Rumour Spreading with a Delaying Scheme

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Abstract

Standard Rumour Spreading (SRS) in a connected graph G, is defined as follows. At the beginning, there is only one particular vertex that knows the rumour. For each vertex in G, we assign a Poisson clock with rate 1. When the clock of a vertex rings, the vertex chooses a uniformly random neighbour to exchange information with. The running time of the SRS is the time needed to let every vertex in G learn the rumour.

In this thesis, we introduce a new variant of the SRS, called Rumour Spreading with a Delaying Scheme (RSDS). In this model, each informed vertex has two possible statuses: either active or dormant, which continuously flipping back and forth with a certain rate called the switching rate. The switching rate is one of the main parameters studied in this work. The same settings as the SRS apply in the RSDS except that dormant vertices are unable to exchange information.

We compare the expected running times between the RSDS and SRS for paths, stars, and complete graphs. In the context of rumour spreading in complete graphs, we provide a threshold for the switching rate function in the following sense. The expected running time of the RSDS asymptotically equals to that of the SRS if and only if the switching rate is significantly smaller than the threshold function. In addition, we provide a more accurate analysis on a specific case, that is the RSDS with unit rate.

Declaration

This thesis contains no material which has been accepted for the award of any other degree or diploma at any university or equivalent institution and that, to the best of my knowledge and belief, this thesis contains no material previously published or written by another person, except where due reference is made in the text of the thesis.

[redacted]

Yozef Tjandra
29 June 2017

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Experiencing a researcher's life for these two years has provided me incredibly diverse wisdom. It shapes not only my mathematical knowledge and skills, but also my character and attitude. I cannot disagree with those advising me that the most valuable gain from a research degree does not come from the subject of the research itself, but rather from the resilience that appears as a result of persistence. I am grateful that my study in this most liveable city has led me not only to enjoy the many convenience of everyday living, but also to progress towards an entirely upgraded phase of life.

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Chapter 1

Introduction

Nowadays, a large number of structures appearing in various real-world situations can be described in the form of networks. In many settings, we can understand a system as a collection of well separated elements where some particular pairs of them are interacting in a certain way. To mention a few of real-world examples, we can find network structures in communication patterns within online social media, scientific paper co-authorship studies, spreading of an infectious disease, and interactions between discrete particles in a crystal.

Many prominent scholars, for example Barabási [4], Newman [37] and Watts [46], conducted influential works exposing the emergence of large networks in many real-world settings. Their works reveal that we are living in an extremely interconnected network that is still growing over time. It is apparent that easy communication appears as a result of effortless access to cheap computers and the Internet. This leads modern people to be much more connected to each other than any kind of society throughout the whole history.

A network, in its simplest form, is often described in terms of a collection of vertices joined together in pairs by the so-called edges, that we will call the *graph*. The vertices of the graph represent the members of the network. On the other hand, when two vertices are joined by an edge in the graph, the edge represents an access of communication between these vertices.

Many aspects of large networks are worthy of study. A deep understanding of the communication patterns occurring in large networks can provide many great benefits to solve various problems that follow from the rapid growth of networks. The connections in a social network, for example, affect the behaviour of a certain community to collect news and form their public opinion. When we have no knowledge about the structure of large networks, we cannot hope to fully understand how the corresponding systems work.

Often the enormous size of the networks makes the analysis very challenging. In particular, some methods used in small deterministic networks can no longer be applicable in the context of larger networks. In fact, some giant networks are very difficult to be accurately determined in real life, for example the network showing transmissions of sexual diseases and the network of friendships. Much literature, for instance Lovász [35] and Newman [37], suggests a stochastic approach to model such networks. From probability theory, we often find that many properties of the stochastic model appear almost surely when the size of the networks tends to infinity. As a result, probabilistic analysis becomes an essential tool in contemporary network studies.

One of the fundamental schemes studied in probabilistic analysis of large networks is the *Randomised Rumour Spreading* scheme. Various problems occurring in large networks can be reduced to the problem of finding the time needed to disseminate information within the members of the network, where the dissemination is performed in a random fashion. For instance, the rumour spreading scheme represents the behaviour of a propaganda propagation in online social networks [12], analysis of epidemic spreading [33], and design of distributed computing [20].

In the rumour spreading scheme, the information passing can only occur between pairs of neighbouring individuals. In the usual setting, the model picks a particular vertex in the network to be the initial rumour spreader. Throughout the time, the informed vertices call their random neighbours and pass the rumour to them. This procedure is repeated until all members of the network know the rumour.

1.1 Existing Results on Rumour Spreading Models

One of the earliest models of rumour spreading was introduced by Frieze and Grimmett in [25]. Their model considered a random rumour spreading scheme in a town having n residents, each of whom possesses a private telephone access to all other people in the town. In a series of discrete rounds, every individual that knows the rumour chooses a uniformly random person in the town to pass the rumour to. This model represents the rumour spreading occurring in a complete graph with n vertices, since all pairs of vertices are able to communicate with each other.

Later on, the rumour spreading procedure that Frieze and Grimmett introduced is also known as the *push protocol*. This is the spreading scheme in which informed individuals push the message to the uninformed ones. The reversed version of the scheme also exists, namely the *pull protocol*. In this scheme, each uninformed individual also uniformly calls its random neighbours to ask the message from them. The first model which involves both push & pull protocols was introduced by Demers, Greene, Hauser, Irish, Larson, Shenker, Sturgis, Swinehart, and Terry in [16] and popularised in [33] by Karp, Schindelhauer, Shenker, and Vöcking.

The scheme by which the messages are passed in discrete rounds is also known as the *synchronous rumour spreading* model. The most studied parameter for this scheme is the number of rounds needed to let everyone learn the rumour. Extensive results on the synchronous model exist for some families of deterministic graphs, including complete graphs [33, 41] and hypercube graphs [21]. Results for various classes of random graphs are also present: Erdős-Rényi random graphs [21, 22, 38], random regular graphs [5, 23], preferential attachment graphs [18], and Chung-Lu random graphs [24]. Since these graph families are not directly relevant to the main topic of this thesis, interested readers are advised to consult the references for precise definitions of each random graph family.

The analysis of the synchronous rumour spreading scheme for general n-vertex graphs also exists. Most of the results of the analysis are expressed asymptotically, usually in terms of n, the order of the graph. We say that an event whose probability depends on n, occurs *with high probability* (abbreviated *w.h.p.*) if its probability converges to 1 as $n \to \infty$. Let G be an arbitrary connected graph with n vertices. Suppose that $T(G)$ denotes the number of rounds needed to let every vertex in the graph G learn the rumour. Feige, Peleg, Raghavan, and Upfal in [21], stated that $\log n \leq T(G) \leq 12n \log n$ w.h.p. for any G. Moreover, this bound is tight in the sense that there exist connected graphs G_1 and G_2 such that $T(G_1) = O(\log n)$ and $T(G_2) = \Omega(n \log n)$ w.h.p.

Further studies which relate the synchronous rumour spreading scheme to some well known graph properties are also present. For any connected graph G, let $\Delta(G)$ denote the maximum degree of the vertices of G. The *diameter* of G, denoted by $\mathrm{diam}(G)$, is the maximum of all distances between all distinct pairs of the vertices if G. The *conductance* of graph G, denoted by $\phi(G)$, is a value in range $0 < \phi(G) \leq 1$ which measures the connectedness of G. Roughly, the smaller conductance that a graph has, the more we find "bottleneck" structures in the graph. The known relationship between $T(G)$ and $\mathrm{diam}(G)$ is $T(G) = O(\Delta(G)(\mathrm{diam}(G) + \log n))$, for any connected graph G, as thoroughly discussed in [21]. There is much literature discussing the relationship between $T(G)$ and $\phi(G)$, for example in [13, 14, 26]. The best known bound for $T(G)$ in terms of $\phi(G)$, is that $T(G) = O(\phi(G)^{-1} \log n)$ as reported in [13].

Several attempts to make more realistic models were made. Boyd, Ghosh, Prabhakar, and Shah in [10] introduced the *asynchronous* version of the rumour spreading scheme. In this variant, the uniformly random neighbour callings are allowed to occur at arbitrary continuous times. This improves the modelling of many real-world rumour spreading phenomena, since real life communication never occurs only in segmented time points. The scheme has been well studied for some classes of deterministic graphs: complete graphs [30] and hypercube graphs [8]. The studies of the scheme for various random graph classes also exist. These include Erdős-Rényi random graphs [40], preferential attachment random graphs [19], Chung-Lu random graphs [24], and random regular graphs [2].

The relationship between the synchronous and asynchronous rumour spreading was studied in a paper by Acan, Collevecchio, Mehrabian, and Wormald [1]. The paper discussed the comparison between the running time of both versions of rumour spreading applied in the same underlying graph. They proved that for any connected graph G, the ratio between the running times of the synchronous and asynchronous rumour spreadings in graph G is bounded below by $\Omega(1/\log n)$ and bounded above by $O(n^{2/3})$. Moreover, the given lower bound is best possible, up to some constant factors.

In [39], Panagiotou, Pourmiri, and Sauerwald introduced a variation of the rumour spreading scheme to accelerate the running time of the spreading using a multiple call scheme. In the model, any informed vertex is able to pass the message to R other individuals at once, where R is a random variable whose support is the positive integers. In particular, the paper discussed how the multiple call scheme applied in the rumour spreading model in complete graphs decreases the running time. As long as the mean and variance of R are bounded, they proved that the multiple call scheme gives insignificant effect to the running time. However, when R follows a power law distribution with exponent β, the running time is significantly quicker if $2 < \beta \leq 3$. Note that in this case, the variance of R is unbounded whereas the mean is still bounded.

The rumour spreading model also has a close relationship to some other probabilistic models, for example the epidemic spreading model, the first-passage percolation problem and the minimal weighted path problem in a graph having random edge weights. In this thesis, we will primarily utilise the strong relationship between the rumour spreading and epidemic models in many technical parts of the analysis.

Despite many similarities of the phenomena modelled by both epidemic and rumour spreading schemes, they have different motivations and studied parameters. In the epidemic modelling, an infectious disease propagates from some initially infected individuals throughout other susceptible members of the network. As references, Miller and Kiss [36] as well as Daley and Gani [15] provide extensive surveys on the models. The main motivation for the epidemic spreading analysis is to prevent the disease from infecting the entire members of the community. This leads the researchers to concentrate their studies on finding the probability that the infection spreads throughout the entire network in the long run. For this reason, the quantification of the spreading time receives relatively less attention in epidemic study. In contrast to most rumour spreading models, every individual in the network will receive the rumour in a finite time with probability 1. This implies that such a long run probability is mostly trivial in the rumour spreading model.

Another critical difference between the rumour spreading and the epidemic models is the rate of the spreading. In the rumour spreading model, the informed individuals pass the message via random neighbour callings whereas the infection model has no such scheme. The different schemes of the spreading imply different spreading rate characterisations of each model. Consider the rate of the spreading conducted from a particular vertex. In the rumour spreading model, this rate is the same for all vertices in the graph. However, in the epidemic model, the spreading rate depends on the degree of the vertex. The more friends that an infected individual has, the more likely that the disease spreads to its friends. Interested readers can find a more precise definition of the infectious spreading method, for instance, in [9]. On the other hand, when we observe the spreading occurring in a certain edge of the graph, the spreading rate within the edge in the rumour spreading model is determined by the degree of the end vertices of the edge. Since a neighbour is called in a uniformly random fashion, the rate of the spreading occurring in a given edge is reciprocal to the degree of the incident vertices. In contrast, the spreading rates between the pairs of adjacent vertices have the same value in the epidemic model.

When the underlying network has a special property, however, the rumour spreading model has an exact equivalence to the epidemic model. Since the distinguishing feature of both models comes from the degree of the vertices, the regular graphs (whose vertices have the same degree) become a special network in which both models have the same characteristic. In this case, the spreading rate is constant when we observe all edges as well as when we consider the vertices. This special feature will be of particular interest in many parts of the thesis.

The first passage percolation (FPP) model in networks, the minimal weighted path problem in a randomly edge weighted graph, and the asynchronous rumour spreading model are also equivalent to each other under certain condition. To provide a brief overview, the FPP model is essentially a probabilistic model describing the flow of fluid which passes through a random medium. To find a historical exposition and extensive survey on the models, one can consult [3]. Often the models employ discrete graphs as the representation of the flowing points of the fluid. The vertices of the graph represent the pores of the medium at which the fluid has access to flow, while an edge between two vertices represents the fluid pathway between them. In the FPP models in networks, the fluid moves within vertices according to random capacities of the edges. In this sense, the FPP problem on a network can be viewed as a graph with random edge weights. On the other hand, the random weights on the edges can be associated with the waiting times of the message passing within the end vertices of the edge in the rumour spreading model. When the random weight is exponentially distributed, the FPP model behaves exactly the same as the rumour spreading model. Extensive works on this topic can be found in [6, 7, 29, 30, 45, 44]. The equivalence of these models is due to the memorylessness property of the exponential distributions so that the random weights can be appropriately shifted in order to be associated with the message passing waiting times in the rumour spreading models. By this correspondence, the length of the minimal weighted path from a given vertex to its "farthest" vertex in the graph depicts the running time of the asynchronous version of rumour spreading.

1.2 Purpose of the Thesis

The main purpose of this thesis is to study a particular variation applied to the classical asynchronous version of rumour spreading. So far, the existing variations of rumour spreading procedures are mostly intended to expedite the spreading time. An attempt to observe the rumour spreading in a different setting compared to ones that have been studied can lead to interesting and useful results. This motivates us to construct another variation in order to obtain a contrasting result from the existing ones.

In this work, we have designed a new rumour spreading model with a delaying scheme that allows some possibilities to decelerate the running time of the standard rumour spreading model. In particular, we will investigate a delaying scheme at which the spreading agents can be absent from the spreading activity for a while. Specifically, we will apply the asynchronous time and the push & pull protocol in our new rumour spreading model. In this thesis, we will study the rumour spreading model in three well known elementary families of graphs: paths, stars, and complete graphs. However, we will primarily concentrate on the rumour spreading analysis in complete graphs.

The delaying scheme models the spreading of messages in a network where the individuals are not constantly available to pass messages. This provides a good representation of the rumour dissemination on the Internet where there are times at which some informed users go offline and are unable to spread the rumour. For example, although the information updates in a Twitter network are almost continuously added, a particular user is very unlikely to receive all of the information exhaustively. During some period, some users log out and miss the current news updates as well as have no access to post new information. Those offline users need to get back online in order to participate in the spreading process. On the other hand, the rumour spreading scheme also models the information dissemination in a network whose broadcaster servers can possibly be damaged with a certain rate. Unless they are repaired, the information cannot pass through them. This leads to a reduced spreading rate.

We will call the new model the *Rumour Spreading with a Delaying Scheme* (RSDS). On the other hand, we call the original asynchronous rumour spreading process which has no delaying scheme the *Standard Rumour Spreading* (SRS). In the RSDS process, the informed vertices have two possible statuses, either *active* or *dormant*. Once a vertex receives the message, it immediately receives an active status. However, we allow an active vertex to be dormant with a certain rate, which means that they refrain from passing the rumour. Although they know the rumour already, dormant vertices are unable to pass the information to their neighbours. On the other hand, a dormant vertex can become active again so that it regains its capability to spread the rumour. The more precise and formal definition of the models will be discussed in Chapter 3.

In the RSDS model, any informed individual in the network has the same rate to switch from active to dormant or vice versa. We will call this rate the *switching* rate, as the status of the informed ones are continuously switching back and forth between active and dormant statuses throughout the entire process. However, we generally specify the message passing rate to be different from the switching rate. In this thesis, we normalise the ratio of these two rates by setting the spreading rate to be 1 while the switching rate is expressed as a function of n, the network size.

The main quantity of interest in this work is the additional time gained by applying the delaying scheme, which we will call the *delay time*. We intend to study how far the application of the delaying scheme affects the spreading time in terms of the switching rate. First, we will briefly discuss the expected delay time of the RSDS for path and star graphs. Afterwards, we will devote the rest of the thesis to analyse the delay time of the RSDS in complete graphs. Moreover, we will also review more thoroughly the phenomena occurring in an RSDS process with unit switching rate.

To give an overview of the main results on the RSDS in complete graphs, we briefly give a mathematical definition of the *delay time*. Suppose that $T(X')$ and $T(X)$ are the random times needed to let every individual receive the rumour in SRS and RSDS processes respectively, applied in a complete graph with n vertices. The *delay time* of the RSDS process, denoted by $D(X)$, is defined to be $T(X) - \mathbb{E}T(X')$. Here, $D(X)$ is a random variable which shares the same probability space as the RSDS process X. Later on, we will see that $\mathbb{E}D(X)$ always has a nonnegative value since we can couple the SRS and RSDS processes in such a way that $T(X') \leq T(X)$.

This work mainly focuses on the asymptotic expectation of $D(X)$ in terms of the switching rate. Note that $\mathbb{E}D(X)$ is the difference between expected running times of SRS and RSDS. We first observe that $\mathbb{E}T(X') = 2\log n/n + O(1/n)$, as described in [30]. This means that if $\mathbb{E}D(X) = o(\log n/n)$, then the delaying scheme only contributes to negligible expected additional time to delay the SRS process. We say that an RSDS process has a *noteworthy* delay time if $\mathbb{E}D(X) = \Omega(\log n/n)$. Hence, one of the main objectives of this work is to characterise the switching rate functions that bring a noteworthy delay time.

We observed that if the switching rate is slow enough, then the expected delay time is insignificant compared to $\mathbb{E}T(X')$. When the switching rate is slower than the spreading rate, an informed vertex has a higher probability to pass the rumour to a new vertex than to become dormant. Thus, the spreading process will finish before a significant number of dormant vertices are able to slow down the process. In other words, the slower the switching rate that an RSDS process has, the more likely its running time behaves similarly to that of the SRS process.

In our main results, we state that the delay time is noteworthy if and only if the switching rate is $\Omega(n/\log n)$. This result divides the switching rate functions into two classes of function in terms of the noteworthiness of the delay time. We will later call $n/\log n$ the *threshold function* for the noteworthiness of the delay time. However, we also show that no matter how fast the switching rate is, the magnitude of the expected delay time is always bounded above by $O(\log n/n)$. This means that the delaying scheme can only extend the expected running time of the rumour spreading process up to a constant factor.

We will also pay attention to a special instance of the RSDS process with a slow switching rate; that is when the rate is 1. In this setting, we analysed the sources of the delay. We say that a rumour spreading process is in a *vacuum state* if there is no active vertex during that time. We will show that the most significant delay comes from the event where the process enters a vacuum state before the initial rumour spreader informs any other vertices. The event becomes the source of the most significant expected delay in the sense that if it is given that the event does not occur, then the expected delay time drops dramatically.

In addition, we will discuss some lower and upper bounds for the RSDS running time that hold w.h.p. The result appears as a direct consequence of the earlier results on the expected delay times of the RSDS. When the switching rate is slow, we proved that the running time of the RSDS is asymptotically the same as that of the SRS, which is $2\log n/n$. On the other hand, for any positive constant ε and any function $\omega = \omega(n)$ tending to infinity arbitrarily slowly, we have that $2(1-\varepsilon)\log n/n < T(X) < \omega \log n/n$ w.h.p. whenever the switching rate is fast.

1.3 Organisation of the Thesis

Chapter 1 provides an introductory exposition on the rumour spreading models, some reviews of existing literature on the model, the purpose of the thesis, and the organisation of the thesis. In Chapter 2, we will establish the standard definitions and notations that will be used throughout the whole thesis as well as some preliminary theories which serve as methods to obtain the main results of this thesis. The two models of rumour spreading, the SRS and RSDS, are defined formally in Chapter 3. Also in this chapter, we provide simple demonstration of the rumour spreading model in some elementary classes of graphs. Chapter 4 discusses general properties of these two rumour spreading processes occurring in a complete graph with n vertices. These properties will serve as important lemmas that will be used extensively in the next chapter. The exposition of the main results are given in Chapter 5. As a preparation for the chapter, we will describe an important property possessed by the so-called compressed version of the RSDS process. In the next two sections of the chapter, we will concentrate on the expected delay time of the RSDS process for two classes of switching rate functions. Then, in the last section, we give a more detailed analysis of a special case of the RSDS where the switching rate is 1. Lastly, in Chapter 6, we will have the concluding statements and discussions about some possible future work.

At the end of this thesis, we provide a glossary and a list of symbols to help the reader to keep track of the definitions of some technical terms and non-standard symbols used in this thesis.

Chapter 2

Preliminaries

This chapter consists of the standard notations and definitions that will be used throughout the whole thesis and some useful elementary observations related to the main results.

2.1 Definitions and Notations

In this section, we provide some conventions regarding definitions and notations, mainly from graph theory and probability theory.

2.1.1 Graph Theory Notations

Most graph theoretic notations in this thesis follow standard notations introduced in [17].

A *simple graph* G is a pair of finite sets $G = (V, E)$ with $V \neq \emptyset$ and E a collection of 2-subsets of V. We call the elements of V the *vertices* of G and the elements of E the *edges* of G. Later on, for any edge $\{u, v\} \in E$, we will just refer to it as uv instead of its set form. The *order* of a graph G is the number of its vertices. Unless otherwise stated, we always let n denote the order of G.

For any $u \in V$ and $e \in E$, we say that u is *incident* to e if $u \in e$. Moreover, if $e = uv$, then we say that u and v are the *ends* of e. Also, for any pair of distinct vertices $u, v \in V$, we say that u is *adjacent* to v if $uv \in E$. We say a graph is *complete*, denoted by K_n, if all pairs of distinct vertices are adjacent.

The *degree* of a vertex v, denoted by $\deg(v)$, is the number of vertices adjacent to v. Suppose $X, Y \subseteq V$. We define $E(X, Y)$ as the set of all edges xy where $x \in X$ and $y \in Y$. We will also call the edge xy an *X-Y edge*. Also, we define $e(X, Y) := |E(X, Y)|$, the total number of X-Y edges. We say that a graph is *regular* if all of its vertices have the same degree.

For any two graphs $G = (V, E)$ and $G' = (V', E')$, we say that G' is a *subgraph* of G, denoted by $G' \subseteq G$, if $V' \subseteq V$ and $E' \subseteq E$. If $G' \subseteq G$, and G' contains all edges $xy \in E$ with $x, y \in V'$, then G' is an *induced subgraph* of G. Also, we say that V' *induces* G' in G, and write $G' := G[V']$.

A *path* is a graph $P_n = (V, E)$ of the form

$$V = \{v_1, \dots, v_n\}, \qquad E = \{v_1v_2, v_2v_3, \dots, v_{n-1}v_n\}.$$

We call the vertices v_1 and v_n the *ends* of P, while $v_2, \dots, v_{n-1}$ are the *internal vertices* of P. The *length* of a path is the number of edges in it. Suppose $P \subseteq G$. We say that v_1 and v_n are *linked* in G by P, or equivalently, P is a v_1-v_n *path* in G. Figure 2.1 visualises the path P_n.

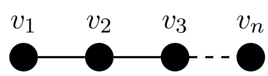

Figure 2.1: The path graph P_n

A *star* is a graph $S_n = (V, E)$ (commonly also known with notation $K_{1,n-1}$) of the form

$$V = \{v, v_1, \dots, v_{n-1}\}, \qquad E = \{vv_1, vv_2 \dots, vv_{n-1}\}.$$

We call v the *centre* vertex, that is the only vertex which is adjacent to every other vertex. On the other hand, we call the other vertices having degree 1 the *leaves*. Figure 2.2 illustrates the star graph S_n.

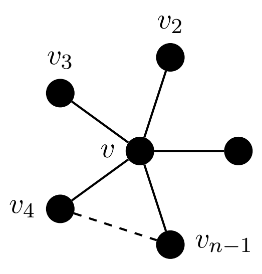

Figure 2.2: The star graph S_n

For any two vertices $x, y \in V$, the *distance* of x and y, denoted by $d(x, y)$, is the length of the shortest x-y path in G. If no such path exists, then we set $d(x, y) = \infty$. We say that G is *connected* if for every pair of vertices $x, y \in V$, we have that $d(x, y) < \infty$. We define the *diameter* of G, denoted by $\operatorname{diam}(G)$ to be $\max_{u,v \in V(G)} d(u, v)$.

2.1.2 Probability and Random Processes Notations

For any two random variables X and Y lying on the same probability space, we say that *X is equal to Y in distribution*, denoted by $X \stackrel{d}{=} Y$, if they share the same probability distribution function.

We say that an event A_n, which depends on a parameter n, occurs *with high probability* (abbreviated *w.h.p.*), if $\mathbb{P}(A_n) \to 1$ as $n \to \infty$. Mainly in this thesis, n refers to the order of the graph. We will often see an event that occurs w.h.p. as a property of a very large graph that appears almost surely.

We use the following shorthands to denote some probability distributions. For $n \in \mathbb{Z}^+$ and $p \in [0,1]$, let $\mathcal{B}(n,p)$ denote a *binomial distribution* corresponding to the number of successes out of n independent Bernoulli trials where each trial has success probability p. Thus, if $X \stackrel{d}{=} \mathcal{B}(n,p)$, then

$$\mathbb{P}(X = k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, 1 \dots, n.$$

We also have that $\mathbb{E}X = np$.

We use the notation $\mathcal{G}(p)$ to denote a *geometric distribution* with success probability $p \in (0,1)$. Suppose that $Y \stackrel{d}{=} \mathcal{G}(p)$. Consider a sequence of independent Bernoulli trials with success probability p. Then, Y essentially counts how many failures occurring in the sequence until it achieves its first success. In other words, the first success of the sequence occurs in the $(Y+1)$-th trial. It means that

$$\mathbb{P}(Y = k) = p(1-p)^k, \quad k = 0, 1, \dots.$$

Note that $\mathbb{E}Y = \frac{1-p}{p}$.

Also we use the notation $\mathcal{E}(r)$ with $r > 0$ to denote an *exponential* random variable with rate r. Suppose that $W \stackrel{d}{=} \mathcal{E}(r)$. The probability density function of W is given as follows.

$$f_W(w) = re^{-rw}, \quad w \geq 0.$$

Note also that $\mathbb{E}W = r^{-1}$. Note that we use the notation $\mathcal{E}(\cdot)$ to denote the exponential random variables, whereas $\exp(\cdot)$ represents the exponential function. We can think of the exponential distribution as the continuous analogue of the geometric distribution since it often represents the waiting time of the first successful occurrence of a certain event.

Both geometric and exponential distributions enjoy an interesting property, namely the *memorylessness* property. For geometric distributions, we have the following formulation for the property. For any $m, n \in \{0, 1, \dots\}$,

$$\mathbb{P}(Y > m + n \mid Y > m) = \mathbb{P}(Y > n).$$

Whereas, the following formulation applies for the exponential distributions. For every $s, t > 0$, we have that

$$\mathbb{P}(W > s + t \mid W > t) = \mathbb{P}(W > s).$$

By interpreting the random variable as the waiting time for the first occurrence of some event, the property says that the waiting time does not depend on how much time has elapsed already. In other words, the information about how long we have waited for the event is unhelpful to predict the future occurrence of the event.

In this thesis, we will use the terms *random processes* and *stochastic processes* interchangeably, referring to the collections of random variables. Throughout the chapters, we will sometimes also simply use the term *processes* to refer to the random processes.

Now we define several important random processes. The random processes definitions used in this thesis are adapted from [27].

Definition 2.1. *A* Poisson process *of rate λ is an integer-valued stochastic process $\{X(t) : t \geq 0\}$ satisfying the following properties:*

1. $X(0) = 0$.
2. *For any time points $0 = t_0 < t_1 < \cdots < t_n$, the increments of the process*

   $$X(t_1) - X(t_0), X(t_2) - X(t_1), \ldots, X(t_n) - X(t_{n-1})$$

   are independent integer-valued random variables.
3. *For all $s \geq 0$ and $t > 0$, the random variable $X(s+t) - X(s)$ has Poisson distribution with mean $t\lambda$, that is*

   $$\mathbb{P}[X(s+t) - X(s) = k] = e^{-t\lambda}\frac{(t\lambda)^k}{k!}, \quad k = 0, 1, \ldots.$$

Corresponding to the Poisson process $\{X(t) : t \geq 0\}$, we also introduce its *Poisson clock* $C_X = (c_1, c_2, \ldots)$ where $0 < c_1 < c_2 < \ldots$. Here, for all $i \geq 1$, c_i denotes the ringing time of the clock corresponding to the time at which the value of the process increases. In other words, a Poisson clock only captures the times at which the process jumps, without paying much attention to the current state of the process. Observe that the value of $X(t)$ represents how many times the Poisson clock has rung up to time t.

It is worth noting the following facts. The time gaps between two consecutive ringing times $\tau_1, \tau_2, \ldots$ of a Poisson clock with rate λ are independent and identical exponential random variables with rate λ. This means that for all $j \geq 1$,

$$\mathbb{P}(\tau_j \geq x) = e^{-\lambda x}, \quad x \geq 0.$$

We state the following important properties of Poisson processes. Suppose that we have two independent Poisson clocks C_1 and C_2 with rate λ and μ, respectively. Then, we define C to be another clock which rings whenever any of those two clocks ring, written as $C = C_1 \wedge C_2$. We have that C is also a Poisson clock with rate $\lambda + \mu$. Moreover, whenever C rings, the probability that the ringing comes from C_1 is $\frac{\lambda}{\lambda+\mu}$. We call this the *superposition property* of a Poisson process and C the *superposition* clock of C_1 and C_2. Now suppose that a Poisson clock with rate λ has two types of ringings. Let p be the probability that a ringing of the clock belongs to the first type. Then, when we create a new clock whose ringing follows only the first type ringing of the original clock, the new clock is also a Poisson clock with rate $p\lambda$. This is called the *thinning property* of a Poisson process.

Next, we define the other well known random processes, namely the *Markov Chains*, which serve as excellent models in diverse ranges of real-world applications. There are two versions of Markov chains in regards to their time indices: *the discrete-time Markov chains* and *the continuous-time Markov chains*. The formal definitions are described as follows.

Definition 2.2 (Discrete-time Markov Chains)**.** *Let S be a countable set. A random process $\{X_n : n \geq 0\}$ whose indices take values in the set of nonnegative integers, is called a* discrete-time Markov chain *if X satisfies the following properties.*

1. Markov property.

$$\mathbb{P}(X_{n+1} = j \mid X_0, X_1, \ldots, X_n) = \mathbb{P}(X_{n+1} = j \mid X_n),$$

for all $n \geq 0$ and $j \in S$.

2. Homogeneity property.

$$\mathbb{P}(X_{n+1} = j \mid X_n = i) = \mathbb{P}(X_1 = j \mid X_0 = i),$$

for all $n \geq 0$ and $i, j \in S$.

Markov property roughly says that, given a history of the process behaviour, the probability distribution of the process in the future only depends on the most recent part of the history. Sometimes this property is also known as the *memoryless* property, since it conveys the fact that all memories possessed by the process in the past are no longer relevant to determine its upcoming performance.

On the other hand, the second property says that the process evolution does not depend on the time taken so far. It is worth remarking that many other literature exclude the homogeneity property in their definition of a general Markov chains. Instead, they call the one possessing the property the homogeneous Markov chains. However, we put the property in our definition since all Markov chains discussed in this thesis possess the property.

The homogeneity property leads us to define the *transition probabilities*, which are

$$p_{i,j} = \mathbb{P}(X_{n+1} = j \mid X_n = i),$$

for all $n \geq 0$. Note that unless the homogeneity property is satisfied, the transition probabilities are not well-defined. Moreover we call the $|S| \times |S|$ matrix $\mathbf{P} = [p_{i,j}]_{i,j \in S}$ the *transition matrix*. Note that $\mathbf{P}$ is a stochastic matrix, which means that all of its entries are nonnegative and the sum of each row is 1. The discrete-time Markov chains are completely determined by its transition matrices.

On the other hand, the continuous-time Markov chain is described as follows.

Definition 2.3 (Continuous-time Markov Chains)**.** *A continuous-time Markov chain* $\{X(t) : t \geq 0\}$ *is a stochastic process indexed by the real half-line* $[0, \infty)$ *and taking values in some countable state space* $\mathcal{S}$ *that satisfies the following properties.*

1. The Markov property*.*

$$\mathbb{P}(X(t_n) = j \mid X(t_1), \ldots, X(t_{n-1})) = \mathbb{P}(X(t_n) = j \mid X(t_{n-1})), \tag{2.1}$$

for any $j \in \mathcal{S}, n \geq 1,$ *and any sequence* $t_1 < t_2 < \cdots < t_n$ *of times.*

2. The Homogeneity property*.*

$$\mathbb{P}(X(t+s) = j \mid X(s) = i) = \mathbb{P}(X(t) = j \mid X(0) = i),$$

for all $0 \leq s \leq t$ *and* $i, j \in \mathcal{S}$*.*

Both properties possessed by the continuous-time Markov chains are the continuous analogues of the discrete version. In particular, note that the Poisson process satisfies the Markov property as described by property 2 of Definition 2.1. Also, a simple application of property 2 and 3 of Definition 2.1 implies that the Poisson process satisfies the Homogeneity property. Thus, we can think of the continuous-time Markov Chain as a generalisation of the Poisson process.

The main difference of these two versions of Markov chains is obviously the time indices of the processes. While the discrete Markov chains can only experience state transitions on given discrete time slots, the state transitions of the continuous version can occur at arbitrary times. However, unlike the discrete version, it is irrelevant to say that a continuous Markov chain performs a state transition to the same state at a given time. In the discrete version, at any given round n and current state i, the process has a chance to stay in state i at the next step as long as $p_{i,i} > 0$. In contrast, when a continuous Markov chain stays at the same state at a given period, it does not perform any transition. To explain this more precisely, we introduce the notion of the generator matrix.

A continuous-time Markov chain $\{X(t) : t \geq 0\}$ is completely determined by its so-called *generator matrix* $\mathbf{Q} = [q_{i,j}]_{i,j \in \mathcal{S}}$ satisfying the following properties:

1. $q_{i,j} \geq 0$ with $\sum\limits_{j \neq i} q_{i,j} > 0$ for all $i \neq j$, and
2. $q_{i,i} = -\sum\limits_{j \neq i} q_{i,j}$ for all $i \in \mathcal{S}$.

For every distinct $i, j \in \mathcal{S}$, we define $p^*_{i,j} = -q_{i,j}/q_{i,i}$. Then, in regards to $\mathbf{Q}$, we can understand the behaviour of X as described by the following rules.

Rules 2.4 ([42] in Definition 4.1.1)**.** *For all $i \in \mathcal{S}$ and $t \geq 0$,*

1. *If $X(t) = i$, then it will stay at state i for an exponentially distributed time with mean $-1/q_{i,i}$.*
2. *If the process leaves state i at time t, it will enter state $j \neq i$ with probability $p^*_{i,j}$.*

Suppose that the process is currently in state i. Then for all $j \in \mathcal{S} - \{i\}$, we call $q_{i,j}$ the *transition rate* from i to j. Equivalently, we say that the process transitions from i to j with rate $q_{i,j}$. On the other hand, we call $p^*_{i,j}$ the *instantaneous transition probability* from i to j.

Alternatively, we can also view the process X in terms of Poisson clocks. Suppose that X is currently at state i. For all $j \in \mathcal{S}$ for which $q_{i,j} > 0$, we define independent Poisson clocks C_j with rate $q_{i,j}$. Then the process will stay at state i until one of these clocks rings. Suppose that the first ringing clock is $C_{j'}$ for some $j' \in \mathcal{S}/\{i\}$. Then, the ringing instructs X to leave state i and enter state j'. By this setting, we have that the time spent by X in state i is equal to the waiting time of the first ringing of the Poisson clocks C_j's. By the superposition property, the waiting time has an exponential distribution whose rate is the sum of the clocks rate, that is $-q_{i,i}$. On the other hand, the probability that the process moves to state j' in the next transition, is $-q_{i,j'}/q_{i,i}$, which is basically $p^*_{i,j}$. This establishes the equivalence of the Poisson clocks point of view to the generator matrix point of view.

We can also construct a discrete-time Markov chain from a continuous-time Markov chain, that will be called the *embedded Markov chain.*

Definition 2.5 (Embedded Markov Chains)**.** *Let $\{X(t) : t \geq 0\}$ be a continuous-time Markov chain and τ_j denote the time at which X experiences the j-th state transition. Suppose that τ_j^+ is an instantaneous time right after the j-th state transition. Then, the* embedded Markov chain *of X is the discrete-time Markov chain $\{X_n^* : X_n^* = X(\tau_n^+), n = 0, 1, \dots\}$.*

Here, the embedded Markov chain only captures the information regarding the history of the state jumpings that X experiences, without taking note of the exact times of the state transitions. Suppose that the transition matrix of X^* is $\mathbf{P}^*$. Then for all $i, j \in \mathcal{S}$, one can check that the (i, j) entry of $\mathbf{P}^*$ is $p_{i,j}^*$, the instantaneous transition probability of X.

2.2 Stochastic Orderings

In this thesis, we will often need to bound the expectation of many random variables. One of the most convenient ways to do this is to find their stochastic orderings to the other random variables.

The notion of *stochastic orderings* is mainly used to describe a condition when a certain random variable is 'typically greater' than another random variable. The precise definition is given as follows.

Definition 2.6 (Stochastic Orderings). *Suppose that X and Y are real-valued random variables, not necessarily lying on the same probability space. We say that X is* stochastically smaller *than Y or equivalently, X is bounded by Y from above stochastically, if for every $t \in \mathbb{R}$,*

$$\mathbb{P}(X > t) \le \mathbb{P}(Y > t).$$

We denote this by $X \le_{ST} Y$, or equivalently $Y \ge_{ST} X$.

One way to show that two random variables satisfy a stochastic ordering relationship is to construct a so-called coupling of them. First, we give the definition of a *coupling* of random variables.

Definition 2.7 (Coupling). *Suppose that $X_1, \dots, X_n$ are any real-valued random variables. We say that $(\hat{X}_1, \dots, \hat{X}_n)$ is a coupling of $X_1, \dots, X_n$ if for all $i = 1, \dots, n$, the marginal distribution of $\hat{X}_i$ is the same as X_i. In other words, for any measurable set A of $\mathbb{R}$,*

$$\mathbb{P}(\hat{X}_i \in A) = \mathbb{P}(X_i \in A) \qquad i = 1, \dots, n.$$

Note that the random variables $X_1, \dots, X_n$ possibly lie on various unrelated probability spaces. However, once they are coupled, they now lie on the same probability space, appearing as a random vector. This marks the critical point of the notion of coupling.

The incredibly simple definition of couplings will lead us to a very useful lemma regarding stochastic ordering, described as follows.

Lemma 2.8. *For any real-valued random variables X and Y, we have that $X \leq_{ST} Y$ if and only if there exists a coupling $(\hat{X}, \hat{Y})$ of X and Y such that*

$$\mathbb{P}(\hat{X} \leq \hat{Y}) = 1.$$

The proof of the lemma is adapted from a work by van der Hofstad [43].

Proof. Suppose that $\mathbb{P}(\hat{X} \leq \hat{Y}) = 1$ for some coupling $(\hat{X}, \hat{Y})$ of X and Y. Hence, for all $t \in \mathbb{R}$,

$$\mathbb{P}(X > t) = \mathbb{P}(\hat{X} > t) = \mathbb{P}(\hat{Y} \geq \hat{X} > t) \leq \mathbb{P}(\hat{Y} > t) = \mathbb{P}(Y > t).$$

Thus, $X \leq_{ST} Y$.

Now suppose that $X \leq_{ST} Y$. First we define the notion of the generalised inverse of a distribution function. For any real-valued random variable Z, let F_Z be the distribution function of Z. Then we define

$$F_Z^{-1}(u) = \inf\{x \in \mathbb{R} : F_Z(x) \geq u\},$$

where $0 \leq u \leq 1$. Note that by the definition, $F_Z^{-1}(u) > x$ if and only if $F_Z(x) < u$. Suppose that U is a uniform random variable on unit interval. Then for any $u \in [0, 1]$,

$$\mathbb{P}(F_Z^{-1}(U) \leq u) = \mathbb{P}(U \leq F_Z(u)) = F_Z(u). \tag{2.2}$$

This means that $F_Z^{-1}(U)$ has distribution function F_Z.

Now we construct a coupling $(\hat{X}, \hat{Y})$ of X and Y. Suppose that F_X and F_Y are the distribution functions of X and Y, respectively. We specify that $\hat{X} = F_X^{-1}(U)$ and $\hat{Y} = F_Y^{-1}(U)$. Notice that by (2.2), $\hat{X}$ and $\hat{Y}$ have the same distribution function as X and Y respectively. Hence, $\hat{X} \overset{d}{=} X$ and $\hat{Y} \overset{d}{=} Y$.

On the other hand, $X \leq_{ST} Y$ is equivalent to $F_X(t) \geq F_Y(t)$ for all $t \in \mathbb{R}$. Observe that we can rewrite this into $F_X^{-1}(u) \leq F_Y^{-1}(u)$ for all $0 \leq u \leq 1$. Hence, we obtain that

$$\mathbb{P}(\hat{X} \leq \hat{Y}) = \mathbb{P}(F_X^{-1}(U) \leq F_Y^{-1}(U)) = 1,$$

and we are done. □

The lemma has a very useful corollary that provides a bound on the expected value of random variables.

Corollary 2.9. *Suppose that $X \leq_{ST} Y$. Then*

$$\mathbb{E}X \leq \mathbb{E}Y.$$

Proof. By Lemma 2.8, there exists a coupling $(\hat{X}, \hat{Y})$ of X and Y such that $X \leq Y$. Then

$$\mathbb{E}X = \mathbb{E}\hat{X} \leq \mathbb{E}\hat{Y} = \mathbb{E}Y. \qquad \square$$

This corollary will be a fundamental tool in the upcoming chapters to bound the expected value of a particular random variable by the expected value of another entirely unrelated random variable.

2.3 Sharp Concentration Inequalities

Throughout the entire thesis, we will often apply the concentration inequality of sums of independent random variables. In this section, we will review some well known results on large deviation probability bounds for sums of independent random variables. These include the prominent Chernoff's bound, which is well known for the sums of random variables having elementary distributions, as well as its generalisations adapted to our settings.

In general, an outcome of a random variable usually deviates relatively little from its expected value. Inequalities which express such phenomenon are usually called the concentration inequalities. There are many existing concentration inequalities, for example, the classical Markov's inequality and Chebyshev's inequality.

When the observed random variable is a sum of other independent random variables, an extremely powerful concentration inequality exists, namely Chernoff's bound. This bound substantially outperforms both Markov's and Chebyshev's inequalities in this specific context. This is due to the exponentially small bound yielded by the Chernoff's bound. Suppose that X is a random variable of the form $X = \sum_{i=1}^{n} X_i$ where the X_i's are independent. Let ε be any positive real constant. Then, the Chernoff's bound shows that both $\mathbb{P}(X > (1+\varepsilon)\mathbb{E}X)$ and $\mathbb{P}(X < (1-\varepsilon)\mathbb{E}X)$ decrease exponentially as ε grows.

The main technique used in the Chernoff's bound is the application of Markov's inequality to the random variable e^{tX} for some appropriate constant t. Note that $\mathbb{E}e^{tX}$ is the moment generating function of X. Here, the independence of the summands comes into practise, since the moment generation function of a sum of independent random variables is equal to the product of the moment generating function of each summand. For all $t > 0$, we have that

$$\begin{aligned}\mathbb{P}(X > (1+\varepsilon)\mathbb{E}X) &= \mathbb{P}\left(e^{tX} > e^{t(1+\varepsilon)\mathbb{E}X}\right) \\ &\le \frac{\mathbb{E}e^{tX}}{\exp(t(1+\varepsilon)\mathbb{E}X)} \\ &= \frac{\prod_{i=1}^{n} \mathbb{E}e^{tX_i}}{\exp(t(1+\varepsilon)\mathbb{E}X)},\end{aligned} \tag{2.3}$$

where the inequality above comes from Markov's inequality and the last equation comes from the independence of the summands.

The analogous version of (2.3) for $\mathbb{P}(X < (1-\varepsilon)\mathbb{E}X)$ can be achieved by taking $t < 0$. Then we have that

$$\mathbb{P}(X < (1-\varepsilon)\mathbb{E}X) \le \frac{\prod_{i=1}^{n} \mathbb{E}e^{tX_i}}{\exp(t(1-\varepsilon)\mathbb{E}X)}, \qquad t < 0.$$

For the final step, we particularly choose the constant t to optimise the result.

The most notable result on this technique is probably the sharp concentration inequality for binomial random variables. Suppose that $X \stackrel{d}{=} \mathcal{B}(n,p)$ for some integer n and $0 \le p \le 1$. Note that we can express $X = \sum_{i=1}^{n} X_i$ where (X_i) are identical and independent Bernoulli random variables with success probability p. We present the following lemma.

Lemma 2.10 (Sums of Bernoulli Trials). *Suppose that* $X \stackrel{d}{=} \mathcal{B}(n,p)$ *and* $\phi(x) = (1+x)\log x - x$. *Then*

$$\mathbb{P}\left(|X - \mathbb{E}X| \ge \varepsilon \mathbb{E}X\right) \le 2\exp(-\phi(\varepsilon)\mathbb{E}X). \tag{2.4}$$

Moreover, if $\varepsilon \in (0, \frac{3}{2})$, *then*

$$\mathbb{P}\left(|X - \mathbb{E}X| \ge \varepsilon \mathbb{E}X\right) \le 2\exp\left(-\frac{\varepsilon^2 \mathbb{E}X}{3}\right). \tag{2.5}$$

The proof written below is adapted from a work of Janson, Łuczak, and Rucinski [32].

Proof. First we calculate the moment generating function of X_i and X. Observe that for all $t > 0$,

$$\mathbb{E}e^{tX_i} = 1 + p(e^t - 1),$$

and

$$\mathbb{E}e^{tX} = \mathbb{E}e^{\left(t\sum_{i=1}^{n} X_i\right)} = \prod_{i=1}^{n} \mathbb{E}e^{tX_i} = (1 + p(e^t-1))^n \le \exp(\mathbb{E}X(e^t - 1)),$$

where the second equation comes from the independence of the summands.

Now, from (2.3), we have that

$$\mathbb{P}(X > (1+\varepsilon)\mathbb{E}X) \le \frac{\mathbb{E}e^{tX}}{e^{t(1+\varepsilon)\mathbb{E}X}} \le \exp\{\mathbb{E}X(e^t - 1 - t(1+\varepsilon))\}. \tag{2.6}$$

Then we choose $t = \log(1+\varepsilon)$, which is optimal in (2.6). We write

$$\mathbb{P}(X > (1+\varepsilon)\mathbb{E}X) \le \exp(-\phi(\varepsilon)\mathbb{E}X).$$

Moreover, by performing analogous steps with $t < 0$, we obtain

$$\mathbb{P}(X < (1-\varepsilon)\mathbb{E}X) \le \exp(-\phi(\varepsilon)\mathbb{E}X).$$

This results in (2.4). Furthermore, when $0 < \varepsilon < \frac{3}{2}$, we have that $\phi(x) \ge \frac{x^2}{3}$ and thus justifies (2.5). $\square$

Now we describe the concentration inequality for sums of independent exponential random variables. Define $W = \sum_{i=1}^n W_i$ where $W_i \overset{d}{=} \mathcal{E}(r_i)$ and is independent of each other with $r_i > 0$ for all i. Define $r^* := \min_{1\le i\le n} r_i$. Then we have the following lemma.

Lemma 2.11 (Sums of Exponentials). *For any $r_1, \dots, r_n > 0$ and any $0 < u \le 1 \le t$, we have that*

$$\mathbb{P}(W \ge t\mathbb{E}W) \le e^{-r^*\mathbb{E}W(t-1-\log t)}, \tag{2.7}$$

and

$$\mathbb{P}(W \le u\mathbb{E}W) \le e^{-r^*\mathbb{E}W(u-1-\log u)}. \tag{2.8}$$

The following proof uses the same procedure as the proof of Theorem 2.1 in Janson's unrefereed technical report [31].

Proof. We only show (2.7) in this proof since (2.8) can be derived in an analogous way. Suppose that $t \ge 1$. First, we calculate the moment generating function $\mathbb{E}e^{vW}$ as follows. If $0 \le v \le r^*$, we have that

$$\mathbb{E}e^{vW_i} = \frac{r_i}{r_i - v} = \left(1 - \frac{v}{r_i}\right)^{-1}.$$

It follows that

$$\mathbb{E}e^{vW} = \prod_{i=1}^n \left(1 - \frac{v}{r_i}\right)^{-1}.$$

From (2.3), we have that

$$\mathbb{P}(W \ge t\mathbb{E}W) \le e^{-tv\mathbb{E}W}\mathbb{E}e^{vW} \le \exp\left(-tv\mathbb{E}W - \sum_{i=1}^n \log\left(1 - \frac{v}{r_i}\right)\right). \tag{2.9}$$

On the other hand, consider the function $\phi(x) = -\log(1-x)/x$ on the domain $(0,1)$. Observe that $\phi(x)$ is an increasing function. This implies that for all $0 < x \le y < 1$,

$$-\log(1-x) \le -\frac{x}{y}\log(1-y). \tag{2.10}$$

Now by taking $x = \frac{v}{r_i}$ and $y = \frac{v}{r^*}$ and putting them into (2.9) and (2.10), we obtain

$$\begin{aligned}\mathbb{P}(W \geq t\mathbb{E}W) &\leq \exp\left(-tv\mathbb{E}W - \log\left(1 - \frac{v}{r^*}\right)\sum_{i=1}^{n}\frac{r^*}{r_i}\right) \\ &= \exp\left(-tv\mathbb{E}W - r^*\mathbb{E}W\log\left(1 - \frac{v}{r^*}\right)\right)\end{aligned}$$

By choosing $v = r^*(1 - t^{-1})$, we get the desired result. $\square$

The same concentration inequality also applies for sums of geometric random variables. Suppose $Y = \sum_{i=1}^{n} Y_i$ where for each $i \in [n]$, Y_i are independent geometric random variables with possibly different parameters. Suppose for all $i \in [1, n]$, $Y_i \overset{d}{=} \mathcal{G}(p_i)$. We define $p^* := \min_{1 \leq i \leq n} p_i$. Then, we have the following lemma.

Lemma 2.12 (Sums of Geometrics)**.** *For any $p_1, \ldots, p_n \in (0, 1)$ and any $0 < u \leq 1 \leq t$, we have that*

$$\mathbb{P}(Y \geq t\mathbb{E}Y) \leq e^{-p^*\mathbb{E}Y(t-1-\log t)},$$

and

$$\mathbb{P}(Y \leq u\mathbb{E}Y) \leq e^{-p^*\mathbb{E}Y(u-1-\log u)}.$$

We omit the proof of the lemma since we can prove it by adapting the steps of the proof of Lemma 2.11.

In the case when the Y_i's are identical for all i, we can obtain a better bound. Let $Y = \sum_{i=1}^{n} Y_i$ where for each $i \in [n]$, we have that $Y_i \overset{d}{=} \mathcal{G}(p)$ for some identical success probability $p \in (0, 1)$. In fact, Y has a well known distribution, which is the negative binomial. It is usually understood as the number of trials needed to achieve the n-th success in a series of Bernoulli trials with a given success probability. The following lemma gives an upper tail concentration inequality.

Lemma 2.13 (Sums of Identical Geometrics, Theorem 1 of [11])**.** *Let $\{X_i\}_{i \in [1,m]}$ be a set of identical and independent geometric random variables with success probability p. Let $X = \sum_{i=1}^{m} X_i$. Then for all $t > 1$,*

$$\mathbb{P}\left(X > tm\frac{1-p}{p}\right) \leq \exp\left(-\left(1 - \frac{1}{t(1-p)}\right)^2 \frac{mt(1-p)}{3}\right).$$

Proof. Consider a biased coin whose tail probability is p and we observe the sequence of the biased coin tossing. We can view X_i as the number of heads needed to get the first tail in the sequence. On the other hand, X is the number of heads appearing in the sequence until we get the m-th tail. Hence, the event $X > tm\frac{1-p}{p}$ implies that in the sequence of $tm\frac{1-p}{p}$ tossing, there are less than m tails. Suppose that Y is the number of tails in the outcome of the first $tm\frac{1-p}{p}$ coin flippings. Note that Y is a binomial random variable whose mean is $tm(1-p)$. By Lemma 2.10, we have that for all positive real $\varepsilon \in (0,1)$,

$$\mathbb{P}(Y < (1-\varepsilon)\mathbb{E}Y) \leq \exp\left(-\frac{\epsilon^2 \mathbb{E}Y}{3}\right).$$

Now by choosing $\varepsilon = 1 - \frac{1}{t(1-p)}$, we obtain

$$\begin{aligned}\mathbb{P}\left(X > tm\frac{1-p}{p}\right) &= \mathbb{P}(Y < m) \\ &\leq \exp\left(-\left(1 - \frac{1}{t(1-p)}\right)^2 \frac{mt(1-p)}{3}\right).\end{aligned}$$ □

Chapter 3

Mathematical Models and Some Examples

In this chapter, we present the precise mathematical models of the rumour spreading schemes and demonstrate some examples of how they behave in two families of elementary graphs.

3.1 Mathematical Models

In this section, we present the models of two rumour spreading processes: *Standard Rumour Spreading* (SRS) process and *Rumour Spreading with a Delaying Scheme* (RSDS) process. The former is the standard process with no delaying scheme while the latter is the model for our primary rumour spreading scheme that we study. In the entire discussion of the thesis, we will always compare the performance of the RSDS to the SRS in order to see how far the delaying scheme affects the spreading time. In this section, we will always denote $G = (V, E)$ as a simple and connected graph with n vertices.

First, we describe the general model of *Standard Rumour Spreading* (SRS) process in G. As briefly mentioned in Chapter 1, in the SRS process, we will employ the push & pull scheme to spread the rumour and use the asynchronous version of the spreading times. In the scheme, the rumour passes through uniformly random neighbour callings that occur with a certain rate.

In the SRS process, there are two possible statuses of the vertices of the graph. We call a vertex *informed* if it already holds the rumour and *uninformed* otherwise. The precise mathematical process is described as follows.

Definition 3.1 (SRS Model)**.** SRS model *in graph G with* initial rumour spreader $v \in V$

is a continuous time Markov Chain $\{X'_G(t) : t \geq 0\}$ *whose state space is* 2^V *with the following properties. The initial state is deterministically* $X'_G(0) = \{v\}$*. For every vertex in* G*, we assign to it a Poisson clock, which we will call the* spreading clocks*, with rate 1 independently of each other. Suppose that for all* $x \in V$*, the spreading clock of* x *rings at time* t *for some* $t \geq 0$*. At time* t*,* x *chooses a random neighbour, say* $y \in V$*. If* $x \in X'_G(t)$ *and* $y \notin X'_G(t)$*, then the ringing of the clock instructs* X'_G *to move to state* $X'_G \cup \{y\}$*. On the other hand, if* $x \notin X'_G(t)$ *and* $y \in X'_G(t)$*, then* X'_G *moves state* $X'_G \cup \{x\}$ *at that time. The process halts when* $X'_G = V(G)$*.*

We provide an interpretation of the definition above as follows. The current state of the process represents the set of vertices that are already informed at that time. The initial state portrays the fact that there is only one vertex knowing the rumour initially. The ringing of the spreading clock associated with x marks the time at which an information exchange occurs between x and a random neighbour y. In the first case, that is when $x \in X'_G(t)$ and $y \notin X'_G(t)$, we can understand that the informed vertex x pushes the rumour to y. On the other hand, when $x \notin X'_G(t)$ and $y \in X'_G(t)$, the uninformed vertex x pulls the information from y.

In addition, we remark that some clock ringings can have no effect in the rumour spreading under certain conditions. Suppose that at a given time $t \geq 0$, x calls y where x and y is in $X'_G(t)$. In this case, the process stays at its current state and experiences no state transition. This portrays an event where two informed vertices exchange information at which the rumour spreading is not progressing. The same condition also applies when both x and y are not in $X'_G(t)$, that is an event where two uninformed vertices communicate.

Now we describe the model of *Rumour Spreading with a Delaying Scheme* (RSDS) in a connected graph G. Unlike the SRS process, each vertex has a status which has three possible values: *uninformed*, *dormant* or *active* in this model. We define a vertex to be *informed* if it is either active or dormant. In other words, the vertices that know the rumour can be in two different situations. These two situations will correspond to their ability to participate in an information exchange.

Similar to the SRS process, the rumour passing method in the RSDS process also applies the push & pull scheme. However, an additional restriction is applied in this model. In order to either push or pull a rumour, the informed vertex involved in the communication needs to be active. In other words, a dormant vertex knows the rumour already but is unable to pass the rumour to other vertices. We describe the precise model as follows.

Definition 3.2 (RSDS Model)**.** *Let* $\mathcal{S}$ *be the collection of all partitions of* V *into three classes* $(\mathcal{A}, \mathcal{D}, \mathcal{U})$*. RSDS model in graph* G *with* initial rumour spreader $v \in V$*, is a continuous time Markov Chain* $\{X_G(t) : t \geq 0\}$ *whose state space is* $\mathcal{S}$ *with the following* *properties. Initially we have that* $X_G(0) = (\{v\}, \emptyset, V - \{v\})$ *deterministically. For each vertex in* G*, we assign to it two Poisson clocks, which we will call the* spreading clocks *and the* switching clocks*. We specify that all spreading clocks have rate 1 whereas the rate of the switching clocks is* $s(n)$*. These clocks are independent of each other. Suppose that for some time* $t \geq 0$*, we have that* $X_G(t) = (A, D, U)$ *for some vertex set partition* (A, D, U)*. When a switching clock assigned to a vertex* $a \in A$ *rings at time* t*, it instructs* X_G *to transition to state* $(A - \{a\}, D \cup \{a\}, U)$*. Analogously, a ringing of a switching clock assigned to a vertex* $d \in D$ *at time* t *indicates the transition to* $(A \cup \{d\}, D - \{d\}, U)$*. Now for any* $x \in V$*, suppose that a spreading clock of* x *rings. Then* x *chooses a uniformly random neighbour, say* $y \in V$*. If* $x \in A$ *and* $y \in U$*, then* X_G *transitions to state* $(A \cup \{y\}, D, U - \{y\})$*. Also, if* $x \in U$ *and* $y \in A$*, then* X_G *moves to* $(A \cup \{x\}, D, U - \{x\})$*. The process stops when it arrives at any state of the form* $(A, D, \emptyset)$*.*

The state of the RSDS process described above represents the statuses of the vertices. Suppose that $X_G(t) = (A, D, U)$ for some $t \geq 0$ and (A, D, U) a partition of V. Then A, D, and U denote the current set of active, dormant, and uninformed vertices respectively. The initial state of the process tells us that there is only one active vertex in the beginning of the RSDS process.

We describe the interpretation of the ringings of the clocks as follows. When a switching clock of an informed (either active or dormant) vertex rings, it immediately flips its status, from dormant to active or vice versa. We remark that the ringing of a switching clock associated to an uninformed vertex has no relevance to the rumour spreading process. On the other hand, the spreading clocks work almost in the same way as that of the SRS process, except that we ignore all random callings that involve any dormant vertices.

In particular, we call the rate of the switching clocks $s(n)$, the *switching rate*. We specify that the switching rate is a function of n, the order of the graph. The switching rate will be the central interest of the upcoming chapters in this thesis. We will study how different choices of the switching rate can substantially affect the running time of the rumour spreading.

On the other hand, when a rumour spreading process takes a non unit spreading rate, there is a simple transformation of the process into another process with a unit spreading rate and a shifted switching rate. For this reason, as briefly mentioned in Chapter 1, we always assume that the rate of the spreading clocks is 1 for the sake of the easiness of the analysis.

Later on, when the context of the underlying graph G is clearly understood, we will address the SRS and RSDS process simply by X' and X respectively, instead of X'_G and X_G.

There are other perspectives to understand the SRS and RSDS models. Before we discuss the perspectives more precisely, we provide the following observation. We say that two adjacent vertices x and y (without paying attention to their statuses) *exchange information* if either the spreading clock of x rings and x calls y, or the spreading clock of y rings and y calls x.

Observation 3.3. *Let x and y be two adjacent vertices in G and C be a clock that rings every time x exchanges information with y. Then C is a Poisson clock with rate $1/\deg(x) + 1/\deg(y)$.*

Note that the observation above is a direct application of the thinning and superposition properties of Poisson clocks. When the spreading clock of x rings, x calls y with probability $\deg(x)^{-1}$ since the random neighbour is uniformly chosen. Then, x informs y with rate $\deg(x)^{-1}$ by the thinning property. By symmetry, we also have that y calls x with rate $\deg(y)^{-1}$. Now, by the superposition property, we have that x and y communicate with rate $\deg(x)^{-1} + \deg(y)^{-1}$.

Now we restate the SRS and RSDS models with another point of view. We will call the new models the *edge clock models*. In these models, we put the spreading clocks on the edges. For each edge xy in G, we associate a spreading clock with rate $1/\deg(x)+1/\deg(y)$, independently of each other. We specify that whenever the clock of xy rings, x and y exchange information. On the other hand, we still apply the same switching clock rules to the edge clock RSDS model. We call such spreading clocks the *edge spreading clocks*. Then, Observation 3.3 implies that the edge clock models are exactly the same as the original rumour spreading models, for both SRS and RSDS processes.

In the light of this notion, we will sometimes call the original SRS and RSDS models in Definition 3.1 and Definition 3.2, the *vertex clock models* of the SRS and RSDS respectively, since their spreading clocks are associated with the vertices of the graph. Also, we will also call their spreading clocks the *vertex spreading clocks*.

A particularly interesting case is when G is a regular graph. Suppose that d is the degree of the vertices in a regular graph G. Then the edge clock rumour spreading models possess a nice property. In these models, all spreading clocks have the same rate, that is $2/d$. This leads to a very handy analysis of rumour spreading when we analyse the edge clock version of the models since we can disregard the random neighbour callings scheme. We will utilise it in the case of complete graphs (which is also a regular graph) in the upcoming chapters.

The rumour spreading edge clock models in regular graphs are equivalent to the infection models, up to some normalisation. Recall that the rumour spreading model differs from the infection model primarily in that the rate of the spreading occurring in a particular edge depends on the degree of its ends vertices. However, when all vertices of the graph have the same degree, the distinguishing factor between these two models simply vanishes.

In particular, when the underlying graph is regular, the edge clock SRS model is equivalent to the SI infection model. For a more detailed definition of such a process, one can consult [28]. In the SI model, a disease spreads from a certain infected vertex to its susceptible neighbour with a constant rate. These infected vertices have no chance to be susceptible again. This leads the scheme to behave exactly the same as the SRS. Then, the time when every individual in the network receives the infection corresponds to the running time of the SRS process.

On the other hand, the edge clock RSDS model has many correspondences with the SIR infection model. To find a formal definition of the SIR model in networks, an interested reader can take [9] and [34] as references. The uninformed, active, and dormant vertices of the RSDS model correspond to the susceptible, infected, and recovered vertices of the SIR model respectively. Similarly to the SI model, an infection propagates from the infected individuals to the susceptible ones with a constant rate for each adjacent pair of them. However, each infected individual has a rate to be recovered. In the SIR model, once a vertex is recovered, it is no longer able to suffer from the disease. This marks the substantial difference between the SIR and RSDS models. Suppose that the SIR model is modified such that a recovered vertex is considered to have a more fragile health than the susceptible vertices. These recovered vertices have a positive rate to get infected again without having a contact with any infected vertices due to their history of infection. However, we specify that when a vertex is in a recovered state, it cannot infect its susceptible neighbours. Then this modified SIR is equivalent to the RSDS model when the graph is regular.

We define the following terms. Let X be a rumour spreading process (either SRS or RSDS) on graph G. We say that X is in *stage* i if there are i informed vertices at that time. For $i = 1, \ldots, n$, we introduce the monotone subsets sequence $I_1(X) \subseteq I_2(X) \subseteq \cdots \subseteq I_n(X) = V(G)$ where $I_i(X)$ is the set of all informed vertices during stage i. For convenience, we define $I_0(X) = \emptyset$. Analogously, we also introduce the monotone subset sequence $(U_i(X))_{0 \le i \le n}$ where $U_i(X) := V(G) - I_i(X)$. We call $I_i(X)$ and $U_i(X)$ the *informed* and *uninformed* sets of stage i respectively.

We also define the *potential set* of stage i, denoted by $P_i(X)$, as

$$P_i(X) = \{w \in U_i : xw \in E(G) \text{ for some } x \in I_i\}.$$

In other words, $P_i(X)$ contains all uninformed vertices during stage i that are potential to be informed at the next stage. We call elements in a potential set the *potential* vertices.

Similarly, we define the *effective set* of stage i, denoted by $F_i(X)$, as

$$F_i(X) = \{x \in I_i : xw \in E(G) \text{ for some } w \in U_i\}.$$

The vertices in an effective set are called *the effective* vertices. Only effective vertices have a significant role in sending the rumour to a new vertex, since all neighbours of a non-effective vertex are informed already. We call a vertex spreading clock associated with an effective vertex the *vertex effective clock*. Similarly, we call an edge spreading clock of an edge whose one of the ends is an effective vertex the *edge effective clock*.

In the context of the RSDS process, we call a vertex *effectual* if it is both effective and active. Also, we say that a vertex effective clock is *effectual* if the associated effective vertex is effectual as well as an edge effective clock of an edge is *effectual* if the associated edge is incident to an effectual vertex.

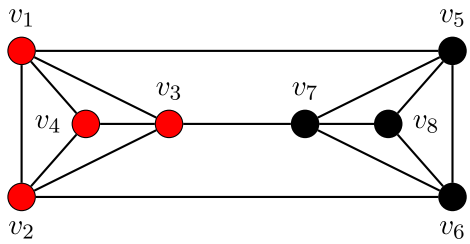

Figure 3.1: Rumour spreading in the graph G

To exhibit these notions better, consider a rumour spreading process X occurring in graph G as illustrated in Figure 3.1. Suppose at a given time, we have that the current informed and uninformed vertices of G are respectively shown by the red and black vertices at the figure above. Notice that X is now in stage 4 since we have 4 red vertices in the graph. Hence,

$$\begin{aligned}
I_4(X) &= \{v_1, v_2, v_3, v_4\},\\
U_4(X) &= \{v_5, v_6, v_7, v_8\},\\
F_4(X) &= \{v_1, v_2, v_3\},\\
P_4(X) &= \{v_5, v_6, v_7\}.
\end{aligned}$$

Now, since v_4 is not effective, it can no longer spread the rumour to any new vertex. It is obvious that the spreading clock associated to v_4 has no effect for the forthcoming spreading process. On the other hand, since v_8 is non-potential, it cannot be an informed vertex in stage 5.

Next, we define the *captured vertex* of stage i, denoted by w_i, as the only vertex in the set $I_i(X) - I_{i-1}(X)$. In other words, the captured vertex w_i is the i-th informed vertex during the whole spreading process. Also, we write that $I_i = \{w_1, \ldots, w_i\}$ for all $1 \le i \le n-1$.

The following terms apply when X is an RSDS process. We define $\mathcal{A}(X,t)$ and $\mathcal{D}(X,t)$ as the set of active and dormant vertices in X at time t respectively. Observe that if X is in stage i at time t, then $\mathcal{A}(X,t)$ and $\mathcal{D}(X,t)$ form a partition of $I_i(X)$. Unlike $I_i(X)$, the random subset $\mathcal{A}(X,t)$ and $\mathcal{D}(X,t)$ are changing during the entire stage i as the informed vertices are switching. Let $\mathrm{Do}(X,t)$ and $\mathrm{In}(X,t)$ be the number of dormant and informed vertices of X at time t respectively.

For $i \in [1, n-1]$, define

$$V_i(X) = \inf\{t \geq 0 : \mathrm{In}(X,t) = i\},$$

that is the time at which X enters stage i for the first time. For $i \in [1, n-1]$, define

$$T_i(X) = V_{i+1}(X) - V_i(X),$$

that is the duration of time spent by X during stage i. Finally, define

$$T(X) = V_n = \sum_{i=1}^{n-1} T_i(X),$$

that is, the time needed for the rumour spreads until every vertex learns the rumour. We will also call $T(X)$ the *running time* of X.

When the process X is clearly understood from the context, we sometimes simply write I_i, U_i, P_i, F_i ,T_i, V_i, and T instead of $I_i(X), U_i(X), P_i(X), F_i(X), T_i(X), V_i(X)$, and $T(X)$.

It is worth describing some basic observations on the nature of both SRS and RSDS. Most of the observations are based on the basic properties of random processes described in Chapter 2. For this reason, some of the observations only require some brief explanation instead of detailed proofs.

The following observations describe some phenomena occurring in an SRS process. Suppose that X' denotes an SRS process in a connected graph G.

Observation 3.4 (Lemma 1 of [40]). *For all $i = 1, \ldots, n-1$, conditional on $I_i(X')$, we have that $T_i(X')$ has an exponential distribution with rate*

$$r_i := \sum_{u \in F_i} \frac{e(\{u\}, P_i)}{\deg(u)} + \sum_{w \in P_i} \frac{e(\{w\}, F_i)}{\deg(v)}. \tag{3.1}$$

Moreover, we have that $T_i(X')$ is independent of $T_1(X'), \ldots, T_{i-1}(X')$.

The observation above is a result of an appropriate application of the superposition and thinning properties. Recall that by Observation 3.3, for a particular pair of adjacent vertices u and w, where $u \in I_i$ and $w \in U_i$, we have that u and w exchange information with rate $1/\deg(u) + 1/\deg(v)$. Recall that $T_i(X')$ is the waiting time for the first rumour passing among all possible pairs of $u \in F_i$ and $w \in P_i$. It follows that by the superposition property, the waiting time is exponentially distributed with rate

$$\sum_{u \in F_i} \sum_{w \in P_i} \left(\frac{1}{\deg(u)} + \frac{1}{\deg(w)} \right) \mathbf{1}\{uw \in E\}$$

which is essentially the same as (3.1).

The next observation describes the probability distribution of the captured vertex of stage i.

Observation 3.5. *For all $i = 1, \ldots, n-1$, conditional on $I_i(X')$, suppose that for any potential vertex $w \in P_i(X')$, we define*

$$\begin{aligned} F_i^{(w)} &= \{u \in F_i : uw \in E\}, \\ r_i^{(w)} &= \sum_{u \in F_i^{(w)}} \left(\frac{1}{\deg(w)} + \frac{1}{\deg(u)} \right). \end{aligned} \tag{3.2}$$

Also define r_i to be the same as in (3.1)*. Then, the probability that a potential vertex $w \in P_i(X')$ is chosen from the set $P_i(X')$ to be the captured vertex, is $r_i^{(w)}/r_i$. In other words,*

$$\mathbb{P}(w_i = w) = \frac{r_i^{(w)}}{r_i}, \qquad \textit{for all } w \in P_i(X').$$

We provide a brief explanation for the observation above. Consider the edge clock version of the rumour spreading model. Note that the term in (3.2) is the sum of the rates of all effective clocks associated with the edges incident to w. In other words, $r_i^{(w)}$ is the rate of informing w. Note that $r_i = \sum_{w \in P_i} r_i^{(w)}$. It follows that the probability that w is the captured vertex, is proportional to $r_i^{(w)}$ where the normalising factor is r_i.

The analogous version of the observations above for the RSDS process is less straightforward than the SRS version. Similar to the SRS, non-effective informed vertices in RSDS process also play void role to pass the rumour to new vertices. However, when an effective vertex in the RSDS process is dormant, it is also unable to pass the message. At an instantaneous time, the rate of informing a new vertex depends on the number of effectual vertices (instead of effective vertices as described in Observation 3.4) at that time. Yet, as the process runs during a particular stage, the effective vertices are switching and this changes the number of effectual vertices. This implies that the running time of stage i is not necessarily exponentially distributed and requires more intricate analysis.

To describe the analogous observations more precisely, we define several additional terms. Suppose that X is an RSDS process occurring in a connected graph G. For $i = 1, \ldots, n-1$, we define SW_i as a non negative integer-valued random variable that counts how many times the vertices in F_i of X switch during stage i.

We define the sequence of time points $V_i(X) = s_i^0 < s_i^1 < s_i^2 < \cdots$ with the following specification. Starting from $V_i(X)$, we observe the times at which the vertices in F_i switch. For $j \geq 1$, starting from the beginning of stage i, define s_i^j as the j-th earliest switching time. Here we just capture the times at which these vertices switch without paying substantial attention on the rumour spreading process. We call s_i^j the *j-th switching* *time* of F_i. Also, for all $j \geq 0$ we define C_i^j to be the set of all effectual vertices at time s_i^{j+}, an instantaneous time after the j-th switching time. Similarly, $(C_i^j)_{j\geq 0}$ retrieves the information about the statuses of the effective vertices for the upcoming times without paying attention to the rumour spreading outcomes.

The following observations describe the distribution of the running time of stage i in X as an RSDS-adapted version of Observation 3.4. Recall that $s(n)$ is the rate of the switching clock.

Observation 3.6. *Conditional on SW_i and $(C_i^j)_{j\geq 0}$, suppose that for $j \geq 0$, we define*

$$q_j := \sum_{u \in C_i^j} \frac{e(\{u\}, P_i)}{\deg(u)} + \sum_{w \in P_i} \frac{e(\{w\}, C_i^j)}{\deg(w)} \tag{3.3}$$

and E_j to be an exponential random variable with rate $s(n)|F_i| + q_j$, independently of each other. Then

$$T_i(X) \mid_{SW_i,(C_i^j)} \stackrel{d}{=} \sum_{j=0}^{SW_j} E_j.$$

Note that the derivation of Observation 3.6 from Observation 3.4 is less straightforward. To accommodate this, we provide the proof of the observation as follows.

Proof. Define a Poisson clock B at the beginning of stage i to be the superposition of all switching clocks of the effective vertices in F_i and all effective edge spreading clocks of the stage. Let $b_0 = V_i(X)$ and $b_1 < b_2 < \dots$ be the ascending ringing times of B. Note that for all $1 \leq j \leq SW_i$, the clock that rings at time b_j is a switching clock, and b_{SW_i+1} marks the first ringing of a spreading clock. It follows that

$$T_i(X) \mid_{SW_i} = \sum_{i=0}^{SW_i} b_{j+1} - b_j. \tag{3.4}$$

Now observe that the distribution of $b_{j+1} - b_j$ is independent of what type of clock that rings at time b_{j+1}. In fact, $b_j = s_i^j$ for $j = 0, \dots, SW_i$. Thus, conditional on C_i^j, starting from time b_j, the spreading rate is the sum of the rates of all effectual spreading clocks at that time. A simple observation reveals that this equals to q_j. Note also that q_j in (3.3) is a slight modification of (3.1) where we replace F_i with C_i^j. On the other hand, we have that the switching rate is $s(n)|F_i|$ since all non-effective informed vertices play void role in the rest of the rumour spreading. This means that

$$(b_{j+1} - b_j) \mid_{C_i^j} \stackrel{d}{=} \mathcal{E}(s(n)|F_i| + q_j). \tag{3.5}$$

By combining (3.4) and (3.5), we have the desired result. $\square$

Another adapted observation describing the distribution of the captured vertices is given as follows.

Observation 3.7. *Conditional on SW_i and $\{C_i^j\}_{j\geq 0}$, suppose that for any potential vertex $w \in P_i(X)$, we define*

$$C_i^{(w)} = \{u \in C_i^{SW_i} : uw \in E\},$$
$$\hat{r}_i^{(w)} = \sum_{u \in C_i^{(w)}} \left(\frac{1}{\deg(u)} + \frac{1}{\deg(w)} \right),$$
$$\hat{r}_i = \sum_{w \in P_i} s_i^{(w)}.$$

Then for any $w \in P_i(X)$, the probability that w is the captured vertex is $\hat{r}_i^{(w)}/\hat{r}_i$, that is

$$\mathbb{P}(w_i = w) = \frac{\hat{r}_i^{(w)}}{\hat{r}_i}, \qquad \text{for all } w \in P_i(X).$$

Again, Observation 3.7 is the RSDS-adapted version of Observation 3.5. However, unlike Observation 3.6, the adaptation follows directly from its SRS version by considering the effectual set in an instantaneous time before an effectual spreading clock rings.

Next, we discuss the formal definition of *delay time* of an RSDS graph. For any connected graph G, let X'_G and X_G respectively denote the SRS and RSDS process conducted in the same underlying graph G. As being briefly introduced in Chapter 1, the delay time serves as a measure to compare the running times of these two schemes. Recall that $T(X'_G)$ and $T(X)$ respectively denote the running time of the SRS and RSDS process in G. We define the delay time of X_G to be

$$D(X_G) = T(X_G) - \mathbb{E}T(X'_G).$$

We also decompose the delay time into stages. For all $i = 1, \dots n-1$, define

$$D_i(X_G) = T_i(X_G) - \mathbb{E}T_i(X'_G). \tag{3.6}$$

We call $D_i(X_G)$ the *delay time of stage i*. Then, we can express

$$D(X_G) = \sum_{i=1}^{n-1} D_i(X_G).$$

Note that $D(X_G)$ is a random variable that shares the same probability space as the stochastic process X_G. We can understand $D(X_G)$ as the measure of how strong the delaying scheme slows the running time of the standard process. In this thesis, we will mostly discuss the expectation of $D(X_G)$ as the main parameter in the analysis of the RSDS process.

We now show that $\mathbb{E}D(X_G)$ is always non negative. For $i = 1, \dots, n-1$, we couple $T_i(X_G)$ and $T_i(X'_G)$ in such a way that $T_i(X_G) \geq T_i(X'_G)$. Then, by Lemma 2.8, the existence of such coupling implies that

$$T_i(X'_G) \leq_{ST} T_i(X_G). \tag{3.7}$$

We specify that once stage i begins, these two processes X_G and X'_G share the edge spreading clocks to govern stage i of both processes. At the time when stage i of X_G finishes, a spreading clock associated to an effectual vertex rings. It follows that the ringing of the spreading clock necessarily terminates stage i of X'_G. Thus, (3.7) is satisfied. As a result, we have that $\mathbb{E}T_i(X_G) \geq \mathbb{E}T_i(X'_G)$. By summing this inequality for all $i \in [1, n-1]$, we obtain that $\mathbb{E}T(X_G) \geq \mathbb{E}T(X'_G)$. This establishes the non-negativity of $\mathbb{E}D(X_G)$.

3.2 Some Examples of SRS and RSDS

This section discusses the behaviour of the running time of SRS and RSDS processes in two elementary families of graphs: paths and stars. The main aim for this section is to illustrate how the RSDS process works in a simple setting. We will present the expected running time of both rumour spreading processes and their comparisons in paths and stars.

In this section, we will use the edge clock perspective in order to analyse the rumour spreading processes in paths and stars. The term 'spreading clocks' in this section refers to the edge spreading clocks.

3.2.1 Rumour Spreading Processes in Paths

Recall that P_n denotes the path graph with n vertices where $V(P_n) = \{v_1, \ldots, v_n\}$ and $E(P_n) = \{v_1v_2, \ldots, v_{n-1}v_n\}$. In this subsection, we let Y' and Y be the SRS and RSDS process in P_n, respectively.

We choose the initial rumour spreader, that is a leaf vertex of the path. Note that if we choose a vertex of degree 2 to be the initial rumour spreader, then the running time is the maximum between the running times of two other rumour spreading processes on the other smaller paths where the initial rumour spreader is a leaf vertex. For this reason, we will be only interested in the analysis of the case where a leaf becomes the initial rumour spreader. Without losing of generality, let v_1 be the initial rumour spreader in our analysis. In Figure 3.2, the red vertex indicates the initial rumour spreader.

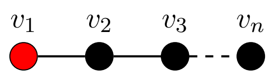

Figure 3.2: Initial rumour spreading setting in P_n

For all $i \in [1, n-1]$, observe that we deterministically have that $P_i(Y') = \{v_{i+1}\}$. Also, the effective set of stage i is also deterministic, that is $F_i = \{v_i\}$. Figure 3.3 visualises stage i of Y' where the red vertices represent the informed vertices while the black ones represent the uninformed vertices. The only spreading clock whose ringing affects the rumour spreading progress is the one lying on v_iv_{i+1}. It follows that the i-th captured vertex is chosen deterministically, that is $w_i = v_i$.

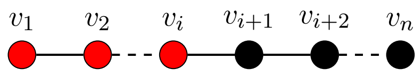

Figure 3.3: Stage i of Y'

Now we apply Observation 3.4 to find the distribution of $T_i(Y')$ for all i. Note that the only vertices with degree 1 are v_1 and v_n. So, the running time distribution of stage 1 and stage $n-1$ will receive a special attention since they involve the spreading contacts with these vertices. Now observe that for all i, we have that $E(F_i, P_i) = \{v_iv_{i+1}\}$ deterministically. Hence, we can write (3.1) as

$$r_i = \frac{1}{\deg(v_i)} + \frac{1}{\deg(v_{i+1})}.$$

Then, Observation 3.4 implies that $T_i(Y')$ is an exponential random variable with rate r_i where

$$r_i = \begin{cases} 3/2, & \text{if } i \in \{1, n-1\}, \\ 1, & \text{otherwise,} \end{cases}$$

and

$$\mathbb{E}T_i(Y') = \frac{1}{r_i} = \begin{cases} 2/3, & \text{if } i \in \{1, n-1\}, \\ 1, & \text{otherwise.} \end{cases}$$

It follows that

$$\mathbb{E}T(Y') = \sum_{i=1}^{n-1} \mathbb{E}T_i(Y') = \frac{2}{3} + \sum_{i=2}^{n-2} 1 + \frac{2}{3} = n - \frac{5}{3}.$$

To calculate $\mathbb{E}T_i(Y)$ we analyse the distribution of SW_i, that is the number of switchings experienced by v_i during stage i of Y.

The fact that $F_i = \{v_i\}$ deterministically means that during stage i of Y, the process can either have one effectual vertex or none at all. At the former case, we say that Y is *alive* and *dead* otherwise.

When Y is alive during stage i, it has two possible state transitions: either v_i passes the rumour to v_{i+1} or v_i becomes dormant. With this in hand, we can understand the alive condition of Y as a Bernoulli trial where the success refers to the spreading of the rumour to v_{i+1}. Observe that when Y is alive, in order to make a state transition, Y waits until either the switching clock of v_i or the spreading clock of v_iv_{i+1} rings. Note that the switching clock has rate $s(n)$ whereas the spreading clock has rate r_i. This means that the success probability is equivalent to the probability that the spreading clock rings before the switching clock, that is

$$\frac{r_i}{r_i + s(n)}$$

In the case when Y becomes dead, that is when v_i switches before it passes the rumour, the spreading is paused since no vertex is able to spread the rumour to the potential vertex. To continue the spreading, Y has to wait for v_i to come back active, that is when the switching clock of v_i rings again. Then, after v_i switches for the second time, Y comes back to be alive. Thus, the failure of the Bernoulli trial means that v_i experiences two consecutive switchings which leads to another identical and independent Bernoulli trial. These Bernoulli trials are repeated until we get the first successful trial. It follows that

$$SW_i \stackrel{d}{=} 2\,\mathcal{G}\left(\frac{r_i}{r_i + s(n)}\right),$$

and

$$\mathbb{E}SW_i = \frac{2s(n)}{r_i}.$$

The factor 2 above comes from the fact that each failure of the trial contributes to two switchings of v_i.

Observe that $C_i^j = F_i$ when j is even and $C_i^j = \emptyset$ otherwise deterministically. It follows that

$$e(C_i^j, P_i) = \begin{cases} 1, & \text{if } j \text{ is even}, \\ 0, & \text{if } j \text{ is odd}. \end{cases}$$

Now we apply Observation 3.6. Whenever Y is alive, it waits for an exponentially distributed time with rate $r_i + s(n)$ until either it spreads the message or it goes dormant. On the other hand, when Y is dead, the process will wait for an exponentially distributed with rate $s(n)$ until v_i comes back active. Now, for all $j \geq 0$, let E_j be an exponential random variable with rate $r_i + s(n)$ if j is even and with rate $s(n)$ otherwise, independently of each other. Thus, we have that

$$\begin{aligned}
\mathbb{E}T_i(Y) &= \mathbb{E}\left[\mathbb{E}\left(T_i(Y) \mid SW_i, (C_i^j)_{j\geq 0}\right)\right] \\
&= \mathbb{E}\left[\mathbb{E}\left(\sum_{j=0}^{SW_i} E_j \middle| SW_i\right)\right] \\
&= \mathbb{E}\left[\mathbb{E}\left(\sum_{k=0}^{SW_i/2-1} (E_{2k} + E_{2k+1}) + E_{SW_i} \middle| SW_i\right)\right] \\
&= \frac{\mathbb{E}SW_i}{2}\left(\mathbb{E}E_0 + \mathbb{E}E_1\right) + \mathbb{E}E_0 \\
&= \frac{s(n)}{r_i}\left(\frac{1}{r_i + s(n)} + \frac{1}{s(n)}\right) + \frac{1}{r_i + s(n)} \\
&= \frac{2}{r_i} = 2\mathbb{E}T_i(Y').
\end{aligned}$$

This leads to

$$\mathbb{E}T(Y) = \sum_{i=1}^{n-1} \mathbb{E}T_i(Y) = 2\mathbb{E}T(Y') = 2\left(n - \frac{2}{3}\right),$$

and

$$\mathbb{E}D(Y) = n - \frac{2}{3}.$$

In path graphs, we showed that the expected delay time is unaffected by the choice of the switching rate. No matter how fast the switching rate is, the expected running time of an RSDS process in P_n will always be twice as much as the expected running time of its SRS version.

3.2.2 Rumour Spreading Processes in Stars

Recall that S_n denotes the star graph with n vertices, where $V(S_n) = \{v, v_1, \ldots, v_{n-1}\}$ and $E(S_n) = \{vv_1, vv_2, \ldots, vv_{n-1}\}$. Throughout this subsection, we let Z' and Z be the SRS and RSDS process running in S_n, respectively.

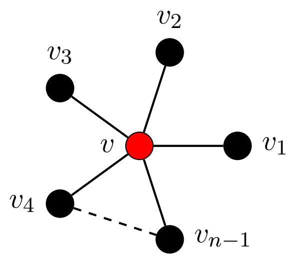

Figure 3.4: Initial rumour spreading setting in S_n

Similar to the paths, we also make a particular choice for the first informed vertex in S_n. In the stars, we choose the centre vertex v to be the initial rumour spreader. Figure 3.4 visualises this, where the red vertex represents the initial rumour spreader. Observe that when the rumour starts from a leaf vertex, the rumour spreading can be decomposed into two rumour spreading processes of two smaller stars. To illustrate this, we suppose that v_1 is the initial rumour holder. Then, the next informed vertex is the centre vertex deterministically. The process that pass the rumour from v_1 to v is essentially the rumour spreading process in S_2 (whose both vertices can act as the centre of the star). On the other hand, once v is informed, v_1 is no longer an effective vertex and the remaining process is the rumour spreading in S_{n-1} whose initial rumour holder is the centre vertex. Thus, if a leaf vertex becomes the initial rumour holder, then the running time of the process is the sum of the same spreading process occurring in an edge (star with 2 vertices) and a smaller star obtained by deleting the leaf (see figure 3.5). For this reason, we will always assume that the initial rumour spreader is the centre vertex.

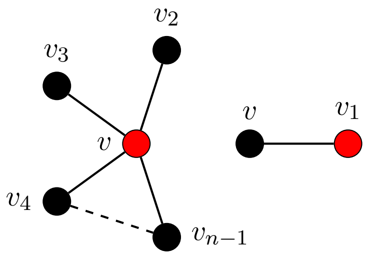

Figure 3.5: Decomposing rumour spreading

The star graphs also enjoy the deterministic choice of effective sets similarly to the path graphs. In fact, throughout the entire process, v is always effective and every leaf vertex is always non-effective, that is, $F_i = \{v\}$ for all i. This is because all leaves have only one neighbour, the centre vertex, which has become informed already since the beginning of the process.

However, unlike the paths, the informed and uninformed sets within stages are not deterministic in stars. To see this, note that all uninformed vertices are potential since they are always adjacent to the informed centre vertex. This implies that $U_i = P_i$ for all i. Next, one can check by simple calculation that the value of $r_i^{(w)}$ in (3.2) is the same for all $w \in U_i = P_i$. Hence, Observation 3.5 implies that the captured vertex w_i is uniformly distributed among all uninformed vertices. This means that $I_i = \{v\} \cup S$ where S is a random subset uniformly picked from all $(i-1)$-subset of $\{v_1, v_2, \ldots, v_{n-1}\}$.

Without losing of generality, we assume that $w_i = v_i$ in order to simplify the analysis. We can write that $I_i = \{v, v_1, \ldots v_{i-1}\}$ and $U_i = \{v_i, \ldots, v_{n-1}\}$ for all i.

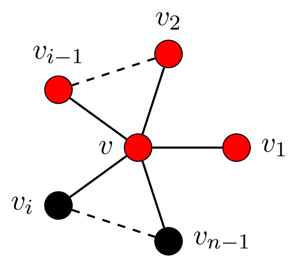

Figure 3.6: Stage i of Z'

Although P_i is a random set, we have the following deterministic facts. Since F_i has only one member at all times, we have that $e(\{u\}, F_i) = 1$ for all $u \in P_i$. On the other hand, since $v \in F_i$ is adjacent to all other vertices in the graph, we have that $e(\{v\}, P_i) = |P_i| = n - i$. This means that r_i in (3.1) is

$$r_i = \frac{n-i}{n-1} + n - i \tag{3.8}$$

deterministically.

By Observation 3.4, we have that $T_i(Z')$ is an exponential random variable with rate r_i and thus

$$\mathbb{E}T_i(Z') = r_i^{-1} = \left(1 - \frac{1}{n}\right)\frac{1}{n-i}.$$

Therefore,

$$\mathbb{E}T(Z') = \sum_{i=1}^{n-1} \mathbb{E}T_i(Z') = \sum_{i=1}^{n-1} \frac{1-n^{-1}}{n-i} = \log n + O(1).$$

Now we calculate $\mathbb{E}T_i(Z)$. We begin by looking for the distribution of $SW_i(Z)$, the number of switching experienced by v during stage i.

Since v is the only effective vertex during the whole process, the performance of the whole rumour spreading heavily depends on the status of v. It implies that the switching clocks associated to the leaves play no role during the entire process and our running time analysis requires no attention to these clocks.

Suppose that Z is in stage i. Similar to the analysis of the paths, we say that the process is *dead* when v is dormant and *alive* when v is active. We will analyse the running time of stage i based on these two conditions.

Now, suppose that Z is alive. Then, there are two options for the next state transition. Observe that Z will stay at its current position until either one of the effective spreading clocks or the switching clock of v rings. When the earliest ringing clock is a spreading clock, Z enters stage i. Otherwise, v goes dormant and Z becomes dead.

With this in hand, we can associate the alive state of Z in stage i with a Bernoulli trial, where the success corresponds to the entering of the next stage. The success probability of the trial is the same as the probability that the first ringing clock is a spreading clock. Observe that all spreading clocks have rate $1 + (n-1)^{-1}$ since all edges are incident to a leaf and the centre vertex. Note also that the rate of the spreading clocks is the same as the value $r_i^{(w)}$ described in (3.2) for all $u \in P_i$. Hence, the spreading rate is

$$(n-i)r_i^{(w)} = n - i + \frac{n-i}{n-1} = r_i$$

where r_i here refers to (3.8). On the other hand, recall that the switching clock has rate $s(n)$. It follows that the success probability is

$$\frac{r_i}{r_i + s(n)}.$$

When the Bernoulli trial fails, Z becomes dead. In this state, Z has no other transition option but to wait for v to switch again to become active and thus Z is alive again. This means that when the trial fails, Z performs two consecutive switchings for v and faces another identical and independent Bernoulli trial. In other words, v experiences an even number of switchings until it finally passes the rumour to a new vertex. Thus, we have that

$$SW_i \overset{d}{=} 2\,\mathcal{G}\left(\frac{r_i}{r_i + s(n)}\right), \qquad \text{and} \qquad \mathbb{E}SW_i = \frac{2s(n)}{r_i}.$$

Now we observe the set of effectual vertices C_i^j. In the case of stars, C_i^j is a deterministic set for all i and j. When Z first enters stage i, the centre vertex is necessarily effectual. Since, there is only one effective vertex at all times, we have that $C_i^j = \{v\}$ when j is even and $C_i^j = \emptyset$ otherwise. One can check that q_j in (3.3) has value

$$q_j = \begin{cases} r_i, & \text{if } j \text{ is even,} \\ 0, & \text{otherwise.} \end{cases}$$

Now, for all $j = 0, \ldots, SW_i$, let E_j be an exponential random variable with rate $s(n) + q_j$, independently of each other and SW_i. Therefore, by Observation 3.6, we have that

$$\begin{aligned} \mathbb{E}T_i(Z) &= \mathbb{E}\left[\mathbb{E}\left(T_i(Z) \mid SW_i, (C_i^j)_{j\geq 0}\right)\right] \\ &= \mathbb{E}\left[\mathbb{E}\left(\sum_{j=0}^{SW_i} E_j \,\middle|\, SW_i\right)\right] \\ &= \mathbb{E}\left[\mathbb{E}\left(\sum_{k=0}^{SW_i/2-1} (E_{2k} + E_{2k+1}) + E_{SW_i} \,\middle|\, SW_i\right)\right] \\ &= \frac{\mathbb{E}SW_i}{2}\left(\mathbb{E}E_0 + \mathbb{E}E_1\right) + \mathbb{E}E_0 \\ &= \frac{s(n)}{r_i}\left(\frac{1}{s(n) + r_i} + \frac{1}{s(n)}\right) + \frac{1}{s(n) + r_i} \\ &= \frac{2}{r_i}. \end{aligned}$$

This leads to

$$\mathbb{E}T(Z) = \sum_{i=1}^{n-1} \mathbb{E}T_i(Y) = \sum_{i=1}^{n-1} \frac{2}{r_i} = 2\log n + O(1),$$

and

$$\mathbb{E}D(Z) = \log n + O(1).$$

3.2.3 Some Remarks

Although the expected running time of the rumour spreading process in stars is much faster than in paths, the application of the delaying scheme gives the same effect on both graphs. The expected delay time of the RSDS conducted in both graphs are the same as the expected running time of their SRS versions. This means that by enabling the delaying scheme, the expected running time of the rumour spreading is doubled when the underlying graph is either a path or a star.

Observe also that the delay times of the RSDS process in paths and stars are independent of the switching rates. At first, this might be surprising since the slow switching rates setting should suggest that the dormancy events are very rare. Thus, this should lead the RSDS model to behave in strong similarity to the SRS. However, the results tell us that the expected running time is still doubled whichever slow rates that we pick. This is due to the large amount of delay times contributed by the rare events. Once the only effective vertex of the stage becomes dormant (which occurs with a very low probability), the process needs to wait for a very long time until the next ringing of the slow switching clock. This phenomenon, however, is absent in the complete graphs as we will investigate more thoroughly in the upcoming chapters.

Chapter 4

Spreading a Rumour in Complete Graphs

Although complete graphs are simple to define and seemingly elementary, the analysis of the RSDS running time in K_n is fairly intricate. In the previous chapter, determining the exact expected running times in paths and stars is relatively easy since their graph structures allow us to observe many deterministic conditions. For instance, in paths and stars, there are always one effective vertex (the most recent informed vertex in paths and the central vertex in stars) at all stages.

This property is absent when we conduct the rumour spreading in complete graphs. In complete graphs, once an informed vertex is dormant during a particular stage $i \geq 2$, there could be many other active vertices which are able to continue the spreading. However, the number of active vertices in a given period is varying in a random fashion during stages. This is one of many factors that complicates the RSDS running time analysis in complete graphs.

In this chapter, we will provide additional terminology to describe the RSDS occurring in complete graphs more precisely. We will also expose some useful general lemmas in a specific setting of rumour spreading in K_n, that will be frequently used throughout the following chapters.

Throughout this whole chapter, we let X' and X respectively be the SRS and RSDS process in K_n.

4.1 Model Reformulations

As described in Chapter 3, the running time distribution of a rumour spreading process for a general graph depends on the choice of the initial rumour spreader. For example, a rumour spreading process (either the SRS or RSDS) in a path starting from a leaf vertex is intuitively slower than the one which starts from an internal vertex of the path. Nevertheless, this phenomenon is absent in complete graphs. Since K_n is vertex-transitive, the running time distribution of the rumour spreading processes in K_n is always the same regardless of the choice of the initial rumour spreader.

Considering this, we will reformulate the rumour spreading models without paying attention to the identity of the initial rumour spreader. Instead of specifying the model in terms of the choices of the initial rumour spreader, the new models will simply assume an arbitrary vertex to be the initial rumour spreader without losing of generality.

As we briefly mention in Chapter 3, the rate of each edge spreading clock is constant when the underlying graph is regular. This leads the edge clock model to bring more advantages for the rumour spreading analysis for complete graphs since we do not need to consider the neighbour random calling scheme. For this reason, unless otherwise stated, we will always refer to the edge clock version of the rumour spreading models in K_n.

Note that in K_n, all vertices have degree $n-1$. By Observation 3.3, the setting where the spreading clocks with unit rate are assigned on the vertices, is equivalent to the one where the spreading clocks are assigned on the edges, each of which has rate $2/(n-1)$. Now we can associate the vertex clock models to the edge clock models via a certain normalisation. To be more precise, suppose that a vertex clock rumour spreading model has switching rate $s(n)$. Then, the vertex clock model is equivalent to an edge clock model with switching rate $\hat{s}(n) = (n-1)s(n)/2$.

With this in hand, from now on, we only consider the edge clock model of rumour spreading where the spreading clocks have the unit rate in the context of K_n. We choose this model to make the analysis convenient.

Next, we define the simplified models in K_n, that we will call the *unlabelled rumour spreading models*. Again, by the strong symmetries possessed by K_n, it is unnecessary to have complete information regarding the identities of the informed vertices at all times throughout the entire process. The running time distribution of each stage of the rumour spreading processes in K_n is always the same despite various choices of the informed set at that time.

This motivates us to introduce the notion of *unlabelled spreading clocks* as well as *unlabelled switching clocks* to accompany the unlabelled model. Unlike the ordinary definition of spreading and switching clocks in Definition 3.1 and Definition 3.2, the unlabelled clocks are not associated with any edges and vertices of the graph. The unlabelled clocks will interact with the rumour spreading model only in accordance to the number of informed and active vertices at a given time without paying much attention to their specific identities. This will be elaborated more in the later parts of the section.

Based on this, we will also sometimes call the original clocks (both spreading and switching clocks) in Definition 3.1 and Definition 3.2, the *labelled clocks* since they are associated with particular vertices and edges of the graph. We will present the model description without involving any Poisson clocks for the sake of simplicity. However, for each definition, we will provide another perspective which involves the unlabelled clocks. In the later parts of the thesis, we will mainly use the unlabelled perspective of the model to perform the analysis of the RSDS.

Now we define the unlabelled SRS model for complete graphs as follows.

Definition 4.1 (Unlabelled SRS Model)**.** *An* unlabelled SRS *process in a complete graph with n vertices is a continuous time Markov Chain $\{X'(t) : t \geq 0\}$ whose state space is $\{1, \ldots, n\}$ with the following initial condition and transition rates. The initial condition is deterministic, that is $X'(0) = 1$. Suppose that $\boldsymbol{Q} = [q_{i,j}]$ is the generator matrix of X'. For all $i = 1, \ldots, n-1$, we specify that $q_{i,j} = i(n-i)$ if $j = i+1$ and $q_{i,j} = 0$ otherwise. The process halts when it enters state n.*

We define the following unlabelled clocks to accompany the unlabelled SRS model as follows. Suppose that $i \in \{1, \ldots, n-1\}$. Once X' enters state i, we introduce i independent Poisson clocks with rate $n-i$, that will be called the *unlabelled spreading clocks of stage i*. Once an unlabelled spreading clock rings, we specify that X' moves to state $i+1$.

The interpretation of the model description in Definition 4.1 is described as follows. The state of X' at a given time represents the number of informed vertices at that time, without giving detailed information about the identities of the informed vertices. It also means that for all $1 \leq i \leq n$, we have that $X'(t) = i$ if and only if X' is in stage i at time t. The initial state of the process depicts the fact that there is only one initial rumour spreader without revealing its identity. The first ringing of an unlabelled spreading clock of stage i indicates the time at which a new vertex receives the rumour.

The model in Definition 4.1 is equivalent to the edge clock SRS model. Since every pair of vertices are adjacent in K_n, we also have that every informed vertex is adjacent to every uninformed vertex as long as the process is still running. This means that every uninformed vertex is potential and every informed vertex is effective. Observe that the choices of both potential and effective sets in stage i can vary arbitrarily among the collection of all $(n-i)$-subset and i-subset of the vertex set respectively. However, the size of the potential and effective sets in stage i are always $(n-i)$ and i respectively, since

$P_i = U_i$ and $F_i = I_i$. Furthermore, with probability 1, there are $i(n-i)$ pairs of informed and uninformed vertices during stage i and thus we always have that

$$e(F_i(X'), P_i(X')) = i(n-i)$$

deterministically regardless of the varied choices of $F_i(X')$ and $P_i(X')$. In the edge clock model, these $F_i(X')$-$P_i(X')$ edges correspond to $i(n-i)$ spreading clocks whose ringing terminates stage i. Since each clock has rate 1, the waiting time for the ringing of these clocks is exponentially distributed with rate $i(n-i)$. On the other hand, we can associate each unlabelled spreading clock (in the unlabelled model) with a superposition of the $n-i$ edge spreading clocks (in the edge clock model) corresponding to the edges in $E(\{w\}, P_i)$ for a particular $w \in F_i$. Hence, the earliest ringing among these i unlabelled spreading clocks indicates that a new vertex becomes informed in both rumour spreading processes. This means that the waiting time for the ringing of the unlabelled spreading clocks is also exponentially distributed with rate $i(n-i)$. This establishes the equivalence between the edge clock model and the model introduced Definition 4.1.

The analogous property is also present in the RSDS model. Suppose that X has d dormant vertices at a given time during stage i. Then, the choice of the set of dormant vertices can be taken arbitrarily from the collection of all d-subsets of the informed set. Nevertheless, at that time, there are always $(i-d)(n-i)$ spreading clocks in the edge clock model whose ringing leads X to end stage i. This means that we can omit the way the process chooses the dormant vertices within stages. The only information that we need is just the number of dormant vertices at any given time, instead of the exhaustive information about the list of dormant vertices.

We now provide the definition of the unlabelled RSDS model for K_n as follows.

Definition 4.2 (Unlabelled RSDS Model). *Let $\mathcal{S} = \{(d,i) : 0 \le d \le i \le n-1\}$. An* unlabelled RSDS *process on a complete graph with n vertices is a continuous time Markov Chain $\{X(t) : t \ge 0\}$ whose state space is $\mathcal{S}$ with the following properties. The initial condition is deterministic, that is $X(0) = (0,1)$. Suppose that X is in state (d,i) with $1 \le d \le i \le n-1$ at a given time. Then X has the following transition rates.*

1. (Dormancy transition). *If $d < i$, then X moves to state $(d+1, i)$ with rate $s(n)(i-d)$.*
2. (Waking up transition). *If $d > 0$, then X moves to state $(d-1, i)$ with rate $s(n)d$.*
3. (Spreading transition). *If $d < i$, then X moves to state $(d, i+1)$ with rate $(i-d)(n-i)$.*

Lastly, we specify that X halts when it enters any state of the form (d,n) with $d < n$.

Again, we can think of this model in terms of Poisson clocks. However, to describe how the clocks govern the process more precisely, we need to introduce an additional term, the *dormancy process*, throughout each stage. Suppose that for all $i \in \{1, \dots, n-1\}$, we have that (d_i, i) is the first state of the form $(\cdot, i)$ that X enters, for some (random) $d_i \le i$. Once X enters state (d_i, i), we introduce i independent *unlabelled switching clocks* $\{S_1, \dots, S_i\}$ and i independent *unlabelled spreading clocks* $\{R_1, \dots, R_i\}$. Each unlabelled switching clock has rate $s(n)$, whereas the rate of each unlabelled spreading clock is $(n-i)$. Additionally, we introduce an integer-valued process $\{d(t) : t \ge 0\}$ at the start of stage i, which we will call *the dormancy process* of stage i. We specify that $d(0) = d_i$. Define $c_0 = 0$ and for all $j \ge 1$, starting from time $V_i(X)$, define c_j to be the waiting time for the j-th earliest ringing among the clocks in $\{S_1, \dots S_i, R_1, \dots, R_i\}$. For all $j \ge 0$, we define $d_j := d(c_j^-)$ where c_j^- denotes an instantaneous time before the ringing time c_j. In addition, define C_j to be the clock ringing at time c_j. For all $j \ge 1$, the ringing of C_j governs the realisation of stage i of X according to the following rules.

Rules 4.3. *The ringing of C_j gives the following effects to stage i of X.*

1. *If $C_j \in \{S_1, \dots, S_{d_j}\}$, then $d(c_j) = d_j - 1$.*
2. *If $C_j \in \{S_{d_j+1}, \dots, S_i\}$, then $d(c_j) = d_j + 1$.*
3. *If $C_j \in \{R_1, \dots, R_{d_j}\}$, then $d(c_j) = d_j$. In particular, we call the clocks in this set the* futile clocks.
4. *If $C_j \in \{R_{d_j+1}, \dots, R_i\}$, then X moves to state $(d(c_j), i+1)$ at time c_j. In particular, we call the clocks in the set the* terminating clocks.

The unlabelled RSDS model has the following interpretation. For any given time $t \ge 0$, the process state $X(t) = (d, i)$ means that at time t, the process has d dormant vertices and i informed vertices. Also, $X(t) = (d, i)$ if and only if X is in stage i at time t. The initial state means that in the beginning, there is only one active vertex and the rest of the vertices are uninformed. The dormancy process of stage i, $d(t)$ counts the dormant vertices in X at time $V_i(X) + t$.

Suppose that X is now in stage i. We define three types of transition events in X: the *waking up*, *dormancy*, and *spreading* events, that are marked by the names of the transition rates in Definition 4.2. These three transitions correspond to the ringing of C_j described in rule 1, 2, and 4 in Rules 4.3 respectively. At the moment when X undergoes a dormancy transition (waking up transition), it sends an active (dormant) vertex to be dormant (active). Again, the exact identity of the switching vertex is irrelevant here. On the other hand, the ringing of a terminating clock indicates that a new vertex is informed and thus X enters stage $i+1$. Meanwhile, the ringing of a futile clock in rule 3 depicts the ringing of a non-effective spreading clock at which there is no progress of the rumour spreading process.

It is worth noting that at some particular states, the process has less transition options. Suppose that X is in state (d, i). If $0 < d < i$, then it can move to either $(d-1, i)$,

$(d+1, i)$, or $(d, i+1)$ at the next transition, which represent the waking up, dormancy, and spreading events respectively. However, when $d = 0$, the waking up transition is not possible since all vertices are active at that time. On the other hand, if $d = i$, then the only possible transition is the waking up transition. In this case, we have no active vertex, which means that no individual is able to spread the rumour and thus a spreading event is impossible to occur. Also, since all vertices are dormant, it is also not possible to conduct another dormancy transition. This is a rather special case at which the process will transition to state $(i-1, i)$ deterministically after waiting for a while. Later, we will put special attention to these special states that we will call the vacuum states.

4.2 SRS Process in Complete Graphs

Most topics covered in this section are mainly taken from a work of Janson [30]. The paper studied some properties of the complete graphs with random exponential edge weights. In particular, Janson considered the problem of finding the minimal weighted path between vertices in the graph and derived some asymptotic values of such minimal weighted paths for various settings. As mentioned briefly in Chapter 1, the problem is equivalent to the SRS model in complete graphs.

To begin the discussion about the SRS process in complete graphs, we will first describe the minimal weighted path problem in an exponentially edge-weighted graph. Then, we will exhibit the relationship between this problem and the SRS process before analysing the running time of the SRS process in K_n.

Now we describe the exponentially edge-weighted complete graphs model. Suppose that we assign to each edge $e \in E(K_n)$, an random weight W_e. For all $e \in E(K_n)$, the random variables W_e are independent to each other and are exponentially distributed with unit rate. For every pair of distinct vertices u, w, let $\mathcal{P}(u, w)$ be the set of all paths whose ends are u and w. For each path $P \in \mathcal{P}(u, w)$, we define $d_W(P) = \sum_{e \in E(P)} W_e$, the weight of the path P. Finally we define $d_W(u, w) = \min_{P \in \mathcal{P}(u,w)} d_W(P)$. We can think of $d_W(u, w)$ as the shortest distance between u and w.

We have a nice relationship between the complete graphs with exponentially distributed edge weights and the SRS process run in K_n. First we pick a particular vertex v as the rumour spreader. We set the exponential edge weights as the waiting time for the ringing of spreading clocks on edges. Now we expose one by one the vertices having increasing weighted distance from v. Let $(v = v_1, v_2, \ldots, v_n)$ be this ordering, that is $0 = d_W(v, v_1) < d_W(v, v_2) < \cdots < d_W(v, v_n)$. We claim that this ordering also denotes the ordering of the captured vertices in the SRS process. We show our claim by induction. The base case is trivial by the definition. Now assume that

$(v_1, \ldots, v_k)$ is the correct ordering for the sequence of captured vertices in the SRS. Define $E_k = \{v_i v_j : 1 \le i \le k, k+1 \le j \le n\}$ and $u_i u_j := \min_{e \in E_k} W_e$. Then, $v_{k+1} = u_j$ by the ordering definition. Now, by memorylessness property, we reset all spreading clocks lying on all edges in E_k at time $V_i(X')$. We then couple the waiting time of these reinitialised spreading clocks with the random exponential weights. It follows that the spreading clock lying on $u_i u_j$ is the fastest among all significant spreading clocks whose ringing can potentially lead to the next stage. Hence, u_j is the next captured vertex in the SRS process.

By the explanation above, we can see that $d_W(u, w)$ defined in the weighted graph corresponds to the time needed to let w learn the rumour in the SRS process where u is the initial rumour spreader. Thus, $\max_w d_W(v, w)$ corresponds to the running time of the SRS process, for an arbitrary choice of v.

Now we calculate $\mathbb{E}T(X')$. As described in Definition 4.1 in the previous section, we have that

$$T_i(X') \stackrel{d}{=} \mathcal{E}(i(n-i)).$$

Thus,

$$\mathbb{E}T_i(X') = \frac{1}{i(n-i)}.$$

This leads to

$$\begin{aligned}
\mathbb{E}T(X') &= \mathbb{E}\left(\sum_{i=1}^{n-1} T_i(X')\right) \\
&= \sum_{i=1}^{n-1} \frac{1}{i(n-i)} \\
&= \frac{1}{n}\left(\sum_{i=1}^{n-1}\left[\frac{1}{i} + \frac{1}{n-i}\right]\right) \\
&= \frac{2\log n}{n} + O\left(\frac{1}{n}\right). \qquad (4.1)
\end{aligned}$$

The term $2\log n/n$ in (4.1) will be an important term to which we will often compare the expected running time of the RSDS process.

The running time of the SRS process is also sharply concentrated around its expected value as $n \to \infty$. To show this, we first calculate $\mathrm{Var}(T(X))$. Note that since $T_i(X') \stackrel{d}{=} \mathcal{E}(i(n-i))$, we have that

$$\mathrm{Var}\, T_i(X') = \frac{1}{i^2(n-i)^2}.$$

Also, observe that $T_1(X'), T_2(X'), \ldots, T_{n-1}(X')$ are independent of each other since they depict the length of disjoint time intervals. It means that $\mathrm{Var}\, T(X') = \sum_{i=1}^{n-1} \mathrm{Var}\, T_i(X')$.

Hence,

$$\begin{aligned}
\operatorname{Var} T(X') &= \sum_{i=1}^{n-1} \operatorname{Var} T_i(X') \\
&= \sum_{i=1}^{n-1} \frac{1}{i^2(n-i)^2} \\
&= \sum_{i=1}^{\lfloor n/2 \rfloor - 1} \frac{1}{i^2(n-i)^2} + \sum_{i=\lfloor n/2 \rfloor}^{n-1} \frac{1}{i^2(n-i)^2} \\
&\le \sum_{i=1}^{\lfloor n/2 \rfloor - 1} \frac{1}{i^2(n/2)^2} + \sum_{i=\lfloor n/2 \rfloor}^{n-1} \frac{1}{(n/2)^2(n-i)^2} \\
&\le \frac{8}{n^2} \left(\sum_{i=1}^{\lfloor n/2 \rfloor} \frac{1}{i^2} \right) \\
&= O\left(\frac{1}{n^2}\right). \qquad (4.2)
\end{aligned}$$

From (4.2) and an application of Chebyshev's inequality on $T(X')$, we have that, for any arbitrary positive real ε,

$$\mathbb{P}\left(\left|T(X') - \mathbb{E}T(X')\right| > \varepsilon \mathbb{E}T(X')\right) \le \frac{\operatorname{Var} T(X')}{\varepsilon^2 \mathbb{E}T(X')^2} = O\left(\frac{1}{\log^2 n}\right).$$

This means that for any $\varepsilon > 0$,

$$\mathbb{P}\left(\frac{2(1-\varepsilon)\log n}{n} \le T(X') \le \frac{2(1+\varepsilon)\log n}{n}\right) = 1 - O\left(\frac{1}{\log^2 n}\right),$$

which is essentially equivalent to

$$\frac{T(X')}{\log n/n} \xrightarrow{p} 2. \qquad (4.3)$$

4.3 Compressed Version of the RSDS

We introduce the notion of the *compressed version* of rumour spreading processes to begin a more detailed analysis of the RSDS. We provide the intuition behind this notion as follows. As briefly mentioned in Section 4.1, the state at which the RSDS has no active vertex brings a special meaning in the delay time analysis. This is the only state at which the ringing of any unlabelled spreading clocks gives no effect to the process. However, this is also a condition at which the spreading process is not progressing. The spreading process is paused until one of the unlabelled switching clocks rings to bring back an active vertex.

This motivates us to define the compressed version of the RSDS. Roughly, the compressed RSDS is the same RSDS process which excludes those "bad" time periods. In the compressed version, the process has no chance to experience such pausing moments, as there is at least one active vertex at all times.

To be more precise, we define the following terms. For all $i = 1, \dots, n-1$, we say that (i, i) is the *vacuum state*, that is the state when all of the i informed vertices are dormant at that time. We say that X is *vacuum* if it is currently in a vacuum state.

We define the *Compressed Rumour Spreading with a Delaying Scheme* (CRSDS) process, denoted by $\{X^C(t) : t \geq 0\}$, as the same process of the RSDS process X with the removal of all time periods at which X is vacuum. This means that at all times, CRSDS process always has at least one active vertices.

In terms of unlabelled clocks, we can also define the same unlabelled switching and spreading clocks having almost the same rules as described in Rules 4.3. The only difference is described as follows. Suppose that X^C is in stage i for some $i \in [1, n-1]$. When the dormancy process has value $d(t) = i - 1$ at a given time, we abandon rule 2. This also means that the unlabelled switching clock S_i has no effect during the entire stage. By this additional rule, we have a guarantee that $d(t) < i$ for all $t \geq 0$.

Next, recall from the definitions introduced in Chapter 3, we let $T(X^C)$ and $T_i(X^C)$ denote the running time of the CRSDS process and its stage i, respectively. The same analogies also apply for $D_i(X^C)$, and $D(X)$, the delay time of stage i of X^C and the delay time of the whole process. Recall also that, $V_i(X^C)$ is the time at which X^C enters stage i for the first time.

We define $W(X)$ to be the total length of time periods at which X is vacuum. Then we have that

$$T(X) = W(X) + T(X^C).$$

Also, we can express

$$D(X) = T(X) - \mathbb{E}T(X') = W(X) + T(X^C) - \mathbb{E}T(X') = W(X) + D(X^C). \tag{4.4}$$

Here, $W(X)$ can be understood as a part of the delay time contributed by the vacuum condition. We call $W(X)$ the *vacuum delay time.*

Next, we provide some general lemmas regarding the stochastic orderings of the running time of stages in the CRSDS. The usefulness of the lemmas comes from the fact that the stochastic orderings provided by the lemmas are independent of the switching rates. Many arguments in the later discussions regarding the expected delay time of the RSDS will apply the lemmas extensively.

First we introduce some new notations regarding the running time of stage i in X^C in terms of a given stopping time. Suppose that τ is any stopping time with respect to the stochastic process X^C. For all $i \in [1, n-1]$, we define

$$T_i^{<\tau} = (\min\{V_{i+1}, \tau\} - V_i)\, \mathbf{1}\{\tau > V_i\}, \tag{4.5}$$
$$T_i^{\geq\tau} = (V_{i+1} - \max\{V_i, \tau\})\, \mathbf{1}\{\tau \leq V_{i+1}\}. \tag{4.6}$$

We can understand $T_i^{<\tau}$ as the total time spent by the process during stage i and before time τ. Similarly, $T_i^{\geq\tau}$ quantifies the total time spent during stage i after time τ. Observe that the following equation holds.

$$T_i = T_i^{<\tau} + T_i^{\geq\tau}.$$

Now we state the lemmas regarding stochastic orderings of $T_i(X^C)$ in terms of $T_i^{<\tau}$ and $T_i^{\geq\tau}$. We will call the first lemma the *Strong Bound.* First we introduced a specific stopping time. Let $\tau(k, P)$ be the time at which X^C has at least P dormant vertices for the first time since stage k starts. Formally,

$$\tau(k, P) = \inf\{t \geq V_k(X^C) : \mathrm{Do}(X^C, t) \geq P\}.$$

In other words, when we start observing X^C from time $V_k(X^C)$, there will be less than P dormant vertices before time $\tau(k, P)$. The strong bound provides a stochastic upper bound for $T_i^{<\tau(k,P)}$ as follows.

Lemma 4.4 (The Strong Bound)**.** *Let P, k and i be positive integers with $1 \leq P \leq k \leq i \leq n-1$. Then,*

$$T_i^{<\tau(k,P)}(X^C) \leq_{ST} \mathcal{E}((i-P)(n-i)).$$

Proof. We construct a coupling (X^C, F_i) where X^C is a CRSDS process and F_i is a copy of $\mathcal{E}((i-P)(n-i))$. We aim to show that

$$T_i^{<\tau(k,P)}(X^C) \leq F_i. \tag{4.7}$$

Suppose that $\{S_j\}_{1\leq j\leq i}$ and $\{R_j\}_{1\leq j\leq i}$ are respectively the sets of unlabelled switching and spreading clocks of stage i in X^C. Let $\{d(t) : t \geq 0\}$ be the dormancy process of stage i in X^C. We specify F_i to be the waiting time from the start of stage i of X^C until one among the unlabelled spreading clocks in $\{R_{P+1}, \ldots, R_i\}$ rings. We will show that stage i necessarily will have finished by time F_i. Note that in this case, $d(F_i) < P$. Hence, if one of the clocks in $\{R_{P+1}, \ldots, R_i\}$ rings, then the ringing also indicates the termination of stage i, accordingly to Rules 4.3. This establishes (4.7).

To complete the proof, observe that F_i is the waiting time for the first ringing among $(i-P)$ Poisson clocks where each of them has rate $(n-i)$. Therefore, F_i has an exponential distribution with rate $(i-P)(n-i)$ by the superposition property. $\square$

On the other hand, we also define another much looser bound, which we will call the *worst case bound.* The bound uses an essential feature possessed by X^C, which is the guarantee that there is at least one active vertex at all times. In the lemma, we bound $T_i(X^C)$ by considering its worst possible case, that is when the process only has one active vertex at all times.

Lemma 4.5 (Worst Case Bound)**.** *Suppose that $\mathcal{H}$ is an arbitrary history of X^C until the end of stage $i-1$. Then,*

$$T_i(X^C)\ |_{\mathcal{H}} \leq_{ST} \mathcal{E}(n-i).$$

Proof. We construct a coupling $(X^C|_{\mathcal{H}}, F)$ where $X^C|_{\mathcal{F}}$ and F denote the CRSDS process X^C under the probability space conditioned on $\mathcal{H}$, and an exponential random variable with rate $(n-i)$, respectively. We aim to show that

$$T_i(X^C)\ |_{\mathcal{H}} \leq F. \tag{4.8}$$

Let $\{S_j\}_{1\leq j\leq i}$ and $\{R_j\}_{1\leq j\leq i}$ be the sets of unlabelled switching and spreading clocks of stage i in $X^C\ |_{\mathcal{H}}$ respectively. Observe that the performances of these clocks are independent of $\mathcal{H}$. Also let $\{d(t) : t \geq 0\}$ be the dormancy process of stage i of X^C. Observe that the conditioning on the history can affect the distribution of the initial value of the dormancy process. However, in this coupling, the varying value of d throughout the stage is unimportant as it will not be involved in the construction of the coupling. The only fact needed is that $d(t) \leq i-1$ for all t since we consider the compressed process.

We specify F to be the waiting time from the starting time of stage i until the unlabelled spreading clock R_i rings. Since $d(t) \leq i-1$ for all t, the ringing of R_i will lead X^C to terminate stage i. This establishes (4.8).

To complete the proof, we have that F is exponentially distributed with rate $(n-i)$ since it is the waiting time of the ringing of a Poisson clock with rate $n-i$. ☐

In particular, the worst case bound is useful to bound $T_i^{\geq\tau}(X^C)$ for any stopping time τ. The worst case bound implies the following corollary.

Corollary 4.6. *Let τ be any stopping time and A be the event that X^C is in stage i at time τ. Then,*

$$\mathbb{E}T_i^{\geq\tau}(X^C) \leq \frac{\mathbb{P}(A)}{n-i}.$$

Proof. Note that when A does not occur, we have that $T_i^{\geq\tau} = 0$. Now we condition on the occurrence of A. Suppose that S is the state of the process at time τ. By the memorylessness property, we reset all of the Poisson clocks involved during stage i at time τ. We can think of the time τ as the zero time of stage i conditioned on a specific setting. Suppose that A' is the event that X^C is in state S when stage i starts. This means that by conditioning on A, we can see $T_i^{\geq\tau}$ as the running time of stage i of a CRSDS process conditioned on A'. Now, by Lemma 4.5, we have that $T_i \mid_{A'}$ is stochastically smaller than $\mathcal{E}(n-i)$. It follows that

$$T_i^{\geq \tau} \mid_A \overset{d}{=} T_i \mid_{A'} \leq_{ST} \mathcal{E}(n-i).$$

Therefore,

$$\begin{aligned} \mathbb{E}T_i^{\geq \tau} &= \mathbb{E}\left(T_i^{\geq \tau} \mid A\right)\mathbb{P}(A) + \mathbb{E}\left(T_i^{\geq \tau} \mid \bar{A}\right)\mathbb{P}(\bar{A}) \\ &\leq \frac{\mathbb{P}(A)}{n-i}. \end{aligned}$$

□

We will use Corollary 4.6 for many times in the context of bounding the running time of a particular stage at which a very rare event occurs, that is when $\mathbb{P}(A)$ is extremely small.

Next, we provide an upper bound for the number of dormant vertices in a given stage at a fixed time.

Lemma 4.7. *Suppose $i \in [1, n-1]$ and $\{d(t) : t \geq 0\}$ is the dormancy process for stage i of X^C. Then, for all positive fixed function $t = t(n) > 0$ and $\varepsilon \in (0,1)$, there are at most $\frac{i(1+\varepsilon)}{2}$ dormant vertices out of the i informed vertices in $I_i(X^C)$ at time $V_i + t$ with probability $1 - \exp(-\varepsilon^2 i/6)$. Equivalently,*

$$\mathbb{P}\left(d(t) > \frac{i(1+\varepsilon)}{2}\right) < \exp\left(-\frac{\varepsilon^2 i}{6}\right).$$

Proof. In this proof, we will refer to the edge clock RSDS model. All switching and spreading clocks discussed here refer to the labelled switching and spreading clocks, associated to particular vertices and edges of the graph, respectively.

Note that the process has i informed vertices at time $V_i(X^C)$. We aim to bound the number of dormant vertices among them at time $V_i(X^C) + t$. In this proof, we ignore all newly informed vertices that might have appeared between times $V_i(X^C)$ and $V_i(X^C) + t$. Let $Y'(t)$ be the number of dormant vertices in I_i (ignoring new informed vertices which may go dormant, if there are any) at time $V_i(X^C) + t$.

To bound $Y'(t)$, we construct a modification of RSDS. In this modified version, we allow all switching clocks to work from the beginning, even for the uninformed vertices. Let $Y(t)$ capture the same information as $Y'(t)$ occurring in the modified version. We aim to couple $Y(t)$ and $Y'(t)$ in such a way to obtain that $\mathbb{E}Y'(t) \leq \mathbb{E}Y(t)$.

For any informed vertices $w \in I_i$, let $Y'_w(t)$ and $Y_w(t)$ respectively denote the indicator that w is dormant at time $V_i(X^C) + t$ on the original and the modified process. Hence, we can write that

$$Y'(t) = \sum_{w \in I_i} Y'_w(t), \qquad Y(t) = \sum_{w \in I_i} Y_w(t).$$

Note that in both processes, w is dormant if and only if an odd number of switchings is applied to w since the switching clock of associated to w begins to give effect to w. Suppose that Q_w is the (random) time at which w first receives the rumour. Conditioned on V_i and Q_w, we have that $Y_w(t)$ and $Y'_w(t)$ are Poisson random variables with mean $(t+V_i)s(n)$ and $(t+V_i-Q_w)s(n)$, respectively. Observe that $\{Y'_w(t)\}_w$ are not independent of each other since the starting time of the switching clock of a particular vertex depends on the starting time of the other's. However $\{Y_w(t)\}_w$ are independent since all switching clocks start at the same time and thus their ringings are independent within each other. Also observe that $Q_w \leq V_i$ by definition. Now suppose that S is a Poisson random variable with mean $\lambda > 0$. Then, we have that

$$\mathbb{P}(S \text{ is odd}) = e^{-\lambda}\sum_{k=0}^{\infty}\frac{\lambda^{2k+1}}{(2k+1)!} = e^{-\lambda}\sinh(\lambda) = \frac{1}{2}(1-e^{-2\lambda}). \tag{4.9}$$

From this, we have that the following inequality

$$\begin{aligned}\mathbb{E}\left(Y'_w(t) \mid V_i, Q_w\right) &= \frac{1}{2}\left(1-e^{-2(t+V_i-Q_w)s(n)}\right)\\ &\leq \frac{1}{2}\left(1-e^{-2(t+V_i)s(n)}\right)\\ &= \mathbb{E}\left(Y_w(t) \mid V_i, Q_w\right)\end{aligned}$$

holds with probability 1. This implies that

$$\mathbb{P}(Y'_w(t)=1) \leq \mathbb{P}(Y_w(t)=1).$$

In other words,

$$Y'_w(t) \leq_{ST} Y_w(t)$$

which results in

$$Y'(t) \leq_{ST} Y(t).$$

Now notice that each $Y_w(t)$ is a Bernoulli trial with success probability at most $\frac{1}{2}$. This means that $Y_w(t)$ is stochastically dominated by a Bernoulli random variable with success probability $\frac{1}{2}$. Hence, we also have that

$$Y'(t) \leq_{ST} Y(t) \leq_{ST} \mathcal{B}\left(i, \frac{1}{2}\right).$$

Therefore, by Lemma 2.10, for any positive $\varepsilon \in (0,1)$, we have that

$$\mathbb{P}\left(Y'(t) > (1+\varepsilon)\frac{i}{2}\right) \leq \exp\left(-\frac{\varepsilon^2 i}{6}\right).$$

This concludes the proof. ☐

The lemma has a stronger result when the switching rate is relatively slow. While the original lemma provides a bound on the number of dormant vertices only at a given time, the stronger version states that during an entire stage with relatively many informed vertices, the dormancies bound holds with considerably high probability.

Corollary 4.8. *Suppose that $s(n) = o(n/\log n)$. Then, for every positive constant $\varepsilon \in (0,1)$, and every $i \in [M \log n, n-1]$ with $M = 60/\varepsilon^2$, we have that X^C has at most $\frac{i(1+\varepsilon)}{2} + 10$ dormant vertices with probability $1 - O(n^{-6})$ during the whole stage i.*

Proof. Suppose that $\{d(t) : t \geq 0\}$ is the dormancy process of stage i in X^C.

Note that by the worst case bound given by Lemma 4.5, $T_i(X^C)$ is stochastically smaller than $\mathcal{E}(n-i)$. Hence, we have that

$$\mathbb{P}(T_i(X^C) > 6 \log n) \leq \mathbb{P}(\mathcal{E}(n-i) > 6 \log n) = e^{-6(n-i)\log n} = O(n^{-6}). \tag{4.10}$$

This give a rather loose upper bound for T_i with high probability.

Based on this bound, we partition the time interval $[V_i^C, V_i^C + 6 \log n]$ into disjoint subintervals of size $\delta = (ns(n)i)^{-1}$. Hence, there are $K = 6\delta^{-1} \log n = O(n^4)$ subintervals which decompose $[V_i^C, V_i^C + 6 \log n]$. For $k \in [1, K]$, we will call $[V_i^C + (k-1)\delta, V_i^C + k\delta]$ the k-th interval.

By Lemma 4.7, for all $k \in [1, K]$, we have that

$$\begin{aligned} \mathbb{P}\left(d((k-1)\delta) > \frac{i(1+\varepsilon)}{2}\right) &< \exp\left(-\frac{\varepsilon^2 i}{6}\right) \\ &\leq \exp\left(-\frac{\varepsilon^2 M \log n}{6}\right) \\ &= n^{-M\varepsilon^2/6} \\ &= O(n^{-10}). \end{aligned} \tag{4.11}$$

In other words, if we observe the process at the time where the k-th interval starts, we will find at most $\frac{(1+\varepsilon)i}{2}$ dormant vertices with probability $1 - O(n^{-10})$.

We say that an interval is *bad*, if either it starts with more than $\frac{i(1+\varepsilon)}{2}$ dormant vertices, or it undergoes at least 10 switchings during the interval. Note that if there exists a time point at which there are more than $\frac{i(1+\varepsilon)}{2} + 10$ dormant vertices during the k-th interval, then it is necessary that the interval is bad. Observe that the switching rate during stage i is $s(n)i$. Suppose that Z denotes the number of switchings occurring in the k-th interval. Then, Z is a Poisson random variable with parameter $\delta s(n)i$. Observe that

$$\mathbb{P}(Z \geq 10) = e^{-\delta s(n)i} \sum_{k=10}^{\infty} \frac{(\delta s(n)i)^k}{k!} = O((\delta s(n)i)^{10}) = O(n^{-10}). \tag{4.12}$$

We obtain the dominating term of the infinite series above from the fact that $\delta s(n)i = o(1)$.

Now, by (4.11), (4.12) and the union bound, we obtain that for all $0 < t \leq 6 \log n$,

$$\mathbb{P}\left(d(t) > \frac{i(1+\varepsilon)}{2} + 10\right) \leq \mathbb{P}(\text{the } k\text{-th interval is bad}) = O(n^{-10}). \tag{4.13}$$

In other words, during the whole time of the k-th interval, X^C always has at most $\frac{i(1+\varepsilon)}{2}+10$ dormant vertices with probability $1-O(n^{10})$.

Now, we find a bound for the probability that there is a time point during stage i at which there are more than $\frac{i(1+\varepsilon)}{2}+10$ dormant vertices. Note that by the union bound and (4.13), the probability that there is some bad interval k in stage i for $k \in [1, K]$ is $O(Kn^{-10}) = O(n^{-6})$. From this and (4.10), we have that

$$\mathbb{P}\left(\left\{\exists t \in (0, T_i] \text{ such that } d(t) > \frac{i(1+\varepsilon)}{2} + 10\right\}\right) = O(n^{-6})$$

and we get our desired result. □

Chapter 5

RSDS in Complete Graphs

This entire chapter is devoted to provide a thorough analysis of the running time of the RSDS processes in K_n and its relationship to the switching rate.

To begin this chapter, we will give the overview of the main results and state our three main theorems. In the first section, we will discuss the vacuum delay time of the RSDS process, as a preparation for the following sections. Then, the next three sections will provide the complete proofs of the main theorems. In this chapter, we will always let X', X and X^C denote the SRS, RSDS and CRSDS process in K_n respectively.

First, we will introduce the threshold function of the noteworthiness of the delay time. We say that a delay time of an RSDS process X is *noteworthy* if $D(X) = \Omega(\log n/n)$. The term $\log n/n$ here comes from the dominating term in (4.1), which states the expected running time of the SRS process, that is

$$\mathbb{E}T(X') = \frac{2\log n}{n} + O\left(\frac{1}{n}\right).$$

So, if the delaying scheme is not noteworthy, that is when $\mathbb{E}D(X) = o(\log n/n)$, then the expected delay time is negligible compared to $\mathbb{E}T(X)$. As briefly mentioned in Chapter 1, if the switching rate is fast enough, then the delay time is noteworthy. However, when the switching rate is too slow, the delaying scheme only brings negligible impact to the rumour spreading process. We say that f is the threshold function for the noteworthiness of the delay time if $s(n) = \Omega(f(n))$ becomes the necessary and sufficient conditions for the noteworthiness of the delay time. In other words, we can see that when the switching rate grows around the threshold function, a sudden significant amount of expected delay time emerges.

Theorem 5.1 (Expected Delay of the RSDS with a slow rate)**.** *Suppose that X is an RSDS process on K_n with switching rate $s(n) = o(\log n/n)$. Then,*

$$\mathbb{E}D(X) = O\left(\frac{s(n)\log^2 n}{n^2}\right) + O\left(\frac{\log\log n}{n}\right).$$

Theorem 5.2 (Expected Delay of the RSDS with a fast rate)**.** *Suppose that X is an RSDS process on K_n with switching rate $s(n) = \Omega(\log n/n)$. Then,*

$$\mathbb{E}D(X) = \Theta\left(\frac{\log n}{n}\right).$$

Observe that for all $s(n) = o(n/\log n)$, we have that $\mathbb{E}D(X) = o(\log n/n)$ by Theorem 5.1. In this case, the delay time is not noteworthy. Again, this means that the expected additional time given by the delaying scheme is so small that the expected running time of the RSDS differs very little from the SRS.

On the other hand, $s(n) = \Omega(n/\log n)$ becomes a condition for which the delay time is noteworthy, as stated in Theorem 5.2. However, for any choices of the switching rate, the expected delay time of the RSDS process can only differ up to some constant factor from $\mathbb{E}T(X')$. This means that the delaying scheme is incapable of providing a dramatic jump for the rumour spreading running time. As a result, the expected running time of an RSDS process will always be in the same order as the SRS even though we let the switching rate grow infinitely fast.

By these two theorems, we infer that $f(n) = n/\log n$ is the threshold function for the noteworthiness of the delay time. We say that a switching rate is *slow* if it is significantly smaller than the threshold function, that is $s(n) = o(n/\log n)$. Otherwise, we call it *fast*.

Next, we also analyse the RSDS process with unit rate more thoroughly. As a member of the class of slow switching rates, the unit rate will certainly lead the RSDS delay time to follow the same fashion, that is, $\mathbb{E}D(X) = o(\log n/n)$, according to Theorem 5.1. However, we will take a closer look at the process to find a more accurate value of the expected delay time by analysing various sources of the delay time.

We present the following theorem to exhibit the behaviour of the RSDS process with unit rate.

Theorem 5.3 (Expected Delay of the RSDS with unit rate)**.** *Suppose that X is an RSDS process on K_n with unit switching rate, that is $s(n) = 1$. Then,*

$$\mathbb{E}D(X) = \frac{1}{n} + O\left(\frac{1}{n\log n}\right). \tag{5.1}$$

Moreover, suppose that $\mathcal{A}$ is the event where X is never vacuum during the first stage of X. Then

$$\mathbb{E}D(X \mid \mathcal{A}) = O\left(\frac{1}{n\log n}\right). \tag{5.2}$$

The theorem above gives not only an improved bound for the expected delay compared to Theorem 5.1, but also the exact dominating term for expected delay. Furthermore, the dominating term is contributed by the total vacuum time of the process only during the first stage. As signified in (5.2), when we remove all vacuum times of the first stage, we get a much smaller expected delay time. This means that the most significant portion of the delay time comes from the total time spent in the vacuum state during the first stage.

By these results on the expected delay time of the RSDS, we derive a corollary describing the upper and lower bounds for $T(X)$ that hold w.h.p. Recall from (4.3) that the SRS running time is sharply concentrated around its expected value, that is for all $\delta > 0$,

$$\mathbb{P}\left(\left|T(X') - \frac{2\log n}{n}\right| > \frac{2\delta\log n}{n}\right) = o(1). \tag{5.3}$$

We present the corollary as follows. We will provide the proof of the corollary in this part as it is relatively short compared to the proofs of the theorems.

Corollary 5.4. *Let ε be any positive constant and $\omega = \omega(n)$ be an unbounded function growing arbitrarily slowly. Then we have that w.h.p.*

$$(1-\varepsilon)\frac{2\log n}{n} < T(X) < \frac{\omega\log n}{n}. \tag{5.4}$$

Moreover, when $s(n) = o(n/\log n)$, we have that w.h.p.

$$(1-\varepsilon)\frac{2\log n}{n} < T(X) < (1+\varepsilon)\frac{2\log n}{n} \tag{5.5}$$

or equivalently

$$\frac{T(X)}{\log n/n} \xrightarrow{p} 2.$$

Proof. Suppose that $\varepsilon > 0$ arbitrarily. By the stochastic ordering stated in (3.7) and (5.3), we have that

$$\mathbb{P}\left(T(X) < (1-\varepsilon)\frac{2\log n}{n}\right) \leq \mathbb{P}\left(T(X') < (1-\varepsilon)\frac{2\log n}{n}\right) = o(1).$$

The last equation comes from (5.3) by choosing $\delta = \varepsilon$. This shows that the left hand side of inequality (5.4) holds w.h.p.

On the other hand, by Markov's inequality, Theorem 5.1 and Theorem 5.2, we have that

$$\mathbb{P}\left(D(X) > \frac{\omega\log n}{2n}\right) = \frac{O(\log n/n)}{\omega\log n/2n} = o(1)$$

holds for all switching rates. Combining the equation above and (5.3).

$$\mathbb{P}\left(T(X) > \frac{\omega\log n}{n}\right) \leq \mathbb{P}\left(T(X') > \frac{\omega\log n}{2n}\right) + \mathbb{P}\left(D(X) > \frac{\omega\log n}{2n}\right) = o(1).$$

This establishes that the right hand side of inequality (5.4) holds w.h.p.

Now, to show the right hand side of (5.5), we use the fact that $\mathbb{E}D(X) = o(\log n/n)$ when the switching rate is slow. By applying Markov's inequality to $D(X)$ we have that

$$\mathbb{P}\left(D(X) > \frac{\varepsilon\log n}{n}\right) = \frac{\mathbb{E}D(X)}{\varepsilon\log n/n} = o(1). \tag{5.6}$$

Thus, by (5.3) with $\delta = \varepsilon/2$ and (5.6), we obtain that

$$\begin{aligned}\mathbb{P}\left(T(X) > (1+\varepsilon)\frac{2\log n}{n}\right) &\le \mathbb{P}\left(T(X') > \left(1+\frac{\varepsilon}{2}\right)\frac{2\log n}{n}\right) + \mathbb{P}\left(D(X) > \frac{\varepsilon\log n}{n}\right)\\ &= o(1).\end{aligned}$$

□

One interpretation of the corollary is as follows. If the switching rate is slow, then the corollary strengthens the statement that the RSDS process behaves almost the same as the SRS. This is shown not only by the same order of their expected running times but also by the fact that the running time of both processes is strongly concentrated around $2\log n/n$. On the other hand, we also have the same lower bound that holds w.h.p. when the switching rate is fast. This means that for any switching rates, it is very unlikely that the RSDS finishes before time $2\log n/n$. The upper bound for the running time in the case of a fast rate is much looser. However, the result says that the typical running time is a constant multiple of $\log n/n$ where the constant is at least 2. This again affirms that the delaying scheme is incapable of leading the RSDS running time to be much greater than that of the SRS.

5.1 Vacuum Delay Time Analysis

In this section, we will measure the expected vacuum delay time $W(X)$ occurring in the RSDS process. We will mainly discuss the vacuum delay time when the switching rate is slow.

The running time analysis of the RSDS process with a slow switching rate will depend heavily on the result of this section. We will show that as long as the switching rate is $o(n/\log n)$, the expected vacuum delay time is insignificant relative to the expected running time of the SRS, that is, $\mathbb{E}W(X) = o(\log n/n)$. After showing this, we can concentrate our attention on the compressed version of the process alone. Recall that from (4.4), we have that

$$\mathbb{E}D(X) = \mathbb{E}D(X^C) + \mathbb{E}W(X).$$

Hence, showing that $\mathbb{E}D(X^C) = o(\log n/n)$ is sufficient to prove that the expected delay time of X is insignificant. Later in the next section where we will prove Theorem 5.1, we will only pay attention to the analysis of the CRSDS and combine it with the result of this section to complete the whole proof.

To begin the analysis, we define additional terminology as follows. Let $i \in [1, n-1]$. Let $C_i(X)$ denote a random variable which counts how many times X becomes vacuum during stage i. Also, let $W_i(X)$ be the total time that X spends in the vacuum state during stage i. Observe that the total time spent in vacuum condition satisfies

$$W(X) = \sum_{i=1}^{n-1} W_i(X).$$

Now we are ready to present the following proposition.

Proposition 5.5 (Vacuum time of the RSDS with a slow rate)**.** *For any switching rate satisfying $s(n) = o(n/\log n)$, the expected vacuum delay time of the RSDS process X is*

$$\mathbb{E}W(X) = O\left(\frac{\log\log n}{n}\right).$$

Proof. In this proof, we refer to the edge clock RSDS model. All switching and spreading clocks mentioned in this proof are labelled. In other words, the switching and spreading clocks are associated with the vertices and edges of the graph respectively.

To begin, we stochastically bound $C_i(X)$ from above by constructing the following coupling. We introduce a variant of RSDS, denoted by X^*, and couple it to the original process X in such a way that

$$C_i(X) \le C_i(X^*) \tag{5.7}$$

holds. We specify that both processes share the same spreading and switching clocks. They follow the same performance until stage i begins. Among the i informed vertices existing at the beginning of stage i, we pick one particular active vertex and call it the *transient* vertex. In X^*, once stage i starts, we set all informed vertices but the transient vertex to be always dormant afterwards. Also, we ignore all switching clocks associated to them. However, the status of the transient vertex is still flipping around according to its switching clock. Hence, whenever X^* is not vacuum, the spreading rate is always $n-i$ and the dormancy rate is always $s(n)$. Note that whenever X is vacuum, X^* is also necessarily vacuum. This means that (5.7) is satisfied.

Now we examine the distribution of $C_i(X^*)$. The vacuum condition of X^* is completely determined by the status of the transient vertex. When X^* is not vacuum, it waits for the ringing of either the switching clock on the transient vertex or the spreading clocks on the edges incident to the transient vertex. Hence, the probability that it goes vacuum before a new informed vertex appears is $\frac{s(n)}{s(n)+n-i}$. On the other hand, when the transient vertex is dormant (which means that X^* is vacuum), it will deterministically transition back to the state where X^* first enters stage i. Suppose that for every time at which the transient vertex is active, we view the condition as an independent Bernoulli trial where the success refers to the rumour passing to a new vertex. Hence, $C_i(X^*)$ is the number of failures performed until it achieves the first successful trial. Thus, $C_i(X^*) \overset{d}{=} \mathcal{G}\left(1 - \frac{s(n)}{s(n)+n-i}\right)$. It follows that

$$C_i(X) \le_{ST} \mathcal{G}\left(1 - \frac{s(n)}{s(n)+n-i}\right), \tag{5.8}$$

and

$$\mathbb{E}C_i(X) \le \frac{s(n)}{n-i}.$$

We also require to bound $C_i(X)$ in the probability space conditioned on $\{W_i(X) > 0\}$ for the sake of later computations. Note that the conditioning event is exactly the same as $\{C_i(X) \ge 1\}$. In the rest of this paragraph, all probability measures are conditioned on $\{W_i(X) > 0\}$. To get the bound, we again construct a modified RSDS process, X^{**} and couple it with the original process. The process X^{**} is constructed in almost the same way as X^*. The only difference is that the dormancies of non-transient vertices are fixed after the process becomes vacuum for the first time during stage i (instead of the beginning of the stage i). Before the process enters a vacuum state in stage i, we specify that X^{**} and X follow the same transitions scenario. Observe that after X^{**} recovers from the first vacuum condition, that is when the transient vertex wakes up, X^{**} enters a state that is exactly the same as the state at which X^* enters stage i. By the memorylessness property of exponential distributions, the number of occurrences of the next vacuum condition follows the same distribution as the $C_i(X^*)$. Thus, $C_i(X^{**}) \overset{d}{=} 1 + \mathcal{G}\left(1 - \frac{s(n)}{s(n)+n-i}\right)$ where the additional term 1 comes from the first vacuum condition that appears automatically from the conditioning. It follows that

$$\mathbb{E}(C_i(X) \mid W_i(X) > 0) \le 1 + \frac{s(n)}{n-i}. \tag{5.9}$$

Now we bound $W_i(X)$ stochastically. For all i, let F_i be exponential random variables with rate $s(n)i$ independently of each other. Observe that whenever X goes vacuum, it needs to wait for the first ringing of the i switching clocks lying on the informed vertices in order to continue the spreading. This waiting time is exactly a copy of F_i since each of these switching clocks has rate $s(n)$. By noting that X always arrives in the same state whenever it goes in vacuum condition and by the memorylessness property of exponential distributions, we have that

$$W_i(X) = \sum_{j=1}^{C_i(X)} F_i^{(j)} \tag{5.10}$$

where each $F_i^{(j)}$ is an independent copy of F_i.

We use two ways to bound $W_i(X)$ stochastically, by using the couplings (X, X^*) and (X, X^{**}). First, we bound $W_i(X)$ by using $C_i(X^*)$ as the stochastic upper bound on $C_i(X)$ for $1 \le i < \log^2 n$. From (5.8) and (5.10), we have that

$$W_i(X) \le_{ST} \sum_{j=1}^{\hat{H}_i} F_i^{(j)} \tag{5.11}$$

where $\hat{H}_i \overset{d}{=} \mathcal{G}\left(1 - \frac{s(n)}{s(n)+n-i}\right)$ independently from all $(F_i^{(j)})$. Hence, from (5.11), we obtain that

$$\mathbb{E}W_i(X) \le \mathbb{E}\left(\sum_{j=1}^{\hat{H}_i} F_i^{(j)}\right) = \mathbb{E}\left(\mathbb{E}\left[\sum_{j=1}^{\hat{H}_i} F_i^{(j)} \middle| \hat{H}_i\right]\right) = \mathbb{E}\hat{H}_i \mathbb{E}F_i = \frac{1}{i(n-i)}.$$

Note that the last equality comes from the fact that $\mathbb{E}\hat{H}_i = s(n)/(n-i)$ and $\mathbb{E}F_i = (s(n)i)^{-1}$. Hence, we obtain that

$$\sum_{i=1}^{\lfloor \log^2 n \rfloor - 1} \mathbb{E}W_i(X) \le \sum_{i=1}^{\lfloor \log^2 n \rfloor - 1} \frac{1}{i(n-i)} = O\left(\frac{\log\log n}{n}\right) \tag{5.12}$$

Second, we use the coupling of X and X^{**} conditioned on $W_i(X) > 0$ for $\log^2 n \le i \le n-1$. By (5.9) and (5.10), in the probability space conditioned on $\{W_i(X) > 0\}$, we have that

$$W_i(X) \mid_{\{W_i(X)>0\}} \le_{ST} \sum_{j=1}^{H_i} F_i^{(j)}$$

where $H_i \overset{d}{=} 1 + \mathcal{G}\left(1 - \frac{s(n)}{s(n)+n-i}\right)$ independently from all $(F_i^{(j)})$.

Next, we bound the probability of the event $\{W_i(X) > 0\}$. Recall that X^C is the compressed version of X. Observe that if X ever enters a vacuum state during stage i, then it is necessary that there is a time t during stage i where $\mathrm{Do}(X^C, t) > \frac{2i}{3}$. Thus, when $i \ge \log^2 n$, Lemma 4.8 states that

$$\mathbb{P}(W_i(X) > 0) = O(n^{-6}). \tag{5.13}$$

Note that the lemma applies since we only consider the slow rate setting.

Now we bound $\mathbb{E}W_i$. We have that

$$\begin{aligned}
\mathbb{E}W_i(X) &= \mathbb{P}(W_i(X) > 0)\mathbb{E}[W_i(X) \mid W_i(X) > 0] \\
&\le \mathbb{P}(W_i(X) > 0)\mathbb{E}\left(\sum_{j=1}^{H_i} F_i^{(j)}\right) \\
&\le \mathbb{P}(W_i(X) > 0)\mathbb{E}\left(\mathbb{E}\left[\sum_{j=1}^{H_i} F_i^{(j)} \middle| H_i\right]\right) \\
&= \mathbb{P}(W_i(X) > 0)\mathbb{E}H_i\mathbb{E}F_i. \qquad (5.14)
\end{aligned}$$

To bound (5.14), we break into two cases. First, when $n^3 s(n) \to \infty$, we use (5.13) to obtain that $\mathbb{P}(W_i(X) > 0) = O(n^{-6})$ for all $i \ge \log^2 n$. Hence, in this case we have that

$$\begin{aligned}
\mathbb{P}(W_i(X) > 0)\mathbb{E}H_i\mathbb{E}F_i &= O(n^{-6})\left(1 + \frac{s(n)}{(n-i)}\right)\frac{1}{s(n)i} \\
&= \frac{O(n^{-6})}{i(n-i)}.
\end{aligned}$$

It follows that when $n^3 s(n) \to \infty$,

$$\sum_{i=\lfloor \log^2 n \rfloor}^{n-1} \mathbb{E} W_i(X) = \sum_{i=\lfloor \log^2 n \rfloor}^{n-1} \frac{O(n^{-6})}{i(n-i)} = O\left(\frac{\log n}{n^7}\right). \tag{5.15}$$

Now we examine the other case, that is $s(n) = O(n^{-3})$. In this case, we need a smaller bound for $\mathbb{P}(W_i(X) > 0)$. Note that when $i \geq 2$, stage i always starts with at least two active vertices, which are the latest captured vertex and another active vertex from which the captured vertex receives the rumour. This means that the event $\{W_i(X) > 0\}$ implies that X needs to experience switching transitions at least twice (to switch those two active vertices to be dormant). Suppose that $\mathcal{A}$ is the event where at least two switching transitions occur during the whole process. We bound the probability of $\mathcal{A}$ by considering the $n-1$ first state transitions of the process. When $\mathcal{A}$ occurs, it is necessary that there are at least two switching transitions occurring among the first $n-1$ transitions. Now we bound the probability that a given transition is a switching transition. Suppose that X is in stage i when the given transition occurs. Note that there are i switching clocks (where each of them has rate $s(n)$) and at most $i(n-i)$ spreading clocks (with rate 1) whose ringings determine a transition event. Thus, the probability that the earliest ringing clock is a switching clock, is at least

$$\frac{s(n)i}{s(n)i + i(n-i)} = \frac{s(n)}{s(n) + n - i} \geq \frac{s(n)}{2n}.$$

Thus, we have that

$$\mathbb{P}(W_i(X) > 0) \leq \mathbb{P}(\mathcal{A}) \leq \mathbb{P}\left(\mathcal{B}\left(n-1, \frac{s(n)}{2n}\right) \geq 2\right) = O\left(s(n)^2\right).$$

Now, continuing (5.14), we obtain that

$$\begin{aligned}
\mathbb{P}(W_i(X) > 0)\mathbb{E} H_i \mathbb{E} F_i &= O(s(n)^2)\left(1 + \frac{s(n)}{(n-i)}\right)\frac{1}{s(n)i} \\
&= O\left(\frac{s(n)}{i}\right) = O(n^{-3}).
\end{aligned}$$

Hence, when $s(n) = O(n^{-3})$,

$$\sum_{i=\lfloor \log^2 n \rfloor}^{n-1} \mathbb{E} W_i(X) = \sum_{i=\lfloor \log^2 n \rfloor}^{n-1} O(n^{-3}) = O(n^{-2}). \tag{5.16}$$

Now, from (5.15) and (5.16), we have that

$$\sum_{i=\lfloor \log^2 n \rfloor}^{n-1} \mathbb{E} W_i(X) = O(n^{-2}) \tag{5.17}$$

holds for all cases.

Finally, by combining (5.12) and (5.17), the result follows. □

5.2 Proof of Theorem 5.1

In this section, we will always assume that the switching rate is slow, that is $s(n) = o(n/\log n)$.

Also, we will only discuss the running time of the CRSDS process X^C here. As stated in Proposition 5.5, the expected vacuum delay time is $O(\log\log n/n)$. Hence, we can rewrite the expression of the delay time in (4.4) as

$$\mathbb{E}D(X) = \mathbb{E}D(X^C) + O\left(\frac{\log\log n}{n}\right). \tag{5.18}$$

We aim to show that $\mathbb{E}D(X^C) = O(s(n)\log^2 n/n^2) + O(1/n)$ for the case of a slow switching rate. By showing this and considering (5.18), we will get the proof of Theorem 5.1.

We provide the overview of the analysis as follows. We divide the stages of the process into three categories: the *early*, *intermediate*, and *late* stages. For each category, we will analyse the distribution of the number of dormant vertices found during the stages. Although the distributions are varied for different stage categories, they have the same important thing in common. We will show that for each stage, the number of dormant vertices is always insignificant compared to the number of active vertices during the entire stage w.h.p. In the case that there are a small enough number of dormant vertices (occurring with probability close to 1), we apply the strong bound to obtain a stochastic upper bound for $D_i(X)$. In the other case, which occurs with a very low probability, we apply the worst case bound to bound $\mathbb{E}D_i(X)$. Then we analyse the expected delay times of the stages from each category and sum them together to obtain the total expected delay time.

We define the following terms in order to classify the stages. The definitions are based on the switching rates. They have special expressions when $s(n) = O(1)$. Suppose that when $s(n) = O(1)$, M is a positive real constant such that $s(n) \le M$ for all sufficiently large n. Then we define

$$\begin{aligned}
R &= \lfloor 1200\log n\rfloor,\\
P_1 &= \begin{cases} \lfloor 22s(n)\log n\rfloor, & \text{if } s(n)\to\infty,\\ \lfloor (22M+7)\log n\rfloor, & \text{if } s(n) = O(1),\end{cases}\\
P_2 &= \begin{cases} \lfloor 23s(n)\log n\rfloor, & \text{if } s(n)\to\infty,\\ \lfloor (22M+808)\log n\rfloor, & \text{if } s(n) = O(1),\end{cases}\\
Q &= 2P_2
\end{aligned}$$

The constants included in the terms above are chosen for the sake of computational convenience, as we will later discuss in the upcoming parts.

We classify the stages of the CRSDS process as follows. For all $i \in [1, n-1]$, we have the following specification.

Range of i	Name of the stages
$1 \leq i \leq R-1$	The *early* stages
$R \leq i \leq Q-1$	The *intermediate* stages
$Q \leq i \leq n-1$	The *late* stages

The main result of this section is presented as the following proposition.

Proposition 5.6 (Expected Delay of the CRSDS with a slow rate)**.** *If $s(n) = o(n/\log n)$, then*

$$\mathbb{E}D(X^C) = O\left(\frac{s(n)\log^2 n}{n^2}\right) + O\left(\frac{1}{n}\right).$$

Proof. We break our measurements of the delay times into three categories of stages described in the table above. To help the readers track the progressing results of the proposition, we summarise the delay time analysis for each stage classification by including three partial conclusions that can be found throughout this proof, namely Statement 5.8, Statement 5.11, and Statement 5.13. They state the total expected delay times of the early, late, and intermediate stages respectively.

To begin, we recall the definition of $D_i(X^C)$, as stated in (3.6). We have that

$$D_i(X^C) = T_i(X^C) - \mathbb{E}T_i(X') = T_i(X^C) - (i(n-i))^{-1}.$$

First, we analyse the running time of the early stages. We show that having a dormancy during the early stages is relatively unlikely as described by the following claim.

Claim 5.7. *During the first $R-1$ stages, X^C has no dormant vertex with probability $1 - O(\frac{s(n)\log n}{n})$.*

Proof. Consider X^M, the embedded Markov chain of the CRSDS process X^C, whose transition matrix is $P^M = [p_{i,j}]$. Let H be the event that no dormant vertex appears during the first $R-1$ stages. Observe that the process experiences no switching transition before stage R starts if and only if the first $R-1$ state transitions of X^M are entirely the spreading transitions. Hence, H is equivalent to the event

$$\{X_0^M = (0,1), X_1^M = (0,2), \ldots, X_{R-1}^M = (0,R)\},$$

whose probability is

$$\prod_{i=1}^{R-1} p_{(0,i),(0,i+1)}.$$

Now we compute the value of $p_{(0,i),(0,i+1)}$ for each $i \in [1, R-1]$. Suppose that X^C arrives at state $(0,i)$. In this state, it waits for the first ringing among all unlabelled spreading and switching clocks of stage i. If the ringing clock belongs to the spreading clocks set, then X^C moves to state $(0, i+1)$. Thus, $p_{(0,i),(0,i+1)}$ is the probability that the no switching clock rings before the spreading clocks. Hence,

$$p_{(0,i),(0,i+1)} = \frac{i(n-i)}{s(n)i + i(n-i)} = \frac{n-i}{s(n)+n-i}. \tag{5.19}$$

It follows that

$$\begin{aligned}
\mathbb{P}(\bar{H}) &= 1 - \prod_{i=1}^{R-1} \frac{n-i}{s(n)+n-i} \\
&= 1 - \prod_{i=1}^{R-1} \left(1 - \frac{s(n)}{s(n)+n-i}\right) \\
&\leq 1 - \prod_{i=1}^{R-1} \left(1 - \frac{2s(n)}{n}\right) \\
&= 1 - \left(1 - \frac{2s(n)}{n}\right)^{R-1} \\
&= 1 - \left(1 - \frac{2Rs(n)}{n} + O\left(\frac{R^2 s(n)^2}{n^2}\right)\right) \\
&= O\left(\frac{Rs(n)}{n}\right) = O\left(\frac{s(n)\log n}{n}\right).
\end{aligned}$$

The inequality above comes from the fact that $s(n)+n-i \geq n/2$ for all $1 \leq i < R = o(n)$ and $s(n) = o(n/\log n)$. In addition, note also that the approximation in the binomial expansion above holds since $Rs(n)/n = s(n)\log n/n = o(1)$.

To complete the proof, we conclude that H occurs with probability $1 - O(s(n)\log n/n)$. □

Note that the probability bound given by the claim above is $o(1)$ for all choices of slow rates. This means that a dormancy transition is a relatively rare event during the early stages. We will use the worst case bound to estimate the running time when a dormancy transition occurs (whose probability is relatively small). On the other hand, if there is no dormancy at all, the CRSDS behaves exactly the same as the SRS. Thus, typically X^C has no delay at all.

Next, we define the following stopping time to bound $\mathbb{E}T_i(X^C)$. Let $\mathcal{S}_1$ be the first time at which either a dormancy occurs or the process enters stage R. Formally, we write that

$$\begin{aligned}
S_1 &:= \inf\{t \geq 0 : \mathrm{Do}(X^C, t) = 1\}. \\
\mathcal{S}_1 &:= \min(S_1, V_R(X^C)).
\end{aligned}$$

For each $i \in [1, R-1]$, we write $T_i = T_i^{<\mathcal{S}_1} + T_i^{\geq \mathcal{S}_1}$ where $T_i^{<\mathcal{S}_1}$ and $T_i^{\geq \mathcal{S}_1}$, respectively denote the times spent during stage i before and after $\mathcal{S}_1$ as stated in (4.5) and (4.6).

Note that before time $\mathcal{S}_1$, we have that X^C runs exactly the same as the SRS process since it has no dormant vertex. Thus, we can couple $T_i(X^C)^{<\mathcal{S}_1}$ and $T_i(X')$, the running time of stage i in the SRS, as follows. From the beginning of both X^C and X', we specify that both processes share the same unlabelled spreading clocks. Notice that no unlabelled switching clock ever rings before time S_1. Thus, for all $i \in [1, R-1]$, as long as the process has no dormant vertex during stage i, that is when the dormancy process of stage i always takes value 0 during the entire stage, the ringing of any unlabelled spreading clocks will terminate of stage i of both X' and X^C. Hence,

$$T_i(X^C)^{<\mathcal{S}_1} \leq_{ST} T_i(X').$$

It follows that from (3.6),

$$\begin{aligned}\mathbb{E}D_i(X^C) &\leq [\mathbb{E}T_i^{<\mathcal{S}_1}(X^C) - \mathbb{E}T_i(X')] + \mathbb{E}T_i^{\geq\mathcal{S}_1}(X^C) \\ &\leq \mathbb{E}T_i^{\geq\mathcal{S}_1}(X^C). \end{aligned} \tag{5.20}$$

Now we find a bound for $\mathbb{E}T_i^{\geq\mathcal{S}_1}(X^C)$. Suppose that A_i is the event that a dormancy occurs in stage i. We apply Corollary 4.6 with $\tau = \mathcal{S}_1$ and $A = A_i$. Now, since $\mathbb{P}(A_i) \leq \mathbb{P}(\bar{H}) = O(s(n)\log n/n)$ by Claim 5.7, we have that

$$\mathbb{E}T_i^{\geq\mathcal{S}_1}(X^C) \leq \frac{P(A_i)}{n-i} = O\left(\frac{s(n)\log n}{n^2}\right). \tag{5.21}$$

Thus, (5.20) and (5.21) lead to

$$\begin{aligned}\sum_{i=1}^{R-1} \mathbb{E}D_i(X^C) &\leq \sum_{i=1}^{R-1} T_i^{>\mathcal{S}_1}(X^C) \\ &= \sum_{i=1}^{R-1} O\left(\frac{s(n)\log n}{n^2}\right) \\ &= O\left(\frac{s(n)\log^2 n}{n^2}\right). \end{aligned} \tag{5.22}$$

From (5.22), we conclude the following statement regarding the analysis of the early stages.

Statement 5.8 (Early stages of the CRSDS with a slow rate)**.** *If $s(n) = o(n/\log n)$, then the expected delay time during the early stages is*

$$\mathbb{E}\left(\sum_{i=1}^{R-1} D_i(X^C)\right) = O\left(\frac{s(n)\log^2 n}{n^2}\right).$$

Next, we will analyse the delay occurring in the intermediate and late stages. Unlike the early stages, we will find some dormant vertices during these stages w.h.p. However, we will stochastically bound the number of dormant vertices and show that w.h.p. it is always substantially smaller than the number of informed vertices so far. This observation will be the key for the running time analysis on these stages.

We notice that when the switching rate is slow, Corollary 4.8 says that if stage i is either intermediate or late, then the number of dormant vertices during the entire time in the stage is bounded above by $\frac{2i}{3}$ w.h.p. However, we will show that the number of dormant vertices are much smaller than $\frac{2i}{3}$, still with reasonably high probability. Precisely, we prove that during these stages, there are at most P_2 dormant vertices w.h.p. This implies that during the late stages, the new bound is better than Corollary 4.8. This bound will be used to estimate the delay time at the late stages, whereas we will analyse the delay at the intermediate stages by another method described later.

Before we go deeper into the analysis of the intermediate and late stages, we make some observations regarding the number of dormant vertices. We will show that in the case of a slow switching rate, switching transitions are uncommon relative to the spreading transitions. This is due to the small switching transition probabilities yielded by the slow switching rate. This means that the process is more likely to pass the rumour on rather than flipping the statuses of informed vertices. Starting from this, we will provide a stochastic bound for the number of dormant vertices throughout the intermediate and late stages. With that bound in hand, we will apply the strong and worst case bounds to obtain an upper bound for the expected delay times for each stage.

To begin, we define the following stopping times.

$$
\begin{aligned}
S^* :&\quad \inf\{t \geq V_R(X^C) : \mathrm{Do}(X^C, t) > \tfrac{2}{3}\,\mathrm{In}(X^C, t)\}.\\
S_{P_2} :&\quad \inf\{t \geq V_R(X^C) : \mathrm{Do}(X^C, t) > P_2\}.\\
\mathcal{S}_2 :&\quad \min\{S^*, S_{P_2}, T(X^C)\}.
\end{aligned}
$$

We provide the interpretations of the stopping times above as follows. S^* is the time at which the number of dormant vertices in X^C exceeds $2/3$ of the number of informed vertices for the first time, since stage R starts. Similarly, we understand S_{P_2} as the time at which X^C has more than P_2 dormant vertices for the first time since stage R begins. Next, $\mathcal{S}_2$ is the stopping time that picks the earliest time between S^*, S_{P_2} and the finishing time of X^C.

The main aim of defining these stopping times is to prove that w.h.p. the process never experiences more than P_2 dormant vertices during the entire intermediate and late stages. Note that in order to prove this, it is sufficient to show that the probability that $S_{P_2} < T(X^C)$ is small. However, we will provide a slightly stronger result, that is to show that $\mathbb{P}(\mathcal{S}_2 < T(X^C))$ is small. In other words, we will also prove that $S^* < T(X^C)$ w.h.p. along with our main aim. This includes an additional feature that w.h.p. the proportion between the dormant and informed vertices is always bounded above by $2/3$ during the whole intermediate and later stages. This feature will be useful to tackle some technical difficulties to achieve our main aim, as we will later discuss.

For every $i \in [R, n-1]$, we define $Z_i^{<S^*}$ to be the number of switchings occurring during stage i before time S^*. In other words, $Z_i^{<S^*}$ counts the switchings occurring during stage i as long as the stopping criterion of S^* is not satisfied yet. Note that if X^C is in a stage earlier than stage i at time S^*, then $Z_i^{<S^*} = 0$. Also we define

$$Y_i \stackrel{d}{=} \mathcal{G}\left(1 - \frac{3s(n)}{n-i+3s(n)}\right)$$

for $i = R, \ldots, n-1$ independently of each other. Then, the following lemma provides a stochastic upper bound for $Z_i^{<S^*}$.

Lemma 5.9. *For all $i = R, \ldots, n-1$, we have that*

$$Z_i^{<S^*} \leq_{ST} Y_i \tag{5.23}$$

and consequently,

$$\mathbb{E}Z_i^{<S^*} \leq \frac{3s(n)}{n-i}.$$

Proof. Suppose that at a certain time during stage i before S^*, a transition occurs in X^C. We now estimate the probability that the transition is a switching (either dormancy or waking up) transition. Since there are at most $\frac{2i}{3}$ dormant vertices at that time, the spreading rate is at least $\frac{i}{3}(n-i)$. On the other hand, the switching transition rate is always $s(n)i$ during the stage. Thus, the probability that a switching transition occurs at a given transition time, is at most

$$\frac{s(n)i}{s(n)i + i(n-i)/3} = \frac{3s(n)}{n-i+3s(n)}.$$

We can think of $Z_i^{<S^*}$ as the number of transitions occurring during stage i in X^C until either the process experiences the spreading transition or the stopping criterion of S^* is met. We can interpret these transitions as independent Bernoulli trials with success probabilities at least $1 - \frac{3s(n)}{n-i+3s(n)}$. Here the successful trial corresponds to the spreading transition. We then couple $Z_i^{<S^*}$ and Y_i by the following specification. We couple each Bernoulli trial performed by both $Z_i^{<S^*}$ and Y_i by U, a uniformly chosen value from interval $(0, 1)$. Consider the success probability of the trials. If the success probability is greater than U, then we specify that the trial is successful. Now, since the success probability of each trial of $Z_i^{<S^*}$ is always at least that of Y_i, we have the following fact: if a trial from $Z_i^{<S^*}$ fails, then the corresponding trial from Y_i also fails. In this setting, we have that $Z_i^{<S^*} \leq Y_i$ with probability 1. Hence, Lemma 2.8 establishes (5.23).

To complete the proof, we have that $\mathbb{E}Y_i = 3s(n)/(n-i)$. □

Next, we will employ the lemma to show that during the whole process, X^C has an insignificant number of dormant vertices w.h.p. The precise statement is given by the following claim in terms of the stopping times.

Claim 5.10. *$\mathcal{S}_2 = T(X^C)$ with probability $1 - O(n^{-5})$.*

Proof. To prove the claim, we bound the probability of the complementary event by breaking it into two disjoint sub-events. First we consider the event $\{\mathcal{S}_2 < T(X^C)\} \cap \{S^* \leq S_{P_2}\}$. Note that this event is contained in the event $\{S^* < T(X^C)\}$. Observe that the event implies that there exists a time t during some stage $i \in [R, n-1]$ such that $\mathrm{Do}(X^C, t) > \frac{2i}{3}$. By applying Corollary 4.8, with $\varepsilon = \frac{1}{4}$, we have that for a given $i \in [R, n-1]$,

$$\mathbb{P}\left(\mathrm{Do}(X^C, V_i(X^C)) > \frac{2}{3}i\right) = O(n^{-6}). \tag{5.24}$$

Now we apply the union bound on (5.24) for all $i \in [R, n-1]$ to obtain that

$$\mathbb{P}(\{\mathcal{S}_2 < T_R\} \cap \{S^* \leq S_{P_2}\}) = O(n^{-5}). \tag{5.25}$$

Now we bound the probability of $\{\mathcal{S}_2 < T(X^C)\} \cap \{S^* > S_{P_2}\}$. This event implies that the process has more than P_2 dormant vertices during some intermediate stages before the process finishes and before time S^*. Let H_1 be the events where stage R starts with more than $\frac{2}{3}R$ dormant vertices. Also let H_2 be the event that before time S^*, X^C experiences more than P_1 dormancy transitions during the intermediate and late stages. Suppose that neither H_1 nor H_2 occurs. First we consider the case when $s(n) \to \infty$. Here, the maximum number of dormant vertices that can possibly be attained by X^C during the entire intermediate and late stages, is $\frac{2}{3}R + P_1 = 22s(n)\log n(1 + o(1)) \leq P_2$ for sufficiently large n. We get this bound by considering the maximum number of dormant vertices at the start of stage R and the maximum number of switchings experienced by X^C, each of which we consider that it contributes to a new dormant vertex. Second, when $s(n) = O(1)$, a similar bound also applies. In this case, X^C always has at most $\frac{2}{3}R + P_1 \leq (807 + 22M)\log n \leq P_2$ dormant vertices during the whole intermediate and late stages. This means that if neither H_1 nor H_2 occurs, then $\{\mathcal{S}_2 < T_R\} \cap \{S^* > S_{P_2}\}$ also cannot occur. This implies that

$$\mathbb{P}(\{\mathcal{S}_2 < T_R\} \cap \{S^* > S_{P_2}\}) \leq \mathbb{P}(H_1) + \mathbb{P}(H_2). \tag{5.26}$$

We apply Lemma 4.7 to bound the probability of H_1 and specifically we choose $\varepsilon = \frac{1}{3}$. For any $i \geq R$, we have that

$$\mathbb{P}(H_1) = \mathbb{P}\left(\mathrm{Do}(X^C, V_R(X^C)) > \frac{2}{3}i\right) < \exp\left(-\frac{\varepsilon^2 i}{6}\right) = O(n^{-5}). \tag{5.27}$$

Now we bound the probability of H_2. Define $Z^{<S^*} = \sum_{i=R}^{n-1} Z_i^{<S^*}$, that is the number of switchings occurring during the intermediate and late stages before time S^*. Similarly,

we define $Y = \sum_{i=R}^{n-1} Y_i$. Observe that the probability of H_2 is bounded above by $\mathbb{P}\{Z^{<S^*} > P_1\}$, since the switchings counted by $Z^{<S^*}$ occur before time S^*. It is worth noting that this signifies the essential of proving a stronger result by involving S^* mentioned in the earlier paragraph discussing the main aim of defining the stopping times. Now, observe that Lemma 5.9 implies that $Z^{<S^*} \leq_{ST} Y$. Note also that for large enough n,

$$\mathbb{E}Y = 3s(n)(\log n + O(1)) \leq \begin{cases} 3s(n)\log n + O(s(n)), & \text{if } s(n) \to \infty, \\ (3M+1)\log n, & \text{if } s(n) = O(1). \end{cases}$$

We apply the sharp concentration inequality for the sums of geometric random variables to Y stated in Lemma 2.12. Observe that the minimum value among all success probabilities of $\{Y_i\}_{R \leq i \leq n-1}$ is $(3s(n))^{-1}$. The minimum value is attained by the success probability of Y_{n-1}. Then by applying Lemma 2.12 to Y, with $p^* = (3s(n))^{-1}$ and $t = 7$, we obtain the following result.

$$\begin{aligned} \mathbb{P}(Y > P_1) \leq \mathbb{P}(Y > 7\mathbb{E}Y) &< \exp(-p^*\mathbb{E}Y(t - 1 - \log t)) \\ &= \exp(-5.155 \log n) \\ &= O(n^{-5}). \end{aligned}$$

One can check that the inequality above applies for any choice of slow switching rates since $7\mathbb{E}Y \leq P_1$ for sufficiently large n.

It follows that

$$\mathbb{P}(H_2) \leq \mathbb{P}(Z^{<S^*} > P_1) \leq \mathbb{P}(Y > P_1) = O(n^{-5}). \tag{5.28}$$

Thus, by putting (5.27) and (5.28) to (5.26), we obtain that

$$\mathbb{P}(\{\mathcal{S}_2 < T_R\} \cap \{S^* > S_{P_2}\}) = O(n^{-5}). \tag{5.29}$$

Therefore, from (5.25) and (5.29), we conclude that

$$\mathbb{P}(\mathcal{S}_2 < T_R) = O(n^{-5}).$$ □

The claim implies that once stage R starts, the process will not have more than P_2 dormant vertices until every vertex is informed with probability $1 - O(n^{-5})$. This provides a much stronger bound on the number of dormant vertices than Corollary 4.8 and still with reasonably high probability.

Now we bound the delay time during the late stages (and will come back to the intermediate stages later). Let $T_i^{<\mathcal{S}_2}$ and $T_i^{\geq \mathcal{S}_2}$ respectively denote the time spent in stage i before and after time $\mathcal{S}_2$ accordingly to the ones defined in (4.5) and (4.6). We will apply the bound provided by Claim 5.10 together with the strong bound in order to obtain the stochastic bound for $T_i^{<\mathcal{S}_2}$. On the other hand, we will again use the worst case bound to bound $\mathbb{E}T_i^{\geq \mathcal{S}_2}$.

For $Q \leq i \leq n-1$, we bound $\mathbb{E}T_i^{\geq \mathcal{S}_2}$ again by the worst case bound. Suppose that B_i denotes the event that X^C is in stage i at time $\mathcal{S}_2$. Observe that from Claim 5.10, we have that

$$\mathbb{P}(B_i) \leq \mathbb{P}(\mathcal{S}_2 < T(X^C)) = O(n^{-5}).$$

Now we apply Corollary 4.6 with $\tau = \mathcal{S}_2$ and $A = B_i$ in order to obtain that

$$\mathbb{E}T_i^{\geq \mathcal{S}_2} \leq \frac{\mathbb{P}(B_i)}{n-i} \leq \frac{O(n^{-5})}{n-i}. \tag{5.30}$$

Hence,

$$\begin{aligned} \sum_{i=Q}^{n-1} \mathbb{E}T_i^{\geq \mathcal{S}_2} &= \sum_{i=Q}^{n-1} \frac{O(n^{-5})}{n-i} \\ &= O\left(\frac{\log n}{n^5}\right). \end{aligned} \tag{5.31}$$

Now, we bound $T_i^{<\mathcal{S}_2}$ stochastically for $Q \leq i \leq n-1$ by the strong bound. We apply Lemma 4.4 to bound $T_i^{<\mathcal{S}_2}$ with $P = P_2$ and $k = R$. The lemma affirms that

$$T_i^{<\mathcal{S}_2} \leq_{ST} \mathcal{E}((i-P_2)(n-i)),$$

and consequently

$$\mathbb{E}T_i^{<\mathcal{S}_2} \leq \frac{1}{(i-P_2)(n-i)}. \tag{5.32}$$

It follows that

$$\begin{aligned} \mathbb{E}T_i^{<\mathcal{S}_2} - \frac{1}{i(n-i)} &\leq \frac{1}{(i-P_2)(n-i)} - \frac{1}{i(n-i)} \\ &= \frac{P_2}{i(i-P_2)(n-i)} \\ &\leq \frac{2P_2}{i^2(n-i)} \end{aligned} \tag{5.33}$$

where the last inequality comes from the fact that $i \geq Q = 2P_2$ and thus $i - P_2 \geq \frac{i}{2}$ for all $i \in [Q, n-1]$.

Therefore, we can bound the expected delay times occurring at late stages as follows.

$$
\begin{aligned}
\sum_{i=Q}^{n-1} \mathbb{E}D_i &= \sum_{i=Q}^{n-1} \left(\mathbb{E}T_i - \frac{1}{i(n-i)} \right) \\
&= \sum_{i=Q}^{n-1} \mathbb{E}T_i^{\geq \mathcal{S}_2} + \sum_{i=Q}^{n-1} \left(\mathbb{E}T_i^{< \mathcal{S}_2} - \frac{1}{i(n-i)} \right) \\
&\leq O\left(\frac{\log n}{n^5} \right) + \sum_{i=2P_2}^{n-1} \frac{2P_2}{i^2(n-i)} \quad \text{(by (5.33) and (5.31))} \\
&\leq O\left(\frac{\log n}{n^5} \right) + \sum_{i=2P_2}^{n/2} \frac{4P_2}{i^2 n} + \sum_{i=n/2}^{n-1} \frac{8P_2}{n^2(n-i)} \qquad (5.34) \\
&= O\left(\frac{1}{n} \right). \qquad (5.35)
\end{aligned}
$$

Observe that the first summation term in (5.34) comes from the fact that $n - i \geq n/2$ when $i \leq n/2$. On the other hand, the approximation in (5.35) is due to the fact that $\sum_{i=M}^{N} i^{-2} = O(M^{-1})$, provided that $M, N \to \infty$ and $M = o(N)$.

Hence, we conclude the following partial conclusion for the late stages.

Statement 5.11 (Late stages of the RSDS with a slow rate)**.** *If $s(n) = o(n/\log n)$, then the expected delay time during the late stages is*

$$
\mathbb{E}\left(\sum_{i=Q}^{n-1} D_i(X^C) \right) = O\left(\frac{1}{n} \right).
$$

Now, we analyse the delay during intermediate stages. We define $L = Q/R$. Now we break the intermediate stages into a set of stage intervals as follows. For $k = 1, 2, \ldots, L-1$, we say that an intermediate stage i is in *interval* k if $kR \leq i < (k+1)R$. Let M_k be the number of switchings occurring during the intermediate stages up to interval k and before S^*, or equivalently

$$
M_k = \sum_{i=R}^{(k+1)R-1} Z_i^*.
$$

The random variable M_k is stochastically bounded nicely from above. Recall from Lemma 5.9 that $Z_i^* \leq Y_i$ where $Y_i \stackrel{d}{=} \mathcal{G}\left(1 - \frac{3s(n)}{n-i+3s(n)}\right)$. Observe that for any $i = o(n)$, the success probability parameter of Y_i can be bounded from above by $\frac{4s(n)}{n}$. Thus, we have that $Y_i \leq_{ST} \mathcal{G}\left(1 - \frac{4s(n)}{n}\right)$. Suppose that for $k = 1, \ldots, L-1$, we define $\hat{Y}_k$ as the sum of kR identical and independent copies of $\mathcal{G}(1 - \frac{4s(n)}{n})$. By Lemma 5.9, we can write that

$$
M_k \leq_{ST} \sum_{i=R}^{(k+1)R-1} Y_i \leq_{ST} \hat{Y}_k.
$$

Now, suppose that for $m = 1, 2 \ldots, kR$, we define $G_m \stackrel{d}{=} \mathcal{G}\left(1 - \frac{4s(n)}{n}\right)$ independently of each other. Then, observe that

$$\mathbb{E}\hat{Y}_k = \mathbb{E}\left(\sum_{m=1}^{kR} G_m\right) = \frac{4kRs(n)}{n} + O\left(\frac{kRs(n)^2}{n^2}\right).$$

Next, we apply the sharp concentration inequality for the sums of identical geometric random variables to $\hat{Y}_k$ given by Lemma 2.13, with $p = 1 - \frac{4s(n)}{n}$ and $m = kR$. For all constant $t > 1$, we have that

$$\mathbb{P}\left(\hat{Y}_k > \frac{5tkRs(n)}{n}\right) \leq \exp\left(-\Omega\left(\frac{nkR}{ts(n)}\right)\right) = e^{-\omega(Rk\log n)} = o(n^{-R}). \tag{5.36}$$

By putting $t = 6/5$ in (5.36), we have that

$$\mathbb{P}\left(M_k > \frac{6kRs(n)}{n}\right) < \mathbb{P}\left(\hat{Y}_k > \frac{6kRs(n)}{n}\right) = o(n^{-R}).$$

For each $k = 1, \ldots, L-1$, we define the following terms. Let $Q_k := \frac{6kRs(n)}{n} + \frac{2R}{3}$. Define the stopping times

$$\begin{aligned} S_{Q_k} &: \inf\{t \geq V_R(X^C) : \mathrm{Do}(X^C, t) > Q_k\}, \\ \mathcal{S}^{(k)} &: \min\{S_{Q_k}, S^*, V_{(k+1)R}\}. \end{aligned}$$

The interpretation of the stopping times above is given as follows. We can think of S_{Q_k} as the time where X^C has more than Q_k dormant vertices for the first time since stage R starts. On the other hand, $\mathcal{S}^{(k)}$ is the earliest time among S^*, S_{Q_k} and the finishing time of interval k.

In analogy to Claim 5.10, we present the following claim in the context of intermediate stages.

Claim 5.12. *$\mathcal{S}^{(k)} = V_{(k+1)R}$ with probability $1 - O(n^{-5})$.*

Proof. First we claim that

$$\mathbb{P}(\{\mathcal{S}^{(k)} < V_{(k+1)R}\} \cap \{S^* \leq S_{Q_k}\}) \leq O(n^{-5}). \tag{5.37}$$

Note that the event above is contained in the event $\{S^* < V_{(k+1)R}\}$. Now, the latter event implies that there exists a time t during an intermediate stage i with $R \leq i < (k+1)R$ such that $\mathrm{Do}(X^C, t) > \frac{2i}{3}$. By Corollary 4.8 with $\varepsilon = \frac{1}{4}$, we have that for a fixed i with $R \leq i < (k+1)R$, the process never has more than $\frac{2i}{3}$ dormant vertices at any times during stage i, with probability $1 - O(n^{-6})$. Then, by applying the union bound, we have that (5.37) holds.

On the other hand, now we bound the probability of the event $\{\mathcal{S}^{(k)} < V_{(k+1)R}\} \cap \{S^* > S_{Q_k}\}$. This event implies that there exists a time t during an intermediate stage

i with $R \leq i < (k+1)R$ and before S^*, such that $\mathrm{Do}(X^C, t) > \frac{6kRs(n)}{n} + \frac{2R}{3}$. Now suppose that C_1 and C_2 are respectively the events that $\mathrm{Do}(X^C, V_R) > \frac{2R}{3}$ and that there are more than $\frac{6kRs(n)}{n}$ dormancy transitions occurring during intermediate stages before interval $k+1$ begins and before time S^*. Observe that if both $\bar{C}_1$ and $\bar{C}_2$ occur, then $\{\mathcal{S}^{(k)} < V_{(k+1)R}\} \cap \{S^* > S_{Q_k}\}$ cannot occur. This means that

$$\mathbb{P}(\{\mathcal{S}^{(k)} < V_{(k+1)R}\} \cap \{S^* > S_{Q_k}\}) \leq \mathbb{P}(\bar{C}_1) + \mathbb{P}(\bar{C}_2). \tag{5.38}$$

Again, C_1 occurs with probability $1 - O(n^{-6})$ by Corollary 4.8. On the other hand, to let $\bar{C}_2$ occur, the process needs to experience at least $\frac{6kRs(n)}{n}$ switchings before time S^*. Thus, $\bar{C}_2$ is contained in the event $\{M_k > \frac{6kRs(n)}{n}\}$. Hence by (5.36), its probability is at most $O(n^{-R})$. Thus, by plugging in the upper bounds for the probability of $\bar{C}_1$ and $\bar{C}_2$ to (5.38), we obtain that

$$\mathbb{P}(\{\mathcal{S}^{(k)} < V_{(k+1)R}\} \cap \{S^* > S_{Q_k}\}) = O(n^{-6}). \tag{5.39}$$

Finally, from (5.37) and (5.39), we conclude that

$$\mathbb{P}(\mathcal{S}^{(k)} < V_{(k+1)R}) \leq O(n^{-5}). \qquad \square$$

This claim guarantees that within the first k intervals of the intermediate stages, the process always has less than Q_k dormant vertices w.h.p. Again, we will employ this bound on the number of dormant vertices together with the strong and worst case bounds in order to bound $\mathbb{E}T_i$.

Suppose that an intermediate stage i is in interval k. We write $T_i = T_i^{<\mathcal{S}^{(k)}} + T_i^{\geq \mathcal{S}^{(k)}}$, where $T_i^{<\mathcal{S}^{(k)}}$ and $T_i^{\geq \mathcal{S}^{(k)}}$ denote the running time of stage i before and after $\mathcal{S}^{(k)}$ accordingly to (4.6) and (4.5), respectively.

Following similar steps to the arguments for the late stages, we bound $\mathbb{E}T_i^{\geq \mathcal{S}^{(k)}}$ by the worst case bound. Let $\tilde{B}_i$ be the event that at time $\mathcal{S}^{(k)}$, the process is in stage i. Note that if $\tilde{B}_i$ occurs, then $\mathcal{S}^{(k)} < V_{(k+1)R}$. Thus, by Claim 5.38, we infer that

$$\mathbb{P}(\tilde{B}_i) = O(n^{-5}).$$

Again, we apply Corollary 4.6 by plugging in $\tau = \mathcal{S}^{(k)}$ and $A = \tilde{B}_i$ to obtain that

$$\begin{aligned} \mathbb{E}T_i^{\geq \mathcal{S}^{(k)}} &\leq \frac{\mathbb{P}(\tilde{B}_i)}{n-i} \\ &\leq \frac{O(n^{-5})}{n-i}. \end{aligned}$$

On the other hand, we bound $T_i^{<\mathcal{S}^{(k)}}$ by the strong bound given in Lemma 4.4. Putting in $P = Q_k$ and $k = R$ into the lemma, we have that

$$T_i^{<\mathcal{S}^{(k)}} \leq_{ST} \mathcal{E}((i - Q_k)(n-i))$$

and

$$\mathbb{E}T_i^{<\mathcal{S}^{(k)}} \le \frac{1}{(1-Q_k)(n-i)}.$$

Thus, we obtain that

$$\begin{aligned}
\mathbb{E}D_i = \mathbb{E}T_i^{<\mathcal{S}^{(k)}} - \frac{1}{i(n-i)} + \mathbb{E}T_i^{\ge\mathcal{S}^{(k)}} &\le \frac{1}{(i-Q_k)(n-i)} - \frac{1}{i(n-i)} + \frac{O(n^{-5})}{n-i} \\
&\le \frac{2}{n}\left(\frac{Q_k}{i(i-Q_k)} + O(n^{-5})\right) \\
&\le \frac{2}{n}\left(\frac{2R}{3i^2} + \frac{6kRs(n)}{ni(kR - \frac{6kRs(n)}{n} - \frac{2R}{3})} + O(n^{-5})\right) \\
&= O\left(\frac{R}{ni^2}\right) + O\left(\frac{s(n)}{in^2}\right) + O(n^{-6}).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\sum_{i=R}^{Q-1} \mathbb{E}D_i &= \sum_{i=R}^{Q-1}\left(O\left(\frac{R}{ni^2}\right) + O\left(\frac{s(n)}{in^2}\right) + O(n^{-6})\right) \\
&= O\left(\frac{1}{n}\right) + O\left(\frac{s(n)\log(Q/R)}{n^2}\right) + O(n^{-5}) \\
&= O\left(\frac{1}{n}\right). \qquad (5.40)
\end{aligned}$$

Having (5.40) in hand, we conclude the analysis of the intermediate stages, summarised by the following statement.

Statement 5.13 (Intermediate stages of the RSDS with a slow rate)**.** *If $s(n) = o(n/\log n)$, then the expected delay time during the intermediate stages is*

$$\mathbb{E}\left(\sum_{i=R}^{Q-1} D_i(X^C)\right) = O\left(\frac{1}{n}\right).$$

Combining together Statement 5.8, Statement 5.11, and Statement 5.13 for all the three categories of stages, we conclude that

$$\mathbb{E}D(X^C) = \mathbb{E}\left(\sum_{i=1}^{n-1} D_i\right) = O\left(\frac{s(n)\log^2 n}{n^2}\right) + O\left(\frac{1}{n}\right).$$

This concludes the delay time analysis for the CRSDS process with a slow switching rate. □

To close this section, we state the following equation. From Proposition 5.5 and Proposition 5.6, we have that the expected delay for the RSDS process with a slow switching rate is

$$\begin{aligned}
\mathbb{E}D(X) &= \mathbb{E}D(X^C) + \mathbb{E}W(X) \\
&= O\left(\frac{s(n)\log^2 n}{n^2}\right) + O\left(\frac{\log\log n}{n}\right).
\end{aligned}$$

This proves Theorem 5.1. □

5.3 Proof of Theorem 5.2

Throughout this section, we assume that $s(n) = \Omega(n/\log n)$. Unlike the slow rate case, all arguments in this section will consider the ordinary RSDS process X, without giving much attention to its compressed version.

We will provide a bound for the expected delay time at the case of a fast rates. We provide both lower and upper bounds for the expected delay of the RSDS process. We aim to show that both bounds are of order $\log n/n$.

In contrast with the case of a slow switching rate, the RSDS process will frequently enter a vacuum state during a given time period if the statuses of the informed vertices switch frequently enough. Recall that when the switching rate is slow, the process never arrives at a vacuum state w.h.p. during a relatively late stage, since the process will be more likely to conduct the spreading transition than switching the informed vertices. The faster the switching rate is, the more frequently the informed vertices switch before a newly informed vertex appears. This provides a reasonable possibility for the process to become vacuum.

To give a rough picture of this situation, we can think of the switchings experienced by the informed vertices as an unbiased random walk on the set of binary strings. Suppose that we represent the statuses of k informed vertices by a binary string with length k. Here, each bit represents the status of a particular vertex and the 1's denote the active vertices. Then a switching experienced by a vertex means that we flip the bit corresponding to the vertex. Thus, we can translate the sequence of the switching transitions as a random walk on the binary strings set $\{0,1\}^k$ where we can only move to the strings having exactly one different binary coordinate to the current position, with an equally likely random choice. It is well known that the stationary distribution of this random walk is the uniform distribution over $\{0,1\}^k$. Hence, if the vertices switch frequently enough, then the process will eventually become vacuum at some point.

Observe that during a given time interval, it is likely that the RSDS becomes vacuum for a very quick subinterval when the switching rate is fast. However, we showed that when we pick a fixed time point, the probability that the RSDS process is vacuum at that time is very small, as described in Lemma 4.7. This will be the key observation for the arguments in the forthcoming parts.

First, we describe the lower bound for the expected delay time as follows.

Proposition 5.14 (The lower bound delay of the RSDS with a fast rate)**.** *If* $s(n) =$

$\Omega(n/\log n)$, *then*

$$\mathbb{E}D(X) = \Omega\left(\frac{\log n}{n}\right).$$

Proof. The key idea to obtain the lower bound is the fact that with a fast switching rate, informed vertices tend to switch for a lot of times before any new vertex is informed. By experiencing a lot of switchings, significant number of informed vertices get chances to become dormant. Once a significant fraction of dormant vertices arises in some stages, the spreading rate also drops in a considerable way. This results in the significant delay time of the process.

To show this more precisely, we construct a modified CRSDS process X_*, coupled to the original process X, to provide a stochastic lower bound for the running time of X. In this coupling, we will use the spreading and switching clocks introduced in the RSDS edge clock model. In X_*, we do not allow any dormancy transition during the first $\lfloor \frac{n}{2} \rfloor - 1$ stages. In other words, before the process has $\lfloor \frac{n}{2} \rfloor$ informed vertices, the switching clocks have no effect. We start linking both processes once stage $\lfloor \frac{n}{2} \rfloor$ of each process starts. For each dormant vertex in X at time $V_{\lfloor n/2 \rfloor}(X)$, we specify the corresponding vertex in X_* to be always active from time $V_{\lfloor n/2 \rfloor}(X_*)$ until X_* ends. On the other hand, for each active vertex in X at time $V_{\lfloor n/2 \rfloor}(X)$, we set the corresponding vertex in X_* to share the same switching clock from time $V_{\lfloor n/2 \rfloor}(X_*)$ until X_* ends. We call these active vertices the *delaying* vertices. Observe that the status of every delaying vertex in both processes is always the same once stage $\lfloor \frac{n}{2} \rfloor$ of each process starts. We specify that every new informed vertex appearing after stage $\lfloor n/2 \rfloor$ will be always active in X_*. In other words, the switching clocks only affect the delaying vertices in X_* after time $V_{\lfloor n/2 \rfloor}(X_*)$, whereas the non-delaying vertices are always active. In this setting, we have that at all times, whenever a vertex is active in X, it is also active in X_*. This implies that whenever a spreading transition occurs in X, it also occurs in X_*. It follows that $T_i(X_*) \leq T_i(X)$. Hence, the coupling implies that

$$\mathbb{E}T(X_*) \leq \mathbb{E}T(X).$$

Now we aim to stochastically bound from below the running time of X_*. Define

$$L = n - \lfloor n^{1/3} \rfloor.$$

Referring to (3.7), for any stage i with $i \leq L$, we simply use $T_i(X')$, the running time in the SRS process, as the stochastic lower bound for $T_i(X_*)$. In other words, we use the very trivial lower bound $\mathbb{E}D_i(X_*) \geq 0$ for all $i \leq L$. On the other hand, we will show that after stage L, w.h.p. there are $\Theta(n)$ dormant vertices until the process ends. Although we only calculate the delay for the last $\lfloor n^{1/3} \rfloor$ stages, we will show that the expected delay time during these stages is significant.

To show this, we consider the time $V_{\lfloor n/2 \rfloor} + \frac{1}{2}\log n/n$. At this time, we bound the number of two types of vertex: dormant vertices among the delaying vertices, and new informed vertices achieved from the spreading actions. Using these bounds, we aim to show that the following two events occur at time $V_{\lfloor n/2 \rfloor} + \frac{\log n}{2n}$ w.h.p: there is a constant fraction of dormant vertices among the delaying vertices, and stage L is not started yet. We present these as the following two observations.

Observation 5.15. *There exists a constant $d_0 \in \left(0, \frac{1}{10}\right)$ such that the following fact holds. Suppose that we list the ringing times of all spreading clocks starting from time $V_{\lfloor n/2 \rfloor} + \frac{1}{2}\log n/n$ in ascending order as follows.*

$$V_{\lfloor n/2 \rfloor} + \frac{\log n}{2n} < r_1 < r_2 < \cdots$$

Then, for all integers $k \geq 1$,

$$\mathbb{P}(\mathrm{Do}(X_*, r_k) \leq d_0 n) = O(n^{-n^{1/3}}).$$

Proof. Observe that we can write $r_k = V_{\lfloor n/2 \rfloor} + \frac{1}{2}\log n/n + t'$ for some random time $t \geq 0$ which depends only on the ringing of the spreading clocks after time $V_{\lfloor n/2 \rfloor} + \frac{1}{2}\log n/n$. Now write $t = t' + \frac{1}{2}\log n/n$. For the rest of the proof, we will focus on observing the statuses of the delaying vertices at time r_k. Note that the statuses on the delaying vertices depend only on the ringing of the switching clocks associated to them after time $V_{\lfloor n/2 \rfloor}$. This means that the source of randomness in the arguments of the proof comes solely from the switching clocks. For this reason, we will regard t as a fixed function in the context of this proof.

Suppose that $Z(t)$ is the number of dormant vertices among the delaying vertices in X_* at time $V_{\lfloor n/2 \rfloor} + t$. For any delaying vertex v, let $Z_v(t)$ be the indicator random variable for v being dormant. Suppose that $V_D \subseteq V(G)$ denotes the set of all delaying vertices. We can write $Z(t) = \sum_{v \in V_D} Z_v(t)$. Note that v is dormant at time $V_{\lfloor n/2 \rfloor} + t$ if and only if v receives an odd number of switchings during the predetermined time interval $[V_{\lfloor n/2 \rfloor}, V_{\lfloor n/2 \rfloor} + t]$. Suppose that $P_v(t)$ counts how many times the switching clock of v rings during $[V_{\lfloor n/2 \rfloor}, V_{\lfloor n/2 \rfloor} + t]$. Then, $P_v(t)$ is a Poisson distributed random variable with mean $ts(n)$. Hence, adapting (4.9), we obtain

$$\mathbb{E}Z_v(t) = \mathbb{P}(P_v(t) \text{ is odd}) = \frac{1}{2}(1 - e^{-2ts(n)}).$$

Now, we notice that by Lemma 4.7, there are at most $\frac{4}{5}n$ dormant vertices at time $V_{\lfloor n/2 \rfloor}(X)$ in the process X with probability $1 - \exp(-\Omega(n))$. It follows that the number of delaying vertices of the process X_* is at least $\frac{n}{5}$ with probability $1 - \exp(-\Omega(n))$. Thus, we have that $Z(t) \geq_{ST} \mathcal{B}(\frac{n}{5}, \frac{1}{2}(1 - e^{-2ts(n)}))$. Now for any $u \geq \frac{1}{2}\log n/n$, it follows that there exists a constant $d_0 \in (0, \frac{1}{10})$ such that $\mathbb{E}Z(u) \geq d_0 n$. By Lemma 2.10, we have that

$$\begin{aligned}\mathbb{P}(Z(u) < d_0 n) &\leq \mathbb{P}\left[\mathcal{B}\left(\frac{n}{5}, \frac{1}{2}(1 - e^{-2us(n)})\right) < d_0 n\right] \\ &\leq \exp(-\Omega(n)),\end{aligned}$$

which completes the proof. □

We remark that the non-standard condition for t described in the observation above will be useful to tackle some upcoming technical issues as we will later discuss.

Observation 5.16. *With probability $1 - O(n^{-n^{1/3}})$, there are at least $n^{1/3}$ uninformed vertices at time $V_{\lfloor n/2 \rfloor} + \frac{1}{2}\log n/n$ or equivalently, $V_L(X_*) \geq V_{\lfloor n/2 \rfloor} + \frac{1}{2}\log n/n$.*

Proof. Let N be the number of uninformed vertices at time $V_{n/2} + \frac{1}{2}\log n/n$. Define $U_k^* := \sum_{i=\lfloor n/2 \rfloor}^{n-k} T_i(X_*)$, that is the time spent for X_* to move from stage $\lfloor \frac{n}{2} \rfloor$ to stage $n-k$. Observe that $N < k$ if and only if $U_k^* < \frac{1}{2}\log n/n$. Let $\{E_i\}_i$ be independent exponential random variables where E_i has rate $i(n-i)$. Recall that E_i has the same distribution as $T_i(X')$, the running time of stage i in the SRS process. Hence, E_i is stochastically smaller that $T_i(X_*)$ and we write that

$$E_i \leq_{ST} T_i(X_*).$$

From this, we also have that

$$\sum_{i=\lfloor n/2 \rfloor}^{n-k} E_i \leq_{ST} U_k^*.$$

We now show that $N \geq n^{1/3}$ with probability $1-O(n^{-n^{1/3}})$. We bound the probability of the complementary event $\{U_{\lfloor n^{1/3} \rfloor}^* < \frac{1}{2}\log n/n\}$. As an upper bound for the probability, we calculate the probability that $\sum\limits_{i=\lfloor n/2 \rfloor}^{L} E_i < \frac{1}{2}\log n/n$. Notice that

$$\begin{aligned}\mathbb{E}\left(\sum_{i=\lfloor n/2 \rfloor}^{L} E_i\right) &= \sum_{i=\lfloor n^{1/3} \rfloor}^{\lfloor n/2 \rfloor} \frac{1}{i(n-i)} \\ &= \frac{1}{n}\sum_{i=\lfloor n^{1/3} \rfloor}^{\lfloor n/2 \rfloor}\left(\frac{1}{i} + \frac{1}{n-i}\right) \\ &\leq \frac{2\log n}{3n} + O\left(\frac{1}{n}\right).\end{aligned}$$

Next, we employ the sharp concentration inequality for the sums of exponential random variables. Specifically, we apply the second inequality of Lemma 2.11 with $u = 3/4$

and $W = \sum\limits_{i=\lfloor n/2 \rfloor}^{L} E_i$. From this, we obtain

$$\mathbb{P}\left(\sum_{i=\lfloor n/2 \rfloor}^{L} E_i < \frac{\log n}{2n}\right) = \exp(-\Omega(n^{1/3} \log n)).$$

It follows that

$$\mathbb{P}(N < n^{1/3}) = \exp(-\Omega(n^{1/3} \log n)). \qquad \square$$

Together, Observation 5.15 and Observation 5.16 imply an important corollary. For the rest of the proof, d_0 refers to the constant whose existence is stated in Observation 5.15.

Corollary 5.17. *Suppose that we list the ringing times of all spreading clocks starting from time V_L in ascending order as follows.*

$$V_L < r_1^* < r_2^* < \cdots$$

Then, for all integers $k \geq 1$,

$$\mathbb{P}(\mathrm{Do}(X_*, r_k^*) \leq d_0 n) = O(n^{-n^{1/3}}). \tag{5.41}$$

Proof. Suppose that A denotes the event described in (5.41) and B denotes the event that $r_k^* \leq V_{\lfloor n/2 \rfloor} + \frac{1}{2} \log n/n$. By Observation 5.16, we have that $\mathbb{P}(B) = O(n^{-n^{1/3}})$. Now we consider the event A conditioned on $\bar{B}$. Observe that when $\bar{B}$ occurs, r_k^* satisfies the condition given by Observation 5.15. Thus, $\mathbb{P}(A \mid \bar{B}) = \exp(-\Omega(n))$.

Therefore,

$$\begin{aligned}
\mathbb{P}(A) &= \mathbb{P}(A \mid B)\mathbb{P}(B) + \mathbb{P}(A \mid \bar{B})\mathbb{P}(\bar{B}) \\
&\leq \mathbb{P}(B) + \mathbb{P}(A \mid \bar{B}) \\
&= O(n^{-n^{1/3}}).
\end{aligned} \qquad \square$$

We will employ this corollary to bound the expected running times of the last $\lfloor n^{1/3} \rfloor$ stages of X.

For all $L \leq i \leq n-1$, we now find a stochastic lower bound for $T_i(X_*)$. To begin, we define the unlabelled versions of the switching and spreading clocks of stage i in X_*. Let $\{S_j\}_{1 \leq j \leq i}$ and $\{R_j\}_{1 \leq j \leq i}$ be respectively the unlabelled spreading and switching clocks of stage i in X_*. Also, we introduce the dormancy process of stage i, denoted by $\{d(t) : t \geq 0\}$. We let the unlabelled clocks and dormancy process work in almost the same way as the rules defined in Rules 4.3 with some exceptions. Suppose that Δ is the (random) number of the delaying vertices. We specify that whenever $d(t) = \Delta$, we abandon Rule 2, which means that we do not allow a dormancy transition since all of the active vertices at that time are the non-delaying vertices (that are supposed to be always active by the definition of X_*). On the other hand, we let all other rules apply in the usual way when $d(t) < \Delta$.

Now, we construct a coupling $(\hat{T}_i, \tilde{T}_i)$ in such a way that

$$\tilde{T}_i \geq \hat{T}_i \tag{5.42}$$

where the exact definitions of $\hat{T}_i$ and $\tilde{T}_i$ will be provided later. With the assistance of (5.42), we will later find a lower bound for $\mathbb{E}T_i(X_*)$.

To define $\hat{T}_i$ and $\tilde{T}_i$, we introduce the following terms. Let $\{X_*^{(k)}\}_{k\geq 1}$ be independent realisations of X_*. For each $k \geq 1$, let $\{S_j^{(k)}\}_{1\leq j\leq i}$ and $\{R_j^{(k)}\}_{1\leq j\leq i}$ respectively be the unlabelled switching and spreading clocks of stage i in $X_*^{(k)}$. Similarly, we let $\{d^{(k)}(t) : t \geq 0\}$ be the dormancy process of $X_*^{(k)}$ for stage i. In addition, for all $k \geq 1$, we define $u_k = \sum_{j=1}^{k} T_i(X_*^{(j)})$.

We define $\tilde{T}_i$ as follows.

$$\tilde{k} = \min\{k \geq 1 : d^{(k)}(T_i(X^{(k)})) > d_0 n\} \tag{5.43}$$

$$\tilde{T}_i = u_{\tilde{k}}. \tag{5.44}$$

Now we define $\hat{T}_i$. For all $1 \leq j \leq i$, we define R_j^* to be another clock that behaves like a 'concatenation' of $\{R_j^{(k)}\}_{k\geq 1}$ in the following sense. Suppose that for all $k \geq 1$, we have that $(t_1^{(k)} < t_2^{(k)} < \cdots < t_{l_k}^{(k)})$ is the sequence of ringing times of $R_j^{(k)}$ during stage i of $X_*^{(k)}$ in an ascending order, where l_k denotes the (random) number of ringing of $R_j^{(k)}$ during stage i of $X_*^{(k)}$. Then, we specify the ringing times of R_j^* to be

$$(t_1^{(1)}, \ldots, t_{l_1}^{(1)}, u_1 + t_1^{(2)}, \ldots, u_1 + t_{l_2}^{(2)}, \ldots u_k + t_1^{(k+1)}, \ldots, u_k + t_{l_k}^{(k+1)}, \ldots).$$

In other words R_j^* consists of the concatenation of the ringing behaviours of $\{R_j^{(k)}\}_{k\geq 1}$ during the time period $[0, T_i(X_*^{(k)})]$. However, since all clocks in $\{R_j^{(k)}\}_{k\geq 1}$ have rate $(n-i)$, we have that R_j^* is simply another Poisson clock with rate $n-i$ by the memorylessness property of the exponential waiting times. Finally, we define

$$\hat{T}_i := \text{the waiting time for the first clock ringing among } \{R^*_{\lfloor d_0 n\rfloor+1}, \ldots, R_i^*\}. \tag{5.45}$$

Now we show that (5.42) holds. We claim that at time $\tilde{T}_i = u_{\tilde{k}}$, one of the clocks in $\{R^*_{\lfloor d_0 n\rfloor+1}, \ldots, R_i^*\}$ must ring. Note that at this time, a spreading clock R necessarily rings for some $R \in \{R_1^{(\tilde{k})}, \ldots, R_i^{(\tilde{k})}\}$ since the time marks the termination of stage i of $X_*^{(\tilde{k})}$. Now, by (5.43), the number of dormant vertices in $X_*^{(\tilde{k})}$ at that time is more than $d_0 n$. This means that $R \in \{R^{(\tilde{k})}_{\lfloor d_0 n\rfloor+1}, \ldots, R_i^{(\tilde{k})}\}$. It follows that when R rings, one of the clocks among $\{R^*_{\lfloor d_0 n\rfloor+1}, \ldots, R_i^*\}$ also rings. Thus, by (5.45), we have that (5.42) is satisfied.

Now we analyse the distribution of $\tilde{T}_i$ and $\hat{T}_i$. By (5.44), $\tilde{T}_i$ is the sum of $\tilde{k}$ independent copies of $T_i(X_*)$. Note that for all $k \geq 1$, at time $T_i(X_*^{(k)})$, a spreading clock rings. Hence, by Corollary (5.17), we have that $d^{(k)}(T_i(X_*^{(k)})) > d_0 n$ with probability $1 - O(n^{-n^{1/3}})$. It follows that $\tilde{k} \stackrel{d}{=} 1 + \mathcal{G}(1 - O(n^{-n^{1/3}}))$ since we can associate the successful criterion of the Bernoulli trials with the criterion described in (5.43). Consequently, $\mathbb{E}\tilde{k} = (1 - O(n^{-n^{1/3}}))^{-1}$. Thus,

$$\begin{aligned}
\mathbb{E}\tilde{T}_i &= \mathbb{E}\left(\sum_{k=1}^{\tilde{k}} T_i^{(k)}\right) \\
&= \mathbb{E}\left[\mathbb{E}\left(\sum_{k=1}^{\hat{k}} T_i^{(k)} \middle| \tilde{k}\right)\right] \\
&= \mathbb{E}\tilde{k}\mathbb{E}T_i(X_*).
\end{aligned}$$

On the other hand, $\hat{T}_i$ simply has an exponential distribution with rate $(i - \lfloor d_0 n \rfloor)(n-i)$ since it waits for the first ringing among $(i - \lfloor d_0 n \rfloor)$ Poisson clocks, where each of them has rate $(n-i)$. It follows that

$$\begin{aligned}
\mathbb{E}T_i(X_*) &\geq \frac{\mathbb{E}\hat{T}_i}{\mathbb{E}\tilde{k}} \\
&\geq \frac{1 - O(n^{-n^{1/3}})}{(i - \lfloor d_0 n \rfloor)(n-i)}.
\end{aligned}$$

Now we bound the delay time of the last $n^{1/3}$ stages of X_*.

$$\begin{aligned}
\sum_{i=L}^{n-1} \mathbb{E}D_i(X_*) &= \sum_{i=L}^{n-1} \left(\mathbb{E}T_i(X_*) - \frac{1}{i(n-i)}\right) \\
&\geq \sum_{i=L}^{n-1} \left(\frac{1 - O(n^{-n^{1/3}})}{(i - d_0 n)(n-i)} - \frac{1}{i(n-i)}\right) \\
&= \sum_{i=L}^{n-1} \frac{d_0 n}{i(i - d_0 n)(n-i)} - O(n^{-n^{1/3}}) \\
&= \Omega\left(\frac{1}{n}\right) \sum_{i=1}^{\lfloor n^{1/3} \rfloor} \frac{1}{i} = \Omega\left(\frac{\log n}{n}\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathbb{E}D(X) &\geq \mathbb{E}D(X_*) \\
&= \sum_{i=1}^{L-1} \mathbb{E}D_i(X_*) + \sum_{i=L}^{n-1} \mathbb{E}D_i(X_*) \\
&= \Omega\left(\frac{\log n}{n}\right).
\end{aligned}$$

□

Now we find an upper bound for the expected delay time. The precise result on the upper bound is summarised by the following proposition.

Proposition 5.18 (The upper bound delay of the RSDS with a fast rate)**.** *If* $s(n) = \Omega(n/\log n)$*, then*

$$\mathbb{E}D(X) = O\left(\frac{\log n}{n}\right).$$

Proof. For all $1 \leq i \leq n-1$, we aim to provide a stochastic upper bound for $T_i(X)$. To obtain the bound, we construct a coupling (X, X^*) where X^* is another rumour spreading process having slightly different rules from those of X. Specifically, X^* modifies stage i so that during the entire stage, it always has a fixed number of dormant vertices. We let X^* be an exact copy of X until stage i begins. We couple these two processes in such a way that at the time where stage i of X^* finishes, X will have finished by that time as well. In other words,

$$T_i(X) \leq T_i(X^*). \tag{5.46}$$

In this coupling, we refer to the unlabelled model of the RSDS introduced in Definition 4.2. All spreading and switching clocks employed in this coupling are unlabelled.

Suppose that $\{S_j\}_{1\leq j\leq i}$ and $\{R_j\}_{1\leq j\leq i}$ are respectively the sets of unlabelled switching and spreading clocks of stage i in X. Also let $\{d(t) : t \geq 0\}$ be the dormancy process of stage i in X. We will employ these clocks and dormancy process to govern the realisation of stage i in X^*. In this coupling, we allow the dormancy process to extend its role to capture the number of dormant vertices for time $t > T_i(X)$. This means that even though stage i of X has ended, we still let the ringing of the unlabelled switching clocks determine the value of $d(t)$.

We define the clock R_i' to be the superposition of all clocks in $\{R_{\lfloor 2i/3\rfloor+1}, \ldots, R_i\}$. Suppose that $s_0 = 0$ and $s_1 < s_2 < \ldots$ denote the ringing times of R_i'. For $j \geq 1$, we define s_j as the waiting time from $V_i(X)$ until the j-th ringing time of R_i'. In other words, R_i' considers $V_i(X)$ as the zero time. Now, we define

$$k^* := \min\left\{k \geq 1 : d(s_k) \leq \frac{2}{3}i\right\}. \tag{5.47}$$

Then, we specify that stage i of X^* terminates at time s_{k^*}. This means that during stage i of X^*, whenever R_i' rings, the process examines if there are more than $\frac{2}{3}i$ dormant vertices. If more than $\frac{2}{3}i$ vertices are dormant, then stage i of the process continues. Otherwise, it terminates the stage there.

Now we show that the running time of stage i in X is at most r_{k^*}. By (5.47), we have that $d(r_{k^*}) \leq \frac{2}{3}i$. This means that any ringing among the clocks in $\{R_{\lfloor 2i/3\rfloor+1}, \ldots, R_i\}$ implies the termination of stage i in X by the general rules of the unlabelled clocks described in Rules 4.3. Thus, this justifies (5.46).

Next, we examine the distribution of $T_i(X^*)$. Note that k^* denotes the number of the ringings of S_i until the stopping criterion in (5.47) is satisfied. Recall that from Lemma 4.7, for any $t > 0$, we have that

$$\mathbb{P}\left[\mathrm{Do}(X^C, V_i(X^C) + t) > \frac{2}{3}i\right] \leq e^{-0.01i}.$$

Hence, for each s_j, the ringing time of R_i', the probability that X^* terminates stage i at time s_j is at least $1 - e^{-0.01i}$. This implies that $k^* \leq_{ST} 1 + \mathcal{G}(1 - e^{-0.01i})$ since we can associate the successful trial to the termination of stage i event in X^*. Note that

$$\mathbb{E}k^* = (1 - e^{-0.01i})^{-1}.$$

On the other hand, by the memorylessness property of exponential random variables, for all $j \geq 1$, we have that $s_j - s_{j-1}$ has an exponential distribution with rate $\frac{1}{3}i(n-i)$ and are independent of each other. Let $\{H_i^{(k)}\}_{k\geq 1}$ be independent copies of $\mathcal{E}(\frac{1}{3}i(n-i))$. Then, we have that

$$T_i(X^*) = \sum_{k=1}^{k^*} H_i^{(k)}.$$

Now, we have that

$$\begin{aligned}
\mathbb{E}T_i(X) &\leq \mathbb{E}T_i(X^*) \\
&\leq \mathbb{E}\left(\sum_{k=1}^{k^*} H_i^{(k)}\right) \\
&\leq \mathbb{E}\left[\mathbb{E}\left(\sum_{k=1}^{k^*} H_i^{(k)} \middle| k^*\right)\right] \\
&\leq \mathbb{E}k^* \mathbb{E}H_i^{(k)} \\
&\leq (1 - e^{-0.01i})^{-1} \frac{3}{i(n-i)}.
\end{aligned}$$

We are now ready to bound $\mathbb{E}D(X^C)$ from above.

$$\begin{aligned}
\mathbb{E}D(X) &= \sum_{i=1}^{n-1} \mathbb{E}T_i(X^C) - \frac{1}{i(n-i)} \\
&\leq \sum_{i=1}^{n-1} (1 - e^{-0.01i})^{-1} \frac{3}{i(n-i)} - \frac{1}{i(n-i)} \\
&\leq \sum_{i=1}^{\log^2 n - 1} \frac{200}{i(n-i)} + \sum_{i=\log^2 n}^{n-1} \frac{2}{i(n-i)}(1 + O(n^{-10})) \\
&= O\left(\frac{\log n}{n}\right).
\end{aligned}$$

□

To conclude the section, Proposition 5.14 as well as Proposition 5.18 provide the direct proof for Theorem 5.2. □

5.4 Proof of Theorem 5.3

In this section, we will always assume that $s(n) = 1$. Since the unit rate is a specific instance of a slow switching rate, many arguments in this section will frequently refer to the proof of both Proposition 5.5 and Theorem 5.1.

We provide an overview of the proof as follows. We specify three sources of the delay times of which the total delay time is composed: the vacuum time of the first stage, the vacuum time of the rest of the stages, and the delay time of the compressed version.

To be more precise, we define the following terms. Recall that $W_i(X)$ denotes the total time length for which X is vacuum during stage i. Define $W_{>1}(X) = \sum_{i=2}^{n-1} W_i(X)$, that is the total vacuum time experienced by X after stage 2 starts. Then, we rewrite (4.4) as follows.

$$D(X) = W_1(X) + W_{>1}(X) + D(X^C). \tag{5.48}$$

Note that this equation breaks $W(X)$ in (4.4) into two terms, $W_1(X)$ and $W_{>1}(X)$ in order to show that $\mathbb{E}W_1(X)$ has significantly greater value than $\mathbb{E}W_{>1}(X)$. Furthermore we will also show that $\mathbb{E}D(X^C)$ is negligible compared to $\mathbb{E}W_1(X)$.

For this entire section, we let X', X and X^C be the SRS, RSDS, and CRSDS processes on K_n with switching rate 1. Also, we will refer to the labelled edge clock version of the SRS and RSDS models. This means that all switching and spreading clocks in this section are labelled and are assumed to lie on the vertices and edges of the graph, respectively.

First, we calculate the expectation of $W_1(X)$.

Proposition 5.19. *If the switching rate is 1, then*

$$\mathbb{E}W_1(X) = \frac{1}{n-1}.$$

Proof. Recall that X is vacuum for C_1 times during the first stage. Now for $1 \le j \le C_1$, let R_j be the waiting time starting from the time where X becomes vacuum for the j-th time, until the dormant vertex switches and becomes active again. $\{R_j\}$ are then identical and independent exponential random variables with unit rate since they are the waiting times for the ringing of the switching clock lying on the initial rumour spreader (whose rate is 1). It follows that

$$W_1(X) = \sum_{i=1}^{C_1(X)} R_i. \tag{5.49}$$

Now we find the distribution of $C_1(X)$. During the first stage, there are only two possible states for the process, either it is vacuum or not. When X is not vacuum, it has two possible state transitions: either going vacuum or terminating the stage by informing a new vertex. On the other hand, when it is vacuum, it will deterministically transition back to the non-vacuum state after waiting for a while. Now, every time X becomes non-vacuum, we interpret this condition as an independent Bernoulli trial with identical success probability q for some $q \in (0,1)$, where the success corresponds to the spreading transition. Hence, $C_1(X)$ is the number of failures occurring in the sequence of the Bernoulli trials until it achieves the first successful trial. It follows that $C_i(X) \stackrel{d}{=} \mathcal{G}(q)$. Now we find q. When X is not vacuum, it will stay in the current stage until the first ringing of either the switching clock on the initial rumour spreader, or the spreading clocks lying on the edges which are incident to the initial rumour spreader. Note that all of these clocks have rate 1. Thus, q is the probability that the first ringing comes from a spreading clock, that is $q = \frac{n-1}{n}$. It follows that

$$\mathbb{E}C_i = \mathbb{E}\,\mathcal{G}\left(1 - \frac{1}{n}\right) = \frac{1}{n-1}. \tag{5.50}$$

Therefore, from (5.49) and (5.50), we obtain

$$\mathbb{E}W_1(X) = \mathbb{E}[\mathbb{E}(W_1(X) \mid C_1)] = \mathbb{E}\left[\mathbb{E}\left(\sum_{i=1}^{C_1} R_i \middle| C_1\right)\right] = \mathbb{E}C_1\mathbb{E}R_1 = \frac{1}{n-1}$$

and the we have proved our proposition. □

It is worth noting that the proof of the proposition above adapts the arguments for finding the expected RSDS running time of the first stage in S_n. These arguments can be found in Subsection 3.2.2. This is due to the following fact: in the first stage, when we remove all edges whose ends are both uninformed in K_n (as the ringing of their spreading clocks give no effect during the first stage), we will get a star subgraph where the centre vertex is the initial rumour spreader.

Next, we present the following proposition to describe the expected vacuum delay time for the later stages.

Proposition 5.20. *If the switching rate is 1, then*

$$\mathbb{E}W_{>1}(X) = O\left(\frac{\log^2 n}{n^2}\right).$$

Proof. First, we notice that during stages where the process has large enough number of informed vertices, the expected vacuum delay time is extremely small. From Section 5.1, by applying $s(n) = 1$ in (5.15), we have that

$$\sum_{i=\lfloor \log^2 n \rfloor}^{n-1} \mathbb{E}W_i = O\left(\frac{\log n}{n^7}\right). \tag{5.51}$$

Now we find a bound for the expected delay times for stage i with $2 \le i \le \lfloor \log^2 n \rfloor - 1$. To carry this out, we find a stochastic upper bound for $W_i(X)$ for $i = 2, 3, \ldots, \lfloor \log^2 n \rfloor - 1$.

We introduce $\hat{X}$, a variant of X that customises the rules governing stage i. We design the coupling of $\hat{X}$ and X in such a way that

$$W_i(X) \leq W_i(\hat{X}) \tag{5.52}$$

holds, so that $W_i(\hat{X})$ becomes a stochastic upper bound for $W_i(X)$.

Observe that for each stage i with $i \geq 2$, there are always at least 2 active vertices at the beginning of stage i. Recall that w_i is the captured vertex of stage i, that is the most recent informed vertex at the start of stage i. Suppose that x_i is an informed vertex from which w_i receives the rumour. Then, at time $V_i(X)$, we have that w_i and x_i are necessarily active by their definitions. This will become the key observation for the coupling construction.

We specify that $\hat{X}$ follows the same state transitions as X until the end of stage $i-1$. This means that their transitions are governed by the same switching and spreading clocks. When stage i begins, we specify that all vertices but w_i and x_i, are always dormant in $\hat{X}$. During stage i of $\hat{X}$, we will only pay attention to the switching clocks on both w_i and x_i and all spreading clocks lying on the edges which are incident to these vertices. Essentially, during stage i of $\hat{X}$, we restrict the process to have at most two active vertices.

By the coupling construction, the statuses of both w_i and x_i in both X and $\hat{X}$ are always the same during stage i, since they still share the same switching clocks. This means that if an informed vertex is dormant in $\hat{X}$, then it is also dormant in X. Thus, when X is vacuum, so is $\hat{X}$. Hence, this shows that (5.52) holds.

Now we look at the distribution of $C_i(\hat{X})$. Observe that we can think of the transitions occurring during stage i of $\hat{X}$ as an embedded Markov chain with four states. Suppose that Z is the embedded Markov chain with state space $\{0, 1, 2, 3\}$. Each state of Z corresponds to various conditions of $\hat{X}$, stated as follows.

State	Interpretation
0	Both w_i and x_i are active
1	Exactly one of w_i or x_i is active
2	Both w_i and x_i are dormant
3	Stage i finishes

Note that state 2 is the vacuum state of $\hat{X}$. Also, state 3 is an absorbing state since once $\hat{X}$ arrives at this state, it leaves stage i and will never visit other states defined above. On the other hand, we start stage i of $\hat{X}$ with both w_i and x_i being active deterministically. This means that $Z_0 = 0$. Hence, $C_i(X)$ is the number of times Z visits state 2 before it enters the absorbing state 3, given that $Z_0 = 0$.

Suppose that $\mathbf{P} = [p_{\ell,j}]$ is the transition matrix of Z. It follows that $\mathbf{P}$ is in the form

$$\mathbf{P} = \begin{pmatrix} 0 & \dfrac{1}{n-i+1} & 0 & \dfrac{n-i}{n-i+1} \\ \dfrac{1}{n-i+2} & 0 & \dfrac{1}{n-i+2} & \dfrac{n-i}{n-i+2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The nonzero entries of the first row of $\mathbf{P}$ comes from the fact that the two currently active vertices are waiting for the ringing of $2(n-i)$ spreading clocks and 2 switching clocks in other to change their statuses. On the other hand, when there is only exactly one active vertex, the process waits for the ringing of $(n-i)$ spreading clocks and 2 switching clocks to experience a the next state transition. This explains the nonzero entries of the second row of $\mathbf{P}$. Next, define

$$\mathbf{Q} = \begin{pmatrix} 0 & \dfrac{1}{n-i+1} & 0 \\ \dfrac{1}{n-i+2} & 0 & \dfrac{1}{n-i+2} \\ 0 & 1 & 0 \end{pmatrix}, \qquad \mathbf{R} = \begin{pmatrix} \dfrac{n-i}{n-i+1} \\ \dfrac{n-i}{n-i+2} \\ 0 \end{pmatrix},$$

and write that

$$\mathbf{P} = \begin{pmatrix} \mathbf{Q} & \mathbf{R} \\ \mathbf{0} & 1 \end{pmatrix}.$$

We now calculate $\mathbb{E}C_i(\hat{X})$. Consider the so-called fundamental matrix of Z, that is

$$\mathbf{B} := \sum_{k=0}^{\infty} \mathbf{Q}^k = (\mathbf{I} - \mathbf{Q})^{-1}.$$

Suppose that $b_{\ell,j}$ and $p_{\ell,j}^k$ are the (ℓ, j) entries of $\mathbf{B}$ and $\mathbf{P}^k$ respectively, for all $\ell, j \in \{0, 1, 2\}$ and $k \geq 0$. It is well known that

$$b_{\ell,j} = \sum_{k=0}^{\infty} p_{\ell,j}^k.$$

Hence,

$$\begin{aligned} \mathbb{E}C_i(X) &= \mathbb{E}\left(\sum_{k=0}^{\infty} \mathbf{1}\{Z_k = 2\}\right) \\ &= \sum_{k=0}^{\infty} p_{0,2}^k \\ &= b_{0,2} \end{aligned} \tag{5.53}$$

where the second equation comes from a well known fact that

$$p_{\ell,j}^k = \mathbb{P}(Z_{m+k} = j \mid Z_m = \ell), \qquad \text{for all } \ell, j \in \{0, 1, 2\}, m \geq 0.$$

By some calculations, we obtain that

$$\mathbf{B} = \begin{pmatrix} \frac{N_1(N_2-1)}{M} & \frac{N_2}{M} & \frac{1}{M} \\ \frac{N_1}{M} & \frac{N_1N_2}{M} & \frac{N_1}{M} \\ \frac{N_1}{M} & \frac{N_1N_2}{M} & \frac{N_1N_2-1}{M} \end{pmatrix}$$

where

$$\begin{aligned} N_1 &= n-i+1, \\ N_2 &= n-i+2, \\ M &= N_1N_2 - N_1 - 1. \end{aligned}$$

Hence, from (5.53) we have that

$$\mathbb{E}C_i(\hat{X}) = \frac{1}{M} = O\left(\frac{1}{(n-i)^2}\right). \tag{5.54}$$

Now we find the distribution of $W_i(\hat{X})$. When $\hat{X}$ is vacuum, the process stays in the vacuum state until the first ringing among the switching clocks associated to either w_i or x_i. It follows that once $\hat{X}$ is vacuum, the waiting time until it leaves the vacuum state is exponentially distributed with rate 2. For $k \geq 1$, define $H_i^{(k)} \stackrel{d}{=} \mathcal{E}(2)$. Then we have that

$$W_i(\hat{X}) = \sum_{k=1}^{C_i(\hat{X})} H_i^{(k)}.$$

Therefore,

$$\begin{aligned} \mathbb{E}W_i(X) \leq \mathbb{E}W_i(\hat{X}) &= \mathbb{E}\left(\sum_{k=1}^{C_i(\hat{X})} H_i^{(k)}\right) \\ &= \mathbb{E}\left[\mathbb{E}\left(\sum_{k=1}^{C_i(\hat{X})} H_i^{(k)} \middle| C_i(\hat{X})\right)\right] \\ &= \mathbb{E}C_i(\hat{X})\mathbb{E}H_i^{(1)} \\ &= O\left(\frac{1}{(n-i)^2}\right) \end{aligned}$$

where the last equality comes from (5.54).

Therefore,

$$\begin{aligned} \mathbb{E}\left(\sum_{i=2}^{\lfloor \log^2 n\rfloor -1} W_i(X)\right) &= \sum_{i=2}^{\lfloor \log^2 n\rfloor -1} O\left(\frac{1}{(n-i)^2}\right) \\ &= O\left(\frac{\log^2 n}{n^2}\right). \end{aligned} \tag{5.55}$$

Finally, we have that (5.51) and (5.55) conclude the proof. $\square$

The proposition above suggests that we only need to pay attention to the total vacuum time spent during stage 1. This is because the total expected vacuum delay times on the later stages is significantly smaller and can be neglected compared to $\mathbb{E}W_1(X)$.

Now we analyse the other sources of the delay times, that is the delay times contributed by the compressed process. First, by directly applying Proposition 5.6 with $s(n) = 1$, we observe that

$$\mathbb{E}D(X^C) = O\left(\frac{1}{n}\right). \tag{5.56}$$

This term is of the same order as $\mathbb{E}W_i(X)$. However, if we take a finer analysis of the delay time for the unit rate, we can get a slight improvement for the upper bound in (5.56). We present this in the form of the following proposition.

Proposition 5.21 (Delay time of the CRSDS with the unit rate)**.** *If the switching rate is 1, then*

$$\mathbb{E}D(X^C) = O\left(\frac{1}{n\log n}\right).$$

The proof of the proposition is a direct modification of the proof of Proposition 5.6. In the following proof, we will often refer back to some notions stated in Proposition 5.6.

Proof. We again divide the stages into categories. However, unlike the arguments in Proposition 5.6, we abandon the notion of an intermediate stage. In this proof, for every $i \in [1, n-1]$, we simply call stage i *early* if $i < \lfloor \log^2 n \rfloor$ and *late* otherwise. In other words, we redefine $R = Q = \lfloor \log^2 n \rfloor$.

We reformulate Claim 5.7 in the context of the unit rate as follows.

Claim 5.22. *When the switching rate* $s(n)$ *is* 1*, during the first* $\lfloor \log^2 n \rfloor - 1$ *stages,* X^C *has no dormant vertex with probability* $1 - O(\log^2 n/n)$*.*

Proof. Let X^M and H be the embedded Markov chain and the event introduced in Claim 5.7. We apply (5.19) with $s(n) = 1$, to obtain

$$p_{(0,i),(0,i+1)} = 1 - \frac{1}{n-i+1}.$$

This means that

$$\begin{aligned}
\mathbb{P}(\bar{H}) &= 1 - \prod_{i=1}^{\lfloor \log^2 n \rfloor - 1} \left(1 - \frac{1}{n-i+1}\right) \\
&\leq 1 - \prod_{i=1}^{\lfloor \log^2 n \rfloor - 1} \left(1 - \frac{2}{n}\right) \\
&= 1 - \left(1 - \frac{2}{n}\right)^{\lfloor \log^2 n \rfloor - 1} \\
&= 1 - \left(1 - O\left(\frac{\log^2 n}{n}\right)\right) \\
&= O\left(\frac{\log^2 n}{n}\right).
\end{aligned}$$

This proves our claim. □

The claim above provides a stronger condition for the absence of a dormant vertex, however with a slightly lower probability compared to Claim 5.7. Nevertheless, we still have that the bound holds with sufficiently high probability and will employ this together with the worst case bound.

Now by employing the same notations of $\mathcal{S}_1, T_i^{\geq \mathcal{S}_1}$, and A_i as introduced in Proposition 5.6, we have that (5.20) holds, that is

$$\mathbb{E}D_i(X^C) \leq \mathbb{E}T_i^{\geq \mathcal{S}_1}(X^C).$$

On the other hand, by Claim 5.22, we have that $\mathbb{P}(A_i) = O(\log^2 n/n)$. Hence, by the worst case bound in Corollary 4.6, we obtain that

$$\mathbb{E}D_i(X^C) \leq \mathbb{E}T_i^{\geq \mathcal{S}_1}(X^C) \leq \frac{P(A_i)}{n-i} = O\left(\frac{\log^2 n}{n^2}\right).$$

Therefore, the expected delay time during the early stages is

$$\begin{aligned}
\sum_{i=1}^{\lfloor \log^2 n \rfloor - 1} \mathbb{E}D_i(X^C) &\leq \sum_{i=1}^{\lfloor \log^2 n \rfloor - 1} O\left(\frac{\log^2 n}{n^2}\right) \\
&= O\left(\frac{\log^4 n}{n^2}\right).
\end{aligned} \tag{5.57}$$

Now we bound the expected delay time for late stages. Again, we redefine the terms $P_1 = \lfloor 22 \log n \rfloor$ and $P_2 = \lfloor 23 \log n \rfloor$. However, we employ the same notion of $S^*, \mathcal{S}_2, Z_i^{<S^*}$ and Y_i with $s(n) = 1$. While S^* has exactly the same interpretation as that of Proposition 5.6, we can think of $\mathcal{S}_2$ as the waiting time from the start of stage $\lfloor \log^2 n \rfloor$, until either time S^* is achieved or the process has at least $P_2 = \lfloor 22 \log n \rfloor$ dormant vertices for the first time.

In analogy to Claim 5.10, we present the following claim.

Claim 5.23. *If the switching rate $s(n) = 1$, then $\mathcal{S}_2 = T(X^C)$ with probability $1 - O(n^{-5})$.*

Proof. By imitating the proof of Claim 5.10, we have that (5.25) holds in the case of the unit rate, that is

$$\mathbb{P}(\{\mathcal{S}_2 < T_R\} \cap \{S^* \leq S_{P_2}\}) = O(n^{-5}). \tag{5.58}$$

On the other hand, to bound the probability of $\{\mathcal{S}_2 < T_R\} \cap \{S^* > S_{P_2}\}$, we consider two events $\mathcal{H}_1$ and $\mathcal{H}_2$ which slightly modify H_1 and H_2 in Claim 5.10. Define $\mathcal{H}_1$ to be the event that stage $\lfloor \log^2 n \rfloor$ starts with at least six dormant vertices whereas we let $\mathcal{H}_2$ be the event where at least $\lfloor 22 \log n \rfloor$ switching transitions occur during the late stages and before time S^*. Observe that whenever both $\mathcal{H}_1$ and $\mathcal{H}_2$ do not occur, $\{\mathcal{S}_2 < T_R\} \cap \{S^* > S_{P_2}\}$ also cannot occur. Hence, analogous to (5.26), we have the following similar result.

$$\mathbb{P}(\{\mathcal{S}_2 < T_R\} \cap \{S^* > S_{P_2}\}) \leq \mathbb{P}(\mathcal{H}_1) + \mathbb{P}(\mathcal{H}_2). \tag{5.59}$$

Now we show that $\mathcal{H}_1$ occurs with a very low probability. We consider the first $\lfloor \log^2 n \rfloor + 5$ state transitions of the process. Note that when stage $\lfloor \log^2 n \rfloor$ starts with at least 6 dormant vertices, there must be at least 6 switching transitions among the first $\lfloor \log^2 n \rfloor + 5$ state transitions. Now we bound the probability that a particular state transition is a switching transition. For $i \in [1, \lfloor \log^2 n \rfloor - 1]$, suppose that X^C is currently in stage i. During this stage, there are i switching clocks and at most $i(n-i)$ spreading clocks where their ringings lead to a state transition. Note that all of these clocks have rate 1. The probability that the earliest clock ringing among them is a switching clock is at least

$$\frac{i}{i + i(n-i)} = \frac{1}{n - i + 1} \geq \frac{1}{n}.$$

It follows that

$$\mathbb{P}(\mathcal{H}_1) \leq \mathbb{P}\left(\mathcal{B}\left(\lfloor \log^2 n \rfloor + 5, \frac{1}{n}\right) \geq 6\right) = O\left(\frac{\log^{12} n}{n^6}\right).$$

On the other hand, $\mathcal{H}_2$ is contained in the event $\{Z_i^{<S^*} \geq 22 \log n\}$. Observe that Lemma 5.9 holds in the unit rate setting. Thus,

$$\sum_{i=\lfloor \log^2 n \rfloor}^{n-1} Z_i^{<S^*} \leq_{ST} \sum_{i=\lfloor \log^2 n \rfloor}^{n-1} Y_i.$$

Note also that in the case of the unit rate, we have that

$$\mathbb{E}Y = 3 \log n + O(1).$$

It follows that

$$\mathbb{P}(\mathcal{H}_2) \leq \mathbb{P}(Z_i^{<S^*} \geq 22 \log n) \leq \mathbb{P}(Y \geq 7\mathbb{E}Y) = O(n^{-5})$$

in analogy to (5.28). Thus, continuing from (5.59), we have that

$$\mathbb{P}(\{\mathcal{S}_2 < T_R\} \cap \{S^* > S_{P_2}\}) = O(n^{-5}). \tag{5.60}$$

Finally, (5.58) and (5.60) conclude the proof of the claim. $\square$

For all $i \in [\lfloor \log^2 n \rfloor, n-1]$, let $T_i^{<\mathcal{S}_2}, T_i^{\geq \mathcal{S}_2}$ and B_i denote the same notion as in Proposition 5.6.

Now we bound $\mathbb{E}T_i^{\geq \mathcal{S}_2}$ by the worst case bound. By Claim 5.23, we have that $\mathbb{P}(B_i) = O(n^{-5})$. Hence, by applying Corollary 4.6 with $\tau = \mathcal{S}_2$ and $A = B_i$ and following the steps described in (5.30) and (5.31), we obtain that

$$\sum_{i=\lfloor \log^2 n \rfloor}^{n-1} \mathbb{E}T_i^{\geq \mathcal{S}_2} = \sum_{i=\lfloor \log^2 n \rfloor}^{n-1} \frac{O(n^{-5})}{n-i} = O\left(\frac{\log n}{n^5}\right). \tag{5.61}$$

The similar steps in Proposition 5.6 for bounding $\mathbb{E}T_i^{<\mathcal{S}_2}$ can also be applied. Note that (5.32) and (5.33) hold with $P_2 = \lfloor 23 \log n \rfloor$. Hence, we can write that

$$\mathbb{E}T_i^{<\mathcal{S}_2} - \frac{1}{i(n-i)} \leq \frac{46 \log n}{i^2(n-i)}. \tag{5.62}$$

Therefore, the expected delay time for late stages is

$$\begin{aligned}
\sum_{i=\lfloor \log^2 n \rfloor}^{n-1} \mathbb{E}D_i &= \sum_{i=\lfloor \log^2 n \rfloor}^{n-1} \left(\mathbb{E}T_i - \frac{1}{i(n-i)}\right) \\
&= \sum_{i=\lfloor \log^2 n \rfloor}^{n-1} \mathbb{E}T_i^{\geq \mathcal{S}_2} + \sum_{i=\lfloor \log^2 n \rfloor}^{n-1} \left(\mathbb{E}T_i^{<\mathcal{S}_2} - \frac{1}{i(n-i)}\right) \\
&\leq O\left(\frac{\log n}{n^5}\right) + \sum_{i=\lfloor \log^2 n \rfloor}^{n-1} \frac{46 \log n}{i^2(n-i)} \quad \text{(by (5.62) and (5.61))} \\
&= O\left(\frac{\log n}{n^5}\right) + \sum_{i=\lfloor \log^2 n \rfloor}^{\lfloor n/2 \rfloor - 1} O\left(\frac{\log n}{i^2 n}\right) + \sum_{i=\lfloor n/2 \rfloor}^{n-1} O\left(\frac{\log n}{n^2(n-i)}\right) \\
&= O\left(\frac{1}{n \log n}\right).
\end{aligned} \tag{5.63}$$

Therefore, (5.57) and (5.63) establish the proposition. $\square$

Although the bound given by Proposition 5.21 only improves the bound in (5.56) by a logarithmic factor, it turns out that it is essential to identify the most significant source of the expected delay time.

Now we summarise the sources of the expected delay time in the unit rate case. From Proposition 5.19, Proposition 5.20, and Proposition 5.21, we have that

$$\begin{aligned}\mathbb{E}W_1(X) &= \frac{1}{n},\\ \mathbb{E}W_{>1}(X) &= O\left(\frac{\log^2 n}{n^2}\right),\\ \mathbb{E}D(X^C) &= O\left(\frac{1}{n\log n}\right).\end{aligned}$$

From (5.48), recall that the sum of these three terms equals to the expected total delay time of the RSDS process with unit rate. This establishes (5.1), the first equation of Theorem 5.3.

The second equation of the theorem only requires a brief further explanation. Recall that $\mathcal{A}$ is the event where X is never vacuum during the first stage of X. We condition on the event that $\mathcal{A}$ occurs. By this condition, we have that $\mathbb{E}W_1(X) = 0$. On the other hand, the random variable $W_{>1}(X)$ is independent of $\mathcal{A}$ by Markov property. Now, observe that $D_1(X^C) = 0$ by definition, while X^C starts stage 2 with a deterministic state regardless the history of the first stage of the process. This means that $D(X^C)$ is also independent of $\mathcal{A}$. Thus,

$$\mathbb{E}(D(X) \mid \mathcal{A}) = \mathbb{E}(W_1(X) \mid \mathcal{A}) + \mathbb{E}W_{>1}(X) + \mathbb{E}D(X^C) = O\left(\frac{1}{n\log n}\right).$$

This establishes (5.2) and concludes the proof.

□

Chapter 6

Conclusion

6.1 Summary of the Main Results

By applying a delaying scheme to a rumour spreading process, the expected running time of the process is increased. We first proved this for two elementary families of graphs, the paths and the stars. Although the expected running times of the RSDS processes in both graphs are substantially different, we have shown that the delaying scheme doubles their expected running times. Furthermore, this result is completely independent of the switching rate functions.

In the context of complete graphs, we have proved that the delaying scheme has a significant impact on the rumour spreading process if and only if the switching rate is fast enough. Recall that from (4.1), the expected running time of the SRS process in complete graphs is

$$\mathbb{E}T(X') = \frac{2\log n}{n} + O\left(\frac{1}{n}\right).$$

As the first result, Theorem 5.1 tells us that when the switching rate is $s(n) = o(n/\log n)$, the expected delay time is

$$\mathbb{E}D(X) = O\left(\frac{s(n)\log^2 n}{n^2}\right) + O\left(\frac{\log\log n}{n}\right).$$

Note that if $s(n) = o(n\log\log n/\log^2 n)$, then the dominating term of the expression above is $O(\log\log n/n)$. Otherwise, $O\left(\frac{s(n)\log^2 n}{n^2}\right)$ becomes the dominating term. However, for both cases, the dominating term is always $o(\mathbb{E}T(X'))$. This means that the delaying scheme gives no essential effect to the rumour spreading process since the expected extra time yielded by the scheme is significantly smaller than the expected running time of the SRS.

The proof of the result is based on the fact that with high probability, the process only has very few dormant vertices, relative to the number of informed vertices so far, during the entire process. Since the underlying network is a complete graph, the small number of dormant vertices is an insignificant hindrance to spreading the rumour. There are many other active vertices in the graph which are adjacent to all uninformed vertices and are able to pass information. This provides the reason why the spreading progress is substantially unaffected by the delaying scheme.

When the switching rate is fast, on the other hand, the delaying scheme has a significant effect to the rumour spreading process. To be precise, as stated in Theorem 5.2, if $s(n) = \Omega(n/\log n)$, then the expected delay time is

$$\mathbb{E}D(X) = \Theta\left(\frac{\log n}{n}\right).$$

This bound says that by applying the delaying scheme, the expected running time of a rumour spreading process is multiplied by a constant. On the one hand, this shows that the scheme is now able to demonstrate a significant deceleration. However, on the other hand, the result also indicates the inability of the delaying scheme to provide a dramatic difference compared to the SRS process. Even if we choose the switching rate to be extremely fast, both the SRS and RSDS processes will always have the same order of expected running time.

The simple reason behind this result is that by having a fast switching rate, the proportion between the dormant and informed vertices is mainly constant throughout the entire process w.h.p. This results in the decreasing of the spreading rate by some constant factor, which eventually leads to the increase of the expected running time, also up to a constant factor.

We also have shown a more accurate result on the expected delay time when the switching rate is 1, that is

$$\mathbb{E}D(X) = \frac{1}{n} + O\left(\frac{1}{n\log n}\right).$$

The term $1/n$ comes from the expected total vacuum time of X during the first stage. Moreover, we also proved that provided that the process is never vacuum during stage 1, the expected delay time is $O(1/(n\log n))$. This shows that the vacuum condition during stage 1 contributes most significantly to the expected delay time for the RSDS with unit rate.

From our results on the expected delay time, we derive a lower and upper bound for the asymptotic value of $T(X)$. From (4.3), recall that the running time of the SRS satisfies $nT(X')/\log n \xrightarrow{p} 2$. We proved that the same convergence in probability also applies for $T(X)$, that is $nT(X)/\log n \xrightarrow{p} 2$, when the switching rate is slow. Again, this signifies that the delaying scheme with a slow rate is inefficacious. On the other hand, we also showed that the following inequality holds w.h.p. For any positive real ε and unbounded function growing arbitrarily slowly $\omega = \omega(n)$, we have that

$$(1-\varepsilon)\frac{2\log n}{n} < T(X) < \frac{\omega \log n}{n}.$$

In other words, the running time of the RSDS with a fast rate is typically around $c \log n/n$ where $c \geq 2$.

6.2 Future Possible Work

We close the thesis by discussing some possible tasks to do in the future in the context of the extension of this research.

In the analysis of the RSDS in complete graphs, we separate the switching rate functions into two categories, which are the slow (when $s(n) = o(n/\log n)$) and fast rates (when $s(n) = \Omega(n/\log n)$). We define the *slowdown ratio* of an RSDS process X to be $\mathbb{E}D(X)/\mathbb{E}T(X')$, that is, the proportion of the expected delay time and the running time of the SRS version. By our results, the slowdown ratio is bounded away from 0 if and only if the switching rate is fast. Recall that the function $f(n) = n/\log n$ is the *threshold function*, in the sense that the slowdown ratio of the RSDS is $o(1)$ if and only if the switching rate is $s(n) = o(f(n))$.

One interesting possible future work will be to take a closer look at the case when the switching rate is very close to the threshold function. It is interesting to understand how the sudden 'jump' of the expected delay time emerges when the switching rate grows approaching the threshold function. To answer the problem, one can start by analysing the slowdown ratio when the switching rate is equal to some constant multiples of the threshold function. The behaviour of the slowdown ratio as the constant tending to 0 can be the interesting object of study. Next, we can also analyse the slowdown ratio for $s(n) = n/\log n - \omega(n)$ where $\omega(n) = o(n/\log n)$ to examine a finer window of the switching rate functions. To summarise the suggested tasks, we present following open questions.

Open Question 6.1. *Suppose that c is an arbitrary positive real and $\omega(n)$ is an unbounded function which satisfies $\omega(n) = o(n/\log n)$.*

1. *Find the slowdown ratio for an RSDS process in complete graphs with switching rate $s(n) = cn/\log n$, in terms of c.*
2. *Find the slowdown ratio for an RSDS process in complete graphs with switching rate $s(n) = n/\log n - \omega(n)$ for various choices of $\omega(n)$ satisfying $\omega(n) = o(n/\log n)$.*

Next, as mentioned in the previous section, there are important similarities between the RSDS expected delay time analysis in the graphs K_n, P_n and S_n. Our results show that the slowdown ratio of the RSDS process in these three families of graphs is always bounded. The natural question arising from these results is whether the constant factor bound also appears in arbitrary connected graphs. We formulate the open question as follows.

Open Question 6.2. *Let G be a connected graph and X' and X be the SRS and RSDS processes in graph G with the switching rate $s(n)$. Does there exist a positive constant $c > 0$ such that*

$$\mathbb{E}D(X) \leq c\mathbb{E}T(X')$$

for all choices of switching rate $s(n)$?

This is interesting because if such a constant is found, then we can strongly infer that the delaying scheme can only extend the spreading running time up to a constant factor for any topology of the network. On the other hand, it is also interesting if a certain graph structure yielding a significantly longer expected RSDS running time exists. If such graph structures exist, then the next task to do would be to find their characterisations.

A further investigation of the noteworthiness of the delay time for general graphs can lead to many interesting objects of study. We generalise the notion of noteworthiness as follows. We say that a delay time of an RSDS in graph G is *noteworthy* if its slowdown ratio is $\Omega(1)$. This means that the expected delay time of the process is not negligible compared to the expected running time of the SRS version. Unlike the complete graphs, the delay times for the paths and stars are always noteworthy regardless the choice of the switching rate. In this case, the notion of the threshold function for noteworthiness is irrelevant.

Based on this observation, some possible future tasks will be to characterise all families of graphs where the delay time is always noteworthy. On the other hand, when the noteworthiness depends on the switching rate, it is also interesting to find a formula for determining the switching rate threshold function. For example, in the context of complete graphs, the threshold switching rate function is reciprocal of the expected running time of the SRS version. Will the same phenomenon also apply in other classes of graphs? We state the following questions.

Open Question 6.3. *Find all classes of graphs such that the delay times of the RSDS conducted in them are always noteworthy.*

Open Question 6.4. *Suppose that G is a connected graph such that the noteworthiness of the delay time of the RSDS conducted in it is dependent on the switching rate. Find a formula to determine the threshold function of the switching rate.*

Lastly, as another suggestion for future work, we can vary the delaying scheme. In the switching clock setting of the RSDS, we use the same rate for both dormancy and waking up transitions. It is possible to let these two rates differ. The RSDS processes having different dormancy and waking up rates can be considered to represent a more realistic modelling. Consider a rumour spreading scheme where the vertices represent computer servers sending information within a network. In the scheme, the computers can possibly be broken at a certain rate so that they are unable to send or receive the message. On the other hand, the broken computers also have a certain rate to be fixed. In real life application, it is unlikely that the repairing rate equals to the breakdown rates. In this way, the new model can be utilised in the first attempt to study such rumour spreading scheme. We provide the research question as follows.

Open Question 6.5. *How does the different dormancy and waking up rates affect the rumour spreading process?*

A mathematically interesting case would be when the dormancy rate is much faster than the waking up rate. By this setting, we expect a greater amount of the delay time, since the informed vertices are more likely to be dormant. As the varying scheme is able to substantially slow down the spreading time, the expected running time of the new scheme, in terms of the waking up rate, can be a quantity of interest to be investigated in the future.

Glossary

active vertex An informed vertex that is able to pass the rumour to other vertices. 25

captured vertex The most recent informed vertex at that time. 29

Compressed Rumour Spreading with a Delaying Scheme A modified RSDS process that removes all time periods at which the process is vacuum. 50

coupling A method to couple many random variables, not necessarily lying on the same probability space, into a random vector that is defined under the same probability space. 17

CRSDS see Compressed Rumour Spreading with a Delaying Scheme. 50

delay time The difference between the running time of an RSDS process and its expected SRS version. 33

dormancy process A stochastic process defined at the start of stage i in an unlabelled RSDS, that captures the information about the number of dormant vertices. 46

dormancy transition A state transition of a rumour spreading process at which an active vertex goes dormant. 45

dormant vertex An informed vertex that is incapable of spreading the rumour to other vertices. 25

edge A 2-subset of vertex set in a graph. 10

edge clock model A rumour spreading model that employs edge spreading clocks. 27

edge spreading clock A Poisson clock put in the edges of the graph whose ringing serves as a mark for the information exchange between the ends of the edge. 27

effective clock A spreading clock associated with an edge joining an effective vertex to an uninformed vertex. 29

effective vertex An informed vertex that has an uninformed neighbour. 29

effectual clock A spreading clock associated with an edge joining an effectual vertex to an uninformed vertex. 29

effectual vertex An informed vertex that is both effective and active. 29

embedded Markov chain A discrete-time Markov chain that captures the state transitions history of a continuous-time Markov chain at its transition times. 16

fast switching rate A switching rate satisfying $s(n) = \Omega(n/\log n)$. 58

informed vertex A vertex which has already learned the rumour. 24

initial rumour spreader The only vertex that knows the rumour at the beginning of a rumour spreading process. 24

labelled clock A Poisson clock (either switching or spreading) which has an association with an edge or vertex of the graph. 44

Markov Chain A stochastic process whose state space is countable, that enjoys the Markov property. 14

Markov property A property of a stochastic process saying that the only helpful information of the process' history to calculate the probability of a future event, is the most recent part of the history. 14

memorylessness A property of some probability distributions which roughly says that the probability of an upcoming event in the future does not depend on the events occurring in the past. 12

noteworthy A delay time of an RSDS process X is noteworthy if $\mathbb{E}D(X) = \Omega(\log n/n)$. 57

Poisson clock A list of increasing time points at which an associated Poisson process jumps. 13

potential vertex An uninformed vertex that has an informed neighbour. 28

RSDS see Rumour Spreading with a Delaying Scheme. 24, 25

Rumour Spreading with a Delaying Scheme The main rumour spreading scheme studied in this thesis, where the informed vertices are possibly dormant for some period of time. 24, 25

running time The time spent by a rumour spreading process to let every vertex learn the rumour. 30

simple graph A pair of finite sets $G = (V, E)$ with $V \neq \emptyset$ and E a collection of 2-subsets of V. 10

slow switching rate A switching rate satisfying $s(n) = o(n/\log n)$. 58

spreading transition A state transition of a rumour spreading process at which a new vertex receives the rumour. 45

SRS see Standard Rumour Spreading. 24

stage A rumour spreading process is in stage i if it currently has i informed vertices. 28

Standard Rumour Spreading The original rumour spreading scheme with no delaying scheme. 24

superposition A superposition of two Poisson clocks is another Poisson clock which captures all ringing times of these two clocks. 14

switching clock A Poisson clock put in the informed vertices of the graph whose ringing switches the status of the informed vertex, from active to dormant and vice versa. 26

switching rate The rate of the switching clocks. 26

transition matrix An $|S| \times |S|$ matrix specifying the transition probabilities of a Markov chain. 15

transition probability The probability that a stochastic process moves to other states at a given time. 15

uninformed vertex A vertex which has not learned the rumour yet. 24, 25

unlabelled RSDS A version of the RSDS model using unlabelled clocks in the context of complete graphs. 45

unlabelled spreading clock A spreading clock employed in the unlabelled SRS and RSDS models, that is not associated with any edges of the graph. 43, 44, 46

unlabelled SRS A version of the SRS model using unlabelled clocks in the context of complete graphs. 44

unlabelled switching clock A switching clock employed in the unlabelled RSDS model, that is not associated with any vertices of the graph. 43, 46

vacuum A state of an RSDS process where all informed vertices are currently dormant. 50

vacuum delay time The total period of time during which an RSDS process is vacuum. 50

vertex An element of the vertex set V in graph $G = (V, E)$. 10

vertex clock model A rumour spreading model that employs vertex spreading clocks. 27

vertex spreading clock A Poisson clock associated with a particular vertex of the graph whose ringing tell the vertex to call a uniformly random neighbour in order to exchange information between them. 25–27

w.h.p. The abbreviation of "with high probability", occurring with probability tending to 1 as $n \to \infty$. 12

waking up transition A state transition of a rumour spreading process at which a dormant vertex goes active. 45

List of Symbols

$\mathcal{B}(n,p)$	The binomial distribution with parameter n and p
$\mathrm{Do}(X,t)$	The number of dormant vertices in X at time t
$\exp(\cdot)$	The exponential function
$\mathcal{E}(r)$	The exponential distribution with rate r
$\mathcal{G}(p)$	The geometric distribution with success probability p
$\mathrm{In}(X,t)$	The number of informed vertices in X at time t
$C_i(X)$	The number of times that X becomes vacuum during stage i
$D(X)$	The delay time of X
$D_i(X)$	The delay time of stage i of X
E	The set of edges in graph G
$E(X,Y)$	The set of all X-Y edges in G
$e(X,Y)$	The number of X-Y edges in G
$F_i(X)$	The set of effective vertices during stage i of X
G	The graph
$I_i(X)$	The set of informed vertices during stage i of X
K_n	The complete graph with n vertices
$P_i(X)$	The set of potential vertices during stage i of X
P_n	The path with n vertices
$s(n)$	The rate of the switching clocks
S_n	The star with n vertices

$SW_i(X)$	The number of switchings experienced by the effective vertices during stage i of X
$T(X)$	The running time of X
$T_i(X)$	The duration of stage i of X
$T_i^{<\tau}(X)$	The running time of stage i of X before time τ
$T_i^{\geq\tau}(X)$	The running time of stage i of X after time τ
$U_i(X)$	The set of uninformed vertices during stage i of X
V	The set of vertices in graph G
$V_i(X)$	The time at which X enters stage i
$W(X)$	The total vacuum delay time of X
$W_i(X)$	The total length of vacuum delay time during stage i of X
$w_i(X)$	The captured vertex at the start of stage i of X
X'	The SRS version of X
X^C	The CRSDS version of X
$Y \leq_{ST} Z$	Y is stochastically smaller that Z, that is $\mathbb{P}(Y > t) \leq \mathbb{P}(Z > t)$ for all t

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