3.2.24) can be written as

$$-\bar{g}^{\alpha\beta}\bar{\nabla}_\alpha\bar{\nabla}_\beta\bar{g}^{\mu\nu} + 2\bar{\nabla}^{(\mu}\bar{Y}^{\nu)} - 2\bar{P}^{\mu\nu} - 2\bar{Q}^{\mu\nu} + \bar{Y}^\mu\bar{Y}^\nu - \bar{X}^\mu\bar{X}^\nu = -4(\bar{\nabla}^\mu\bar{\nabla}^\nu\Psi - \bar{\underline{\nabla}}^\mu\bar{\underline{\nabla}}^\nu\Psi)$$
$$+ 4(\bar{\nabla}^\mu\Psi\bar{\nabla}^\nu\Psi - \bar{\underline{\nabla}}^\mu\Psi\bar{\underline{\nabla}}^\nu\Psi) - 2e^{2\Psi}(1+\epsilon^2 K)\left(\bar{\rho}\bar{v}^\mu\bar{v}^\nu - \mu\frac{\Lambda}{3}\delta^\mu_0\delta^\nu_0\right)$$
$$- 2\left[\bar{\Box}\Psi + 2|\bar{\nabla}\Psi|^2 + \left(\frac{1-\epsilon^2 K}{2}\bar{\rho} + \Lambda\right)e^{2\Psi}\right]\bar{g}^{\mu\nu} + 2\big[\bar{\underline{\Box}}\Psi + 2|\bar{\underline{\nabla}}\Psi|^2_{\bar{h}} + \big(\frac{1-\epsilon^2 K}{2}\mu + \Lambda\big)e^{2\Psi}\big]\bar{h}^{\mu\nu}. \qquad (3.2.25)$$

In the following proposition, we list various formulas for crucial terms in (3.2.25). These formulas can be established via direct computation; we omit the details.

Proposition 3.2.2. *If the wave gauge* (3.1.38) *is satisfied, and* Ψ *and* $\bar{\gamma}^\nu$ *are as given by* (3.1.19) *and* (3.1.25), *respectively, then the following relations hold:*

$$\bar{\nabla}^\mu\Psi = -\bar{g}^{\mu 0}\frac{1}{t}, \qquad \bar{\underline{\nabla}}^\mu\Psi = -\bar{h}^{\mu 0}\frac{1}{t} = \frac{\Lambda}{3t}\delta^\mu_0, \qquad \bar{\nabla}_\mu\Psi = \bar{\underline{\nabla}}_\mu\Psi = -\delta^0_\mu\frac{1}{t}, \qquad \bar{\Box}\Psi = \frac{1}{t^2}\bar{g}^{00} - \frac{1}{t}\bar{Y}^0 + \frac{1}{t}\bar{\gamma}^0,$$
$$\bar{\underline{\Box}}\Psi = \frac{1}{t^2}\bar{h}^{00} + \frac{1}{t}\bar{\gamma}^0 = -\frac{\Lambda}{3t^2} + \frac{1}{t}\bar{\gamma}^0, \qquad |\bar{\nabla}\Psi|^2_{\bar{g}} = \frac{1}{t^2}\bar{g}^{00}, \qquad |\bar{\underline{\nabla}}\Psi|^2_{\bar{h}} = \frac{1}{t^2}\bar{h}^{00} = -\frac{\Lambda}{3t^2},$$
$$\bar{Y}^\mu\bar{Y}^\nu = 4\bar{\nabla}^\mu\Psi\bar{\nabla}^\nu\Psi + \frac{8\Lambda}{3t^2}\delta^{(\mu}_0\bar{g}^{\nu)0} + \frac{4\Lambda^2}{9t^2}\delta^\mu_0\delta^\nu_0, \qquad 4\bar{\underline{\nabla}}^\mu\bar{\underline{\nabla}}^\nu\Psi = \frac{4\Lambda^2}{9t^2}\delta^\mu_0\delta^\nu_0 + \frac{4\Lambda}{3t^2}\Omega\bar{h}^{ij}\delta^\mu_j\delta^\nu_i$$

and

$$\bar{\nabla}^{(\mu}\bar{Y}^{\nu)} = -2\bar{\nabla}^\mu\bar{\nabla}^\nu\Psi - \frac{2\Lambda}{3t^2}\bar{g}^{0(\mu}\delta^{\nu)}_0 - \frac{\Lambda}{3t}\delta^\sigma_0\bar{\nabla}_\sigma\bar{g}^{\mu\nu} + \frac{2\Lambda}{3t^2}\Omega\bar{g}^{i(\mu}\delta^{\nu)}_i.$$

Using Proposition 3.2.2, we see that (3.2.25) can be written as

$$-\bar{g}^{\alpha\beta}\bar{\partial}_\alpha\bar{\nabla}_\beta(\bar{g}^{\mu\nu}-\bar{h}^{\mu\nu})+\bar{\mathcal{E}}^{\mu\nu}-2\bar{P}^{\mu\nu}-\bar{\mathcal{Q}}^{\mu\nu}=\frac{2\Lambda}{3t}\delta^\sigma_0\bar{\nabla}_\sigma(\bar{g}^{\mu\nu}-\bar{h}^{\mu\nu})-\frac{4\Lambda}{3t^2}\left(\bar{g}^{0\lambda}+\frac{\Lambda}{3}\delta^\lambda_0\right)\delta^{(\mu}_\lambda\delta^{\nu)}_0$$
$$-\frac{2}{t^2}\left(\bar{g}^{00}+\frac{\Lambda}{3}\right)\bar{g}^{\mu\nu}-\frac{2}{t^2}(1+\epsilon^2K)\left((\bar{\rho}-\mu)\bar{v}^\mu\bar{v}^\nu+\mu\big(\bar{v}^\mu\bar{v}^\nu-\frac{\Lambda}{3}\delta^\mu_0\delta^\nu_0\big)\right)-\frac{2\Omega}{t^2}\Lambda(\bar{g}^{\mu\nu}-\bar{h}^{\mu\nu})$$
$$-\frac{1-\epsilon^2K}{t^2}(\bar{\rho}-\mu)\bar{g}^{\mu\nu}-\frac{1-\epsilon^2K}{t^2}\mu(\bar{g}^{\mu\nu}-\bar{h}^{\mu\nu})-\frac{4\Lambda}{3t^2}\Omega\left[(\bar{g}^{ij}-\bar{h}^{ij})\delta^{(\mu}_j\delta^{\nu)}_i+\bar{g}^{i0}\delta^{(\mu}_0\delta^{\nu)}_i\right] \quad (3.2.26)$$

where

$$\underline{\bar{\mathcal{Q}}^{\mu\nu}}:=\underline{2\bar{Q}^{\mu\nu}+\bar{X}^\mu\bar{X}^\nu}=\epsilon^2\mathscr{W}^{\mu\nu}(\epsilon,t,\mathbf{u})$$

and

$$\bar{\mathcal{E}}^{\mu\nu}=\bar{g}^{\alpha\beta}\bar{\gamma}^\lambda_{\alpha\beta}\bar{\nabla}_\lambda(\bar{g}^{\mu\nu}-\bar{h}^{\mu\nu})-\bar{g}^{\alpha\beta}\bar{\gamma}^\mu_{\alpha\lambda}\bar{\nabla}_\beta(\bar{g}^{\lambda\nu}-\bar{h}^{\lambda\nu})-\bar{g}^{\alpha\beta}\bar{\gamma}^\nu_{\alpha\lambda}\bar{\nabla}_\beta(\bar{g}^{\mu\lambda}-\bar{h}^{\mu\lambda}).$$

Contracting both sides of (3.2.26) with δ^0_ν, we find that

$$-\bar{g}^{\alpha\beta}\bar{\partial}_\alpha\delta^0_\nu\bar{\nabla}_\beta(\bar{g}^{\mu\nu}-\bar{h}^{\mu\nu})+\bar{\mathcal{E}}^{\mu0}-2\bar{P}^{\mu0}-\bar{\mathcal{Q}}^{\mu0}=\frac{2\Lambda}{3t}\delta^\sigma_0\delta^0_\nu\bar{\nabla}_\sigma(\bar{g}^{\mu\nu}-\bar{h}^{\mu\nu})-\frac{4\Lambda}{3t^2}\left(\bar{g}^{00}+\frac{\Lambda}{3}\right)\delta^\mu_0$$
$$-\frac{4\Lambda}{3t^2}\bar{g}^{0k}\delta^{(\mu}_k\delta^{0)}_0-\frac{2}{t^2}\left(\bar{g}^{00}+\frac{\Lambda}{3}\right)\bar{g}^{\mu0}-\frac{2}{t^2}(1+\epsilon^2K)\left((\bar{\rho}-\mu)\bar{v}^\mu\bar{v}^0+\mu\big(\bar{v}^\mu\bar{v}^0-\frac{\Lambda}{3}\delta^\mu_0\big)\right)$$
$$-\frac{2\Omega}{t^2}\Lambda(\bar{g}^{\mu0}-\bar{h}^{\mu0})-\frac{1-\epsilon^2K}{t^2}(\bar{\rho}-\mu)\bar{g}^{\mu0}-\frac{1-\epsilon^2K}{t^2}\mu(\bar{g}^{\mu0}-\bar{h}^{\mu0})-\frac{2\Lambda}{3t^2}\Omega\bar{g}^{i0}\delta^\mu_i \quad (3.2.27)$$

where

$$\underline{\bar{\mathcal{E}}^{\mu0}}:=\delta^0_\nu\underline{\bar{\mathcal{E}}^{\mu\nu}}=-2\epsilon E^{-2}\frac{\Omega}{t}\delta^{km}\delta^\mu_k u^{00}_m+\epsilon\mathscr{L}^{\mu0}(\epsilon,t,\mathbf{u})+\epsilon^2\mathscr{J}^{\mu0}(\epsilon,t,\mathbf{u}).$$

To proceed, we need to express (3.2.27) in terms of evolution variables (3.1.41)-(3.1.50). In order to achieve this, we first derive, using(3.1.41), (3.1.42), (3.1.43) and (3.2.8), the identities

$$u^{0\mu}_0=\frac{1}{\epsilon}(\delta^0_\nu\underline{\bar{\nabla}_0\bar{g}^{\mu\nu}}-3\epsilon u^{0\mu})=-u^{0\mu}+2t\partial_0u^{0\mu}+2\Omega\delta^\mu_k u^{0k}, \quad (3.2.28)$$
$$u^{0\mu}_i=\frac{1}{\epsilon}\delta^\nu_0\underline{\bar{\nabla}_i\bar{g}^{\mu\nu}}=2t\frac{1}{\epsilon}\partial_iu^{0\mu}+\frac{4\Lambda}{3}E^2\Omega\delta_{ik}u^{k0}\delta^\mu_0+\frac{\Lambda}{3t}E^2\Omega\delta_{ik}\delta^\mu_l\Theta^{kl}+2\Omega\delta^\mu_iu^{00}. \quad (3.2.29)$$

From these identities, we get

$$-3\epsilon\underline{\bar{g}^{00}}\partial_0u^{0\mu}-12\epsilon tu^{0k}\partial_ku^{0\mu}=-\epsilon\frac{3}{2t}\underline{\bar{g}^{00}}(u^{0\mu}_0+u^{0\mu})+\epsilon\mathscr{L}^{0\mu}(\epsilon,t,\mathbf{u})+\epsilon^2\mathscr{J}^{0\mu}(\epsilon,t,\mathbf{u}). \quad (3.2.30)$$

Using this, we can write (3.2.27) as

$$-\underline{\bar{g}^{00}}\partial_0u^{0\mu}_0-4tu^{0k}\partial_ku^{0\mu}_0-\frac{1}{\epsilon}\underline{\bar{g}^{kl}}\partial_ku^{0\mu}_l+\mathscr{L}^{0\mu}(\epsilon,t,\mathbf{u})+\epsilon\mathscr{J}^{0\mu}(\epsilon,t,\mathbf{u})$$
$$=-\frac{1}{2t}\underline{\bar{g}^{00}}(u^{0\mu}_0+u^{0\mu})+2E^{-2}\frac{\Omega}{t}\delta^{km}\delta^\mu_ku^{00}_m+4\epsilon u^{00}u^{0\mu}_0-4\epsilon u^{00}u^{0\mu}-\frac{4\Lambda}{3t}\Omega\delta^\mu_iu^{i0}-2\frac{1-\epsilon^2K}{t}\mu u^{0\mu}$$
$$-\frac{2}{t^2}\frac{1}{\epsilon}(1+\epsilon^2K)\left((\rho-\mu)\underline{\bar{v}^\mu\bar{v}^0}+\mu\big(\underline{\bar{v}^\mu\bar{v}^0}-\frac{\Lambda}{3}\delta^\mu_0\big)\right)-\frac{4\Omega}{t}\Lambda u^{\mu0}-\frac{1-\epsilon^2K}{t^2}\frac{1}{\epsilon}(\rho-\mu)\underline{\bar{g}^{\mu0}}, \quad (3.2.31)$$

while we see that

$$-\underline{\bar{g}^{00}}\partial_0u^{0\mu}=-\frac{1}{2t}\underline{\bar{g}^{00}}u^{0\mu}_0-\frac{1}{2t}\underline{\bar{g}^{00}}u^{0\mu}+\underline{\bar{g}^{00}}\frac{1}{t}\Omega\delta^\mu_ku^{0k} \quad (3.2.32)$$

follows from (3.2.28). We see also that

$$\partial_l u_0^{0\mu} = \frac{1}{\epsilon}\left(\delta^0_\nu \partial_l \underline{\bar{\nabla}_0 \bar{g}^{\mu\nu}} - \frac{3}{2t}\partial_l \underline{\bar{g}^{0\mu}}\right) = \epsilon \partial_0 u_l^{0\mu} - \frac{3}{2t}\epsilon u_l^{0\mu} + \epsilon \mathscr{L}_l^{\mu}(\epsilon, t, \mathbf{u}),$$

follows from (3.2.8), from which we deduce that

$$\underline{\bar{g}^{kl}}\partial_0 u_k^{0\mu} - \frac{1}{\epsilon}\underline{\bar{g}^{kl}}\partial_k u_0^{0\mu} = \frac{3}{2t}\underline{\bar{g}^{kl}} u_k^{0\mu} + \mathscr{L}^{\mu l}(\epsilon, t, \mathbf{u}) + \epsilon \mathscr{J}^{\mu l}(\epsilon, t, \mathbf{u}). \tag{3.2.33}$$

Together, (3.2.31), (3.2.32), (3.2.33) for a system of evolution equations whose principal part involves the metric variables $\{u_0^{0\mu}, u_l^{0\mu}, u^{0\mu}\}$.

Next, we contract both sides of (3.2.26) with $\delta^k_\mu \delta^l_\nu$ to get

$$\begin{aligned} &-\bar{g}^{\alpha\beta}\delta^k_\mu \delta^l_\nu \bar{\partial}_\alpha \bar{\nabla}_\beta(\bar{g}^{\mu\nu} - \bar{h}^{\mu\nu}) + \bar{\mathcal{E}}^{kl} - 2\bar{P}^{kl} - \bar{\mathcal{Q}}^{kl} = \frac{2\Lambda}{3t}\delta^k_\mu \delta^l_\nu \bar{\nabla}_0(\bar{g}^{\mu\nu} - \bar{h}^{\mu\nu}) - \frac{2}{t^2}\left(\bar{g}^{00} + \frac{\Lambda}{3}\right)\bar{g}^{kl} \\ &\quad - \frac{2}{t^2}(1+\epsilon^2 K)\bar{\rho}\bar{v}^k \bar{v}^l - \frac{1-\epsilon^2 K}{t^2}(\bar{\rho} - \mu)\bar{g}^{kl} - \frac{1-\epsilon^2 K}{t^2}\mu(\bar{g}^{kl} - \bar{h}^{kl}) - \frac{10\Lambda}{3t^2}\Omega(\bar{g}^{kl} - \bar{h}^{kl}), \end{aligned} \tag{3.2.34}$$

where

$$\underline{\bar{\mathcal{E}}^{kl}} := \delta^k_\mu \delta^l_\nu \underline{\bar{\mathcal{E}}^{\mu\nu}} = \epsilon \mathscr{L}^{kl}(\epsilon, t, \mathbf{u}) + \epsilon^2 \mathscr{J}^{kl}(\epsilon, t, \mathbf{u}).$$

Using the identity

$$\breve{g}_{kl}\delta^k_\sigma \delta^l_\nu \bar{\nabla}_\mu \bar{g}^{\sigma\nu} = 3\bar{\nabla}_\mu \ln\alpha + 2\breve{g}_{kl}\delta^k_\mu \bar{g}^{l0}\frac{\Omega}{t}, \tag{3.2.35}$$

where we recall that $(\breve{g}_{kl}) = (\bar{g}^{kl})^{-1}$, and the definitions (3.1.41) and (3.1.47), a direct calculation, using the identity

$$\bar{\partial}_\alpha \breve{g}_{kl} = -\breve{g}_{ki}(\bar{\partial}_\alpha \bar{g}^{ij})\breve{g}_{jl}, \tag{3.2.36}$$

shows that

$$\frac{\Lambda}{3}\frac{2}{3}\underline{\bar{g}^{\alpha\beta}\bar{\partial}_\alpha\left(\breve{g}_{kl}\delta^k_\beta \bar{g}^{l0}\frac{\Omega}{t}\right)} = \epsilon \mathscr{L}(\epsilon, t, \mathbf{u}) + \epsilon^2 \mathscr{J}(\epsilon, t, \mathbf{u})$$

and

$$\delta^k_\mu \delta^l_\nu \underline{\bar{g}^{\alpha\beta}(\bar{\partial}_\alpha \breve{g}_{kl})\bar{\nabla}_\beta\big(\bar{g}^{\mu\nu} - \bar{h}^{\mu\nu}\big)} = \epsilon \mathscr{L}(\epsilon, t, \mathbf{u}) + \epsilon^2 \mathscr{J}(\epsilon, t, \mathbf{u}).$$

We then observe that these two expressions can be used to write (3.2.34) as

$$\begin{aligned} &-\underline{\bar{g}^{00}}\partial_0 u_0 - 4t u^{0k}\partial_k u_0 - \frac{1}{\epsilon}\underline{\bar{g}^{kl}}\partial_k u_l + \mathscr{L}(\epsilon, t, \mathbf{u}) + \epsilon \mathscr{J}(\epsilon, t, \mathbf{u}) = -\frac{2}{t}\underline{\bar{g}^{00}}u_0 + 4\epsilon u^{00}u_0 \\ &\quad - 8\epsilon(u^{00})^2 + \frac{\Lambda}{3}\frac{1}{\epsilon}\frac{2}{3t^2}(1+\epsilon^2 K)\rho \underline{\bar{v}^i \bar{v}^j}\underline{\breve{g}_{ij}} - \frac{4\Omega}{t}\Lambda u^{00} - \frac{1}{\epsilon}\frac{1-\epsilon^2 K}{t^2}(\rho - \mu)\left(\underline{\bar{g}^{00}} - \frac{\Lambda}{3}\right) + \frac{\Lambda}{3}\frac{10\Lambda}{9t^2}\Omega\Theta^{ij}\underline{\breve{g}_{ij}} \\ &\quad - \frac{1}{\epsilon}\frac{2}{t^2}(1+\epsilon^2 K)\left((\rho - \mu)\underline{\bar{v}^0 \bar{v}^0} + \mu\big(\underline{\bar{v}^0 \bar{v}^0} - \frac{\Lambda}{3}\big)\right) - 2\frac{1-\epsilon^2 K}{t}\mu u^{00} + \frac{\Lambda}{3}\frac{1-\epsilon^2 K}{3t^2}\mu \Theta^{ij}\underline{\breve{g}_{ij}}. \end{aligned} \tag{3.2.37}$$

We also observe that the equations

$$\underline{\bar{g}^{kl}}\partial_0 u_l - \frac{1}{\epsilon}\underline{\bar{g}^{kl}}\partial_l u_0 = \frac{2\Lambda}{3t}E^2\Omega \delta_{lj}u_0^{0j}\underline{\bar{g}^{kl}} + \frac{2\Lambda}{3t}(E^2\Omega + 2E^2\Omega^2 + 2Et\partial_t \Omega)\delta_{lj}u^{0j}\underline{\bar{g}^{kl}} \tag{3.2.38}$$

and

$$-\underline{\bar{g}^{00}}\partial_0 u = -\underline{\bar{g}^{00}}u_0 \tag{3.2.39}$$

follow easily from differentiating u_l and u, see (3.1.46)-(3.1.47), with respect to t. Together (3.2.37), (3.2.38) and (3.2.39) form a system of evolution equations whose principal part involves the metric variables $\{u_0, u_l, u\}$.

To obtain evolution equations for the remaining metric variables $\{u^{ij}_0, u^{ij}_l, u^{ij}\}$, we define

$$\mathfrak{L}^{ij}_{kl} = \delta^i_k\delta^j_l - \frac{1}{3}\breve{g}_{kl}\bar{g}^{ij},$$

and contract $\frac{1}{\alpha}\mathfrak{L}^{ij}_{lm}$ on both sides of (3.2.34). A calculation using the identities (the first identity can be derived with the help of (3.2.35))

$$\alpha^{-1}\mathfrak{L}^{ij}_{lm}\delta^l_\mu\delta^m_\nu\bar{\nabla}_\sigma\bar{g}^{\mu\nu} = \delta^i_\mu\delta^j_\nu\bar{\nabla}_\sigma(\alpha^{-1}\bar{g}^{\mu\nu}) - \frac{2}{3}\bar{\mathfrak{g}}^{ij}\breve{g}_{kl}\delta^k_\sigma\bar{g}^{l0}\frac{\Omega}{t} \qquad \text{and} \qquad \mathfrak{L}^{ij}_{lm}\bar{g}^{lm} = 0,$$

where we recall that $\bar{\mathfrak{g}}^{ij}$ is defined by (3.1.51), then shows that the following equation holds:

$$\begin{aligned} &-\bar{g}^{\alpha\beta}\bar{\partial}_\alpha\big(\delta^i_\mu\delta^j_\nu\bar{\nabla}_\beta(\alpha^{-1}\bar{g}^{\mu\nu})\big) + \frac{2}{3}\bar{g}^{\alpha\beta}\bar{\partial}_\alpha\Big(\bar{\mathfrak{g}}^{ij}\breve{g}_{kl}\delta^k_\beta\bar{g}^{l0}\frac{\Omega}{t}\Big) + \bar{g}^{\alpha\beta}\bar{\partial}_\alpha(\alpha^{-1}\mathfrak{L}^{ij}_{lm})\delta^l_\mu\delta^m_\nu\bar{\nabla}_\beta\bar{g}^{\mu\nu} + \alpha^{-1}\mathfrak{L}^{ij}_{lm}\mathcal{E}^{lm} \\ &-\frac{2}{\alpha}\mathfrak{L}^{ij}_{lm}\bar{P}^{lm}(\bar{g}^{-1}) - \frac{1}{\alpha}\mathfrak{L}^{ij}_{lm}\mathcal{Q}^{lm}(\bar{g},\bar{\nabla}\bar{g}^{-1}) \\ =&\frac{2\Lambda}{3t}\delta^i_\mu\delta^j_\nu\bar{\nabla}_0(\alpha^{-1}\bar{g}^{\mu\nu}) - \frac{2}{t^2}(1+\epsilon^2K)\bar{\rho}\frac{1}{\alpha}\mathfrak{L}^{ij}_{lm}\bar{v}^l\bar{v}^m + \frac{1-\epsilon^2K}{t^2}\bar{\mu}\frac{1}{\alpha}\mathfrak{L}^{ij}_{lm}\bar{h}^{lm} + \frac{10\Lambda}{3t^2}\Omega\frac{1}{\alpha}\mathfrak{L}^{ij}_{lm}\bar{h}^{lm}. \end{aligned} \tag{3.2.40}$$

Furthermore, using identity (3.2.36), it is not difficult via a straightforward calculation to verify that

$$\underline{\mathfrak{L}^{ij}_{lm}}\bar{h}^{lm} = \epsilon\mathscr{L}^{ij}(\epsilon,t,\mathbf{u}) + \epsilon^2\mathscr{J}^{ij}(\epsilon,t,\mathbf{u}), \tag{3.2.41}$$

$$\underline{\frac{2}{3}\bar{g}^{\alpha\beta}\bar{\partial}_\alpha\big(\bar{\mathfrak{g}}^{ij}\breve{g}_{kl}\delta^k_\beta\bar{g}^{l0}\frac{\Omega}{t}\big)} = \epsilon\mathscr{L}^{ij}(\epsilon,t,\mathbf{u}) + \epsilon^2\mathscr{J}^{ij}(\epsilon,t,u^{\alpha\beta},\mathbf{u}) \tag{3.2.42}$$

and

$$\underline{\bar{g}^{\alpha\beta}\bar{\partial}_\alpha(\alpha^{-1}\mathfrak{L}^{ij}_{lm})\delta^l_\mu\delta^m_\nu\bar{\nabla}_\beta\bar{g}^{\mu\nu}} = \epsilon^2\mathscr{J}^{ij}(\epsilon,t,\mathbf{u}). \tag{3.2.43}$$

Using (3.2.41)-(3.2.43), we can then rewrite (3.2.40) as

$$\begin{aligned} &-\underline{\bar{g}^{00}}\partial_0 u^{ij}_0 - 4tu^{0k}\partial_k u^{ij}_0 - \frac{1}{\epsilon}\underline{\bar{g}^{kl}}\partial_k u^{ij}_l + \mathscr{L}^{ij}(\epsilon,t,\mathbf{u}) + \epsilon\mathscr{J}^{ij}(\epsilon,t,\mathbf{u}) \\ &\qquad = -\frac{2}{t}\underline{\bar{g}^{00}}u^{ij}_0 + 4\epsilon u^{00}u^{ij}_0 - \frac{1}{\epsilon}\frac{2}{t^2}(1+\epsilon^2K)\rho\underline{\frac{1}{\alpha}\mathfrak{L}^{ij}_{lm}\bar{v}^l\bar{v}^m}, \end{aligned} \tag{3.2.44}$$

while differentiating u^{ij}_l and u^{ij}, (3.1.44)-(3.1.45), with respect to t yields

$$\underline{\bar{g}^{kl}}\partial_0 u^{ij}_l - \frac{1}{\epsilon}\underline{\bar{g}^{kl}}\partial_l u^{ij}_0 = \mathscr{L}^{kij}(\epsilon,t,\mathbf{u}) + \epsilon\mathscr{J}^{kij}(\epsilon,t,\mathbf{u}) \tag{3.2.45}$$

and

$$-\underline{\bar{g}^{00}}\partial_0 u^{ij} = -\underline{\bar{g}^{00}}u^{ij}_0 + \underline{\bar{g}^{00}}\frac{2}{t}\Omega u^{ij}. \tag{3.2.46}$$

Together (3.2.44), (3.2.45) and (3.2.46) form a system of evolution equations whose principal term involves the remaining metric variables $\{u^{ij}_0, u^{ij}_l, u^{ij}\}$.

Gathering (3.2.31)-(3.2.33), (3.2.37)-(3.2.39) and (3.2.44)-(3.2.46) together, we arrive at the following formulation of the reduced conformal Einstein equations:

$$\tilde{B}^0\partial_0\begin{pmatrix} u^{0\mu}_0 \\ u^{0\mu}_k \\ u^{0\mu} \end{pmatrix} + \tilde{B}^k\partial_k\begin{pmatrix} u^{0\mu}_0 \\ u^{0\mu}_l \\ u^{0\mu} \end{pmatrix} + \frac{1}{\epsilon}\tilde{C}^k\partial_k\begin{pmatrix} u^{0\mu}_0 \\ u^{0\mu}_l \\ u^{0\mu} \end{pmatrix} = \frac{1}{t}\tilde{\mathfrak{B}}\mathbb{P}_2\begin{pmatrix} u^{0\mu}_0 \\ u^{0\mu}_l \\ u^{0\mu} \end{pmatrix} + \hat{S}_1, \tag{3.2.47}$$

$$\tilde{B}^0\partial_0\begin{pmatrix}u^{ij}_0\\u^{ij}_k\\u^{ij}\end{pmatrix}+\tilde{B}^k\partial_k\begin{pmatrix}u^{ij}_0\\u^{ij}_l\\u^{ij}\end{pmatrix}+\frac{1}{\epsilon}\tilde{C}^k\partial_k\begin{pmatrix}u^{ij}_0\\u^{ij}_l\\u^{ij}\end{pmatrix}=-\frac{2E^2\underline{\bar{g}^{00}}}{t}\breve{\mathbb{P}}_2\begin{pmatrix}u^{ij}_0\\u^{ij}_l\\u^{ij}\end{pmatrix}+\tilde{S}_2+\tilde{G}_2, \tag{3.2.48}$$

$$\tilde{B}^0\partial_0\begin{pmatrix}u_0\\u_k\\u\end{pmatrix}+\tilde{B}^k\partial_k\begin{pmatrix}u_0\\u_l\\u\end{pmatrix}+\frac{1}{\epsilon}\tilde{C}^k\partial_k\begin{pmatrix}u_0\\u_l\\u\end{pmatrix}=-\frac{2E^2\underline{\bar{g}^{00}}}{t}\breve{\mathbb{P}}_2\begin{pmatrix}u_0\\u_l\\u\end{pmatrix}+\tilde{S}_3+\tilde{G}_3, \tag{3.2.49}$$

where

$$\tilde{B}^0=E^2\begin{pmatrix}-\underline{\bar{g}^{00}} & 0 & 0\\ 0 & \underline{\bar{g}^{kl}} & 0\\ 0 & 0 & -\underline{\bar{g}^{00}}\end{pmatrix},\qquad \tilde{B}^k=E^2\begin{pmatrix}-4tu^{0k} & -\Theta^{kl} & 0\\ -\Theta^{kl} & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, \tag{3.2.50}$$

$$\tilde{C}^k=\begin{pmatrix}0 & -\delta^{kl} & 0\\ -\delta^{kl} & 0 & 0\\ 0 & 0 & 0\end{pmatrix},\qquad \tilde{\mathfrak{B}}=E^2\begin{pmatrix}-\underline{\bar{g}^{00}} & 0 & 0\\ 0 & \frac{3}{2}\underline{\bar{g}^{ki}} & 0\\ 0 & 0 & -\underline{\bar{g}^{00}}\end{pmatrix}, \tag{3.2.51}$$

$$\mathbb{P}_2=\begin{pmatrix}\frac{1}{2} & 0 & \frac{1}{2}\\ 0 & \delta^l_i & 0\\ \frac{1}{2} & 0 & \frac{1}{2}\end{pmatrix},\qquad \breve{\mathbb{P}}_2=\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, \tag{3.2.52}$$

$$\hat{S}_1=E^2\begin{pmatrix}2E^{-2}\frac{\Omega}{t}\delta^{km}\delta^{\mu}_k u^{00}_m-\frac{2(1-\epsilon^2K)}{t}\delta\rho u^{0\mu}+\hat{f}^{0\mu}+\mathscr{L}^{0\mu}+\epsilon\mathscr{J}^{0\mu}\\ \mathscr{L}^{0\mu l}+\epsilon\mathscr{J}^{0\mu l}\\ \mathscr{L}^{00\mu}+\epsilon\mathscr{J}^{00\mu}\end{pmatrix}, \tag{3.2.53}$$

$$\begin{aligned}\hat{f}^{0\mu}=&-\frac{\Lambda}{3}\frac{1}{t^2}\frac{1}{\epsilon}\delta\rho\delta^{\mu}_0-\frac{4}{t^2}\frac{1}{\epsilon}\sqrt{\frac{\Lambda}{3}}\delta\rho\delta^{\mu}_0\Big(\underline{\bar{v}^0}-\sqrt{\frac{\Lambda}{3}}\Big)-\frac{2}{t^2}\frac{1}{\epsilon}\delta\rho\delta^{\mu}_0\Big(\underline{\bar{v}^0}-\sqrt{\frac{\Lambda}{3}}\Big)^2-\frac{2}{t^2}\delta\rho\delta^{\mu}_i z^i\underline{\bar{v}^0}-\frac{2}{t^2}\mu\delta^{\mu}_i z^i\underline{\bar{v}^0}\\&-\frac{2}{t^2}\frac{1}{\epsilon}\mu\delta^{\mu}_0\Big(\underline{\bar{v}^0}-\sqrt{\frac{\Lambda}{3}}\Big)\Big(\underline{\bar{v}^0}+\sqrt{\frac{\Lambda}{3}}\Big)-\frac{2}{t^2}\epsilon K\left(\delta\rho\underline{\bar{v}^{\mu}\bar{v}^0}+\mu\Big(\underline{\bar{v}^{\mu}\bar{v}^0}-\frac{\Lambda}{3}\delta^{\mu}_0\Big)\right)-\frac{\Lambda}{3}\frac{\epsilon K}{t^2}\delta\rho\delta^{\mu}_0,\end{aligned}$$

$$\tilde{G}_2=E^2\begin{pmatrix}-\epsilon\frac{2}{t^2}(1+\epsilon^2K)\rho\underline{\alpha}^{-1}\underline{\mathfrak{L}^{ij}_{lm}}z^lz^m+\mathscr{L}^{ij}\\ \mathscr{L}^{ijl}\\ \mathscr{L}^{0ij}\end{pmatrix}, \tag{3.2.54}$$

$$\tilde{G}_3=E^2\begin{pmatrix}\epsilon\frac{2(1+\epsilon^2K)\Lambda}{9t^2}\rho z^iz^j\underline{\breve{g}_{ij}}-\frac{(1+\epsilon^2K)}{t^2}\rho\Big(\underline{v^0}+\sqrt{\frac{\Lambda}{3}}\Big)\frac{\underline{\bar{v}^0}-\sqrt{\frac{\Lambda}{3}}}{\epsilon}-\epsilon\frac{4\Lambda}{3t^2}K\delta\rho-\frac{2(1-\epsilon^2K)}{t}\delta\rho u^{00}+\mathscr{L}\\ \mathscr{L}^l\\ \mathscr{L}^0\end{pmatrix}, \tag{3.2.55}$$

$$\tilde{S}_2=\epsilon(\mathscr{J}^{ij},\mathscr{J}^{ijl},\mathscr{J}^{0ij})^T\quad\text{and}\quad\tilde{S}_3=\epsilon(\mathscr{J},\mathscr{J}^l,\mathscr{J}^0)^T. \tag{3.2.56}$$

3.2.4 The conformal Euler equations

In this section, we turn to the problem of transforming the conformal Euler equations. We begin by noting that it follows from the computation in [51, 65] that conformal Euler equations (3.1.36), when expressed in Newtonian coordinates, are given by

$$\bar{B}^0\partial_0\begin{pmatrix}\zeta\\ z^i\end{pmatrix}+\bar{B}^k\partial_k\begin{pmatrix}\zeta\\ z^i\end{pmatrix}=\frac{1}{t}\bar{\mathcal{B}}\hat{\mathbb{P}}_2\begin{pmatrix}\zeta\\ z^i\end{pmatrix}+\bar{S}, \tag{3.2.57}$$

where

$$\begin{aligned}
\bar{B}^0 &= \begin{pmatrix} 1 & \epsilon\frac{L^0_i}{\underline{\bar{v}^0}} \\ \epsilon\frac{L^0_j}{\underline{\bar{v}^0}} & K^{-1}M_{ij}\end{pmatrix},\\
\bar{B}^k &= \begin{pmatrix} \frac{1}{\epsilon}\frac{\underline{\bar{v}^k}}{\underline{\bar{v}^0}} & \frac{L^k_i}{\underline{\bar{v}^0}} \\ \frac{L^k_j}{\underline{\bar{v}^0}} & K^{-1}M_{ij}\frac{1}{\epsilon}\frac{\underline{\bar{v}^k}}{\underline{\bar{v}^0}}\end{pmatrix} = \begin{pmatrix} \frac{1}{\underline{\bar{v}^0}}z^k & \frac{1}{\underline{\bar{v}^0}}\delta^k_i \\ \frac{1}{\underline{\bar{v}^0}}\delta^k_j & K^{-1}\frac{1}{\underline{\bar{v}^0}}M_{ij}z^k\end{pmatrix},\\
\bar{\mathfrak{B}} &= \begin{pmatrix} 1 & 0 \\ 0 & -K^{-1}(1-3\epsilon^2K)\frac{\underline{\bar{g}_{ik}}}{\underline{\bar{v}_0}\,\underline{\bar{v}^0}}\end{pmatrix},\\
\hat{\mathbb{P}}_2 &= \begin{pmatrix} 0 & 0 \\ 0 & \delta^k_j\end{pmatrix},\\
\bar{S} &= \begin{pmatrix} -L^\mu_i\underline{\bar{\Gamma}^i_{\mu\nu}}\,\underline{\bar{v}^\nu}\frac{1}{\underline{\bar{v}^0}} \\ -K^{-1}(1-3\epsilon^2K)\frac{1}{\underline{\bar{v}_0}}\underline{\bar{g}_{0j}}-K^{-1}M_{ij}\underline{\bar{v}^\mu}\frac{1}{\epsilon}\underline{\bar{\Gamma}^i_{\mu\nu}}\,\underline{\bar{v}^\nu}\frac{1}{\underline{\bar{v}^0}}\end{pmatrix},\\
L^\mu_i &= \delta^\mu_i-\frac{\underline{\bar{v}_i}}{\underline{\bar{v}_0}}\delta^\mu_0
\end{aligned}$$

and

$$M_{ij}=\underline{\bar{g}_{ij}}-\frac{\underline{\bar{v}_i}}{\underline{\bar{v}_0}}\underline{\bar{g}_{0j}}-\frac{\underline{\bar{v}_j}}{\underline{\bar{v}_0}}\underline{\bar{g}_{0i}}+\frac{\underline{\bar{g}_{00}}}{(\underline{\bar{v}_0})^2}\underline{\bar{v}_i}\,\underline{\bar{v}_j}.$$

In order to bring (3.2.57) into the required form, we perform a change of variables from z^i to z_j, which are related via the map $z^i=z^i(z_j,\underline{\bar{g}^{\mu\nu}})$ given by (3.2.23). Denoting the Jacobian of the transformation by

$$J^{im}:=\frac{\partial z^i}{\partial z_m},$$

we observe that

$$\partial_\sigma z^i=J^{im}\partial_\sigma z_m+\delta^0_\sigma\frac{\partial z^i}{\partial\underline{\bar{g}^{\mu\nu}}}\underline{\bar{\partial}_0\bar{g}^{\mu\nu}}+\epsilon\delta^j_\sigma\frac{\partial z^i}{\partial\underline{\bar{g}^{\mu\nu}}}\underline{\bar{\partial}_j\bar{g}^{\mu\nu}}.$$

Multiplying (3.2.57) by the block matrix $\mathrm{diag}\,(1,J^{jl})$ and changing variables from (ζ,z^i) to $(\delta\zeta,z_j)$, where we recall from (3.1.50) that $\delta\zeta=\zeta-\zeta_H$, it is not difficult to verify that we can write (3.2.57) as

$$B^0\partial_0\begin{pmatrix}\delta\zeta\\ z_m\end{pmatrix}+B^k\partial_k\begin{pmatrix}\delta\zeta\\ z_m\end{pmatrix}=\frac{1}{t}\mathfrak{B}\hat{\mathbb{P}}_2\begin{pmatrix}\delta\zeta\\ z_m\end{pmatrix}+\hat{S} \tag{3.2.58}$$

where

$$\begin{aligned}
B^0 &= \begin{pmatrix} 1 & \epsilon\frac{L^0_i}{\underline{\bar{v}^0}}J^{im} \\ \epsilon\frac{L^0_j}{\underline{\bar{v}^0}}J^{jl} & K^{-1}M_{ij}J^{jl}J^{im}\end{pmatrix},\\
B^k &= \begin{pmatrix} \frac{1}{\underline{\bar{v}^0}}z^k & \frac{1}{\underline{\bar{v}^0}}J^{km} \\ \frac{1}{\underline{\bar{v}^0}}J^{kl} & K^{-1}\frac{1}{\underline{\bar{v}^0}}M_{ij}J^{jl}J^{im}z^k\end{pmatrix},\\
\mathfrak{B} &= \begin{pmatrix} 1 & 0 \\ 0 & -K^{-1}(1-3\epsilon^2K)\frac{1}{\underline{\bar{v}_0}\,\underline{\bar{v}^0}}J^{ml}\end{pmatrix}
\end{aligned}$$

and

$$\hat{S}=\begin{pmatrix}-L^0_i\underline{\bar{\Gamma}^i_{00}}-L^\mu_i\underline{\bar{\Gamma}^i_{\mu j}}\,\underline{\bar{v}^j}\frac{1}{\underline{\bar{v}^0}}+(\bar{\gamma}^i_{i0}-\underline{\bar{\Gamma}^i_{i0}})\\ -K^{-1}J^{jl}M_{ij}\underline{\bar{v}^\mu}\frac{1}{\epsilon}\underline{\bar{\Gamma}^i_{\mu\nu}}\,\underline{\bar{v}^\nu}\frac{1}{\underline{\bar{v}^0}}+\epsilon\frac{L^0_j}{\underline{\bar{v}^0}}J^{jl}\bar{\gamma}^i_{i0}\end{pmatrix}-\begin{pmatrix}\epsilon\frac{L^0_i}{\underline{\bar{v}^0}}\frac{\partial z^i}{\partial\bar{g}^{\mu\nu}}\underline{\bar{\partial}_0\bar{g}^{\mu\nu}}+\epsilon\frac{\delta^k_i}{\underline{\bar{v}^0}}\frac{\partial z^i}{\partial\bar{g}^{\mu\nu}}\underline{\bar{\partial}_k\bar{g}^{\mu\nu}}\\ K^{-1}M_{ij}J^{jl}\frac{\partial z^i}{\underline{\partial\bar{g}^{\mu\nu}}}\underline{\bar{\partial}_0\bar{g}^{\mu\nu}}+\epsilon K^{-1}\bar{M}_{ij}\frac{z^k}{\underline{\bar{v}^0}}J^{jl}\frac{\partial z^i}{\partial\underline{\bar{g}^{\mu\nu}}}\underline{\bar{\partial}_k\bar{g}^{\mu\nu}}\end{pmatrix}.$$

A direct calculation employing (3.2.23) and the expansions (3.2.6) and (3.2.7) shows that

$$J^{ik}=E^{-2}\delta^{ik}+\epsilon\Theta^{ik}+\epsilon^2\hat{\mathscr{S}}^{ik}(\epsilon,t,u,u^{\mu\nu},z_j),\tag{3.2.59}$$

where $\hat{\mathscr{S}}^{ik}(\epsilon,t,0,0,0)=0$. Similarly, it is not difficult to see from (3.2.23) and the expansions (3.2.6) and (3.2.7)-(3.2.10) that

$$\frac{\partial z^i}{\underline{\partial\bar{g}^{\mu\nu}}}\underline{\bar{\partial}_\sigma\bar{g}^{\mu\nu}}=-2\Bigg(\delta^0_\sigma E^{-2}\frac{\Omega}{t}z_j\delta^{ij}+\sqrt{\frac{3}{\Lambda}}\Big(\delta^0_\sigma(u^{0i}_0+3u^{0i})+\delta^j_\sigma u^{0i}_j-2\delta^0_\sigma\Omega u^{0i}\Big)\Bigg)+\epsilon\mathscr{S}^i_\sigma(\epsilon,t,\mathbf{u},z_j)\tag{3.2.60}$$

and

$$\epsilon\frac{\delta^k_i}{\underline{\bar{v}^0}}\frac{\partial z^i}{\partial\bar{g}^{\mu\nu}}\bar{\partial}_k\bar{g}^{\mu\nu}=-\epsilon\frac{6}{\Lambda}u^{0i}_k\delta^k_i+\epsilon^2\mathscr{S}(\epsilon,t,\mathbf{u},z_j),\tag{3.2.61}$$

where $\mathscr{S}^i_\sigma(\epsilon,t,0,0)=\mathscr{S}(\epsilon,t,0,0)=0$. We further note that the term $-K^{-1}J^{jl}M_{ij}\underline{\bar{v}^\mu}\frac{1}{\epsilon}\underline{\bar{\Gamma}^j_{\mu\nu}}\,\underline{\bar{v}^\nu}\frac{1}{\underline{\bar{v}^0}}$ found in $\hat{S}$ above is not singular in ϵ. This can be seen from the expansions (3.2.6), (3.2.7), (3.2.12) and (3.2.14), which can be used to derive

$$\begin{aligned}\frac{1}{\epsilon}\underline{\bar{\Gamma}^j_{\mu\nu}v^\mu v^\nu}=&2\underline{\bar{\Gamma}^j_{0i}}\underline{v^0}z^i+\epsilon\underline{\bar{\Gamma}^j_{ik}}z^iz^k+\frac{1}{\epsilon}\underline{\bar{\Gamma}^j_{00}}\underline{v^0v^0}\\ =&\sqrt{\frac{\Lambda}{3}}\frac{2\Omega}{t}E^{-2}z_i\delta^{ij}+u^{0j}_0+3u^{0j}-\frac{1}{2}\left(\frac{3}{\Lambda}\right)E^{-2}\delta^{jk}u^{00}_k+\epsilon\mathscr{S}^j(\epsilon,t,\mathbf{u},z_j),\end{aligned}\tag{3.2.62}$$

where $\mathscr{S}^j(\epsilon,t,0,0)=0$. Moreover, using the expansions (3.2.59), (3.2.60) and (3.2.62) in conjunction with (3.2.6), (3.2.7), (3.2.12), (3.2.13), (3.2.20), (3.2.21) and (3.2.22), we observe that the matrices $\{B^0,B^k,\mathfrak{B}\}$ and source term $\hat{S}$ can be expanded as

$$B^0=\begin{pmatrix}1&0\\0&K^{-1}E^{-2}\delta^{lm}\end{pmatrix}+\epsilon\begin{pmatrix}0&0\\0&K^{-1}\Theta^{lm}\end{pmatrix}+\epsilon^2\hat{\mathscr{S}}^0(\epsilon,t,\mathbf{u},z_j),\tag{3.2.63}$$

$$\begin{aligned}B^k=&\sqrt{\frac{3}{\Lambda}}\begin{pmatrix}z^k&E^{-2}\delta^{km}\\E^{-2}\delta^{kl}&K^{-1}E^{-2}\delta^{lm}z^k\end{pmatrix}+\epsilon\sqrt{\frac{3}{\Lambda}}\begin{pmatrix}\frac{3}{\Lambda}tu^{00}z^k&\Theta^{km}+\frac{3}{\Lambda}tu^{00}E^{-2}\delta^{km}\\ \Theta^{kl}+\frac{3}{\Lambda}tu^{00}E^{-2}\delta^{kl}&K^{-1}(\Theta^{lm}+\frac{3}{\Lambda}tu^{00}E^{-2}\delta^{lm})z^k\end{pmatrix}\\ &+\epsilon^2\hat{\mathscr{S}}^k(\epsilon,t,\mathbf{u},z_j),\end{aligned}\tag{3.2.64}$$

$$\mathfrak{B}=\begin{pmatrix}1&0\\0&K^{-1}(1-3\epsilon^2K)E^{-2}\delta^{lm}\end{pmatrix}+\epsilon\begin{pmatrix}0&0\\0&K^{-1}\Theta^{lm}\end{pmatrix}+\epsilon^2\hat{\mathscr{S}}(\epsilon,t,\mathbf{u},z_j)\tag{3.2.65}$$

and

$$\begin{aligned}\hat{S}=&\begin{pmatrix}0\\-K^{-1}\left[\sqrt{\frac{3}{\Lambda}}(-u^{0l}_0+(-3+4\Omega)u^{0l})+\frac{1}{2}\left(\frac{3}{\Lambda}\right)^{\frac{3}{2}}E^{-2}\delta^{lk}u^{00}_k\right]\end{pmatrix}\\ &+\epsilon\begin{pmatrix}\frac{1}{2}E^2\delta_{ij}\big(u^{ij}_0+\frac{3}{\Lambda}E^{-2}(3u^{00}+u^{00}_0-u_0)\delta^{ij}\big)+\frac{6}{\Lambda}u^{0i}_k\delta^k_i\\ \mathscr{S}_1(\epsilon,t,\mathbf{u},z_j)\end{pmatrix}+\epsilon^2\mathscr{S}(\epsilon,t,\mathbf{u},z_j),\end{aligned}\tag{3.2.66}$$

where all the remainder terms $\hat{\mathscr{S}}^\mu$, $\hat{\mathscr{S}}$, $\mathscr{S}_1$ and $\mathscr{S}$ vanish for $(\mathbf{u},z_j)=(0,0)$.

3.2.5 The reduced conformal Einstein-Euler equations

Collecting (3.2.47), (3.2.48), (3.2.49) and (3.2.58) together and setting

$$\hat{\mathbf{U}} = (\hat{\mathbf{U}}_1, \mathbf{U}_2)^T, \tag{3.2.67}$$

where

$$\hat{\mathbf{U}}_1 = (u_0^{0\mu}, u_k^{0\mu}, u^{0\mu}, u_0^{ij}, u_k^{ij}, u^{ij}, u_0, u_k, u)^T \quad \text{and} \quad \mathbf{U}_2 = (\delta\zeta, z_i)^T, \tag{3.2.68}$$

we obtain the following symmetric hyperbolic formulation of the reduced conformal Einstein-Euler equations:

$$\mathbf{B}^0\partial_t\hat{\mathbf{U}} + \mathbf{B}^i\partial_i\hat{\mathbf{U}} + \frac{1}{\epsilon}\mathbf{C}^i\partial_i\hat{\mathbf{U}} = \frac{1}{t}\mathbf{B}\mathbf{P}\hat{\mathbf{U}} + \hat{\mathbf{H}} \tag{3.2.69}$$

where

$$\mathbf{B}^0 = \begin{pmatrix} \tilde{B}^0 & 0 & 0 & 0 \\ 0 & \tilde{B}^0 & 0 & 0 \\ 0 & 0 & \tilde{B}^0 & 0 \\ 0 & 0 & 0 & B^0 \end{pmatrix}, \quad \mathbf{B}^i = \begin{pmatrix} \tilde{B}^i & 0 & 0 & 0 \\ 0 & \tilde{B}^i & 0 & 0 \\ 0 & 0 & \tilde{B}^i & 0 \\ 0 & 0 & 0 & B^i \end{pmatrix}, \quad \mathbf{C}^i = \begin{pmatrix} \tilde{C}^i & 0 & 0 & 0 \\ 0 & \tilde{C}^i & 0 & 0 \\ 0 & 0 & \tilde{C}^i & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{3.2.70}$$

$$\mathbf{B} = \begin{pmatrix} \tilde{\mathfrak{B}} & 0 & 0 & 0 \\ 0 & -2E^2\underline{\bar{g}^{00}}I & 0 & 0 \\ 0 & 0 & -2E^2\underline{\bar{g}^{00}}I & 0 \\ 0 & 0 & 0 & \mathfrak{B} \end{pmatrix}, \quad \mathbf{P} = \begin{pmatrix} \mathbb{P}_2 & 0 & 0 & 0 \\ 0 & \breve{\mathbb{P}}_2 & 0 & 0 \\ 0 & 0 & \breve{\mathbb{P}}_2 & 0 \\ 0 & 0 & 0 & \hat{\mathbb{P}}_2 \end{pmatrix}, \tag{3.2.71}$$

$$\hat{\mathbf{H}} = (\hat{S}_1, \tilde{S}_2 + \tilde{G}_2, \tilde{S}_3 + \tilde{G}_3, \hat{S})^T,$$

and $\hat{\mathbf{H}}$ vanishes for $\hat{\mathbf{U}} = 0$. The importance of this formulation of the reduced conformal Einstein-Euler equations is that it is now of the form analyzed in [66]. As a consequence, we could, for fixed $\epsilon > 0$, use the results of [66] to obtain the global existence to the future under a suitable small initial data assumption. What the above formulation is not yet suitable for is analyzing the limit $\epsilon \searrow 0$. To bring the system into a form that is suitable requires a further non-local transformation, which is carried out in §3.5.

3.3 Initial data

Before continuing on with the analysis of the evolution equations, we will, in this section, turn to the problem of selecting initial data. It is well known that the initial data for the reduced conformal Einstein-Euler equations cannot be chosen freely on the initial hypersurface

$$\Sigma = \{1\} \times \mathbb{R}^3 \subset M = (0, 1] \times \mathbb{R}^3$$

due to constraint equations that must satisfied on Σ. To solve these constraints, we employ a variation of Lottermoser's method [52] (also see [51, 61, 62, 64]), which we use to construct 1-parameter families of ϵ-dependent solutions to the constraint equations that behave appropriately in the limit $\epsilon \searrow 0$. In order to use Lottermoser's method, we represent the gravitation field in terms of the variables $\hat{\mathfrak{u}}^{\mu\nu}$ and $\hat{\mathfrak{u}}^{\mu\nu}_\sigma$ that are defined via the formulas

$$\hat{g}^{\mu\nu} := \theta\bar{g}^{\mu\nu} = \bar{h}^{\mu\nu} + \epsilon^2\hat{\mathfrak{u}}^{\mu\nu} \qquad \text{and} \qquad \hat{\mathfrak{u}}^{\mu\nu}_\sigma := \bar{\partial}_\sigma\hat{\mathfrak{u}}^{\mu\nu}, \tag{3.3.1}$$

respectively, where

$$\theta = \frac{\sqrt{|\bar{g}|}}{\sqrt{|\bar{h}|}} = E^{-3}\sqrt{\frac{\Lambda}{3}|\bar{g}|}, \quad |\bar{g}| = -\det \bar{g}_{\mu\nu} \quad \text{and} \quad |\bar{h}| = -\det \bar{h}_{\mu\nu} = \frac{3}{\Lambda}E^6. \tag{3.3.2}$$

The complete set of constraints that we must solve on Σ are:

$$(\bar{G}^{0\mu} - \bar{T}^{0\mu})|_\Sigma = 0 \quad \text{(gravitational constraints)}, \tag{3.3.3}$$

$$\bar{Z}^\mu|_\Sigma = 0 \quad \text{(gauge constraints)} \tag{3.3.4}$$

and

$$(\bar{v}^\mu \bar{v}_\mu + 1)|_\Sigma = 0 \quad \text{(velocity normalization)}. \tag{3.3.5}$$

3.3.1 Transformation formulas

Before proceeding, we first establish some transformation formulas that will be used repeatedly in our analysis of the constraint equations. In the following, we will freely use the notation set out in §3.1.1 for analytic remainder terms.

Lemma 3.3.1.

$$\theta = E^3\sqrt{-\frac{3}{\Lambda}\det(\bar{h}^{\mu\nu} + \epsilon^2 \hat{\mathfrak{u}}^{\mu\nu})} = 1 + \frac{1}{2}\epsilon^2\left(-\frac{3}{\Lambda}\hat{\mathfrak{u}}^{00} + E^2\hat{\mathfrak{u}}^{ij}\delta_{ij}\right) + \epsilon^4 \breve{\mathscr{S}}(\epsilon, t, E, \hat{\mathfrak{u}}^{\mu\nu}),$$

where $\breve{\mathscr{S}}$ vanishes to second order in $\hat{\mathfrak{u}}^{\mu\nu}$.

Proof. The proof follows from a direct calculation. □

Using the above lemma, we obtain the related formulas

$$\frac{1}{\theta} - 1 = -\frac{1}{2}\epsilon^2\left(-\frac{3}{\Lambda}\hat{\mathfrak{u}}^{00} + E^2\hat{\mathfrak{u}}^{ij}\delta_{ij}\right) + \epsilon^4\breve{\mathscr{S}}(\epsilon, t, E, \hat{\mathfrak{u}}^{\mu\nu}) = \frac{3}{2\Lambda}\epsilon^2\hat{\mathfrak{u}}^{00} - \frac{1}{2}\epsilon^2E^2\hat{\mathfrak{u}}^{ij}\delta_{ij} + \epsilon^4\breve{\mathscr{S}}(\epsilon, t, E, \hat{\mathfrak{u}}^{\mu\nu}) \tag{3.3.6}$$

and

$$\frac{\theta - 1}{\epsilon^2} = \frac{1}{2}\left(-\frac{3}{\Lambda}\hat{\mathfrak{u}}^{00} + E^2\hat{\mathfrak{u}}^{ij}\delta_{ij}\right) + \epsilon^2\breve{\mathscr{S}}(\epsilon, t, E, \hat{\mathfrak{u}}^{\mu\nu}) = -\frac{3}{2\Lambda}\hat{\mathfrak{u}}^{00} + \frac{1}{2}E^2\hat{\mathfrak{u}}^{ij}\delta_{ij} + \epsilon^2\breve{\mathscr{S}}(\epsilon, t, E, \hat{\mathfrak{u}}^{\mu\nu}), \tag{3.3.7}$$

where as above the remainder terms $\breve{\mathscr{S}}$ vanish to second order in $\hat{\mathfrak{u}}^{\mu\nu}$.

Lemma 3.3.2. *The metric variables $u^{0\mu}$, u^{ij} and u can be expressed in terms of the $\hat{\mathfrak{u}}^{\mu\nu}$ via the transformation formulas*

$$u^{0\mu} = \frac{\epsilon}{2t}\left(\frac{1}{2}\underline{\hat{\mathfrak{u}}^{00}}\delta^\mu_0 + \underline{\hat{\mathfrak{u}}^{0k}}\delta^\mu_k + \frac{\Lambda}{6}E^2\underline{\hat{\mathfrak{u}}^{ij}}\delta_{ij}\delta^\mu_0\right) + \epsilon^3\breve{\mathscr{S}}(\epsilon, t, E, \Omega/t, \underline{\hat{\mathfrak{u}}^{\alpha\beta}}), \tag{3.3.8}$$

$$u = \epsilon\frac{2\Lambda}{9}E^2\underline{\hat{\mathfrak{u}}^{ij}}\delta_{ij} + \epsilon^3\breve{\mathscr{S}}(\epsilon, t, E, \Omega/t, \underline{\hat{\mathfrak{u}}^{\alpha\beta}}), \tag{3.3.9}$$

$$u^{ij} = \epsilon\left(\underline{\hat{\mathfrak{u}}^{ij}} - \frac{1}{3}\underline{\hat{\mathfrak{u}}^{kl}}\delta_{kl}\delta^{ij}\right) + \epsilon^3\breve{\mathscr{S}}(\epsilon, t, E, \Omega/t, \underline{\hat{\mathfrak{u}}^{\alpha\beta}}), \tag{3.3.10}$$

where all of the remainder terms $\breve{\mathscr{S}}$ vanish to second order in $\underline{\hat{\mathfrak{u}}^{\mu\nu}}$. Moreover, the 0-component of the conformal fluid four-velocity $\bar{v}^\mu$ can be written as

$$\underline{\bar{v}^0} = \sqrt{\frac{\Lambda}{3}} + \epsilon^2\breve{\mathscr{T}}(\epsilon, t, E, \Omega/t, \underline{\hat{\mathfrak{u}}^{\alpha\beta}}, z_j)$$

where $\breve{\mathscr{T}}$ *vanishes to first order in* $(\underline{\hat{\mathfrak{u}}^{\alpha\beta}}, z_j)$.

Proof. First, we observe that the first formula in the statement of the lemma follows directly from (3.2.7) and Lemma 3.3.1. Next, using (3.3.1), it is not difficult to verify that

$$\det(\bar{g}^{kl}) = \theta^{-3}(E^{-6} + \epsilon^2 E^{-4}\hat{\mathfrak{u}}^{ij}\delta_{ij}) + \epsilon^4\breve{\mathscr{S}} = E^{-6} + \frac{1}{2}\epsilon^2 E^{-6}\left(\frac{9}{\Lambda}\hat{\mathfrak{u}}^{00} - E^2\hat{\mathfrak{u}}^{ij}\delta_{ij}\right) + \epsilon^4\breve{\mathscr{S}}(\epsilon, t, E, \Omega/t, \hat{\mathfrak{u}}^{\alpha\beta}),$$

from which, with the help of (3.1.51), we get

$$\alpha = 1 + \frac{1}{6}\epsilon^2\left(\frac{9}{\Lambda}\hat{\mathfrak{u}}^{00} - E^2\hat{\mathfrak{u}}^{ij}\delta_{ij}\right) + \epsilon^4\breve{\mathscr{S}}(\epsilon, t, E, \Omega/t, \hat{\mathfrak{u}}^{\alpha\beta}). \tag{3.3.11}$$

Then by (3.1.46), (3.1.52) and (3.3.11), we obtain

$$u = 2tu^{00} - \frac{1}{\epsilon}\frac{\Lambda}{3}\ln[1 + (\underline{\alpha} - 1)] = \epsilon\frac{2\Lambda}{9}E^2\underline{\hat{\mathfrak{u}}^{ij}}\delta_{ij} + \epsilon^3\breve{\mathscr{S}}(\epsilon, t, E, \Omega/t, \underline{\hat{\mathfrak{u}}^{\alpha\beta}}),$$

while we see that

$$u^{ij} = \frac{1}{\epsilon}\left((\underline{\alpha\theta})^{-1}\underline{\hat{g}^{ij}} - \bar{h}^{ij}\right) = \epsilon\left(\underline{\hat{\mathfrak{u}}^{ij}} - \frac{1}{3}\underline{\hat{\mathfrak{u}}^{kl}}\delta_{kl}\delta^{ij}\right) + \epsilon^3\breve{\mathscr{S}}(\epsilon, t, E, \Omega/t, \underline{\hat{\mathfrak{u}}^{\alpha\beta}})$$

follows from (3.2.6), (3.3.1), (3.3.11) and $(\alpha\theta)^{-1} = 1 - \epsilon^2\frac{1}{3}E^2\hat{\mathfrak{u}}^{ij}\delta_{ij} + \epsilon^4\breve{\mathscr{S}}(\epsilon, t, E, \Omega/t, \hat{\mathfrak{u}}^{\mu\nu})$; this establishes the second and third formulas from the statement of the lemma. Finally, we observe that last formula is a consequence of (3.2.7), (3.2.21) and the first three formulas. □

3.3.2 Reformulation of the constraint equations

We start the process of expressing the constraint equations (3.3.3)-(3.3.5) in terms of the variables (3.3.1) by noting that

$$\bar{g}_{\lambda\sigma}\bar{\nabla}_\nu\bar{g}^{\lambda\sigma} = -\frac{2}{\theta}\bar{\nabla}_\nu\theta \quad \text{and} \quad \hat{g}_{\lambda\sigma}\bar{\nabla}_\nu\hat{g}^{\lambda\sigma} = \frac{2}{\theta}\bar{\nabla}_\nu\theta \tag{3.3.12}$$

where $(\hat{g}_{\lambda\sigma}) = (\hat{g}^{\alpha\beta})^{-1}$. Using this, we can express the vector fields $\bar{X}^\mu$ and $\bar{Y}^\mu$, recall $\bar{Z}^\mu = \bar{X}^\mu + \bar{Y}^\mu$ by (3.1.38), in terms of the variables (3.3.1) by

$$\bar{X}^\mu = -\bar{\nabla}_\nu\bar{g}^{\mu\nu} + \frac{1}{2}\bar{g}^{\mu\nu}\bar{g}_{\alpha\beta}\bar{\nabla}_\nu\bar{g}^{\alpha\beta} = -\bar{\nabla}_\nu\bar{g}^{\mu\nu} - \bar{g}^{\mu\nu}\frac{1}{\sqrt{|\bar{g}|}}\bar{\partial}_\nu\sqrt{|\bar{g}|} + \bar{g}^{\mu\nu}\bar{\gamma}^\alpha_{\nu\alpha} = -\frac{1}{\theta}\bar{\nabla}_\nu\hat{g}^{\mu\nu} = -\epsilon^2\frac{1}{\theta}\bar{\nabla}_\nu\hat{\mathfrak{u}}^{\mu\nu}$$

and

$$\bar{Y}^\mu = -2\bar{\nabla}^\mu\Psi + \frac{2\Lambda}{3t}\delta^\mu_0 = -2(\bar{g}^{\mu\nu} - \bar{h}^{\mu\nu})\bar{\nabla}_\nu\Psi = -2\bar{\nabla}^\mu\Psi + 2\bar{\nabla}^\mu\Psi = \frac{2}{t}\left(\bar{g}^{\mu0} + \frac{\Lambda}{3}\delta^\mu_0\right),$$

respectively, which in turn, allows us to express the gauge constraint equations (3.3.4) as

$$\bar{\nabla}_0\hat{\mathfrak{u}}^{\mu0} = -\bar{\nabla}_i\hat{\mathfrak{u}}^{\mu i} + \frac{2}{t}\left(\hat{\mathfrak{u}}^{0\mu} + \frac{\Lambda}{3}\frac{\theta - 1}{\epsilon^2}\delta^\mu_0\right) = -\bar{\partial}_i\hat{\mathfrak{u}}^{\mu i} - \bar{\gamma}^\mu_{i\lambda}\hat{\mathfrak{u}}^{\lambda i} - \bar{\gamma}^i_{i\lambda}\hat{\mathfrak{u}}^{\lambda\mu} + \frac{2}{t}\left(\hat{\mathfrak{u}}^{0\mu} + \frac{\Lambda}{3}\frac{\theta - 1}{\epsilon^2}\delta^\mu_0\right). \tag{3.3.13}$$

Using (3.3.7), it not difficult to verify that (3.3.13) is equivalent to the pair of equations

$$\partial_0\underline{\hat{\mathfrak{u}}^{00}} = -\frac{1}{\epsilon}\partial_i\underline{\hat{\mathfrak{u}}^{0i}} + \frac{1}{t}(1 - 3\Omega)\underline{\hat{\mathfrak{u}}^{00}} + \frac{\Lambda}{3t}E^2(1 - \Omega)\delta_{ij}\underline{\hat{\mathfrak{u}}^{ji}} + \epsilon^2\breve{\mathscr{S}}(\epsilon, t, E, \underline{\hat{\mathfrak{u}}^{\mu\nu}}), \tag{3.3.14}$$

$$\partial_0\underline{\hat{\mathfrak{u}}^{k0}} = -\frac{1}{\epsilon}\partial_i\underline{\hat{\mathfrak{u}}^{ki}} + \frac{1}{t}(2 - 5\Omega)\underline{\hat{\mathfrak{u}}^{0k}}, \tag{3.3.15}$$

where $\breve{\mathscr{S}}$ vanishes to second order in $\underline{\hat{\mathfrak{u}}^{\mu\nu}}$.

The importance of the equations (3.3.14)-(3.3.15) is that they allow us to determine the time derivatives $\partial_0\underline{\hat{\mathfrak{u}}^{\mu 0}}$ from metric variables $\underline{\hat{\mathfrak{u}}^{\mu\nu}}$ and their spatial derivatives on the initial hypersurface Σ. As an application, we see after taking the time derivative of (3.3.6) and then using (3.3.14) to replace $\partial_0\underline{\hat{\mathfrak{u}}^{00}}$ with the right hand side of (3.3.14) that

$$\partial_t\left(\frac{\theta-1}{\epsilon^2}\right)=\frac{3}{2\Lambda}\frac{1}{\epsilon}\partial_i\underline{\hat{\mathfrak{u}}^{0i}}-\frac{3}{2\Lambda t}(1-3\Omega)\underline{\hat{\mathfrak{u}}^{00}}+\breve{\mathscr{L}}(\epsilon,t,E,\Omega/t,\underline{\hat{\mathfrak{u}}^{kl}},\underline{\hat{\mathfrak{u}}^{ij}_0})+\epsilon\breve{\mathscr{B}}(\epsilon,t,E,\Omega/t,\underline{\hat{\mathfrak{u}}^{\mu\nu}},D\underline{\hat{\mathfrak{u}}^{\lambda\sigma}})$$
$$+\epsilon^2\breve{\mathscr{R}}(\epsilon,t,E,\Omega/t,\underline{\hat{\mathfrak{u}}^{\mu\nu}},\underline{\hat{\mathfrak{u}}^{ij}_0})+\epsilon^2\breve{\mathscr{S}}(\epsilon,t,E,\underline{\hat{\mathfrak{u}}^{\mu\nu}}),$$

where $\breve{\mathscr{L}}$ is linear in $(\underline{\hat{\mathfrak{u}}^{kl}},\underline{\hat{\mathfrak{u}}^{ij}_0})$, $\breve{\mathscr{S}}$ vanishes to second order in $\underline{\hat{\mathfrak{u}}^{\mu\nu}}$, and $\breve{\mathscr{R}}$ and $\breve{\mathscr{B}}$ both vanish to first order in $\underline{\hat{\mathfrak{u}}^{\mu\nu}}$ and are linear in $\underline{\hat{\mathfrak{u}}^{ij}_0}$ and $D\underline{\hat{\mathfrak{u}}^{\lambda\sigma}}$, respectively. Furthermore, differentiating (3.3.14)-(3.3.15) with respect to t, we find, after using (3.3.14)-(3.3.15) to replace the time derivatives $\partial_0\underline{\hat{\mathfrak{u}}^{\mu 0}}$, that the second time derivatives $\partial_0^2\underline{\hat{\mathfrak{u}}^{\mu 0}}$ can be, on the initial hypersurface, expressed as

$$\partial_0^2\underline{\hat{\mathfrak{u}}^{00}}=\frac{1}{\epsilon^2}\partial_i\partial_j\underline{\hat{\mathfrak{u}}^{ij}}+\frac{1}{\epsilon}\frac{1}{t}(8\Omega-3)\partial_i\underline{\hat{\mathfrak{u}}^{0i}}+\frac{1}{t^2}(9\Omega^2-6\Omega-3t\partial_t\Omega+1)\underline{\hat{\mathfrak{u}}^{00}}+\breve{\mathscr{L}}(\epsilon,t,E,\Omega/t,\underline{\hat{\mathfrak{u}}^{kl}},\underline{\hat{\mathfrak{u}}^{kl}_0})$$
$$+\epsilon\breve{\mathscr{B}}(\epsilon,t,E,\Omega/t,\underline{\hat{\mathfrak{u}}^{\mu\nu}},D\underline{\hat{\mathfrak{u}}^{\lambda\sigma}})+\epsilon^2\breve{\mathscr{R}}(\epsilon,t,E,\Omega/t,\underline{\hat{\mathfrak{u}}^{\mu\nu}},\underline{\hat{\mathfrak{u}}^{kl}_0})+\epsilon^2\breve{\mathscr{S}}(\epsilon,t,E,\underline{\hat{\mathfrak{u}}^{\mu\nu}}), \tag{3.3.16}$$

$$\partial_0^2\underline{\hat{\mathfrak{u}}^{j0}}=-\frac{1}{\epsilon}\partial_i\underline{\hat{\mathfrak{u}}^{ij}_0}-\frac{1}{\epsilon}\frac{1}{t}(2-5\Omega)\partial_i\underline{\hat{\mathfrak{u}}^{ij}}+\frac{1}{t^2}(25\Omega^2-15\Omega-5t\partial_t\Omega+2)\underline{\hat{\mathfrak{u}}^{0j}}, \tag{3.3.17}$$

where $\breve{\mathscr{L}}$, $\breve{\mathscr{B}}$, $\breve{\mathscr{R}}$ and $\breve{\mathscr{S}}$ are defined as above.

With the reformulation of the gauge constraints complete, we turn our attention to the gravitational constraint equations (3.3.3). We begin the reformulation process by observing the Ricci scalar $\bar{R}$ is given by

$$\bar{R}=\bar{g}_{\mu\nu}\bar{R}^{\mu\nu}\overset{(3.1.32)}{=}\frac{1}{2}\bar{g}_{\mu\nu}\bar{g}^{\alpha\beta}\bar{\nabla}_\alpha\bar{\nabla}_\beta\bar{g}^{\mu\nu}+\bar{\nabla}_\lambda\bar{X}^\lambda+\bar{g}_{\mu\nu}\bar{\mathcal{R}}^{\mu\nu}+\bar{g}_{\mu\nu}\bar{P}^{\mu\nu}+\bar{g}_{\mu\nu}\bar{Q}^{\mu\nu}. \tag{3.3.18}$$

Using (3.1.32), (3.3.12) and (3.3.18) in conjunction with the identities

$$\bar{\nabla}_\lambda\bar{g}^{\alpha\beta}=\frac{1}{\theta}\bar{\nabla}_\lambda\hat{g}^{\alpha\beta}-\frac{1}{2\theta}\hat{g}^{\alpha\beta}\hat{g}_{\mu\sigma}\bar{\nabla}_\lambda\hat{g}^{\mu\sigma}, \tag{3.3.19}$$

and

$$\bar{\nabla}_\lambda\bar{g}_{\alpha\beta}=\theta\bar{\nabla}_\lambda\hat{g}_{\alpha\beta}+\frac{1}{2}\theta\hat{g}_{\alpha\beta}\hat{g}_{\mu\sigma}\bar{\nabla}_\lambda\hat{g}^{\mu\sigma}=-\theta\hat{g}_{\alpha\mu}\hat{g}_{\beta\nu}\bar{\nabla}_\lambda\hat{g}^{\mu\nu}+\frac{1}{2}\theta\hat{g}_{\alpha\beta}\hat{g}_{\mu\sigma}\bar{\nabla}_\lambda\hat{g}^{\mu\sigma}, \tag{3.3.20}$$

which follow from (3.3.12) and relation $-\bar{\nabla}_\lambda\hat{g}_{\alpha\beta}=\hat{g}_{\alpha\mu}\hat{g}_{\beta\nu}\bar{\nabla}_\lambda\hat{g}^{\mu\nu}$, we see that the Einstein tensor is given by

$$\bar{G}^{\mu\nu}=\frac{1}{2\theta^2}\hat{g}^{\alpha\beta}\bar{\nabla}_\alpha\bar{\nabla}_\beta\hat{g}^{\mu\nu}+\bar{\nabla}^{(\mu}\bar{X}^{\nu)}-\frac{1}{2\theta}\hat{g}^{\mu\nu}\bar{\nabla}_\lambda\bar{X}^\lambda+\bar{\mathcal{R}}^{\mu\nu}-\frac{1}{2}\bar{h}^{\mu\nu}\bar{\mathcal{R}}+\tilde{\mathcal{P}}^{\mu\nu}+\tilde{\mathcal{Q}}^{\mu\nu}-\frac{1}{2}\bar{X}^\mu\bar{X}^\nu \tag{3.3.21}$$

where

$$\tilde{\mathcal{P}}^{\mu\nu}=\frac{1}{2}\bar{h}^{\mu\nu}\bar{h}_{\alpha\beta}\bar{\mathcal{R}}^{\alpha\beta}-\frac{1}{2}\hat{g}^{\mu\nu}\hat{g}_{\alpha\beta}\bar{\mathcal{R}}^{\alpha\beta}+\bar{P}^{\mu\nu}-\frac{1}{2}\hat{g}^{\mu\nu}\hat{g}_{\alpha\beta}\bar{P}^{\alpha\beta}, \tag{3.3.22}$$

$$\tilde{\mathcal{Q}}^{\mu\nu}=\frac{1}{8}\bar{g}^{\alpha\beta}\bar{g}_{\lambda\sigma}\bar{g}^{\mu\nu}\bar{g}_{\gamma\delta}\bar{\nabla}_\beta\bar{g}^{\lambda\sigma}\bar{\nabla}_\alpha\bar{g}^{\gamma\delta}-\frac{1}{4}\bar{g}^{\alpha\beta}\bar{g}^{\mu\nu}\bar{\nabla}_\alpha\bar{g}_{\lambda\sigma}\bar{\nabla}_\beta\bar{g}^{\lambda\sigma}+\bar{Q}^{\mu\nu}-\frac{1}{2}\hat{g}^{\mu\nu}\hat{g}_{\alpha\beta}\bar{Q}^{\alpha\beta}+\frac{1}{2}\bar{X}^\mu\bar{X}^\nu, \tag{3.3.23}$$

and $\bar{P}^{\mu\nu}$ and $\bar{Q}^{\mu\nu}$ are defined previously by (3.1.34) and (3.1.35), respectively.

To proceed, we use (3.3.19) and (3.3.20) to express $\bar{\nabla}_\lambda\bar{g}^{\alpha\beta}$ and $\bar{\nabla}_\lambda\bar{g}_{\alpha\beta}$ in (3.3.23) in terms of $\bar{\nabla}_\lambda\hat{g}^{\mu\nu}$ followed by replacing $\hat{g}^{\mu\nu}$ with $\hat{\mathfrak{u}}^{\mu\nu}$ using (3.3.1). This allows us to write $\tilde{\mathcal{Q}}^{\mu\nu}$ as

$$\tilde{\mathcal{Q}}^{\mu\nu}=\epsilon^2\breve{\mathscr{W}}^{\mu\nu}(\epsilon,t,E,\Omega/t,\hat{\mathfrak{u}}^{\lambda\sigma},D\hat{\mathfrak{u}}^{\alpha\beta},\hat{\mathfrak{u}}^{ij}_0)$$

where

$$\begin{aligned}\breve{\mathscr{W}}^{\mu\nu}(\epsilon,t,E,\Omega/t,\hat{\mathfrak{u}}^{\mu\nu},D\hat{\mathfrak{u}}^{\alpha\beta},\hat{\mathfrak{u}}^{ij}_0) =&\epsilon^2\breve{\mathscr{S}}^{\mu\nu}(\epsilon,t,E,\Omega/t,\hat{\mathfrak{u}}^{\alpha\beta})+\epsilon\breve{\mathscr{R}}^{\mu\nu}(\epsilon,t,E,\Omega/t,\epsilon\hat{\mathfrak{u}}^{\lambda\sigma},\epsilon\hat{\mathfrak{u}}^{ij}_0,D\hat{\mathfrak{u}}^{\alpha\beta},\hat{\mathfrak{u}}^{ij}_0)\\ &+\breve{\mathscr{Q}}^{\mu\nu}(\epsilon,t,E,\Omega/t,\hat{\mathfrak{u}}^{\lambda\sigma},D\hat{\mathfrak{u}}^{\alpha\beta})+\epsilon\breve{\mathscr{B}}^{\mu\nu}(\epsilon,t,E,\Omega/t,\hat{\mathfrak{u}}^{\alpha\beta},D\hat{\mathfrak{u}}^{\lambda\sigma}),\end{aligned} \tag{3.3.24}$$

and in this expression, $\breve{\mathscr{S}}^{\mu\nu}$ vanishes to second order in $\hat{\mathfrak{u}}^{\alpha\beta}$, $\breve{\mathscr{Q}}^{\mu\nu}$ vanishes to second order in $D\hat{\mathfrak{u}}^{\lambda\sigma}$, $\breve{\mathscr{R}}^{\mu\nu}$vanishes to first order in $(\epsilon\hat{\mathfrak{u}}^{\lambda\sigma},\epsilon\hat{\mathfrak{u}}^{ij}_0,D\hat{\mathfrak{u}}^{\mu\nu})$ and is linear in $\hat{\mathfrak{u}}^{ij}_0$, and $\breve{\mathscr{B}}^{\mu\nu}$ vanishes to first order in $\hat{\mathfrak{u}}^{\alpha\beta}$ and is linear in $D\hat{\mathfrak{u}}^{\lambda\sigma}$.

Remark 3.3.3. For the remainder of this section, we will use the following notation unless otherwise stated: $\breve{\mathscr{L}}(\epsilon,t,E,\Omega/t,\hat{\mathfrak{u}}^{kl})$ and $\breve{\mathscr{L}}^j(\epsilon,t,E,\Omega/t,\hat{\mathfrak{u}}^{kl})$ will denote remainder terms that are linear in $\hat{\mathfrak{u}}^{kl}$, while $\breve{\mathscr{S}}(\epsilon,t,E,\Omega/t,\hat{\mathfrak{u}}^{\alpha\beta})$, $\breve{\mathscr{S}}^j(\epsilon,t,E,\Omega/t,\hat{\mathfrak{u}}^{\alpha\beta})$, $\breve{\mathscr{S}}^{\mu\nu}(\epsilon,t,E,\Omega/t,\hat{\mathfrak{u}}^{\alpha\beta})$ and $\breve{\mathscr{S}}_{\alpha\beta}(\epsilon,t,E,\Omega/t,\hat{\mathfrak{u}}^{\alpha\beta})$ will denote remainder terms that vanish to second order in $\hat{\mathfrak{u}}^{\alpha\beta}$.

Next, we express $\bar{P}^{\mu\nu}$, see (3.1.34), in terms of $\hat{\mathfrak{u}}^{\mu\nu}$ by using the expansion

$$\begin{aligned}\bar{g}^{\mu\lambda}-\bar{h}^{\mu\lambda}=&\hat{g}^{\mu\lambda}-\bar{h}^{\mu\lambda}+\bar{h}^{\mu\lambda}\left(\frac{1}{\theta}-1\right)+(\hat{g}^{\mu\lambda}-\bar{h}^{\mu\lambda})\left(\frac{1}{\theta}-1\right)\\ =&\left[\epsilon^2\hat{\mathfrak{u}}^{\mu\lambda}+\epsilon^2\frac{3}{2\Lambda}\bar{h}^{\mu\lambda}\hat{\mathfrak{u}}^{00}\right]+\left[\bar{h}^{\mu\lambda}\left(\frac{1}{\theta}-1-\frac{3}{2\Lambda}\epsilon^2\hat{\mathfrak{u}}^{00}\right)+(\hat{g}^{\mu\lambda}-\bar{h}^{\mu\lambda})\left(\frac{1}{\theta}-1\right)\right],\end{aligned}$$

together with (3.3.6) to get

$$\begin{aligned}\bar{P}^{\mu\nu}=&-\frac{1}{2}\left[\epsilon^2\hat{\mathfrak{u}}^{\mu\lambda}+\epsilon^2\frac{3}{2\Lambda}\bar{h}^{\mu\lambda}\hat{\mathfrak{u}}^{00}\right]\bar{h}^{\alpha\beta}\bar{\mathcal{R}}_{\lambda\alpha\beta}{}^{\nu}-\frac{1}{2}\left[\epsilon^2\hat{\mathfrak{u}}^{\alpha\beta}+\epsilon^2\frac{3}{2\Lambda}\bar{h}^{\alpha\beta}\hat{\mathfrak{u}}^{00}\right]\bar{h}^{\mu\lambda}\bar{\mathcal{R}}_{\lambda\alpha\beta}{}^{\nu}\\ &-\frac{1}{2}\left[\epsilon^2\hat{\mathfrak{u}}^{\nu\lambda}+\epsilon^2\frac{3}{2\Lambda}\bar{h}^{\nu\lambda}\hat{\mathfrak{u}}^{00}\right]\bar{h}^{\alpha\beta}\bar{\mathcal{R}}_{\lambda\alpha\beta}{}^{\mu}-\frac{1}{2}\left[\epsilon^2\hat{\mathfrak{u}}^{\alpha\beta}+\epsilon^2\frac{3}{2\Lambda}\bar{h}^{\alpha\beta}\hat{\mathfrak{u}}^{00}\right]h^{\nu\lambda}\bar{\mathcal{R}}_{\lambda\alpha\beta}{}^{\mu}\\ &+\epsilon^2\breve{\mathscr{L}}^{\mu\nu}(\epsilon,t,E,\Omega/t,\hat{\mathfrak{u}}^{kl})+\epsilon^4\breve{\mathscr{S}}^{\mu\nu}(\epsilon,t,E,\Omega/t,\hat{\mathfrak{u}}^{\alpha\beta}),\end{aligned}$$

which, with the help of (3.1.26)-(3.1.27), we can write as

$$\bar{P}^{00}=\epsilon^2\breve{\mathscr{L}}(\epsilon,t,E,\Omega/t,\hat{\mathfrak{u}}^{ij})+\epsilon^4\breve{\mathscr{S}}(\epsilon,t,E,\Omega/t,\hat{\mathfrak{u}}^{\mu\nu}), \tag{3.3.25}$$

$$\bar{P}^{j0}=-\epsilon^2\frac{\Lambda}{3t^2}(\Omega-2\Omega^2-t\partial_t\Omega)\hat{\mathfrak{u}}^{j0}+\epsilon^2\breve{\mathscr{L}}^j(\epsilon,t,E,\Omega/t,\hat{\mathfrak{u}}^{kl})+\epsilon^4\breve{\mathscr{S}}^j(\epsilon,t,E,\Omega/t,\hat{\mathfrak{u}}^{\mu\nu}) \tag{3.3.26}$$

and

$$\bar{P}^{ij}=\epsilon^2E^{-2}\frac{2}{t^2}\Omega^2\delta^{ij}\hat{\mathfrak{u}}^{00}+\epsilon^2\breve{\mathscr{L}}^{ij}(\epsilon,t,E,\Omega/t,\hat{\mathfrak{u}}^{kl})+\epsilon^4\breve{\mathscr{S}}^{ij}(\epsilon,t,E,\Omega/t,\hat{\mathfrak{u}}^{\mu\nu}). \tag{3.3.27}$$

By Lemma D.1.2 in Appendix D.1, we see that

$$\hat{g}_{\alpha\beta}-\bar{h}_{\alpha\beta}=-\epsilon^2\bar{h}_{\alpha\lambda}\hat{\mathfrak{u}}^{\lambda\sigma}\bar{h}_{\sigma\beta}+\epsilon^4\breve{\mathscr{S}}_{\alpha\beta}(\epsilon,t,E,\Omega/t,\hat{\mathfrak{u}}^{\mu\nu}),$$

which together with (3.3.1) gives

$$\begin{aligned}-\frac{1}{2}\hat{g}^{\mu0}\hat{g}_{\alpha\beta}\bar{P}^{\alpha\beta}=&-\frac{1}{2}(\hat{g}^{\mu0}-\bar{h}^{\mu0})(\hat{g}_{\alpha\beta}-\bar{h}_{\alpha\beta})\bar{P}^{\alpha\beta}-\frac{1}{2}(\hat{g}^{\mu0}-\bar{h}^{\mu0})\bar{h}_{\alpha\beta}\bar{P}^{\alpha\beta}\\ &-\frac{1}{2}\bar{h}^{\mu0}(\hat{g}_{\alpha\beta}-\bar{h}_{\alpha\beta})\bar{P}^{\alpha\beta}-\frac{1}{2}\bar{h}^{\mu0}\bar{h}_{\alpha\beta}\bar{P}^{\alpha\beta}\\ =&-\frac{1}{2}\bar{h}^{\mu0}\bar{h}_{\alpha\beta}\bar{P}^{\alpha\beta}+\epsilon^4\breve{\mathscr{S}}^{0\mu}(\epsilon,t,E,\Omega/t,\hat{\mathfrak{u}}^{\mu\nu}).\end{aligned} \tag{3.3.28}$$

Further, we observe that

$$\bar{h}_{\alpha\beta}\bar{P}^{\alpha\beta}=\bar{h}_{00}\bar{P}^{00}+\bar{h}_{ij}\bar{P}^{ij}=\epsilon^2\frac{6}{t^2}\Omega\hat{\mathfrak{u}}^{00}+\epsilon^2\breve{\mathscr{L}}(\epsilon,t,E,\Omega/t,\hat{\mathfrak{u}}^{ij})+\epsilon^4\breve{\mathscr{S}}(\epsilon,t,E,\Omega/t,\hat{\mathfrak{u}}^{\mu\nu}) \tag{3.3.29}$$

by (3.3.25), (3.3.27). Then recalling the definition (3.3.22) of $\tilde{\mathcal{P}}^{\mu\nu}$, we can, with the help of (3.3.28), write $\tilde{\mathcal{P}}^{\mu 0}$ as

$$\begin{aligned}\tilde{\mathcal{P}}^{\mu 0} =& -\frac{1}{2}\bar{\mathcal{R}}^{\alpha\beta}[(\hat{g}^{\mu 0}-\bar{h}^{\mu 0})\bar{h}_{\alpha\beta}+(\hat{g}^{\mu 0}-\bar{h}^{\mu 0})(\hat{g}_{\alpha\beta}-\bar{h}_{\alpha\beta})+\bar{h}^{\mu 0}(\hat{g}_{\alpha\beta}-\bar{h}_{\alpha\beta})]+\bar{P}^{\mu 0}-\frac{1}{2}\hat{g}^{\mu 0}\hat{g}_{\alpha\beta}\bar{P}^{\alpha\beta}\\ =& -\frac{1}{2}\bar{\mathcal{R}}^{\alpha\beta}[\epsilon^2\hat{\mathfrak{u}}^{\mu 0}\bar{h}_{\alpha\beta}-\epsilon^2\bar{h}^{\mu 0}\bar{h}_{\alpha\lambda}\hat{\mathfrak{u}}^{\lambda\sigma}\bar{h}_{\sigma\beta}]+\bar{P}^{\mu 0}-\frac{1}{2}\bar{h}^{\mu 0}\bar{h}_{\alpha\beta}\bar{P}^{\alpha\beta}+\epsilon^4\mathscr{S}^{\mu 0}(\epsilon,t,E,\Omega/t,\hat{\mathfrak{u}}^{\mu\nu})\end{aligned}$$

from which we see, by (3.3.25) and (3.3.29), that the $\mu=0$ and $\mu=j$ components of $\mathcal{P}^{\mu 0}$ can be expressed as

$$\tilde{\mathcal{P}}^{00} =\epsilon^2\frac{\Lambda}{2t^2}(3\Omega-3\Omega^2-t\partial_t\Omega)\hat{\mathfrak{u}}^{00}+\epsilon^2\breve{\mathscr{L}}(\epsilon,t,E,\Omega/t,\hat{\mathfrak{u}}^{ij})+\epsilon^4\breve{\mathscr{S}}(\epsilon,t,E,\Omega/t,\hat{\mathfrak{u}}^{\mu\nu}) \tag{3.3.30}$$

and

$$\tilde{\mathcal{P}}^{j0} =\epsilon^2\frac{2\Lambda}{3t^2}(\Omega-2\Omega^2-t\partial_t\Omega)\hat{\mathfrak{u}}^{j0}+\epsilon^2\breve{\mathscr{L}}^j(\epsilon,t,E,\Omega/t,\hat{\mathfrak{u}}^{kl})+\epsilon^4\breve{\mathscr{S}}^j(\epsilon,t,E,\Omega/t,\hat{\mathfrak{u}}^{\alpha\beta}), \tag{3.3.31}$$

respectively.

On the initial hypersurface Σ, we know from the above arguments that we can satisfy the constraint equations $\bar{Z}^\mu=0$ by choosing $\partial_0\underline{\hat{\mathfrak{u}}^{j0}}$ according to (3.3.14) and (3.3.15). Doing so, we find using (3.3.21) that we can write the conformal Einstein equations (3.1.17) as

$$\begin{aligned}\frac{1}{2\theta^2}\hat{g}^{\alpha\beta}\bar{\nabla}_\alpha\bar{\nabla}_\beta\hat{g}^{\mu\nu}-\bar{\nabla}^{(\mu}\bar{Y}^{\nu)}+\frac{1}{2\theta}\hat{g}^{\mu\nu}\bar{\nabla}_\lambda\bar{Y}^\lambda+\bar{\mathcal{R}}^{\mu\nu}-\frac{1}{2}\bar{h}^{\mu\nu}\bar{\mathcal{R}}+\tilde{\mathcal{P}}^{\mu\nu}+\tilde{\mathcal{Q}}^{\mu\nu}-\frac{1}{2}\bar{Y}^\mu\bar{Y}^\nu\\ =e^{4\Psi}\tilde{T}^{\mu\nu}-\frac{1}{\theta}e^{2\Psi}\Lambda\hat{g}^{\mu\nu}+2(\bar{\nabla}^\mu\bar{\nabla}^\nu\Psi-\bar{\nabla}^\mu\Psi\bar{\nabla}^\nu\Psi)-\frac{1}{\theta}(2\bar{\Box}\Psi+|\bar{\nabla}\Psi|^2_{\bar{g}})\hat{g}^{\mu\nu}.\end{aligned} \tag{3.3.32}$$

We also note that conformal Einstein equations for the conformal FLRW metric (3.1.20) are given by

$$\begin{aligned}&\bar{\mathcal{R}}^{\mu\nu}-\frac{1}{2}\bar{\mathcal{R}}\bar{h}^{\mu\nu}=e^{4\Psi}\bar{\mathcal{T}}^{\mu\nu}-e^{2\Psi}\Lambda\bar{h}^{\mu\nu}+2(\bar{\nabla}^\mu\bar{\nabla}^\nu\Psi-\bar{\nabla}^\mu\Psi\bar{\nabla}^\nu\Psi)-(2\bar{\boxminus}\Psi+|\bar{\nabla}\Psi|^2_{\bar{h}})\bar{h}^{\mu\nu}\\ =&(1+\epsilon^2K)\mu\frac{\Lambda}{3}\delta^\mu_0\delta^\nu_0e^{2\Psi}+\epsilon^2K\mu\bar{h}^{\mu\nu}e^{2\Psi}-e^{2\Psi}\Lambda\bar{h}^{\mu\nu}+2(\bar{\nabla}^\mu\bar{\nabla}^\nu\Psi-\bar{\nabla}^\mu\Psi\bar{\nabla}^\nu\Psi)-(2\bar{\boxminus}\Psi+|\bar{\nabla}\Psi|^2_{\bar{h}})\bar{h}^{\mu\nu}.\end{aligned} \tag{3.3.33}$$

In order to expand (3.3.32) further, we list some key calculations below. First, with the help of (3.1.33), (3.1.40) and Proposition 3.2.2, we see from a direct calculation that

$$\bar{\nabla}_\lambda\bar{Y}^\lambda=-2\bar{\Box}\Psi+2\bar{\boxminus}\Psi+2\bar{X}^\lambda{}_{\lambda\sigma}\bar{\nabla}^\sigma\Psi=\frac{2}{t^2}\left(\bar{g}^{00}+\frac{\Lambda}{3}\right)+\frac{2\Lambda}{3t}\bar{X}^\lambda{}_{\lambda 0}, \tag{3.3.34}$$

where, using (3.3.12), we note $\bar{X}^\lambda{}_{\lambda 0}$ can be expressed as

$$\bar{X}^\lambda{}_{\lambda 0}=-\frac{1}{2}\left(\bar{g}_{\sigma 0}\bar{\nabla}_\lambda\bar{g}^{\lambda\sigma}+\bar{g}_{\lambda\sigma}\bar{\nabla}_0\bar{g}^{\lambda\sigma}-\bar{g}^{\lambda\sigma}\bar{g}_{\lambda\delta}\bar{g}_{0\gamma}\bar{\nabla}_\sigma\bar{g}^{\delta\gamma}\right)=\frac{1}{2}\hat{g}_{\lambda\sigma}\bar{\nabla}_0\hat{g}^{\lambda\sigma}=\frac{1}{\theta}\bar{\nabla}_0\theta.$$

Using Proposition 3.2.2 again, we see that

$$2\bar{\Box}\Psi+|\bar{\nabla}\Psi|^2_{\bar{g}}=\frac{3}{t^2}\bar{g}^{00}-\frac{4}{t^2}\left(\bar{g}^{00}+\frac{\Lambda}{3}\right)+\frac{2\Lambda\Omega}{t^2}\quad\text{and}\quad 2\bar{\boxminus}\Psi+|\bar{\nabla}\Psi|^2_{\bar{h}}=-\frac{\Lambda}{t^2}+\frac{2\Lambda\Omega}{t^2}, \tag{3.3.35}$$

which together can be used to show that

$$\begin{aligned}-2\theta(2\bar{\Box}\Psi+|\bar{\nabla}\Psi|^2_{\bar{g}})\hat{g}^{\mu\nu}+2\theta^2(2\bar{\boxminus}\Psi+|\bar{\nabla}\Psi|^2_{\bar{h}})\bar{h}^{\mu\nu}=\frac{2\Lambda}{t^2}(\hat{g}^{\mu\nu}-\bar{h}^{\mu\nu})+\frac{2\Lambda}{t^2}(1-\theta^2)\bar{h}^{\mu\nu}\\ +\frac{2}{t^2}\left(\hat{g}^{00}+\frac{\Lambda}{3}\right)\hat{g}^{\mu\nu}-\frac{8\Lambda}{3t^2}(1-\theta)\hat{g}^{\mu\nu}-\frac{4\Lambda\Omega}{t^2}\theta(\hat{g}^{\mu\nu}-\bar{h}^{\mu\nu})-\frac{4\Lambda\Omega}{t^2}\theta(1-\theta)\bar{h}^{\mu\nu}.\end{aligned} \tag{3.3.36}$$

Furthermore, by direct calculation, it is not difficult to verify that

$$\bar{\rho}\bar{g}^{\mu\nu}-\mu\bar{h}^{\mu\nu}=(\bar{\rho}-\mu)\frac{1}{\theta}\hat{g}^{\mu\nu}+\mu\bar{h}^{\mu\nu}\left(\frac{1}{\theta}-1\right)(1-\theta)+\mu\frac{1}{\theta}(1-\theta)(\hat{g}^{\mu\nu}-\bar{h}^{\mu\nu}) \\ +\mu(\hat{g}^{\mu\nu}-\bar{h}^{\mu\nu})+\mu(1-\theta)\bar{h}^{\mu\nu}. \tag{3.3.37}$$

Inserting (3.3.33)–(3.3.37) into (3.3.32) yields the following representation of conformal Einstein equations:

$$\hat{g}^{\alpha\beta}\bar{\nabla}_\alpha\bar{\nabla}_\beta(\hat{g}^{\mu\nu}-\bar{h}^{\mu\nu})+\theta\hat{g}^{\mu\nu}\frac{2}{t^2}\left(\bar{g}^{00}+\frac{\Lambda}{3}\right)+2\theta^2\tilde{\mathcal{P}}^{\mu\nu}+2\theta^2\tilde{\mathcal{Q}}^{\mu\nu} \\ =-\theta\frac{2\Lambda}{3t}\delta^\sigma_0\bar{\nabla}_\sigma(\hat{g}^{\mu\nu}-\bar{h}^{\mu\nu})+\theta\frac{4\Lambda}{3t^2}\left(\hat{g}^{0\lambda}+\frac{\Lambda}{3}\delta^\lambda_0\right)\delta^{(\mu}_\lambda\delta^{\nu)}_0+\theta(\theta-1)\frac{4\Lambda^2}{9t^2}\delta^\lambda_0\delta^{(\mu}_\lambda\delta^{\nu)}_0 \\ -\theta\frac{4\Lambda}{3t^2}\Omega(\theta\bar{h}^{ij}\delta^\mu_j\delta^\nu_i-\hat{g}^{ij}\delta^{(\mu}_j\delta^{\nu)}_i)+\theta\frac{4\Lambda}{3t^2}\Omega\hat{g}^{i0}\delta^{(\mu}_0\delta^{\nu)}_i+2\theta^2(1+\epsilon^2K)\frac{1}{t^2}\left[(\bar{\rho}-\mu)\bar{v}^\mu\bar{v}^\nu+\mu(\bar{v}^\mu\bar{v}^\nu-\frac{\Lambda}{3}\delta^\mu_0\delta^\nu_0)\right] \\ +2\theta^2\epsilon^2K\frac{1}{t^2}\left((\bar{\rho}-\mu)\frac{1}{\theta}\hat{g}^{\mu\nu}+\mu\bar{h}^{\mu\nu}(\frac{1}{\theta}-1)(1-\theta)+\mu\frac{1}{\theta}(1-\theta)(\hat{g}^{\mu\nu}-\bar{h}^{\mu\nu})+\mu(\hat{g}^{\mu\nu}-\bar{h}^{\mu\nu})+\mu(1-\theta)\bar{h}^{\mu\nu}\right) \\ -\frac{2\Lambda}{t^2}(\theta-1)(\hat{g}^{\mu\nu}-\bar{h}^{\mu\nu})-\frac{2\Lambda}{t^2}(1-\theta)\bar{h}^{\mu\nu}+\frac{2}{t^2}\left(\hat{g}^{00}+\frac{\Lambda}{3}\right)\hat{g}^{\mu\nu}-\frac{8\Lambda}{3t^2}(1-\theta)\hat{g}^{\mu\nu} \\ -\frac{4\Lambda\Omega}{t^2}\theta(\hat{g}^{\mu\nu}-\bar{h}^{\mu\nu})-\frac{4\Lambda\Omega}{t^2}\theta(1-\theta)\bar{h}^{\mu\nu}. \tag{3.3.38}$$

Next, with the help of (3.3.15)-(3.3.17), we observe that the $(\mu,\nu)=(0,0)$ and $(\mu,\nu)=(j,0)$ components of the principal term $\hat{g}^{\alpha\beta}\bar{\nabla}_\alpha\bar{\nabla}_\beta(\hat{g}^{\mu\nu}-\bar{h}^{\mu\nu})$ of (3.3.38) can be expressed as

$$\delta^0_\mu\delta^0_\nu\hat{g}^{\alpha\beta}\bar{\nabla}_\alpha\bar{\nabla}_\beta(\hat{g}^{\mu\nu}-\bar{h}^{\mu\nu})=\epsilon^2\delta^0_\mu\delta^0_\nu\hat{g}^{\alpha\beta}\bar{\nabla}_\alpha\bar{\nabla}_\beta\hat{\mathfrak{u}}^{\mu\nu} \\ =E^{-2}\Delta\hat{\mathfrak{u}}^{00}+\epsilon^2\hat{\mathfrak{u}}^{ij}\partial_i\partial_j\hat{\mathfrak{u}}^{00}+\epsilon^2\hat{\mathfrak{u}}^{00}\partial_i\partial_j\hat{\mathfrak{u}}^{ij}-2\epsilon^2\hat{\mathfrak{u}}^{0i}\partial_i\partial_j\hat{\mathfrak{u}}^{0j}-\frac{\Lambda}{3}\partial_i\partial_j\hat{\mathfrak{u}}^{ij}+\epsilon\frac{\Lambda}{3t}(3-\Omega)\partial_i\hat{\mathfrak{u}}^{0i} \\ +\epsilon^2\frac{\Lambda}{3t^2}(6\Omega^2+3\Omega+3t\partial_t\Omega-1)\hat{\mathfrak{u}}^{00}+\epsilon^2\breve{\mathscr{L}}(\epsilon,t,E,\Omega/t,t\partial_t\Omega,\hat{\mathfrak{u}}^{kl},\hat{\mathfrak{u}}^{kl}_0)+\epsilon^4\breve{\mathscr{S}}(\epsilon,t,E,\Omega/t,t\partial_t\Omega,\hat{\mathfrak{u}}^{\alpha\beta}) \\ +\epsilon^3\breve{\mathscr{R}}(\epsilon,t,E,\Omega/t,t\partial_t\Omega,\epsilon\hat{\mathfrak{u}}^{\alpha\beta},\hat{\mathfrak{u}}^{kl}_0)+\epsilon^3\breve{\mathscr{B}}(\epsilon,t,E,\Omega/t,t\partial_t\Omega,\hat{\mathfrak{u}}^{\alpha\beta},D\hat{\mathfrak{u}}^{\lambda\sigma}) \tag{3.3.39}$$

and

$$\delta^j_\mu\delta^0_\nu\hat{g}^{\alpha\beta}\bar{\nabla}_\alpha\bar{\nabla}_\beta(\hat{g}^{\mu\nu}-\bar{h}^{\mu\nu})=\epsilon^2\delta^j_\mu\delta^0_\nu\hat{g}^{\alpha\beta}\bar{\nabla}_\alpha\bar{\nabla}_\beta\hat{\mathfrak{u}}^{\mu\nu} \\ =E^{-2}\Delta\hat{\mathfrak{u}}^{j0}+\epsilon\frac{\Lambda}{3}\partial_i\hat{\mathfrak{u}}^{ij}_0+\epsilon^2\hat{\mathfrak{u}}^{kl}\partial_k\partial_l\hat{\mathfrak{u}}^{j0}-\epsilon^3\hat{\mathfrak{u}}^{00}\partial_i\hat{\mathfrak{u}}^{ij}_0-2\epsilon^2\hat{\mathfrak{u}}^{0i}\partial_i\partial_k\hat{\mathfrak{u}}^{jk}+2\epsilon E^{-2}\delta^{kl}\frac{\Omega}{t}\partial_k\hat{\mathfrak{u}}^{00} \\ -\epsilon^2\frac{\Lambda}{3t^2}(-2\Omega^2-6\Omega-4t\partial_t\Omega+2)\hat{\mathfrak{u}}^{0j}+\epsilon^2\breve{\mathscr{L}}^j(\epsilon,t,E,\Omega/t,t\partial_t\Omega,\hat{\mathfrak{u}}^{kl},\partial_k\hat{\mathfrak{u}}^{kl})+\epsilon^4\breve{\mathscr{S}}^j(\epsilon,t,E,\Omega/t,t\partial_t\Omega,\hat{\mathfrak{u}}^{\alpha\beta}) \\ +\epsilon^3\breve{\mathscr{R}}^j(\epsilon,t,E,\Omega/t,t\partial_t\Omega,\epsilon\hat{\mathfrak{u}}^{\alpha\beta},\hat{\mathfrak{u}}^{kl}_0)+\epsilon^3\breve{\mathscr{B}}^j(\epsilon,t,E,\Omega/t,t\partial_t\Omega,\hat{\mathfrak{u}}^{\alpha\beta},D\hat{\mathfrak{u}}^{\lambda\sigma}). \tag{3.3.40}$$

respectively, where $\breve{\mathscr{L}}$ and $\breve{\mathscr{L}}^j$ are linear in $(\hat{\mathfrak{u}}^{kl},\hat{\mathfrak{u}}^{kl}_0)$ and $(\hat{\mathfrak{u}}^{kl},\partial_k\hat{\mathfrak{u}}^{kl})$ respectively, $\breve{\mathscr{S}}$ and $\breve{\mathscr{S}}^j$ vanish to second order in $\hat{\mathfrak{u}}^{\mu\nu}$, $\breve{\mathscr{R}}$ and $\breve{\mathscr{R}}^j$ vanish to first order in $\epsilon\hat{\mathfrak{u}}^{\alpha\beta}$ and are linear in $\hat{\mathfrak{u}}^{ij}_0$, and $\breve{\mathscr{B}}$ and $\breve{\mathscr{B}}^j$ vanish to first order in $\hat{\mathfrak{u}}^{\alpha\beta}$ and are linear in $D\hat{\mathfrak{u}}^{\lambda\sigma}$. Furthermore, we observe, using (3.1.25), (3.3.1) and (3.3.14)–(3.3.15) that

$$-\delta^0_\mu\delta^0_\nu\frac{2\Lambda}{3t}\delta^\sigma_0\bar{\nabla}_\sigma(\hat{g}^{\mu\nu}-\bar{h}^{\mu\nu})=\epsilon\frac{2\Lambda}{3t}\partial_i\hat{\mathfrak{u}}^{0i}-\epsilon^2\frac{2\Lambda}{3t^2}(1-3\Omega)\hat{\mathfrak{u}}^{00}+\epsilon^2\breve{\mathscr{L}}(\epsilon,t,E,\Omega/t,\hat{\mathfrak{u}}^{kl})+\epsilon^4\breve{\mathscr{S}}(\epsilon,t,E,\Omega/t,\hat{\mathfrak{u}}^{\mu\nu}) \tag{3.3.41}$$

and

$$-\delta^j_\mu\delta^0_\nu\frac{2\Lambda}{3t}\delta^\sigma_0\bar{\nabla}_\sigma(\hat{g}^{\mu\nu}-\bar{h}^{\mu\nu})=\epsilon\frac{2\Lambda}{3t}\partial_i\hat{\mathfrak{u}}^{ji}-\epsilon^2\frac{2\Lambda}{3t^2}(2-4\Omega)\hat{\mathfrak{u}}^{0j}. \tag{3.3.42}$$

Since the gravitational constraint equations (3.3.3) only involve the $(\mu,\nu)=(0,0)$ and $(\mu,\nu)=(j,0)$ components of the conformal Einstein equations, we separate these out from (3.3.38) to get

$$\delta^0_\mu\delta^0_\nu\hat{g}^{\alpha\beta}\bar{\nabla}_\alpha\bar{\nabla}_\beta(\hat{g}^{\mu\nu}-\bar{h}^{\mu\nu})+2\theta^2\tilde{\mathcal{P}}^{00}+2\theta^2\tilde{\mathcal{Q}}^{00}=-\theta\delta^0_\mu\delta^0_\nu\frac{2\Lambda}{3t}\delta^\sigma_0\bar{\nabla}_\sigma(\hat{g}^{\mu\nu}-\bar{h}^{\mu\nu})+\theta\frac{4\Lambda}{3t^2}\left(\hat{g}^{00}+\frac{\Lambda}{3}\right)$$
$$+2\theta^2(1+\epsilon^2K)\frac{1}{t^2}\left[(\bar{\rho}-\bar{\mu})\bar{v}^0\bar{v}^0+\bar{\mu}(\bar{v}^0\bar{v}^0-\frac{\Lambda}{3})\right]+2\theta\epsilon^2K\frac{1}{t^2}(\bar{\rho}-\bar{\mu})\hat{g}^{00}+2\theta\epsilon^2K\frac{1}{t^2}\mu\bar{h}^{00}(1-\theta)^2$$
$$+2\theta\epsilon^2K\frac{1}{t^2}\mu(1-\theta)(\hat{g}^{00}-\bar{h}^{00})+2\theta^2\epsilon^2K\frac{1}{t^2}\mu(\hat{g}^{00}-\bar{h}^{00})+2\theta^2\epsilon^2K\frac{1}{t^2}\mu(1-\theta)\bar{h}^{00}+\frac{4\Lambda}{t^2}(\theta-1)\bar{h}^{00}$$
$$+\theta(\theta-1)\frac{4\Lambda^2}{9t^2}-\frac{4\Lambda\Omega}{t^2}\theta(\hat{g}^{00}-\bar{h}^{00})-\frac{4\Lambda\Omega}{t^2}\theta(1-\theta)\bar{h}^{00} \tag{3.3.43}$$

and

$$\delta^j_\mu\delta^0_\nu\hat{g}^{\alpha\beta}\bar{\nabla}_\alpha\bar{\nabla}_\beta(\hat{g}^{\mu\nu}-\bar{h}^{\mu\nu})+2\theta^2\tilde{\mathcal{P}}^{j0}+2\theta^2\tilde{\mathcal{Q}}^{j0}=-\theta\delta^j_\mu\delta^0_\nu\frac{2\Lambda}{3t}\delta^\sigma_0\bar{\nabla}_\sigma(\hat{g}^{\mu\nu}-\bar{h}^{\mu\nu})+\theta\frac{2\Lambda}{3t^2}(1-5\Omega)\hat{g}^{0j}$$
$$+2\theta^2(1+\epsilon^2K)\frac{1}{t^2}\left[(\bar{\rho}-\mu)\bar{v}^j\bar{v}^0+\mu\bar{v}^j\bar{v}^0\right]+2\epsilon^2K\frac{1}{t^2}\mu\theta(1-\theta)(\hat{g}^{j0}-\bar{h}^{j0})$$
$$+2\theta\epsilon^2K\frac{1}{t^2}(\bar{\rho}-\mu)\hat{g}^{j0}+2\theta^2\epsilon^2K\frac{1}{t^2}\mu(\hat{g}^{j0}-\bar{h}^{j0}). \tag{3.3.44}$$

To continue, we introduce the following notation for the initial data:

$$\breve{\mathrm{u}}^{ij}(\mathbf{x})=\frac{1}{\epsilon}\underline{\hat{\mathrm{u}}^{ij}}(1,\mathbf{x}),\quad \breve{\mathrm{u}}^{ij}_0(\mathbf{x})=\underline{\hat{\mathrm{u}}^{ij}_0}(1,\mathbf{x}),\quad \breve{\mathrm{u}}^{0\mu}(\mathbf{x})=\underline{\hat{\mathrm{u}}^{0\mu}}(1,\mathbf{x}),\quad \breve{\mathrm{u}}^{0\mu}_0(\mathbf{x})=\underline{\hat{\mathrm{u}}^{0\mu}_0}(1,\mathbf{x}) \tag{3.3.45}$$

and

$$\delta\breve{\rho}(\mathbf{x})=\delta\rho(1,\mathbf{x}),\quad \breve{z}^j(\mathbf{x})=z^j(1,\mathbf{x}), \tag{3.3.46}$$

where $\mathbf{x}=(x^i)$. We then find after a long, but straightforward calculation using (3.1.24), (3.3.1), (3.3.24), (3.3.30)-(3.3.31), (3.3.39)-(3.3.42), Lemma 3.3.1 and the expansion (which is a consequence of (3.2.19) and $\bar{v}^0=\bar{g}^{00}\bar{v}_0+\bar{g}^{0i}\bar{v}_i$)

$$\underline{\bar{v}^0\bar{v}^0}=\frac{\Lambda}{3}-\epsilon^2\frac{1}{2}\breve{\mathrm{u}}^{00}+\epsilon^2\breve{\mathscr{S}}(\epsilon,\breve{\mathrm{u}}^{\mu\nu})+\epsilon^2\breve{\mathscr{F}}_1(\epsilon^2,\breve{\mathrm{u}}^{\mu\nu},\breve{z}_k)+\epsilon^2\breve{\mathscr{F}}_2(\epsilon^2,\breve{\mathrm{u}}^{\mu\nu},\breve{z}_k)+\epsilon^2\breve{\mathscr{L}}(\epsilon\breve{\mathrm{u}}^{ij}),$$

where the remainder terms $\breve{\mathscr{F}}_1$ and $\breve{\mathscr{F}}_2$ vanish to first and second order in $\breve{z}_k$, respectively, that (3.3.43)-(3.3.44), when written in terms of Newtonian coordinates, take the following form on the initial hypersurface Σ:

$$\Delta\breve{\mathrm{u}}^{00}=\epsilon^2E^2(1)\frac{\Lambda}{3}\big(7-6\Omega(1)\big)\breve{\mathrm{u}}^{00}+\epsilon\frac{\Lambda}{3}\big(\Omega(1)-1\big)E^2(1)\partial_i\breve{\mathrm{u}}^{0i}$$
$$+\epsilon E^2(1)\frac{\Lambda}{3}\partial_i\partial_j\breve{\mathrm{u}}^{ij}+\frac{2\Lambda}{3}E^2(1)\delta\breve{\rho}+\epsilon^2\mathtt{A}^0(\epsilon,\breve{\mathrm{u}}^{00},\breve{\mathrm{u}}^{0k},\breve{\xi}), \tag{3.3.47}$$
$$\Delta\breve{\mathrm{u}}^{0j}=2\Lambda\epsilon^2E^2(1)(1+\epsilon^2K)\big(\Omega(1)-2\big)\Omega(1)\breve{\mathrm{u}}^{0j}-2\epsilon\Omega(1)\delta^{jl}\partial_l\breve{\mathrm{u}}^{00}$$
$$-\epsilon E^2(1)\frac{\Lambda}{3}\partial_i\breve{\mathrm{u}}^{ji}_0+2\epsilon E^2(1)\sqrt{\frac{\Lambda}{3}}\breve{\rho}\breve{z}^j+\epsilon^2\mathtt{A}^j(\epsilon,\breve{\mathrm{u}}^{00},\breve{\mathrm{u}}^{0k},\breve{\xi}) \tag{3.3.48}$$

where

$$\mathtt{A}^0(\epsilon,\breve{\mathrm{u}}^{00},\breve{\mathrm{u}}^{0j},\breve{\xi})=\epsilon E^2(1)\breve{\mathrm{u}}^{00}\partial_i\partial_j\breve{\mathrm{u}}^{ij}+\epsilon E^2(1)\breve{\mathrm{u}}^{ij}\partial_i\partial_j\breve{\mathrm{u}}^{00}+E^2(1)\partial_i\partial_j\big(\breve{\mathrm{u}}^{i0}\breve{\mathrm{u}}^{0j}\big)+\breve{\mathscr{L}}_1(\epsilon,\breve{\mathrm{u}}^{kl}_0,\epsilon\breve{\mathrm{u}}^{kl})$$
$$+\epsilon^2\breve{\mathscr{S}}_1(\epsilon,\breve{\mathrm{u}}^{\alpha\beta})+\epsilon\breve{\mathscr{R}}_1(\epsilon,\epsilon\breve{\mathrm{u}}^{\mu\nu},\epsilon\breve{\mathrm{u}}^{kl}_0,D\breve{\mathrm{u}}^{\alpha\beta},\breve{\mathrm{u}}^{kl}_0)+\breve{\mathscr{Q}}_1(\epsilon,\breve{\mathrm{u}}^{\alpha\beta},D\breve{\mathrm{u}}^{\alpha\beta})$$
$$+\epsilon\breve{\mathscr{B}}_1(\epsilon,\breve{\mathrm{u}}^{\alpha\beta},D\breve{\mathrm{u}}^{\alpha\beta})+\breve{\mathscr{F}}_1(\epsilon,\breve{\mathrm{u}}^{\alpha\beta},\breve{z}^k,\delta\breve{\rho})+\epsilon^2\breve{\mathscr{G}}_1(\epsilon,\breve{\mathrm{u}}^{\alpha\beta},\delta\breve{\rho},\breve{z}^k), \tag{3.3.49}$$

$$\mathtt{A}^j(\epsilon,\breve{\mathrm{u}}^{00},\breve{\mathrm{u}}^{0j},\breve{\xi})=2\epsilon E^2(1)\breve{\mathrm{u}}^{0i}\partial_i\partial_k\breve{\mathrm{u}}^{kj}-\epsilon E^2(1)\breve{\mathrm{u}}^{kl}\partial_k\partial_l\breve{\mathrm{u}}^{0j}+\epsilon E^2(1)\breve{\mathrm{u}}^{00}\partial_i\breve{\mathrm{u}}^{ij}_0+\breve{\mathscr{L}}^j_2(\epsilon,\epsilon\breve{\mathrm{u}}^{kl},D\breve{\mathrm{u}}^{kl})$$

$$+ \epsilon^2 \breve{\mathscr{S}}_2^j(\epsilon, \breve{\mathrm{u}}^{\alpha\beta}) + \epsilon \breve{\mathscr{R}}_2^j(\epsilon, \epsilon\breve{\mathrm{u}}^{\alpha\beta}, \epsilon\breve{\mathrm{u}}_0^{kl}, D\breve{\mathrm{u}}^{\alpha\beta}, \breve{\mathrm{u}}_0^{kl}) + \breve{\mathscr{Q}}_2^j(\epsilon, \breve{\mathrm{u}}^{\alpha\beta}, D\breve{\mathrm{u}}^{\alpha\beta})$$
$$+ \epsilon \breve{\mathscr{B}}_2^j(\epsilon, \breve{\mathrm{u}}^{\alpha\beta}, D\breve{\mathrm{u}}^{\alpha\beta}) + \epsilon^2 \breve{\mathscr{F}}_2^j(\epsilon, \breve{\mathrm{u}}^{\alpha\beta}, \breve{z}^k, \delta\breve{\rho}) + \epsilon \breve{\mathscr{G}}_2^j(\epsilon, \breve{\mathrm{u}}^{\mu\nu}, \delta\breve{\rho}, \breve{z}^k), \tag{3.3.50}$$

and

$$\breve{\xi} = (\breve{\mathrm{u}}^{ij}, \breve{\mathrm{u}}_0^{ij}, \breve{z}^k, \delta\breve{\rho}) \tag{3.3.51}$$

denotes collectively the free initial data. Here, the remainder terms $\breve{\mathscr{L}}_1$ and $\breve{\mathscr{L}}_2^j$ are linear in $(\breve{\mathrm{u}}_0^{kl}, \epsilon\breve{\mathrm{u}}^{kl})$ and $(\epsilon\breve{\mathrm{u}}^{kl}, D\breve{\mathrm{u}}^{kl})$ respectively, $\breve{\mathscr{S}}_1$ and $\breve{\mathscr{S}}_2^j$ vanish to second order in $\breve{\mathrm{u}}^{\alpha\beta}$, $\breve{\mathscr{R}}_1$ and $\breve{\mathscr{R}}_2^j$ vanish to first order in $(\epsilon\breve{\mathrm{u}}^{\alpha\beta}, \epsilon\breve{\mathrm{u}}_0^{kl}, D\breve{\mathrm{u}}^{\alpha\beta})$ and are linear in $\breve{\mathrm{u}}_0^{kl}$, $\breve{\mathscr{B}}_1$ and $\breve{\mathscr{B}}_2^j$ vanish to first order in $\breve{\mathrm{u}}^{\alpha\beta}$ and are linear in $D\breve{\mathrm{u}}^{\lambda\sigma}$, $\breve{\mathscr{Q}}_1$ and $\breve{\mathscr{Q}}_2^j$ vanish to second order in $D\breve{\mathrm{u}}^{\alpha\beta}$, $\breve{\mathscr{F}}_1$ and $\breve{\mathscr{F}}_2^j$ are linear in $\delta\breve{\rho}$, and $\breve{\mathscr{G}}_1$ and $\breve{\mathscr{G}}_2^j$ vanish to first order in $\breve{z}^k$.

As a final observation, we note that $\Omega(1) < 0$ is a consequence of the definition (3.1.23) of $\Omega(t)$ from which it follows that $7 - 6\Omega(1) > 0$, $(\Omega(1) - 2)\Omega(1) > 0$, $\Omega(1) - 1 < 0$ and $-\Omega(1) > 0$; this observation will be important for the analysis carried out in §3.3.5.

3.3.3 Yukawa potentials

The *Yukawa potential operator of order* s, denoted $(\kappa^2 - \Delta)^{-\frac{s}{2}}$, is one of the main technical tools we employ for solving the constraints. It is defined for $0 < s < \infty$ and $\kappa \geq 0$, and it acts on function f via the formula

$$(\kappa^2 - \Delta)^{-\frac{s}{2}}(f) = (\widehat{\mathcal{Y}}_{s,\kappa}\widehat{f})^\vee = \mathcal{Y}_{s,\kappa} * f$$

where

$$\mathcal{Y}_{s,\kappa}(x) = \big((\kappa^2 + 4\pi^2|\xi|^2)^{-\frac{s}{2}}\big)^\vee(x).$$

In the special case $n = 3$ and $s = 2$, see [58, §3.2], we have the closed form convolution representation

$$(\kappa^2 - \Delta)^{-1}(f)(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{e^{-\kappa|x-y|}}{|x-y|} f(y) d^n y.$$

Note also that the Yukawa potential operator coincides with the Riesz potential operator and Bessel potential operator when $\kappa = 0$ and $\kappa = 1$, respectively; see Appendices B.0.1 and B.0.2.

Before moving forward, we recall the following well known fact concerning convolution operators

Lemma 3.3.4. *[31, Exercise 1.2.9] Let $T(f) = f * K$, where K is a positive $L^1(\mathbb{R}^n)$ function and f is in $L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$. Then the operator norm of $T : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ is equal to $\|K\|_{L^1}$.*

An important property of the Yukawa potential operator $(\kappa^2 - \Delta)^{-\frac{s}{2}}$ is that it maps L^p to itself whenever $\kappa > 0$. The following proposition gives a precise statement of this mapping property and it should be viewed as a generalization of the mapping property for the Bessel potential operator from Theorem B.0.4.(1).

Proposition 3.3.5. *For $0 < s < \infty$, $\kappa > 0$ and $1 \leq p \leq \infty$, the operator $\kappa^s(-\Delta + \kappa^2)^{-\frac{s}{2}}$ maps $L^p(\mathbb{R}^n)$ to itself with norm 1, that is,*

$$\|\kappa^s(-\Delta + \kappa^2)^{-\frac{s}{2}} f\|_{L^p} \leq \|f\|_{L^p}$$

for all $f \in L^p(\mathbb{R}^n)$ and

$$\|\kappa^s(\kappa^2 - \Delta)^{-\frac{s}{2}}\|_{\mathrm{op}} = \|\kappa^s \mathcal{Y}_s\|_{L^1} = 1.$$

Proof. Let[2] $S_\lambda(f)(x) = f(\lambda x)$ denote the scaling operator. Then from the identity $S_{\frac{1}{\kappa}}(\hat{G}_s) = \kappa^s \hat{\mathcal{Y}}_{s,\kappa}$, see Appendix B.0.2 for the definition of $\mathcal{G}_s$, we find that $S_{\frac{1}{\kappa}}(\hat{\mathcal{G}}_s)\hat{f} = S_{\frac{1}{\kappa}}(\hat{\mathcal{G}}_s S_\kappa \hat{f}) = \kappa^s \hat{\mathcal{Y}}_{s,\kappa}\hat{f}$. Taking the inverse Fourier transform of this expression gives

$$\kappa^s(\kappa^2 - \Delta)^{-\frac{s}{2}} f = \kappa^s \mathcal{Y}_{s,\kappa} * f = \kappa^n S_\kappa(\mathcal{G}_s) * f.$$

[2] In the following, we will use the well known identities $\widehat{S_\lambda(f)} = \lambda^{-n} S_{\frac{1}{\lambda}}(\hat{f})$ and $\|S_\lambda(f)\|_{L^p} = \lambda^{-\frac{n}{p}}\|f\|_{L^p}$.

The proof now follows from Lemma 3.3.4 since $\|\kappa^s\mathcal{Y}_{s,\kappa}\|_{L^1} = \|\kappa^n S_\kappa(\mathcal{G}_s)\|_{L^1} = \|\mathcal{G}_s\|_{L^1} = 1$ by Theorem B.0.4.(1). □

For applications in this articles, we single out the inequalities from Proposition 3.3.5 on $\mathbb{R}^3$ corresponding to $s = 1$ and $s = 2$, which are given by

$$\|\kappa(-\Delta + \kappa^2)^{-\frac{1}{2}} f\|_{L^p} \leq \|f\|_{L^p} \tag{3.3.52}$$

and

$$\|\kappa^2(\kappa^2 - \Delta)^{-1} f\|_{L^p} \leq \|f\|_{L^p}, \tag{3.3.53}$$

respectively.

Next, we obtain estimates for the operators $\partial_j(\kappa^2 - \Delta)^{-1} f$ and $\partial_j\partial_k(\kappa^2 - \Delta)^{-1} f$ on $\mathbb{R}^3$ that are uniform in κ.

Proposition 3.3.6. *Suppose $s \in \mathbb{Z}_{\geq 0}$, $k \in \mathbb{Z}_{\geq 1}$, $\kappa \geq 0$ and $1 < p < q < \infty$. Then there exists a constant $C > 0$, independent of κ, such that:*

1. $$\|\partial_j(\kappa^2 - \Delta)^{-1} f\|_{L^q} \leq C\|f\|_{L^p}$$

   for all $f \in L^p(\mathbb{R}^n)$ provided that p and q also satisfy $\frac{1}{p} - \frac{1}{q} = \frac{1}{n}$,

2. $$\|\partial_j\partial_k(\kappa^2 - \Delta)^{-1} f\|_{W^{s,p}} \leq C\|f\|_{W^{s,p}}$$

   for all $f \in W^{s,p}(\mathbb{R}^n)$, and

3. $$\|\partial_k(\kappa^2 - \Delta)^{-1} f\|_{R^k} \leq C\|f\|_{H^{k-1}}$$

   for all $f \in H^{s-1}(\mathbb{R}^3)$.

Proof. First, for $\kappa = 0$, we note that the above estimates are a direct consequence of Proposition B.0.3. Therefore, we assume that $\kappa > 0$. Then differentiating the identity

$$(\kappa^2 - \Delta)^{-1} f = (\kappa^2 - \Delta)^{-1}(\kappa^2 - \Delta - \kappa^2)(-\Delta)^{-1} f = (-\Delta)^{-1} f - \kappa^2(\kappa^2 - \Delta)^{-1}(-\Delta)^{-1} f$$

gives

$$\partial_j(\kappa^2 - \Delta)^{-1} f = -\mathfrak{R}_j(-\Delta)^{-\frac{1}{2}} f + \kappa^2\mathfrak{R}_j(-\Delta)^{-\frac{1}{2}}(\kappa^2 - \Delta)^{-1} f, \tag{3.3.54}$$

where $\mathfrak{R}_j$ is the Riesz transform, see Appendix B.0.1. Taking the L^q norm of both sides, where q and p are related by $\frac{1}{p} - \frac{1}{q} = \frac{1}{n}$, we find that

$$\begin{aligned}\|\partial_j(\kappa^2 - \Delta)^{-1} f\|_{L^q} &\lesssim \|\mathfrak{R}_j(-\Delta)^{-\frac{1}{2}} f\|_{L^q} + \kappa^2\|\mathfrak{R}_j(-\Delta)^{-\frac{1}{2}}(\kappa^2 - \Delta)^{-1} f\|_{L^q} \\ &\lesssim \|f\|_{L^p} + \|\kappa^2(\kappa^2 - \Delta)^{-1} f\|_{L^p} \\ &\lesssim \|f\|_{L^p} \end{aligned} \tag{3.3.55}$$

by Theorems B.0.1 and B.0.2, and (3.3.53). This proves the first inequality. To prove the second inequality, we differentiate (3.3.54) again to get

$$\partial_j\partial_k(\kappa^2 - \Delta)^{-1} f = \mathfrak{R}_j\mathfrak{R}_k f - \kappa^2\mathfrak{R}_j\mathfrak{R}_k(\kappa^2 - \Delta)^{-1} f.$$

From this it then follows that

$$\begin{aligned}\|\partial_j\partial_k(\kappa^2-\Delta)^{-1}f\|_{W^{s,p}} \lesssim&\|\mathfrak{R}_j\mathfrak{R}_k f\|_{W^{s,p}} + \|\kappa^2\mathfrak{R}_j\mathfrak{R}_k(\kappa^2-\Delta)^{-1}f\|_{W^{s,p}}\\ \lesssim&\|f\|_{W^{s,p}} + \|\kappa^2(\kappa^2-\Delta)^{-1}f\|_{W^{s,p}}\\ \lesssim&\|f\|_{W^{s,p}} \end{aligned} \tag{3.3.56}$$

by Propositions 3.3.5 and B.0.3, and the fact that $[D^\alpha, (\kappa^2-\Delta)^{-1}] = 0$. Finally, we observe that the last inequality follows from (3.3.54) and the inequalities (3.3.55), with $(n, q, p) = (3, 6, 2)$, and (3.3.56):

$$\|\partial_k(\kappa^2-\Delta)^{-1}f\|_{R^s} = \|\partial_k(\kappa^2-\Delta)^{-1}f\|_{L^6} + \|D\partial_k(\kappa^2-\Delta)^{-1}f\|_{H^{s-1}} \lesssim \|f\|_{H^{s-1}}.$$

□

3.3.4 Relation between the Riesz and Yukawa potential operators

In the following proposition, we establish an estimate, uniform in κ, that quantifies the relation between the Riesz and Yukawa potential operators, which is a variation on Lemma 2 from [80, §3.2].

Proposition 3.3.7. *Suppose* $0 < s < \infty$, $\kappa > 0$, *and* $1 \le p \le \infty$. *Then there exists a constant* $C > 0$, *independent of* κ, *such that*

$$\|(-\Delta+\kappa^2)^{-\frac{s}{2}}(-\Delta)^{\frac{s}{2}}(f)\|_{L^p} \le C\|f\|_{L^p}$$

for all $f \in L^p(\mathbb{R}^n)$.

Proof. By Young's inequality for convolutions, see Proposition C.2.9, we get

$$\begin{aligned}\|(-\Delta+\kappa^2)^{-\frac{s}{2}}(-\Delta)^{\frac{s}{2}}f\|_{L^p} &= \|\mathcal{Y}_{s,\kappa} * \big((4\pi^2|\xi|^2)^{\frac{s}{2}}\big)^\vee * f\|_{L^p}\\ &\le \|\mathcal{Y}_{s,\kappa} * \big((4\pi^2|\xi|^2)^{\frac{s}{2}}\big)^\vee\|_{L^1}\|f\|_{L^p} = \|\big(\widehat{\mathcal{Y}}_{s,\kappa}\cdot\big((4\pi^2|\xi|^2)^{\frac{s}{2}}\big)\big)^\vee\|_{L^1}\|f\|_{L^p}.\end{aligned}$$

By Lemma 3.3.4, the proof would then follow from a bound on $\|\big(\widehat{\mathcal{Y}}_{s,\kappa}\cdot\big((4\pi^2|\xi|^2)^{\frac{s}{2}}\big)\big)^\vee\|_{L^1}$. To see that this bound holds, we first observe that

$$\|\big(\widehat{\mathcal{Y}}_s\cdot\big((4\pi^2|\xi|^2)^{\frac{s}{2}}\big)\big)^\vee\|_{L^1} = \left\|\left(\frac{(4\pi^2|\xi|^2)^{\frac{s}{2}}}{(\kappa^2+4\pi^2|\xi|^2)^{\frac{s}{2}}}\right)^\vee\right\|_{L^1} = \left\|\left[\left(\mathbb{1}-\frac{\kappa^2}{\kappa^2+4\pi^2|\xi|^2}\right)^{\frac{s}{2}}\right]^\vee\right\|_{L^1}.$$

Since $s > 0$, we see from $\widehat{\mathcal{Y}}_{2m,\kappa} = (\kappa^2+4\pi^2|\xi|^2)^{-\frac{2m}{2}}$ and expanding $(1-y)^{\frac{s}{2}}$ in a power series that

$$\left(\mathbb{1}-\frac{\kappa^2}{\kappa^2+4\pi^2|\xi|^2}\right)^{\frac{s}{2}} = \mathbb{1} + \sum_{m=1}^\infty A_{m,s}\kappa^{2m}(\kappa^2+4\pi^2|\xi|^2)^{-\frac{2m}{2}} = \mathbb{1} + \sum_{m=1}^\infty A_{m,s}\kappa^{2m}\widehat{\mathcal{Y}}_{2m,\kappa}$$

holds for $|\xi| > 0$. But since $\sum_{m=1}^\infty |A_{m,s}| < \infty$ by Raabe's test[3], see [6], and

$$\left[\left(\mathbb{1}-\frac{\kappa^2}{\kappa^2+4\pi^2|\xi|^2}\right)^{\frac{s}{2}}\right]^\vee = \Big[\mathbb{1} + \sum_{m=1}^\infty A_{m,s}\kappa^{2m}\widehat{\mathcal{Y}}_{2m,\kappa}\Big]^\vee = \delta_0 + \sum_{m=1}^\infty A_{m,s}\kappa^{2m}\mathcal{Y}_{2m,\kappa},$$

[3] To apply Raabe's test, consider the series $(1-y)^s = 1 + \sum_{m=1}^\infty \binom{s}{m}(-1)^m y^m$, where $\binom{s}{m} := \frac{s!}{m!(s-m)!}$ and $s > 0$. Since for m is large enough, $m\left(\frac{\left|\binom{s}{m}\right|}{\left|\binom{s}{m+1}\right|} - 1\right) = \frac{m(1+s)}{m-s} \to 1 + s > 1$ as $m \to \infty$, the sum $\sum_{m=1}^\infty \binom{s}{m}(-1)^m$ is absolutely convergent.

we deduce that

$$\|(\widehat{\mathcal{Y}}_{s,\kappa}\cdot((4\pi^2|\xi|^2)^{\frac{s}{2}}))^{\vee}\|_{L^1} = \left\|\left[\left(\mathbb{1}-\frac{\kappa^2}{\kappa^2+4\pi^2|\xi|^2}\right)^{\frac{s}{2}}\right]^{\vee}\right\|_{L^1}$$
$$\leq 1+\sum_{m=1}^{\infty}|A_{m,s}|\|\kappa^{2m}\mathcal{Y}_{2m,\kappa}\|_{L^1} = 1+\sum_{m=1}^{\infty}|A_{m,s}| \leq \infty,$$

where in deriving the final equality we used $\|\kappa^{2m}\mathcal{Y}_{2m,\kappa}\|_{L^1}=1$, which is a consequence of Proposition 3.3.5. □

We proceed by using Proposition 3.3.7 to obtain estimates for the operators $(-\Delta+\kappa^2)^{-\frac{1}{2}}\partial_j$ and $\kappa(-\Delta+\kappa^2)^{-1}\partial_j$ that are uniform in κ.

Proposition 3.3.8. *Suppose $s\in\mathbb{Z}_{\geq 0}$, $\kappa>0$ and $1\leq p\leq\infty$. Then there exists a constant $C>0$, independent of κ, such that*

$$\|(-\Delta+\kappa^2)^{-\frac{1}{2}}\partial_j f\|_{W^{s,p}}+\|\kappa(-\Delta+\kappa^2)^{-1}\partial_j f\|_{W^{s,p}} \leq C\|f\|_{W^{s,p}}$$

for all $f\in W^{s,p}(\mathbb{R}^n)$.

Proof. The proof follows directly from the identities

$$\kappa(-\Delta+\kappa^2)^{-1}\partial_j f = -\kappa(-\Delta+\kappa^2)^{-\frac{1}{2}}(-\Delta+\kappa^2)^{-\frac{1}{2}}\partial_j f$$

and

$$(-\Delta+\kappa^2)^{-\frac{1}{2}}\partial_j f = -(-\Delta+\kappa^2)^{-\frac{1}{2}}(-\Delta)^{\frac{1}{2}}\mathfrak{R}_j f,$$

and an application of Propositions 3.3.5 and 3.3.7, and Theorem B.0.2. □

We will also need the following generalization of Theorems B.0.1 and B.0.4.(2) for the operator $(\kappa^s-\Delta)^{-\frac{s}{2}}$ with estimates that are uniform in κ.

Proposition 3.3.9. *Suppose $0<s<n$ and $1<p<q<\infty$ satisfy $\frac{1}{p}-\frac{1}{q}=\frac{s}{n}$, and $\kappa>0$. Then there exists a constant $C>0$, independent of κ, such that*

$$\|(\kappa^2-\Delta)^{-\frac{s}{2}}f\|_{L^q}\leq C\|f\|_{L^p}$$

for all $f\in L^p(\mathbb{R}^n)$.

Proof. Noting the identity $(\kappa^2-\Delta)^{-\frac{s}{2}}f=(\kappa^2-\Delta)^{-\frac{s}{2}}(-\Delta)^{\frac{s}{2}}(-\Delta)^{-\frac{s}{2}}f$, it is clear that the proof follows directly from an application of Proposition 3.3.7 and Theorem B.0.1. □

For convenience, we list the following special cases of the estimate from Proposition 3.3.9 on $\mathbb{R}^3$, which we will use frequently in subsequence sections:

$$\|(\kappa^2-\Delta)^{-\frac{1}{2}}f\|_{L^6}\lesssim\|f\|_{L^2}, \tag{3.3.57}$$
$$\|(\kappa^2-\Delta)^{-1}f\|_{L^6}\lesssim\|f\|_{L^{6/5}} \tag{3.3.58}$$

and

$$\|(\kappa^2-\Delta)^{-\frac{1}{2}}f\|_{L^2}\lesssim\|f\|_{L^{6/5}}. \tag{3.3.59}$$

3.3.5 Solving the constraint equations

Having established the necessary estimates for the Yukawa potential operator, we now turn to solving the constraint equations (3.3.47)-(3.3.48). For this, we will use a fixed point method by first specifying the *free data* $(\breve{\mathtt{u}}^{ij}, \breve{\mathtt{u}}^{ij}_0, \delta\breve{\rho}, \breve{z}^l)$ and then rewriting the constraint equations (3.3.47)-(3.3.48) as a contraction mapping whose fixed point yields a solution $(\breve{\mathtt{u}}^{00}, \breve{\mathtt{u}}^{0j})$ to the constraint equations.

The contraction mapping

To streamline the set up of the contraction mapping, we set

$$\phi = \breve{\mathtt{u}}^{00} \quad \text{and} \quad \psi^j = \breve{\mathtt{u}}^{0j},$$

and we define the constants

$$\mathtt{a} = \frac{\Lambda}{3}E^2(1)\big(7 - 6\Omega(1)\big) > 0, \quad \mathtt{b} = \frac{\Lambda}{3}\big(\Omega(1) - 1\big)E^2(1) < 0,$$
$$\mathtt{c} = 2\Lambda E^2(1)(1+\epsilon^2 K)\big(\Omega(1) - 2\big)\Omega(1) > 0 \quad \text{and} \quad \mathtt{d} = -2\Omega(1) > 0,$$

where $\Omega(t)$ is as defined previously by (3.2.1). Further, we fix the free data according to

$$\breve{\mathtt{u}}^{ij} \in R^{s+1}(\mathbb{R}^3, \mathbb{S}_3), \quad \breve{\mathtt{u}}^{ij}_0 \in H^s(\mathbb{R}^3, \mathbb{S}_3), \quad \breve{z}^j \in L^{6/5} \cap K^s(\mathbb{R}^3, \mathbb{R}^3) \quad \text{and} \quad \delta\breve{\rho} \in L^{6/5} \cap K^s(\mathbb{R}^3, \mathbb{R}). \tag{3.3.60}$$

With the above definition, the constraint equations (3.3.47)-(3.3.48) become

$$\begin{pmatrix} \Delta - \epsilon^2\mathtt{a} & -\epsilon\mathtt{b}\partial_j \\ -\epsilon\mathtt{d}\partial^j & \Delta - \epsilon^2\mathtt{c} \end{pmatrix} \begin{pmatrix} \phi \\ \psi^j \end{pmatrix} = \begin{pmatrix} f(\epsilon, \phi, \psi^j, \breve{\xi}) \\ g^j(\epsilon, \phi, \psi^k, \breve{\xi}) \end{pmatrix} \tag{3.3.61}$$

where

$$f(\epsilon, \phi, \psi^k, \breve{\xi}) = \epsilon E^2(1)\frac{\Lambda}{3}\partial_i\partial_j\breve{\mathtt{u}}^{ij} + \frac{2\Lambda}{3}E^2(1)\delta\breve{\rho} + \epsilon^2\mathtt{A}^0(\epsilon, \phi, \psi^k, \breve{\xi}), \tag{3.3.62}$$

$$g^j(\epsilon, \phi, \psi^k, \breve{\xi}) = -\epsilon E^2(1)\frac{\Lambda}{3}\partial_i\hat{\mathtt{u}}^{ji}_0 + 2\epsilon E^2(1)\sqrt{\frac{\Lambda}{3}}(\breve{\rho}\breve{z}^j) + \epsilon^2\mathtt{A}^j(\epsilon, \phi, \psi^k, \breve{\xi}). \tag{3.3.63}$$

Apart from the regularity requirements (3.3.60) on the free initial data , we also require the following smallness condition on the initial density, see (3.1.6), of the background FRLW solution given by

$$0 < \mu(1) < \frac{1}{8}(19 + 5\sqrt{29})\Lambda, \tag{3.3.64}$$

which, by (3.1.23), implies that

$$\frac{1}{4}(-1-\sqrt{29}) < \Omega(1) < 0 \quad \text{and} \quad -\frac{\mathtt{a}}{\mathtt{bd}} > 2. \tag{3.3.65}$$

Remark 3.3.10. It is probable that the condition (3.3.64) is not necessary for establishing the existence of 1-parameter families of ϵ-dependent initial data that satisfy the constraint equations. Indeed, in the article [65], a similar method was used to establish the existence of 1-parameter families of ϵ-dependent initial data satisfying the constraint equations without a similar smallness condition on the background FLRW solution. However, the gauge condition used in this article, which is suited to the long time evolution problem, is different from the gauge used in [65], and the analysis of the constraint equations in [65] employed a more complicated conformal decomposition. Consequently, it is not clear if the choice of gauge or the particular representation of the constraint equations used in this article is responsible for the requirement (3.3.64). In any case, we stress that (3.3.64) is only needed to establish the existence of suitable 1-parameter families of ϵ-dependent initial data; this restriction is not needed for the evolution of the initial data where a smallness condition is only needed on the perturbed part of the initial data, see Theorem 3.1.6 for details, and not on the background FLRW solution.

Replacing ψ^j with ϑ^j defined by

$$\vartheta^j = \psi^j - \epsilon \mathtt{d}\partial^j(\Delta - \epsilon^2\mathtt{c})^{-1}\phi \tag{3.3.66}$$

allows us to write (3.3.61) as

$$\begin{pmatrix} \Delta - \epsilon^2(\mathtt{a}+\mathtt{bd}) & -\epsilon\mathtt{b}\partial_j \\ 0 & \Delta - \epsilon^2\mathtt{c} \end{pmatrix}\begin{pmatrix}\phi \\ \vartheta^j\end{pmatrix} = \begin{pmatrix}\tilde{f}(\epsilon,\phi,\vartheta^k,\breve{\xi}) \\ \tilde{g}^j(\epsilon,\phi,\vartheta^k,\breve{\xi})\end{pmatrix} \tag{3.3.67}$$

where

$$\tilde{f}(\epsilon,\phi,\vartheta^k,\breve{\xi}) = f(\epsilon,\phi,\vartheta^k + \epsilon\mathtt{d}\partial^k(\Delta-\epsilon^2\mathtt{c})^{-1}\phi,\breve{\xi}) + \epsilon^4\mathtt{bcd}(\Delta-\epsilon^2\mathtt{c})^{-1}\phi \tag{3.3.68}$$

and

$$\tilde{g}^j(\epsilon,\phi,\vartheta^k,\breve{\xi}) = g^j(\epsilon,\phi,\vartheta^k + \epsilon\mathtt{d}\partial^k(\Delta-\epsilon^2\mathtt{c})^{-1}\phi,\breve{\xi}). \tag{3.3.69}$$

The advantage of (3.3.67) over (3.3.61) is that the linear operator on the left hand side of (3.3.67) is invertible with the inverse given by[4]

$$\begin{pmatrix} \Delta - \epsilon^2(\mathtt{a}+\mathtt{bd}) & \epsilon\mathtt{b}\partial_j \\ 0 & \Delta - \epsilon^2\mathtt{c} \end{pmatrix}^{-1} = \begin{pmatrix} \big(\Delta - \epsilon^2(\mathtt{a}+\mathtt{bd})\big)^{-1} & \epsilon\mathtt{b}\partial_j(\Delta-\epsilon^2\mathtt{c})^{-1}\big(\Delta - \epsilon^2(\mathtt{a}+\mathtt{bd})\big)^{-1} \\ 0 & (\Delta-\epsilon^2\mathtt{c})^{-1} \end{pmatrix}.$$

Acting on both sides of (3.3.67) with the inverse operator yields the equations

$$\begin{pmatrix}\phi \\ \vartheta^j\end{pmatrix} = \begin{pmatrix} \big(\Delta - \epsilon^2(\mathtt{a}+\mathtt{bd})\big)^{-1} & \epsilon\mathtt{b}\partial_j(\Delta-\epsilon^2\mathtt{c})^{-1}\big(\Delta - \epsilon^2(\mathtt{a}+\mathtt{bd})\big)^{-1} \\ 0 & (\Delta-\epsilon^2\mathtt{c})^{-1} \end{pmatrix}\begin{pmatrix}\tilde{f}(\epsilon,\phi,\vartheta^k,\breve{\xi}) \\ \tilde{g}^j(\epsilon,\phi,\vartheta^k,\breve{\xi})\end{pmatrix}, \tag{3.3.70}$$

which, we stress, are completely equivalent to the constraint equations (3.3.47)-(3.3.48).

In order to solve the constraint equations (3.3.70) using a fixed point method, we let the right hand side of (3.3.70) define a map $\mathbf{G}$ which takes elements $\acute{\iota} = (\acute{\phi},\acute{\vartheta}^k)$ and maps them to $\grave{\iota} = \mathbf{G}(\acute{\iota})$, where $\grave{\iota} = (\grave{\phi},\grave{\vartheta}^k)$, according to

$$\grave{\phi} = \big(\Delta - \epsilon^2(\mathtt{a}+\mathtt{bd})\big)^{-1}\tilde{f}(\epsilon,\acute{\phi},\acute{\vartheta}^k,\breve{\xi}) + \epsilon\mathtt{b}\partial_j(\Delta-\epsilon^2\mathtt{c})^{-1}\big(\Delta - \epsilon^2(\mathtt{a}+\mathtt{bd})\big)^{-1}\tilde{g}^j(\epsilon,\acute{\phi},\acute{\vartheta}^k,\breve{\xi}), \tag{3.3.71}$$

$$\grave{\vartheta}^k = (\Delta-\epsilon^2\mathtt{c})^{-1}\tilde{g}^k(\epsilon,\acute{\phi},\acute{\vartheta}^j,\breve{\xi}). \tag{3.3.72}$$

Before considering the map $\mathbf{G}$ further, we first establish the following technical lemma that will allows us to estimate terms of the form $\|(\Delta-\epsilon^2\lambda)^{-1}\tilde{g}^j(\epsilon,\acute{\phi},\acute{\vartheta}^k,\breve{\xi})\|_{R^{s+1}}$ and $\|(\Delta-\epsilon^2\lambda)^{-1}\tilde{f}(\epsilon,\acute{\phi},\acute{\vartheta}^j,\breve{\xi})\|_{R^{s+1}}$, where $\lambda > 0$ is some constant. These estimates will be needed below in the proof of Proposition 3.3.12 where we show that $\mathbf{G}$ defines a contraction map.

Lemma 3.3.11. *Suppose $s \in \mathbb{Z}_{\geq 3}$, $0 < \epsilon < \epsilon_0$, $\lambda \in \mathbb{R}_{>0}$, and F is defined by*

$$\begin{aligned} F =& \epsilon^4 H_1(\epsilon,f_1,f_2) + \epsilon\partial_i\partial_j f_3 + f_4 + \epsilon^3 H_5(\epsilon,f_5,\partial_i\partial_j f_6) + \epsilon^3 H_7(\epsilon,f_7,\partial_i f_8) \\ &+ \epsilon^3 H_0(\epsilon,f_0,f_8) + \epsilon^3 f_9 + \epsilon^2 f_{10} + \epsilon\partial_i f_{11} + \epsilon f_{12} \end{aligned}$$

where $f_1,f_2,f_3,f_5,f_6,f_7,f_9 \in R^{s+1}(\mathbb{R}^3)$, $f_4,f_{12} \in L^{\frac{6}{5}} \cap K^s(\mathbb{R}^3)$, $f_0 \in R^s(\mathbb{R}^3)$, $f_8,f_{10},f_{11} \in H^s(\mathbb{R}^3)$, and the maps $H_\ell(\epsilon,u,v)$, $\ell = 0,1,5,7$, are smooth, vanish to first order in u, and are linear in v. Then

[4] Note that $a + bd > -bd > 0$ by (3.3.65).

$(\epsilon^2\lambda-\Delta)^{-1}F \in R^{s+1}$ *and*

$$\|(\epsilon^2\lambda-\Delta)^{-1}F\|_{R^{s+1}} \leq C_0\Big[\epsilon^2\|f_1\|_{R^{s+1}}\|f_2\|_{R^{s+1}} + \epsilon\|f_3\|_{R^{s+1}} + \|f_4\|_{L^{\frac{6}{5}}\cap K^s} + \epsilon\|f_5\|_{R^{s+1}}\|f_6\|_{R^{s+1}} + \epsilon\|f_9\|_{R^{s+1}} + \epsilon(\|f_7\|_{R^{s+1}} + \|f_0\|_{R^s})\|f_8\|_{H^s} + \epsilon\|f_{10}\|_{H^s} + \epsilon\|f_{11}\|_{H^s} + \epsilon\|f_{12}\|_{L^{\frac{6}{5}}\cap K^s}\Big] \quad (3.3.73)$$

where $C_0 = C_0\big(\|f_0\|_{R^s}, \|f_1\|_{R^{s+1}}, \|f_2\|_{R^{s+1}}, \|f_5\|_{R^{s+1}}, \|f_6\|_{R^{s+1}}, \|f_7\|_{R^{s+1}}, \|f_8\|_{H^s}\big)$.

Furthermore, if $f_{10} = \mathtt{G}(f)g$, *where* $f \in K^s(\mathbb{R}^3)$, $g \in H^s(\mathbb{R}^3)$ *and* $\mathtt{G}(u)$ *is smooth, then*

$$\|f_{10}\|_{H^s} \leq C(\|f\|_{K^s})\|g\|_{H^s},$$

and, in the case $\mathtt{G}(u)$ *also vanishes to first order in* u,

$$\|f_{10}\|_{H^s} \leq C(\|f\|_{K^s})\|f\|_{K^s}\|g\|_{H^s}.$$

Proof. Since

$$\|(\epsilon^2\lambda-\Delta)^{-1}F\|_{R^{s+1}} = \|(\epsilon^2\lambda-\Delta)^{-1}F\|_{L^6} + \|(\epsilon^2\lambda-\Delta)^{-1}DF\|_{H^s},$$

we proceed by estimating each term separately starting with $\|(\epsilon^2\lambda-\Delta)^{-1}F\|_{L^6}$. Using (3.3.53), (3.3.58), Proposition 3.3.8 and Theorem C.2.6, we find that

$$\begin{aligned}&\|(\epsilon^2\lambda-\Delta)^{-1}\big(\epsilon^4\mathtt{H}_1(\epsilon,f_1,f_2) + f_4 + \epsilon^3\mathtt{H}_5(\epsilon,f_5,\partial_i\partial_j f_6) + \epsilon^3\mathtt{H}_7(\epsilon,f_7,\partial_i f_8) + \epsilon^3\mathtt{H}_0(\epsilon,f_0,f_8) + \epsilon^3 f_9\big)\|_{L^6}\\ &\leq C_0\Big[\epsilon^2\|f_1\|_{L^\infty}\|f_2\|_{L^6} + \|f_4\|_{L^{\frac{6}{5}}} + \epsilon\|f_5\|_{L^\infty}\|\partial_i\partial_j f_6\|_{L^6} + \epsilon\|f_7\|_{L^\infty}\|\partial_i f_8\|_{L^6} + \epsilon\|f_0\|_{L^\infty}\|f_8\|_{L^6} + \epsilon\|f_9\|_{L^6}\Big],\end{aligned} \quad (3.3.74)$$

where here and for the rest of the proof, we take C_0 to be a constant of the form

$$C_0 = C_0\big(\|f_0\|_{R^s}, \|f_1\|_{R^{s+1}}, \|f_2\|_{R^{s+1}}, \|f_5\|_{R^{s+1}}, \|f_6\|_{R^{s+1}}, \|f_7\|_{R^{s+1}}, \|f_8\|_{H^s}\big).$$

Next, we have that

$$\|\epsilon(\epsilon^2\lambda-\Delta)^{-1}\partial_i\partial_j f_3\|_{L^6} \lesssim \epsilon\|f_3\|_{L^6} \quad (3.3.75)$$

by Proposition 3.3.6.(2), while

$$\|\epsilon(\epsilon^2\lambda-\Delta)^{-1}\partial_i f_{11}\|_{L^6} \lesssim \epsilon\|f_{11}\|_{L^2} \quad (3.3.76)$$

follows from Proposition 3.3.6.(1). Furthermore, we see that

$$\|\epsilon^2(\epsilon^2\lambda-\Delta)^{-1}f_{10}\|_{L^6} \lesssim \|\epsilon(\epsilon^2\lambda-\Delta)^{-\frac{1}{2}}f_{10}\|_{L^6} \lesssim \epsilon\|f_{10}\|_{L^2} \quad (3.3.77)$$

by (3.3.52) and (3.3.57), as well as

$$\|\epsilon(\epsilon^2\lambda-\Delta)^{-1}f_{12}\|_{L^6} \lesssim \epsilon\|f_{12}\|_{L^{6/5}}$$

by (3.3.58). Combining (3.3.74)-(3.3.77) yields the estimate

$$\begin{aligned}\|(\epsilon^2\lambda-\Delta)^{-1}F\|_{L^6} \leq &C_0\Big[\epsilon^2\|f_1\|_{L^\infty}\|f_2\|_{L^6} + \epsilon\|f_3\|_{L^6} + \|f_4\|_{L^{\frac{6}{5}}} + \epsilon\|f_5\|_{L^\infty}\|\partial_i\partial_j f_6\|_{L^6} + \epsilon\|f_7\|_{L^\infty}\|\partial_i f_8\|_{L^6}\\ &+ \epsilon\|f_0\|_{L^\infty}\|f_8\|_{L^6} + \epsilon\|f_9\|_{L^6} + \epsilon\|f_{11}\|_{L^2} + \epsilon\|f_{10}\|_{L^2} + \epsilon\|f_{12}\|_{L^{6/5}}\Big]\end{aligned} \quad (3.3.78)$$

for the term $\|(\epsilon^2\lambda-\Delta)^{-1}F\|_{L^6}$.

Next, we turn to estimating $\|(\epsilon^2\lambda-\Delta)^{-1}DF\|_{H^s}$. First, using the Leibniz's rule, we see, with the help of Theorem C.2.7 and Proposition C.2.8, that the estimate

$$\|D(\mathtt{G}(f)g)\|_{H^s} \lesssim \|\mathtt{G}(f)\|_{W^{1,\infty}}\|Dg\|_{H^s} + \|D\mathtt{G}(f)\|_{H^s}\|g\|_{W^{1,\infty}}$$

$$\lesssim \|\mathtt{G}(f)\|_{R^{s+1}}\|g\|_{R^{s+1}} \leq C(\|f\|_{R^{s+1}})\|f\|_{R^{s+1}}\|g\|_{R^{s+1}} \tag{3.3.79}$$

holds for smooth functions $\mathtt{G}(u)$ that vanish to first order in u. Then from this estimate and (3.3.53), it follows that

$$\|(\epsilon^2\lambda-\Delta)^{-1}D(\epsilon^4\mathtt{H}_1(\epsilon,f_1,f_2))\|_{H^s} \lesssim \epsilon^2\|D(\mathtt{H}_1(\epsilon,f_1,f_2))\|_{H^s} \leq C_0\epsilon^2\|f_1\|_{R^{s+1}}\|f_2\|_{R^{s+1}}, \tag{3.3.80}$$

and

$$\|(\epsilon^2\lambda-\Delta)^{-1}D(\epsilon^3 f_9)\|_{H^s} \lesssim \epsilon\|Df_9\|_{H^s} \lesssim \epsilon\|f_9\|_{R^{s+1}}. \tag{3.3.81}$$

Continuing on, we note by Proposition 3.3.6 and Theorem C.2.2 that

$$\|D(\Delta-\epsilon^2\lambda)^{-1}f_4\|_{H^s} \lesssim \|D(\Delta-\epsilon^2\lambda)^{-1}f_4\|_{L^2} + \|D^2(\Delta-\epsilon^2\lambda)^{-1}f_4\|_{H^{s-1}} \lesssim \|f_4\|_{L^{\frac{6}{5}}\cap K^s} \tag{3.3.82}$$

and

$$\|D(\Delta-\epsilon^2\lambda)^{-1}\epsilon f_{12}\|_{H^s} \lesssim \epsilon\|f_{12}\|_{L^{\frac{6}{5}}\cap K^s}. \tag{3.3.83}$$

We further observe by (3.1.13), Proposition 3.3.6.(2), Proposition 3.3.8 and Theorems C.2.7 and C.2.8 that the inequality

$$\begin{aligned}
\|D(\Delta-\epsilon^2\lambda)^{-1}\epsilon(\mathtt{G}(f)Dg)\|_{H^s} = {}& \|D(\Delta-\epsilon^2\lambda)^{-1}\epsilon D(\mathtt{G}(f)g) - D(\Delta-\epsilon^2\lambda)^{-1}\epsilon(D\mathtt{G}(f)g)\|_{H^s} \\
\lesssim{}& \epsilon\|\mathtt{G}(f)g\|_{H^s} + \|D\mathtt{G}(f)g\|_{H^s} \\
\lesssim{}& \epsilon\|\mathtt{G}(f)\|_{L^\infty}\|g\|_{H^s} + \epsilon\|D\mathtt{G}(f)\|_{H^{s-1}}\|g\|_{L^\infty} + \|D\mathtt{G}(f)\|_{H^s}\|g\|_{L^\infty} + \|D\mathtt{G}(f)\|_{L^\infty}\|Dg\|_{H^{s-1}} \\
\lesssim{}& \|\mathtt{G}(f)\|_{R^{s+1}}\|g\|_{H^s} \\
\leq{}& C(\|f\|_{R^{s+1}})\|f\|_{R^{s+1}}\|g\|_{H^s}
\end{aligned}$$

holds for smooth functions $\mathtt{G}(u)$ that vanish to first order in u. Making use of this inequality, we find that

$$\|D(\Delta-\epsilon^2\lambda)^{-1}\epsilon^3(\mathtt{H}_5(\epsilon,f_5,\partial_i\partial_j f_6))\|_{H^s} \leq C_0\epsilon^2\|f_5\|_{R^{s+1}}\|f_6\|_{R^{s+1}} \tag{3.3.84}$$

and

$$\|D(\Delta-\epsilon^2\lambda)^{-1}\epsilon^3\big(\mathtt{H}_7(\epsilon,f_7,\partial_i f_8)\big)\|_{H^s} \lesssim \epsilon^2\|f_7\|_{R^{s+1}}\|f_8\|_{H^s}. \tag{3.3.85}$$

By Proposition 3.3.6.(2), we deduce that

$$\|D(\Delta-\epsilon^2\lambda)^{-1}\epsilon\partial_i\partial_j f_3\|_{H^s} \lesssim \epsilon\|Df_3\|_{H^s} \quad \text{and} \quad \|D(\Delta-\epsilon^2\lambda)^{-1}\epsilon\partial_i f_{11}\|_{H^s} \lesssim \epsilon\|f_{11}\|_{H^s} \tag{3.3.86}$$

while it is clear that

$$\|\epsilon^2(\epsilon^2\lambda-\Delta)^{-1}Df_{10}\|_{H^s} \lesssim \epsilon\|f_{10}\|_{H^s} \tag{3.3.87}$$

is a direct consequence of Proposition 3.3.8.

With the help of Proposition 3.3.8, Theorem C.2.7 and C.2.8, we see that

$$\|(\epsilon^2\lambda-\Delta)^{-1}D(\epsilon\mathtt{G}(f)g)\|_{H^s} \lesssim \|\mathtt{G}(f)g\|_{H^s} \lesssim \|\mathtt{G}(f)\|_{L^\infty}\|g\|_{H^s} + \|D\mathtt{F}(f)\|_{H^{s-1}}\|g\|_{L^\infty} \leq C(\|f\|_{R^s})\|f\|_{R^s}\|g\|_{H^s}$$

holds for smooth functions $\mathtt{G}(u)$ that vanish to first order in u. Using this inequality, we obtain the estimate

$$\|(\epsilon^2\lambda-\Delta)^{-1}D(\epsilon^3\mathtt{H}_0(\epsilon,f_0,f_8))\|_{H^s} \leq C_0\epsilon^2\|f_0\|_{R^s}\|f_8\|_{H^s}. \tag{3.3.88}$$

Gathering the estimates (3.3.80)-(3.3.88), we arrive at the estimate

$$\|(\epsilon^2\lambda-\Delta)^{-1}\partial_m F\|_{H^s} \lesssim C_0\Big[\epsilon^2\|f_1\|_{R^{s+1}}\|f_2\|_{R^{s+1}} + \epsilon\|Df_3\|_{H^s} + \|f_4\|_{L^{\frac{6}{5}}\cap K^s} + \epsilon^2\|f_5\|_{R^{s+1}}\|f_6\|_{R^{s+1}}$$

$$+\epsilon\|f_9\|_{R^{s+1}}+\epsilon^2(\|f_7\|_{R^{s+1}}+\|f_0\|_{R^s})\|f_8\|_{H^s}+\epsilon\|f_{11}\|_{H^s}+\epsilon\|f_{10}\|_{H^s}+\epsilon\|f_{12}\|_{L^{\frac{6}{5}}\cap K^s}\Big]. \quad (3.3.89)$$

The estimate (3.3.73) is then a direct consequence of (3.3.78), (3.3.89), the triangle inequality, and the inequality $\|f_8\|_{W^{1,6}}\lesssim\|f_8\|_{R^s}\lesssim\|f_8\|_{H^s}$, which follows from (3.1.13)-(3.1.14).

Finally, if $\mathtt{G}(u)$ is a smooth function, it follows directly from Theorems C.2.2, C.2.6 and C.2.7 that

$$\|f_{10}\|_{H^s}=\|\mathtt{G}(f)g\|_{H^s}\lesssim\|\mathtt{G}(f)\|_{L^\infty}\|g\|_{H^s}+\|D\mathtt{G}(f)\|_{H^{s-1}}\|g\|_{L^\infty}\le C(\|f\|_{K^s})\|g\|_{H^s},$$

which can be improved to

$$\|f_{10}\|_{H^s}=\|\mathtt{F}(f)g\|_{H^s}\lesssim\|\mathtt{G}(f)\|_{L^\infty}\|g\|_{H^s}+\|D\mathtt{G}(f)\|_{H^{s-1}}\|g\|_{L^\infty}\le C(\|f\|_{K^s})\|f\|_{K^s}\|g\|_{H^s}$$

if $\mathtt{G}(u)$ also vanishes to first order in u. □

Next, we use the above lemma to prove that $\mathbf{G}$ is a contraction mapping in the following proposition.

Proposition 3.3.12. *Suppose $s\in\mathbb{Z}_{\ge3}$, $r>0$, $\mathbf{G}$ is a map defined by (3.3.71)-(3.3.72), $\mu(1)$ satisfies (3.3.64) and the free data $\breve{\xi}=(\breve{\mathfrak{u}}^{ij},\breve{\mathfrak{u}}^{ij}_0,\breve{z}^k,\delta\breve{\rho})$ is bounded by*

$$\|\xi\|_s:=\|\breve{\mathfrak{u}}^{ij}\|_{R^{s+1}}+\|\breve{\mathfrak{u}}^{ij}_0\|_{H^s}+\|\delta\breve{\rho}\|_{L^{\frac{6}{5}}\cap K^s}+\|\breve{z}^j\|_{L^{\frac{6}{5}}\cap K^s}\le r. \quad (3.3.90)$$

Then for

$$l>\frac{2\Lambda}{3}\frac{7-4\mathring{\Omega}(1)-2\mathring{\Omega}^2(1)}{7-2\mathring{\Omega}(1)-4\mathring{\Omega}^2(1)}\mathring{E}^2(1)r,$$

there exists constants $\epsilon_0>0$ and $k\in(0,1)$ such that for every $\epsilon\in(0,\epsilon_0)$, $\mathbf{G}$ maps the closed ball $\overline{B_l\big(R^{s+1}(\mathbb{R}^3,\mathbb{R}^4)\big)}$ to itself and satisfies

$$\|\mathbf{G}(\acute{\iota}_1)-\mathbf{G}(\acute{\iota}_2)\|_{R^{s+1}}\le k\|\acute{\iota}_1-\acute{\iota}_2\|_{R^{s+1}}$$

for all $\acute{\iota}_1$, $\acute{\iota}_2\in\overline{B_l\big(R^{s+1}(\mathbb{R}^3,\mathbb{R}^4)\big)}$.

Proof. Suppose $\acute{\iota}=(\acute{\phi},\acute{\vartheta}^j)\in\overline{B_l\big(R^{s+1}(\mathbb{R}^3,\mathbb{R}^4)\big)}$ with the radius $l>0$ to be chosen later. Then by definition, $\grave{\iota}=\mathbf{G}(\acute{\iota})$, where $\grave{\iota}=(\grave{\phi},\grave{\vartheta}^k)$ with $\grave{\phi}$ and $\grave{\vartheta}^k$ given by (3.3.71) and (3.3.72), respectively. Differentiating $\grave{\phi}$ and $\grave{\vartheta}^k$ yields

$$\partial_m\grave{\phi}=\partial_m\big(\Delta-\epsilon^2(\mathtt{a}+\mathtt{b}\mathtt{d})\big)^{-1}\tilde{f}(\epsilon,\acute{\phi},\acute{\vartheta}^k,\breve{\xi})+\epsilon\mathtt{b}\partial_m\partial_j(\Delta-\epsilon^2\mathtt{c})^{-1}\big(\Delta-\epsilon^2(\mathtt{a}+\mathtt{b}\mathtt{d})\big)^{-1}\tilde{g}^j(\epsilon,\acute{\phi},\acute{\vartheta}^k,\breve{\xi}), \quad (3.3.91)$$

$$\partial_m\grave{\vartheta}^j=\partial_m(\Delta-\epsilon^2\mathtt{c})^{-1}\tilde{g}^j(\epsilon,\acute{\phi},\acute{\vartheta}^k,\breve{\xi}). \quad (3.3.92)$$

Then taking L^6 norm of $\grave{\phi}$ and $\grave{\vartheta}^j$ and the H^s norm of $\partial_m\grave{\phi}$ and $\partial_m\grave{\vartheta}^j$, we obtain, with the help of Proposition 3.3.8, the estimates

$$\|\grave{\phi}\|_{R^{s+1}}\le\frac{-\mathtt{b}\mathtt{d}}{\mathtt{a}+\mathtt{b}\mathtt{d}}\|\acute{\phi}\|_{R^{s+1}}+\|\big(\Delta-\epsilon^2(\mathtt{a}+\mathtt{b}\mathtt{d})\big)^{-1}f\big(\epsilon,\acute{\phi},\acute{\vartheta}^k+\epsilon\mathtt{d}\partial^k(\Delta-\epsilon^2\mathtt{c})^{-1}\acute{\phi},\breve{\xi}\big)\|_{R^{s+1}}$$
$$+C\|(\Delta-\epsilon^2\mathtt{c})^{-1}\tilde{g}^j(\epsilon,\acute{\phi},\acute{\vartheta}^j,\breve{\xi})\|_{R^{s+1}}, \quad (3.3.93)$$

$$\|\grave{\vartheta}^j\|_{R^{s+1}}\lesssim\|(\Delta-\epsilon^2\mathtt{c})^{-1}\tilde{g}^k(\epsilon,\acute{\phi},\acute{\vartheta}^j,\breve{\xi})\|_{R^{s+1}}, \quad (3.3.94)$$

where, by (3.3.65), $-\mathtt{b}\mathtt{d}/(\mathtt{a}+\mathtt{b}\mathtt{d})<1$. By looking at the expressions (3.3.49)-(3.3.50), (3.3.62)-(3.3.63) and (3.3.68)-(3.3.69), it is not difficult to see, by making the identifications

$$\mathtt{H}_1(\epsilon,f_1,f_2)=\breve{\mathscr{S}}_1(\epsilon,\breve{\mathfrak{u}}^{\mu\nu})+\breve{\mathscr{S}}_2^j(\epsilon,\breve{\mathfrak{u}}^{\mu\nu}),\quad f_3=E^2(1)\frac{\Lambda}{3}\breve{\mathfrak{u}}^{ij}+\epsilon E^2(1)\breve{\mathfrak{u}}^{i0}\breve{\mathfrak{u}}^{0j},\quad f_4=\frac{2\Lambda}{3}E^2(1)\delta\breve{\rho},$$

$$f_{11} = -E^2(1)\frac{\Lambda}{3}\breve{\mathfrak{u}}^{ji}_0, \quad f_{12} = 2E^2(1)\sqrt{\frac{\Lambda}{3}}\breve{\rho}\breve{z}^j, \quad \mathtt{H}_7(\epsilon, f_7, \partial_i f_8) = E^2(1)\breve{\mathfrak{u}}^{00}\partial_i\breve{\mathfrak{u}}^{ij}_0, \quad f_9 = \alpha\breve{\mathfrak{u}}^{kl},$$
$$\mathtt{H}_0(\epsilon, f_0, f_8) = \breve{\mathscr{R}}_1(\epsilon, \epsilon\breve{\mathfrak{u}}^{\mu\nu}, \epsilon\breve{\mathfrak{u}}^{ij}_0, D\breve{\mathfrak{u}}^{\mu\nu}, \breve{\mathfrak{u}}^{ij}_0) + \breve{\mathscr{R}}^j_2(\epsilon, \epsilon\breve{\mathfrak{u}}^{\mu\nu}, \epsilon\breve{\mathfrak{u}}^{ij}_0, D\breve{\mathfrak{u}}^{\mu\nu}, \breve{\mathfrak{u}}^{ij}_0),$$
$$\mathtt{H}_5(\epsilon, f_5, \partial_i\partial_j f_6) = E^2(1)(\breve{\mathfrak{u}}^{00}\partial_i\partial_j\breve{\mathfrak{u}}^{ij} + \breve{\mathfrak{u}}^{ij}\partial_i\partial_j\breve{\mathfrak{u}}^{00} + \breve{\mathfrak{u}}^{0i}\partial_i\partial_k\breve{\mathfrak{u}}^{kj} + \breve{\mathfrak{u}}^{kl}\partial_k\partial_l\breve{\mathfrak{u}}^{0j})$$

and

$$f_{10} = \beta\breve{\mathfrak{u}}^{kl}_0 + \sigma D\breve{\mathfrak{u}}^{kl} + \breve{\mathscr{Q}}_1 + \breve{\mathscr{Q}}^j_2 + \epsilon(\breve{\mathscr{B}}_1 + \breve{\mathscr{B}}^j_2) + (\breve{\mathscr{F}}_1 + \epsilon^2\breve{\mathscr{F}}^j_2) + (\epsilon^2\breve{\mathscr{G}}_1 + \epsilon\breve{\mathscr{G}}^j_2)$$

for appropriate constants α, β and σ, that we can use Lemma 3.3.11 to estimate the terms $\|\big(\Delta - \epsilon^2(\mathtt{a}+\mathtt{bd})\big)^{-1} f\big(\epsilon, \acute{\phi}, \acute{\vartheta}^k + \epsilon\mathtt{d}\partial^k(\Delta - \epsilon^2\mathtt{c})^{-1}\acute{\phi}, \breve{\xi}\big)\|_{R^{s+1}}$ and $\|(\Delta - \epsilon^2\mathtt{c})^{-1}\tilde{g}^j(\epsilon, \acute{\phi}, \acute{\vartheta}^j, \breve{\xi})\|_{R^{s+1}}$ that appear on the right hand side of (3.3.93) and (3.3.94). Doing so, we find, with the help of Theorems C.2.2.(3) and C.2.7, that

$$\begin{aligned}\|\grave{\phi}\|_{R^{s+1}} \leq& \Big(\frac{-\mathtt{bd}}{\mathtt{a}+\mathtt{bd}} + C(l,r,\epsilon)\epsilon\Big)\|\acute{\phi}\|_{R^{s+1}} + C(l,r,\epsilon)\epsilon(\|\acute{\vartheta}^j\|_{R^{s+1}} + \|\breve{\xi}\|_s) + \frac{2\Lambda}{3}E^2(1)\|\delta\breve{\rho}\|_{L^{6/5}\cap K^s} \\ \leq& \Big(\frac{-\mathtt{bd}}{\mathtt{a}+\mathtt{bd}} + C(l,r,\epsilon)\epsilon\Big)l + \Big(\frac{2\Lambda}{3}E^2(1) + C(l,r,\epsilon)\epsilon\Big)r\end{aligned}$$

and

$$\|\grave{\vartheta}^k\|_{R^{s+1}} \leq C(l,r,\epsilon)\epsilon(\|\acute{\vartheta}^j\|_{R^{s+1}} + \|\acute{\phi}\|_{R^{s+1}} + \|\breve{\xi}\|_s) \leq C(l,r,\epsilon)\epsilon l.$$

From this we see that

$$\|(\grave{\phi}, \grave{\vartheta}^k)\|_{R^{s+1}}\big|_{\epsilon=0} = \big(\|\grave{\phi}\|_{R^{s+1}} + \|\grave{\vartheta}^k\|_{R^{s+1}}\big)\big|_{\epsilon=0} \leq \Big(\frac{-\mathtt{bd}}{\mathtt{a}+\mathtt{bd}}\Big)\Big|_{\epsilon=0} l + \frac{2\Lambda}{3}E^2(1)\big|_{\epsilon=0} r,$$

which, recalling that $-\mathtt{bd}/(\mathtt{a}+\mathtt{bd}) < 1$ by (3.3.65), we can satisfy

$$\|(\grave{\phi}, \grave{\vartheta}^k)\|_{R^{s+1}}\big|_{\epsilon=0} < l$$

by choosing l so that

$$l > \Big(\frac{2\Lambda E^2(1)(\mathtt{a}+\mathtt{bd})}{3(\mathtt{a}+2\mathtt{bd})}\Big)\Big|_{\epsilon=0} r = \frac{2\Lambda}{3}\frac{7 - 4\mathring{\Omega}(1) - 2\mathring{\Omega}^2(1)}{7 - 2\mathring{\Omega}(1) - 4\mathring{\Omega}^2(1)}\mathring{E}^2(1) r.$$

It then follows from the continuous dependence of the constants on ϵ that there exists an $\epsilon_0 = \epsilon_0(l,r)$ such that $\|(\grave{\phi}, \grave{\vartheta}^k)\|_{R^{s+1}} < l$ for all $\epsilon \in (0, \epsilon_0)$, or in other words, $\mathbf{G}$ maps the closed ball $\overline{B_l\big(R^{s+1}(\mathbb{R}^3, \mathbb{R}^4)\big)}$ to itself for all $\epsilon \in (0, \epsilon_0)$.

Due to the linearity of Yukawa potential operator, derivatives and the Riesz transform, calculation similar to those used to derive (3.3.93)-(3.3.94) show that

$$\begin{aligned}\|\grave{\phi}_1 - \grave{\phi}_2\|_{R^{s+1}} \leq& \|\big(\Delta - \epsilon^2(\mathtt{a}+\mathtt{bd})\big)^{-1}\big[\epsilon^2\big(\mathtt{A}^0(\epsilon, \acute{\phi}_1, \acute{\psi}^j_1, \breve{\xi}) - \mathtt{A}^0(\epsilon, \acute{\phi}_2, \acute{\psi}^j_2, \breve{\xi})\big)\big]\|_{R^{s+1}} \\ &+ \frac{-\mathtt{bd}}{\mathtt{a}+\mathtt{bd}}\|\acute{\phi}_1 - \acute{\phi}_2\|_{R^{s+1}} + \|(\Delta - \epsilon^2 c)^{-1}\epsilon^2\big(\mathtt{A}^j(\epsilon, \acute{\phi}_1, \acute{\psi}^j_1, \breve{\xi}) - \mathtt{A}^j(\epsilon, \acute{\phi}_2, \acute{\psi}^j_2, \breve{\xi})\big)\|_{R^{s+1}},\end{aligned} \tag{3.3.95}$$

$$\|\grave{\vartheta}^j_1 - \grave{\vartheta}^j_2\|_{R^{s+1}} \lesssim \|(\Delta - \epsilon^2\mathtt{c})^{-1}\epsilon^2\big(\mathtt{A}^j(\epsilon, \acute{\phi}_1, \acute{\psi}^j_1, \breve{\xi}) - \mathtt{A}^j(\epsilon, \acute{\phi}_2, \acute{\psi}^j_2, \breve{\xi})\big)\|_{R^{s+1}}, \tag{3.3.96}$$

where $-\mathtt{bd}/(\mathtt{a}+\mathtt{bd}) < 1$ by (3.3.65). Defining maps $\mathtt{B}^\mu(\epsilon, \acute{\phi}_2, \acute{\psi}^j_2, \acute{\phi}_1 - \acute{\phi}_2, \acute{\psi}^j_1 - \acute{\psi}^j_2, \breve{\xi})$ by

$$\mathtt{B}^\mu(\epsilon, \acute{\phi}_2, \acute{\psi}^j_2, \acute{\phi}_1 - \acute{\phi}_2, \acute{\psi}^j_1 - \acute{\psi}^j_2, \breve{\xi}) = \epsilon^2\big(\mathtt{A}^\mu(\epsilon, \acute{\phi}_1, \acute{\psi}^j_1, \breve{\xi}) - \mathtt{A}^\mu(\epsilon, \acute{\phi}_2, \acute{\psi}^j_2, \breve{\xi})\big),$$

which we note are analytic in all variables and vanish to first order in $(\acute{\phi}_1 - \acute{\phi}_2, \acute{\psi}^j_1 - \acute{\psi}^j_2)$, we can use Lemma 3.3.11 in a similar fashion as above, although this time with $f_4 = f_{11} = f_{12} = f_9 = 0$, to obtain from (3.3.95)-(3.3.96) the estimate

$$\|(\grave{\phi}_1-\grave{\phi}_2,\grave{\psi}^j_1-\grave{\psi}^j_2)\|_{R^{s+1}}\leq k\|(\acute{\phi}_1-\acute{\phi}_2,\acute{\psi}^j_1-\acute{\psi}^j_2)\|_{R^{s+1}}$$

for all $(\acute{\phi}_a,\acute{\psi}^j_a)\in\overline{B_l\big(R^{s+1}(\mathbb{R}^3,\mathbb{R}^4)\big)}$, $a=1,2$, where $k=\max\{C(l,r,\epsilon)\epsilon,-\mathtt{bd}/(\mathtt{a}+\mathtt{bd})+C(l,r,\epsilon)\epsilon\}$. Since $0<-\mathtt{bd}/(\mathtt{a}+\mathtt{bd})<1$, it is clear that by shrinking ϵ_0, if necessary, that we can arrange $k\in(0,1)$ for all $\epsilon\in(0,\epsilon_0)$. □

Existence

We now use the contraction map **G** to establish the existence of 1-parameter families of initial data that solve the constraint equations.

Remark 3.3.13. All solutions in this article, whether they are solutions of the constraint equations or the evolution equations, depend on the singular parameter ϵ and the free data. Depending on context and what we want to emphasize, we will either make the dependence on ϵ explicit by including an ϵ subscript, e.g. $u^{\mu\nu}_\epsilon$, or treat the ϵ dependence as implicit, e.g. $u^{\mu\nu}$. We will also use the subscript notation to make explicit the dependence of the solution on other initial data parameters, e.g. $u^{\mu\nu}_{\epsilon,\vec{\mathbf{y}}}$.

Theorem 3.3.14. *Suppose $s\in\mathbb{Z}_{\geq 3}$, $r>0$, $\mu(1)$ satisfies (3.3.64), and the free initial data $\breve{\xi}=(\breve{\mathfrak{u}}^{ij},\breve{\mathfrak{u}}^{ij}_0,\breve{z}^k,\delta\breve{\rho})$ is bounded by*

$$\|\breve{\xi}\|_s=\|\breve{\mathfrak{u}}^{ij}\|_{R^{s+1}}+\|\breve{\mathfrak{u}}^{ij}_0\|_{H^s}+\|\delta\breve{\rho}\|_{L^{\frac{6}{5}}\cap K^s}+\|\breve{z}^j\|_{L^{\frac{6}{5}}\cap K^s}\leq r.$$

Then there exists an $\epsilon_0>0$ and a family of one parameter maps $(\breve{\mathfrak{u}}^{0\mu}_\epsilon,\breve{\mathfrak{u}}^{0\mu}_{0,\epsilon})\in R^{s+1}(\mathbb{R}^3,\mathbb{R}^4)\times R^s(\mathbb{R}^3,\mathbb{R}^4)$, $0<\epsilon<\epsilon_0$, such that

$$\underline{\hat{\mathfrak{u}}^{\mu\nu}_\epsilon}|_\Sigma=\begin{pmatrix}\breve{\mathfrak{u}}^{00}_\epsilon & \breve{\mathfrak{u}}^{0j}_\epsilon\\ \breve{\mathfrak{u}}^{i0}_\epsilon & \epsilon\breve{\mathfrak{u}}^{ij}\end{pmatrix}\quad\text{and}\quad\underline{\hat{\mathfrak{u}}^{\mu\nu}_{0,\epsilon}}|_\Sigma=\begin{pmatrix}\breve{\mathfrak{u}}^{00}_{0,\epsilon} & \breve{\mathfrak{u}}^{0j}_{0,\epsilon}\\ \breve{\mathfrak{u}}^{i0}_{0,\epsilon} & \breve{\mathfrak{u}}^{ij}_0\end{pmatrix},$$

where the $\underline{\breve{\mathfrak{u}}^{0\mu}_{0,\epsilon}}$ are determined by (3.3.14)-(3.3.15), solve the constraint equations (3.3.3)-(3.3.5) for $0<\epsilon<\epsilon_0$. Moreover, $\{\breve{\mathfrak{u}}^{00}_\epsilon,\breve{\mathfrak{u}}^{0j}_\epsilon\}$ and $\{\partial_m\breve{\mathfrak{u}}^{00}_\epsilon,\partial_m\breve{\mathfrak{u}}^{0j}_\epsilon\}$ can be expanded as

$$\breve{\mathfrak{u}}^{00}_\epsilon=\frac{2\Lambda}{3}E^2(1)\big(\Delta-\epsilon^2(\mathtt{a}+\mathtt{bd})\big)^{-1}\delta\breve{\rho}+\mathcal{R}(\epsilon,\breve{\mathfrak{u}}^{kl},\breve{\mathfrak{u}}^{kl}_0,\delta\breve{\rho},\breve{z}^l),\tag{3.3.97}$$

$$\breve{\mathfrak{u}}^{0j}_\epsilon=\frac{2\Lambda}{3}\epsilon\mathtt{d}E^2(1)\partial^j(\Delta-\epsilon^2\mathtt{c})^{-1}\big(\Delta-\epsilon^2(\mathtt{a}+\mathtt{bd})\big)^{-1}\delta\breve{\rho}+\mathcal{R}^j(\epsilon,\breve{\mathfrak{u}}^{kl},\breve{\mathfrak{u}}^{kl}_0,\delta\breve{\rho},\breve{z}^l)\tag{3.3.98}$$

and

$$\partial_m\breve{\mathfrak{u}}^{00}_\epsilon=\frac{2\Lambda}{3}E^2(1)\partial_m\Delta^{-1}\delta\breve{\rho}+\epsilon\mathcal{S}_m(\epsilon,\breve{\mathfrak{u}}^{kl},\breve{\mathfrak{u}}^{kl}_0,\delta\breve{\rho},\breve{z}^l)\quad\text{and}\quad\partial_m\breve{\mathfrak{u}}^{0j}_\epsilon=\epsilon\mathcal{S}^j_m(\epsilon,\breve{\mathfrak{u}}^{kl},\breve{\mathfrak{u}}^{kl}_0,\delta\breve{\rho},\breve{z}^l),\tag{3.3.99}$$

respectively, where the remainder terms are bounded by

$$\|\mathcal{R}(\epsilon,\breve{\mathfrak{u}}^{kl},\breve{\mathfrak{u}}^{kl}_0,\delta\breve{\rho},\breve{z}^l)\|_{R^{s+1}}+\|\mathcal{R}^j(\epsilon,\breve{\mathfrak{u}}^{kl},\breve{\mathfrak{u}}^{kl}_0,\delta\breve{\rho},\breve{z}^l)\|_{R^{s+1}}+\|\mathcal{S}_m(\epsilon,\breve{\mathfrak{u}}^{kl},\breve{\mathfrak{u}}^{kl}_0,\delta\breve{\rho},\breve{z}^l)\|_{R^s}+\|\mathcal{S}^j_m(\epsilon,\breve{\mathfrak{u}}^{kl},\breve{\mathfrak{u}}^{kl}_0,\delta\breve{\rho},\breve{z}^l)\|_{R^s}\lesssim\|\breve{\xi}\|_s,\tag{3.3.100}$$

and $\{\breve{\mathfrak{u}}^{0\mu}_\epsilon,\breve{\mathfrak{u}}^{0\mu}_{0,\epsilon}\}$ satisfy the uniform estimates

$$\|\breve{\mathfrak{u}}^{0\mu}_\epsilon\|_{R^{s+1}}+\|\breve{\mathfrak{u}}^{0\mu}_{0,\epsilon}\|_{R^s}\lesssim\|\breve{\xi}\|_s\tag{3.3.101}$$

for $\epsilon\in(0,\epsilon_0)$.

Proof. Given $r>0$, choose $l>\frac{2\Lambda}{3}\frac{7-4\mathring{\Omega}(1)-2\mathring{\Omega}^2(1)}{7-2\mathring{\Omega}(1)-4\mathring{\Omega}^2(1)}\mathring{E}^2(1)r>0$ and let $\epsilon_0>0$ and $k\in(0,1)$ be as in Proposition 3.3.12. Since we know by Proposition 3.3.12 that **G** is a contraction mapping on

$\overline{B_l(R^{s+1}(\mathbb{R}^3,\mathbb{R}^4))}$, it follows from Banach's fixed point theorem that $\mathbf{G}$ has a unique fixed point $\acute{\iota}_* = (\phi,\vartheta^k) = (\breve{\mathtt{u}}^{00}_\epsilon, \breve{\mathtt{u}}^{0k}_\epsilon - \epsilon d\partial^j(\Delta-\epsilon^2\mathtt{c})^{-1}\breve{\mathtt{u}}^{00}_\epsilon) \in \overline{B_l(R^{s+1}(\mathbb{R}^3,\mathbb{R}^4))}$, that is, $\mathbf{G}(\acute{\iota}_*) = \acute{\iota}_*$. Furthermore, we know that the successive approximations $\acute{\iota}_m = \mathbf{G}^m(\acute{\iota}_0)$ starting from any seed $\acute{\iota}_0 \in \overline{B_l(R^{s+1}(\mathbb{R}^3,\mathbb{R}^4))}$ converge to $\acute{\iota}_*$ and satisfy

$$\|\acute{\iota}_0 - \acute{\iota}_*\|_{R^{s+1}} \leq \frac{1}{1-k}\|\mathbf{G}(\acute{\iota}_0) - \acute{\iota}_0\|_{R^{s+1}}. \tag{3.3.102}$$

In the following, we consider the seed $\acute{\iota}_0 = (\phi_{\text{seed}}, \vartheta^j_{\text{seed}})$ defined by

$$\phi_{\text{seed}} = \frac{2\Lambda}{3}E^2(1)\big(\Delta - \epsilon^2(\mathtt{a}+\mathtt{bd})\big)^{-1}\delta\breve{\rho} \quad \text{and} \quad \vartheta^j_{\text{seed}} = 0.$$

Since $\delta\breve{\rho} \in L^{\frac{6}{5}} \cap K^s$, it follows from Proposition 3.3.6.(2), (3.3.58)-(3.3.59) and (3.3.65) that

$$\|\big(\Delta - \epsilon^2(\mathtt{a}+\mathtt{bd})\big)^{-1}\delta\breve{\rho}\|_{L^6} \lesssim \|\delta\breve{\rho}\|_{L^{\frac{6}{5}}\cap K^s}$$

and

$$\begin{aligned}\|\partial_m\phi_{\text{seed}}\|_{H^s} \lesssim & \|\partial_m\big(\Delta - \epsilon^2(\mathtt{a}+\mathtt{bd})\big)^{-1}\delta\breve{\rho}\|_{H^s} \\ \lesssim & \|D\big(\Delta - \epsilon^2(\mathtt{a}+\mathtt{bd})\big)^{-1}\delta\breve{\rho}\|_{L^2} + \|D^2\big(\Delta - \epsilon^2(\mathtt{a}+\mathtt{bd})\big)^{-1}\delta\breve{\rho}\|_{H^{s-1}} \\ \lesssim & \|\delta\breve{\rho}\|_{L^{\frac{6}{5}}} + \|\delta\breve{\rho}\|_{H^{s-1}} \lesssim \|\delta\breve{\rho}\|_{L^{\frac{6}{5}}\cap K^s}\end{aligned}$$

from which we deduce that $\acute{\iota}_0 \in \overline{B_l(R^{s+1}(\mathbb{R}^3,\mathbb{R}^4))}$ provided l is chosen large enough.

Next, we estimate $\|\mathbf{G}(\acute{\iota}_0) - \acute{\iota}_0\|_{R^{s+1}}$. Before doing so, we let $\mathbf{G}(\acute{\iota}) = (\mathbf{G}^0(\acute{\iota}), \mathbf{G}^j(\acute{\iota}))$ denote the decomposition of $\mathbf{G}$ into components, where $\acute{\iota} = (\acute{\phi}, \acute{\vartheta}^j)$, and the components $\mathbf{G}^0(\acute{\iota})$ and $\mathbf{G}^j(\acute{\iota})$ are given by the formulas (3.3.71) and (3.3.72), respectively. We then find via a direct calculation involving (3.3.62) and (3.3.71)-(3.3.72) that difference $\mathbf{G}(\acute{\iota}_0) - \acute{\iota}_0$ is given by

$$\begin{aligned}\mathbf{G}^0(\phi_{\text{seed}}, \vartheta^j_{\text{seed}}) - \phi_{\text{seed}} = & \big(\Delta - \epsilon^2(\mathtt{a}+\mathtt{bd})\big)^{-1}\Big[\epsilon E^2(1)\frac{\Lambda}{3}\partial_i\partial_j\breve{\mathtt{u}}^{ij} + \epsilon^2\mathtt{A}^0(\epsilon, \phi_{\text{seed}}, \psi^j_{\text{seed}}, \breve{\xi}) \\ & + \epsilon^4\mathtt{bcd}(\Delta-\epsilon^2\mathtt{c})^{-1}\phi_{\text{seed}}\Big] + \epsilon\mathtt{b}\partial_j(\Delta-\epsilon^2\mathtt{c})^{-1}\big(\Delta - \epsilon^2(\mathtt{a}+\mathtt{bd})\big)^{-1}\tilde{g}^j(\epsilon, \phi_{\text{seed}}, \vartheta^k_{\text{seed}})\end{aligned}$$

and

$$\mathbf{G}^j(\phi_{\text{seed}}, \vartheta^k_{\text{seed}}) - \vartheta^j_{\text{seed}} = (\Delta-\epsilon^2\mathtt{c})^{-1}\tilde{g}^k(\epsilon, \phi_{\text{seed}}, \vartheta^j_{\text{seed}})$$

where

$$\psi^j_{\text{seed}} := \vartheta^j_{\text{seed}} + \epsilon\mathtt{d}\partial^j(\Delta-\epsilon^2\mathtt{c})^{-1}\phi_{\text{seed}} = \epsilon\frac{2\Lambda}{3}E^2(1)\mathtt{d}\partial^j(\Delta-\epsilon^2\mathtt{c})^{-1}\big(\Delta - \epsilon^2(\mathtt{a}+\mathtt{bd})\big)^{-1}\delta\breve{\rho}.$$

By similar arguments used to derive (3.3.93)-(3.3.94), we can estimate $\|\mathbf{G}(\acute{\iota}_0) - \acute{\iota}_0\|_{R^{s+1}}$ using Lemma 3.3.11, with f_4 set to zero, to get

$$\|\mathbf{G}(\acute{\iota}_0) - \acute{\iota}_0\|_{R^{s+1}} \lesssim \|\breve{\mathtt{u}}^{ij}\|_{R^{s+1}} + \|\breve{\mathtt{u}}^{ij}_0\|_{H^s} + \|\delta\breve{\rho}\|_{L^{\frac{6}{5}}\cap K^s} + \|\breve{z}^j\|_{L^{\frac{6}{5}}\cap K^s} = \|\breve{\xi}\|_s,$$

from which it follows that

$$\|\acute{\iota}_0 - \acute{\iota}_*\|_{R^{s+1}} \lesssim \|\breve{\xi}\|_s$$

by (3.3.102). Since $\acute{\iota}_* = (\breve{\mathtt{u}}^{00}_\epsilon, \breve{\mathtt{u}}^{0k}_\epsilon - \epsilon d\partial^j(\Delta-\epsilon^2\mathtt{c})^{-1}\breve{\mathtt{u}}^{00}_\epsilon)$ and $\|\acute{\iota}_0\|_{R^{s+1}} \lesssim \|\breve{\xi}\|_s$, it is clear that (3.3.97)–(3.3.98) follows from the above estimate.

In order to bound $\partial_m \breve{u}^{0j}_\epsilon$, we first note that the estimate

$$\begin{aligned}\|(\Delta-\epsilon^2\lambda)^{-1}\delta\breve{\rho}\|_{R^s} \lesssim & \|(\Delta-\epsilon^2\lambda)^{-1}\delta\breve{\rho}\|_{L^6}+\sum_l\|(\Delta-\epsilon^2\lambda)^{-1}\partial_l\delta\breve{\rho}\|_{L^2}+\sum_{k,l}\|(\Delta-\epsilon^2\lambda)^{-1}\partial_l\partial_k\delta\breve{\rho}\|_{H^{s-2}}\\ \lesssim & \|\delta\breve{\rho}\|_{L^{\frac{6}{5}}}+\|\delta\breve{\rho}\|_{H^{s-2}},\end{aligned}$$

which holds for any constant $\lambda\geq 0$, follows from (3.3.58), (3.3.59), and Propositions 3.3.6.(1) and 3.3.6.(2). From this, (3.3.66) and (3.3.91)–(3.3.92), we then get from an application of Propositions 3.3.5, 3.3.6, 3.3.8 and B.0.3, and Theorem B.0.1 the estimates

$$\begin{aligned}&\left\|\partial_m\breve{u}^{00}_\epsilon-\frac{2\Lambda}{3}E^2(1)\partial_m\Delta^{-1}\delta\breve{\rho}\right\|_{R^s}=\left\|\partial_m\phi-\frac{2\Lambda}{3}E^2(1)\partial_m\Delta^{-1}\delta\breve{\rho}\right\|_{R^s}\\
\lesssim&\|\partial_m\big(\Delta-\epsilon^2(\mathtt{a}+\mathtt{bd})\big)^{-1}f(\epsilon,\phi,\psi^k,\breve{\xi})-\frac{2\Lambda}{3}E^2(1)\partial_m\Delta^{-1}\delta\breve{\rho}\|_{R^s}\\
&+\|\partial_m\big(\Delta-\epsilon^2(\mathtt{a}+\mathtt{bd})\big)^{-1}\epsilon^4\mathtt{bcd}(\Delta-\epsilon^2\mathtt{c})^{-1}\phi\|_{R^s}\\
&+\|\epsilon\mathtt{b}\partial_m\partial_j(\Delta-\epsilon^2\mathtt{c})^{-1}\big(\Delta-\epsilon^2(\mathtt{a}+\mathtt{bd})\big)^{-1}\tilde{g}^j(\epsilon,\phi,\vartheta^k,\breve{\xi})\|_{R^s}\\
\lesssim&\Big\|\partial_m\big(\Delta-\epsilon^2(\mathtt{a}+\mathtt{bd})\big)^{-1}\Big(f(\epsilon,\phi,\psi^k,\breve{\xi})-\frac{2\Lambda}{3}E^2(1)\delta\breve{\rho}\Big)\\
&+\epsilon^2(\mathtt{a}+\mathtt{bd})\frac{2\Lambda}{3}E^2(1)\partial_m\big(\Delta-\epsilon^2(\mathtt{a}+\mathtt{bd})\big)^{-1}\Delta^{-1}\delta\breve{\rho}\Big\|_{R^s}\\
&+\epsilon\|\phi\|_{R^s}+\|(\Delta-\epsilon^2\mathtt{c})^{-1}\tilde{g}^j(\epsilon,\phi,\vartheta^k,\breve{\xi})\|_{R^{s+1}}\\
\lesssim&\Big\|\big(\Delta-\epsilon^2(\mathtt{a}+\mathtt{bd})\big)^{-1}\Big(f(\epsilon,\phi,\psi^k,\breve{\xi})-\frac{2\Lambda}{3}E^2(1)\delta\breve{\rho}\Big)\Big\|_{R^{s+1}}+\epsilon\|\delta\breve{\rho}\|_{L^{\frac{6}{5}}}+\epsilon\|\delta\breve{\rho}\|_{H^{s-2}}+\epsilon\|\phi\|_{R^s}\\
&+\|(\Delta-\epsilon^2\mathtt{c})^{-1}\tilde{g}^j(\epsilon,\phi,\vartheta^k,\breve{\xi})\|_{R^{s+1}}\end{aligned}$$

and

$$\|\partial_m\breve{u}^{0j}_\epsilon\|_{R^s}=\|\partial_m\psi^j\|_{R^s}\lesssim\|\partial_m\vartheta^j\|_{R^s}+\|\epsilon\mathtt{d}\partial_m\partial^j(\Delta-\epsilon^2\mathtt{c})^{-1}\phi\|_{R^s}\lesssim\|(\Delta-\epsilon^2\mathtt{c})^{-1}\tilde{g}^j(\epsilon,\phi,\vartheta^k,\breve{\xi})\|_{R^{s+1}}+\epsilon\|\phi\|_{R^s}$$

Finally, (3.3.99), (3.3.100) and (3.3.101) follow directly from Lemma 3.3.11 and (3.3.97)–(3.3.98). This completes the proof of the theorem. □

3.3.6 Bounding initial evolution variables

For the evolution problem, we will need to translate the ϵ-independent bound on the initial data from Theorem 3.3.14 to an ϵ-independent bound on the initial data $\hat{\mathbf{U}}|_\Sigma$ for the formulation (3.2.69) of the reduced conformal Einstein-Euler equations. The following proposition serves this purpose.

Proposition 3.3.15. *Suppose that the hypotheses of Theorem 3.3.14 hold and that $\breve{\xi}$, $\|\breve{\xi}\|_s$, and the maps $\{\breve{u}^{0\mu},\breve{u}^{0\mu}_0\}$ are as given in Theorem 3.3.14. Then on the initial hypersurface Σ, the collection*

$$\{u^{\mu\nu}_\epsilon,u^{ij}_{\gamma,\epsilon},u^{0\mu}_{i,\epsilon},u^{0\mu}_{0,\epsilon},u_\epsilon,u_{\gamma,\epsilon},z_{j,\epsilon},\delta\zeta_\epsilon\}$$

of gravitational and matter fields can be written as

$$u^{0\mu}_\epsilon|_\Sigma=\epsilon\mathcal{S}^\mu(\epsilon,\breve{u}^{kl},\breve{u}^{kl}_0,\delta\breve{\rho},\breve{z}^l), \tag{3.3.103}$$

$$u_\epsilon|_\Sigma=\epsilon^2\frac{2\Lambda}{9}E^2(1)\breve{u}^{ij}\delta_{ij}+\epsilon^3\mathcal{S}(\epsilon,\breve{u}^{kl},\breve{u}^{kl}_0,\delta\breve{\rho},\breve{z}^l), \tag{3.3.104}$$

$$u^{ij}_\epsilon|_\Sigma=\epsilon^2\left(\breve{u}^{ij}-\frac{1}{3}\breve{u}^{kl}\delta_{kl}\delta^{ij}\right)+\epsilon^3\mathcal{S}^{ij}(\epsilon,\breve{u}^{kl},\breve{u}^{kl}_0,\delta\breve{\rho},\breve{z}^l), \tag{3.3.105}$$

$$z_{j,\epsilon}|_\Sigma=E^2(1)\delta_{kl}\breve{z}^k+\epsilon\mathcal{R}_j(\epsilon,\breve{u}^{kl},\breve{u}^{kl}_0,\delta\breve{\rho},\breve{z}^l), \tag{3.3.106}$$

$$\delta\zeta_\epsilon|_\Sigma=\frac{1}{1+\epsilon^2K}\ln\left(1+\frac{\delta\breve{\rho}}{\mu(1)}\right), \tag{3.3.107}$$

$$u^{0\mu}_{i,\epsilon}|_\Sigma=\frac{\Lambda}{3}E^2(1)\delta^\mu_0\partial_i\Delta^{-1}\delta\breve{\rho}+\epsilon\mathcal{S}^\mu_i(\epsilon,\breve{u}^{kl},\breve{u}^{kl}_0,\delta\breve{\rho},\breve{z}^l), \tag{3.3.108}$$

$$u_{0,\epsilon}^{0\mu}|_\Sigma = \epsilon\mathcal{S}_0^\mu(\epsilon, \breve{\mathfrak{u}}^{kl}, \breve{\mathfrak{u}}_0^{kl}, \delta\breve{\rho}, \breve{z}^l), \tag{3.3.109}$$

$$u_{\gamma,\epsilon}|_\Sigma = \epsilon\mathcal{S}_\gamma(\epsilon, \breve{\mathfrak{u}}^{kl}, \breve{\mathfrak{u}}_0^{kl}, \delta\breve{\rho}, \breve{z}^l) \tag{3.3.110}$$

and

$$u_{\gamma,\epsilon}^{ij}|_\Sigma = \epsilon\mathcal{S}_\gamma^{ij}(\epsilon, \breve{\mathfrak{u}}^{kl}, \breve{\mathfrak{u}}_0^{kl}, \delta\breve{\rho}, \breve{z}^l) \tag{3.3.111}$$

where the remainder terms satisfy bounds of the form

$$\begin{aligned}
&\|\mathcal{S}^\mu(\epsilon, \breve{\mathfrak{u}}^{kl}, \breve{\mathfrak{u}}_0^{kl}, \delta\breve{\rho}, \breve{z}^l)\|_{R^{s+1}} + \|\mathcal{S}(\epsilon, \breve{\mathfrak{u}}^{kl}, \breve{\mathfrak{u}}_0^{kl}, \delta\breve{\rho}, \breve{z}^l)\|_{R^{s+1}} + \|\mathcal{S}^{ij}(\epsilon, \breve{\mathfrak{u}}^{kl}, \breve{\mathfrak{u}}_0^{kl}, \delta\breve{\rho}, \breve{z}^l)\|_{R^{s+1}} \\
&+\|\mathcal{R}_j(\epsilon, \breve{\mathfrak{u}}^{kl}, \breve{\mathfrak{u}}_0^{kl}, \delta\breve{\rho}, \breve{z}^l)\|_{R^{s+1}} + \|\mathcal{S}_i^\mu(\epsilon, \breve{\mathfrak{u}}^{kl}, \breve{\mathfrak{u}}_0^{kl}, \delta\breve{\rho}, \breve{z}^l)\|_{R^{s+1}} + \|\mathcal{S}_0^\mu(\epsilon, \breve{\mathfrak{u}}^{kl}, \breve{\mathfrak{u}}_0^{kl}, \delta\breve{\rho}, \breve{z}^l)\|_{R^{s+1}} \\
&+\|\mathcal{S}_\gamma(\epsilon, \breve{\mathfrak{u}}^{kl}, \breve{\mathfrak{u}}_0^{kl}, \delta\breve{\rho}, \breve{z}^l)\|_{R^{s+1}} + \|\mathcal{S}_\gamma^{ij}(\epsilon, \breve{\mathfrak{u}}^{kl}, \breve{\mathfrak{u}}_0^{kl}, \delta\breve{\rho}, \breve{z}^l)\|_{R^{s+1}} \lesssim \|\breve{\xi}\|_s.
\end{aligned}$$

Moreover, the estimates

$$\|u_\epsilon^{\mu\nu}|_\Sigma\|_{R^{s+1}} + \|u_\epsilon|_\Sigma\|_{R^{s+1}} + \|u_{i,\epsilon}^{0k}|_\Sigma\|_{R^s} + \|u_{0,\epsilon}^{0\mu}|_\Sigma\|_{R^s} + \|u_{\mu,\epsilon}|_\Sigma\|_{R^s} + \|u_{\mu,\epsilon}^{ij}|_\Sigma\|_{R^s} \lesssim \epsilon\|\breve{\xi}\|_s \tag{3.3.112}$$

and

$$\|u_{i,\epsilon}^{00}|_\Sigma\|_{R^s} + \|z_{j,\epsilon}|_\Sigma\|_{R^s} + \|\delta\zeta_\epsilon|_\Sigma\|_{L^{\frac{6}{5}}\cap K^s} \lesssim \|\breve{\xi}\|_s \tag{3.3.113}$$

hold uniformly for $\epsilon \in (0, \epsilon_0)$.

Proof. To start, we observe that (3.3.103)-(3.3.105) are a direct consequence of Lemma 3.3.2 and the expansions (3.3.97)-(3.3.98), while (3.3.106) follows from (3.2.11), (3.2.20), (3.2.22) and (3.3.103)-(3.3.105). Next, we deduce (3.3.107) directly from (3.1.49), (3.1.50) and (3.2.18), and we observe by (3.2.8), (3.3.1), (3.3.12), (3.3.99) and Lemma 3.3.1 that

$$u_{i,\epsilon}^{0\mu}|_\Sigma = \frac{1}{2}\delta_0^\mu\partial_i\breve{\mathfrak{u}}_\epsilon^{00} + \delta_k^\mu\partial_i\breve{\mathfrak{u}}_\epsilon^{0k} + \epsilon\mathcal{T}_i^\mu(\epsilon, \breve{\mathfrak{u}}^{kl}, \breve{\mathfrak{u}}_0^{kl}, \breve{\rho}_0, \breve{\nu}^l) = \frac{\Lambda}{3}E^2(1)\delta_0^\mu\partial_i\Delta^{-1}\delta\breve{\rho} + \epsilon\mathcal{S}_i^\mu(\epsilon, \breve{\mathfrak{u}}^{kl}, \breve{\mathfrak{u}}_0^{kl}, \delta\breve{\rho}, \breve{z}^l),$$

where $\mathcal{S}_i^\mu(\epsilon, 0, 0, 0, 0) = 0$, which gives (3.3.108). Furthermore, similar calculations using (3.2.8), (3.3.1), (3.3.8), Lemma 3.3.1 and Theorem 3.3.14 give (3.3.109). From (3.3.11), (3.3.103), (3.3.108), (3.3.109) and Theorem 3.3.14, we find, with the help of (3.2.9), that

$$u_{\beta,\epsilon} = 3u_\epsilon^{00}\delta_\beta^0 + u_{\beta,\epsilon}^{00} - \frac{1}{\epsilon}\frac{\Lambda}{3}\underline{\frac{1}{\alpha}\bar{\partial}_\beta\alpha}, \tag{3.3.114}$$

from which (3.3.110) follows via a straightforward calculation. We also find, with the help of (3.3.114), Lemma 3.3.1 and Theorem 3.3.14, that

$$u_{\gamma,\epsilon}^{ij}|_\Sigma = \frac{1}{\epsilon}\underline{\bar{\nabla}_\gamma(\alpha^{-1}\theta^{-1}\hat{g}^{ij})} = \epsilon\mathcal{S}_\gamma^{ij}(\epsilon, \breve{\mathfrak{u}}^{kl}, \breve{\mathfrak{u}}_0^{kl}, \breve{\rho}_0, \breve{\nu}^l),$$

where $\mathcal{S}^{ij}(\epsilon, 0, 0, 0, 0) = 0$, follows from (3.1.45) and (3.1.51), which establishes (3.3.111). Finally, it is not difficult to verify that the estimates (3.3.112) and (3.3.113) are a direct consequence of the expansions (3.3.103)-(3.3.111) and Theorem 3.3.14. □

3.3.7 Matter fluctuations away from homogeneity

As discussed in the introduction, we are interested in initial data where the density and velocity fluctuations away from homogeneity are of the form

$$\delta\breve{\rho}_{\epsilon,\vec{\mathbf{y}}}(\mathbf{x}) = \sum_{\lambda=1}^N \delta\breve{\rho}_\lambda\left(\mathbf{x} - \frac{\mathbf{y}_\lambda}{\epsilon}\right) \quad \text{and} \quad \breve{z}_{\epsilon,\vec{\mathbf{y}}}^j(\mathbf{x}) = \sum_{\lambda=1}^N \breve{z}_\lambda^j\left(\mathbf{x} - \frac{\mathbf{y}_\lambda}{\epsilon}\right), \tag{3.3.115}$$

where $\vec{\mathbf{y}} = (\mathbf{y}_1, \cdots, \mathbf{y}_N) \in \mathbb{R}^{3N}$. This fixes part of the free initial data. We will assume that the remainder of the free initial data $\{\breve{\mathfrak{u}}^{ij}_\epsilon, \breve{\mathfrak{u}}^{ij}_{0,\epsilon}\}$ is bounded as $\epsilon \searrow 0$ with the simplest choice being $\breve{\mathfrak{u}}^{ij}_\epsilon = \breve{\mathfrak{u}}^{ij}_{0,\epsilon} = 0$. Noting that the bounds

$$\|\delta\breve{\rho}_{\epsilon,\vec{\mathbf{y}}}\|_{L^{\frac{6}{5}}\cap K^s} \leq \sum_{\lambda=1}^{N} \|\delta\breve{\rho}_\lambda\|_{L^{\frac{6}{5}}\cap K^s} \quad \text{and} \quad \|\breve{z}^j_{\epsilon,\vec{\mathbf{y}}}\|_{L^{\frac{6}{5}}\cap K^s} \leq \sum_{\lambda=1}^{N} \|\breve{z}^j_\lambda\|_{L^{\frac{6}{5}}\cap K^s} \tag{3.3.116}$$

follow immediately from the triangle inequality and the translation invariance of the norms $L^{\frac{6}{5}} \cap K^s$, it is clear that we can apply Theorem 3.3.14 and Proposition 3.3.15 to this class of free initial data to obtain the following result.

Theorem 3.3.16. *Suppose $s \in \mathbb{Z}_{\geq 3}$, $r > 0$, $\epsilon_1 > 0$, $\vec{\mathbf{y}} = (\mathbf{y}_1, \cdots, \mathbf{y}_N) \in \mathbb{R}^{3N}$, $\breve{\mathfrak{u}}^{ij}_\epsilon \in R^{s+1}(\mathbb{R}^3, \mathbb{S}_3)$ and $\breve{\mathfrak{u}}^{ij}_{0,\epsilon} \in R^s(\mathbb{R}^3, \mathbb{S}_3) \cap L^2(\mathbb{R}^3, \mathbb{S}_3)$ for $\epsilon \in (0, \epsilon_1)$, $\delta\breve{\rho}_\lambda \in L^{\frac{6}{5}} \cap K^s(\mathbb{R}^3, \mathbb{R})$ and $\breve{z}^j_\lambda \in L^{\frac{6}{5}} \cap K^s(\mathbb{R}^3, \mathbb{R}^3)$ for $\lambda = 1, \cdots, N$, $\delta\breve{\rho}_{\epsilon,\vec{\mathbf{y}}}$ and $\breve{z}^j_{\epsilon,\vec{\mathbf{y}}}$ are defined by (3.3.115) and $\mu(1)$ satisfies (3.3.64). Then there exists a constant $\epsilon_0 \in (0, \epsilon_1)$ such that if the free initial data satisfies*

$$\|\breve{\xi}_\epsilon\|_s := \|\breve{\mathfrak{u}}^{ij}_\epsilon\|_{R^{s+1}} + \|\breve{\mathfrak{u}}^{ij}_{0,\epsilon}\|_{H^s} + \sum_{\lambda=1}^{N} \|\delta\breve{\rho}_\lambda\|_{L^{\frac{6}{5}}\cap K^s} + \sum_{\lambda=1}^{N} \|\breve{z}^j_\lambda\|_{L^{\frac{6}{5}}\cap K^s} \leq r, \quad 0 < \epsilon < \epsilon_0,$$

then there exists a family $(\epsilon, \vec{\mathbf{y}})$-dependent maps

$$\hat{\mathbf{U}}_{\epsilon,\vec{\mathbf{y}}}|_\Sigma = \{u^{\mu\nu}_{\epsilon,\vec{\mathbf{y}}}, u_{\epsilon,\vec{\mathbf{y}}}, u^{ij}_{\gamma,\epsilon,\vec{\mathbf{y}}}, u^{0\mu}_{i,\epsilon,\vec{\mathbf{y}}}, u^{0\mu}_{0,\epsilon,\vec{\mathbf{y}}}, u_{\gamma,\epsilon,\vec{\mathbf{y}}}, z_{j,\epsilon,\vec{\mathbf{y}}}, \delta\zeta_{\epsilon,\vec{\mathbf{y}}}\}|_\Sigma, \quad (\epsilon, \vec{\mathbf{y}}) \in (0, \epsilon_0) \times \mathbb{R}^{3N},$$

such that $\hat{\mathbf{U}}_{\epsilon,\vec{\mathbf{y}}}|_\Sigma \in X^s(\mathbb{R}^3)$, $\hat{\mathbf{U}}_{\epsilon,\vec{\mathbf{y}}}|_\Sigma$ determines a solution of the constraint equations (3.3.3)-(3.3.5), and the components of $\hat{\mathbf{U}}_{\epsilon,\vec{\mathbf{y}}}|_\Sigma$ can be expressed as

$$u^{0\mu}_{\epsilon,\vec{\mathbf{y}}}|_\Sigma = \epsilon\mathcal{S}^\mu(\epsilon, \breve{\mathfrak{u}}^{kl}_\epsilon, \breve{\mathfrak{u}}^{kl}_{0,\epsilon}, \delta\breve{\rho}_{\epsilon,\vec{\mathbf{y}}}, \breve{z}^l_{\epsilon,\vec{\mathbf{y}}}), \tag{3.3.117}$$

$$u_{\epsilon,\vec{\mathbf{y}}}|_\Sigma = \epsilon^2 \frac{2\Lambda}{9} E^2(1)\breve{\mathfrak{u}}^{ij}_\epsilon \delta_{ij} + \epsilon^3 \mathcal{S}(\epsilon, \breve{\mathfrak{u}}^{kl}_\epsilon, \breve{\mathfrak{u}}^{kl}_{0,\epsilon}, \delta\breve{\rho}_{\epsilon,\vec{\mathbf{y}}}, \breve{z}^l_{\epsilon,\vec{\mathbf{y}}}), \tag{3.3.118}$$

$$u^{ij}_{\epsilon,\vec{\mathbf{y}}}|_\Sigma = \epsilon^2 \left(\breve{\mathfrak{u}}^{ij}_\epsilon - \frac{1}{3}\breve{\mathfrak{u}}^{kl}_\epsilon \delta_{kl}\delta^{ij}\right) + \epsilon^3 \mathcal{S}^{ij}(\epsilon, \breve{\mathfrak{u}}^{kl}_\epsilon, \breve{\mathfrak{u}}^{kl}_{0,\epsilon}, \delta\breve{\rho}_{\epsilon,\vec{\mathbf{y}}}, \breve{z}^l_{\epsilon,\vec{\mathbf{y}}}), \tag{3.3.119}$$

$$z_{j,\epsilon,\vec{\mathbf{y}}}|_\Sigma = E^2(1)\delta_{kl}\breve{z}^k_{\epsilon,\vec{\mathbf{y}}} + \epsilon\mathcal{R}_j(\epsilon, \breve{\mathfrak{u}}^{kl}_\epsilon, \breve{\mathfrak{u}}^{kl}_{0,\epsilon}, \delta\breve{\rho}_{\epsilon,\vec{\mathbf{y}}}, \breve{z}^l_{\epsilon,\vec{\mathbf{y}}}), \tag{3.3.120}$$

$$\delta\zeta_{\epsilon,\vec{\mathbf{y}}}|_\Sigma = \frac{1}{1 + \epsilon^2 K} \ln\left(1 + \frac{\delta\breve{\rho}_{\epsilon,\vec{\mathbf{y}}}}{\mu(1)}\right), \tag{3.3.121}$$

$$u^{0\mu}_{i,\epsilon,\vec{\mathbf{y}}}|_\Sigma = \frac{\Lambda}{3} E^2(1)\delta^\mu_0 \partial_i \Delta^{-1}\delta\breve{\rho}_{\epsilon,\vec{\mathbf{y}}} + \epsilon\mathcal{S}^\mu_i(\epsilon, \breve{\mathfrak{u}}^{kl}_\epsilon, \breve{\mathfrak{u}}^{kl}_{0,\epsilon}, \delta\breve{\rho}_{\epsilon,\vec{\mathbf{y}}}, \breve{z}^l_{\epsilon,\vec{\mathbf{y}}}), \tag{3.3.122}$$

$$u^{0\mu}_{0,\epsilon,\vec{\mathbf{y}}}|_\Sigma = \epsilon\mathcal{S}^\mu_0(\epsilon, \breve{\mathfrak{u}}^{kl}_\epsilon, \breve{\mathfrak{u}}^{kl}_{0,\epsilon}, \delta\breve{\rho}_{\epsilon,\vec{\mathbf{y}}}, \breve{z}^l_{\epsilon,\vec{\mathbf{y}}}), \tag{3.3.123}$$

$$u_{\gamma,\epsilon,\vec{\mathbf{y}}}|_\Sigma = \epsilon\mathcal{S}_\gamma(\epsilon, \breve{\mathfrak{u}}^{kl}_\epsilon, \breve{\mathfrak{u}}^{kl}_{0,\epsilon}, \delta\breve{\rho}_{\epsilon,\vec{\mathbf{y}}}, \breve{z}^l_{\epsilon,\vec{\mathbf{y}}}), \tag{3.3.124}$$

and

$$u^{ij}_{\gamma,\epsilon,\vec{\mathbf{y}}}|_\Sigma = \epsilon\mathcal{S}^{ij}_\gamma(\epsilon, \breve{\mathfrak{u}}^{kl}_\epsilon, \breve{\mathfrak{u}}^{kl}_{0,\epsilon}, \delta\breve{\rho}_{\epsilon,\vec{\mathbf{y}}}, \breve{z}^l_{\epsilon,\vec{\mathbf{y}}}), \tag{3.3.125}$$

where the remainders are bounded by

$$\begin{aligned} &\|\mathcal{S}^\mu(\epsilon, \breve{\mathfrak{u}}^{kl}_\epsilon, \breve{\mathfrak{u}}^{kl}_{0,\epsilon}, \delta\breve{\rho}_{\epsilon,\vec{\mathbf{y}}}, \breve{z}^l_{\epsilon,\vec{\mathbf{y}}})\|_{R^{s+1}} + \|\mathcal{S}(\epsilon, \breve{\mathfrak{u}}^{kl}_\epsilon, \breve{\mathfrak{u}}^{kl}_{0,\epsilon}, \delta\breve{\rho}_{\epsilon,\vec{\mathbf{y}}}, \breve{z}^l_{\epsilon,\vec{\mathbf{y}}})\|_{R^{s+1}} + \|\mathcal{S}^{ij}(\epsilon, \breve{\mathfrak{u}}^{kl}_\epsilon, \breve{\mathfrak{u}}^{kl}_{0,\epsilon}, \delta\breve{\rho}_{\epsilon,\vec{\mathbf{y}}}, \breve{z}^l_{\epsilon,\vec{\mathbf{y}}})\|_{R^{s+1}} \\ &+ \|\mathcal{R}_j(\epsilon, \breve{\mathfrak{u}}^{kl}_\epsilon, \breve{\mathfrak{u}}^{kl}_{0,\epsilon}, \delta\breve{\rho}_{\epsilon,\vec{\mathbf{y}}}, \breve{z}^l_{\epsilon,\vec{\mathbf{y}}})\|_{R^{s+1}} + \|\mathcal{S}^\mu_i(\epsilon, \breve{\mathfrak{u}}^{kl}_\epsilon, \breve{\mathfrak{u}}^{kl}_{0,\epsilon}, \delta\breve{\rho}_{\epsilon,\vec{\mathbf{y}}}, \breve{z}^l_{\epsilon,\vec{\mathbf{y}}})\|_{R^{s+1}} + \|\mathcal{S}^\mu_0(\epsilon, \breve{\mathfrak{u}}^{kl}_\epsilon, \breve{\mathfrak{u}}^{kl}_{0,\epsilon}, \delta\breve{\rho}_{\epsilon,\vec{\mathbf{y}}}, \breve{z}^l_{\epsilon,\vec{\mathbf{y}}})\|_{R^{s+1}} \\ &+ \|\mathcal{S}_\gamma(\epsilon, \breve{\mathfrak{u}}^{kl}_\epsilon, \breve{\mathfrak{u}}^{kl}_{0,\epsilon}, \delta\breve{\rho}_{\epsilon,\vec{\mathbf{y}}}, \breve{z}^l_{\epsilon,\vec{\mathbf{y}}})\|_{R^{s+1}} + \|\mathcal{S}^{ij}_\gamma(\epsilon, \breve{\mathfrak{u}}^{kl}_\epsilon, \breve{\mathfrak{u}}^{kl}_{0,\epsilon}, \delta\breve{\rho}_{\epsilon,\vec{\mathbf{y}}}, \breve{z}^l_{\epsilon,\vec{\mathbf{y}}})\|_{R^{s+1}} \lesssim \|\breve{\xi}\|_s \end{aligned}$$

for all $(\epsilon, \vec{\mathbf{y}}) \in (0, \epsilon_0) \times \mathbb{R}^{3N}$. Moreover, the components of $\hat{\mathbf{U}}_{\epsilon,\vec{\mathbf{y}}}|_\Sigma$ satisfy the uniform bounds

$$\|u^{\mu\nu}_{\epsilon,\vec{\mathbf{y}}}|_\Sigma\|_{R^{s+1}} + \|u_{\epsilon,\vec{\mathbf{y}}}|_\Sigma\|_{R^{s+1}} + \|u^{0k}_{i,\epsilon,\vec{\mathbf{y}}}|_\Sigma\|_{R^s} + \|u^{0\mu}_{0,\epsilon,\vec{\mathbf{y}}}|_\Sigma\|_{R^s} + \|u_{\mu,\epsilon,\vec{\mathbf{y}}}|_\Sigma\|_{R^s} + \|u^{ij}_{\mu,\epsilon,\vec{\mathbf{y}}}|_\Sigma\|_{R^s} \lesssim \epsilon\|\breve{\xi}_\epsilon\|_s$$

and

$$\|u^{00}_{i,\epsilon,\vec{\mathbf{y}}}|_\Sigma\|_{R^s} + \|\breve{z}_{j,\epsilon,\vec{\mathbf{y}}}|_\Sigma\|_{R^s} + \|\delta\zeta_{\epsilon,\vec{\mathbf{y}}}|_\Sigma\|_{L^{\frac{6}{5}}\cap K^s} \lesssim \|\breve{\xi}_\epsilon\|_s$$

for all $(\epsilon, \vec{\mathbf{y}}) \in (0, \epsilon_0) \times \mathbb{R}^{3N}$.

3.4 Local existence and continuation

3.4.1 Reduced conformal Einstein-Euler equations

The formulation (3.2.69) of the reduced conformal Einstein-Euler equations is symmetric hyperbolic. Consequently, we can apply standard results, e.g. [54, §2.3], to obtain the local-in-time existence and uniqueness of solutions in uniformly local Sobolev spaces $H^s_{\text{ul}}(\mathbb{R}^3)$, $s \in \mathbb{Z}_{\geq 3}$, along with a continuation principle; see Proposition 3.4.1 below for the precise statement. However, in order to obtain the global existence of solutions to the future that exist for all parameter values $\epsilon \in (0, \epsilon_0)$ and all $t \in (0, 1]$, we cannot use the formulation (3.2.69) of the conformal Einstein-Euler equations. Instead, we rely on a non-local formulation, which is defined below by (3.5.23). Due to the non-locality, it is not enough to have local existence and continuation in the uniformly local Sobolev spaces. Instead, we need to establish the local-in-time existence of solutions and a continuation principle in the spaces $R^s(\mathbb{R}^3)$, $s \in \mathbb{Z}_{\geq 3}$, which we do below in Corollary 3.4.2.

Proposition 3.4.1. *Suppose* $s \in \mathbb{Z}_{\geq 3}$ *and*

$$\hat{\mathbf{U}}_{\epsilon,\vec{\mathbf{y}}}|_\Sigma = \{u^{\mu\nu}_{\epsilon,\vec{\mathbf{y}}}, u_{\epsilon,\vec{\mathbf{y}}}, u^{ij}_{\gamma,\epsilon,\vec{\mathbf{y}}}, u^{0\mu}_{i,\epsilon,\vec{\mathbf{y}}}, u^{0\mu}_{0,\epsilon,\vec{\mathbf{y}}}, u_{\gamma,\epsilon,\vec{\mathbf{y}}}, z_{j,\epsilon,\vec{\mathbf{y}}}, \delta\zeta_{\epsilon,\vec{\mathbf{y}}}\}|_\Sigma \in X^s(\mathbb{R}^3), \quad (\epsilon, \vec{\mathbf{y}}) \in (0, \epsilon_0) \times \mathbb{R}^{3N},$$

is the initial data from Theorem 3.3.16. Then there exists a constant $T > 0$ *and a unique classical solution*

$$\hat{\mathbf{U}}_{\epsilon,\vec{\mathbf{y}}} \in C((T, 1], H^s_{\text{loc}}(\mathbb{R}^3, \mathbb{K})) \bigcap C^1((T, 1], H^{s-1}_{\text{loc}}(\mathbb{R}^3, \mathbb{K})) \bigcap L^\infty((T, 1], H^s_{\text{ul}}(\mathbb{R}^3, \mathbb{K})),$$

where $\mathbb{K} = \mathbb{S}_4 \times \mathbb{R} \times \mathbb{S}_3 \times (\mathbb{R}^3)^2 \times \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}$, *to* (3.2.69) *on the spacetime region* $(T, 1] \times \mathbb{R}^3$ *that agrees with the initial data from Theorem 3.3.16 on the initial hypersurface* Σ. *Moreover, there exists a constant* $\sigma > 0$, *independent of the initial data, such that if* $\hat{\mathbf{U}}_{\epsilon,\vec{\mathbf{y}}}$ *exists for* $t \in (T_1, 1]$ *with the same regularity as above and satisfies* $\|\hat{\mathbf{U}}_{\epsilon,\vec{\mathbf{y}}}\|_{L^\infty((T_1,1],W^{1,\infty})} < \sigma$, *then the solution* $\hat{\mathbf{U}}_{\epsilon,\vec{\mathbf{y}}}$ *can be uniquely continued as a classical solution with the same regularity to the larger spacetime region* $(T^*, 1] \times \mathbb{R}^3$ *for some* $T^* \in (0, T_1)$.

Proof. First, we observe by Theorem C.2.2, (3.2.6)-(3.2.7), (3.2.16), (3.2.19)–(3.2.23) and (3.3.5) that there exists a constant $\sigma > 0$ such that if $\hat{\mathbf{U}}(t, x)$ satisfies $\|\hat{\mathbf{U}}(t)\|_{W^{1,\infty}} < \sigma$, then the metric $\bar{g}^{\mu\nu}$, conformal four-velocity $\bar{v}^\mu$ and proper energy density $\bar{\rho}$ will remain non-degenerate, future directed, and strictly positive, respectively. We also observe by Theorem 3.3.16 that there exists initial data $\hat{\mathbf{U}}_{\epsilon,\vec{\mathbf{y}}}|_\Sigma$ for the evolution equation (3.2.69) that satisfies[5] $\|\hat{\mathbf{U}}_{\epsilon,\vec{\mathbf{y}}}|_\Sigma\|_{W^{1,\infty}} < \sigma$. It then follows from[6] Theorem 2.1 from [54, §2.3] that there exists a $T \in (0, 1)$ and a unique classical solution

$$\hat{\mathbf{U}}_{\epsilon,\vec{\mathbf{y}}} \in C((T, 1], H^s_{\text{loc}}(\mathbb{R}^3, \mathbb{K})) \bigcap C^1((T, 1], H^{s-1}_{\text{loc}}(\mathbb{R}^3, \mathbb{K})) \bigcap L^\infty((T, 1], H^s_{\text{ul}}(\mathbb{R}^3, \mathbb{K}))$$

to (3.2.69) that satisfies $\|\hat{\mathbf{U}}_{\epsilon,\vec{\mathbf{y}}}\|_{L^\infty((T,1],W^{1,\infty})} < \sigma$ and agrees with the initial data $\hat{\mathbf{U}}_{\epsilon,\vec{\mathbf{y}}}|_\Sigma$ at $t = 1$. This proves the existence and uniqueness part of the statement. The continuation part of the statement[7] is a direct consequence of Theorem 2.2 from [54, §2.3] since the bound $\|\hat{\mathbf{U}}\|_{L^\infty((T_1,1],W^{1,\infty}(\mathbb{R}^3))} < \sigma$ together with the equations of motion (3.2.69) imply a bound of the form $\|\hat{\mathbf{U}}\|_{W^{1,\infty}((T_1,1]\times\mathbb{R}^3))} < C(\sigma)$. □

[5] We can always arrange this by shrinking ϵ_0, if necessary, to guarantee via Sobolev's inequality that the bound $\|\hat{\mathbf{U}}_{\epsilon,\vec{\mathbf{y}}}|_\Sigma\|_{W^{1,\infty}} < \sigma$ is satisfied for any particular choice of σ.

[6] Here, we are using the inclusion $X^s(\mathbb{R}^3) \subset H^s_{\text{ul}}(\mathbb{R}^3, \mathbb{K})$ which is a direct consequence of Sobolev's inequality and the definition of the space $X^s(\mathbb{R}^3)$.

[7] This is similar to the corresponding proof of Proposition 2.3.1. The second and the third alternatives of Majda's criterion can not occur as we mentioned in the Remark 2.3.2 of Proposition 2.3.1. We omit the details here.

Corollary 3.4.2. *Suppose* $s \in \mathbb{Z}_{\geq 3}$*,*

$$\hat{\mathbf{U}}_{\epsilon,\vec{\mathbf{y}}}|_\Sigma = \{u^{\mu\nu}_{\epsilon,\vec{\mathbf{y}}}, u_{\epsilon,\vec{\mathbf{y}}}, u^{ij}_{\gamma,\epsilon,\vec{\mathbf{y}}}, u^{0\mu}_{i,\epsilon,\vec{\mathbf{y}}}, u^{0\mu}_{0,\epsilon,\vec{\mathbf{y}}}, u_{\gamma,\epsilon,\vec{\mathbf{y}}}, z_{j,\epsilon,\vec{\mathbf{y}}}, \delta\zeta_{\epsilon,\vec{\mathbf{y}}}\}|_\Sigma \in X^s(\mathbb{R}^3), \quad (\epsilon, \vec{\mathbf{y}}) \in (0,\epsilon_0) \times \mathbb{R}^{3N},$$

is the initial data from Theorem 3.3.16. Then there exists a constant $T > 0$ *and a unique classical solution*

$$\hat{\mathbf{U}}_{\epsilon,\vec{\mathbf{y}}} \in \bigcap_{\ell=0}^{1} C^\ell((T,1], R^{s-\ell}(\mathbb{R}^3,\mathbb{K}))$$

to (3.2.69) on the spacetime region $(T,1]\times\mathbb{R}^3$ *that agrees with the initial data from Theorem 3.3.16 on the initial hypersurface* Σ*. Moreover, there exists a constant* $\sigma > 0$*, independent of the initial data, such that if* $\hat{\mathbf{U}}_{\epsilon,\vec{\mathbf{y}}}$ *exists for* $t \in (T_1, 1]$ *with the same regularity as above and satisfies* $\|\hat{\mathbf{U}}_{\epsilon,\vec{\mathbf{y}}}\|_{L^\infty((T_1,1],R^s))} < \sigma$*, then the solution* $\hat{\mathbf{U}}_{\epsilon,\vec{\mathbf{y}}}$ *can be uniquely continued as a classical solution with the same regularity to the larger spacetime region* $(T^*, 1] \times \mathbb{R}^3$ *for some* $T^* \in (0, T_1)$*.*

Proof. While the solution $\hat{\mathbf{U}}_{\epsilon,\vec{\mathbf{y}}}$ to (3.2.69) from Proposition 3.4.1 satisfies $\hat{\mathbf{U}}_{\epsilon,\vec{\mathbf{y}}}(1) \in X^s(\mathbb{R}^3) \subset R^s(\mathbb{R}^3,\mathbb{K})$, we only know that $\hat{\mathbf{U}}_{\epsilon,\vec{\mathbf{y}}}(t) \in H^s_{\text{ul}}(\mathbb{R}^3,\mathbb{K})$ for $t < 1$. Thus, the first step involves showing that $\hat{\mathbf{U}}_{\epsilon,\vec{\mathbf{y}}}(t)$ remains in $R^s(\mathbb{R}^3,\mathbb{K})$ for $t < 1$. To accomplish this, we use energy estimates. In fact, it will be enough to show that solutions $\hat{\mathbf{U}}_{\epsilon,\vec{\mathbf{y}}}(t)$ that stay in $R^s(\mathbb{R}^3,\mathbb{K})$ satisfy an energy estimate that yield a bound on the norm $\|\hat{\mathbf{U}}_{\epsilon,\vec{\mathbf{y}}}(t)\|_{R^s}$. While this may seem circular, it is easy to justify using the finite speed of propagation to first prove energy estimates on truncated spacetime cones, which are well defined for solutions $\hat{\mathbf{U}}_{\epsilon,\vec{\mathbf{y}}}(t)$ that lie in $H^s_{\text{ul}}(\mathbb{R}^3,\mathbb{K})$, followed by letting the width of the cone go to infinity to obtain estimates on the spacetime slab of the form $(T_1, 1] \times \mathbb{R}^3$. For reasons of economy, we omit these easily reproducible details.

In the following, we will suppress the subscripts and just write $\hat{\mathbf{U}}$ for the solution. We then derive energy estimates for $D\hat{\mathbf{U}}$ by differentiating the evolution equations (3.2.69) to get

$$\begin{aligned}\mathbf{B}^0 D^\alpha \partial_t \hat{\mathbf{U}} = &- \mathbf{B}^i D^\alpha \partial_i \hat{\mathbf{U}} - \frac{1}{\epsilon}\mathbf{C}^i D^\alpha \partial_i \hat{\mathbf{U}} + \frac{1}{t}\mathbf{B}^0 D^\alpha (\mathbf{B}^0)^{-1}\mathbf{B}\mathbb{P}\hat{\mathbf{U}} + \mathbf{B}^0[(\mathbf{B}^0)^{-1}\mathbf{B}^i, D^\alpha]\partial_i\hat{\mathbf{U}} \\ &+ \frac{1}{\epsilon}\mathbf{B}^0[(\mathbf{B}^0)^{-1}, D^\alpha]\mathbf{C}^i\partial_i\hat{\mathbf{U}} + \mathbf{B}^0 D^\alpha\big((\mathbf{B}^0)^{-1}\hat{\mathbf{H}}\big) \quad (3.4.1)\end{aligned}$$

while differentiating $\mathbf{B}^0$ with respect to t yields

$$\partial_t \mathbf{B}^0 = D_t\mathbf{B}^0 + D_{\hat{\mathbf{U}}}\mathbf{B}^0 \cdot \left(-(\mathbf{B}^0)^{-1}\mathbf{B}^i\partial_i\hat{\mathbf{U}} - \frac{1}{\epsilon}(\mathbf{B}^0)^{-1}C^i\partial_i\hat{\mathbf{U}} + \frac{1}{t}(\mathbf{B}^0)^{-1}\mathbf{B}\mathbb{P}\hat{\mathbf{U}} + (\mathbf{B}^0)^{-1}\hat{\mathbf{H}}\right). \quad (3.4.2)$$

Multiplying (3.4.1) by the transpose of $D^\alpha\hat{\mathbf{U}}$, $|\alpha| \geq 1$, followed by integrating over $\mathbb{R}^3$ and summing over α for $1 \leq |\alpha| \leq s+1$, we obtain, with the help of (3.4.2), integration by parts and the calculus inequalities from Appendix C, the following variation on the standard energy estimate for symmetric hyperbolic systems:

$$\begin{aligned}-\partial_t\|D\hat{\mathbf{U}}\|^2_{H^{s-1}} &= -\sum_{1\leq|\alpha|\leq s}\langle D^\alpha\hat{\mathbf{U}}, (\partial_t\mathbf{B}^0)D^\alpha\hat{\mathbf{U}}\rangle - 2\sum_{1\leq|\alpha|\leq s}\langle D^\alpha\hat{\mathbf{U}}, \mathbf{B}^0 D^\alpha\partial_t\hat{\mathbf{U}}\rangle \\ &\leq C(\|\hat{\mathbf{U}}\|_{W^{1,\infty}}, \epsilon^{-1})\|D\hat{\mathbf{U}}\|_{H^{s-1}}(\|D\hat{\mathbf{U}}\|_{H^{s-1}} + \|\hat{\mathbf{U}}\|_{L^\infty}),\end{aligned}$$

which we note is equivalent to

$$-\partial_t\|D\hat{\mathbf{U}}\|_{H^{s-1}} \leq C(\|\hat{\mathbf{U}}\|_{W^{1,\infty}})(\|D\hat{\mathbf{U}}\|_{H^{s-1}} + \|\hat{\mathbf{U}}\|_{L^\infty}). \quad (3.4.3)$$

Assuming that $T_0 \in (0,1]$, we obtain from integrating (3.4.3) the estimate

$$\|D\hat{\mathbf{U}}(t)\|_{H^{s-1}} \leq \|D\hat{\mathbf{U}}(T_0)\|_{H^{s-1}} + \int_t^{T_0} C(\|\hat{\mathbf{U}}(\tau)\|_{W^{1,\infty}})(\|D\hat{\mathbf{U}}(\tau)\|_{H^{s-1}} + \|\hat{\mathbf{U}}(\tau)\|_{L^\infty})d\tau \quad (3.4.4)$$

for $0 < T_1 < t \leq T_0 \leq 1$. On the other hand, multiplying the evolution equation (3.2.69) on the left by $(\mathbf{B}^0)^{-1}$ followed by integrating in time, we find, after taking the L^6 norm, that

$$\begin{aligned}\|\hat{\mathbf{U}}(t)\|_{L^6} \leq & \|\hat{\mathbf{U}}(T_0)\|_{L^6} + \int_t^{T_0} \bigg(\|(\mathbf{B}^0)^{-1}\mathbf{B}^i\partial_i\hat{\mathbf{U}}\|_{L^6} + \frac{1}{\epsilon}\|(\mathbf{B}^0)^{-1}C^i\partial_i\hat{\mathbf{U}}\|_{L^6} \\ & \qquad + \frac{1}{t}\|(\mathbf{B}^0)^{-1}\mathbf{B}\mathbb{P}\hat{\mathbf{U}}\|_{L^6} + \|(\mathbf{B}^0)^{-1}\hat{\mathbf{H}}\|_{L^6}\bigg) d\tau \\ \leq & \|\hat{\mathbf{U}}(T_0)\|_{L^6} + \int_t^{T_0} C(\|\hat{\mathbf{U}}(\tau)\|_{L^\infty})\|\hat{\mathbf{U}}(\tau)\|_{W^{1,6}} d\tau \end{aligned} \tag{3.4.5}$$

for $0 < T_1 < t \leq T_0 \leq 1$. Adding the two inequalities (3.4.4) and (3.4.5), we get, with the help of (3.1.13), that

$$\|\hat{\mathbf{U}}(t)\|_{R^s} \leq \|\hat{\mathbf{U}}(T_0)\|_{R^s} + \int_t^{T_0} C(\|\hat{\mathbf{U}}(\tau)\|_{R^s})\|\hat{\mathbf{U}}(\tau)\|_{R^s} d\tau, \quad 0 < T_1 < t \leq T_0 \leq 1.$$

Then by the Grönwall's inequality, there exists a $T_* \in [T_1, T_0)$ such that

$$\|\hat{\mathbf{U}}(t)\|_{R^s} \leq C(\|\hat{\mathbf{U}}(T_0)\|_{R^s}), \quad 0 < T_* < t \leq T_0 \leq 1.$$

From this inequality and Proposition 3.4.1, we deduce that the space $R^s(\mathbb{R}^3, \mathbb{K})$ is preserved under evolution, and hence there exists a $T \in (0, 1]$ such that

$$\hat{\mathbf{U}}_{\epsilon,\vec{\mathbf{y}}} \in \bigcap_{\ell=0}^{1} C^\ell((T, 1], R^{s-\ell}(\mathbb{R}^3, \mathbb{K})). \tag{3.4.6}$$

Moreover, since $\|\hat{\mathbf{U}}_{\epsilon,\vec{\mathbf{y}}}\|_{L^\infty((T_1,1],W^{1,\infty})} \lesssim \|\hat{\mathbf{U}}_{\epsilon,\vec{\mathbf{y}}}\|_{L^\infty((T_1,1],R^s)}$ by (3.1.13), it follows from the continuation principle from Proposition 3.4.1 that there exists a constant $\sigma > 0$, independent of the initial data, such that if $\hat{\mathbf{U}}_{\epsilon,\vec{\mathbf{y}}}$ exists for $t \in (T_1, 1]$ with the same regularity as (3.4.6) and satisfies $\|\hat{\mathbf{U}}_{\epsilon,\vec{\mathbf{y}}}\|_{L^\infty((T_1,1],R^s)} < \sigma$, then the solution $\hat{\mathbf{U}}_{\epsilon,\vec{\mathbf{y}}}$ can be uniquely continued as a classical solution with the same regularity to the larger spacetime region $(T^*, 1] \times \mathbb{R}^3$ for some $T^* \in (0, T_1)$. □

Remark 3.4.3. While it is clear from the energy estimates that the time of existence T from the above corollary does not depend on the parameter $\vec{\mathbf{y}}$ since the norm of the initial data $\|\hat{\mathbf{U}}_{\epsilon,\vec{\mathbf{y}}}|_\Sigma\|_{R^s}$ is independent of $\vec{\mathbf{y}}$ due to the translational invariance of the norm $\|\cdot\|_{R^s}$, the time of existence does appear to depend on ϵ due to the appearance of ϵ^{-1} in the energy estimates. To show that the time of existence does not, in fact, depend on ϵ and that for small enough data the solution exists on the whole time interval $(0, 1]$ relies on the non-local version of the reduced conformal Einstein equations defined by (3.5.23).

3.4.2 Conformal Poisson-Euler equations

In this section, we consider the local-in-time existence and uniqueness of solutions to the conformal cosmological Poisson-Euler equations, and we establish a continuation principle based on bounding the R^s norm of $(\delta\mathring{\zeta}, \mathring{z}^j)$. For convenience, we define

$$\varpi^j := \frac{\mathring{E}^3}{t^3}\mathring{\rho}\mathring{z}^j, \tag{3.4.7}$$

and let

$$\delta\mathring{\zeta} = \mathring{\zeta} - \mathring{\zeta}_H \quad \text{and} \quad \mathring{z}_j = \mathring{E}^2\delta_{ij}\mathring{z}^i \tag{3.4.8}$$

where, see (3.1.9), (3.1.57) and (3.2.1),

$$\mathring{\zeta}_H = \ln(t^{-3}\mathring{\mu}). \tag{3.4.9}$$

For each positive constant $\beta > 0$, we further define the quantity

$$\mathring{\Upsilon} = \epsilon\beta\frac{\Lambda}{3t^3}\mathring{E}^3(\Delta - \epsilon^2\beta)^{-1}\delta\mathring{\rho}, \tag{3.4.10}$$

which will be used to simplify $\mathring{\mathbf{F}}$ later in §3.7.

Proposition 3.4.4. *Suppose $s \in \mathbb{Z}_{\geq 3}$, $\epsilon_0 > 0$, $\epsilon \in (0, \epsilon_0)$, $\vec{\mathbf{y}} \in \mathbb{R}^{3N}$, $\delta\breve{\rho}_\lambda \in L^{\frac{6}{5}} \cap K^s(\mathbb{R}^3, \mathbb{R})$ and $\breve{z}^j_\lambda \in L^{\frac{6}{5}} \cap K^s(\mathbb{R}^3, \mathbb{R}^3)$ for $\lambda = 1, \cdots, N$, $\delta\breve{\rho}_{\epsilon,\vec{\mathbf{y}}}$ and $\breve{z}^j_{\epsilon,\vec{\mathbf{y}}}$ are defined by (3.3.115), and $\mathring{\mu}(1) > 0$. Then there exists a $T \in (0, 1]$ and a unique classical solution $(\mathring{\zeta}_{\epsilon,\vec{\mathbf{y}}}, \mathring{z}^i_{\epsilon,\vec{\mathbf{y}}}, \mathring{\Phi}_{\epsilon,\vec{\mathbf{y}}})$ of the conformal cosmological Poisson-Euler equations, given by (3.1.65)-(3.1.67), such that*

$$(\delta\mathring{\zeta}_{\epsilon,\vec{\mathbf{y}}}, \mathring{z}^i_{\epsilon,\vec{\mathbf{y}}}, \mathring{\Phi}_{\epsilon,\vec{\mathbf{y}}}) \in \bigcap_{\ell=0}^{1} C^\ell((T, 1], H^{s-\ell}(\mathbb{R}^3)) \times \bigcap_{\ell=0}^{1} C^\ell((T, 1], H^{s-\ell}(\mathbb{R}^3, \mathbb{R}^3)) \times \bigcap_{\ell=0}^{1} C^\ell((T, 1], R^{s+2-\ell}(\mathbb{R}^3))$$

on the spacetime region $(T, 1] \times \mathbb{R}^3$ that satisfies

$$(\delta\mathring{\zeta}_{\epsilon,\vec{\mathbf{y}}}, \mathring{z}^i_{\epsilon,\vec{\mathbf{y}}})|_\Sigma = \left(\ln\left(1 + \frac{\delta\breve{\rho}_{\epsilon,\vec{\mathbf{y}}}}{\mathring{\mu}(1)}\right), \breve{z}^i_{\epsilon,\vec{\mathbf{y}}}\right) \in L^{\frac{6}{5}} \cap K^s(\mathbb{R}^3, \mathbb{R}^4) \subset H^s(\mathbb{R}^3, \mathbb{R}^4) \tag{3.4.11}$$

on the initial hypersurface Σ, and the estimates

$$\|\varpi^j_{\epsilon,\vec{\mathbf{y}}}\|_{H^s} + \|\partial_t\mathring{\Phi}_{\epsilon,\vec{\mathbf{y}}}\|_{R^{s+1}} + \|(-\Delta)^{-\frac{1}{2}}\mathfrak{R}_k\varpi^j_{\epsilon,\vec{\mathbf{y}}}\|_{R^{s+1}} + \|\partial_t\mathring{\Phi}_{i,\epsilon,\vec{\mathbf{y}}}\|_{H^s} \leq C\big(\|\delta\mathring{\zeta}_{\epsilon,\vec{\mathbf{y}}}\|_{L^\infty((t,1],H^s)}\big)\|\mathring{z}^j_{\epsilon,\vec{\mathbf{y}}}\|_{H^s}, \tag{3.4.12}$$

$$\|\mathring{\Phi}_{\epsilon,\vec{\mathbf{y}}}\|_{R^{s+1}} + \|\mathring{\Phi}_{i,\epsilon,\vec{\mathbf{y}}}\|_{H^s} \leq C\|\delta\breve{\rho}_{\epsilon,\vec{\mathbf{y}}}\|_{L^{\frac{6}{5}}\cap H^s} + C\big(\|\delta\mathring{\zeta}_{\epsilon,\vec{\mathbf{y}}}\|_{L^\infty((t,1],H^s)}\big)\int_t^1 \|\mathring{z}^k_{\epsilon,\vec{\mathbf{y}}}(\tau)\|_{H^s}d\tau, \tag{3.4.13}$$

$$\begin{aligned}&\|t\partial_t\mathfrak{R}_j(-\Delta)^{-\frac{1}{2}}\varpi^j_{\epsilon,\vec{\mathbf{y}}}\|_{R^s} + \|t\partial_t^2\mathring{\Phi}_{\epsilon,\vec{\mathbf{y}}}\|_{R^s} + \|t\partial_t\varpi^j_{\epsilon,\vec{\mathbf{y}}}\|_{R^{s-1}} \\ &\leq C\big(\|\delta\mathring{\zeta}_{\epsilon,\vec{\mathbf{y}}}\|_{L^\infty((t,1],H^s)}, \|\mathring{z}^j_{\epsilon,\vec{\mathbf{y}}}\|_{L^\infty((t,1],H^s)}\big)\Big(\|\delta\mathring{\zeta}_{\epsilon,\vec{\mathbf{y}}}\|_{R^s} + \|\mathring{z}^j_{\epsilon,\vec{\mathbf{y}}}\|_{H^s} + \|\delta\breve{\rho}_{\epsilon,\vec{\mathbf{y}}}\|_{L^{\frac{6}{5}}\cap H^s} + \int_t^1 \|\mathring{z}^k_{\epsilon,\vec{\mathbf{y}}}(\tau)\|_{H^s}d\tau\Big),\end{aligned} \tag{3.4.14}$$

$$\|\mathring{\Upsilon}_{\epsilon,\vec{\mathbf{y}}}\|_{H^s} \leq C\|\delta\breve{\rho}_{\epsilon,\vec{\mathbf{y}}}\|_{L^{\frac{6}{5}}\cap H^s} + C\big(\|\delta\mathring{\zeta}_{\epsilon,\vec{\mathbf{y}}}\|_{L^\infty((t,1],H^s)}\big)\int_t^1 \|\mathring{z}^j_{\epsilon,\vec{\mathbf{y}}}(\tau)\|_{H^s}d\tau, \tag{3.4.15}$$

and

$$\|\partial_t\mathring{\Upsilon}_{\epsilon,\vec{\mathbf{y}}}\|_{R^s} \leq C\big(\|\delta\mathring{\zeta}_{\epsilon,\vec{\mathbf{y}}}\|_{L^\infty((t,1],H^s)}\big)\|\mathring{z}^j_{\epsilon,\vec{\mathbf{y}}}\|_{H^s} \tag{3.4.16}$$

for all $t \in (T, 1]$. Furthermore, there exists a constant $\sigma > 0$, independent of the initial data and $T_1 \in (0, 1)$, such that if $(\delta\mathring{\zeta}_{\epsilon,\vec{\mathbf{y}}}, \mathring{z}^i_{\epsilon,\vec{\mathbf{y}}}, \mathring{\Phi}_{\epsilon,\vec{\mathbf{y}}})$ exists for $t \in (T_1, 1]$ with the same regularity as above and satisfies $\|(\delta\mathring{\zeta}_{\epsilon,\vec{\mathbf{y}}}, \mathring{z}^i_{\epsilon,\vec{\mathbf{y}}})\|_{L^\infty((T_1,1],R^s)} < \sigma$, then the solution $(\delta\mathring{\zeta}_{\epsilon,\vec{\mathbf{y}}}, \mathring{z}^i_{\epsilon,\vec{\mathbf{y}}}, \mathring{\Phi}_{\epsilon,\vec{\mathbf{y}}})$ can be uniquely continued as a classical solution with the same regularity to the larger spacetime region $(T^*, 1] \times \mathbb{R}^3$ for some $T^* \in (0, T_1)$.

Proof. With the help of (3.2.3) and (3.4.8)-(3.4.9), we can write the first two equations from the conformal cosmological Poisson-Euler system, see (3.1.65)-(3.1.66), as

$$\partial_t\delta\mathring{\zeta} + \sqrt{\frac{3}{\Lambda}}(\mathring{z}^j\partial_j\delta\mathring{\zeta} + \partial_j\mathring{z}^j) = 0, \tag{3.4.17}$$

$$\sqrt{\frac{\Lambda}{3}}\partial_t\mathring{z}^j + \mathring{z}^i\partial_i\mathring{z}^j + K\frac{\delta^{ji}}{\mathring{E}^2}\partial_i\mathring{\zeta} = \sqrt{\frac{\Lambda}{3}}\frac{1}{t}\mathring{z}^j - \frac{1}{2}\frac{3}{\Lambda}t\mathring{E}^{-3}\delta^{ij}\mathring{\Phi}_i, \tag{3.4.18}$$

where we have set

$$\mathring{\Phi}_i = \partial_i \mathring{\Phi}. \tag{3.4.19}$$

Rather than considering the Poisson equation (3.1.61) (see also (3.1.67)) directly, we find it convenient instead to consider the equation satisfied by $\mathring{\Phi}_i$. To derive this equation, we write (3.1.61) as

$$\mathring{\Phi} = \frac{\Lambda}{3t^3}\mathring{E}^3 \Delta^{-1}\delta\mathring{\rho}. \tag{3.4.20}$$

Applying $\frac{\Lambda}{3t^3}\mathring{E}^3\Delta^{-1}$ to (3.1.59) we derive, with the help of (3.1.63) and (3.4.20), the equation

$$\partial_t \mathring{\Phi} = -\sqrt{\frac{\Lambda}{3}}\frac{\mathring{E}^3}{t^3}\partial_k \Delta^{-1}\left(\mathring{\rho}\mathring{z}^k\right) = -\sqrt{\frac{\Lambda}{3}}\mathfrak{R}_j(-\Delta)^{-\frac{1}{2}}\varpi^j. \tag{3.4.21}$$

In a similar fashion, we derive

$$\partial_t \mathring{\Upsilon} = -\epsilon\beta\sqrt{\frac{\Lambda}{3}}\partial_j(\Delta - \epsilon^2\beta)^{-1}\varpi^j \tag{3.4.22}$$

by applying $\epsilon\beta\frac{\Lambda}{3t^3}\mathring{E}^3(\Delta-\epsilon^2\beta)^{-1}$ to (3.1.59). Applying the spatial partial derivative ∂_j to (3.4.21) then yields the desired evolution equation

$$\partial_t \mathring{\Phi}_i = \sqrt{\frac{\Lambda}{3}}\frac{\mathring{E}^3}{t^3}\mathfrak{R}_i\mathfrak{R}_j\left(\mathring{\rho}\mathring{z}^j\right) = \sqrt{\frac{\Lambda}{3}}\mathfrak{R}_i\mathfrak{R}_j\varpi^j \tag{3.4.23}$$

for $\mathring{\Phi}_i$.

Together, the equations (3.4.17), (3.4.18) and (3.4.23) can be cast into a non-local symmetric hyperbolic form in the unknowns $\{\delta\mathring{\zeta}, \mathring{z}^i, \mathring{\Phi}_i\}$ by multiplying (3.4.18) by $\mathring{E}^2K^{-1}\sqrt{\frac{3}{\Lambda}}$. Moreover, we observe that the initial data is bounded by

$$\|\mathring{\Phi}_{i,\epsilon,\vec{\mathbf{y}}}(1)\|_{H^{s+1}} = \left\|\frac{\Lambda}{3}\mathring{E}^3(1)\partial_i\Delta^{-1}\delta\mathring{\rho}_{\epsilon,\vec{\mathbf{y}}}(1)\right\|_{H^{s+1}} \lesssim \|(-\Delta)^{-\frac{1}{2}}\delta\mathring{\rho}_{\epsilon,\vec{\mathbf{y}}}(1)\|_{H^{s+1}} \lesssim \|\delta\breve{\rho}_{\epsilon,\vec{\mathbf{y}}}\|_{L^{\frac{6}{5}}\cap H^s}$$

and $\|\delta\mathring{\zeta}_{\epsilon,\vec{\mathbf{y}}}\|_{H^s} + \|\mathring{z}^i_{\epsilon,\vec{\mathbf{y}}}\|_{H^s} \leq \|\delta\mathring{\zeta}_{\epsilon,\vec{\mathbf{y}}}\|_{L^{\frac{6}{5}}\cap K^s} + \|\mathring{z}^i_{\epsilon,\vec{\mathbf{y}}}\|_{L^{\frac{6}{5}}\cap K^s}$. We can therefore conclude from standard local-in-time existence and uniqueness results and continuation principles for symmetric hyperbolic systems, e.g. Theorems 2.1 and 2.2 of [54, §2.1], which continue to apply for non-local systems, that there exists, for some time $T \in (0,1)$, a unique local-in-time classical solution

$$(\delta\mathring{\zeta}_{\epsilon,\vec{\mathbf{y}}}, \mathring{z}^i_{\epsilon,\vec{\mathbf{y}}}, \mathring{\Phi}_{i,\epsilon,\vec{\mathbf{y}}}) \in \bigcap_{\ell=0}^{1} C^\ell((T,1], H^{s-\ell}(\mathbb{R}^3)) \times \bigcap_{\ell=0}^{1} C^\ell((T,1], H^{s-\ell}(\mathbb{R}^3,\mathbb{R}^3)) \times \bigcap_{\ell=0}^{1} C^\ell((T,1], H^{s+1-\ell}(\mathbb{R}^3,\mathbb{R}^3)) \tag{3.4.24}$$

of (3.4.17), (3.4.18) and (3.4.23) that agrees with the initial data (3.4.11) on Σ. Moreover, if the solution satisfies $\|\delta\mathring{\zeta}_{\epsilon,\vec{\mathbf{y}}}\|_{L^\infty((T,1],W^{1,\infty})} + \|\mathring{z}^i_{\epsilon,\vec{\mathbf{y}}}\|_{L^\infty((T,1],W^{1,\infty})} < \sigma$, then there exists a time $T^* \in (0,T)$ such that the solution (3.4.24) uniquely extends to the spacetime region $(T^*,1]\times\mathbb{R}^3$ with the same regularity. By (3.1.13), this is clearly implied by the stronger condition $\|(\delta\mathring{\zeta}_{\epsilon,\vec{\mathbf{y}}}, \mathring{z}^i_{\epsilon,\vec{\mathbf{y}}})\|_{L^\infty((T,1],R^s)} < \sigma$.

From the definition (3.4.7) of ϖ^j, it is clear that the bound

$$\|\varpi^j_{\epsilon,\vec{\mathbf{y}}}(t)\|_{H^s} \leq C\big(\|\delta\mathring{\zeta}_{\epsilon,\vec{\mathbf{y}}}\|_{L^\infty((t,1],H^s)}\big)\|\mathring{z}^j_{\epsilon,\vec{\mathbf{y}}}(t)\|_{H^s}, \quad T < t \leq 1, \tag{3.4.25}$$

follows from the calculus inequalities, see Appendix C. Next, integrating (3.4.21) in time and then taking $\|\cdot\|_{L^6}$ norm, we obtain, with the help of the calculus inequalities and the potential theory from Appendix B, the estimate

$$\|\mathring{\Phi}_{\epsilon,\vec{\mathbf{y}}}(t)\|_{L^6} \leq C\|\delta\mathring{\rho}_{\epsilon,\vec{\mathbf{y}}}(1)\|_{L^{\frac{6}{5}}} + C(\|\delta\mathring{\zeta}_{\epsilon,\vec{\mathbf{y}}}\|_{L^\infty((t,1],H^s)})\int_t^1 \|\mathring{z}^k_{\epsilon,\vec{\mathbf{y}}}(\tau)\|_{L^2}d\tau, \quad T < t \leq 1, \tag{3.4.26}$$

while the estimate

$$\|\partial_t\mathring{\Phi}_{\epsilon,\vec{\mathbf{y}}}(t)\|_{L^6}\lesssim\|(-\Delta)^{-\frac{1}{2}}(\mathring{\rho}_{\epsilon,\vec{\mathbf{y}}}(t)\mathring{z}^k_{\epsilon,\vec{\mathbf{y}}}(t))\|_{L^6}\lesssim\|\mathring{\rho}_{\epsilon,\vec{\mathbf{y}}}(t)\mathring{z}^k_{\epsilon,\vec{\mathbf{y}}}(t)\|_{L^2}\leq C\big(\|\delta\mathring{\zeta}_{\epsilon,\vec{\mathbf{y}}}\|_{L^\infty((t,1],H^s)}\big)\|\mathring{z}^k_{\epsilon,\vec{\mathbf{y}}}(t)\|_{L^2}, \tag{3.4.27}$$

for $T<t\leq 1$, follows directly from applying the $\|\cdot\|_{L^6}$ norm to (3.4.21). Integrating (3.4.23), we obtain, after taking $\|\cdot\|_{H^s}$-norm, the estimate

$$\begin{aligned}\|\mathring{\Phi}_{i,\epsilon,\vec{\mathbf{y}}}(t)\|_{H^s}\leq&\|\mathring{\Phi}_{i,\epsilon,\vec{\mathbf{y}}}(1)\|_{H^s}+\int^1_t\|\varpi^j_{\epsilon,\vec{\mathbf{y}}}\|_{H^s}d\tau\\ \leq&C\|\delta\breve{\rho}_{\epsilon,\vec{\mathbf{y}}}\|_{L^{\frac{6}{5}}\cap H^s}+C\big(\|\delta\mathring{\zeta}_{\epsilon,\vec{\mathbf{y}}}\|_{L^\infty((t,1],H^s)}\big)\int^1_t\|\mathring{z}^k_{\epsilon,\vec{\mathbf{y}}}(\tau)\|_{H^s}d\tau\end{aligned} \tag{3.4.28}$$

for $T<t\leq 1$. From this estimate and (3.4.26), we then deduce that

$$\|\mathring{\Phi}_{\epsilon,\vec{\mathbf{y}}}(t)\|_{R^{s+1}}\leq C\|\delta\breve{\rho}_{\epsilon,\vec{\mathbf{y}}}\|_{L^{\frac{6}{5}}\cap H^s}+C\big(\|\delta\mathring{\zeta}_{\epsilon,\vec{\mathbf{y}}}\|_{L^\infty((t,1],H^s)}\big)\int^1_t\|\mathring{z}^k_{\epsilon,\vec{\mathbf{y}}}(\tau)\|_{H^s}d\tau,\quad T<t\leq 1.$$

Applying the norm $\|\cdot\|_{H^s}$-norm to (3.4.23) yields the estimate

$$\|\partial_t\mathring{\Phi}_{i,\epsilon,\vec{\mathbf{y}}}\|_{H^s}\lesssim\|\mathfrak{R}_i\mathfrak{R}_j\varpi^j_{\epsilon,\vec{\mathbf{y}}}\|_{H^s}\lesssim C\big(\|\delta\mathring{\zeta}_{\epsilon,\vec{\mathbf{y}}}\|_{L^\infty((t,1],H^s)}\big)\|\mathring{z}^j_{\epsilon,\vec{\mathbf{y}}}\|_{H^s},\quad T<t\leq 1,$$

which when combined with (3.4.27) gives

$$\|\partial_t\mathring{\Phi}_{\epsilon,\vec{\mathbf{y}}}\|_{R^{s+1}}\leq C\big(\|\delta\mathring{\zeta}_{\epsilon,\vec{\mathbf{y}}}\|_{L^\infty((t,1],H^s)}\big)\|\mathring{z}^j_{\epsilon,\vec{\mathbf{y}}}\|_{H^s},\quad T<t\leq 1.$$

By adding the conformal cosmological Poisson-Euler equations (3.1.59)-(3.1.60) together, we obtain the following equation for $\mathring{\rho}\mathring{z}^j$:

$$\partial_t\left(\mathring{\rho}\mathring{z}^j\right)+\sqrt{\frac{3}{\Lambda}}K\frac{\delta^{ji}}{\mathring{E}^2}\partial_i\mathring{\rho}+\sqrt{\frac{3}{\Lambda}}\partial_i\left(\mathring{\rho}\mathring{z}^i\mathring{z}^j\right)=\frac{4-3\mathring{\Omega}}{t}\mathring{\rho}\mathring{z}^j-\frac{1}{2}\left(\frac{3}{\Lambda}\right)^{\frac{3}{2}}\frac{t}{\mathring{E}}\delta^{ij}\mathring{\rho}\mathring{\Phi}_i.$$

With the help of (3.4.7), it is not difficult to verify that this equation is equivalent to

$$t\partial_t\varpi^j+\frac{\mathring{E}}{t^2}\sqrt{\frac{3}{\Lambda}}K\delta^{ij}\partial_i\delta\mathring{\rho}+\frac{\mathring{E}^3}{t^2}\sqrt{\frac{3}{\Lambda}}\partial_i\left(\mathring{\rho}\mathring{z}^i\mathring{z}^j\right)=\varpi^j-\frac{1}{2}\left(\frac{3}{\Lambda}\right)^{\frac{3}{2}}\frac{\mathring{E}^2}{t}\delta^{ij}(\mathring{\rho}\mathring{\Phi}_i). \tag{3.4.29}$$

From this equation, (3.4.25) and (3.4.28) , we obtain, with the help of calculus inequalities, the estimate

$$\begin{aligned}\|t\partial_t\varpi^j_{\epsilon,\vec{\mathbf{y}}}\|_{R^{s-1}}\leq&C\big(\|\delta\mathring{\zeta}_{\epsilon,\vec{\mathbf{y}}}\|_{L^\infty((t,1],H^s)},\|\mathring{z}^j_{\epsilon,\vec{\mathbf{y}}}\|_{L^\infty((t,1],H^s)}\big)\Big(\|\delta\mathring{\zeta}_{\epsilon,\vec{\mathbf{y}}}\|_{R^s}+\|\mathring{z}^j_{\epsilon,\vec{\mathbf{y}}}\|_{H^s}+\|\delta\breve{\rho}_{\epsilon,\vec{\mathbf{y}}}\|_{L^{\frac{6}{5}}\cap H^s}\\&+\int^1_t\|\mathring{z}^k_{\epsilon,\vec{\mathbf{y}}}(\tau)\|_{H^s}d\tau\Big).\end{aligned}$$

Next, applying the operator $\mathfrak{R}_j(-\Delta)^{-\frac{1}{2}}$ to (3.4.29) gives

$$\begin{aligned}t\partial_t\mathfrak{R}_j(-\Delta)^{-\frac{1}{2}}\varpi^j-\frac{\mathring{E}}{t^2}\sqrt{\frac{3}{\Lambda}}K\delta^{ij}\mathfrak{R}_j\mathfrak{R}_i\delta\mathring{\rho}-\frac{\mathring{E}^3}{t^2}\sqrt{\frac{3}{\Lambda}}\mathfrak{R}_j\mathfrak{R}_i\left(\mathring{\rho}\mathring{z}^i\mathring{z}^j\right)&\\=\mathfrak{R}_j(-\Delta)^{-\frac{1}{2}}\varpi^j-\frac{1}{2}\left(\frac{3}{\Lambda}\right)^{\frac{3}{2}}\frac{\mathring{E}^2}{t}\delta^{ij}\mathfrak{R}_j(-\Delta)^{-\frac{1}{2}}(\mathring{\rho}\mathring{\Phi}_i).&\end{aligned} \tag{3.4.30}$$

Using the potential theory estimates from Appendix B and (3.4.25), we observe that the bound

$$\|(-\Delta)^{-\frac{1}{2}}\mathfrak{R}_k\varpi^j\|_{R^{s+1}}\lesssim\|(-\Delta)^{\frac{1}{2}}\varpi^j\|_{L^6}+\sum^3_{l=1}\|\mathfrak{R}_l\mathfrak{R}_k\varpi^j\|_{H^s}\lesssim\|\varpi^j\|_{H^s}\leq C\big(\|\delta\mathring{\zeta}\|_{L^\infty((t,1],H^s)}\big)\|\mathring{z}^j\|_{H^s}$$

holds for $T < t \leq 1$. It is then not difficult to verify that the estimate

$$\|t\partial_t \mathfrak{R}_j(-\Delta)^{-\frac{1}{2}}\varpi^j_{\epsilon,\vec{\mathbf{y}}}\|_{R^s} \leq C\big(\|\delta\mathring{\zeta}_{\epsilon,\vec{\mathbf{y}}}\|_{L^\infty((t,1],H^s)}, \|\mathring{z}^j_{\epsilon,\vec{\mathbf{y}}}\|_{L^\infty((t,1],H^s)}\big)\big(\|\delta\mathring{\zeta}_{\epsilon,\vec{\mathbf{y}}}\|_{R^s} + \|\mathring{z}^j_{\epsilon,\vec{\mathbf{y}}}\|_{H^s} + \|\delta\breve{\rho}_{\epsilon,\vec{\mathbf{y}}}\|_{L^{\frac{6}{5}}\cap H^s} + \int_t^1 \|\mathring{z}^k_{\epsilon,\vec{\mathbf{y}}}(\tau)\|_{H^s} d\tau\big) \tag{3.4.31}$$

follows from (3.4.28), (3.4.30), the potential theory estimates, and the calculus inequalities.

Differentiating (3.4.21) with respect to t shows that $\partial_t^2\mathring{\Phi}_{\epsilon,\vec{\mathbf{y}}} = -\sqrt{\frac{\Lambda}{3}}\mathfrak{R}_j(-\Delta)^{-\frac{1}{2}}\partial_t\varpi^j_{\epsilon,\vec{\mathbf{y}}}$. Using this, we obtain, with the help of (3.4.31), the estimate

$$\|t\partial_t^2\mathring{\Phi}_{\epsilon,\vec{\mathbf{y}}}\|_{R^s} \lesssim \|\mathfrak{R}_j(-\Delta)^{-\frac{1}{2}}t\partial_t\varpi^j_{\epsilon,\vec{\mathbf{y}}}\|_{R^s} \leq C\big(\|\delta\mathring{\zeta}_{\epsilon,\vec{\mathbf{y}}}\|_{L^\infty((t,1],H^s)}, \|\mathring{z}^j_{\epsilon,\vec{\mathbf{y}}}\|_{L^\infty((t,1],H^s)}\big)\Big(\|\delta\mathring{\zeta}_{\epsilon,\vec{\mathbf{y}}}\|_{R^s} + \|\mathring{z}^j_{\epsilon,\vec{\mathbf{y}}}\|_{H^s} + \|\delta\breve{\rho}_{\epsilon,\vec{\mathbf{y}}}\|_{L^{\frac{6}{5}}\cap H^s} + \int_t^1 \|\mathring{z}^k_{\epsilon,\vec{\mathbf{y}}}(\tau)\|_{H^s} d\tau\Big)$$

for $T < t \leq 1$.

Applying the $\|\cdot\|_{R^s}$ norm to (3.4.22), it is clear that the estimate

$$\|\partial_t\mathring{\Upsilon}\|_{R^s} \lesssim \|\varpi^j\|_{R^s} \lesssim \|\varpi^j\|_{H^s} \leq C\big(\|\delta\mathring{\zeta}\|_{L^\infty((t,1],H^s)}\big)\|\mathring{z}^j\|_{H^s}, \quad t \in (T,1],$$

follows directly from (3.4.25) and the Yukawa operator estimate form Proposition 3.3.8. Finally, from the bound

$$\|\mathring{\Upsilon}_{i,\epsilon,\vec{\mathbf{y}}}(1)\|_{H^{s+1}} = \Big\|\epsilon\beta\frac{\Lambda}{3}\mathring{E}^3(1)(\Delta-\epsilon^2\beta)^{-1}\delta\mathring{\rho}_{\epsilon,\vec{\mathbf{y}}}(1)\Big\|_{H^{s+1}} \lesssim \|(\epsilon^2\beta-\Delta)^{-\frac{1}{2}}\delta\mathring{\rho}_{\epsilon,\vec{\mathbf{y}}}(1)\|_{H^{s+1}} \lesssim \|\delta\breve{\rho}_{\epsilon,\vec{\mathbf{y}}}\|_{L^{\frac{6}{5}}\cap H^s},$$

which follows from the Yukawa operator estimates from Proposition 3.3.8 along with (3.3.52) and (3.3.59), we see, after integrating (3.4.22) in time and applying the $\|\cdot\|_{H^s}$ norm, that

$$\begin{aligned}\|\mathring{\Upsilon}_{\epsilon,\vec{\mathbf{y}}}(t)\|_{H^s} &\lesssim \|\mathring{\Upsilon}_{\epsilon,\vec{\mathbf{y}}}(1)\|_{H^s} + \int_t^1 \|\epsilon\partial_j(\Delta-\epsilon^2\beta)^{-1}\varpi^j_{\epsilon,\vec{\mathbf{y}}}\|_{H^s}d\tau \leq C\|\delta\breve{\rho}_{\epsilon,\vec{\mathbf{y}}}\|_{L^{\frac{6}{5}}\cap H^s} + \int_t^1 \|\varpi^j_{\epsilon,\vec{\mathbf{y}}}(\tau)\|_{H^s}d\tau \\ &\leq C\|\delta\breve{\rho}_{\epsilon,\vec{\mathbf{y}}}\|_{L^{\frac{6}{5}}\cap H^s} + C\big(\|\delta\mathring{\zeta}_{\epsilon,\vec{\mathbf{y}}}\|_{L^\infty((t,1],H^s)}\big)\int_t^1 \|\mathring{z}^j_{\epsilon,\vec{\mathbf{y}}}(\tau)\|_{H^s}d\tau\end{aligned}$$

for $T < t \leq 1$, which complete the proof. □

Remark 3.4.5. It follows from (3.3.116) and (3.4.11) that the size of the initial $(\delta\mathring{\zeta}_{\epsilon,\vec{\mathbf{y}}}|_\Sigma, \mathring{z}^i_{\epsilon,\vec{\mathbf{y}}}|_\Sigma)$, as measured with respect to the H^s norm, is independent of the parameters $(\epsilon, \vec{\mathbf{y}})$. An immediate consequence is that the time of existence T from Proposition 3.4.4 is independent of $(\epsilon, \vec{\mathbf{y}})$.

3.5 A non-local formulation of the reduced conformal Einstein-Euler system

3.5.1 Poisson potential estimates

In §3.2.5, we brought the reduced conformal Einstein-Euler equations into a form, see (3.2.69), that is suitable for obtaining the global existence of solutions to the future at fixed $\epsilon > 0$ using the theory developed in [66]. However, due to the singular dependence of the source term $\hat{\mathbf{H}}$ in the evolution equations (3.2.69) on the parameter ϵ, these equations, in their current form, are not useful for analyzing the global existence problem in the limit $\epsilon \searrow 0$. To remedy this situation, we perform a non-local change of variables designed to eliminate the singular term from $\hat{\mathbf{H}}$. We note that a similar transformation was used previously in the articles [51, 59, 60, 61, 62, 63, 64, 65].

The transformation is based on shifting the metric variable $u^{0\mu}_i$, see (3.5.16), by the following non-local term

$$\Phi^\mu_k = \frac{\Lambda}{3}\frac{E^3}{t^3}\delta^\mu_0\partial_k(\Delta-\epsilon^2\beta)^{-1}\big(E^{-3}\sqrt{|\underline{\bar{g}}|}\underline{\bar{v}^0}\varrho - \mu^{\frac{1}{1+\epsilon^2 K}}\big) \tag{3.5.1}$$

where $\varrho = \rho^{\frac{1}{1+\epsilon^2 K}}$ and $|\underline{\bar{g}}| := -\det\big(\underline{\bar{g}_{\mu\nu}}\big)$. We note that this term is closely related to the spatial derivative of the Newtonian potential. We also observe that two equations

$$\partial_l\Phi^\mu_k = \frac{\Lambda}{3t^3}\delta^\mu_0(\Delta-\epsilon^2\beta)^{-1}\partial_k\partial_l\big(\sqrt{|\underline{\bar{g}}|}\underline{\bar{v}^0}\varrho - E^3\mu^{\frac{1}{1+\epsilon^2 K}}\big) \tag{3.5.2}$$

and

$$\delta^{ik}\partial_i\Phi^\mu_k = \frac{\Lambda}{3t^3}\delta^\mu_0\big(\sqrt{|\underline{\bar{g}}|}\underline{\bar{v}^0}\varrho - E^3\mu^{\frac{1}{1+\epsilon^2 K}}\big) + \epsilon^2\beta\frac{\Lambda}{3t^3}\delta^\mu_0(\Delta-\epsilon^2\beta)^{-1}\big(\sqrt{|\underline{\bar{g}}|}\underline{\bar{v}^0}\varrho - E^3\mu^{\frac{1}{1+\epsilon^2 K}}\big) \tag{3.5.3}$$

follow directly from differentiating (3.5.1).

For use below in simplifying the expression $\mathbf{F}$ from §3.7, we define

$$\Upsilon = \frac{\Lambda}{3}\frac{E^3}{t^3}\epsilon\beta(\Delta-\epsilon^2\beta)^{-1}\big(E^{-3}\sqrt{|\underline{\bar{g}}|}\underline{\bar{v}^0}\varrho - \mu^{\frac{1}{1+\epsilon^2 K}}\big), \tag{3.5.4}$$

where $\beta > 0$ is an arbitrary constant. We further note the expansions

$$\big(\rho^{\frac{1}{1+\epsilon^2 K}} - \mu^{\frac{1}{1+\epsilon^2 K}}\big) - (\rho-\mu) = \epsilon^2\mathscr{S}(\epsilon,t,\delta\zeta) \tag{3.5.5}$$

and

$$E^{-3}\sqrt{|\underline{\bar{g}}|}\underline{\bar{v}^0}\varrho - \mu^{\frac{1}{1+\epsilon^2 K}} = t^3 e^{\zeta_H}(e^{\delta\zeta}-1) + \epsilon\mathscr{T}_1(\epsilon,t,u^{\mu\nu},u,\delta\zeta) + \epsilon^2\mathscr{T}_2(\epsilon,t,u^{\mu\nu},u,\delta\zeta,z_j), \tag{3.5.6}$$

where $\mathscr{S}$, $\mathscr{T}_1$, and $\mathscr{T}_2$ vanish to first order in $\delta\zeta$, $(u^{\mu\nu},u)$, and $(u^{\mu\nu},u,\delta\zeta,z_j)$, respectively.

Proposition 3.5.1. *Suppose $s \in \mathbb{Z}_{\geq 3}$ and $\hat{\mathbf{U}}_{\epsilon,\vec{\mathbf{y}}} \in \bigcap_{\ell=0}^1 C^\ell((T,1],R^{s-\ell}(\mathbb{R}^3,\mathbb{K}))$ is the solution to (3.2.69) from Corollary 3.4.2. Then Φ^μ_k and Υ satisfy the estimates*

$$\|\Phi^\mu_{k,\epsilon,\vec{\mathbf{y}}}\|_{R^s} \leq C_0\Big(\|\breve{\xi}_\epsilon\|_s + \int_t^1(\|u^{0i}_{\epsilon,\vec{\mathbf{y}}}(\tau)\|_{R^s} + \|z_{l,\epsilon,\vec{\mathbf{y}}}(\tau)\|_{R^s})d\tau\Big), \tag{3.5.7}$$

$$\|\partial_l\Phi^\mu_{k,\epsilon,\vec{\mathbf{y}}}\|_{R^s} \leq C_0(\|\delta\zeta_{\epsilon,\vec{\mathbf{y}}}\|_{R^s} + \|u^{\mu\nu}_{\epsilon,\vec{\mathbf{y}}}\|_{R^s} + \|u_{\epsilon,\vec{\mathbf{y}}}\|_{R^s} + \|z_{j,\epsilon,\vec{\mathbf{y}}}\|_{R^s}), \tag{3.5.8}$$

$$\|\partial_t\Phi^\mu_{k,\epsilon,\vec{\mathbf{y}}}\|_{R^s} \leq C_0(\|u^{0j}_{\epsilon,\vec{\mathbf{y}}}\|_{R^s} + \|z_{i,\epsilon,\vec{\mathbf{y}}}\|_{R^s}) \tag{3.5.9}$$

and

$$\|\Upsilon_{\epsilon,\vec{\mathbf{y}}}\|_{R^s} \leq C_0\Big(\|\breve{\xi}_\epsilon\|_s + \int_t^1(\|u^{0i}_{\epsilon,\vec{\mathbf{y}}}(\tau)\|_{R^s} + \|z_{l,\epsilon,\vec{\mathbf{y}}}(\tau)\|_{R^s})d\tau\Big) \tag{3.5.10}$$

for $T < t \leq 1$, where $C_0 = C_0\big(\|(u^{\mu\nu}_{\epsilon,\vec{\mathbf{y}}},u_{\epsilon,\vec{\mathbf{y}}},\delta\zeta_{\epsilon,\vec{\mathbf{y}}},z_{j,\epsilon,\vec{\mathbf{y}}})\|_{L^\infty((t,1],R^s)}\big)$.

Proof. To simplify notation, we drop the subscripts involving the parameters $(\epsilon,\vec{\mathbf{y}})$ from all quantities for the remainder of this proof. Fixing a solution $\hat{\mathbf{U}} \in \bigcap_{\ell=0}^1 C^\ell((T,1],R^{s-\ell}(\mathbb{R}^3,\mathbb{K}))$ to the reduced Einstein-Euler system (3.2.69) from Corollary 3.4.2, we let $\bar{v}^\mu$, $\bar{\rho}$ and $\bar{g}_{\mu\nu}$ denote the fluid variables and spacetime metric in relativistic coordinates determined by this solution. Then, contracting the conformal Euler equations (3.1.36) with $\bar{v}_\nu$ yields the conformal continuity equation $\bar{v}^\mu\bar{\nabla}_\mu\bar{\rho} + (1+\epsilon^2 K)\bar{\rho}\bar{\nabla}_\mu\bar{v}^\mu = -3(1+\epsilon^2 K)\bar{\rho}\bar{v}^\mu\bar{\nabla}_\mu\Psi$, which in turn implies that $\bar{\nabla}_\mu(\bar{\varrho}\bar{v}^\mu) = \frac{1}{\sqrt{|\bar{g}|}}\bar{\partial}_\mu(\sqrt{|\bar{g}|}\bar{\varrho}\bar{v}^\mu) = \frac{3}{t}\bar{\varrho}\bar{v}^0$, where $\bar{\varrho} = \bar{\rho}^{\frac{1}{1+\epsilon^2 K}}$. From this equation, we then find that

$$\partial_t(\sqrt{|\underline{\bar{g}}|}\underline{\bar{v}^0}\varrho) + \partial_i(\sqrt{|\underline{\bar{g}}|}\varrho z^i) = \frac{3}{t}\sqrt{|\underline{\bar{g}}|}\underline{\bar{v}^0}\varrho. \tag{3.5.11}$$

Next, we see that the equations

$$\partial_t \Phi^\mu_k = -\frac{\Lambda}{3}(\Delta - \epsilon^2\beta)^{-1}\partial_k\partial_l\big(\delta^\mu_0\sqrt{|\underline{g}|}e^{\zeta}z^l\big) \tag{3.5.12}$$

and

$$\partial_t \Upsilon = -\epsilon\beta\frac{\Lambda}{3}(\Delta - \epsilon^2\beta)^{-1}\partial_l\big(\sqrt{|\underline{g}|}e^{\zeta}z^l\big). \tag{3.5.13}$$

follow from acting on (3.5.11) with the operators $\frac{\Lambda}{3}\frac{1}{t^3}\delta^\mu_0(\Delta-\epsilon^2\beta)^{-1}\partial_k$ and $\epsilon\beta\frac{\Lambda}{3}\frac{1}{t^3}(\Delta-\epsilon^2\beta)^{-1}$ along with the help of (3.1.49) and (3.5.1). Applying the $\|\cdot\|_{R^s}$ norm to (3.5.12), we obtain, with the help of (3.2.22), Proposition 3.3.6.(2) and the calculus inequalities from Appendix C , the estimate

$$\|\partial_t\Phi^\mu_k(t)\|_{R^s} \leq C_0(\|u^{0j}(t)\|_{R^s} + \|z_i(t)\|_{R^s}), \quad T<t\leq 1,$$

where, here and for the remainder of the proof, we let C_0 denote a constant of the form

$$C_0 = C_0\big(\|(u^{\mu\nu}_{\epsilon,\vec{\mathbf{y}}}, u_{\epsilon,\vec{\mathbf{y}}}, \delta\zeta_{\epsilon,\vec{\mathbf{y}}}, z_{j,\epsilon,\vec{\mathbf{y}}})\|_{L^\infty((t,1],R^s)}\big).$$

Integrating (3.5.12) and (3.5.13) in time, we obtain, after taking $\|\cdot\|_{R^s}$ norm and using (3.2.22), (3.3.57), (3.3.121), (3.5.1), (3.5.6), Propositions 3.3.5 and (3.3.8), and Corollary 3.4.2, the estimates

$$\begin{aligned}\|\Phi^\mu_k(t)\|_{R^s} \lesssim & \|\Phi^\mu_k(1)\|_{R^s} + \int_t^1 \Big\|(\Delta-\epsilon^2\beta)^{-1}\partial_k\partial_l\big(\sqrt{|\underline{g}|}e^{\zeta}z^i\delta^\mu_0\big)\Big\|_{R^s} d\tau \\ \leq & C\|(\epsilon^2\beta-\Delta)^{-\frac{1}{2}}\big(t^3e^{\zeta_H}(e^{\delta\zeta}-1) + \epsilon\mathscr{T}_1(\epsilon,t,u^{\mu\nu},u,\delta\zeta) + \epsilon^2\mathscr{T}_2(\epsilon,t,u^{\mu\nu},u,\delta\zeta,z_j)\big)(1)\|_{R^s} \\ & + C_0\int_t^1(\|u^{0i}(\tau)\|_{R^s} + \|z_l(\tau)\|_{R^s})d\tau \\ \leq & C_0\bigg(\|u^{\mu\nu}(1)\|_{R^s} + \|u(1)\|_{R^s} + \|\delta\zeta(1)\|_{R^s} + \|z_j(1)\|_{R^s} + \int_t^1(\|u^{0i}(\tau)\|_{R^s} + \|z_l(\tau)\|_{R^s})d\tau\bigg)\end{aligned}$$

and, similarly,

$$\|\Upsilon(t)\|_{R^s} \leq \bigg(\|u^{\mu\nu}(1)\|_{R^s} + \|u(1)\|_{R^s} + \|\delta\zeta(1)\|_{R^s} + \|z_j(1)\|_{R^s} + \int_t^1(\|u^{0i}(\tau)\|_{R^s} + \|z_l(\tau)\|_{R^s})d\tau\bigg)$$

for $T<t\leq 1$, where in the above derivations, we have used (3.3.52) to conclude that $\|(\epsilon^2\beta-\Delta)^{-\frac{1}{2}}\epsilon\mathscr{T}_1\|_{R^s} \lesssim \|\mathscr{T}_1\|_{R^s}$ and $\|(\epsilon^2\beta-\Delta)^{-\frac{1}{2}}\epsilon^2\mathscr{T}_2\|_{R^s} \lesssim \epsilon\|\mathscr{T}_2\|_{R^s}$. In addition, we deduce from (3.5.2) and (3.5.6) the estimates

$$\begin{aligned}\|\partial_l\Phi^\mu_k(t)\|_{L^6} \lesssim & \|(\Delta-\epsilon^2\beta)^{-1}\partial_k\partial_l\big(\sqrt{|\underline{g}|}\underline{\bar{v}}^0\varrho - E^3\mu^{\frac{1}{1+\epsilon^2K}}\big)(t)\|_{L^6} \\ \leq & C_0(\|\delta\zeta(t)\|_{L^6} + \|u^{\mu\nu}(t)\|_{L^6} + \|u(t)\|_{L^6} + \|z_j(t)\|_{L^6})\end{aligned} \tag{3.5.14}$$

and

$$\begin{aligned}\|\partial_j\partial_l\Phi^\mu_k(t)\|_{H^{s-1}} \lesssim & \|\partial_j(\Delta-\epsilon^2\beta)^{-1}\partial_k\partial_l\big(\sqrt{|\underline{g}|}\underline{\bar{v}}^0\varrho - E^3\mu^{\frac{1}{1+\epsilon^2K}}\big)(t)\|_{H^{s-1}} \\ \leq & C_0\big(\|\partial_j\delta\zeta(t)\|_{H^{s-1}} + \|\partial_j u^{\mu\nu}(t)\|_{H^{s-1}} + \|\partial_j u(t)\|_{H^{s-1}} + \|\partial_j z_j(t)\|_{H^{s-1}}\big),\end{aligned} \tag{3.5.15}$$

which hold for $T<t\leq 1$. Together, (3.5.14) and (3.5.15) imply that

$$\|\partial_l\Phi^\mu_k(t)\|_{R^s} \leq C_0(\|\delta\zeta(t)\|_{R^s} + \|u^{\mu\nu}(t)\|_{R^s} + \|u(t)\|_{R^s} + \|z_j(t)\|_{R^s}), \quad T<t\leq 1,$$

which completes the proof. □

3.5.2 The non-local transformation

From the definition of source term $\hat{H}$, see (3.2.69), it is clear, see (3.2.53)-(3.2.56) and (3.2.66), that the only ϵ-singular terms appear in $\hat{S}_1$ and are of the form $\frac{1}{\epsilon}\left(\underline{\bar{v}}^0-\sqrt{\frac{\Lambda}{3}}\right)$ and $-\frac{1}{\epsilon}\frac{\Lambda}{3}\frac{1}{t^2}E^2\delta\rho\delta^\mu_0$. Noting that $\frac{1}{\epsilon}\left(\underline{\bar{v}}^0-\sqrt{\frac{\Lambda}{3}}\right)$ is actually regular in ϵ as can be seen directly from the expansion (3.2.21), the only ϵ-singular term left to deal with is $-\frac{1}{\epsilon}\frac{\Lambda}{3}\frac{1}{t^2}E^2\delta\rho\delta^\mu_0$. Following the method introduced in [59] and then adapted to the cosmological setting in [61], we can remove this singular term from (3.2.69) while preserving its desirable structure via the introduction of the shifted variable

$$w^{0\mu}_k = u^{0\mu}_k - tE^{-1}\Phi^\mu_k, \tag{3.5.16}$$

where Φ^μ_k is as defined above by (3.5.1).

Under the change of variables (3.5.16), a short calculation using (3.5.3), where we note that

$$\frac{1}{\epsilon}\frac{\Lambda}{3t^2}\delta^\mu_0\left(E^{-3}\sqrt{|\underline{\bar{g}}|}\underline{\bar{v}}^0\varrho-\mu^{\frac{1}{1+\epsilon^2K}}\right)-\frac{1}{\epsilon}\frac{\Lambda}{3t^2}\delta^\mu_0\delta\rho=\mathscr{X}^\mu_1(\epsilon,t,u^{\alpha\beta},u,\delta\zeta)+\epsilon\mathscr{X}^\mu_2(\epsilon,t,u^{\alpha\beta},u,\delta\zeta,z_j)$$

with $\mathscr{X}^\mu_1$ and $\mathscr{X}^\mu_2$ vanishing to first order in $(u^{\alpha\beta},u)$ and $(u^{\alpha\beta},u,\delta\zeta,z_j)$, respectively, shows that equation (3.2.47) transforms into

$$\tilde{B}^0\partial_0\begin{pmatrix}u^{0\mu}_0\\ w^{0\mu}_k\\ u^{0\mu}\end{pmatrix}+\tilde{B}^k\partial_k\begin{pmatrix}u^{0\mu}_0\\ w^{0\mu}_l\\ u^{0\mu}\end{pmatrix}+\frac{1}{\epsilon}\tilde{C}^k\partial_k\begin{pmatrix}u^{0\mu}_0\\ w^{0\mu}_l\\ u^{0\mu}\end{pmatrix}=\frac{1}{t}\tilde{\mathfrak{B}}\mathbb{P}_2\begin{pmatrix}u^{0\mu}_0\\ w^{0\mu}_l\\ u^{0\mu}\end{pmatrix}+\tilde{G}_1+\tilde{S}_1, \tag{3.5.17}$$

where

$$\tilde{G}_1=E^2\begin{pmatrix}2E^{-2}\frac{\Omega}{t}\delta^{kj}\delta^\mu_j w^{00}_k-\frac{2(1-\epsilon^2K)}{t}\delta\rho u^{0\mu}+\tilde{f}^{0\mu}+\mathscr{L}^{0\mu}\\ \mathscr{L}^{0\mu l}\\ \mathscr{L}^{00\mu}\end{pmatrix}, \tag{3.5.18}$$

$$\tilde{S}_1=E^2\begin{pmatrix}2E^{-3}\Omega\delta^{kj}\delta^\mu_j\Phi^0_k+\Theta^{kl}tE^{-1}\partial_k\Phi^\mu_l-\frac{2}{t^2}\rho\delta^\mu_i z^i\sqrt{\frac{\Lambda}{3}}+E^{-3}t\delta^\mu_0\Upsilon+\epsilon\mathscr{J}^{0\mu}\\ \left(\frac{1}{2}+\Omega\right)E^{-1}\underline{\bar{g}}^{kl}\Phi^\mu_k-\underline{\bar{g}}^{kl}tE^{-1}\partial_0\Phi^\mu_k+\epsilon\mathscr{J}^{0\mu l}\\ \epsilon\mathscr{J}^{00\mu},\end{pmatrix},$$

and we have set

$$\begin{aligned}\tilde{f}^{0\mu}=&\mathscr{X}^\mu_1+\epsilon\mathscr{X}^\mu_2-\frac{\Lambda}{3}\frac{\epsilon K}{t^2}\delta\rho\delta^\mu_0-\frac{4}{t^2}\frac{1}{\epsilon}\sqrt{\frac{\Lambda}{3}}\delta\rho\delta^\mu_0\left(\underline{\bar{v}}^0-\sqrt{\frac{\Lambda}{3}}\right)-\frac{2}{t^2}\frac{1}{\epsilon}\delta\rho\delta^\mu_0\left(\underline{\bar{v}}^0-\sqrt{\frac{\Lambda}{3}}\right)^2\\ &-\frac{2}{t^2}\rho\delta^\mu_i z^i\left(\underline{\bar{v}}^0-\sqrt{\frac{\Lambda}{3}}\right)-\frac{2}{t^2}\frac{1}{\epsilon}\mu\delta^\mu_0\left(\underline{\bar{v}}^0-\sqrt{\frac{\Lambda}{3}}\right)\left(\underline{\bar{v}}^0+\sqrt{\frac{\Lambda}{3}}\right)-\frac{2}{t^2}\epsilon K\left(\delta\rho\underline{\bar{v}}^\mu\underline{\bar{v}}^0+\mu\left(\underline{\bar{v}}^\mu\underline{\bar{v}}^0-\frac{\Lambda}{3}\delta^\mu_0\right)\right).\end{aligned}$$

Observe now that the right hand side of this equation is regular in ϵ. For later use in §3.7, we decompose the remainder term $\hat{S}$, see (3.2.66), from the Euler equation (3.2.58) as

$$\hat{S}=G+S,$$

where

$$G=\begin{pmatrix}0\\ -K^{-1}\left[\sqrt{\frac{3}{\Lambda}}(-u^{0l}_0+(-3+4\Omega)u^{0l})+\frac{1}{2}\left(\frac{3}{\Lambda}\right)^{\frac{3}{2}}E^{-2}\delta^{lk}w^{00}_k\right]\end{pmatrix} \tag{3.5.19}$$

and

$$S = \begin{pmatrix} 0 \\ -K^{-1}\frac{1}{2}\left(\frac{3}{\Lambda}\right)^{\frac{3}{2}} E^{-3}\delta^{lk}t\Phi^0_k \end{pmatrix} + \epsilon\mathscr{S}(\epsilon, t, u, u^{\alpha\beta}, u_\gamma, u^{\alpha\beta}_\gamma, z_j). \tag{3.5.20}$$

3.5.3 The complete evolution system

We incorporate the shifted variable (3.5.16) into our set of gravitational variables by defining the vector quantity

$$\mathbf{U}_1 = (u^{0\mu}_0, w^{0\mu}_k, u^{0\mu}, u^{ij}_0, u^{ij}_k, u^{ij}, u_0, u_k, u)^T, \tag{3.5.21}$$

and then combine this with the fluid variables by defining

$$\mathbf{U} = (\mathbf{U}_1, \mathbf{U}_2)^T, \tag{3.5.22}$$

where $\mathbf{U}_2 = (\delta\zeta, z_i)^T$ is as previously defined by (3.2.68). Gathering (3.2.48), (3.2.49), (3.5.17) and (3.2.58) together, we arrive at the following complete evolution equation for $\mathbf{U}$:

$$\mathbf{B}^0\partial_t\mathbf{U} + \mathbf{B}^i\partial_i\mathbf{U} + \frac{1}{\epsilon}\mathbf{C}^i\partial_i\mathbf{U} = \frac{1}{t}\mathbf{B}\mathbf{P}\mathbf{U} + \mathbf{H} + \mathbf{F}, \tag{3.5.23}$$

where we recall that $\mathbf{B}^0$, $\mathbf{B}^i$, $\mathbf{C}^i$, $\mathbf{B}$ and $\mathbf{P}$ are defined by (3.2.70)-(3.2.71) and

$$\mathbf{H} = (\tilde{G}_1, \tilde{G}_2, \tilde{G}_3, G)^T \quad \text{and} \quad \mathbf{F} = (\tilde{S}_1, \tilde{S}_2, \tilde{S}_3, S)^T. \tag{3.5.24}$$

The importance of equation (3.5.23) is twofold. First, it is completely equivalent to the formulation (3.2.69) of the reduced conformal Einstein-Euler equations. Second, it is of the required form so that the a priori estimates established below in §3.6 apply to its solutions. These two properties will be crucial for the proof of Theorem 3.1.6; see §3.7 for details.

Before completing this section, we state the following proposition, which is a direct consequence of Corollary 3.4.2, Proposition 3.5.1 and the change of variables (3.5.16).

Proposition 3.5.2. *Suppose $s \in \mathbb{Z}_{\geq 3}$, $\epsilon_0 > 0$, $\epsilon \in (0, \epsilon_0)$, $\vec{\mathbf{y}} \in \mathbb{R}^{3N}$ and*

$$\hat{\mathbf{U}}_{\epsilon,\vec{\mathbf{y}}}|_\Sigma = \{u^{\mu\nu}_{\epsilon,\vec{\mathbf{y}}}, u_{\epsilon,\vec{\mathbf{y}}}, u^{ij}_{\gamma,\epsilon,\vec{\mathbf{y}}}, u^{0\mu}_{i,\epsilon,\vec{\mathbf{y}}}, u^{0\mu}_{0,\epsilon,\vec{\mathbf{y}}}, u_{\gamma,\epsilon,\vec{\mathbf{y}}}, z_{j,\epsilon,\vec{\mathbf{y}}}, \delta\zeta_{\epsilon,\vec{\mathbf{y}}}\}|_\Sigma \in X^s(\mathbb{R}^3)$$

is the initial data from Theorem 3.3.16. Then

1. *there exists a constant $T > 0$ and a unique classical solution*

$$\mathbf{U}_{\epsilon,\vec{\mathbf{y}}} \in \bigcap_{\ell=0}^{1} C^\ell((T, 1], R^{s-\ell}(\mathbb{R}^3, \mathbb{K}))$$

to (3.5.23) on the spacetime region $(T, 1] \times \mathbb{R}^3$ that agrees, after applying the transformation (3.5.16) to the $w^{0\mu}_{k,\epsilon,\vec{\mathbf{y}}}$ component of $\mathbf{U}_{\epsilon,\vec{\mathbf{y}}}$, with the initial data $\hat{\mathbf{U}}_{\epsilon,\vec{\mathbf{y}}}|_\Sigma$ on the initial hypersurface Σ,

2. *the $w^{0\mu}_{k,\epsilon,\vec{\mathbf{y}}}$ component of $\mathbf{U}_{\epsilon,\vec{\mathbf{y}}}$ can be expanded as*

$$w^{0\mu}_{k,\epsilon,\vec{\mathbf{y}}}|_\Sigma = \epsilon\mathcal{S}^\mu_k(\epsilon, \breve{\mathfrak{u}}^{kl}_{\epsilon,\vec{\mathbf{y}}}, \breve{\mathfrak{u}}^{kl}_{0,\epsilon,\vec{\mathbf{y}}}, \delta\breve{\rho}_{\epsilon,\vec{\mathbf{y}}}, \breve{z}^l_{\epsilon,\vec{\mathbf{y}}})$$

on the initial hypersurface, where $\mathcal{S}^\mu_k$ is defined in Theorem 3.3.16,

3. *and there exists a constant $\sigma > 0$, independent of the initial data and $T_1 \in (0, 1]$, such that if $\mathbf{U}_{\epsilon,\vec{\mathbf{y}}}$ exists for $t \in (T_1, 1]$ with the same regularity as above and satisfies $\|\mathbf{U}_{\epsilon,\vec{\mathbf{y}}}\|_{L^\infty((T_1,1],R^s))} < \sigma$,*

then the solution $\mathbf{U}_{\epsilon,\vec{\mathbf{y}}}$ can be uniquely continued as a classical solution with the same regularity to the larger spacetime region $(T^*, 1] \times \mathbb{R}^3$ for some $T^* \in (0, T_1)$.

Remark 3.5.3. It is worthwhile noting that the time of existence T from the above proposition can be chosen to be independent of ϵ. This follows from the form of the evolution equations (3.5.23), which would allow us to use the method from [12, 40, 41, 45] for deriving ϵ-independent energy estimates. We omit the details since we will establish this and more in the following section; see Theorem 3.6.10 for details.

3.6 Singular Symmetric Hyperbolic Systems

In this section, we establish uniform a priori estimates for solutions to a class of symmetric hyperbolic systems that are jointly singular in ϵ and t, and include both the formulation of the reduced conformal Einstein-Euler equations given by (3.5.23) and the $\epsilon \searrow 0$ limit of these equations. We also establish *error estimates*, that is, a priori estimates for the difference between solutions of the ϵ-dependent singular symmetric hyperbolic systems and their corresponding $\epsilon \searrow 0$ limit equations.

The ϵ-dependent singular terms that appear in the symmetric hyperbolic systems we consider are of a type that have been well studied, see [12, 40, 41, 45], while the t-dependent singular terms are of the type analyzed in [66]. Previously, we analyzed such systems on the torus $\mathbb{T}^n$ [51]. Here, we will generalize the results of [51] in three spatial dimensions from $\mathbb{T}^3$ to $\mathbb{R}^3$.

Remark 3.6.1. In this section, we switch to the standard time orientation, where the future is located in the direction of increasing time, while keeping the singularity located at $t = 0$. We do this in order to make the derivation of the energy estimates in this section as similar as possible to those for non-singular symmetric hyperbolic systems, which we expect will make it easier for readers familiar with such estimates to follow the arguments below. To get back to the time orientation used to formulate the conformal Einstein-Euler equations, we need only apply the trivial time transformation $t \mapsto -t$.

3.6.1 Uniform estimates

The class of singular hyperbolic systems that we will consider are of the following form:

$$A^0\partial_t U + A^i\partial_i U + \frac{1}{\epsilon}C^i\partial_i U = \frac{1}{t}\mathfrak{A}\mathbb{P}U + H \quad \text{in} \quad [T_0, T_1) \times \mathbb{R}^3, \tag{3.6.1}$$

where

$$\begin{aligned}
U &= (w, u)^T, \\
A^0 &= \begin{pmatrix} A_1^0(\epsilon, t, x, w) & 0 \\ 0 & A_2^0(\epsilon, t, x, w) \end{pmatrix}, \\
A^i &= \begin{pmatrix} A_1^i(\epsilon, t, x, w) & 0 \\ 0 & A_2^i(\epsilon, t, x, w) \end{pmatrix}, \\
C^i &= \begin{pmatrix} C_1^i & 0 \\ 0 & C_2^i \end{pmatrix}, \quad \mathbb{P} = \begin{pmatrix} \mathbb{P}_1 & 0 \\ 0 & \mathbb{P}_2 \end{pmatrix}, \\
\mathfrak{A} &= \begin{pmatrix} \mathfrak{A}_1(\epsilon, t, x, w) & 0 \\ 0 & \mathfrak{A}_2(\epsilon, t, x, w) \end{pmatrix}, \\
H &= \begin{pmatrix} H_1(\epsilon, t, x, w) \\ H_2(\epsilon, t, x, w, u) + R_2 \end{pmatrix} + \begin{pmatrix} F_1(\epsilon, t, x) \\ F_2(\epsilon, t, x) \end{pmatrix}, \\
R_2 &= \frac{1}{t}M_2(\epsilon, t, x, w, u)\mathbb{P}_3 U,
\end{aligned}$$

and the following assumptions hold for fixed constants $\epsilon_0, R > 0$, $T_0 < T_1 < 0$ and $s \in \mathbb{Z}_{\geq 3}$:

Assumptions 3.6.2.

1. *The* C_a^i*,* $i = 1, \dots, n$ *and* $a = 1, 2$*, are constant, symmetric* $N_a \times N_a$ *matrices.*

2. *The* $\mathbb{P}_a$*,* $a = 1, 2$*, are constant, symmetric* $N_a \times N_a$ *projection matrices, i.e.* $\mathbb{P}_a^2 = \mathbb{P}_a$*. We use* $\mathbb{P}_a^\perp = \mathbb{1} - \mathbb{P}_a$ *to denote the complementary projection matrix.*

3. $\mathbb{P}_4$ *is a constant, symmetric* $N_1 \times N_1$ *projection matrix that commutes with* A_1^μ*,* $\mathfrak{A}_1$*,* $\mathbb{P}_1$ *and* C_1^i*, that is,*
$$[\mathbb{P}_4, A_1^\mu] = [\mathbb{P}_4, \mathfrak{A}_1] = [\mathbb{P}_4, \mathbb{P}_1] = [\mathbb{P}_4, C_1^i] = 0.$$

4. *The source terms* $H_a(\epsilon, t, x, w)$*,* $a = 1, 2$*,* $F_a(\epsilon, t, x)$*,* $a = 1, 2$*, and* $M_2(\epsilon, t, x, w, u)$ *satisfy* $H_1 \in E^0\big((0, \epsilon_0) \times (2T_0, 0) \times \mathbb{R}^3 \times B_R(\mathbb{R}^{N_1}), \mathbb{R}^{N_1}\big)$*,* $H_2 \in E^0\big((0, \epsilon_0) \times (2T_0, 0) \times \mathbb{R}^3 \times B_R(\mathbb{R}^{N_1}) \times B_R(\mathbb{R}^{N_2}), \mathbb{R}^{N_2}\big)$*,* $F_a \in C^0\big((0, \epsilon_0) \times [T_0, T_1), H^s(\mathbb{R}^3, \mathbb{R}^{N_a})\big)$*,* $M_2 \in E^0\big((0, \epsilon_0) \times (2T_0, 0) \times \mathbb{R}^3 \times B_R(\mathbb{R}^{N_1}) \times B_R(\mathbb{R}^{N_2}), \mathbb{M}_{N_2 \times N_2}\big)$*, and*
$$\mathbb{P}_4 H_1(\epsilon, t, x, \mathbb{P}_4^\perp w) = 0, \quad H_1(\epsilon, t, x, 0) = 0, \quad H_2(\epsilon, t, x, 0, 0) = 0 \quad \text{and} \quad M_2(\epsilon, t, x, 0, 0) = 0$$
for all $(\epsilon, t, x) \in (0, \epsilon_0) \times (2T_0, 0) \times \mathbb{R}^3$*.*

5. *The matrix valued maps* $A_a^\mu(\epsilon, t, x, w)$*,* $\mu = 0, \dots, 3$ *and* $a = 1, 2$*, satisfy* $A_a^\mu \in E^0\big((0, \epsilon_0) \times (2T_0, 0) \times \mathbb{R}^3 \times B_R(\mathbb{R}^{N_a}), \mathbb{S}_{N_a}\big)$*.*

6. *The matrix valued maps* $A_a^0(\epsilon, t, x, w)$*,* $a = 1, 2$*, and* $\mathfrak{A}_a(\epsilon, t, x, w)$*,* $a = 1, 2$*, can be decomposed as*
$$A_a^0(\epsilon, t, x, w) = \mathring{A}_a^0(t) + \epsilon \tilde{A}_a^0(\epsilon, t, x, w), \tag{3.6.2}$$
$$\mathfrak{A}_a(\epsilon, t, x, w) = \mathring{\mathfrak{A}}_a(t) + \epsilon \tilde{\mathfrak{A}}_a(\epsilon, t, x, w), \tag{3.6.3}$$
where $\mathring{A}_a^0 \in E^1\big((2T_0, 0), \mathbb{S}_{N_a}\big)$*,* $\mathring{\mathfrak{A}}_a \in E^1\big((2T_0, 0), \mathbb{M}_{N_a \times N_a}\big)$*,* $\tilde{A}_a^0 \in E^1\big((0, \epsilon_0) \times (2T_0, 0) \times \mathbb{R}^3 \times B_R(\mathbb{R}^{N_1}), \mathbb{S}_{N_a}\big)$*,* $\tilde{\mathfrak{A}}_a \in E^0\big((0, \epsilon_0) \times (2T_0, 0) \times \mathbb{R}^3 \times B_R(\mathbb{R}^{N_1}), \mathbb{M}_{N_a \times N_a}\big)$*, and*[8]
$$D_x \tilde{\mathfrak{A}}_a(\epsilon, t, x, 0) = D_x \tilde{A}_a^0(\epsilon, t, x, 0) = 0 \tag{3.6.4}$$
for all $(\epsilon, t, x) \in (0, \epsilon_0) \times (2T_0, 0) \times \mathbb{R}^3$*.*

7. *For* $a = 1, 2$*, the matrix* $\mathfrak{A}_a$ *commutes with* $\mathbb{P}_a$*, i.e.*
$$[\mathbb{P}_a, \mathfrak{A}_a(\epsilon, t, x, w)] = 0 \tag{3.6.5}$$
for all $(\epsilon, t, x, w) \in (0, \epsilon_0) \times (2T_0, 0) \times \mathbb{R}^3 \times B(\mathbb{R}^{N_1})$*.*

8. $\mathbb{P}_3$ *is a symmetric* $(N_1 + N_2) \times (N_1 + N_2)$ *projection matrix that satisfies*
$$\mathbb{P}\mathbb{P}_3 = \mathbb{P}_3 \mathbb{P} = \mathbb{P}_3, \tag{3.6.6}$$
$$\mathbb{P}_3 A^i(\epsilon, t, x, w) \mathbb{P}_3^\perp = \mathbb{P}_3 C^i \mathbb{P}_3^\perp = \mathbb{P}_3 \mathfrak{A}(\epsilon, t, x, w) \mathbb{P}_3^\perp = 0 \tag{3.6.7}$$
and
$$[\mathbb{P}_3, A^0(\epsilon, t, x, w)] = 0 \tag{3.6.8}$$
for all $(\epsilon, t, x, w) \in (0, \epsilon_0) \times (2T_0, 0) \times \mathbb{R}^3 \times B_R(\mathbb{R}^{N_1})$*, where* $\mathbb{P}_3^\perp = \mathbb{1} - \mathbb{P}_3$ *defines the complementary projection matrix.*

9. *There exists constants* κ*,* γ_1*,* $\gamma_2 > 0$*, such that*
$$\frac{1}{\gamma_1} \mathbb{1} \leq A_a^0(\epsilon, t, x, w) \leq \frac{1}{\kappa} \mathfrak{A}_a(\epsilon, t, x, w) \leq \gamma_2 \mathbb{1} \tag{3.6.9}$$
for all $(\epsilon, t, x, w) \in (0, \epsilon_0) \times (2T_0, 0) \times \mathbb{R}^3 \times B(\mathbb{R}^{N_1})$ *and* $a = 1, 2$*.*

[8] Or in other words, the matrices $\tilde{\mathfrak{A}}_a|_{w=0}$ and $\tilde{A}_a^0|_{w=0}$ depend only on (ϵ, t).

10. *For $a = 1, 2$, the matrix A^0_a satisfies*

$$\mathbb{P}^\perp_a A^0_a(\epsilon, t, x, \mathbb{P}^\perp_1 w)\mathbb{P}_a = \mathbb{P}_a A^0_a(\epsilon, t, x, \mathbb{P}^\perp_1 w)\mathbb{P}^\perp_a = 0 \tag{3.6.10}$$

for all $(\epsilon, t, x, w) \in (0, \epsilon_0) \times (2T_0, 0) \times \mathbb{R}^3 \times B(\mathbb{R}^{N_1})$.

11. *For $a = 1, 2$, the matrix $\mathbb{P}^\perp_a[D_w A^0_a \cdot (A^0_1)^{-1}\mathfrak{A}_1\mathbb{P}_1 w]\mathbb{P}^\perp_a$ can be decomposed as*

$$\mathbb{P}^\perp_a\big[D_w A^0_a(\epsilon, t, x, w) \cdot \big(A^0_1(\epsilon, t, x, w)\big)^{-1}\mathfrak{A}_1(\epsilon, t, x, w)\mathbb{P}_1 w\big]\mathbb{P}^\perp_a = t\mathscr{S}_a(\epsilon, t, x, w) + \mathscr{T}_a(\epsilon, t, x, w, \mathbb{P}_1 w) \tag{3.6.11}$$

for some $\mathscr{S}_a \in E^0\big((0, \epsilon_0) \times (2T_0, 0) \times \mathbb{R}^3 \times B_R(\mathbb{R}^{N_1}), \mathbb{M}_{N_a\times N_a}\big)$, $a = 1, 2$, and $\mathscr{T}_a \in E^0\big((0, \epsilon_0) \times (2T_0, 0) \times \mathbb{R}^3 \times B_R(\mathbb{R}^{N_1}) \times \mathbb{R}^{N_1}, \mathbb{M}_{N_a\times N_a}\big)$, $a = 1, 2$, where the $\mathscr{T}_a(\epsilon, t, x, w, \xi)$ vanish to second order in ξ.

Before proceeding with the analysis, we take a moment to make a few observations about the structure of the singular system (3.6.1). First, if $\mathfrak{A} = 0$, then the singular term $\frac{1}{t}\mathfrak{A}\mathbb{P}U$ disappears from (3.6.1) and it becomes a regular symmetric hyperbolic system. Uniform ϵ-independent a priori estimates that are valid for $t \in [T_0, 0)$ would then follow, under a suitable small initial data assumption, as a direct consequence of the energy estimates from [12, 40, 41, 45]. When $\mathfrak{A} \neq 0$, the positivity assumption (3.6.9) guarantees that the singular term $\frac{1}{t}\mathfrak{A}\mathbb{P}U$ acts like a friction term. This allows us to generalize the energy estimates from [12, 40, 41, 45] in such a way as to obtain, under a suitable small initial data assumption, uniform ϵ-independent a priori estimates that are valid on the time interval $[T_0, 0)$; see (3.6.50), (3.6.51), (3.6.52) and (3.6.53) for the key differential inequalities used to derive these a priori estimates.

Remark 3.6.3. The equation for w decouples from the system (3.6.1) and is given by

$$A^0_1\partial_t w + A^i_1\partial_i w + \frac{1}{\epsilon}C^i_1\partial_i w = \frac{1}{t}\mathfrak{A}_1\mathbb{P}_1 w + H_1 + F_1 \quad \text{in} \quad [T_0, T_1) \times \mathbb{R}^3. \tag{3.6.12}$$

Remark 3.6.4.

1. By Taylor expanding $A^0_a(\epsilon, t, x, \mathbb{P}^\perp_1 w + \mathbb{P}_1 w)$ in the variable $\mathbb{P}_1 w$, it follows from (3.6.10) that there exist matrix valued maps $\hat{A}^0_a, \breve{A}^0_a \in E^1\big((0, \epsilon_0)\times(2T_0, 0)\times\mathbb{R}^3\times B_R\big(\mathbb{R}^{N_1}\big), \mathbb{M}_{N_a\times N_a}\big)$, $a = 1, 2$, such that

$$\mathbb{P}^\perp_a A^0_a(\epsilon, t, x, w)\mathbb{P}_a = \mathbb{P}^\perp_a[\hat{A}^0_a(\epsilon, t, x, w) \cdot \mathbb{P}_1 w]\mathbb{P}_a \tag{3.6.13}$$

and

$$\mathbb{P}_a A^0_a(\epsilon, t, x, w)\mathbb{P}^\perp_a = \mathbb{P}_a[\breve{A}^0_a(\epsilon, t, x, w) \cdot \mathbb{P}_1 w]\mathbb{P}^\perp_a \tag{3.6.14}$$

for all $(\epsilon, t, x, w) \in (0, \epsilon_0) \times (2T_0, 0) \times \mathbb{R}^3 \times B(\mathbb{R}^{N_1})$.

2. It is not difficult to see that the assumptions (3.6.9) and (3.6.10) imply that

$$\mathbb{P}^\perp_a\big(A^0_a(\epsilon, t, x, \mathbb{P}^\perp_1 w)\big)^{-1}\mathbb{P}_a = \mathbb{P}_a\big(A^0_a(\epsilon, t, x, \mathbb{P}^\perp_1 w)\big)^{-1}\mathbb{P}^\perp_a = 0$$

for all $(\epsilon, t, x, w) \in (0, \epsilon_0)\times(2T_0, 0)\times\mathbb{R}^3\times B(\mathbb{R}^{N_1})$. By Taylor expanding $\big(A^0_a(\epsilon, t, x, \mathbb{P}^\perp_1 w+\mathbb{P}_1 w)\big)^{-1}$ in the variable $\mathbb{P}_1 w$, it follows that there exist matrix valued maps $\hat{B}^0_a, \breve{B}^0_a \in E^1\big((0, \epsilon_0)\times(2T_0, 0)\times \mathbb{R}^3 \times B_R\big(\mathbb{R}^{N_1}\big), \mathbb{M}_{N_a\times N_a}\big)$, $a = 1, 2$, such that

$$\mathbb{P}^\perp_a\big(A^0_a(\epsilon, t, x, w)\big)^{-1}\mathbb{P}_a = \mathbb{P}^\perp_a[\hat{B}^0_a(\epsilon, t, x, w) \cdot \mathbb{P}_1 w]\mathbb{P}_a \tag{3.6.15}$$

and

$$\mathbb{P}_a\big(A^0_a(\epsilon, t, x, w)\big)^{-1}\mathbb{P}^\perp_a = \mathbb{P}_a[\breve{B}^0_a(\epsilon, t, x, w) \cdot \mathbb{P}_1 w]\mathbb{P}^\perp_a \tag{3.6.16}$$

for all $(\epsilon, t, x, w) \in (0, \epsilon_0) \times (2T_0, 0) \times \mathbb{R}^3 \times B(\mathbb{R}^{N_1})$.

To facilitate the statement and proof of our a priori estimates for solutions of the system (3.6.1), we introduce the following energy norms:

Definition 3.6.5. Suppose $w \in L^\infty([T_0, T_1) \times \mathbb{R}^3, \mathbb{R}^{N_1})$, $k \in \mathbb{Z}_{\geq 0}$, and $\{\mathbb{P}_a, A^0_a\}$, $a = 1, 2$, are as defined above. Then for maps f_a, $a = 1, 2$, and U from $\mathbb{R}^3$ into R^{N_a} and $R^{N_1} \times R^{N_2}$, respectively, the *energy norms* of f_a and U are defined by

$$\begin{aligned} |||f_a|||^2_{a,H^k} &= \sum_{0 \leq |\alpha| \leq k} \langle D^\alpha f_a, A^0_a\big(\epsilon, t, \cdot, w(t, \cdot)\big) D^\alpha f_a \rangle, \\ |||f_a|||^2_{a,R^k} &= |||Df_a|||^2_{a,H^{k-1}} + \|f_a\|^2_{L^6}, \\ |||U|||^2_{H^k} &= \sum_{0 \leq |\alpha| \leq k} \langle D^\alpha U, A^0\big(\epsilon, t, \cdot, w(t, \cdot)\big) D^\alpha U \rangle, \end{aligned}$$

and

$$|||U|||^2_{R^k} = |||DU|||^2_{H^{k-1}} + \|U\|^2_{L^6}.$$

In addition to the energy norms, we also define, for $T_0 < T \leq T_1$, the spacetime norm of maps f_a, $a = 1, 2$, from $[T_0, T) \times \mathbb{R}^3$ to R^{N_a} defined by

$$\|f_a\|_{M^\infty_{\mathbb{P}_a,k}([T_0,T)\times\mathbb{R}^3)} = \|f_a\|_{L^\infty([T_0,T),R^k(\mathbb{R}^3))} + \left(-\int_{T_0}^{T} \frac{1}{t} \|\mathbb{P}_a f_a(t)\|^2_{R^k(\mathbb{R}^3)} dt \right)^{\frac{1}{2}}.$$

Remark 3.6.6. For $w \in L^\infty([T_0, T_1) \times \mathbb{R}^3, \mathbb{R}^{N_1})$ satisfying $\|w\|_{L^\infty([T_0,T_1)\times\mathbb{R}^3)} < R$, we observe, by (3.6.9), that the standard Sobolev norm $\|\cdot\|_{H^k}$ and the energy norms $|||\cdot|||_{a,H^k}$, $a = 1, 2$, are equivalent since they satisfy

$$\frac{1}{\sqrt{\gamma_1}} \|\cdot\|_{H^k} \leq |||\cdot|||_{a,H^k} \leq \sqrt{\gamma_2} \|\cdot\|_{H^k}. \tag{3.6.17}$$

Furthermore, if $k \geq 2$, we have that

$$\|f_a\|_{R^k} \lesssim |||f_a|||_{a,R^k} \lesssim \|f_a\|_{R^k} \quad \text{and} \quad \|U\|_{R^k} \lesssim |||U|||_{R^k} \lesssim \|U\|_{R^k}. \tag{3.6.18}$$

These norm equivalences will be used without comment throughout this section.

With the preliminaries out of the way, we are now ready to state and prove a priori estimates for solutions of the system (3.6.1) that are uniform in ϵ.

Theorem 3.6.7. *Suppose $R > 0$, $s \in \mathbb{Z}_{\geq 3}$, $T_0 < T_1 < 0$, $\epsilon_0 > 0$, $\epsilon \in (0, \epsilon_0)$, Assumption 3.6.2 holds, the map*

$$U = (w, u) \in \bigcap_{\ell=0}^{1} C^\ell([T_0, T_1), R^{s-\ell}(\mathbb{R}^3, \mathbb{R}^{N_1})) \times \bigcap_{\ell=0}^{1} C^\ell([T_0, T_1), R^{s-1-\ell}(\mathbb{R}^3, \mathbb{R}^{N_2})),$$

defines a solution of the system (3.6.1), $\mathbb{P}_4 w \in \bigcap_{\ell=0}^{1} C^\ell([T_0, T_1), H^{s-\ell}(\mathbb{R}^3, \mathbb{R}^{N_1}))$, and for $t \in [T_0, T_1)$, the source terms F_a, $a = 1, 2$, satisfy the estimates

$$\|\mathbb{P}_4 F_1(\epsilon, t)\|_{L^2} + \|F_1(\epsilon, t)\|_{R^s} \leq C(\|w\|_{L^\infty([T_0,t),R^s)}, \|\mathbb{P}_4 w\|_{L^\infty([T_0,t),H^s)})(\mathcal{C}_*(t) + \mathcal{C}^* + \|w(t)\|_{R^s} + \|\mathbb{P}_4 w(t)\|_{H^s}) \tag{3.6.19}$$

and

$$\|F_2(\epsilon, t)\|_{R^{s-1}} \leq C\big(\|w\|_{L^\infty([T_0,t),R^s)}, \|\mathbb{P}_4 w\|_{L^\infty([T_0,t),H^s)}, \|u\|_{L^\infty([T_0,t),R^{s-1})}\big)(\mathcal{C}_*(t) + \mathcal{C}^* + \|w(t)\|_{R^s}$$

$$+ \|\mathbb{P}_4 w(t)\|_{H^s} + \|u(t)\|_{Q^{s-1}}), \tag{3.6.20}$$

where $\mathcal{C}_*(t) = \int_{T_0}^t \big(\|w(\tau)\|_{R^s} + \|\mathbb{P}_4 w(\tau)\|_{H^s} + \|u(\tau)\|_{R^{s-1}}\big)d\tau$ *and the constants* $\mathcal{C}^*$, $C\big(\|w\|_{L^\infty([T_0,t),R^s)}, \|\mathbb{P}_4 w\|_{L^\infty([T_0,t),H^s)}\big)$ *and* $C\big(\|w\|_{L^\infty([T_0,t),R^s)}, \|\mathbb{P}_4 w\|_{L^\infty([T_0,t),H^s)}, \|u\|_{L^\infty([T_0,t),R^{s-1})}\big)$ *are independent of* $\epsilon \in (0,\epsilon_0)$ *and* $T_1 \in (T_0, 0]$. *Then there exists a* $\sigma > 0$ *independent of* $\epsilon \in (0,\epsilon_0)$ *and* $T_1 \in (T_0, 0)$, *such that if initially*

$$\|w(T_0)\|_{R^s} + \|\mathbb{P}_4 w(T_0)\|_{H^s} + \|u(T_0)\|_{R^{s-1}} + \mathcal{C}^* \leq \sigma,$$

then

$$\|w\|_{L^\infty([T_0,T_1)\times\mathbb{R}^3)} \leq \frac{R}{2}$$

and there exists a constant $C > 0$, *independent of* $\epsilon \in (0,\epsilon_0)$ *and* $T_1 \in (T_0, 0)$, *such that*

$$\|\mathbb{P}_4 w\|_{L^\infty([T_0,t),L^2)} + \Big(-\int_{T_0}^t \frac{1}{\tau}\|\mathbb{P}_1\mathbb{P}_4 w(\tau)\|^2_{L^2}\, d\tau\Big)^{\frac{1}{2}} + \|w\|_{M^\infty_{\mathbb{P}_1,s}([T_0,t)\times\mathbb{R}^3)} + \|u\|_{M^\infty_{\mathbb{P}_2,s-1}([T_0,t)\times\mathbb{R}^3)} - \int_{T_0}^t \frac{1}{\tau}\|\mathbb{P}_3 U(\tau)\|_{Q^{s-1}}\, d\tau \leq C\sigma$$

for $T_0 \leq t < T_1$.

Proof. According to the definition of R^s and (3.1.13), there exists a constant C_1, such that

$$\|w(T_0)\|_{L^\infty} \leq C_1\big(\|w(T_0)\|_{R^s} + \|\mathbb{P}_4 w(T_0)\|_{L^2}\big) \leq C_1\sigma.$$

We then choose σ to satisfy

$$\sigma \leq \min\Big\{1, \frac{\hat{R}}{4}\Big\}, \tag{3.6.21}$$

where $\hat{R} = \frac{R}{2C_1}$, so that

$$\|w(T_0)\|_{L^\infty} \leq \frac{R}{8}.$$

Next, we define

$$K_1(t) = \|w\|_{L^\infty([T_0,t),R^s)}, \quad K_2(t) = \|u\|_{L^\infty([T_0,t),R^{s-1})} \quad \text{and} \quad K_3(t) = \|\mathbb{P}_4 w\|_{L^\infty([T_0,t),H^s)},$$

and we observe that $K_1(T_0) + K_2(T_0) + K_3(T_0) \leq \hat{R}/2$, and hence, by continuity, either $K_1(t) + K_2(t) + K_3(t) < \hat{R}$ for all $t \in [T_0, T_1)$, or else there exists a first time $T_* \in (T_0, T_1)$ such that $K_1(T_*) + K_2(T_*) + K_3(T_*) = \hat{R}$. Letting $T_* = T_1$ if the first case holds, we then have that

$$K_1(t) + K_2(t) + K_3(t) < \hat{R}, \quad 0 \leq t < T_*, \tag{3.6.22}$$

where $T_* = T_1$ or else T_* is the first time in (T_0, T_1) for which $K_1(T_*) + K_2(T_*) + K_3(T_*) = \hat{R}$.

Before proceeding the proof, we first establish a number of estimates that will be needed in the proof; we collect them together in the following lemma.

Lemma 3.6.8. *There exists constants* $C(K_1(t))$ *and* $C(K_1(t), K_2(t))$, *both independent of* $\epsilon \in (0,\epsilon_0)$ *and* $T_* \in (T_0, T_1]$, *such that the following estimates hold for* $T_0 \leq t < T_* < 0$:

$$-\frac{2}{t}\sum_{1\leq|\alpha|\leq s}\langle D^\alpha w, A^0_1[(A^0_1)^{-1}\mathfrak{A}_1, D^\alpha]\mathbb{P}_1 w\rangle \leq -\frac{1}{t}C(K_1)\|w\|_{R^s}|||\mathbb{P}_1 w|||^2_{1,R^s}, \tag{3.6.23}$$

$$-\frac{2}{t}\sum_{1\leq|\alpha|\leq s-1}\langle D^\alpha u, A^0_2[(A^0_2)^{-1}\mathfrak{A}_2, D^\alpha]\mathbb{P}_2 u\rangle \leq -\frac{1}{t}C(K_1)(\|u\|_{Q^{s-1}} + \|w\|_{R^s})(|||\mathbb{P}_2 u|||^2_{2,R^{s-1}} + |||\mathbb{P}_2 w|||^2_{1,R^s}), \tag{3.6.24}$$

$$-\sum_{1\leq|\alpha|\leq s}\langle D^\alpha w, A_1^0[D^\alpha,(A_1^0)^{-1}A_1^i]\partial_i w\rangle \leq C(K_1)\|w\|^2_{R^s}, \tag{3.6.25}$$

$$-\sum_{1\leq|\alpha|\leq s-1}\langle D^\alpha u, A_2^0[D^\alpha,(A_2^0)^{-1}A_2^i]\partial_i u\rangle \leq C(K_1)\|u\|^2_{Q^{s-1}}, \tag{3.6.26}$$

$$-\sum_{1\leq|\alpha|\leq s}\langle D^\alpha w, [\tilde{A}_1^0, D^\alpha](A_1^0)^{-1}C_1^i\partial_i w\rangle \leq C(K_1)\|w\|^2_{R^s}, \tag{3.6.27}$$

$$-\sum_{1\leq|\alpha|\leq s-1}\langle D^\alpha u, [\tilde{A}_2^0, D^\alpha](A_2^0)^{-1}C_2^i\partial_i u\rangle \leq C(K_1)\|u\|^2_{Q^{s-1}}, \tag{3.6.28}$$

$$\sum_{1\leq|\alpha|\leq s}\langle D^\alpha w, (\partial_t A_1^0)D^\alpha w\rangle \leq C(K_1)\|w\|^2_{R^s} - \frac{1}{t}C(K_1)\|w\|_{R^s}|||\mathbb{P}_1 w|||^2_{1,R^s}, \tag{3.6.29}$$

$$\sum_{1\leq|\alpha|\leq s-1}\langle D^\alpha u, (\partial_t A_2^0)D^\alpha u\rangle \leq C(K_1)\|u\|^2_{Q^{s-1}} - \frac{1}{t}C(K_1,K_2)(\|u\|_{Q^{s-1}}+\|w\|_{R^s})(|||\mathbb{P}_2 u|||^2_{2,R^{s-1}}+|||\mathbb{P}_1 w|||^2_{1,R^s}) \tag{3.6.30}$$

and

$$\sum_{1\leq|\alpha|\leq s-1}\langle D^\alpha\mathbb{P}_3 U, (\partial_t A^0)D^\alpha\mathbb{P}_3 U\rangle \leq -\frac{1}{t}C(K_1)\|\mathbb{P}_1 w\|_{R^s}|||\mathbb{P}_3 U|||^2_{R^{s-1}} + C(K_1)\|\mathbb{P}_3 U\|^2_{Q^{s-1}}. \tag{3.6.31}$$

Proof. Using the properties $\mathbb{P}_1^2=\mathbb{P}_1$, $\mathbb{P}_1+\mathbb{P}_1^\perp=\mathbb{1}$, $\mathbb{P}_1^{\mathrm{T}}=\mathbb{P}_1$, and $D\mathbb{P}_1=0$ of the projection matrix $\mathbb{P}_1$ repeatedly, we compute

$$\begin{aligned}
&-\frac{2}{t}\sum_{1\leq|\alpha|\leq s}\langle D^\alpha w, A_1^0[(A_1^0)^{-1}\mathfrak{A}_1, D^\alpha]\mathbb{P}_1 w\rangle\\
&=-\frac{2}{t}\sum_{1\leq|\alpha|\leq s}\langle D^\alpha\mathbb{P}_1 w, A_1^0[(A_1^0)^{-1}\mathfrak{A}_1, D^\alpha]\mathbb{P}_1 w\rangle - \frac{2}{t}\sum_{1\leq|\alpha|\leq s}\langle D^\alpha\mathbb{P}_1^\perp w, \mathbb{P}_1^\perp A_1^0[(A_1^0)^{-1}\mathfrak{A}_1, D^\alpha]\mathbb{P}_1 w\rangle\\
&\overset{by (3.6.5)}{=}-\frac{2}{t}\sum_{1\leq|\alpha|\leq s}\langle D^\alpha\mathbb{P}_1 w, A_1^0[(A_1^0)^{-1}\mathfrak{A}_1, D^\alpha]\mathbb{P}_1 w\rangle - \frac{2}{t}\sum_{1\leq|\alpha|\leq s}\langle D^\alpha\mathbb{P}_1^\perp w, \mathbb{P}_1^\perp A_1^0[(A_1^0)^{-1}\mathbb{P}_1\mathfrak{A}_1, D^\alpha]\mathbb{P}_1 w\rangle\\
&=-\frac{2}{t}\sum_{1\leq|\alpha|\leq s}\langle D^\alpha\mathbb{P}_1 w, A_1^0[(A_1^0)^{-1}\mathfrak{A}_1, D^\alpha]\mathbb{P}_1 w\rangle - \frac{2}{t}\sum_{1\leq|\alpha|\leq s}\langle D^\alpha\mathbb{P}_1^\perp w, \mathbb{P}_1^\perp A_1^0\mathbb{P}_1^\perp[\mathbb{P}_1^\perp(A_1^0)^{-1}\mathbb{P}_1\mathfrak{A}_1, D^\alpha]\mathbb{P}_1 w\rangle\\
&\qquad-\frac{2}{t}\sum_{1\leq|\alpha|\leq s}\langle D^\alpha\mathbb{P}_1^\perp w, \mathbb{P}_1^\perp A_1^0\mathbb{P}_1[\mathbb{P}_1(A_1^0)^{-1}\mathbb{P}_1\mathfrak{A}_1, D^\alpha]\mathbb{P}_1 w\rangle.
\end{aligned}$$

From this expression, we obtain, with the help of the Cauchy-Schwarz inequality, the calculus inequalities (C.2.2) and Proposition C.2.8, the expansions (3.6.2)-(3.6.3), the relations (3.6.4), (3.6.13), and (3.6.15), the inequality (3.6.22) and the equivalence of norms (3.6.18), the estimate

$$\begin{aligned}
&-\frac{1}{t}\sum_{1\leq|\alpha|\leq s}\langle D^\alpha w, A_1^0[(A_1^0)^{-1}\mathfrak{A}_1, D^\alpha]\mathbb{P}_1 w\rangle\\
&\lesssim -\frac{1}{t}\big[\|A_1^0\|_{L^\infty}\|\mathbb{P}_1 w\|_{R^s}\|D\big((A_1^0)^{-1}\mathfrak{A}_1\big)\|_{R^{s-1}\cap L^2} + \|A_1^0\|_{L^\infty}\|\mathbb{P}_1^\perp w\|_{R^s}\|D\big(\mathbb{P}_1^\perp(A_1^0)^{-1}\mathbb{P}_1\mathfrak{A}_1\big)\|_{R^{s-1}\cap L^2}\\
&+\|\mathbb{P}_1^\perp A_1^0\mathbb{P}_1\|_{L^\infty}\|\mathbb{P}_1^\perp w\|_{R^s}\|D\big(\mathbb{P}_1(A_1^0)^{-1}\mathbb{P}_1\mathfrak{A}_1\big)\|_{R^{s-1}\cap L^2}\big]\|\mathbb{P}_1 w\|_{R^{s-1}}\\
\leq&-C(K_1)\frac{1}{t}\|w\|_{R^s}\|\mathbb{P}_1 w\|^2_{R^s} \leq -\frac{1}{t}C(K_1)\|w\|_{R^s}|||\mathbb{P}_1 w|||^2_{1,R^s}
\end{aligned}$$

for $T_0\leq t<T_*$, where the constant $C(K_1)$ is independent of $\epsilon\in(0,\epsilon_0)$ and $T_*\in(T_0,T_1]$. This

establishes the first estimate (3.6.23). By a similar calculation, we find that

$$
\begin{aligned}
-\frac{2}{t}\sum_{1\le|\alpha|\le s-1}\langle D^\alpha u, A_2^0[(A_2^0)^{-1}\mathfrak{A}_2, D^\alpha]\mathbb{P}_2 u\rangle = & -\frac{2}{t}\sum_{1\le|\alpha|\le s-1}\langle D^\alpha \mathbb{P}_2 u, A_2^0[(A_2^0)^{-1}\mathfrak{A}_2, D^\alpha]\mathbb{P}_2 u\rangle \\
& -\frac{2}{t}\sum_{1\le|\alpha|\le s-1}\langle D^\alpha \mathbb{P}_2^\perp u, \mathbb{P}_2^\perp A_2^0\mathbb{P}_2^\perp[\mathbb{P}_2^\perp(A_2^0)^{-1}\mathbb{P}_2\mathfrak{A}_2, D^\alpha]\mathbb{P}_2 u\rangle \\
& -\frac{2}{t}\sum_{1\le|\alpha|\le s-1}\langle D^\alpha \mathbb{P}_2^\perp u, \mathbb{P}_2^\perp A_2^0\mathbb{P}_2[\mathbb{P}_2(A_2^0)^{-1}\mathbb{P}_2\mathfrak{A}_2, D^\alpha]\mathbb{P}_2 u\rangle \\
\le -\frac{1}{t}C(K_1)\|w\|_{R^s}\|\mathbb{P}_2 u\|^2_{Q^{s-1}} - \frac{1}{t}C(K_1)\|u\|_{Q^{s-1}}\|\mathbb{P}_1 w\|_{R^s}\|\mathbb{P}_2 u\|_{Q^{s-1}} & - \frac{1}{t}C(K_1)\|u\|_{Q^{s-1}}\|\mathbb{P}_1 w\|_{R^s}\|\mathbb{P}_2 u\|_{Q^{s-1}} \\
\le -\frac{1}{t}C(K_1)(\|u\|_{Q^{s-1}} + \|w\|_{R^s})(|\!|\!|\mathbb{P}_2 u|\!|\!|^2_{2,R^{s-1}} & + |\!|\!|\mathbb{P}_2 w|\!|\!|^2_{1,R^s}),
\end{aligned}
$$

which establishes the second estimate (3.6.24).

Next, using the calculus inequalities (C.2.2) and Proposition C.2.8, we observe that

$$
\sum_{1\le|\alpha|\le s-1}\langle D^\alpha u, -A_2^0[D^\alpha,(A_2^0)^{-1}A_2^i]\partial_i u\rangle \lesssim \|A_2^0\|_{L^\infty}\|u\|^2_{Q^{s-1}}\|D((A_2^0)^{-1}A_2^i)\|_{Q^{s-1}\cap L^2} \le C(K_1)\|u\|^2_{Q^{s-1}},
$$

which establishes the fourth estimate (3.6.26). Since the estimates (3.6.25), (3.6.27) and (3.6.28) can be obtained in a similar fashion, we omit the details.

Finally, we consider the estimates (3.6.29)-(3.6.30). We begin establishing these estimates by writing (3.6.12) as

$$
\epsilon\partial_t w = \epsilon\frac{1}{t}(A_1^0)^{-1}\mathfrak{A}_1\mathbb{P}_1 w - \epsilon(A_1^0)^{-1}A_1^i\partial_i w - (A_1^0)^{-1}C_1^i\partial_i w + \epsilon(A_1^0)^{-1}H_1 + \epsilon(A_1^0)^{-1}F_1.
$$

Using this and the expansion (3.6.2), we can express the time derivatives $\partial_t A_a^0$, $a=1,2$, as

$$
\begin{aligned}
\partial_t A_a^0 = & D_w A_a^0\cdot\partial_t w + D_t A_a^0 \\
= & -D_w A_a^0\cdot(A_1^0)^{-1}A_1^i\partial_i w - [D_w\tilde{A}_a^0\cdot(A_1^0)^{-1}C_1^i\partial_i w] \\
& + [D_w A_a^0\cdot(A_1^0)^{-1}H_1] + D_t A_a^0 + [D_w A_a^0\cdot(A_1^0)^{-1}F_1] + \frac{1}{t}[D_w A_a^0\cdot(A_1^0)^{-1}\mathfrak{A}_1\mathbb{P}_1 w].
\end{aligned} \tag{3.6.32}
$$

Using (3.6.32) with $a=2$, we see, with the help of the calculus inequalities from Appendix C, the Cauchy-Schwarz inequality, the estimate (3.6.19), and the expansion (3.6.11) for $a=2$, that

$$
\begin{aligned}
\sum_{1\le|\alpha|\le s-1}\langle D^\alpha u,(\partial_t A_2^0)D^\alpha u\rangle \le & \sum_{1\le|\alpha|\le s-1}\Big[\langle D^\alpha u,\mathbb{P}_2^\perp(\partial_t A_2^0)\mathbb{P}_2^\perp D^\alpha u\rangle + \langle D^\alpha u,\mathbb{P}_2^\perp(\partial_t A_2^0)\mathbb{P}_2 D^\alpha u\rangle \\
& + \langle D^\alpha u,\mathbb{P}_2(\partial_t A_2^0)\mathbb{P}_2^\perp D^\alpha u\rangle + \langle D^\alpha u,\mathbb{P}_2(\partial_t A_2^0)\mathbb{P}_2 D^\alpha u\rangle\Big] \\
\le & C(K_1)\|u\|^2_{Q^{s-1}} - \frac{2}{t}\|u\|_{Q^{s-1}}\|(A_1^0)^{-1}\mathfrak{A}_1\|_{L^\infty}\|D_w A_2^0\|_{L^\infty}\|\mathbb{P}_2 u\|_{Q^{s-1}}\|\mathbb{P}_1 w\|_{Q^{s-1}} \\
& - \frac{1}{t}\|\mathbb{P}_1 w\|_{R^s}\|(A^0)^{-1}\mathfrak{A}\|_{L^\infty}\|D_w A_2^0\|_{L^\infty}\|\mathbb{P}_2 u\|^2_{H^{s-1}} - \frac{1}{t}\|u\|^2_{Q^{s-1}}C(K_1)\|\mathbb{P}_1 w\|^2_{Q^{s-1}} \\
\le & C(K_1)\|u\|^2_{Q^{s-1}} - \frac{1}{t}C(K_1,K_2)(\|u\|_{Q^{s-1}} + \|w\|_{R^s})(\|\mathbb{P}_2 u\|^2_{Q^{s-1}} + \|\mathbb{P}_1 w\|^2_{R^s}).
\end{aligned}
$$

With the help of (3.6.18), this establishes the estimate (3.6.30). Since the estimate (3.6.29) can be established using similar arguments, we omit the details. The last estimate (3.6.31) can also be established using similar arguments with the help of the identity $\mathbb{P}_3\mathbb{P} = \mathbb{P}\mathbb{P}_3 = \mathbb{P}_3$. We again omit the details. □

Applying $A^0D^\alpha(A^0)^{-1}$ to both sides of (3.6.1), we find that

$$A^0\partial_t D^\alpha U + A^i\partial_i D^\alpha U + \frac{1}{\epsilon}C^i\partial_i D^\alpha U = - A^0[D^\alpha,(A^0)^{-1}A^i]\partial_i U - [\tilde{A}^0, D^\alpha](A^0)^{-1}C^i\partial_i U + \frac{1}{t}\mathfrak{A}D^\alpha\mathbb{P}U + \frac{1}{t}A^0[D^\alpha,(A^0)^{-1}\mathfrak{A}]\mathbb{P}U + A^0D^\alpha[(A^0)^{-1}H], \tag{3.6.33}$$

where in deriving this we have used

$$\frac{1}{\epsilon}[A^0,D^\alpha](A^0)^{-1}C^i\partial_i U \overset{(3.6.2)}{=} \frac{1}{\epsilon}[\mathring{A}^0+\epsilon\tilde{A}^0,D^\alpha](A^0)^{-1}C^i\partial_i U = [\tilde{A}^0,D^\alpha](A^0)^{-1}C^i\partial_i U$$

and

$$\begin{aligned} A^0[D^\alpha,(A^0)^{-1}]C^i\partial_i U &= A^0D^\alpha\big((A^0)^{-1}C^i\partial_i U\big) - D^\alpha\big(C^i\partial_i U\big) \\ &= A^0D^\alpha\big((A^0)^{-1}C^i\partial_i U\big) - D^\alpha\big(A^0(A^0)^{-1}C^i\partial_i U\big) = [A^0,D^\alpha](A^0)^{-1}C^i\partial_i U. \end{aligned}$$

Writing A^0_a, $a=1,2$, as $A^0_a=(A^0_a)^{\frac{1}{2}}(A^0_a)^{\frac{1}{2}}$, which we can do since A^0_a is a real symmetric and positive-definite, we see from (3.6.9) that

$$(A^0_a)^{-\frac{1}{2}}\mathfrak{A}_\mathfrak{a}(A^0_a)^{-\frac{1}{2}} \geq \kappa\mathbb{1}. \tag{3.6.34}$$

Since, by (3.6.5),

$$\frac{2}{t}\langle D^\alpha f,\mathfrak{A}_a D^\alpha\mathbb{P}_a f\rangle = \frac{2}{t}\langle D^\alpha\mathbb{P}_a f,(A^0)^{\frac{1}{2}}[(A^0_a)^{-\frac{1}{2}}\mathfrak{A}_a(A^0_a)^{-\frac{1}{2}}](A^0_a)^{\frac{1}{2}}D^\alpha\mathbb{P}_a f\rangle, \quad a=1,2,$$

it follows immediately from (3.6.34) that

$$\frac{2}{t}\sum_{1\leq|\alpha|\leq s-1}\langle D^\alpha u,\mathfrak{A}_2D^\alpha\mathbb{P}_2u\rangle \leq \frac{2\kappa}{t}|\!|\!|D\mathbb{P}_2u|\!|\!|^2_{2,H^{s-2}} \quad\text{and}\quad \frac{2}{t}\sum_{1\leq|\alpha|\leq s}\langle D^\alpha w,\mathfrak{A}_1D^\alpha\mathbb{P}_1w\rangle \leq \frac{2\kappa}{t}|\!|\!|D\mathbb{P}_1w|\!|\!|^2_{1,H^{s-1}}. \tag{3.6.35}$$

Letting f_1 denote one of $\mathbb{P}_1w$, w or $\mathbb{P}_4w$, and f_2 denote one of $\mathbb{P}_2u$ or u , we have, by Theorem C.2.2.(1) and (3.6.17), that

$$\|f_a\|_{L^6} \leq C_{\mathrm{S}}\|Df_a\|_{L^2} \leq C_{\mathrm{S}}\|Df_a\|_{H^{s-a}} \leq C_{\mathrm{S}}\sqrt{\gamma_1}|\!|\!|Df_a|\!|\!|_{a,H^{s-a}}, \tag{3.6.36}$$

for $a=1,2$, which yields that

$$|\!|\!|f_a|\!|\!|^2_{a,R^{s-a+1}} \leq (C^2_{\mathrm{S}}\gamma_1+1)|\!|\!|Df_a|\!|\!|^2_{a,H^{s-a}} \tag{3.6.37}$$

and

$$\frac{\kappa}{t}|\!|\!|Df_a|\!|\!|^2_{a,H^{s-a}} \leq \frac{\kappa}{t}\frac{1}{C^2_{\mathrm{S}}\gamma_1}\|f_a\|^2_{L^6}. \tag{3.6.38}$$

Adding $\frac{\kappa}{t}|\!|\!|Df_a|\!|\!|^2_{a,H^{s-a}}$ on both sides of above (3.6.38), recall $t<0$, yields

$$\frac{2\kappa}{t}|\!|\!|Df_a|\!|\!|^2_{a,H^{s-a}} \leq \frac{\kappa}{t}\frac{1}{C^2_{\mathrm{S}}\gamma_1}\|f_a\|^2_{L^6} + \frac{\kappa}{t}|\!|\!|Df_a|\!|\!|^2_{a,H^{s-a}} \leq \frac{2\hat{\kappa}}{t}|\!|\!|f_a|\!|\!|^2_{a,R^{s-a+1}} \tag{3.6.39}$$

where we have set

$$\hat{\kappa} = \frac{1}{2}\kappa\min\Big(\frac{1}{C^2_{\mathrm{S}}\gamma_1},1\Big).$$

Using (3.6.39), it is clear that the inequalities (3.6.35) imply that

$$\frac{2}{t}\sum_{1\leq|\alpha|\leq s-1}\langle D^\alpha u, \mathfrak{A}_2 D^\alpha \mathbb{P}_2 u\rangle \leq \frac{2\hat{\kappa}}{t}|||\mathbb{P}_2 u|||^2_{2,R^{s-1}} \quad \text{and} \quad \frac{2}{t}\sum_{1\leq|\alpha|\leq s}\langle D^\alpha w, \mathfrak{A}_1 D^\alpha \mathbb{P}_1 w\rangle \leq \frac{2\hat{\kappa}}{t}|||\mathbb{P}_1 w|||^2_{1,R^s}.$$

Then differentiating $\langle D^\alpha w, A^0_1 D^\alpha w\rangle$ with respect to t, we see, from the identities $\langle D^\alpha w, C^i_1\partial_i D^\alpha w\rangle = 0$ and $2\langle D^\alpha w, A^i_1\partial_i D^\alpha w\rangle = -\langle D^\alpha w, (\partial_i A^i_1)D^\alpha w\rangle$, the block decomposition of (3.6.33), which we can use to determine $D^\alpha\partial_t w$, the estimates (3.6.19) and (3.6.35) together with those from Lemma 3.6.8, the relation (3.6.37), Young's inequality (Proposition C.2.10) and the calculus inequalities from Appendix C, that

$$\begin{aligned}
\partial_t|||Dw|||^2_{1,H^{s-1}} =& \sum_{1\leq|\alpha|\leq s}\langle D^\alpha w, (\partial_t A^0_1)D^\alpha w\rangle + 2\sum_{1\leq|\alpha|\leq s}\langle D^\alpha w, A^0_1 D^\alpha\partial_t w\rangle \\
\leq& C(K_1)\|w\|^2_{R^s} - \frac{1}{t}C(K_1)\|w\|_{R^s}|||\mathbb{P}_1 w|||^2_{1,R^s} + \sum_{1\leq|\alpha|\leq s}\langle D^\alpha w, (\partial_i A^i_1)D^\alpha w\rangle \\
&- \frac{2}{\epsilon}\sum_{1\leq|\alpha|\leq s}\overbrace{\langle D^\alpha w, C^i_1\partial_i D^\alpha w\rangle}^{=0} - 2\sum_{1\leq|\alpha|\leq s}\langle D^\alpha w, A^0_1[D^\alpha, (A^0_1)^{-1}A^i_1]\partial_i w\rangle \\
&- 2\sum_{1\leq|\alpha|\leq s}\langle D^\alpha w, [\tilde{A}^0_1, D^\alpha](A^0_1)^{-1}C^i_1\partial_i w\rangle + \frac{2}{t}\sum_{1\leq|\alpha|\leq s}\langle D^\alpha w, \mathfrak{A}_1 D^\alpha\mathbb{P}_1 w\rangle \\
&+ \frac{2}{t}\sum_{1\leq|\alpha|\leq s}\langle D^\alpha w, A^0_1[(A^0_1)^{-1}\mathfrak{A}_1, D^\alpha]\mathbb{P}_1 w\rangle + 2\sum_{1\leq|\alpha|\leq s}\langle D^\alpha w, A^0_1 D^\alpha[(A^0_1)^{-1}(H_1+F_1)]\rangle \\
\leq& C(K_1)(|||w|||^2_{1,R^s} + |||\mathbb{P}_4 w|||^2_{L^2} + \mathcal{C}^*|||w|||_{1,R^s} + \mathcal{C}_*|||Dw|||_{1,H^{s-1}}) \\
&\qquad + \frac{2}{t}\left[\hat{\kappa} - C_1(K_1)\|w\|_{R^s}\right]|||\mathbb{P}_1 w|||^2_{1,R^s} \\
\leq& C(K_1)(|||Dw|||^2_{1,H^{s-1}} + |||\mathbb{P}_4 w|||^2_{L^2} + \sigma|||Dw|||_{1,H^{s-1}} + \mathcal{C}^2_*) \\
&\qquad + \frac{2}{t}\left[\hat{\kappa} - C_1(K_1)\|w\|_{R^s}\right]|||\mathbb{P}_1 w|||^2_{1,R^s} \qquad (3.6.40)
\end{aligned}$$

for $t\in[T_0, T_*)$, where we note that the last inequality follows Theorem C.2.2.(1). By similar calculation, we obtain, from differentiating $\langle D^\alpha u, A^0_2 D^\alpha u\rangle$ with respect to t, the estimate

$$\begin{aligned}
\partial_t|||Du|||^2_{2,H^{s-2}} =& \sum_{1\leq|\alpha|\leq s-1}\langle D^\alpha u, (\partial_t A^0_2)D^\alpha u\rangle + 2\sum_{1\leq|\alpha|\leq s-1}\langle D^\alpha u, A^0_2 D^\alpha\partial_t u\rangle \\
\leq& C(K_1)\|u\|^2_{Q^{s-1}} - \frac{1}{t}C(K_1,K_2)(\|u\|_{Q^{s-1}} + \|w\|_{R^s})(|||\mathbb{P}_2 u|||^2_{2,R^{s-1}} + |||\mathbb{P}_1 w|||^2_{1,R^s}) \\
&\sum_{1\leq|\alpha|\leq s-1}\langle D^\alpha u, (\partial_i A^i_2)D^\alpha u\rangle - \frac{2}{\epsilon}\sum_{1\leq|\alpha|\leq s-1}\overbrace{\langle D^\alpha u, C^i_2\partial_i D^\alpha u\rangle}^{=0} \\
&- 2\sum_{1\leq|\alpha|\leq s-1}\langle D^\alpha u, A^0_2[D^\alpha, (A^0_2)^{-1}A^i_2]\partial_i u\rangle - 2\sum_{1\leq|\alpha|\leq s-1}\langle D^\alpha u, [\tilde{A}^0_2, D^\alpha](A^0_2)^{-1}C^i_2\partial_i u\rangle \\
&+ \frac{2}{t}\sum_{1\leq|\alpha|\leq s-1}\langle D^\alpha u, \mathfrak{A}_2 D^\alpha\mathbb{P}_2 u\rangle - \frac{2}{t}\sum_{1\leq|\alpha|\leq s-1}\langle D^\alpha u, A^0_2[(A^0_2)^{-1}\mathfrak{A}_2, D^\alpha]\mathbb{P}_2 u\rangle \\
&+ 2\sum_{1\leq|\alpha|\leq s-1}\left\langle D^\alpha u, A^0_2 D^\alpha[(A^0_2)^{-1}(H_2 + \frac{1}{t}M_2\mathbb{P}_3 U + F_2)]\right\rangle \\
\leq& C(K_1,K_2,K_3)(|||Du|||^2_{2,H^{s-2}} + |||Dw|||^2_{1,H^{s-1}} + |||\mathbb{P}_4 w|||^2_{L^2} + \sigma|||Du|||_{2,H^{s-2}} + \mathcal{C}^2_*) \\
&- \frac{1}{4t}C_2(K_1,K_2)(\|u\|_{Q^{s-1}} + \|w\|_{R^s})|||\mathbb{P}_1 w|||^2_{1,R^s} \\
&- C(K_1)\frac{1}{t}(|||Du|||^2_{2,H^{s-2}} + |||Dw|||^2_{1,H^{s-1}})|||\mathbb{P}_3 U|||_{R^{s-1}}
\end{aligned}$$

$$+\frac{2}{t}\big[\hat{\kappa}-C_2(K_1,K_2)(\|u\|_{Q^{s-1}}+\|w\|_{R^s})\big]|||\mathbb{P}_2u|||^2_{2,R^{s-1}} \tag{3.6.41}$$

for $t\in[T_0,T_*)$.

Next, we estimate $|||\mathbb{P}_4w|||_{L^2}$. Acting on both sides of (3.6.12) with $\mathbb{P}_4$, we deduce from Assumption 3.6.2.(3) that

$$A^0_1\partial_t\mathbb{P}_4w+A^i_1\partial_i\mathbb{P}_4w+\frac{1}{\epsilon}C^i_1\partial_i\mathbb{P}_4w=\frac{1}{t}\mathfrak{A}_1\mathbb{P}_1\mathbb{P}_4w+\mathbb{P}_4H_1+\mathbb{P}_4F_1. \tag{3.6.42}$$

Then using (3.6.32), (3.6.37), (3.6.42) and similar energy estimate to derive (3.6.40) and (3.6.41), we find, with the help of the estimate $\|\mathbb{P}_4H_1(\epsilon,t,x,w)\|_{L^2}\leq C(K_1)\|\mathbb{P}_4w\|_{L^2}$, which follows from $\mathbb{P}_4H_1(\epsilon,t,x,\mathbb{P}_4^\perp w)=0$ (see Assumption 3.6.2.(4)), that

$$\begin{aligned}\partial_t|||\mathbb{P}_4w|||^2_{1,L^2}=&2\langle\mathbb{P}_4w,A^0_1\partial_t\mathbb{P}_4w\rangle+\langle\mathbb{P}_4w,(\partial_tA^0_1)\mathbb{P}_4w\rangle\\ \leq& C(K_1,K_3)(|||\mathbb{P}_4w|||^2_{1,L^2}+|||Dw|||^2_{1,H^{s-1}}+\sigma|||\mathbb{P}_4w|||_{1,L^2}+C_*^2)-\frac{1}{4t}C_4(K_1,K_3)\big(\|\mathbb{P}_4w\|_{L^2}\\ &+\|w\|_{R^s}\big)|||\mathbb{P}_1w|||^2_{1,R^s}+\frac{4}{t}\Big(\hat{\kappa}-C_4(K_1,K_3)\big(\|\mathbb{P}_4w\|_{L^2}+\|w\|_{R^s}\big)\Big)|||\mathbb{P}_1\mathbb{P}_4w|||^2_{1,L^2}.\end{aligned} \tag{3.6.43}$$

Applying the operator $A^0D^\alpha\mathbb{P}^3(A^0)^{-1}$ to (3.6.1), we conclude, with the help of (3.6.6)-(3.6.8), that

$$\begin{aligned}A^0\partial_tD^\alpha\mathbb{P}_3U+\mathbb{P}_3A^i\mathbb{P}_3\partial_iD^\alpha\mathbb{P}_3U+\frac{1}{\epsilon}\mathbb{P}_3C^i\mathbb{P}_3\partial_iD^\alpha\mathbb{P}_3U=&-A^0[D^\alpha,(A^0)^{-1}\mathbb{P}_3A^i\mathbb{P}_3]\partial_i\mathbb{P}_3U\\ -[\tilde{A}^0,D^\alpha](A^0)^{-1}\mathbb{P}_3C^i\mathbb{P}_3\partial_i\mathbb{P}_3U+\frac{1}{t}\mathbb{P}_3\mathfrak{A}\mathbb{P}_3D^\alpha\mathbb{P}_3U+&\frac{1}{t}A^0[D^\alpha,(A^0)^{-1}\mathbb{P}_3\mathfrak{A}\mathbb{P}_3]\mathbb{P}_3U+A^0D^\alpha[(A^0)^{-1}\mathbb{P}_3H].\end{aligned} \tag{3.6.44}$$

By similar arguments that were used to derive (3.6.40) and (3.6.41), we obtain from (3.6.44) the estimate

$$\begin{aligned}\partial_t|||D\mathbb{P}_3U|||^2_{H^{s-2}}=&\sum_{1\leq|\alpha|\leq s-1}\langle D^\alpha\mathbb{P}_3U,(\partial_tA^0)D^\alpha\mathbb{P}_3U\rangle+2\sum_{1\leq|\alpha|\leq s-1}\langle D^\alpha\mathbb{P}_3U,\mathbb{P}_3A^0\mathbb{P}_3D^\alpha\partial_t\mathbb{P}_3U\rangle\\ \leq&-\frac{1}{t}C(K_1)\|\mathbb{P}_1w\|_{R^s}|||\mathbb{P}_3U|||^2_{R^{s-1}}+C(K_1)\|\mathbb{P}_3U\|^2_{Q^{s-1}}\\ &+\sum_{1\leq|\alpha|\leq s-1}\langle D^\alpha\mathbb{P}_3U,(\partial_iA^i)D^\alpha\mathbb{P}_3U\rangle-\frac{2}{\epsilon}\sum_{1\leq|\alpha|\leq s-1}\overbrace{\langle D^\alpha\mathbb{P}_3U,C^i\partial_iD^\alpha\mathbb{P}_3U\rangle}^{=0}\\ &-2\sum_{1\leq|\alpha|\leq s-1}\langle D^\alpha\mathbb{P}_3U,A^0[D^\alpha,(A^0)^{-1}A^i]\partial_i\mathbb{P}_3U+[\tilde{A}^0,D^\alpha](A^0)^{-1}C^i\partial_i\mathbb{P}_3U\rangle\\ &+\frac{2}{t}\sum_{1\leq|\alpha|\leq s-1}\langle D^\alpha\mathbb{P}_3U,\mathfrak{A}D^\alpha\mathbb{P}_3U\rangle+\frac{2}{t}\sum_{1\leq|\alpha|\leq s-1}\langle D^\alpha\mathbb{P}_3U,A^0[(A^0)^{-1}\mathfrak{A},D^\alpha]\mathbb{P}_3U\rangle\\ &+2\sum_{1\leq|\alpha|\leq s-1}\big\langle D^\alpha\mathbb{P}_3U,A^0D^\alpha[(A^0)^{-1}\mathbb{P}_3H]\big\rangle\\ \leq& C(K_1)\|\mathbb{P}_3U\|^2_{Q^{s-1}}+C(K_1)\|\mathbb{P}_3U\|_{Q^{s-1}}\Big(\|H_1\|_{Q^{s-1}}+\|H_2\|_{Q^{s-1}}+\|F_1\|_{Q^{s-1}}\\ &+\|F_2\|_{Q^{s-1}}\Big)+\frac{1}{t}\Big(2\hat{\kappa}-C(K_1,K_2)\big(\|w\|_{R^s}+\|u\|_{Q^{s-1}}\big)\Big)|||\mathbb{P}_3U|||^2_{R^{s-1}}\\ \leq& C(K_1)|||\mathbb{P}_3U|||^2_{R^{s-1}}+C(K_1,K_2)\big(|||w|||_{1,R^s}+|||u|||_{2,R^{s-1}}+\sigma+\mathcal{C}_*+|||\mathbb{P}_4w|||_{1,L^2}\big)|||\mathbb{P}_3U|||_{R^{s-1}}\\ &+\frac{1}{t}\Big(2\hat{\kappa}-C(K_1,K_2)\big(\|w\|_{R^s}+\|u\|_{Q^{s-1}}\big)\Big)|||\mathbb{P}_3U|||^2_{R^{s-1}}.\end{aligned} \tag{3.6.45}$$

We also find, by a similar derivation used to establish (3.6.36), that

$$\|\mathbb{P}_3U\|_{L^6}\leq C_{\mathrm{S}}\sqrt{\gamma_1}|||D\mathbb{P}_3U|||_{H^{s-2}}, \tag{3.6.46}$$

from which it follows that

$$|||D\mathbb{P}_3 U|||_{H^{s-2}} \leq |||\mathbb{P}_3 U|||_{R^{s-1}} \leq (1 + C_{\mathrm{S}}\sqrt{\gamma_1})|||D\mathbb{P}_3 U|||_{H^{s-2}} \tag{3.6.47}$$

by adding $|||D\mathbb{P}_3 U|||_{H^{s-2}}$ to both sides of (3.6.46). Furthermore, using (3.6.45) and (3.6.47), we see that $\partial_t |||D\mathbb{P}_3 U|||^2_{H^{s-2}}$ is dominated by

$$\partial_t |||D\mathbb{P}_3 U|||^2_{H^{s-2}} \leq C(K_1, K_2)\big(|||w|||_{1,R^s} + |||u|||_{2,R^{s-1}} + \sigma + \mathcal{C}_* + |||\mathbb{P}_4 w|||_{1,L^2}\big)|||D\mathbb{P}_3 U|||_{H^{s-2}}$$
$$+C(K_1)|||\mathbb{P}_3 U|||_{R^{s-1}}|||D\mathbb{P}_3 U|||_{H^{s-2}} + \frac{1}{t}\Big(2\hat{\kappa} - C(K_1, K_2)\big(\|w\|_{R^s} + \|u\|_{Q^{s-1}}\big)\Big)|||\mathbb{P}_3 U|||_{R^{s-1}}|||D\mathbb{P}_3 U|||_{H^{s-2}}.$$

Using this in conjunction with (3.6.37) yields the estimate

$$\partial_t |||D\mathbb{P}_3 U|||_{H^{s-2}} \leq C(K_1)|||\mathbb{P}_3 U|||_{R^{s-1}} + C(K_1, K_2)\big(|||Dw|||_{1,H^{s-1}} + |||Du|||_{2,H^{s-2}} + \sigma + \mathcal{C}_* + |||\mathbb{P}_4 w|||_{1,L^2}\big)$$
$$+ \frac{1}{t}\Big(\hat{\kappa} - C_2(K_1, K_2)\big(\|w\|_{R^s} + \|u\|_{Q^{s-1}}\big)\Big)|||\mathbb{P}_3 U|||_{R^{s-1}}. \tag{3.6.48}$$

To proceed, we choose $\sigma > 0$ small enough so that the inequality

$$\Big(C_1(\hat{R}) + 2C_2(\hat{R}, \hat{R}) + 2C_4(\hat{R}, \hat{R})\Big)\sigma < \frac{\hat{\kappa}}{4}$$

holds in addition to (3.6.21). Then

$$\hat{\kappa} - \Big(C_1(K_1(T_0))\|w(T_0)\|_{R^s} + C_2(K_1(T_0), K_2(T_0))\big(\|w(T_0)\|_{R^s} + \|u(T_0)\|_{R^{s-1}}\big)$$
$$+ C_4(K_1(T_0), K_3(T_0))\big(\|\mathbb{P}_4 w(T_0)\|_{L^2} + \|w(T_0)\|_{R^s}\big)\Big) > \frac{\hat{\kappa}}{2},$$

and we see by continuity that either

$$\hat{\kappa} - \Big(C_1(K_1(t))\|w(t)\|_{R^s} + C_2(K_1(t), K_2(t))\big(\|w(t)\|_{R^s} + \|u(t)\|_{R^{s-1}}\big)$$
$$+ C_4(K_1(t), K_3(t))\big(\|\mathbb{P}_4 w(t)\|_{L^2} + \|w(t)\|_{R^s}\big)\Big) > \frac{\hat{\kappa}}{2}, \quad 0 \leq t < T_*,$$

or else there exists a first time $T^* \in (0, T_*)$ such that

$$\hat{\kappa} - \Big(C_1(K_1(T^*))\|w(T^*)\|_{R^s} + C_2(K_1(T^*), K_2(T^*))\big(\|w(T^*)\|_{R^s} + \|u(T^*)\|_{R^{s-1}}\big)$$
$$+ C_4(K_1(T^*), K_3(T^*))\big(\|\mathbb{P}_4 w(T^*)\|_{L^2} + \|w(T^*)\|_{R^s}\big)\Big) = \frac{\hat{\kappa}}{2}.$$

Letting $T^* = T_*$ if the first case holds, we then have that

$$\hat{\kappa} - \Big(C_1(K_1(t))\|w(t)\|_{R^s} + C_2(K_1(t), K_2(t))\big(\|w(t)\|_{R^s} + \|u(t)\|_{R^{s-1}}\big)$$
$$+ C_4(K_1(t), K_3(t))\big(\|\mathbb{P}_4 w(t)\|_{L^2} + \|w(t)\|_{R^s}\big)\Big) > \frac{\hat{\kappa}}{2}, \quad 0 \leq t < T^* \leq T_*. \tag{3.6.49}$$

Taken together, the estimates (3.6.22), (3.6.40), (3.6.41), (3.6.43), (3.6.48) and (3.6.49), with the help of (3.6.47) and Young's inequality, imply that

$$\partial_t |||Dw|||^2_{1,H^{s-1}} \leq C(\hat{R})\Big(|||Dw|||^2_{1,H^{s-1}} + |||\mathbb{P}_4 w|||^2_{1,L^2} + \sigma^2 + \mathcal{C}_*^2\Big) + \frac{\hat{\kappa}}{t}|||\mathbb{P}_1 w|||^2_{1,R^s}, \tag{3.6.50}$$

$$\partial_t |||Du|||^2_{2,H^{s-2}} \leq C(\hat{R})\Big(|||Du|||^2_{2,H^{s-2}} + |||Dw|||^2_{1,H^{s-1}} + |||\mathbb{P}_4 w|||^2_{1,L^2} + \sigma^2 + \mathcal{C}_*^2\Big) - \frac{\hat{\kappa}}{8t}|||\mathbb{P}_1 w|||^2_{1,R^s}$$
$$+ \frac{\hat{\kappa}}{t}|||\mathbb{P}_2 u|||^2_{2,R^{s-1}} - \frac{1}{t}C_3(\hat{R})\big(|||Du|||^2_{2,H^{s-2}} + |||Dw|||^2_{1,H^{s-1}}\big)|||D\mathbb{P}_3 U|||_{H^{s-2}}, \tag{3.6.51}$$

$$\partial_t \||\mathbb{P}_4 w\||^2_{1,L^2} \leq C(\hat{R})\Big(\||\mathbb{P}_4 w\||^2_{1,L^2} + \||Dw\||^2_{1,H^{s-1}} + \sigma^2 + \mathcal{C}_*^2\Big) + \frac{2\hat{\kappa}}{t}\||\mathbb{P}_1\mathbb{P}_4 w\||^2_{1,L^2} - \frac{\hat{\kappa}}{8t}\||\mathbb{P}_1 w\||^2_{1,R^s} \tag{3.6.52}$$

and

$$\partial_t \||D\mathbb{P}_3 U\||_{H^{s-2}} \leq C(\hat{R})\Big(\||D\mathbb{P}_3 U\||_{H^{s-2}} + \||Dw\||_{1,H^{s-1}} + \||Du\||_{2,H^{s-2}} + \sigma + \mathcal{C}_* + \||\mathbb{P}_4 w\||_{1,L^2}\Big) + \frac{\hat{\kappa}}{2t}\||D\mathbb{P}_3 U\||_{H^{s-2}} \tag{3.6.53}$$

for $0 \leq t < T^* \leq T_*$.

Next, we set

$$X = \||Dw\||^2_{1,H^{s-1}} + \||Du\||^2_{2,H^{s-2}} + \||\mathbb{P}_4 w\||^2_{1,L^2},$$
$$Y = \||\mathbb{P}_1 w\||^2_{1,R^s} + \||\mathbb{P}_2 u\||^2_{2,R^{s-1}} + \||\mathbb{P}_1\mathbb{P}_4 w\||^2_{1,L^2}, \quad \text{and} \quad Z = \||D\mathbb{P}_3 U\||_{H^{s-2}}.$$

Since $C_3(\hat{R})X(T_0)/\sigma \leq C(\hat{R})\sigma$, we can choose σ small enough so that $C_3(\hat{R})X(T_0)/\sigma < \hat{\kappa}/8$. Then by continuity, either $C_3(\hat{R})X(t)/\sigma \leq \hat{\kappa}/8$ for $t \in [T_0, T^*)$, or else there exists a first time $T \in (T_0, T^*)$ such that $C_3(\hat{R})X(T)/\sigma = \hat{\kappa}/8$. Setting $T = T^*$ if the first case holds, we then have that

$$C_3(\hat{R})\frac{X(t)}{\sigma} < \hat{\kappa}/8, \quad T_0 \leq t < T \leq T^* \leq T_*. \tag{3.6.54}$$

Adding the inequalities (3.6.50), (3.6.51) and (3.6.52) and dividing the results by σ, we obtain, with the aid of (3.6.54), the inequality

$$\partial_t\left(\frac{X}{\sigma}\right) \leq C(\hat{R})\left(\frac{X + \mathcal{C}_*^2}{\sigma} + \sigma\right) - \frac{\hat{\kappa}}{8t}Z + \frac{\hat{\kappa}}{2t}\frac{Y}{\sigma}, \quad T_0 \leq t < T \leq T^* \leq T_*, \tag{3.6.55}$$

while the inequality

$$\partial_t Z \leq C(\hat{R})\left(Z + \sigma + \frac{X + \mathcal{C}_*^2}{\sigma}\right) + \frac{\hat{\kappa}}{4t}Z, \quad T_0 \leq t < T^* \leq T_* \tag{3.6.56}$$

follows from (3.6.53) and Young's inequality. Adding (3.6.55) and (3.6.56) gives

$$\partial_t\left(\frac{X}{\sigma} + Z - \frac{\hat{\kappa}}{8}\int_{T_0}^t \frac{1}{\tau}\left(\frac{Y}{\sigma} + Z\right)d\tau + \sigma\right) \leq C(\hat{R})\left(\frac{X + \mathcal{C}_*^2}{\sigma} + Z - \frac{\hat{\kappa}}{8}\int_{T_0}^t \frac{1}{\tau}\left(\frac{Y}{\sigma} + Z\right)d\tau + \sigma\right). \tag{3.6.57}$$

Noting that $\mathcal{C}_*(t)^2 \lesssim \int_{T_0}^t X(\tau)d\tau$ by the Hölder and Sobolev inequalities, see Theorems C.2.1.(1) and C.2.2.(1), we see, after adding $X(t)/\sigma$ to both sides of (3.6.57), that the inequality

$$\partial_t\left(\frac{X + \int_{T_0}^t X(\tau)d\tau}{\sigma} + Z - \frac{\hat{\kappa}}{8}\int_{T_0}^t \frac{1}{\tau}\left(\frac{Y}{\sigma} + Z\right)d\tau + \sigma\right) \leq C(\hat{R})\left(\frac{X + \int_{T_0}^t X(\tau)d\tau}{\sigma} + Z - \frac{\hat{\kappa}}{8}\int_{T_0}^t \frac{1}{\tau}\left(\frac{Y}{\sigma} + Z\right)d\tau + \sigma\right) \tag{3.6.58}$$

holds for $T_0 \leq t < T \leq T^* \leq T_*$. Since $X(T_0) \leq C(\hat{R})\sigma^2$ and $Z(T_0) \lesssim \sigma$, it follows directly from (3.6.58) and Grönwall's inequality that

$$\frac{1}{\sigma}\left(X(t) + \int_{T_0}^t X(\tau)d\tau\right) + Z(t) - \frac{\hat{\kappa}}{8}\int_{T_0}^t \frac{1}{\tau}\left(\frac{Y(\tau)}{\sigma} + Z(\tau)\right)d\tau + \sigma \leq e^{C(\hat{R})(t-T_0)}C(\hat{R})\sigma, \quad T_0 \leq t < T \leq T^* \leq T_*,$$

from which it follows that

$$\|\mathbb{P}_4 w(t)\|_{L^2} + \left(-\int_{T_0}^t \frac{1}{\tau}\|\mathbb{P}_1\mathbb{P}_4 w\|^2_{L^2}\, d\tau\right)^{\frac{1}{2}} + \|w\|_{M^\infty_{\mathbb{P}_1,s}([T_0,t)\times\mathbb{R}^3)} + \|u\|_{M^\infty_{\mathbb{P}_2,s-1}([T_0,t)\times\mathbb{R}^3)}$$

$$-\int_{T_0}^{t}\frac{1}{\tau}\|\mathbb{P}_3U(\tau)\|_{Q^{s-1}}\,d\tau \le C(\hat{R})\sigma,$$

for $T_0 \le t < T \le T^* \le T_*$, where we stress that the constant $C(\hat{R})$ is independent of ϵ and the times T, T^*, T_*, and T_1. Choosing σ small enough, it is then clear from the estimate (3.6.58) and the definition of the times T, T^*, and T_1 that $T = T^* = T_* = T_1$, which completes the proof. □

3.6.2 Error estimates

In this section, we consider solutions of the singular initial value problem:

$$A_1^0(\epsilon,t,x,w)\partial_t w + A_1^i(\epsilon,t,x,w)\partial_i w + \frac{1}{\epsilon}C_1^i\partial_i w = \frac{1}{t}\mathfrak{A}_1(\epsilon,t,x,w)\mathbb{P}_1 w + H_1 + F_1 \quad \text{in} \quad [T_0,T_1)\times\mathbb{R}^3, \tag{3.6.59}$$

$$w(T_0,x) = \mathring{w}^0(x) + \epsilon s^0(\epsilon,x) \quad \text{in} \quad \{T_0\}\times\mathbb{R}^3, \tag{3.6.60}$$

where the matrices A_1^0, A_1^i, $i=1,\dots,n$, and $\mathfrak{A}_1$ and the source terms H_1 and F_1 satisfy the conditions from Assumption 3.6.2. Our aim is to use the uniform a priori estimates from Theorem 3.6.7 to establish uniform a priori estimates for solutions of (3.6.59)-(3.6.60) and the corresponding *limit equation* defined by

$$\mathring{A}_1^0\partial_t\mathring{w} + \mathring{A}_1^i\partial_i\mathring{w} = \frac{1}{t}\mathring{\mathfrak{A}}_1\mathbb{P}_1\mathring{w} - C_1^i\partial_i v + \mathring{H}_1 + \mathring{F}_1 \quad \text{in} \quad [T_0,T_1)\times\mathbb{R}^3, \tag{3.6.61}$$

$$C_1^i\partial_i\mathring{w} = 0 \quad \text{in} \quad [T_0,T_1)\times\mathbb{R}^3, \tag{3.6.62}$$

$$\mathring{w}(T_0,x) = \mathring{w}^0(x) \quad \text{in} \quad \{T_0\}\times\mathbb{R}^3, \tag{3.6.63}$$

and to establish an error estimate between solutions of (3.6.59)-(3.6.60) and (3.6.61)-(3.6.63).

In the limit equation, $\mathring{A}_1^0$ and $\mathring{\mathfrak{A}}_1$ are defined by (3.6.2) and (3.6.3) with $a=1$, respectively, while $\mathring{A}_1^i$ and $\mathring{H}_1$ are defined by the limits

$$\mathring{A}_1^i(t,x,\mathring{w}) = \lim_{\epsilon\searrow 0} A_1^i(\epsilon,t,x,\mathring{w}) \quad \text{and} \quad \mathring{H}_1(t,x,\mathring{w}) = \lim_{\epsilon\searrow 0} H_1(\epsilon,t,x,\mathring{w}), \tag{3.6.64}$$

respectively. We further assume that the following conditions hold for fixed constants $R>0$, $T_0 < T_1 < 0$ and $s \in \mathbb{Z}_{>n/2+1}$:

Assumptions 3.6.9.

1. *The source terms*[9] $\mathring{F}_1$ *and* v *satisfy* $\mathring{F}_1 \in C^0\big([T_0,T_1),H^s(\mathbb{R}^3,\mathbb{R}^{N_1})\big)$ *and* $v \in \bigcap_{\ell=0}^1 C^\ell\big([T_0,T_1),R^{s+1-\ell}(\mathbb{R}^3,\mathbb{R}^{N_1})\big)$, *respectively.*
2. *The matrices* $\mathring{A}_1^i$, $i=1,\dots,n$ *and the source term* $\mathring{H}_1$ *satisfy*[10] $t\mathring{A}_1^i \in E^1\big((2T_0,0)\times\mathbb{R}^3\times B_R(\mathbb{R}^{N_1}),\mathbb{S}_{N_1}\big)$, $t\mathring{H}_1 \in E^1\big((2T_0,0)\times\mathbb{R}^3\times B_R(\mathbb{R}^{N_1}),\mathbb{R}^{N_1}\big)$, *and* $D_t\big(t\mathring{H}_1(t,x,0)\big)=0$.

We are now ready to state and establish uniform a priori estimates for solutions of the singular initial value problem (3.6.59)-(3.6.60) and the associated limit equation defined by (3.6.61)-(3.6.63).

Theorem 3.6.10. *Suppose* $R>0$, $s\in\mathbb{Z}_{\ge 3}$, $T_0 < T_1 \le 0$, $\epsilon_0 > 0$, $\mathring{w}^0 \in H^s(\mathbb{R}^3,\mathbb{R}^{N_1})$, $s^0 \in L^\infty\big((0,\epsilon_0),R^s(\mathbb{R}^3,\mathbb{R}^{N_1})\big)$, *Assumptions 3.6.2 and 3.6.9 hold, the maps*

$$(w,\mathring{w}) \in \bigcap_{\ell=0}^{1} C^\ell\big([T_0,T_1),R^{s-\ell}(\mathbb{R}^3,\mathbb{R}^{N_1})\big) \times \bigcap_{\ell=0}^{1} C^\ell\big([T_0,T_1),H^{s-\ell}(\mathbb{R}^3,\mathbb{R}^{N_1})\big)$$

[9]The source term $\mathring{F}_1$ should be thought of as the $\epsilon\searrow 0$ limit of F_1. This is made precise by the hypothesis (3.6.67) of Theorem 3.6.10.

[10]From the assumptions 3.6.2.(4)-(5) on A_1^i and H_1, it follows directly from the (3.6.64) that $\mathring{A}_1^i \in E^0\big((2T_0,0)\times\mathbb{R}^3\times B_R(\mathbb{R}^{N_1}),\mathbb{S}_{N_1}\big)$ and $\mathring{H}_1 \in E^0\big((2T_0,0)\times\mathbb{R}^3\times B_R(\mathbb{R}^{N_1}),\mathbb{R}^{N_1}\big)$.

define solutions to the initial value problems (3.6.59)-(3.6.60) and (3.6.61)-(3.6.63), and, for $t \in [T_0, T_1)$*, the following estimates hold:*

$$\|v(t)\|_{R^{s+1}} - \frac{1}{t}\|\mathbb{P}_1 v(t)\|_{R^{s+1}} + \|\partial_t v(t)\|_{R^s} + \|\mathring{F}_1(t)\|_{H^s} + \|t\partial_t \mathring{F}_1(t)\|_{R^{s-1}} + \|F_1(\epsilon,t)\|_{R^s}$$
$$\leq C(\|w\|_{L^\infty([T_0,t),R^s)}, \|\mathring{w}\|_{L^\infty([T_0,t),H^s)})\Big(\mathcal{C}^* + \|w(t)\|_{R^s} + \|\mathring{w}(t)\|_{H^s} + \int_{T_0}^t (\|\mathring{w}(\tau)\|_{H^s} + \|w(\tau)\|_{R^s}) d\tau\Big), \tag{3.6.65}$$

$$\|A_1^i(\epsilon, t, \cdot, \mathring{w}(t)) - \mathring{A}_1^i(t, \cdot, \mathring{w}(t))\|_{R^{s-1}} \leq \epsilon C\big(\|\mathring{w}(t)\|_{L^\infty([T_0,t),R^s)}\big), \tag{3.6.66}$$

and

$$\|H_1(\epsilon,t, \cdot, \mathring{w}(t)) - \mathring{H}_1(t, \cdot, \mathring{w}(t))\|_{R^{s-1}} + \|F_1(\epsilon,t) - \mathring{F}_1(t)\|_{R^{s-1}}$$
$$\leq \epsilon C\big(\|w\|_{L^\infty([T_0,t),R^s)}, \|\mathring{w}\|_{L^\infty([T_0,t),H^s)}\big)\Big(\mathcal{C}^* + \|w(t)\|_{R^s} + \|z(t)\|_{Q^{s-1}} + \|\mathring{w}(t)\|_{H^s}$$
$$+ \int_{T_0}^t (\|w(\tau)\|_{R^s} + \|z(\tau)\|_{Q^{s-1}} + \|\mathring{w}(\tau)\|_{H^s}) d\tau\Big), \tag{3.6.67}$$

where

$$z = \frac{1}{\epsilon}(w - \mathring{w} - \epsilon v),$$

and the constants $\mathcal{C}^*$, $C\big(\|w\|_{L^\infty([T_0,t),R^s)}\big)$, $C\big(\|\mathring{w}\|_{L^\infty([T_0,t),R^s)}\big)$ *and* $C\big(\|w\|_{L^\infty([T_0,T_t),R^s)}, \|\mathring{w}\|_{L^\infty([T_0,t),R^s)}\big)$ *are independent of* $\epsilon \in (0, \epsilon_0)$ *and* $T_1 \in (T_0, 0)$*.*

Then there exists a small constant $\sigma > 0$*, independent of* $\epsilon \in (0, \epsilon_0)$ *and* $T_1 \in (T_0, 0)$*, such that if*

$$\|\mathring{w}^0\|_{H^s} + \|s^0\|_{R^s} + \mathcal{C}^* \leq \sigma, \tag{3.6.68}$$

then

$$\max\{\|w\|_{L^\infty([T_0,T_1)\times\mathbb{R}^3)}, \|\mathring{w}\|_{L^\infty([T_0,T_1)\times\mathbb{R}^3)}\} \leq \frac{R}{2} \tag{3.6.69}$$

and there exists a constant $C > 0$*, independent of* $\epsilon \in (0, \epsilon_0)$ *and* $T_1 \in (T_0, 0)$*, such that*

$$\|\mathring{w}\|_{L^\infty([T_0,t),L^2)} + \|w\|_{M^\infty_{\mathbb{P}_1,s}([T_0,t)\times\mathbb{R}^3)} + \|\mathring{w}\|_{M^\infty_{\mathbb{P}_1,s}([T_0,t)\times\mathbb{R}^3)} + \|t\partial_t \mathring{w}\|_{M^\infty_{\mathbb{1},s-1}([T_0,t)\times\mathbb{R}^3)}$$
$$\left(-\int_{T_0}^t \frac{1}{\tau}\|\mathbb{P}_1 \mathring{w}\|^2_{L^2} d\tau\right)^{\frac{1}{2}} + \int_{T_0}^t \|\partial_t \mathring{w}\|_{Q^{s-1}} d\tau - \int_{T_0}^t \frac{1}{\tau}\|\mathbb{P}_1 \mathring{w}\|_{Q^{s-1}} d\tau \leq C\sigma, \tag{3.6.70}$$

$$\|w - \mathring{w}\|_{L^\infty([T_0,t),R^{s-1})} \leq \epsilon C\sigma \tag{3.6.71}$$

and

$$-\int_{T_0}^t \frac{1}{\tau}\|\mathbb{P}_1(w - \mathring{w})\|^2_{R^{s-1}} d\tau \leq \epsilon^2 C\sigma^2 \tag{3.6.72}$$

for $T_0 \leq t < T_1$*.*

Proof. First, we observe, by (3.6.2) and (3.6.10), that $\mathring{A}_1^0$ satisfies

$$\mathbb{P}_1^\perp \mathring{A}_1^0 \mathbb{P}_1 = \mathbb{P}_1 \mathring{A}_1^0 \mathbb{P}_1^\perp. \tag{3.6.73}$$

Using this, we find, after applying $\mathbb{P}_1$ to the limit equation (3.6.61), that

$$b = \mathbb{P}_1 \mathring{w} \tag{3.6.74}$$

satisfies the equation

$$\mathbb{P}_1\mathring{A}_1^0\mathbb{P}_1\partial_t b+\mathbb{P}_1\mathring{A}_1^i\mathbb{P}_1\partial_i b=\frac{1}{t}\mathbb{P}_1\mathring{\mathfrak{A}}_1\mathbb{P}_1 b+\mathbb{P}_1\mathring{H}_1+\mathbb{P}_1\bar{F}_2, \tag{3.6.75}$$

where $\bar{F}_2=-\mathbb{P}_1\mathring{A}_1^i\mathbb{P}_1^\perp\partial_i\mathring{w}+\mathbb{P}_1\mathring{F}_1-\mathbb{P}_1C_1^i\partial_i v$. Clearly, $\bar{F}_2$ satisfies

$$\|\bar{F}_2(t)\|_{R^{s-1}}\leq C\big(\|\mathring{w}\|_{L^\infty([T_0,t),H^s)},\|w\|_{L^\infty([T_0,t),R^s)}\big)\Big(\mathcal{C}^*+\|w(t)\|_{R^s}+\|\mathring{w}(t)\|_{H^s} \\ +\int_{T_0}^t(\|\mathring{w}(\tau)\|_{H^s}+\|w(\tau)\|_{R^s})d\tau\Big) \tag{3.6.76}$$

for $0\leq t<T_1$ by (3.1.14), (3.6.65) and the calculus inequalities from Appendix C, while

$$\|b(T_0)\|_{R^{s-1}}\leq\|\mathring{w}^0\|_{R^s}\lesssim\|\mathring{w}^0\|_{H^s}\leq\sigma, \tag{3.6.77}$$

follows from the assumption (3.6.68) on the initial data, and we note that $\mathbb{P}_1\mathring{H}_1(t,x,\mathring{w})$ satisfies

$$\mathbb{P}_1\mathring{H}_1(t,x,0)=0 \tag{3.6.78}$$

by Assumption 3.6.2.(3).

Next, we set

$$y=t\partial_t\mathring{w}.$$

In order to derive an evolution equation for y, we apply $t\partial_t$ to (3.6.61) and use the identity

$$t\partial_t f=tD_t f+[D_{\mathring{w}}f\cdot t\partial_t\mathring{w}]=D_t(tf)-f+[D_{\mathring{w}}f\cdot t\partial_t\mathring{w}],\quad f=f(t,x,\mathring{w}(t,x)),$$

to obtain

$$\mathring{A}_1^0\partial_t y+\mathring{A}_1^i\partial_i y=\frac{1}{t}(\mathbb{P}_1\mathring{\mathfrak{A}}_1\mathbb{P}_1+\mathring{A}_1^0)y-\frac{1}{t}\mathring{\mathfrak{A}}_1 b+\tilde{R}_2+\tilde{H}_2+\tilde{F}_2, \tag{3.6.79}$$

where

$$\tilde{H}_2=D_t(t\mathring{H}_1)-\mathring{H}_1+[D_{\mathring{w}}\mathring{H}_1\cdot y]+(D_t\mathring{\mathfrak{A}}_1)b-(D_t\mathring{A}_1^0)y$$

and

$$\tilde{F}_2=-[D_{\mathring{w}}\mathring{A}_1^i\cdot y]\partial_i\mathring{w}-D_t(t\mathring{A}_1^i)\partial_i\mathring{w}+\mathring{A}_1^i\partial_i\mathring{w}+t\partial_t\mathring{F}_1+tC_1^i\partial_i\partial_t v.$$

Note that in deriving the above equation, we have used the identity

$$\mathring{\mathfrak{A}}_1\mathbb{P}_1=\mathbb{P}_1\mathring{\mathfrak{A}}_1=\mathbb{P}_1\mathring{\mathfrak{A}}_1\mathbb{P}_1, \tag{3.6.80}$$

which follows directly from (3.6.3) and (3.6.5). We further note by (3.6.65), Assumption 3.6.2.(4) and Assumption 3.6.9.(2) that $\tilde{F}_2$ and $\tilde{H}_2=\tilde{H}_2(t,x,\mathring{w},b,y)$ satisfy

$$\|\tilde{F}_2(t)\|_{Q^{s-1}}\leq C\big(\|\mathring{w}\|_{L^\infty([T_0,t),H^s)},\|w\|_{L^\infty([T_0,t),R^s)}\big)\Big(\mathcal{C}^*+\|y(t)\|_{Q^{s-1}}+\|w(t)\|_{R^s}+\|\mathring{w}(t)\|_{H^s} \\ +\int_{T_0}^t(\|\mathring{w}(\tau)\|_{H^s}+\|w(\tau)\|_{R^s})d\tau\Big) \tag{3.6.81}$$

for $T_0\leq t<T_1$ and

$$\tilde{H}_2(t,x,0,0,0)=0, \tag{3.6.82}$$

respectively. Using (3.6.61) and (3.6.68), we deduce that

$$y|_\Sigma=\Big[(\mathring{A}_1^0)^{-1}\mathring{\mathfrak{A}}_1\mathbb{P}_1\mathring{w}-t(\mathring{A}_1^0)^{-1}\mathring{A}_1^i\partial_i\mathring{w}-t(\mathring{A}_1^0)^{-1}C_1^i\partial_i v+t(\mathring{A}_1^0)^{-1}\mathring{H}_1+t(\mathring{A}_1^0)^{-1}\mathring{F}_1\Big]\Big|_\Sigma,$$

which in turn, implies, via (3.6.65), (3.6.68), and the calculus inequalities from Appendix C, that

$$\|y(T_0)\|_{Q^{s-1}} \leq C(\sigma)\sigma.$$

Next, a short computation using (3.6.59), (3.6.61) and (3.6.62) shows that

$$A_1^0\partial_t z + A_1^i\partial_i z + \frac{1}{\epsilon}C_1^i\partial_i z = \frac{1}{t}\mathfrak{A}_1\mathbb{P}_1 z + \hat{R}_2 + \hat{F}_2, \tag{3.6.83}$$

where

$$\hat{F}_2 = \frac{1}{\epsilon}(H_1 - \mathring{H}_1) + \frac{1}{\epsilon}(F_1 - \mathring{F}_1) - \frac{1}{\epsilon}(A_1^i - \mathring{A}_1^i)\partial_i\mathring{w} - A_1^i\partial_i v - A_1^0\partial_t v + \frac{1}{t}\mathbb{P}_1\mathfrak{A}_1\mathbb{P}_1 v$$

and

$$\hat{R}_2 = -\frac{1}{t}\tilde{A}_1^0 y + \frac{1}{t}\tilde{\mathfrak{A}}_1 b,$$

and we recall that $\tilde{A}_1^0$ and $\tilde{\mathfrak{A}}_1$ are defined by the expansions (3.6.2)-(3.6.3). To proceed, we estimate

$$\begin{aligned}
&\frac{1}{\epsilon}\|H_1(\epsilon,t,\cdot,w(t)) - \mathring{H}_1(t,\cdot,\mathring{w}(t))\|_{Q^{s-1}} \\
&\quad\leq \frac{1}{\epsilon}\|H_1(\epsilon,t,\cdot,w(t)) - H_1(\epsilon,t,\cdot,\mathring{w}(t))\|_{Q^{s-1}} + \frac{1}{\epsilon}\|H_1(\epsilon,t,\cdot,\mathring{w}(t)) - \mathring{H}_1(t,\cdot,\mathring{w}(t))\|_{Q^{s-1}} \\
&\quad\leq C\big(\|w\|_{L^\infty([T_0,t),R^s)}, \|\mathring{w}\|_{L^\infty([T_0,t),H^s)}\big)\Big(\mathcal{C}^* + \|w(t)\|_{R^s} + \|z(t)\|_{Q^{s-1}} + \|\mathring{w}(t)\|_{H^s} \\
&\qquad + \int_{T_0}^t (\|w(\tau)\|_{R^s} + \|z(\tau)\|_{Q^{s-1}} + \|\mathring{w}(\tau)\|_{H^s}d\tau\Big),
\end{aligned} \tag{3.6.84}$$

for $T_0 \leq t < T_1$, where in deriving the second inequality, we used (3.6.67), Taylor's Theorem (in the last variable), and the calculus inequalities. By similar arguments, we also see that the inequality

$$\begin{aligned}
&\frac{1}{\epsilon}\|(A_1^i(\epsilon,t,\cdot,w(t)) - \mathring{A}_1^i(t,\cdot,\mathring{w}(t))\|_{Q^{s-1}} \\
&\leq C(\|w\|_{L^\infty([T_0,t),R^s)}, \|\mathring{w}\|_{L^\infty([T_0,t),H^s)})\Big(\mathcal{C}^* + \|w(t)\|_{R^s} + \|z(t)\|_{Q^{s-1}} + \|\mathring{w}(t)\|_{H^s} \\
&\quad + \int_{T_0}^t (\|w(\tau)\|_{R^s} + \|z(\tau)\|_{Q^{s-1}} + \|\mathring{w}(\tau)\|_{H^s})d\tau + 1\Big),
\end{aligned} \tag{3.6.85}$$

holds for $T_0 \leq t < T_1$. The estimates (3.6.65), (3.6.67), and (3.6.84)-(3.6.85) together with the calculus inequalities then imply that

$$\begin{aligned}
\|\hat{F}_2(\epsilon,t)\|_{Q^{s-1}} \leq{}& C\big(\|w\|_{L^\infty([T_0,t),R^s)}, \|\mathring{w}\|_{L^\infty([T_0,t),H^s)}, \|z\|_{L^\infty([T_0,t),R^{s-1})}\big)\Big(\mathcal{C}^* + \|w(t)\|_{R^s} + \|z(t)\|_{Q^{s-1}} \\
&+ \|\mathring{w}(t)\|_{H^s} + \int_{T_0}^t (\|w(\tau)\|_{R^s} + \|z(\tau)\|_{Q^{s-1}} + \|\mathring{w}(\tau)\|_{H^s})d\tau\Big)
\end{aligned} \tag{3.6.86}$$

for $T_0 \leq t < T_1$. Furthermore, we see from (3.6.65) and (3.6.68) that we can estimate z at $t = T_0$ by

$$\|z(T_0)\|_{Q^{s-1}} \leq C(\sigma)\sigma. \tag{3.6.87}$$

We can combine the two equations (3.6.59) and (3.6.61) into the single system

$$\begin{aligned}
&\begin{pmatrix} A_1^0 & 0 \\ 0 & \mathring{A}_1^0 \end{pmatrix}\partial_t\begin{pmatrix} w \\ \mathring{w} \end{pmatrix} + \begin{pmatrix} A_1^i & 0 \\ 0 & \mathring{A}_1^i \end{pmatrix}\partial_i\begin{pmatrix} w \\ \mathring{w} \end{pmatrix} + \frac{1}{\epsilon}\begin{pmatrix} C_1^i & 0 \\ 0 & 0 \end{pmatrix}\partial_i\begin{pmatrix} w \\ \mathring{w} \end{pmatrix} \\
&\qquad = \frac{1}{t}\begin{pmatrix} \mathfrak{A}_1 & 0 \\ 0 & \mathring{\mathfrak{A}}_1 \end{pmatrix}\begin{pmatrix} \mathbb{P}_1 & 0 \\ 0 & \mathbb{P}_1 \end{pmatrix}\begin{pmatrix} w \\ \mathring{w} \end{pmatrix} + \begin{pmatrix} H_1 \\ \mathring{H}_1 \end{pmatrix} + \begin{pmatrix} F_1 \\ \mathring{F}_1 - C_1^i\partial_i v \end{pmatrix},
\end{aligned} \tag{3.6.88}$$

and collect the three equations (3.6.75), (3.6.79) and (3.6.83) together to get

$$A_2^0\partial_t\begin{pmatrix}b\\y\\z\end{pmatrix}+A_2^i\partial_i\begin{pmatrix}b\\y\\z\end{pmatrix}+\frac{1}{\epsilon}C_2^i\partial_i\begin{pmatrix}b\\y\\z\end{pmatrix}=\frac{1}{t}\mathfrak{A}_2\mathbb{P}_2\begin{pmatrix}b\\y\\z\end{pmatrix}+H_2+R_2+F_2, \tag{3.6.89}$$

where

$$A_2^0:=\begin{pmatrix}\mathbb{P}_1\mathring{A}_1^0\mathbb{P}_1 & 0 & 0\\ 0 & \mathring{A}_1^0 & 0\\ 0 & 0 & A_1^0\end{pmatrix},\quad A_2^i:=\begin{pmatrix}\mathbb{P}_1\mathring{A}_1^i\mathbb{P}_1 & 0 & 0\\ 0 & \mathring{A}_1^i & 0\\ 0 & 0 & A_1^i\end{pmatrix}, \tag{3.6.90}$$

$$C_2^i:=\begin{pmatrix}0&0&0\\0&0&0\\0&0&C_1^i\end{pmatrix},\quad \mathbb{P}_2:=\begin{pmatrix}\mathbb{P}_1&0&0\\0&\mathbb{1}&0\\0&0&\mathbb{P}_1\end{pmatrix},\quad \mathfrak{A}_2=\begin{pmatrix}\mathbb{P}_1\mathring{\mathfrak{A}}_1\mathbb{P}_1 & 0 & 0\\ -\mathbb{P}_1\mathring{\mathfrak{A}}_1\mathbb{P}_1 & \mathbb{P}_1\mathring{\mathfrak{A}}_1\mathbb{P}_1+\mathring{A}_1^0 & 0\\ 0 & 0 & \mathfrak{A}_1\end{pmatrix}, \tag{3.6.91}$$

$$H_2:=\begin{pmatrix}\mathbb{P}_1\mathring{H}_1\\ \tilde{H}_2\\ 0\end{pmatrix},\quad R_2:=\begin{pmatrix}0\\0\\ \hat{R}_2\end{pmatrix}\quad\text{and}\quad F_2:=\begin{pmatrix}\mathbb{P}_1\bar{F}_2\\ \tilde{F}_2\\ \hat{F}_2\end{pmatrix}. \tag{3.6.92}$$

We remark that due to the projection operator $\mathbb{P}_1$ that appears in the definition (3.6.74) of b and in the top row of (3.6.90), the vector $(b,y,z)^{\mathrm{T}}$ takes values in the vector space $\mathbb{P}_1\mathbb{R}^{N_1}\times\mathbb{R}^{N_1}\times\mathbb{R}^{N_1}$ and (3.6.90) defines a symmetric hyperbolic system, i.e. A_2^0 and A_2^i define symmetric linear operators on $\mathbb{P}_1\mathbb{R}^{N_1}\times\mathbb{R}^{N_1}\times\mathbb{R}^{N_1}$ and A_2^0 is non-degenerate.

Setting

$$\mathbb{P}_3:=\begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&\mathbb{P}_1&0&0\\0&0&0&\mathbb{1}&0\\0&0&0&0&0\end{pmatrix}\quad\text{and}\quad \mathbb{P}_4=\begin{pmatrix}0&0\\0&\mathbb{1}\end{pmatrix},$$

it is then not difficult to verify from the estimates (3.6.65), (3.6.76), (3.6.81) and (3.6.86), the initial bounds (3.6.68), (3.6.77) and (3.6.87), the relations (3.6.73), (3.6.78), (3.6.80) and (3.6.82), and the assumptions on the coefficients $\{A_1^0,\ A_1^i,\ \mathring{A}_1^0,\ \mathring{A}_1^i,\ \mathfrak{A}_1,\ \mathring{\mathfrak{A}}_1,\ H,\ F\}$ (see Assumptions 3.6.2 and 3.6.9) that the system consisting of (3.6.88) and (3.6.89) and the solution $U=(w,\mathring{w},b,y,z)^{\mathrm{T}}$ satisfy the hypotheses of Theorem 3.6.7, and thus, for $\sigma>0$ chosen small enough, there exists a constant $C>0$, independent of $\epsilon\in(0,\epsilon_0)$ and $T_1\in(T_0,0)$, such that

$$\|(w,\mathring{w})\|_{L^\infty([T_0,T_1)\times\mathbb{R}^3)}\leq\frac{R}{2} \tag{3.6.93}$$

and

$$\|\mathring{w}\|_{L^\infty([T_0,t),L^2)}+\left(-\int_{T_0}^t\frac{1}{\tau}\|\mathbb{P}_1\mathring{w}\|_{L^2}^2\,d\tau\right)^{\frac{1}{2}}+\|(w,\mathring{w})\|_{M^\infty_{\mathring{\mathbb{P}}_1,s}([T_0,t)\times\mathbb{R}^3)} \\ +\|(b,y,z)\|_{M^\infty_{\mathbb{P}_2,s-1}([T_0,t)\times\mathbb{R}^3)}-\int_{T_0}^t\frac{1}{\tau}\|\mathbb{P}_3U\|_{Q^{s-1}}\,d\tau\leq C\sigma \tag{3.6.94}$$

for $T_0\leq t<T_1$. This completes the proof since the estimates (3.6.69)-(3.6.72) follow immediately from (3.6.93) and (3.6.94). □

3.7 Proof of the Main Theorem 3.1.6

3.7.1 Transforming the conformal Einstein-Euler equations

The first step of the proof is to observe that the non-local formulation of the conformal Einstein-Euler equations given by (3.5.23) can be transformed into the form (3.6.59) analyzed in §3.6 by making the simple change of time coordinate

$$t \mapsto \hat{t} := -t \tag{3.7.1}$$

and the substitutions

$$w(\hat{t},x) = \mathbf{U}(-\hat{t},x),\ A_1^0(\epsilon,-\hat{t},w) = \mathbf{B}^0(\epsilon,-\hat{t},\mathbf{U}),\ A_1^i(\epsilon,\hat{t},w) = -\mathbf{B}^i(\epsilon,-\hat{t},\mathbf{U}),\ \mathfrak{A}_1(\epsilon,\hat{t},w) = \mathbf{B}(\epsilon,-\hat{t},\mathbf{U}), \tag{3.7.2}$$

$$C_1^i = -\mathbf{C}^i,\ \mathbb{P}_1 = \mathbf{P},\ H_1(\epsilon,\hat{t},w) = -\mathbf{H}(\epsilon,-\hat{t},\mathbf{U}) \textbf{ and } F_1(\epsilon,\hat{t},x) = -\mathbf{F}(\epsilon,-\hat{t},x,\mathbf{U},\partial_k\Phi,\partial_t\partial_k\Phi,\partial_k\partial_l\Phi). \tag{3.7.3}$$

With these choices, we can use the same arguments as in [51, §7] to verify that all the structural assumptions listed in §3.6 hold. We omit the details.

3.7.2 Limit equations

Setting

$$\mathring{\mathbf{U}} = (\mathring{u}_0^{0\mu}, \mathring{w}_k^{0\mu}, \mathring{u}^{0\mu}, \mathring{u}_0^{ij}, \mathring{u}_k^{ij}, \mathring{u}^{ij}, \mathring{u}_0, \mathring{u}_k, \mathring{u}, \delta\mathring{\zeta}, \mathring{z}_i)^{\mathrm{T}}, \tag{3.7.4}$$

the limit equation, see §3.6.2, associated to (3.5.23) on the spacetime region $(T_2,1]\times\mathbb{R}^3$, $0<T_2<1$, is given by

$$\mathring{\mathbf{B}}^0\partial_t\mathring{\mathbf{U}} + \mathring{\mathbf{B}}^i\partial_i\mathring{\mathbf{U}} + \mathbf{C}^i\partial_i\mathbf{V} = \frac{1}{t}\mathring{\mathbf{B}}\mathbf{P}\mathring{\mathbf{U}} + \mathring{\mathbf{H}} + \mathring{\mathbf{F}} \qquad \text{in } (T_2,1]\times\mathbb{R}^3, \tag{3.7.5}$$

$$\mathbf{C}^i\partial_i\mathring{\mathbf{U}} = 0 \qquad \text{in } (T_2,1]\times\mathbb{R}^3, \tag{3.7.6}$$

where

$$\mathring{\mathbf{B}}^\mu(t,\mathring{\mathbf{U}}) := \lim_{\epsilon\searrow 0}\mathbf{B}^\mu(\epsilon,t,\mathring{\mathbf{U}}),\quad \mathring{\mathbf{B}}(t,\mathring{\mathbf{U}}) := \lim_{\epsilon\searrow 0}\mathbf{B}(\epsilon,t,\mathring{\mathbf{U}}),\quad \mathring{\mathbf{H}}(t,\mathring{\mathbf{U}}) := \lim_{\epsilon\searrow 0}\mathbf{H}(\epsilon,t,\mathring{\mathbf{U}}), \tag{3.7.7}$$

and

$$\mathring{\mathbf{F}} := \Bigg(2\mathring{E}^{-1}\mathring{\Omega}\delta^{kj}\delta^\mu_j\mathring{\Phi}_k - 2\sqrt{\frac{\Lambda}{3}}t\mathring{E}^{-1}\delta^\mu_i\varpi^i + t\delta^\mu_0\mathring{E}^{-1}\mathring{\Upsilon}, \left(\frac{1}{2}+\mathring{\Omega}\right)\mathring{E}^{-1}\delta^{kl}\delta^\mu_0\mathring{\Phi}_k - \delta^{kl}t\mathring{E}^{-1}\delta^\mu_0\partial_0\mathring{\Phi}_k,$$
$$0,0,0,0,0,0,0,0,-K^{-1}\frac{1}{2}\left(\frac{3}{\Lambda}\right)^{\frac{3}{2}}\mathring{E}^{-3}\delta^{lk}t\mathring{\Phi}_k\Bigg)^{\mathrm{T}}. \tag{3.7.8}$$

In $\mathring{\mathbf{F}}$, $\mathring{\Phi}$ is the Newtonian potential, see (3.1.67), and $\mathring{E}$, $\mathring{\Omega}$ and ϖ^j are defined previously by (3.1.62), (3.1.64) and (3.4.7), respectively.

We then observe that under the change of time coordinate (3.7.1) and the substitutions

$$\mathring{w}(\hat{t},x) = \mathring{\mathbf{U}}(-\hat{t},x),\quad \mathring{A}_1^0(\hat{t},w) = \mathring{\mathbf{B}}^0(-\hat{t},\mathring{\mathbf{U}}),\ \mathring{A}_1^i(\hat{t},w) = -\mathring{\mathbf{B}}^i(-\hat{t},\mathring{\mathbf{U}}),\ \mathring{\mathfrak{A}}_1(\hat{t},w) = \mathring{\mathbf{B}}(-\hat{t},\mathring{\mathbf{U}}),\ C_1^i = -\mathbf{C}^i, \tag{3.7.9}$$

$$v(\hat{t},x) = \mathbf{V}(-\hat{t},x),\quad \mathbb{P}_1 = \mathbf{P},\quad \mathring{H}_1(\hat{t},w) = -\mathring{\mathbf{H}}(-\hat{t},\mathring{\mathbf{U}}) \quad\text{and}\quad \mathring{F}_1(\hat{t},x) = -\mathring{\mathbf{F}}(-\hat{t},x), \tag{3.7.10}$$

the limit equation (3.7.5)-(3.7.6) transforms into

$$\mathring{A}_1^0\partial_{\hat{t}}\mathring{w} + \mathring{A}_1^i\partial_i\mathring{w} = \frac{1}{\hat{t}}\mathring{\mathfrak{A}}_1\mathbb{P}_1\mathring{w} - C_1^i\partial_i v + \mathring{H}_1 + \mathring{F}_1 \qquad \text{in} \quad [-1,-T_2)\times\mathbb{R}^3,$$

$$C_1^i \partial_i \mathring{w} = 0 \qquad\qquad \text{in} \quad [-1, -T_2) \times \mathbb{R}^3,$$

which is of the form analyzed in §3.6.2, see (3.6.61)-(3.6.62) and (3.6.64). It is also not difficult to verify that the matrices $\mathring{A}_1^i$ and the source term $\mathring{H}_1$ satisfy the Assumption 3.6.9.(1) from §3.6.2.

3.7.3 Local existence and continuation

For fixed $\epsilon \in (0, \epsilon_0)$, we know from Proposition 3.4.1, Corollary 3.4.2 and Proposition 3.5.2 that for $T_1 \in (0,1)$ chosen close enough to 1 there exists a unique solution[11]

$$\mathbf{U} \in \bigcap_{\ell=0}^{1} C^\ell\big((T_1, 1], R^{s-\ell}(\mathbb{R}^3, \mathbb{K})\big)$$

to (3.5.23) satisfying the initial condition

$$\mathbf{U}|_\Sigma = \big(u_0^{0\mu}|_\Sigma, w_k^{0\mu}|_\Sigma, u^{0\mu}|_\Sigma, u_0^{ij}|_\Sigma, u_k^{ij}|_\Sigma, u^{ij}|_\Sigma, u_0|_\Sigma, u_k|_\Sigma, u|_\Sigma, \delta\zeta|_\Sigma, z_i|_\Sigma\big)^{\mathrm{T}},$$

where the initial data is determined from Lemma 3.3.15 and Proposition 3.5.2.(2). Moreover, we know that this solution can be continued beyond T_1 provided that

$$\sup_{t \in (T_1, 1]} \|\mathbf{U}(t)\|_{R^s} < \infty.$$

Next, by Proposition 3.4.4, there exists, for some $T_2 \in (0,1]$, a unique solution $(\mathring{\zeta}, \mathring{z}^i, \mathring{\Phi})$ which verifies

$$(\delta\mathring{\zeta}, \mathring{z}^i, \mathring{\Phi}) \in \bigcap_{\ell=0}^{1} C^\ell\big((T_2, 1], H^{s-\ell}(\mathbb{R}^3)\big) \times \bigcap_{\ell=0}^{1} C^\ell((T_2, 1], H^{s-\ell}(\mathbb{R}^3, \mathbb{R}^3)) \times \bigcap_{\ell=0}^{1} C^\ell((T_2, 1], R^{s+2-\ell}(\mathbb{R}^3)), \tag{3.7.11}$$

to the conformal cosmological conformal Poisson-Euler equations, given by (3.1.65)-(3.1.67), satisfying the initial condition

$$(\delta\mathring{\zeta}, \mathring{z}_i)|_\Sigma = \bigg(\ln\Big(1 + \frac{\delta\breve{\rho}_{\epsilon,\vec{\mathbf{y}}}}{\mathring{\mu}(1)}\Big), \mathring{E}^2 \delta_{ij} \breve{z}^j_{\epsilon,\vec{\mathbf{y}}}\bigg).$$

Setting

$$\mathbf{V} = \Big(V_0^{0\mu}, V_k^{0\mu}, V^{0\mu}, 0, 0, 0, 0, 0, 0, 0, 0\Big)^{\mathrm{T}},$$

where

$$V_0^{0\mu} = -\bigg(\frac{1}{2} + \mathring{\Omega}\bigg)\delta_0^\mu \mathring{E}^{-1}\mathring{\Phi} + t\mathring{E}^{-1}\partial_0\mathring{\Phi}\delta_0^\mu, \tag{3.7.12}$$

$$V_k^{0\mu} = -2\mathring{E}^{-1}\mathring{\Omega}\delta_k^\mu\mathring{\Phi} + 2\delta_j^\mu\sqrt{\frac{\Lambda}{3}}t\mathring{E}^{-1}(-\Delta)^{-\frac{1}{2}}\mathfrak{R}_k\varpi^j, \tag{3.7.13}$$

$$V^{0\mu} = \bigg(\frac{1}{2} + \Omega\bigg)\delta_0^\mu\mathring{E}^{-1}\mathring{\Phi}, \tag{3.7.14}$$

it follows from Proposition 3.4.4 and (3.7.11) that $\mathbf{V}$ is well-defined and lies in the space

$$\mathbf{V} \in \bigcap_{\ell=0}^{1} C^\ell\big((T_2, 1], R^{s+1-\ell}(\mathbb{R}^3, \mathbb{K})\big).$$

It can be verified by a direct calculation that the pair $(\mathbf{V}, \mathring{\mathbf{U}})$, where

$$\mathring{\mathbf{U}} = (0, 0, 0, 0, 0, 0, 0, 0, 0, \delta\mathring{\zeta}, \mathring{z}_i), \tag{3.7.15}$$

[11] Recall that $\mathbb{K}$ is defined in Proposition 3.4.1.

determines a solution of the limit equation (3.7.5)-(3.7.6). Moreover, by Proposition 3.4.4, it is clear that this solution can be continued past T_2 provided that

$$\sup_{t\in(T_2,1]}\|\mathring{\mathbf{U}}(t)\|_{R^s}<\infty.$$

3.7.4 Global existence and error estimates

To complete the proof, we use the a priori estimates from Theorem 3.6.10 to show that the solutions $\mathbf{U}$ and $(\mathbf{V},\mathring{\mathbf{U}})$ to the reduced conformal Einstein-Euler equations and the corresponding limit equation, respectively, can be continued all the way to $t=0$, i.e. $T_1=T_2=0$, with uniform bounds and an error estimate. In order to apply Theorem 3.6.10, we need to verify that the estimates (3.6.65)-(3.6.67) hold for the solutions $\mathbf{U}$ and $(\mathbf{V},\mathring{\mathbf{U}})$. We begin by observing, via routine calculations, that the components of $\partial_t\mathbf{V}$ are given by

$$\partial_t V^{0\mu}_0=\Big(1-2\mathring{\Omega}-\frac{1}{2}\Big)\mathring{E}^{-1}\delta^\mu_0\partial_t\mathring{\Phi}+t\mathring{E}^{-1}\delta^\mu_0\partial_t^2\mathring{\Phi}-\Big(\partial_t\mathring{\Omega}-(\frac{1}{2}+\mathring{\Omega})\frac{\mathring{\Omega}}{t}\Big)\mathring{E}^{-1}\delta^\mu_0\mathring{\Phi}, \tag{3.7.16}$$

$$\partial_t V^{0\mu}=\Big(\partial_t\mathring{\Omega}-(\frac{1}{2}+\mathring{\Omega})\frac{\mathring{\Omega}}{t}\Big)\delta^\mu_0\mathring{E}^{-1}\mathring{\Phi}+\Big(\frac{1}{2}+\mathring{\Omega}\Big)\delta^\mu_0\mathring{E}^{-1}\partial_t\mathring{\Phi} \tag{3.7.17}$$

and

$$\partial_t V^{0\mu}_k=2\mathring{E}^2\sqrt{\frac{\Lambda}{3}}\delta^\mu_j(-\Delta)^{-\frac{1}{2}}\mathfrak{R}_k\mathring{E}^{-3}\big((1-\mathring{\Omega})\varpi^j+t\partial_t\varpi^j\big)+2\mathring{E}^{-1}\delta^\mu_k\Big(\frac{\mathring{\Omega}^2}{t}-\partial_t\mathring{\Omega}\Big)\mathring{\Phi}-2\mathring{E}^{-1}\mathring{\Omega}\delta^\mu_k\partial_t\mathring{\Phi}. \tag{3.7.18}$$

We further compute

$$\frac{1}{t}\mathbf{PV}=\Big(\frac{1}{2t}(V^{0\mu}_0+V^{0\mu}),\frac{1}{t}V^{0\mu}_i,\frac{1}{2t}(V^{0\mu}_0+V^{0\mu}),0,0,0,0,0,0,0,0\Big)^{\mathrm{T}},$$

where the components are given by

$$\frac{1}{2t}(V^{0\mu}_0+V^{0\mu})=\frac{1}{2}\delta^\mu_0\mathring{E}^{-1}\partial_0\mathring{\Phi}, \tag{3.7.19}$$

$$\frac{1}{t}V^{0\mu}_k=-2\mathring{E}^{-1}\frac{\mathring{\Omega}}{t}\delta^\mu_k\mathring{\Phi}+2\delta^\mu_j\mathring{E}^{-1}\sqrt{\frac{\Lambda}{3}}(-\Delta)^{-\frac{1}{2}}\mathfrak{R}_k\varpi^j. \tag{3.7.20}$$

Then by (3.7.12)-(3.7.14) and (3.7.16)-(3.7.20), it is clear that the estimate

$$\begin{aligned}
&\|\mathbf{V}(t)\|_{R^{s+1}}+\|t^{-1}\mathbf{PV}(t)\|_{R^{s+1}}+\|\partial_t\mathbf{V}(t)\|_{R^s}\\
&\quad\leq\|\partial_t\mathring{\Phi}(t)\|_{R^{s+1}}+\|t\partial_t^2\mathring{\Phi}(t)\|_{R^s}+\|\mathring{\Phi}(t)\|_{R^{s+1}}+\|(-\Delta)^{-\frac{1}{2}}\mathfrak{R}_k\varpi^j(t)\|_{R^{s+1}}+\|t\partial_t(-\Delta)^{-\frac{1}{2}}\mathfrak{R}_k\varpi^j(t)\|_{R^s}\\
&\quad\leq C\big(\|\delta\mathring{\zeta}\|_{L^\infty((t,1],H^s)},\|\mathring{z}^j\|_{L^\infty((t,1],H^s)}\big)\Big(\|\delta\mathring{\zeta}(t)\|_{R^s}+\|\mathring{z}^j(t)\|_{H^s}+\|\delta\breve{\rho}\|_{L^{\frac{6}{5}}\cap H^s}+\int_t^1\|\mathring{z}^k(\tau)\|_{H^s}d\tau\Big)\\
&\quad\leq C\big(K_4(t)\big)\Big(\|\mathring{\mathbf{U}}(t)\|_{H^s}+\|\breve{\xi}_\epsilon\|_s+\int_t^1\|\mathring{\mathbf{U}}(\tau)\|_{H^s}d\tau\Big),
\end{aligned} \tag{3.7.21}$$

where

$$K_4(t)=\|\mathring{\mathbf{U}}\|_{L^\infty((t,1],H^s)}+\|\mathbf{U}\|_{L^\infty(((t,1],R^s))},$$

follows from the estimates (3.4.13)-(3.4.14). From similar reasoning and the embedding $H^s\hookrightarrow R^s$, it is also not difficult, using (3.4.15) and (3.4.16) to estimate $\mathring{\Upsilon}$ and $\partial_t\mathring{\Upsilon}$, to verify that $\mathring{\mathbf{F}}$, defined by (3.7.8), satisfies the estimate

$$\|\mathring{\mathbf{F}}(t)\|_{H^s}+\|t\partial_t\mathring{\mathbf{F}}(t)\|_{R^{s-1}}\leq C\big(\|\delta\mathring{\zeta}\|_{L^\infty((t,1],H^s)},\|\mathring{z}^j\|_{L^\infty((t,1],H^s)}\big)\Big(\|\delta\mathring{\zeta}(t)\|_{R^s}+\|\mathring{z}^j(t)\|_{H^s}+\|\delta\breve{\rho}\|_{L^{\frac{6}{5}}\cap H^s}$$

$$+\int_t^1 \|\mathring{z}^k(\tau)\|_{H^s}d\tau\Big) \tag{3.7.22}$$

for $T_2 < t \leq 1$. Furthermore, we see from the definition of $\mathbf{F}$, see (3.5.24), the estimates (3.5.8)-(3.5.9), and the calculus inequalities that $\mathbf{F}$ is bounded by

$$\begin{aligned}\|\mathbf{F}(t)\|_{R^s} \leq & C\big(\|\mathbf{U}\|_{L^\infty((t,1],R^s)}, \|\Phi^\mu_k\|_{L^\infty((t,1],R^s)}\big)(\|\mathbf{U}(t)\|_{R^s} + \|D\Phi^\mu_k(t)\|_{R^s} + \|\Phi^\mu_k(t)\|_{R^s} + \|t\partial_t\Phi^\mu_k(t)\|_{R^s}) \\ \leq & C\big(\|\mathbf{U}\|_{L^\infty((t,1],R^s)}\big)\Big(\|\breve{\xi}_\epsilon\|_s + \|\mathbf{U}(t)\|_{R^s} + \int_t^1 (\|u^{0i}_{\epsilon,\vec{\mathbf{y}}}(\tau)\|_{R^s} + \|z_{l,\epsilon,\vec{\mathbf{y}}}(\tau)\|_{R^s})d\tau\Big)\end{aligned} \tag{3.7.23}$$

for $T_1 < t < 1$. Together, (3.7.21), (3.7.22) and (3.7.23) show that source terms $\{F_1, \mathring{F}_1, v\}$, as defined by (3.7.3) and (3.7.10), satisfy the estimates (3.6.65) from Theorem 3.6.10 for times $-1 \leq \hat{t} < -T_3$, where

$$T_3 = \max\{T_1, T_2\}.$$

Next, we verify that the Lipschitz estimates (3.6.66)-(3.6.67) from Theorem 3.6.10 are satisfied. We start by noticing, with the help of (3.2.50), (3.2.64) and (3.7.15), that

$$\begin{aligned}&\tilde{B}^i(\epsilon, t, \mathring{\mathbf{U}}) = 0, \\ &B^i(\epsilon, t, \mathring{\mathbf{U}}) = \sqrt{\frac{3}{\Lambda}}\begin{pmatrix} \mathring{z}^i & E^{-2}\delta^{im} \\ E^{-2}\delta^{il} & K^{-1}E^{-2}\delta^{lm}\mathring{z}^i\end{pmatrix} + \epsilon^2\bar{\mathscr{S}}^i(\epsilon, t, \mathring{\mathbf{U}})\end{aligned}$$

and

$$B^i(0, t, \mathring{\mathbf{U}}) = \sqrt{\frac{3}{\Lambda}}\begin{pmatrix} \mathring{z}^i & \mathring{E}^{-2}\delta^{im} \\ \mathring{E}^{-2}\delta^{il} & K^{-1}\mathring{E}^{-2}\delta^{lm}\mathring{z}^i\end{pmatrix}.$$

From the above expressions, the expansion (3.2.5), and the calculus inequalities, we then obtain the estimate

$$\|\mathbf{B}^i(\epsilon, t, \mathring{\mathbf{U}}) - \mathring{\mathbf{B}}^i(t, \mathring{\mathbf{U}})\|_{R^{s-1}} \leq \epsilon C(\|\mathring{\mathbf{U}}\|_{L^\infty((t,1],R^s)}), \quad T_3 < t \leq 1. \tag{3.7.24}$$

Next, using (3.2.54), (3.2.55), (3.5.18), (3.5.19) and (3.7.15), we can express the components of $\mathbf{H}(\epsilon, t, \mathring{\mathbf{U}})$, see (3.5.24), as follows:

$$\begin{gathered}\tilde{G}_1(\epsilon, t, \mathring{\mathbf{U}}) = \big(\epsilon\mathscr{S}^\mu(\epsilon, t, \mathring{\mathbf{U}}), 0, 0\big)^{\mathrm{T}}, \quad \tilde{G}_2(\epsilon, t, \mathring{\mathbf{U}}) = \big(\epsilon\mathscr{S}^{ij}(\epsilon, t, \mathring{\mathbf{U}}), 0, 0\big)^{\mathrm{T}}, \\ \tilde{G}_3(\epsilon, t, \mathring{\mathbf{U}}) = \big(\epsilon\mathscr{S}(\epsilon, t, \mathring{\mathbf{U}}), 0, 0\big)^{\mathrm{T}}, \quad \text{and} \quad G(\epsilon, t, \mathring{\mathbf{U}}) = (0, 0)^{\mathrm{T}},\end{gathered}$$

where $\mathscr{S}^\mu$, $\mathscr{S}^{ij}$ and $\mathscr{S}$ all vanish for $\mathring{\mathbf{U}} = 0$. It follows immediately from these expressions and the definitions (3.5.24) and (3.7.7) that

$$\mathring{\mathbf{H}}(t, \mathring{\mathbf{U}}) = \Big(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\Big)^T,$$

and, with the help of the calculus inequalities and (3.2.4)-(3.2.5), that

$$\|\mathbf{H}(\epsilon, t, \mathring{\mathbf{U}}) - \mathring{\mathbf{H}}(t, \mathring{\mathbf{U}})\|_{R^{s-1}} \leq \epsilon C(\|\mathring{\mathbf{U}}\|_{L^\infty((t,1],R^s)})\|\mathring{\mathbf{U}}\|_{R^{s-1}}, \quad T_3 < t \leq 1. \tag{3.7.25}$$

To proceed, we define

$$\mathbf{Z} = \frac{1}{\epsilon}(\mathbf{U} - \mathring{\mathbf{U}} - \epsilon\mathbf{V}), \tag{3.7.26}$$

and set

$$z(\hat{t}, x) = \mathbf{Z}(-\hat{t}, x).$$

In view of the definitions (3.5.24) and (3.7.8), it is not difficult to verify that the inequality

$$\begin{aligned}
&\|\mathbf{F}(\epsilon,t,\cdot)-\mathring{\mathbf{F}}(t,\cdot)\|_{R^{s-1}}\leq C\big(\|\mathbf{U}\|_{L^\infty((t,1],R^s)}\big)\Big(\epsilon\|\mathbf{U}(t)\|_{R^{s-1}}+\epsilon\|\Phi^\mu_k(t)\|_{R^{s-1}}+\epsilon\|D\Phi^\mu_k(t)\|_{R^{s-1}}\\
&\quad+\epsilon\|\partial_0\Phi^\mu_k(t)\|_{R^{s-1}}+\|E^{-1}\Upsilon(t)-\mathring{E}^{-1}\mathring{\Upsilon}(t)\|_{R^{s-1}}+\epsilon\|\mathbf{Z}(t)\|_{R^{s-1}}+\epsilon\|\mathbf{V}(t)\|_{R^{s-1}}\\
&\quad+\|\Phi^0_k(t)-\mathring{\Phi}_k(t)\|_{R^{s-1}}+\|\partial_0(\Phi^0_k(t)-\mathring{\Phi}_k(t))\|_{R^{s-1}}+\|\rho z^j(t)-\mathring{\rho}\mathring{z}^j(t)\|_{R^{s-1}}\Big)\\
&\leq C\big(\|\mathbf{U}\|_{L^\infty((t,1],R^s)}\big)\Big(\epsilon\|\breve{\xi}_\epsilon\|_s+\epsilon\int^1_t(\|u^{0i}_{\epsilon,\vec{\mathbf{y}}}(\tau)\|_{R^s}+\|z_{l,\epsilon,\vec{\mathbf{y}}}(\tau)\|_{R^s})d\tau+\epsilon\|\mathbf{U}(t)\|_{R^s}\\
&\quad+\epsilon\|\mathbf{Z}(t)\|_{R^{s-1}}+\epsilon\|\mathring{\mathbf{U}}(t)\|_{H^s}+\epsilon\int^1_t\|\mathring{\mathbf{U}}(\tau)\|_{H^s}d\tau+\|\Phi^0_k(t)-\mathring{\Phi}_k(t)\|_{R^{s-1}}+\|\Upsilon(t)-\mathring{\Upsilon}(t)\|_{R^{s-1}}\\
&\quad+\|\partial_0\Phi^0_k(t)-\partial_0\mathring{\Phi}_k(t)\|_{R^{s-1}}\,d\tau\Big),
\end{aligned}\tag{3.7.27}$$

follows from (3.2.4)-(3.2.5), the estimates (3.5.8)-(3.5.9) and (3.7.21), and the calculus inequalities, and holds for $T_3<t\leq 1$. To complete the Lipschitz estimate for $\mathbf{F}$, we require the estimates from the following lemma, which are an extension of the estimates from Propositions 3.4.4 and 3.5.1.

Lemma 3.7.1. *The estimates*

$$\|\Phi^0_k-\mathring{\Phi}_k\|_{R^{s-1}}+\|\Upsilon-\mathring{\Upsilon}\|_{R^{s-1}}\leq\epsilon C(K_4)\Big(\|\breve{\xi}_\epsilon\|_s+\int^1_t(\|\mathbf{Z}(\tau)\|_{R^{s-1}}+\|\mathbf{U}(\tau)\|_{R^s}+\|\mathring{\mathbf{U}}(\tau)\|_{H^s})d\tau\Big)$$

and

$$\|\partial_t(\Phi^0_k-\mathring{\Phi}_k)\|_{R^{s-1}}\leq\epsilon C(K_4)\Big(\|\breve{\xi}_\epsilon\|_s+\|\mathbf{Z}\|_{R^{s-1}}+\|\mathring{\mathbf{U}}\|_{H^{s-1}}+\|\mathbf{U}\|_{R^{s-1}}+\int^1_t\|\mathring{\mathbf{U}}(\tau)\|_{H^s}d\tau\Big)$$

hold for $t\in(T_3,1]$.

Proof. Noting that

$$\sqrt{|\underline{g}|}e^\zeta z^l-\sqrt{\frac{3}{\Lambda}}\mathring{E}^3e^{\mathring{\zeta}}\mathring{z}^l=\sqrt{\frac{3}{\Lambda}}\mathring{E}^3\big[e^\zeta(z^l-\mathring{z}^l)+e^{\mathring{\mu}}\mathring{z}^le^{\delta\mathring{\zeta}}\big(e^{\delta\zeta-\delta\mathring{\zeta}}-1\big)\big]+\epsilon\mathscr{S}^l(\epsilon,t,u^{\mu\nu},u,\delta\zeta,z_j),$$

where $\mathscr{S}^l(\epsilon,t,0,0,\delta\zeta,0)=0$, we have by (3.4.22)-(3.4.23) and (3.5.12)-(3.5.13) that

$$\begin{aligned}
\partial_t\big(\Phi^0_k-\mathring{\Phi}_k\big)=&\frac{\Lambda}{3}\partial_k\partial_l(\Delta-\epsilon^2\beta)^{-1}\left[\Delta^{-1}(\Delta-\epsilon^2\beta)\left(\sqrt{\frac{3}{\Lambda}}\mathring{E}^3e^{\mathring{\zeta}}\mathring{z}^l\right)-\big(\sqrt{|\underline{g}|}e^\zeta z^l\big)\right]\\
=&-\sqrt{\frac{\Lambda}{3}}\mathring{E}^3\partial_k\partial_l(\Delta-\epsilon^2\beta)^{-1}\Big[e^\zeta(z^l-\mathring{z}^l)+e^{\mathring{\mu}}\mathring{z}^le^{\delta\mathring{\zeta}}\big(e^{\delta\zeta-\delta\mathring{\zeta}}-1\big)+\epsilon\mathscr{S}^l(\epsilon,t,u^{\mu\nu},u,\delta\zeta,z^j)\Big]\\
&+\sqrt{\frac{\Lambda}{3}}\epsilon^2\beta(\Delta-\epsilon^2\beta)^{-1}\mathfrak{R}_k\mathfrak{R}_l\mathring{E}^3e^{\mathring{\zeta}}\mathring{z}^l
\end{aligned}\tag{3.7.28}$$

and

$$\begin{aligned}
\partial_t\big(\Upsilon-\mathring{\Upsilon}\big)=&\frac{\Lambda}{3}\epsilon\beta\partial_l(\Delta-\epsilon^2\beta)^{-1}\left[\left(\sqrt{\frac{3}{\Lambda}}\mathring{E}^3e^{\mathring{\zeta}}\mathring{z}^l\right)-\big(\sqrt{|\underline{g}|}e^\zeta z^l\big)\right]\\
=&-\sqrt{\frac{\Lambda}{3}}\mathring{E}^3\partial_k\partial_l(\Delta-\epsilon^2\beta)^{-1}\Big[e^\zeta(z^l-\mathring{z}^l)+e^{\mathring{\mu}}\mathring{z}^le^{\delta\mathring{\zeta}}\big(e^{\delta\zeta-\delta\mathring{\zeta}}-1\big)+\epsilon\mathscr{S}^l(\epsilon,t,u^{\mu\nu},u,\delta\zeta,z^j)\Big].
\end{aligned}\tag{3.7.29}$$

We also observe that

$$\|\epsilon^2\beta(\Delta-\epsilon^2\beta)^{-1}\big(e^{\mathring{\zeta}}\mathring{z}^l\big)\|_{R^{s-1}}\leq\epsilon\sqrt{\beta}\|(\Delta-\epsilon^2\beta)^{-\frac{1}{2}}\big(e^{\mathring{\zeta}}\mathring{z}^l\big)\|_{R^{s-1}}\leq\epsilon C\|e^{\mathring{\zeta}}\mathring{z}^l\|_{H^{s-2}}\tag{3.7.30}$$

follows from the inequality (3.1.14) and an application of Proposition 3.3.5. Then applying the R^{s-1}

norm on both sides of (3.7.28), we find, with the help of Theorem B.0.2, Propositions 3.3.5 and 3.3.6, and (3.3.57), (3.7.21) and (3.7.30), that the inequality

$$\|\partial_t\big(\Phi^0_k-\mathring{\Phi}_k\big)\|_{R^{s-1}}\leq\epsilon C(K_4)\Big(\|\breve{\xi}_\epsilon\|_s+\|\mathbf{Z}\|_{R^{s-1}}+\|\mathring{\mathbf{U}}\|_{H^{s-1}}+\|\mathbf{U}\|_{R^{s-1}}+\int^1_t\|\mathring{\mathbf{U}}(\tau)\|_{H^s}d\tau\Big).$$

holds for $t\in(T_3,1]$. This concludes the proof of the second estimate in the statement of the lemma.

Turning to the first estimate, we start by estimating the initial values $\|(\Phi^0_k-\mathring{\Phi}_k)|_\Sigma\|_{R^{s-1}}$ and $\|(\Upsilon-\mathring{\Upsilon})|_\Sigma\|_{R^{s-1}}$. From there, the desired estimates for $\|\Phi^0_k-\mathring{\Phi}_k\|_{R^{s-1}}$ and $\|\Upsilon-\mathring{\Upsilon}\|_{R^{s-1}}$ follow from integrating (3.7.28) and (3.7.29) in time and then applying the R^{s-1} norm.

Using the expansion

$$\sqrt{|\underline{\bar{g}}|}e^\zeta\underline{\bar{v}}^0=E^3e^\zeta+\epsilon\mathscr{T}_1(\epsilon,t,u^{\mu\nu},u,\delta\zeta)+\epsilon^2\mathscr{T}_2(\epsilon,t,u^{\mu\nu},u,\delta\zeta,z_j),\tag{3.7.31}$$

where $\mathscr{T}_1(\epsilon,t,0,0,\delta\zeta)=0$ and $\mathscr{T}_2(\epsilon,t,0,0,0,0)=0$, along with the identity

$$\Delta^{-1}(\mathring{E}^3e^{\mathring{\zeta}}-\mathring{E}^3e^{\mathring{\zeta}_H})=(\Delta-\epsilon^2\beta)^{-1}(\mathring{E}^3e^{\mathring{\zeta}}-\mathring{E}^3e^{\mathring{\zeta}_H})-\epsilon^2\beta\Delta^{-1}(\Delta-\epsilon^2\beta)^{-1}(\mathring{E}^3e^{\mathring{\zeta}}-\mathring{E}^3e^{\mathring{\zeta}_H}),$$

and (3.2.4), (3.2.5), (3.2.18) to expand $\delta\rho$, and (3.5.5), we derive the following expansion for $\Phi^0_k-\mathring{\Phi}_k$:

$$\begin{aligned}\Phi^0_k-\mathring{\Phi}_k=&\frac{\Lambda}{3}\partial_k\big[(\Delta-\epsilon^2\beta)^{-1}\big(\sqrt{|\underline{\bar{g}}|}e^\zeta\underline{\bar{v}}^0-E^3e^{\zeta_H}\big)-\Delta^{-1}(\mathring{E}^3e^{\mathring{\zeta}}-\mathring{E}^3e^{\mathring{\zeta}_H})\big]\\=&\frac{\Lambda}{3}\partial_k(\Delta-\epsilon^2\beta)^{-1}\big(E^3(e^\zeta-e^{\zeta_H})-\mathring{E}^3(e^{\mathring{\zeta}}-e^{\mathring{\zeta}_H})\big)+\frac{\Lambda}{3}\partial_k(\Delta-\epsilon^2\beta)^{-1}\epsilon\mathscr{T}_1(\epsilon,t,u^{\mu\nu},u,\delta\zeta)\\&+\frac{\Lambda}{3}\partial_k(\Delta-\epsilon^2\beta)^{-1}\epsilon^2\mathscr{T}_2(\epsilon,t,u^{\mu\nu},u,\delta\zeta,z_j)+\frac{\Lambda}{3}\epsilon^2\beta(\Delta-\epsilon^2\beta)^{-1}\mathfrak{R}_k(-\Delta)^{-\frac12}(\mathring{E}^3e^{\mathring{\zeta}}-\mathring{E}^3e^{\mathring{\zeta}_H})\\=&\frac{\Lambda E^3}{3t^3}\partial_k(\Delta-\epsilon^2\beta)^{-1}\big(\delta\rho-\delta\mathring{\rho}\big)+\frac{\Lambda}{3}\partial_k(\Delta-\epsilon^2\beta)^{-1}\epsilon[\mathscr{T}_1(\epsilon,t,u^{\mu\nu},u,\delta\zeta)+\mathscr{T}_3(\epsilon,t,\delta\mathring{\zeta})+\epsilon\mathscr{T}_4(\epsilon,t,\delta\zeta)]\\&+\frac{\Lambda}{3}\partial_k(\Delta-\epsilon^2\beta)^{-1}\epsilon^2\mathscr{T}_2(\epsilon,t,u^{\mu\nu},u,\delta\zeta,z_j)+\frac{\Lambda\mathring{E}^3}{3}\epsilon^2\beta(\Delta-\epsilon^2\beta)^{-1}\mathfrak{R}_k(-\Delta)^{-\frac12}(e^{\mathring{\zeta}}-e^{\mathring{\zeta}_H}).\end{aligned}\tag{3.7.32}$$

By similar arguments, we also see that

$$\begin{aligned}\Upsilon-\mathring{\Upsilon}=&\epsilon\beta\frac{\Lambda}{3t^3}E^3(\Delta-\epsilon^2\beta)^{-1}\big(\delta\rho-\delta\mathring{\rho}\big)+\epsilon^2\beta\frac{\Lambda}{3}(\Delta-\epsilon^2\beta)^{-1}\big[\mathscr{T}_1(\epsilon,t,u^{\mu\nu},u,\delta\zeta)+\mathscr{T}_3(\epsilon,t,\delta\mathring{\zeta})\\&+\epsilon\mathscr{T}_4(\epsilon,t,\delta\zeta)\big]+\epsilon^3\beta\frac{\Lambda}{3}(\Delta-\epsilon^2\beta)^{-1}\mathscr{T}_2(\epsilon,t,u^{\mu\nu},u,\delta\zeta,z_j),\end{aligned}\tag{3.7.33}$$

where $\mathscr{T}_3(\epsilon,t,0)=\mathscr{T}_4(\epsilon,t,0)=0$. Since $\delta\rho|_\Sigma=\delta\mathring{\rho}|_\Sigma=\delta\breve{\rho}$, we have, initially,

$$\Big[\frac{\Lambda E^3}{3t^3}\partial_k(\Delta-\epsilon^2\beta)^{-1}\big(\delta\rho-\delta\mathring{\rho}\big)\Big]\Big|_\Sigma=\Big[\frac{\Lambda E^3}{3t^3}\epsilon\beta(\Delta-\epsilon^2\beta)^{-1}\big(\delta\rho-\delta\mathring{\rho}\big)\Big]\Big|_\Sigma=0.$$

Substituting this into (3.7.32), we see that the estimate

$$\begin{aligned}\|(\Phi^0_k-\mathring{\Phi}_k)|_\Sigma\|_{R^{s-1}}\lesssim&\|\mathscr{T}_1(\epsilon,t,u^{\mu\nu},u,\delta\zeta)|_\Sigma\|_{R^{s-1}}+\epsilon\|\mathscr{T}_3(\epsilon,t,\delta\mathring{\zeta})|_\Sigma\|_{H^{s-2}}+\|\epsilon\mathscr{T}_2(\epsilon,t,u^{\mu\nu},u,\delta\zeta,z_j)|_\Sigma\|_{R^{s-1}}\\&+\epsilon\|\mathscr{T}_4(\epsilon,t,\delta\zeta)|_\Sigma\|_{R^{s-1}}+\epsilon\|\delta\mathring{\zeta}|_\Sigma\|_{L^{\frac65}\cap H^{s-2}}\\\lesssim&\|u^{\mu\nu}|_\Sigma\|_{R^{s-1}}+\|u|_\Sigma\|_{R^{s-1}}+\epsilon\|z_j|_\Sigma\|_{R^{s-1}}+\epsilon\|\delta\zeta|_\Sigma\|_{R^{s-1}}+\epsilon\|\delta\mathring{\zeta}|_\Sigma\|_{L^{\frac65}\cap H^{s-2}}\\\lesssim&\epsilon\|\breve{\xi}_\epsilon\|_s\end{aligned}\tag{3.7.34}$$

follows from (3.3.53) and an application of Theorems 3.3.16, B.0.1 and B.0.2, and Propositions 3.3.5 and 3.3.6. Using similar arguments, it is also not difficult to verify the inequality

$$\|(\Upsilon-\mathring{\Upsilon})|_\Sigma\|_{R^{s-1}}\lesssim\epsilon\|\breve{\xi}_\epsilon\|_s.\tag{3.7.35}$$

Integrating (3.7.28) in time and applying the $\|\cdot\|_{R^{s-1}}$ to the result, we see, after recalling the definitions (3.4.19), (3.4.20), (3.5.1) and (3.7.26), and using Proposition 3.3.5, (3.3.52) and (3.3.57) to estimate terms involving the Yukawa potential operators, that inequality

$$\begin{aligned}\|\Phi^0_k - \mathring{\Phi}_k\|_{R^{s-1}} \leq & \|(\Phi^0_k - \mathring{\Phi}_k)|_\Sigma\|_{R^{s-1}} + \int_t^1 \Big(\|e^{\zeta}(z^l - \mathring{z}^l)(\tau)\|_{R^{s-1}} + \|e^{\mathring{\mu}}\mathring{z}^l e^{\delta\mathring{\zeta}}\big(e^{\delta\zeta-\delta\mathring{\zeta}} - 1\big)(\tau)\|_{R^{s-1}} \\ & + \|\epsilon\mathscr{S}^l(\epsilon,\tau,u^{\mu\nu},u,\delta\zeta,z_j)\|_{R^{s-1}} + \|\epsilon^2\beta(\Delta-\epsilon^2\beta)^{-1}\big(e^{\mathring{\zeta}}\mathring{z}^l\big)(\tau)\|_{R^{s-1}}\Big) d\tau \\ \leq & \epsilon C\|\breve{\xi}_\epsilon\|_s + \epsilon C(K_4)\int_t^1 (\|\mathbf{Z}(\tau)\|_{R^{s-1}} + \|\mathbf{V}(\tau)\|_{R^{s-1}} + \|\mathbf{U}(\tau)\|_{R^{s-1}} + \|\mathring{\mathbf{U}}(\tau)\|_{H^{s-1}}) d\tau \\ \leq & \epsilon C(K_4)\Big(\|\breve{\xi}_\epsilon\|_s + \int_t^1 (\|\mathbf{Z}(\tau)\|_{R^{s-1}} + \|\mathbf{U}(\tau)\|_{R^s} + \|\mathring{\mathbf{U}}(\tau)\|_{H^s}) d\tau\Big)\end{aligned}$$

is a direct consequence of the expansion (3.7.31) and the estimates (3.7.21) and (3.7.34). Similar arguments can also be used to verify the estimate

$$\|\Upsilon - \mathring{\Upsilon}\|_{R^{s-1}} \leq \epsilon C(K_4)\Big(\|\breve{\xi}_\epsilon\|_s + \int_t^1 (\|\mathbf{Z}(\tau)\|_{R^{s-1}} + \|\mathbf{U}(\tau)\|_{R^s} + \|\mathring{\mathbf{U}}(\tau)\|_{H^s}) d\tau\Big).$$

Combining the above two estimates together yields the first estimate of the lemma and thus completes the proof. □

From the estimate (3.7.27) and Lemma 3.7.1, it is clear that

$$\begin{aligned}\|\mathbf{F}(\epsilon,t,\cdot) - \mathring{\mathbf{F}}(t,\cdot)\|_{R^{s-1}} \leq & \epsilon C(K_4)\Big(\|\breve{\xi}_\epsilon\|_s + \|\mathbf{U}(t)\|_{R^s} + \|\mathbf{Z}(t)\|_{R^{s-1}} + \|\mathring{\mathbf{U}}(t)\|_{H^s} \\ & + \int_t^1 (\|\mathbf{Z}(\tau)\|_{R^{s-1}} + \|\mathbf{U}(\tau)\|_{R^{s-1}} + \|\mathring{\mathbf{U}}(\tau)\|_{H^{s-1}}) d\tau\Big), \quad T_3 < t \leq 1. \end{aligned} \tag{3.7.36}$$

Together, (3.7.24), (3.7.25) and (3.7.36) show that the Lipschitz estimates (3.6.66)-(3.6.67) are satisfied.

The final condition that we need to verify in order to use Theorem 3.6.10 is the bound (3.6.68) on the initial data. To see that this holds, we observe that the estimate $\|(\mathbf{U} - \mathring{\mathbf{U}})|_\Sigma\|_{R^s} \lesssim \epsilon\|\breve{\xi}_\epsilon\|_s$ follows directly from (3.7.15), Theorem 3.3.16, Proposition 3.5.2.(2), and the expansion $\delta\zeta|_\Sigma = \delta\mathring{\zeta}|_\Sigma + \epsilon^2\mathscr{S}(\epsilon, \delta\breve{\rho})$, which follows by direct calculation.

Having verified that all of the hypotheses of Theorem 3.6.10 are satisfied, we conclude that there exists a constant $\sigma > 0$, independent of $\epsilon \in (0, \epsilon_0)$, such that if the free initial data is chosen so that $\|\breve{\xi}_\epsilon\|_s < \sigma$, then the estimates

$$\|\mathbf{U}\|_{L^\infty((T_3,1],R^s)} \leq C\sigma, \quad \|\mathring{\mathbf{U}}\|_{L^\infty((T_3,1],H^s)} \leq C\sigma \quad \text{and} \quad \|\mathbf{U} - \mathring{\mathbf{U}}\|_{L^\infty((T_3,1],R^{s-1})} \leq \epsilon C\sigma \tag{3.7.37}$$

hold for some constant $C > 0$, independent of $T_3 \in (0,1)$ and $\epsilon \in (0,\epsilon_0)$. Furthermore, from the continuation criterion, it is clear that the bounds (3.7.37) imply that the solutions $\mathbf{U}$ and $\mathring{\mathbf{U}}$ exist globally on $M = (0,1] \times \mathbb{R}^3$ and satisfy the estimates (3.7.37) with $T_3 = 0$ uniformly for $\epsilon \in (0, \epsilon_0)$. In particular, this implies, via the definition of $\mathbf{U}$ and $\mathring{\mathbf{U}}$, see (3.5.22) and (3.7.4), that

$$\begin{gathered}\|\delta\zeta(t) - \delta\mathring{\zeta}(t)\|_{R^{s-1}} \leq \epsilon C\sigma, \quad \|z_j(t) - \mathring{z}_j(t)\|_{R^{s-1}} \leq \epsilon C\sigma, \\ \|u^{\mu\nu}_0(t)\|_{R^{s-1}} \leq \epsilon C\sigma, \quad \|u^{\mu\nu}_k(t) - \delta^\mu_0\delta^\nu_0\partial_k\mathring{\Phi}(t)\|_{R^{s-1}} \leq \epsilon C\sigma, \quad \|u^{\mu\nu}(t)\|_{R^{s-1}} \leq \epsilon C\sigma, \\ \|u_0(t)\|_{R^{s-1}} \leq \epsilon C\sigma, \quad \|u_k(t)\|_{R^{s-1}} \leq \epsilon C\sigma \quad \text{and} \quad \|u(t)\|_{R^{s-1}} \leq \epsilon C\sigma\end{gathered}$$

for $0 < t \leq 1$, while, from (3.2.21), we see that

$$\left\|\underline{v}^0(t) - \sqrt{\frac{\Lambda}{3}}\right\|_{R^{s-1}} \leq C\epsilon\sigma, \quad 0 < t \leq 1.$$

This concludes the proof of Theorem 3.1.6.

Chapter 4
Discussion

Chapter 4

Discussion

> It doesn't matter how beautiful your theory is, it doesn't matter how smart you are. If it doesn't agree with experiment, it's wrong.
>
> Richard Feynman

This thesis contributed to the rigorous proof of the existence of cosmological Newtonian limits and accurately gives the largest space-time region of existence of cosmological Newtonian limits under certain initial data. We establish the existence of 1-parameter families of ϵ-dependent solutions on a specific manifold M to the Einstein-Euler equations with a positive cosmological constant $\Lambda > 0$ and a linear equation of state $p = \epsilon^2 K\rho$, $0 < K \leq 1/3$, for the parameter values $0 < \epsilon < \epsilon_0$. The main purpose of this thesis is to conclude that M can be extended to $[0, \infty) \times \mathbb{R}^3$ in terms of the standard coordinates for the FLRW metric. These solutions exist globally to the future[1], converge as $\epsilon \searrow 0$ to solutions of the cosmological Poison-Euler equations of Newtonian gravity on cosmological scales, and are inhomogeneous non-linear perturbations of FLRW fluid solutions.

Now, let us firmly answer the question proposed in §1.2, under a small initial data and a positive cosmological constant condition, that is, Newtonian cosmological simulations can be trusted to approximate relativistic cosmologies globally to the future on cosmological scales.

The key requirements of this thesis are the positive cosmological constant and the smallness conditions of initial data. The main proofs for the long time issue are based on one type of the conformal transform from the physical Einstein-Euler system to a conformal one which brings the Einstein-Euler system to a singular symmetric hyperbolic equation (see (2.5.1) and (3.6.1)) in the Newtonian coordinates. A key structure of the singular system which enables the analysis to be carried out is that the time singular term has a correct sign due to the positive cosmological constant. The bulk of the work is on the analysis of such singular equations based on various assumptions and function spaces. The differences between these two model systems in Chapter 2 and Chapter 3 come from the distinct initial data setting. The periodic data prescribed in Chapter 2 is a cosmological version of the isolated system instead of the cosmological relevant data because of the period $\sim \epsilon$. We use this simplification of isolated cosmological system in Chapter 2 because we only want to focus on the long time issue by ignoring the real cosmological scales to develop a technique for the long time Newtonian limits

[1]This means these ϵ-dependent solutions are future geodesically complete for every $\epsilon > 0$. This can be seen from the asymptotic properties of the metric $g^{\mu\nu}$ and the future geodesic completeness of the FLRW metric (this is easy via analyzing the line element of this metric, when time approaches the future infinity, the affine parameter of the geodesics goes to infinity). These asymptotic properties have been shown in Theorem 1.3 in [66] or, equivalently, in terms of the physical metric and the original standard FLRW coordinates, the asymptotic results can be found in Theorem 12.1 of [77]. These asymptotic properties imply that the line element of the perturbed metric is dominated by the one of the background FLRW metric. Therefore, by using these asymptotic properties, it is not hard to conclude when time tends to future infinity, the line element goes to infinity too, which, in turn, implies the future geodesic completeness of the ϵ-dependent solutions for every $\epsilon > 0$. We omit the detailed calculations and derivations of this. In the current thesis, we do not intend to understand how the geodesics varies with respect to ϵ. It might be an interesting question to pursue in future and we do not think it will be too difficult.

problem as this gives us enough simplification in Chapter 2 compared with Chapter 3. In Chapter 3, with the techniques of long time issues in hand, we generalized the results in Chapter 2 by taking the cosmological scales into account to derive the complete results on the Newtonian limits on large cosmological spacetime scales.

Another important part of this thesis is the initialization for the cosmological scales. We formulated the constraints to the elliptic system by a variation of the method developed by Lottermoser. Then, in order to analyze this system on $\mathbb{R}^3$, we have to choose the suitable function space with the purpose of employing the Banach's fixed point theory. It turns out that a decent tool to establish our contraction mapping is Yukawa potential operators introduced in §3.3.3 which can be viewed as a generalization of Riesz potential operators and Bessel potential operators. Yukawa potential operators are better than Riesz potential operators since Yukawa potential has better mapping properties compared with the Riesz one. With proper rescalings or composed with spatial derivatives, Yukawa potential operators are mappings from L^p to L^p spaces, the construction of the contraction, thus, becomes easier to achieve.

In this chapter, we review this thesis and remark on possible future directions.

4.1 Summary and conclusion

Chapter 2 provides the first long time result for the rigorous cosmological Newtonian limit problem on the cosmological version of the isolated system. It establishes the long time existence of 1-parameter families of ϵ-dependent solutions to Einstein-Euler systems which are small, non-linear perturbations of FLRW solutions that converge, in certain sense, as $\epsilon \searrow 0$ to solutions of the cosmological Poison-Euler equations of Newtonian gravity. The basic idea is using the conformal singular equation to transform the long time existence problem to a short one, but the cost is that the equations become singular. Due to the "right sign" on the time singular term which is a direct consequence of the positive cosmological constant, such singular equations behave well in the analysis, that is, the existing solution could be extended as $t \to 0$. The constraint equations can be solved by applying the standard method developed by Lottermoser.

Because the result of the long time Newtonian limits in Chapter 2 is built on a cosmological version of the isolated system instead of the authentic cosmological scales, the main purpose of Chapter 3 is to remedy this defect. To do this, we must select cosmologically relevant initial data to ensure the light travel time between the different density spikes remains bounded away from zero in the limit $\epsilon \searrow 0$. Given the free data which verify the requirements of the cosmologically relevant data, the constraint equations are not simple to solve. As we point out, Yukawa potential operators are effective tools to conquer such difficulties. Due to the variations on the type of initial data, our analysis on the singular hyperbolic equations requires corresponding modifications to fit these data. We provide, in this chapter, the answer to the basic question in the cosmological simulation that we have proposed in §1.2, under a small initial data and the positive cosmological constant condition, that is, Newtonian cosmological simulations can be trusted to approximate relativistic cosmologies globally to the future on cosmological scales. We adopt the long time scheme of Chapter 2 in more suitable function spaces to analyze the evolution part of this system and introduce a new tool, the Yukawa potential operator, to analyze the constraint equations. We believe this result lays a firm foundation for large scale cosmological simulations using Newtonian gravity in astrophysics.

4.2 Future directions

There are many potential directions continuing the work of the current thesis. We mention some of these in this section.

4.2.1 Post-Newtonian expansion on large cosmological scales

The direct extension of this thesis is to continue to investigate the next order of the post-Newtonian expansion on large scales. This problem is proposed because there are relativistic effects that are important for precision cosmology and are not captured by the Newtonian solutions. To understand these relativistic effects, higher order post-Newtonian (PN) expansions are required starting with the 1/2-PN expansion, which is, by definition, the ϵ order correction to the Newtonian gravity. In particular, it can be shown [67] that the 1-parameter families of solutions must admit a 1/2-PN expansion in order to view them on large scales as a linear perturbation of FLRW solutions. The importance of this result is that it shows it is possible to have rigorous solutions that fit within the standard cosmological paradigm of linear perturbations of FLRW metrics on large scales while, at the same time, are fully non-linear on small scales of order ϵ. Thus the natural next step is to extend the current results to include the existence of 1-parameter families of ϵ-dependent solutions to the Einstein-Euler equations that admit 1/2-PN expansions globally to the future on cosmological scales. This problem will directly continue this thesis with reasonable difficulties and this work is currently in progress.

4.2.2 The relation between Fuchsian analysis and Oliynyk's singular system

As we mentioned before, we use conformal transform and some specific wave gauge to rewrite the Einstein-Euler system as a singular hyperbolic equation and consider the limiting case that time $t \to 0$ by giving initial data at $t = 1$, which implies the long time behavior by reversing the conformal transform. We also know there is the Fuchsian analysis which also involves singular hyperbolic equations but given asymptotic data initially at the singular time. We want to understand the relations of solutions between these two systems. We do not think this is a very difficult problem but it is worth checking this relationship in future.

4.2.3 Long time behavior of FLRW universe with large data or without cosmological constant

We have repeatedly emphasized that the results of this thesis are obtained under small initial data and positive cosmological constant conditions. Therefore, a natural question arises, which is, if imposing certain large initial data or if there is no positive cosmological constant, what happens to the cosmological Newtonian limits? In fact, before considering cosmological Newtonian limits, we have to investigate the future behavior of the FLRW universe under certain large initial data or $\Lambda = 0$. As we remarked previously, the positive cosmological constant is a crucial structure leading to the long time picture. If this cosmological constant is zero, the whole method of the conformal singular hyperbolic system is destroyed, one has to explore some new way to carry it out. In addition, small initial data ensure the nonlinearities do not deviate too much from the linear systems. If initial data are large, one must seek another way to control the nonlinearities. As all the current methods have failed, this would be a very complicated new question to answer. We believe some new ideas and more delicate conditions must be presented in order to answer it and the results might vary under different assumptions.

4.2.4 About cosmological relevant data selection

In Chapter 3, from our proof in §3.3.5, there is an unpleasant condition (3.3.64) which is

$$0 < \mu(1) < \frac{1}{8}(19 + 5\sqrt{29})\Lambda,$$

restricting the density of the background homogeneous fluid from being too large. We do not know if this is necessary or not, but it is worth investigating. As we noted in Chapter 3, [65] established the existence of 1-parameter families of ϵ-dependent initial data satisfying the constraint equations without a similar smallness condition on the background FLRW solution. However, the gauge condition used in this article, which is suited to the long time evolution problem, is different from the gauge used in [65], and the analysis of the constraint equations in [65] employed a more complicated conformal decomposition. Consequently, it is not clear if the choice of gauge or the particular representation of the constraint equations used in this article is responsible for the requirement (3.3.64). On the other hand, [65] only concludes the local-in-time Newtonian limits on cosmological scales, there is no smallness condition required for the initial perturbations. Therefore, large perturbations are allowed, which means that the density of the background homogeneous fluid does not play an important role because large perturbations give the initial data enough freedom. In contrast, in order to control the nonlinearity for the long time problem, we have to keep the initial smallness condition of the perturbation data in this thesis. This endows the density of the background homogeneous fluid with the role of the center of the perturbations, and only in the long time problem, the importance of the density of the background homogeneous fluid makes a real impact. Thus, it is interesting to investigate the same initialization problem as in Chapter 3 without this condition (3.3.64). We do not know how difficult this would be, and currently, we do not know how to get rid of this condition.

4.2.5 The future behavior of the FLRW solutions and the equations of state

In this thesis, we adopt a linear type of equation of state for the perfect fluid

$$\bar{p} = \epsilon^2 K \bar{\rho}, \qquad 0 < K \leq \frac{1}{3}.$$

In fact, the universe is not so simple due to the complex thermodynamic properties and other complicated factors. One interesting question is to ask how the equation of state affects the future behavior of FLRW fluid solutions in general and how the effects of an equation of state compete with those of the cosmological constant. Some specific questions can be proposed, for example, for a general equation of state

$$p = f(\rho)$$

where $f : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ is a function that $f'(\rho) > 0$ for all $\rho > 0$, can one find as simple as possible conditions on f to ensure the future stability result of FLRW solutions with or without the positive cosmological constant? After understanding the relationship between the future behavior of the FLRW solutions and the equations of state, the corresponding existence problem of Newtonian limits and post-Newtonian expansions could be investigated further. This is a direct and reasonable question for our next step to continue this thesis because by delicately designing the property of f, it seems it is not too difficult to use the equation of state to mimic the behavior of effects of the positive cosmological constant.

In addition, another interesting and reasonable problem is to consider the linear equation of state $p = K\rho$ with $1/3 < K < 1$. To the best of our knowledge, there is no any result in the literature and in this case, instabilities might occur[2].

4.2.6 Applications of the technique of rigorous Newtonian limits and post-Newtonian expansions

What can we do by using the technique of rigorous Newtonian limits and post-Newtonian expansions? There are many potential powerful applications. Physicists have successfully used Newtonian limits and post-Newtonian approximations to solve many difficult questions *approximately* or *numerically*. Once we have proven the existence of rigorous Newtonian limits or post-Newtonian expansion, the next natural and possible issue is to solve and prove those complicated problems in physics analytically and accurately instead of using numerical methods.

One option is to work on the rigorous analysis of the two-body problem in general relativity analytically rather than numerically. However, more foundational work needs to be established before we can understand this problem.

The first step in approaching this problem is to understand the long time behavior of the isolated compact elastic body in the general relativity with the positive cosmological constant $\Lambda > 0$. Because we know $\Lambda > 0$ may lead to long time existence in view of our past and current work where $\Lambda > 0$ gives the equations some nice structure, and due to the work [2, 3, 4, 5, 7] in which some aspects of the

[2]This question is suggested by Professor Uwe Brauer and the author thanks him for this interesting problem.

elastic body and the short time behavior of the elastic body has been researched in general relativity and Newtonian gravity, it is reasonable to push this question into long time behavior and this long time behavior of the one body problem is the first thing we should understand before the two-body problem. We choose the elastic body first because this is simpler than the liquid body, which has more complicated boundary conditions. Our final goal is to understand this question for the liquid body, but in order to avoid the complicated boundary conditions, the elastic body is a good model to explore fundamental ideas.

The difficulties of this project come from several aspects. The outstanding difficulty is because we do not assume any symmetry on the elastic body and there is no suitable exact solution to the Einstein equation as the background (If we know there is some long time exact solution, then we could perturb it and see if this solution is stable; if it is stable, then there is a family of long time solutions. However, this is not true in such case). Therefore, we can only observe the same question in the Newtonian case and try to understand this Newtonian case first. We would then try to apply our Newtonian limit technique to connect the Newtonian case and relativistic one by imposing certain suitable assumptions.

If we could achieve this for the elastic body, we may then turn to the long time behavior of the relativistic liquid body. Another choice is to investigate the two-body problem for elastic bodies.

The two-body problem in general relativity is very difficult to reach currently. But a lot of very fundamental work with this goal is doable now. But even these fundamental work seems difficult too because the corresponding problems in Newtonian gravity or without gravity are also not so easy. However, a lot of progresses on Euler equations and Euler-Poisson equations have been achieved in the recent thirty years. Therefore, it is reasonable to ask these problems in general relativity now.

There are many interesting questions around rigorous Newtonian limits and post-Newtonian expansions, and we expect that these techniques will contribute new ways to rigorously explore general relativity and cosmology.

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Appendix A

The dimensionless version of the Einstein-Euler system

The original Einstein-Euler system with positive cosmological constant $\Lambda > 0$ is

$$\tilde{G}^{\mu\nu} + \frac{1}{c^2}\Lambda\tilde{g}^{\mu\nu} = \frac{8\pi G}{c^4}\tilde{T}^{\mu\nu} \tag{A.0.1}$$

$$\tilde{\nabla}_\mu \tilde{T}^{\mu\nu} = 0 \tag{A.0.2}$$

where

$$\tilde{T}^{\mu\nu} = \left(\rho + \frac{1}{c^2}p\right)\tilde{v}^\mu\tilde{v}^\nu + p\tilde{g}^{\mu\nu} \tag{A.0.3}$$

is the energy-momentum tensor of perfect fluid which determined by the following linear type equation of state:

$$p = K\rho \quad (K \geq 0) \tag{A.0.4}$$

and we have the normalization on velocity:

$$\tilde{v}^\mu \tilde{v}_\mu = -c^2 \tag{A.0.5}$$

We give conventions on the following variables:

$$[x^\mu] = L, \quad [\tilde{g}^{\mu\nu}] = 1, \quad [\rho] = \frac{M}{L^3}, \quad [p] = \frac{M}{LT^2}, \quad [\tilde{v}^\mu] = [c] = \frac{L}{T} \tag{A.0.6}$$

Under above conventions, we can calculate

$$[\tilde{T}^{\mu\nu}] = \frac{M}{LT^2}, \quad [\tilde{G}^{\mu\nu}] = [\tilde{R}^{\mu\nu}] = [\tilde{R}] = \frac{1}{L^2}, \quad [\Lambda] = \frac{1}{T^2} \tag{A.0.7}$$

Then we have

$$[G] = \frac{L^3}{MT^2} \tag{A.0.8}$$

Introduce dimensionless variables:

$$\tilde{v}^\mu = \frac{1}{\epsilon}v_T\hat{v}^\mu, \quad \rho = \rho_T\hat{\rho}, \quad \hat{p} = \epsilon^2\frac{p}{v_T^2\rho_T}, \quad \hat{g}^{\mu\nu} = \tilde{g}^{\mu\nu}, \quad \hat{K} = \frac{K}{v_T^2}, \quad \epsilon = \frac{v_T}{c} \tag{A.0.9}$$

$$\kappa = \frac{8\pi G \rho_T}{v_T^2}, \quad \hat{x}^\mu = \epsilon\sqrt{\kappa} x^\mu, \quad \hat{\Lambda} = \frac{1}{v_T^2 \kappa}\Lambda \tag{A.0.10}$$

where c is the speed of light, $\hat{\rho}_T$ is typical density and v_T is the typical velocity, and $[\kappa] = \frac{1}{L^2}$. Then we can write the Einstein-Euler system as

$$\hat{G}^{\mu\nu} + \hat{\Lambda}\hat{g}^{\mu\nu} = \hat{T}^{\mu\nu} \tag{A.0.11}$$

$$\hat{\nabla}_\mu \hat{T}^{\mu\nu} = 0 \tag{A.0.12}$$

$$\hat{T}^{\mu\nu} = (\hat{\rho} + \hat{p})\hat{v}^\mu \hat{v}^\nu + \hat{p}\hat{g}^{\mu\nu} \tag{A.0.13}$$

$$\hat{v}^\mu \hat{v}_\mu = -1 \tag{A.0.14}$$

$$\hat{p} = \epsilon^2 \hat{K}\hat{\rho} \tag{A.0.15}$$

$$\tag{A.0.16}$$

Therefore, all the dynamical variables and coordinates are dimensionless, and if we choose

$$v_T = 1, \quad \kappa = 1 \quad \text{and} \quad \rho_T = 1, \tag{A.0.17}$$

then, all the units are fixed.

Appendix B

Potential operators

In this section, we state the basic mapping properties of the Riesz and Bessel potentials that will be used throughout this article. We omit the proofs, which can be found in [32].

B.0.1 Riesz potentials

In the following, we use

$$\Delta = \delta^{ij}\partial_i\partial_j \qquad (i, j = 1, \cdots, n)$$

to denote the flat Laplacian on $\mathbb{R}^n$. For $0 < s < \infty$, the *Riesz potential operator of order* s, denoted $(-\Delta)^{-\frac{s}{2}}$, is defined by

$$(-\Delta)^{-\frac{s}{2}}(f) = (\widehat{\mathcal{K}}_s\widehat{f})^\vee = \mathcal{K}_s * f,$$

where

$$\mathcal{K}_s(x) = \big((4\pi^2|\xi|^2)^{-\frac{s}{2}}\big)^\vee(x),$$

or equivalently, by

$$(-\Delta)^{-\frac{s}{2}}(f)(x) = 2^{-s}\pi^{-\frac{n}{2}}\frac{\Gamma(\frac{n-s}{2})}{\Gamma(\frac{s}{2})}\int_{\mathbb{R}^n} f(x-y)|y|^{-n+s}d^ny.$$

The Riesz potential operator satisfies the estimates:

Theorem B.0.1. *[32, Theorem 1.2.3] Suppose* $0 < s < n$ *and* $1 < p < q < \infty$ *satisfy* $\frac{1}{p} - \frac{1}{q} = \frac{s}{n}$. *Then*

$$\|(-\Delta)^{-\frac{s}{2}}(f)\|_{L^q} \lesssim \|f\|_{L^p}$$

for all $f \in L^p(\mathbb{R}^n)$.

From the point of view of applications considered in this article, the following specific cases of the above estimate on $\mathbb{R}^3$ will be of most interest:

$$\|(-\Delta)^{-1}f\|_{L^6} \lesssim \|f\|_{L^{\frac{6}{5}}} \quad \text{and} \quad \|(-\Delta)^{-\frac{1}{2}}f\|_{L^6} \lesssim \|f\|_{L^2}.$$

Next, we recall that the *Riesz transform* $\mathfrak{R}_j$ is defined by

$$\mathfrak{R}_j = -\partial_j(-\Delta)^{-\frac{1}{2}},$$

and satisfies the estimate:

Theorem B.0.2. *[31, Corollary 5.2.8] Suppose* $1 < p < \infty$ *and* $s \in \mathbb{Z}_{\geq 0}$. *Then*

$$\|\mathfrak{R}_j(f)\|_{W^{s,p}} \lesssim \|f\|_{W^{s,p}}$$

for all $f \in W^{s,p}(\mathbb{R}^n)$.

Combining Theorems B.0.1 and B.0.2 gives:

Proposition B.0.3.

1. *If* $1 < p < q < \infty$ *satisfy* $\frac{1}{p} - \frac{1}{q} = \frac{1}{n}$*, then*

$$\|\partial_j \Delta^{-1} f\|_{L^q} = \|\mathfrak{R}_j (-\Delta)^{-\frac{1}{2}} f\|_{L^q} \lesssim \|(-\Delta)^{-\frac{1}{2}} f\|_{L^q} \lesssim \|f\|_{L^p}$$

for all $f \in L^p(\mathbb{R}^n)$.

2. *If* $1 < p < \infty$ *and* $s \in \mathbb{Z}_{\geq 0}$*, then*

$$\|\partial_j \partial_k \Delta^{-1} f\|_{W^{s,p}} = \|\mathfrak{R}_j \mathfrak{R}_k f\|_{W^{s,p}} \lesssim \|f\|_{W^{s,p}}$$

for all $f \in W^{s,p}(\mathbb{R}^n)$.

For applications, the following case on $\mathbb{R}^3$ will be of particular importance:

$$\|\partial_j \Delta^{-1} f\|_{L^6} \lesssim \|f\|_{L^2}.$$

B.0.2 Bessel potentials

For $0 < s < \infty$, the *Bessel potential operator of order* s, denoted $(\mathbb{1} - \Delta)^{-\frac{s}{2}}$, is defined by:

$$(\mathbb{1} - \Delta)^{-\frac{s}{2}}(f) = (\widehat{\mathcal{G}_s} \widehat{f})^{\vee} = \mathcal{G}_s * f,$$

where

$$\mathcal{G}_s(x) = \big((1 + 4\pi^2 |\xi|^2)^{-\frac{s}{2}}\big)^{\vee}(x).$$

It satisfies the following estimates:

Theorem B.0.4. *[32, Corollary 1.2.6]*

1. *Suppose* $0 < s < \infty$ *and* $1 \leq r \leq \infty$*. Then*

$$\|(\mathbb{1} - \Delta)^{-\frac{s}{2}}(f)\|_{L^r} \leq \|f\|_{L^r}$$

for all $f \in L^r(\mathbb{R}^n)$*, and*

$$\|(\mathbb{1} - \Delta)^{-\frac{s}{2}}\|_{\mathrm{op}} = \|\mathcal{G}_s\|_{L^1} = 1.$$

2. *Suppose* $0 < s < n$ *and* $1 < p < q < \infty$ *satisfy* $\frac{1}{p} - \frac{1}{q} = \frac{s}{n}$*. Then*

$$\|(\mathbb{1} - \Delta)^{-\frac{s}{2}}(f)\|_{L^q} \lesssim \|f\|_{L^p}$$

for all $f \in L^p(\mathbb{R}^n)$.

For applications, the following cases on $\mathbb{R}^3$ will be of particular interest:

$$\|(\mathbb{1} - \Delta)^{-1}(f)\|_{L^6} \leq \|f\|_{L^6} \quad \text{and} \quad \|(\mathbb{1} - \Delta)^{-1}(f)\|_{L^6} \lesssim \|f\|_{L^{\frac{6}{5}}}.$$

Appendix C

Calculus Inequalities

In this appendix, we list important calculus inequalities that will be used throughout this thesis. Most of the proofs can be found in the references [1, 42, 83]. Proofs will be given for statements that cannot be found it those references.

C.1 Calculus inequalities for Chapter 2

Theorem C.1.1. [Sobolev's inequality] *If $s \in \mathbb{Z}_{>n/2}$, then*

$$\|f\|_{L^\infty} \lesssim \|f\|_{H^s}$$

for all $f \in H^s(\mathbb{T}^n)$.

Lemma C.1.2. *Suppose $s \in \mathbb{Z}_{\geq 1}$, $l \in \mathbb{Z}_{\geq 2}$, $f_i \in L^\infty(\mathbb{T}^n)$ for $1 \leq i \leq l$, $f_l \in H^s(\mathbb{T}^n)$, and $Df_i \in H^{s-1}(\mathbb{T}^n)$ for $1 \leq i \leq l-1$. Then there exists a constant $C > 0$, depending on s and l, such that*

$$\|f_1 \ldots f_l\|_{H^s} \leq C\biggl(\|f_l\|_{H^s} \prod_{i=1}^{l-1} \|f_i\|_{L^\infty} + \sum_{i=1}^{l-1} \|Df_i\|_{H^{s-1}} \prod_{i \neq j} \|f_j\|_{L^\infty}\biggr).$$

Lemma C.1.3. *Suppose $s \in \mathbb{Z}_{\geq 1}$, $f \in L^\infty(\mathbb{T}^n, V) \cap H^s(\mathbb{T}^n, V) \cap C^0(\mathbb{T}^n, V)$, $W, U \subset V$ are open with U bounded and $\overline{U} \subset W$, $f(x) \in U$ for all $x \in \mathbb{T}^n$ and $F \in C^s(W)$. Then there exists a constant $C > 0$, depending on s, such that*

$$\|D^\alpha(F \circ f)\|_{L^2} \leq C\|DF\|_{W^{s-1,\infty}(U)}\|f\|^{s-1}_{L^\infty} \left(\sum_{|\beta|=s} \|D^\beta f\|_{L^2}\right)^{\frac{1}{2}}$$

for any multi-index α satisfying $|\alpha| = s$.

Lemma C.1.4. *If $s \in \mathbb{Z}_{\geq 1}$ and $|\alpha| \leq s$, then*

$$\|D^\alpha(fg) - fD^\alpha g\|_{L^2} \lesssim \|Df\|_{H^{s-1}}\|g\|_{L^\infty} + \|Df\|_{L^\infty}\|g\|_{H^{s-1}}$$

for all f, g satisfying $Df, g \in L^\infty(\mathbb{T}^n) \cap H^{s-1}(\mathbb{T}^n)$.

C.2 Calculus inequalities for Chapter 3

C.2.1 Sobolev-Gagliardo-Nirenberg inequalities

Theorem C.2.1. [Hölder's inequality]

1. *If* $0 < p, q, r \leq \infty$ *satisfy* $1/p + 1/q = 1/r$, *then*

$$\|uv\|_{L^r} \leq \|u\|_{L^p}\|v\|_{L^q}$$

for all $u \in L^p(\mathbb{R}^n)$ *and* $v \in L^q(\mathbb{R}^n)$.

2. *If* $0 < p, q, r \leq \infty$, $0 \leq \theta \leq 1$ *and* $\frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q}$, *then*

$$\|u\|_{L^r} \leq \|u\|_{L^p}^{\theta}\|u\|_{L^q}^{1-\theta} \lesssim \|u\|_{L^p} + \|u\|_{L^q}$$

for all $u \in L^p \bigcap L^q(\mathbb{R}^n)$.

Theorem C.2.2. [Gagliardo-Nirenberg-Sobolev inequalities]

1. *If* $1 \leq p < \infty$, *then*

$$\|u\|_{L^{p*}} \leq C_{Sob}\|Du\|_{L^p}, \qquad p^* = \frac{np}{n-p},$$

for all $u \in \{v \in L^{p*}(\mathbb{R}^n) \cap W^{1,p}_{\text{loc}}(\mathbb{R}^n) \mid \|Dv\|_{L^p} < \infty\}$.

2. *If* $s \in \mathbb{Z}_{\geq 1}$, $1 \leq p < \infty$ *and* $sp < n$, *then*

$$\|u\|_{L^q} \lesssim \|u\|_{W^{s,p}}, \qquad p \leq q \leq \frac{np}{n-sp}$$

for all $u \in W^{s,p}(\mathbb{R}^n)$.

3. *If* $s \in \mathbb{Z}_{\geq 1}$, $1 \leq p < \infty$ *and* $sp > n$, *then*

$$\|u\|_{L^\infty} \lesssim \|u\|_{W^{s,p}}$$

for all $u \in W^{s,p}(\mathbb{R}^n)$.

C.2.2 Product and commutator inequalities

Theorem C.2.3.

1. *Suppose* $1 \leq p_1, p_2, q_1, q_2 \leq \infty$, $|\alpha| = s \in \mathbb{Z}_{\geq 0}$, *and* $\frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2} = \frac{1}{r}$. *Then*

$$\|D^\alpha(uv)\|_{L^r} \lesssim \|D^s u\|_{L^{p_1}}\|v\|_{L^{q_1}} + \|u\|_{L^{p_2}}\|D^s v\|_{L^{q_2}}$$

and

$$\|[D^\alpha, u]v\|_{L^r} \lesssim \|Du\|_{L^{p_1}}\|D^{s-1}v\|_{L^{q_1}} + \|D^s u\|_{L^{p_2}}\|v\|_{L^{q_2}}$$

for all $u, v \in C_0^\infty(\mathbb{R}^n)$.

2. *If* $s_1, s_2 \geq s_3 \geq 0$, $1 \leq p \leq \infty$, *and* $s_1 + s_2 - s_3 > n/p$, *then*

$$\|uv\|_{W^{s_3,p}} \lesssim \|u\|_{W^{s_1,p}}\|v\|_{W^{s_2,p}}$$

for all $u \in W^{s_1,p}(\mathbb{R}^n)$ *and* $v \in W^{s_2,p}(\mathbb{R}^n)$.

Proposition C.2.4. *If* $s \in \mathbb{Z}_{\geq 1}$, *then*

$$\|D(uv)\|_{R^s} \lesssim \|Du\|_{R^s}\|v\|_{L^\infty} + \|u\|_{L^\infty}\|Dv\|_{R^s}$$

for all $u, v \in C_0^\infty(\mathbb{R}^3)$. *Furthermore, if* $s \in \mathbb{Z}_{\geq 2}$, *then*

$$\|uv\|_{R^s} \lesssim \|u\|_{R^s}\|v\|_{R^s}$$

for all $u, v \in C_0^\infty(\mathbb{R}^3)$*.*

Proof. By Theorem C.2.3.(1) and the definition of R^s norm, we have

$$\begin{aligned}\|D(uv)\|_{R^s} =&\|D(uv)\|_{L^6} + \|D^2(uv)\|_{H^{s-1}}\\ \lesssim&\big(\|Du\|_{L^6} + \|D^2u\|_{H^{s-1}}\big)\|v\|_{L^\infty} + \|u\|_{L^\infty}\big(\|Dv\|_{L^6} + \|D^2v\|_{H^{s-1}}\big)\\ \lesssim&\|Du\|_{R^s}\|v\|_{L^\infty} + \|u\|_{L^\infty}\|Dv\|_{R^s}\end{aligned}$$

for $s \geq 1$, and similarly, with the help of (3.1.13),

$$\begin{aligned}\|uv\|_{R^s} =&\|uv\|_{L^6} + \|D(uv)\|_{H^{s-1}}\\ \lesssim&\big(\|u\|_{L^6} + \|Du\|_{H^{s-1}}\big)\|v\|_{L^\infty} + \|u\|_{L^\infty}\big(\|v\|_{L^6} + \|Dv\|_{H^{s-1}}\big)\\ \lesssim&\|u\|_{R^s}\|v\|_{L^\infty} + \|u\|_{L^\infty}\|v\|_{R^s}\\ \lesssim&\|u\|_{R^s}\|v\|_{R^s}\end{aligned}$$

for $s \geq 2$. □

Proposition C.2.5. *If* $s \geq 3$, $0 \leq |\alpha| \leq s$ *and* $0 \leq |\beta| \leq s-1$*, then*

$$\|[D^\beta, u]v\|_{L^2} \lesssim\|Du\|_{R^{s-2}\cap L^2}\|v\|_{R^{s-1}} \lesssim \|u\|_{R^{s-1}}\|v\|_{R^{s-1}}, \tag{C.2.1}$$

$$\|[D^\alpha, u]v\|_{L^2} \lesssim\|Du\|_{R^{s-1}\cap L^2}\|v\|_{R^{s-1}} \lesssim \|u\|_{R^{s}}\|v\|_{R^{s-1}}, \tag{C.2.2}$$

$$\|[D^\beta, u]Dv\|_{L^2} \lesssim\|Du\|_{R^{s-1}\cap L^2}\|v\|_{R^{s-1}} \lesssim \|u\|_{R^{s}}\|v\|_{R^{s-1}} \tag{C.2.3}$$

and

$$\|[D^\beta, u]wDv\|_{L^2} \lesssim\|Du\|_{R^{s-1}\cap L^2}\|w\|_{R^s}\|v\|_{R^{s-1}} \lesssim \|u\|_{R^{s}}\|w\|_{R^s}\|v\|_{R^{s-1}} \tag{C.2.4}$$

for all $u, v, w \in C_0^\infty(\mathbb{R}^3)$*.*

Proof. We first consider the inequalities (C.2.1)-(C.2.2). Since they are trivial for $|\alpha| = |\beta| = 0$, we start by assuming that $|\alpha| = |\beta| = 1$. Then

$$\|[D, u]v\|_{L^2} = \|(Du)v\|_{L^2} \leq \|v\|_{L^\infty}\|Du\|_{L^2} \overset{(3.1.13)}{\lesssim} \|Du\|_{R^{s-2}\cap L^2}\|v\|_{R^{s-1}}.$$

Next, assuming $|\beta| = l \geq 2$, we see from Theorem C.2.1.(2) and Theorem C.2.3.(1) that

$$\begin{aligned}\|[D^\beta, u]v\|_{L^2} \lesssim&\|D^\ell u\|_{L^2}\|v\|_{L^\infty} + \|Du\|_{L^6}\|D^{l-1}v\|_{L^3}\\ \lesssim&\|D^\ell u\|_{L^2}\|v\|_{L^\infty} + \|Du\|_{L^6}(\|D^{l-1}v\|_{L^2} + \|D^{l-1}v\|_{L^6})\\ \lesssim&\|Du\|_{H^{l-1}}\|v\|_{L^\infty} + \|Du\|_{L^6}(\|v\|_{W^{l-1,6}} + \|Dv\|_{H^{l-2}})\\ \overset{(3.1.13)}{\lesssim}& (\|Du\|_{R^{s-2}} + \|Du\|_{L^2})\|v\|_{R^{s-1}} \lesssim \|u\|_{R^{s-1}}\|v\|_{R^{s-1}}\end{aligned}$$

while if $|\alpha| = s$, then

$$\begin{aligned}\|[D^\alpha, u]v\|_{L^2} \lesssim&\|Du\|_{L^\infty}\|Dv\|_{H^{s-2}} + \|v\|_{L^\infty}\|Du\|_{H^{s-1}}\\ \overset{(3.1.13)}{\lesssim}& (\|Du\|_{R^{s-1}} + \|Du\|_{L^2})\|v\|_{R^{s-1}} \lesssim \|u\|_{R^{s}}\|v\|_{R^{s-1}}.\end{aligned}$$

Together the above three inequalities verify the validity of (C.2.1)-(C.2.2).

Using the identity $[D^\beta, u]Dv = [D^\beta D, u]v - D^\beta\big((Du)v\big)$, it is then clear that

$$\|[D^\beta, u]Dv\|_{L^2} \leq\|[D^\beta D, u]v\|_{L^2} + \|D^\beta\big((Du)v\big)\|_{L^2} \lesssim \|Du\|_{R^{s-1}\cap L^2}\|v\|_{R^{s-1}} \lesssim \|u\|_{R^s}\|v\|_{R^{s-1}}.$$

follows from Theorem C.2.3(1) and the inequality (C.2.2). This establishes the inequality (C.2.3). For the final inequality, we find, using the identity $[D^\beta, u]wDv = [D^\beta, u]D(wv) - [D^\beta, u]\big((Dw)v\big)$, the inequality(C.2.1) and (C.2.3), Proposition C.2.4 and the obvious inequality $\|Dw\|_{R^{s-1}} \lesssim \|w\|_{R^s}$, that

$$\begin{aligned}\|[D^\beta, u]wDv\|_{L^2} \leq& \|[D^\beta, u]D(wv)\|_{L^2} + \|[D^\beta, u]\big((Dw)v\big)\|_{L^2}\\ \lesssim& \|Du\|_{R^{s-1}\cap L^2}\big(\|wv\|_{R^{s-1}} + \|(Dw)v\|_{R^{s-1}}\big)\\ \lesssim& \|Du\|_{R^{s-1}\cap L^2}\big(\|w\|_{R^{s-1}}\|v\|_{R^{s-1}} + \|Dw\|_{R^{s-1}}\|v\|_{R^{s-1}}\big)\\ \lesssim& \|Du\|_{R^{s-1}\cap L^2}\|w\|_{R^s}\|v\|_{R^{s-1}},\end{aligned}$$

which completes the proof. □

C.2.3 Moser estimates

Theorem C.2.6. *Suppose $s \in \mathbb{Z}_{\geq 1}$, $1 \leq p \leq \infty$, $|\alpha| = s$, $f \in C^s(\mathbb{R})$ with $f(0) = 0$, and U is open and bounded in $\mathbb{R}$. Then*

$$\|D^\alpha f(u)\|_{L^p} \leq C\big(\|Df\|_{C^{s-1}(\overline{U})}\big)\|u\|_{L^\infty}^{s-1}\|D^s u\|_{L^p}$$

and

$$\|f(u)\|_{L^p} \leq C\big(\|Df\|_{C^0(\overline{U})}\big)\|u\|_{L^p}$$

for all $u \in C_0^\infty(\mathbb{R}^n)$ with $u(\mathbf{x}) \in U$ for all $\mathbf{x} \in \mathbb{R}^n$.

Theorem C.2.7. *If $s \in \mathbb{Z}_{\geq 1}$ and $l \in \mathbb{Z}_{\geq 2}$, then*

$$\|f_1 \ldots f_l\|_{H^s} \lesssim \left(\|f_l\|_{H^s}\prod_{i=1}^{l-1}\|f_i\|_{L^\infty} + \sum_{i=1}^{l-1}\|Df_i\|_{H^{s-1}}\prod_{i\neq j}\|f_j\|_{L^\infty}\right)$$

for all $f_i \in C_0^\infty(\mathbb{R}^n)$, $1 \leq i \leq l$.

Proposition C.2.8. *Suppose $s \in \mathbb{Z}_{\geq 2}$, $f \in C^s(\mathbb{R})$, $f(0) = 0$ and U is open and bounded in $\mathbb{R}$. Then*

$$\|Df(u)\|_{Q^{s-1}} \leq C\big(\|Df\|_{C^{s-1}(\overline{U})}\big)(1 + \|u\|_{L^\infty}^{s-1})\|Du\|_{R^{s-1}}$$

and

$$\|f(u)\|_{R^s} \leq C\big(\|Df\|_{C^{s-1}(\overline{U})}\big)(1 + \|u\|_{L^\infty}^{s-1})\|u\|_{R^s}$$

for all $u \in C_0^\infty(\mathbb{R}^3)$ with $u(\mathbf{x}) \in U$ for all $\mathbf{x} \in \mathbb{R}^3$.

Proof. From a direct application of Theorem C.2.6 and the definition of R^s norm, we see that

$$\begin{aligned}\|Df(u)\|_{R^{s-1}} = \|Df(u)\|_{L^6} + \|D^2 f(u)\|_{H^{s-2}} \leq& C\big(\|Df\|_{C^{s-1}(\overline{U})}\big)(1 + \|u\|_{L^\infty}^{s-1})(\|Du\|_{L^6} + \|D^2u\|_{H^{s-2}})\\ \leq& C\big(\|Df\|_{C^{s-1}(\overline{U})}\big)(1 + \|u\|_{L^\infty}^{s-1})\|Du\|_{R^{s-1}},\end{aligned}$$

which establishes the first inequality. The second inequality follows from a similar argument. □

C.2.4 Young's inequality

We cite the important inequality for convolution which is known as Young's inequality. The proof of this inequality can be found in various references, for instance, see [1, Corollary 2.25]

Proposition C.2.9. (Young's inequality for convolution) *If $1/p+1/q = 1+1/r$ with $1 \leq p, q, r \leq +\infty$, and if $u \in L^p(\mathbb{R}^n)$ and $v \in L^q(\mathbb{R}^n)$, then $u * v \in L^r(\mathbb{R}^n)$, and*

$$\|u * v\|_{L^r} \leq \|u\|_{L^p}\|v\|_{L^q} \tag{C.2.5}$$

where $u * v$ denotes convolution.

The other Young's inequality spreading around almost every PDE references is for conjugate Hölder exponents which is stated as

Proposition C.2.10. (Young's inequality for conjugate Hölder exponents) *If a and b are nonnegative real numbers and p, q are positive real numbers such that*

$$\frac{1}{p} + \frac{1}{q} = 1, \tag{C.2.6}$$

then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}. \tag{C.2.7}$$

The equality holds if and only if $a^p = b^q$.

Appendix D

Additional tools

In this Appendix, we introduce some necessary basic tools on matrix relations, analyticity and Raabe's test.

D.1 Matrix relations

Lemma D.1.1. *Suppose*

$$A = \begin{pmatrix} a & b \\ b^{\mathrm{T}} & c \end{pmatrix}$$

is an $(n+1)\times(n+1)$ symmetric matrix, where a is an 1×1 matrix, b is an $1\times n$ matrix and c is an $n\times n$ symmetric matrix. Then

$$A^{-1} = \begin{pmatrix} a & b \\ b^{\mathrm{T}} & c \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{a}\left[1 + b\left(c - \frac{1}{a}b^{\mathrm{T}}b\right)^{-1}b^{\mathrm{T}}\right] & -\frac{1}{a}b\left(c - \frac{1}{a}b^{\mathrm{T}}b\right)^{-1} \\ -\left(c - \frac{1}{a}b^{\mathrm{T}}b\right)^{-1}\frac{1}{a}b^{\mathrm{T}} & \left(c - \frac{1}{a}b^{\mathrm{T}}b\right)^{-1} \end{pmatrix}$$

Proof. Follows from direct computation. □

We also recall the well-known Neumann series expansion.

Lemma D.1.2. *If A and B are $n\times n$ matrices with A invertible, then there exists an $\epsilon_0 > 0$ such that the map*

$$(-\epsilon_0, \epsilon_0) \ni \epsilon \longmapsto (A + \epsilon B)^{-1} \in \mathbb{M}_{n\times n}$$

is analytic and can be expanded as

$$(A + \epsilon B)^{-1} = A^{-1} + \sum_{n=1}^{\infty}(-1)^n \epsilon^n (A^{-1}B)^n A^{-1}, \quad |\epsilon| < \epsilon_0.$$

D.2 Analyticity

We list some well-known properties of analytic maps that will be used throughout this article. We refer interested readers to [37] or [61] for the proofs.

Lemma D.2.1. *Let X, Y and Z be Banach spaces with $U \subset X$ and $V \subset Y$ open.*

1. *If $L : X \to Y$ is a continuous linear map, then $L \in C^\omega(X, Y)$;*
2. *If $B : X \times Y \to Z$ is a continuous bilinear map, then $B \in C^\omega(X \times Y, Z)$;*
3. *If $f \in C^\omega(U, Y)$, $g \in C^\omega(V, Z)$ and $ran(f) \subset V$, then $g \circ f \in C^\omega(U, Z)$.*

Lemma D.2.2. *Suppose $s \in \mathbb{Z}_{>n/2}$, $F \in C^\omega\big(B_R(\mathbb{R}^N), \mathbb{R}\big)$, and that*

$$F(y_1, \cdots, y_N) = F_0 + \sum_{|\alpha| \geq 1} c_\alpha y_1^{\alpha_1} \cdots y_N^{\alpha_N}$$

is the powerseries expansion for $F(y)$ about 0. Then there exists a constant C_s such that the map

$$\big(B_{R/C_s}\big(H^s(\mathbb{T}^n)\big)\big)^N \ni (\psi_1, \psi_2, \cdots, \psi_N) \mapsto F(\psi_1, \psi_2, \cdots, \psi_N) \in H^s(\mathbb{T}^n)$$

is in $C^\omega\Big(\big(B_{R/C_s}\big(H^s(\mathbb{T}^n)\big)\big)^N, H^s(\mathbb{T}^n)\Big)$, and

$$F(\psi_1, \cdots, \psi_N) = F_0 + \sum_{|\alpha| \geq 1} c_\alpha \psi_1^{\alpha_1} \psi_2^{\alpha_2} \cdots \psi_N^{\alpha_N}$$

for all $(\psi_1, \cdots, \psi_N) \in \big(B_{R/C_s}\big(H^s(\mathbb{T}^n)\big)\big)^N$.

D.3 Raabe's Test

We recall a simple Lemma first which can be found in many reference, for instance [6].

Lemma D.3.1 (*Raabe's Test*)**.** *Suppose $a_n > 0$, $n = 1, 2, \cdots$. If there exists $r > 1$, such that n large enough, the following inequality holds*

$$n\left(\frac{a_n}{a_{n+1}} - 1\right) \geq r, \tag{D.3.1}$$

Then series $\sum_{n=1}^\infty a_n$ is convergent.

D.4 Bootstrap arguments and continuation principles

Bootstrap arguments and continuation principles are the important tools for the long time issues of the nonlinear hyperbolic equations. One can use them to extend the local existence to the long time one by verifying certain requirements.

Bootstrap arguments is more like the principle of mathematical induction. The key idea is, intuitively, to impose one hypothesis on the target problem, then try to improve the result we have assumed, that is, to prove a better or stronger result than the one in the hypothesis. After this improvement, one can conclude that the target argument is true. Therefore, bootstrap arguments provide some new assumptions "for free" to help the proof just like the principle of mathematical induction. We cite the *abstract bootstrap argument* as follows, for the details, we refer readers to [82]

Proposition D.4.1. *Let I be a time interval and for each $t \in I$ suppose we have two statements, a "hypothesis" $H(t)$ and a "conclusion" $C(t)$. Suppose we can verify the following four assertions:*

1. *(Hypothesis implies conclusion) If $H(t)$ is true for some time $t \in I$, then $C(t)$ is also true for that time t.*
2. *(Conclusion is stronger than hypothesis) If $C(t)$ is true for some time $t \in I$, than $H(t')$ is true for all $t' \in I$ in a neighbourhood of t.*
3. *(Conclusion is closed) If $t_1, t_2, \cdots$ is a sequence of times in I which converges to another time $t \in I$, and $C(t_n)$ is true for all t_n, then $C(t)$ is true.*
4. *(Base case) $H(t)$ is true for at least one time $t \in I$.*

Then, $C(t)$ is true for all $t \in I$.

We usually choose above conditions 2–4 are relatively easy to satisfied, and try to prove condition 1, which implies to improve $H(t)$ to a stronger statement $C(t)$ ($C(t)$ is stronger due to condition 2). The proof of this argument is an easy application of the simple topology statement about the both open and close sets, see [82] for the details.

Next, let us turn to the *continuation principle* which can be found in [54, p. 46] and [83, p. 420] for the proofs and more details. Let us consider the following equation, for instance,

$$\begin{aligned} A^0\partial_t u + A^i\partial_i u =& 0 \\ u(x,0) =& u_0(x) \end{aligned}$$

Theorem D.4.2. *Assume that $u_0 \in H^s$ for some $s > n/2+1$. Let $T > 0$ be some given time. Assume that there are fixed constants M_1, M_2 and a fixed open set G_1 with $\bar{G}_1 \subset\subset G$ (all independent of T^*) so that for any interval of classical existence $[0, T^*]$, $T^* \leq T$ for $u(t)$ of Theorem* 2.1 *(local existence) in [54, p. 35], the following a priori estimates are satisfied:*

1. $|\partial_\mu A^\mu|_{L^\infty} \leq M_1, \qquad 0 \leq t \leq T^*;$
2. $|Du|_{L^\infty} \leq M_2, \qquad 0 \leq t \leq T^*;$
3. $u \in \bar{G}_1 \subset\subset G, \qquad (x,t) \in \mathbb{R}^n \times [0, T^*].$

Then the classical solution $u(t)$ exists on the interval $[0,T]$ with $u(t) \in C^0([0,T], H^s) \cap C^1([0,T], H^{s-1})$. Furthermore, $u(t)$ satisfies the a priori estimate

$$\|u\|_{L^\infty([0,T^*],H^s)} \leq C \exp\left((M_1 + M_2)CT^*\right)\|u_0\|_{H^s} \tag{D.4.1}$$

for T^* with $0 \leq T^* \leq T$ and the two constants C in (D.4.1) depend only on s and $\bar{G}_1$, i. e., $C(s; \bar{G}_1)$.

and

Corollary D.4.3. *Assume that $u_0 \in H^s$ for some $s > n/2 + 1$. Let $[0, T^*)$ is a maximal interval of H^s existence. Then the following alternatives can occur*

1. *either $T^* = \infty$;*
2. *or $T^* < \infty$ and $\int_0^{T^*}(\|\partial_t u\|_{L^\infty} + \|Du\|_{L^\infty})dt = \infty$;*
3. *or $T^* < \infty$ and for $t \to T^*$, u leaves every compact subset $K \subset\subset G$.*

Appendix E

Index of notation

E.1 Index of notation of Chapter 2

$\tilde{g}_{\mu\nu}$	physical spacetime metric; §2.1
$\tilde{v}^{\mu}$	physical fluid four-velocity; §2.1
$\bar{\rho}$	fluid proper energy density; §2.1
$\bar{p} = \epsilon^2 K \bar{\rho}$	fluid pressure; §2.1
$\epsilon = \frac{v_T}{c}$	Newtonian limit parameter; §2.1
$M_\epsilon = (0,1] \times \mathbb{T}^3_\epsilon$	relativistic spacetime manifold; §2.1
$M = M_1$	Newtonian spacetime manifold; §2.1
$a(t)$	FLRW scale factor; §2.1, eqns. (2.1.4) and (2.1.7)
$\tilde{v}_H(t)$	FLRW fluid four-velocity; §2.1, eqn. (2.1.5)
$\rho_H(t)$	FLRW proper energy density; §2.1, eqn. (2.1.6) (see also (2.2.3) and (2.2.39))
$(\bar{x}^\mu) = (t, \bar{x}^i)$	relativistic coordinates; §2.1
$(x^\mu) = (t, x^i)$	Newtonian coordinates; §2.1, eqn. (2.1.8)
$\mathring{a}(t)$	Newtonian limit of $a(t)$; §2.1, eqn. (2.1.10)
$\mathring{\rho}_H(t)$	Newtonian limit of $\rho_H(t)$; §2.1, eqn. (2.1.10)
$\underline{f}(t, x^i)$	evaluation in Newtonian coordinates; §2.1.1, eqn. (2.1.11)
$\bar{X}^s_{\epsilon_0,r}(\mathbb{T}^3)$	free initial data function space; §2.1.1
$\mathcal{S}(\epsilon, t, \xi), \ldots$	remainder terms that are elements of E^0, §2.1.1
$\mathtt{S}(\epsilon, t, \xi), \ldots$	remainder terms that are elements of E^1, §2.1.1
$\bar{g}^{\mu\nu}$	conformal metric; §2.1.2, eqn. (2.1.12)
$\bar{v}^{\mu}$	conformal four-velocity; §2.1.2, eqn. (2.1.13)
Ψ	conformal factor; §2.1.3, eqn. (2.1.18)
$\bar{h}$	conformal FLRW metric; §2.1.3, eqn. (2.1.19)
$E(t)$	modified scale factor; §2.1.3, eqn. (2.1.20) (see also (2.2.4))
$\Omega(t)$	modified density; §2.1.3, eqn. (2.1.22) (see also (2.2.2))
$\bar{\gamma}^0_{ij}$, $\bar{\gamma}^i_{j0}$	non-vanishing Christoffel symbols of $\bar{h}$; §2.1.3, eqn. (2.1.23)
$\bar{\gamma}^{\sigma}$	contracted Christoffel symbols of $\bar{h}$; §2.1.3, eqn. (2.1.24)
$\bar{Z}^{\mu}$	wave gauge vector field; §2.1.4, eqn. (2.1.26)
$\bar{X}^{\mu}$	contracted Christoffel symbols; §2.1.4, eqn. (2.1.27)
$\bar{Y}^{\mu}$	gauge source vector field; §2.1.4, eqn. (2.1.28)
$u^{\mu\nu}$, u	modified conformal metric variables; §2.1.5, eqns. (2.1.29), (2.1.32) and (2.1.34)
$u^{\mu\nu}_{\gamma}$	first order metric field variables; §2.1.5, eqns. (2.1.30), (2.1.31), (2.1.33) and (2.1.35)
z_i	modified lower conformal fluid 3-velocity; §2.1.5, eqn. (2.1.36)
ζ	modified density; §2.1.5, eqn. (2.1.37)
$\delta\zeta$	difference between ζ and ζ_H; §2.1.5, eqn. (2.1.38)
$\bar{\mathfrak{g}}^{ij}$	densitized conformal 3-metric; §2.1.5, eqn. (2.1.39)
α	cube root of conformal 3-metric determinant; §2.1.5, eqn. (2.1.39)

$\breve{g}_{ij}$	inverse of the conformal 3-metric $\bar{g}^{ij}$; §2.1.5, eqn. (2.1.39)
$\bar{\mathfrak{q}}$	modified conformal 3-metric determinant; §2.1.5, eqn. (2.1.40)
$\bar{\eta}$	background Minkowski metric; §2.1.5, eqn. (2.1.41)
$\zeta_H(t)$	FLRW modified density; §2.1.5, eqns. (2.1.42) and (2.1.43)
C_0	FLRW constant; §2.1.5, eqn. (2.1.44)
$\mathring{\zeta}_H(t)$	Newtonian limit of $\zeta_H(t)$, §2.1.5, eqns. (2.1.46) and (2.1.47) (see also (2.4.8))
z^i	modified upper conformal fluid 3-velocity; §2.1.5, eqn. (2.1.48)
$\mathring{\rho}$	Newtonian fluid density; §2.1.6
$\mathring{z}^j$	Newtonian fluid 3-velocity; §2.1.6
$\mathring{\Phi}$	Newtonian potential; §2.1.6
Π	projection operator; §2.1.6, eqn. (2.1.52)
$\mathring{E}(t)$	Newtonian limit of $E(t)$; §2.1.6, eqn. (2.1.53)
$\mathring{\Omega}(t)$	Newtonian limit of $\Omega(t)$; §2.1.6, eqn. (2.1.54)
$\mathring{\zeta}$	modified Newtonian fluid density; §2.1.6
ρ	fluid proper energy density in Newtonian coordinates; §2.2.3, eqn. (2.2.38)
$\delta\rho$	difference between ρ and ρ_H; §2.2.3, eqn. (2.2.40)
$w_k^{0\mu}$	shifted first order gravitational variable; §2.2.4, eqn. (2.2.61)
Φ	gravitational potential; §2.2.4, eqn. 2.2.62
ϕ	renormalized spatially average density; §2.2.4, eqn. 2.2.64
$\mathbf{U}_1$	gravitational field vector; §2.2.6, eqn. (2.2.100)
$\mathbf{U}$	combined gravitational and matter field vector; §2.2.6, eqn. (2.2.101)
$\mathbf{U}_2$	matter field vector; §2.2.6, eqn. (2.2.102)
$\delta\mathring{\zeta}$	difference between $\mathring{\zeta}$ and $\mathring{\zeta}_H$; §2.4, eqn. (2.4.7)
$\vert\vert\vert\cdot\vert\vert\vert_{a,H^k}$, $\vert\vert\vert\cdot\vert\vert\vert_{H^k}$,	energy norms; §2.5.1, Definition 2.5.5
$\Vert\cdot\Vert_{M^\infty_{\mathbb{P}_{a,k}}([T_0,T)\times\mathbb{T}^n)}$	the spacetime norm; §2.5.1, Definition 2.5.5
$\mathscr{Q}(\xi)$, $\mathscr{R}(\xi)$, ...	analytic remainder terms; §2.6

E.2 Index of notation of Chapter 3

$\tilde{g}_{\mu\nu}$	physical spacetime metric; §3.1
$\tilde{v}^\mu$	physical fluid four-velocity; §3.1
$\bar{\rho}$	fluid proper energy density; §3.1
$\bar{p} = \epsilon^2 K \bar{\rho}$	fluid pressure; §3.1
$\epsilon = \frac{v_T}{c}$	Newtonian limit parameter; §3.1
$M = (0,1] \times \mathbb{R}^3$	spacetime manifold; §3.1
$a(t)$	FLRW scale factor; §3.1, eqns. (3.1.4) and (3.1.7)
$\tilde{v}_H(t)$	FLRW fluid four-velocity; §3.1, eqn. (3.1.5)
$\mu(t)$	FLRW proper energy density; §3.1, eqn. (3.1.6) (see also (3.2.1) and (3.2.17))
$(\bar{x}^\mu) = (t, \bar{x}^i)$	relativistic coordinates; §3.1
$(x^\mu) = (t, x^i)$	Newtonian coordinates; §3.1, eqn. (3.1.10)
$\mathring{a}(t)$	Newtonian limit of $a(t)$; §3.1, eqn. (3.1.9)
$\mathring{\mu}(t)$	Newtonian limit of $\mu(t)$; §3.1, eqn. (3.1.9)
$\underline{f}(t, x^i)$	evaluation in Newtonian coordinates; §3.1.1, eqn. (3.1.11)
$\Vert\cdot\Vert_{R^s}$, $\Vert\cdot\Vert_{K^s}$	energy norms; §3.1.1, eqn. (3.1.12)
$X^s_{\epsilon_0}(\mathbb{R}^3)$, $X^s(\mathbb{R}^3)$	free initial data function space; §3.1.1
$\mathscr{S}(\epsilon, t, \xi)$, ...	remainder terms that are elements of E^0, §3.1.1
$\hat{\mathscr{S}}(\epsilon, t, \xi)$, ...	remainder terms that are elements of E^1, §3.1.1
$\bar{g}^{\mu\nu}$	conformal metric; §3.1.2, eqn. (3.1.15)
$\bar{v}^\mu$	conformal four-velocity; §3.1.2, eqn. (3.1.16)
Ψ	conformal factor; §3.1.2, eqn. (3.1.19)
$\bar{h}$	conformal FLRW metric; §3.1.2, eqn. (3.1.20)
$E(t)$	modified scale factor; §3.1.2, eqn. (3.1.21) (see also (3.2.2))

$\Omega(t)$	modified density; §3.1.2, eqn. (3.1.23) (see also (3.2.1))
$\bar{\gamma}^0_{ij}$, $\bar{\gamma}^i_{j0}$	non-vanishing Christoffel symbols of $\bar{h}$; §3.1.2, eqn. (3.1.25)
$\bar{\gamma}^\sigma$	contracted Christoffel symbols of $\bar{h}$; §3.1.2, eqn. (3.1.25)
$\bar{\mathcal{R}}_{\mu\nu\sigma}{}^\lambda$	contracted Riemannian tensor; §3.1.2, eqn. (3.1.26)-(3.1.27)
$\bar{\mathcal{R}}_{\mu\nu}$, $\bar{\mathcal{R}}^{\mu\nu}$, $\bar{\mathcal{R}}$	(inverse of) Ricci tensors and Ricci scalar of $\bar{h}$; §3.1.2, eqn. (3.1.28)-(3.1.30)
$\bar{\nabla\!\!\!\!-}_\mu$, $\bar{\nabla}_\mu$	covariant derivative with respect to metrics $\bar{h}$ and $\bar{g}$, respectively; §3.1.2
$\bar{\boxminus}$, $\bar{\Box}$	d'Alembertian operator with respect to metrics $\bar{h}$ and $\bar{g}$, respectively; §3.1.2
$\bar{Z}^\mu$	wave gauge vector field; §3.1.2, eqn. (3.1.38)
$\bar{X}^\mu$	contracted Christoffel symbols; §3.1.2, eqn. (3.1.39)
$\bar{Y}^\mu$	gauge source vector field; §3.1.2, eqn. (3.1.40)
$u^{\mu\nu}$, u	modified conformal metric variables; §3.1.2, eqns. (3.1.41), (3.1.44) and (3.1.46)
$u^{\mu\nu}_\gamma$	first order metric field variables; §3.1.2, eqns. (3.1.42), (3.1.43), (3.1.45) and (3.1.47)
z_i	modified lower conformal fluid 3-velocity; §3.1.2, eqn. (3.1.48)
ζ	modified density; §3.1.2, eqn. (3.1.49)
$\delta\zeta$	difference between ζ and ζ_H; §3.1.2, eqn. (3.1.50)
$\bar{\mathfrak{g}}^{ij}$	densitized conformal 3-metric; §3.1.2, eqn. (3.1.51)
α	cube root of conformal 3-metric determinant; §3.1.2, eqn. (3.1.51)
$\breve{g}_{ij}$	inverse of the conformal 3-metric $\bar{g}^{ij}$; §3.1.2, eqn. (3.1.51)
$\bar{\mathfrak{q}}$	modified conformal 3-metric determinant; §3.1.2, eqn. (3.1.52)
$\zeta_H(t)$	FLRW modified density; §3.1.2, eqns. (3.1.53) and (3.1.54)
C_0	FLRW constant; §3.1.2, eqn. (3.1.55)
$\mathring{\zeta}_H(t)$	Newtonian limit of $\zeta_H(t)$, §3.1.2, eqns. (3.1.56) and (3.1.57) (see also (3.4.9))
z^i	modified upper conformal fluid 3-velocity; §3.1.2, eqn. (3.1.58)
$\mathring{\rho}$	Newtonian fluid density; §3.1.3
$\mathring{z}^j$	Newtonian fluid 3-velocity; §3.1.3
$\mathring{\Phi}$	Newtonian potential; §3.1.3
$\mathring{E}(t)$	Newtonian limit of $E(t)$; §3.1.3, eqn. (3.1.62)
$\mathring{\Omega}(t)$	Newtonian limit of $\Omega(t)$; §3.1.3, eqn. (3.1.64)
$\mathring{\zeta}$	modified Newtonian fluid density; §3.1.3
ρ	fluid proper energy density in Newtonian coordinates; §3.2.2, eqn. (3.2.16)
$\delta\rho$	difference between ρ and ρ_H; §3.2.2, eqn. (3.2.18)
$\hat{\mathbf{U}}$	combined gravitational and matter field vector; §3.2.5, eqn. (3.2.67)
$\hat{\mathbf{U}}_1$	gravitational field vector; §3.2.5, eqn. (3.2.68)
$\mathbf{U}_2$	matter field vector; §3.2.5, eqn. (3.2.68)
$\hat{g}^{\mu\nu}$	rescaled conformal metric by θ; §3.3, eqn. (3.3.1)
$\hat{\mathfrak{u}}^{\mu\nu}$	modified gravitational variables of $\hat{g}^{\mu\nu}$; §3.3, eqn. (3.3.1)
$\hat{\mathfrak{u}}^{\mu\nu}_\sigma$	first order derivative of modified gravitational variables of $\hat{g}^{\mu\nu}$; §3.3, eqn. (3.3.1)
θ	ratio of $\sqrt{\lvert\bar{g}\rvert}$ and $\sqrt{\lvert\bar{h}\rvert}$; §3.3, eqn. (3.3.2)
$\breve{\mathfrak{u}}^{\mu\nu}$, $\breve{\mathfrak{u}}^{\mu\nu}_0$	initial data of gravitational variables; §3.3, eqn. (3.3.45)
$\delta\breve{\rho}$, $\breve{z}^j$	initial data of matter field variables; §3.3, eqn. (3.3.46)
$\breve{\mathscr{Q}}(\xi)$, ...	analytic remainder terms; §3.3
$\breve{\xi}$	the set of free data; §3.3, eqn. (3.3.51)
$\lVert\xi\rVert_s$	certain norm of free data; §3.3, eqn. (3.3.90)
ϖ^j	quantity related to $\mathring{\rho}\mathring{z}^j$; §3.4.2, eqn. (3.4.7)
$\delta\mathring{\zeta}$	difference between $\mathring{\zeta}$ and $\mathring{\zeta}_H$; §3.4.2, eqn. (3.4.8)
$\mathring{z}_j$	modified lower Newtonian fluid 3-velocity; §3.4.2, eqn. (3.4.8)
$\mathring{\Upsilon}$	modified Newtonian potential in terms of Yukawa potential; §3.4.2, eqn. (3.4.10)
$\mathring{\Phi}_i$	first derivative of Newtonian potential; §3.4.2, eqn. (3.4.19)
Φ^μ_k	first derivative of modified gravitational potential; §3.5, eqn. (3.5.1)
Υ	modified gravitational potential in terms of Yukawa potential; §3.5, eqn. (3.5.4)
$w^{0\mu}_k$	shifted first order gravitational variable; §3.5.2, eqn. (3.5.16)
$\mathbf{U}_1$	gravitational field vector including $w^{0\mu}_k$; §3.5.3, eqn. (3.5.21)
$\mathbf{U}$	combined gravitational and matter field vector; §3.5.3, eqn. (3.5.22)

$\vert\vert\vert\cdot\vert\vert\vert_{a,H^k}$, $\vert\vert\vert\cdot\vert\vert\vert_{a,R^k}$	energy norms; §3.6.1, Definition 3.6.5
$\vert\vert\vert\cdot\vert\vert\vert_{H^k}$, $\vert\vert\vert\cdot\vert\vert\vert_{R^k}$	energy norms; §3.6.1, Definition 3.6.5
$\Vert \cdot \Vert_{M^{\infty}_{\mathbb{P}_{a},k}([T_0,T)\times\mathbb{R}^3)}$	the spacetime norm; §3.6.1, Definition 3.6.5