Counting Subgraphs of Regular Graphs using Spectral Moments

by

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Thesis
Submitted by Marsha Elizabeth Minchenko
for fulfilment of the Requirements for the Degree of

Doctor of Philosophy (3531)

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December, 2013
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Ecclesiastes 1:9b
Contents

List of Tables .................................................. vi
Abstract .......................................................... vii
Acknowledgments .................................................. xi

1 Introduction .................................................... 1
   1.1 Problem and motivation ................................... 1
   1.2 Research contributions ................................... 2
   1.3 Publications .............................................. 6

2 Literature Review ............................................. 7
   2.1 Background ................................................. 7
   2.2 Spectral moment equations ................................. 9
   2.3 Integral graphs ........................................... 13
   2.4 Quartic integral graphs ................................... 18
   2.5 Cayley integral graphs ................................... 20
   2.6 Strongly regular graphs ................................... 22
   2.7 Moore graphs ............................................. 25

3 Spectral Moments of Regular Graphs ......................... 28
   3.1 Spectral moments equations ............................... 28
   3.2 Base walks and extensions ............................... 29
   3.3 Generating walks around subgraphs ....................... 30
   3.4 Equations relating moments to subgraph counts ........ 34
   3.5 Remarks .................................................. 36

4 Quartic Integral Cayley Graphs ............................... 37
   4.1 Vertex-transitive integral graphs ....................... 37
   4.2 Vertex-transitive cases ................................... 38
   4.3 The algorithm ............................................ 40
   4.4 Quartic integral graphs ................................... 41
      4.4.1 Bipartite Cayley integral graphs .................. 41
      4.4.2 Bipartite arc-transitive integral graphs .......... 49
      4.4.3 Integral graphs as quotients ....................... 50
4.4.4 Non-bipartite Cayley integral graphs ................................. 52
4.4.5 Non-bipartite arc-transitive integral graphs ......................... 55
4.4.6 The only vertex-transitive graph on 32 vertices ...................... 57
4.4.7 Other quartic integral graphs ......................................... 58
4.5 Remarks ................................................................. 59

5 Counting Subgraphs of Strongly Regular Graphs ......................... 61
  5.1 Counting subgraphs in terms of smaller subgraphs ...................... 61
  5.2 Subgraph counts in terms of SR parameters ............................ 63
  5.3 Equations for subgraphs of a strongly regular graph .................. 66
  5.4 Subgraph counts for some specific graph parameter sets ............... 70
  5.5 Remarks ................................................................. 72

6 Conclusion ................................................................. 74
  6.1 Results summary ....................................................... 74
    6.1.1 Spectral moment equations ...................................... 74
    6.1.2 Quartic integral graphs ......................................... 75
    6.1.3 Subgraph counts of strongly regular graphs .................... 75
  6.2 Future work ........................................................... 76

Appendix A Feasible Vertex-Transitive Spectra ............................ 79
List of Tables

2.1 Connected integral graphs with $n$ vertices ............................... 16
2.2 Connected regular integral graphs with $n$ vertices ................................. 16
2.3 Connected Cayley integral graphs with $n$ vertices ................................. 20
2.4 Strongly regular graph non-existence results for $n \leq 100$ ......................... 24

4.1 Finding the set of possible spectra for vertex transitive graphs .................. 39
4.2 Results at each step of our computations for finding Cayley graphs .............. 42
4.3 Quartic bipartite integral Cayley graphs ........................................... 48
4.4 Drawings of quartic bipartite integral Cayley graphs $G_1$ to $G_{14}$ .............. 49
4.5 Quartic bipartite arc-transitive non-Cayley integral graphs ....................... 50
4.6 Non-bipartite graphs found for $G_i$ ................................................. 52
4.7 Quartic non-bipartite integral Cayley graphs ....................................... 55
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Abstract

This thesis introduces a novel approach to counting subgraphs of regular graphs. This is useful for relating the algebraic properties of a graph $G$ to its structural properties; namely the eigenvalues of the adjacency matrix of $G$ to the counts of various subgraphs of $G$.

It was previously known that the $\ell$th spectral moment of $G$ is equal to the number of closed walks of length $\ell$ in $G$. The first four spectral moments were found to relate the eigenvalues of $G$ to the number of vertices, edges, triangles, and quadrilaterals in $G$. For bipartite regular graphs, the 6th spectral moment was found to relate the eigenvalues of $G$ to the degree and the number of vertices, quadrilaterals, and hexagons in $G$.

We present a general method for counting all closed walks of a given length in connected regular graphs. We describe a set of base walks and their possible extensions to the set of all closed walks of $G$. All the extensions are handled using generating functions, resulting in a substantial reduction in the amount of direct enumeration required. Consequently, we are able to derive equations relating the $\ell$th spectral moment to the degree, the number of vertices, and the counts of various subgraphs in a regular graph.

Such equations have been used to help in the search for members of interesting families of graphs. One such graph family is the integral graphs. These have only integers as eigenvalues. We build on previous results that give a list of spectra (the eigenvalues with their multiplicities) that are possible for a graph that is connected, 4-regular, bipartite, and integral. From this list, we consider the subgraph configurations necessary to eliminate spectra that cannot be realized by a vertex-transitive graph. Using this refined list, we are able to find all connected, 4-regular, integral Cayley graphs.

We present a method for counting connected subgraphs $H$ of a strongly regular graph. Strongly regular graphs are $r$-regular with $e$ common neighbours between adjacent vertices, and $f$ common neighbours between non-adjacent vertices. For some subgraphs $H$, we are able to express the number of copies of $H$ in terms of $r$, $e$, $f$, and the number of vertices. Counts of other subgraphs are calculated in terms of the counts of smaller subgraphs. We use basic counting arguments to prove these results and outline the algorithm which applies these results recursively on subgraphs. We give examples of the equations obtained for several cases of parameter values where it is yet to be established whether strongly regular graphs exist.
We look at how both sets of equations derived in this thesis can be used to investigate the ‘missing’ Moore graph. Moore Graphs are strongly regular graphs with $e = 0$ and $f = 1$. Some well known examples are the Petersen graph and the Hoffman-Singleton graph. It is not known if a 57-regular Moore graph exists. If it does exist then it is integral with spectrum \{57, 7^{1729}, -8^{1520}\}. Our equations give further insights into the structure of the ‘missing’ Moore graph such as the numbers of various cycles and theta graphs.

It is our hope that our equations for counting subgraphs in regular graphs and for counting subgraphs in strongly regular graphs will be useful in various ways. These include finding other specified families of graphs, finding the numbers of subgraphs of interest, investigating co-spectral pairs of graphs, and proving the non-existence of graphs.
Counting Subgraphs of Regular Graphs using Spectral Moments

Declaration

I declare that this thesis is my own work and has not been submitted in any form for another degree or diploma at any university or other institute of tertiary education. Information derived from the published and unpublished work of others has been acknowledged in the text and a list of references is given.

Marsha Elizabeth Minchenko
Counting Subgraphs of Regular Graphs using Spectral Moments

Declaration (Co-Authored Papers)

Material from the following joint papers, with the relative author contributions indicated, has been incorporated into the thesis.

<table>
<thead>
<tr>
<th>Title</th>
<th>Authors</th>
<th>Location in thesis</th>
</tr>
</thead>
<tbody>
<tr>
<td>Spectral moments of regular graphs in terms of subgraph counts</td>
<td>M. Minchenko (80%) I. Wanless (20%)</td>
<td>Chapter 3</td>
</tr>
<tr>
<td>Connected quartic bipartite integral Cayley graphs</td>
<td>M. Minchenko (80%) I. Wanless (20%)</td>
<td>Sections 4.1 – 4.4.5, 4.4.7 – 4.5</td>
</tr>
<tr>
<td>Counting subgraphs of strongly regular graphs</td>
<td>M. Minchenko (80%) I. Wanless (20%)</td>
<td>Chapter 5</td>
</tr>
</tbody>
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Marsha Elizabeth Minchenko

Prof. Ian M. Wanless
Acknowledgments

Firstly, I’d like to thank the most brilliant person I know, my supervisor Ian Wanless, who was pivotal to my success in every area of my PhD research and thesis. His expertise, attention to detail, time, and wealth of ideas offered the guidance and support I needed throughout this endeavour. He went above and beyond to get me started on this journey and settled in Australia, going so far as to offer to pick me up from the airport by public transport! Thanks also to Daniel Horsley for his contributions to my thesis and Daniel Delbourgo for his advice and willingness to answer any queries I had at the beginning of my candidature.

Most importantly, I’d like to thank my parents, Lisa and Greg Minchenko. Mom and Dad, you were there for me even though you were in a different continent. When I found this PhD to be emotionally trying, you were somehow present in a timely matter, often via skype, and able to remind me that there are aspects of life beyond work, and that it is not “the end of the world”. My sisters, Tanya and Sara, you’re always my best friends and even if there were just the few rare moments that we managed to catch up, those were some of the greater ones. For my aunt and uncle, Lynda and Michael Cunningham, for letting me ‘gate-crash’ for an extended period of over four years, for feeding me, and giving me more than I could ask for in so many ways. I’ll never forget these years of my life getting to know you guys as well as many other relatives in Australia. Thank you so much to my partner Thomas Kleinbauer for close friendship and counsel, for conveniently completing your PhD before me and thus having fresh understanding; for offering experience, guidance, support, love; for listening to my whinging. You were a major part of this journey, and I’m looking forward to many more years together with you.

Thank you to my super wonderful group of friends Marc Cheong, Kerri Morgan, Lily Zang, Sheikh Mohammad Rokonuzzaman, Amiza Amir, Minh Duc Cao, Subrata Chakraborty, Dhananjay Thiruvady, Arun Mani, Masud & Tavoos Moshtagh, Andrew Cookson, Rotem Aharon, members of The Lunch Bunch and the Badminton Club. Patricia Jusuf, you were a great friend and mentor. Thanks also to my office-mates and friends from the mathematics department: Sangeeta, Zohreh, Sevvandi, and Judi.

A special thanks to Brendan McKay for hosting me; it was a great honour and an inspiration – you are a wealth of information and ideas in this area as well, Brendan! Your feedback on my first paper on this topic was much appreciated and Section 4.4.6 was all you. A special thanks to Marston Condor for his guidance and support regarding all things MAGMA and regarding arc-transitive graphs. Also to Heiko Dietrich and Csaba Schneider for their helpful advice and for confirming some of the computational results on quartic integral graphs.

Marsha Elizabeth Minchenko

Monash University

December 2013
Chapter 1

Introduction

1.1 Problem and motivation

Graphs are used to model and investigate various problems and systems, where objects and the connections between the objects are described by vertices and edges respectively. Atoms and their bonds, cities and the roads between them, computers and information that is being sent and received are all examples of situations in which graphs can encapsulate the underlying relationships.

Matrices can be used to represent graphs. The eigenvalues of a matrix are a special set of scalars associated with the matrix. The set of all eigenvalues corresponding to the matrix representation of a graph is known as its spectrum. Spectral graph theory seeks to relate the spectrum of a graph to the structural properties of the graph.

The origins of spectral graph theory can be traced to the papers of Günthard and Primas [90] and Cvetković and Gutman [58]. In these papers it was recognized that Hückel’s model in quantum chemistry [107] was also a mathematical theory of graph spectra relating the underlying graph to the eigenvalues, which in this case represented energy levels of certain electrons. Another early problem was posed as “can one hear the shape of a drum?” [114] and this was translated to a problem in spectral graph theory [76] with the graph modelling the shape of a drum while the eigenvalues corresponded to the sound of that drum. Eigenvalues associated with graphs have since arisen in many applications. Many mathematicians worked in this area in the 1950s and 1960s and by the year 1970 there were over 80 papers published on eigenvalues in graphs. In 1980, the book, Spectra of Graphs by Cvetković et al. [62], covered most of the results in the area at the time. It is still one of the best resources as an introduction to this growing field of mathematics.

The paper by Günthard and Primas [90], also raised the question, “which graphs are determined by their spectrum?”. It was thought that the spectrum uniquely determined the graph until a pair of graphs with the same set of eigenvalues was presented in [46]. The answer given instinctively by a survey paper on this area [179] was “almost all graphs”, but as they state more factually “for almost all graphs the answer to this question is still unknown”.

The eigenvalues of a graph are always real numbers. The more restricted class of graphs where the eigenvalues are all integer valued are the integral graphs. The task of determining the graphs with this property was initiated in the early seventies [98] but the problem of finding all integral graphs has been called intractable [98]. At the beginning of the 21st century a use for integral graphs
as a certain quantum spin network in quantum information processing was determined [43] and this
lead to a new emphasis on this topic. There are situations in which integral graphs are preferable to
other graphs because they have the advantage of involving only integer arithmetic [64].

1.2 Research contributions

This thesis makes contributions to the area of spectral graph theory. We prove results that give
information about the subgraphs of a graph based on the set of eigenvalues of the graph. We are
able to use these results to find previously unknown integral graphs. Other discoveries in this thesis
are useful for investigating strongly regular graphs.

Our main research contributions in the thesis can be summarized as follows.

In Chapter 3, we find equations that relate the eigenvalues of a regular graph to the number of
isomorphic copies of certain small graphs which appear as subgraphs of the graph.

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ denote the eigenvalues of the adjacency matrix of $G$. The sum, $w_\ell = \sum_{j=1}^{n} \lambda_j^\ell$, of the $\ell$-th powers of the eigenvalues is known as the $\ell$-th spectral moment and is equal to the number of
closed walks of length $\ell$ in $G$. It is folklore that $w_1 = 0$, $w_2$ is two times the number of edges of $G$, and $w_3$ is six times the number of triangles in $G$. By using generating functions to count a certain
type of closed walk, we extend this list of equations to beyond what appears in the current literature.

Let $C_i$ denote the $i$-cycle. For any graph $H$, let $[H]$ denote the number of (not necessarily induced)
subgraphs in $G$ that are isomorphic to $H$, where the parent graph $G$ will be implicitly specified by
the context. For 4-regular bipartite $G$, equations for $w_\ell$ up to $\ell = 6$ were given by Cvetković et
al. [63]:

$$
\begin{align*}
w_0 &= n, \\
w_4 &= 28n + 8[C_4], \\
w_2 &= 4n, \\
w_6 &= 232n + 144[C_4] + 12[C_6].
\end{align*}
$$

(1.1)

Stevanović et al. [176] added an inequality for $\ell = 8$:

$$
w_8 \geq 2092n + 2024[C_4] + 288[C_6].
$$

(1.2)

We give a general method for finding equations such as (1.1) for spectral moments of regular
graphs, whether or not they are bipartite.

The subgraphs that appear in our equations will be called contributors (in (1.1) all of the con-
tributors are cycles, but this is not true in general). Our methods build on those of Friedland et
al. [78] and Wanless [195] who relate the numbers of matchings to the counts of contributors. We
use a special class of closed walks, called the not totally-reducible walks, similar to the “primitive
circuits” used by McKay [141] to count spanning trees.

There is a result in the literature that can be used to count the totally-reducible walks in a
graph [140]. We count closed walks as the sum of totally-reducible walks and closed not totally-
reducible walks. The ideas of base walks and their extensions are introduced. We prove that given
a closed and not totally-reducible walk $W'$, there exists a unique base walk $W$ such that $W$ extends
to $W'$.

A suitable generating function is derived to count all not totally-reducible walks that extend a
base walk of length $\ell$. Suppose $W = v_0v_1 \cdots v_\ell$ is a closed walk of length $\ell$ in an $(r + 1)$-regular
graph, $G$. It has been shown for an infinite rooted tree $T$ where the root has degree $k_1$ and every other vertex has degree $k_2 + 1$, that the generating function for closed rooted walks in $T$ is

$$T_{k_1} = \frac{2k_2}{2k_2 - k_1 + k_1 \sqrt{1 - 4x^2k_2}}.$$ 

We prove that the generating function for walks in an $r$-regular graph $G$ that are extensions of $W$ is

$$\psi(\ell) = x^\ell T_r T_{r+1} \left( \frac{1 - x^2 T_r^2}{1 - r x^2 T_r^2} \right).$$

Algorithms are used to take a contributor $H$ and find $b_i$, the number of base walks of length $i$ up to $\ell$ that induce $H$; to calculate $\eta(H) = \sum_{i=1}^{\ell} b_i \psi(i)$; as well as to determine $\mathcal{H}_\ell$, the set of all contributors that can be induced by the base walks of length up to $\ell$. Finally, it is shown that the generating function for closed and not totally-reducible walks of length at most $\ell$ in a graph $G$ is

$$\sum_{H \in \mathcal{H}_\ell} \eta(H)[H] + O(x^{\ell+1}).$$

In this way we derive generating functions that give expressions for the numbers of closed walks in terms of the counts of contributors.

The chapter ends by characterising the contributors that can occur in the resulting equations.

In Chapter 4, we use a variety of methods to find new quartic integral graphs. We produce exhaustive lists of the connected 4-regular integral Cayley graphs and the connected 4-regular integral arc-transitive graphs.

The spectrum of a connected 4-regular bipartite integral graph has the form $\{4, 3^x, 2^y, 1^z, 0^w, -1^z, -2^y, -3^x, -4\}$; which can be abbreviated by the quadruple $[x, y, z, w]$. Cvetković et al. [63] showed that the number of vertices in a connected $r$-regular bipartite graph with radius $R$ is bounded above by $(2(r - 1)^R - 2)/(r - 2)$. Thus, it can be shown that connected 4-regular bipartite integral graphs have at most 6560 vertices.

Cvetković et al. [63] found quadruples $[x, y, z, w]$ that are candidates for the spectrum of a bipartite quartic integral graph. They called these possible spectra. Research activities regarding the set of possible spectra fall into two streams: eliminate possible spectra based on new information and/or techniques, or find graphs that realize a possible spectrum. Useful tools include an identity by Hoffman [104], spectral moment equations like (1.1), and inequality (1.2). All quartic integral graphs (QIGs) that avoid eigenvalues of $\pm 3$ and realize a possible spectrum are found in [173]. Stevanović [171] eliminates spectra using equations arising from graph angles. In the same paper he determines that the possible values for the order $n$ of the graph are between 8 and 1260, except for 5 identified spectra. Stevanović et al. [176] use their inequality for $w_8$ to determine that $8 \leq n \leq 560$. All of the bipartite QIGs with $n \leq 24$ that realize one of the possible spectra were found and are listed with drawings in [176].

The equations for the spectral moments from Chapter 3, when specialized to bipartite 4-regular graphs, are sufficient to reproduce the list of possible spectra presented in [176]. In Section 4.2, we show that some spectra on this list cannot be achieved by any vertex-transitive graph.
The Cayley QIGs on finite Abelian groups have been studied by Abdollahi et al. [2]. They showed that for an Abelian group, $\Gamma$, if $\text{Cay}(\Gamma, S)$ is a Cayley QIG then

$$|\Gamma| \in \{5, 6, 8, 9, 10, 12, 16, 18, 20, 24, 25, 32, 36, 40, 48, 50, 60, 64, 72, 80, 96, 100, 120, 144\},$$

but they did not establish whether Cayley QIGs of these orders exist. We find that the set of orders of Cayley QIGs on Abelian groups is precisely $\{5, 6, 8, 9, 10, 12, 16, 18, 24, 36\}$. More generally, we consider all groups and find that many Cayley QIGs are on non-Abelian groups. We show that for any group $\Gamma$, if $\text{Cay}(\Gamma, S)$ is a Cayley QIG then

$$|\Gamma| \in \{5, 6, 8, 9, 10, 12, 16, 18, 20, 24, 30, 32, 36, 40, 48, 60, 72, 120\}.$$ 

Furthermore, for each of these orders we give the Cayley QIGs that exist.

Our findings are that up to isomorphism, there are exactly

- 32 connected 4-regular integral Cayley graphs,
- 17 connected 4-regular bipartite integral Cayley graphs,
- 27 connected 4-regular integral graphs that are arc-transitive, and
- 16 connected 4-regular bipartite integral graphs that are arc-transitive.

Upon investigating the relationship between two of the bipartite arc-transitive non-Cayley integral graphs $F_1$ and $F_4$, we uncovered the method for constructing new integral graphs that are quotients of existing integral graphs. This leads to our finding additional connected 4-regular integral graphs that are not Cayley and not arc-transitive.

The graphs with only 3 distinct eigenvalues are the strongly regular graphs. In Chapter 5, we present equations for the counts of subgraphs of a strongly regular graph in terms of its parameters and the counts of a few small subgraphs. In this case, the parameters are the standard ones: the number of vertices $n$, the regularity $r$, the number of common neighbours between adjacent vertices $e$, and the number of common neighbours between non-adjacent vertices $f$. We denote a strongly regular graph by $\text{SR}(n, r, e, f)$.

Necessary conditions have been established for the existence of a strongly regular graph with parameter set $(n, r, e, f)$, and where these conditions are met, a parameter set is considered feasible. For a feasible parameter set, it is of interest to determine firstly whether it is realizable by a graph and secondly to find all graphs $\text{SR}(n, r, e, f)$. A list of feasible parameter sets and whether or not they are realizable, including references, can be found in [30]. Currently, the smallest feasible parameter set for which the existence question remains unanswered is $(65, 32, 15, 16)$. For $n \leq 100$, the existence question is still unanswered for 16 sets of feasible parameters (and their complements).

We present results for determining the number of copies of a subgraph $H$ in a strongly regular graph in terms of the counts of subgraphs with strictly fewer vertices. When the existence question for a feasible parameter set is unanswered, our methods give subgraph counts that a graph with that parameter set would have to have. Thus, this tool is useful for both studying subgraphs of existing graphs as well as considering the existence question for feasible parameter sets.
We summarize our results here but the details and an explanation of the notation used can be found in Sections 5.1 and 5.2. Consider a strongly regular graph $SR(n, r, e, f)$ and a graph $H$ containing a vertex $s$ of degree one. Let $u$ denote the neighbour of $s$ in $H$ and let $N$ denote the neighbours of $u$ in $H \setminus \{s\}$. Let $X$ denote a set of vertices of $H \setminus \{u, s\} \cup N$ that are orbit representatives under the action of $Aut(H \setminus \{s\}_u)$. Then,

$$[H] = \frac{(r - |N|)[H \setminus \{s\}, u] - \sum_{x \in X}[H \setminus \{s\} + ux, u, x]}{|Aut(H)u||Aut(H)_\{u\}u||Aut(H)_{\{u\}u}s|}.$$

Consider a strongly regular graph $SR(n, r, e, f)$ and a graph $H$ containing a vertex $s$ of degree two, such that $s$ is not a cut-vertex of $H$. Let $u$ and $v$ denote the two neighbours of $s$ in $H$ and let $C$ denote the common neighbours of $u$ and $v$ in $H \setminus \{s\}$. Let $X$ denote a set of vertices of $H \setminus \{(u, v, s) \cup C\}$ that are orbit representatives under the action of $Aut(H \setminus \{s\}_{\{u,v\}})$. If $u$ is adjacent to $v$ in $H$ then,

$$[H] = \frac{(e - |C|)[H \setminus \{s\}, u, v] - \sum_{x \in X}[H \setminus \{s\} + ux + xv, u, v, x]}{|Aut(H)u||Aut(H)_{\{u\}v}||Aut(H)_{\{u,v\}s}|}.$$

Otherwise,

$$[H] = \frac{(f - |C|)[H \setminus \{s\}, u, v] + (e - f)[H \setminus \{s\} + uv, u, v] - \sum_{x \in X}[H \setminus \{s\} + ux + xv, u, v, x]}{|Aut(H)u||Aut(H)_{\{u\}v}||Aut(H)_{\{u,v\}s}|}.$$

An algorithm is used to recursively call these theorems for successively smaller graphs until the graphs consist of a single vertex or a graph with minimum degree 3. We thereby express the number of copies of some subgraphs in terms of the standard parameters of strongly regular graphs. Counts of other subgraphs are calculated in terms of the counts of smaller subgraphs.

One of the most famous open problems in algebraic graph theory is whether or not a 57-regular Moore graph exists. A Moore graph is a regular graph with degree $\Delta$ and diameter $D$ for which $n = 1 + \Delta \sum_{i=1}^{D} (\Delta - 1)^{i-1}$. A Moore graph of diameter 2 is a strongly regular graph with no common neighbours between adjacent vertices and exactly 1 common neighbour between non-adjacent vertices. Hoffman and Singleton [105] prove that any Moore graph with diameter 2 must have degree 2, 3, 7, or 57. They also show that unique graphs exist for the first 3 values: the pentagon, the Petersen graph, and the Hoffman-Singleton graph. The 4th value corresponds to the strongly regular graph $G = SR(3250, 57, 0, 1)$. If such a graph exists, $|V(G)| = 3250$, $Sp(G) = \{57^1, 7^{1729}, -8^{1520}\}$, $Aut(G)$ is not a rank 3 group [7], and $G$ is not vertex-transitive (see for example [40]). More recent work on the missing Moore graph has been done in investigating the symmetries and the automorphism group of $G$ [135, 138, 139].

At the end of the chapter we present a few case studies. We implement our algorithm to count small subgraphs of various strongly regular graphs which are not known to exist. Among these case studies, we present the counts of small subgraphs of the famous “missing” Moore graph.
1.3 Publications

Publications arising from this thesis include:


Minchenko, M. and Wanless, I. M. (2013), Spectral moments of regular graphs in terms of subgraph counts. Accepted to Linear Algebra and its Applications.


Chapter 2

Literature Review

2.1 Background

All graphs, $G = \{V(G), E(G)\}$, in this document are simple and undirected with edge set $E(G)$ and vertex set $V(G)$ of size $n$.

The adjacency matrix, $A = [a_{ij}]$, of $G$ is the $n \times n$ matrix defined as

$$a_{ij} = \begin{cases} 1 & \text{if } i \text{ is adjacent to } j, \\ 0 & \text{otherwise.} \end{cases}$$

The eigenvalues of $A$ are the values $\lambda$ satisfying the equation $Av = \lambda v$ for non-zero vectors $v$. The adjacency matrix is real and symmetric. Hence, all its eigenvalues are real and $A$ is similar to a diagonal matrix with diagonal consisting of the eigenvalues of $A$ [132]. The trace of $A$, $\text{Tr}(A)$ is the sum of the diagonal entries. Now since similar matrices have the same trace;

$$\text{Tr}(A) = \sum_{k=1}^{n} \lambda_k.$$  \hfill (2.1)

A walk $W$ of length $\ell$ in $G$ is a sequence of vertices $v_0v_1 \cdots v_\ell$, where $v_i$ is adjacent to $v_{i-1}$ for each $i = 1, 2, \ldots, \ell$. When $v_0 = v_\ell$ we say that $W$ is a closed walk. If we consider the matrix $A^2$ and look at one entry, we have

$$A^2_{i,j} = a_{i,1}a_{1,j} + a_{i,2}a_{2,j} + \cdots + a_{i,n}a_{n,j}$$

which is the number of walks of length 2 from vertex $i$ to vertex $j$. Similarly $A^\ell_{i,j}$ gives the number of walks of length $\ell$ from vertex $i$ to vertex $j$.

Thus the entries along the diagonal in $A^\ell$ give the number of walks of length $\ell$ from a given vertex to itself; and the trace gives the total number of closed walks of length $\ell$ in $G$. By (2.1), $\text{Tr}(A^\ell) = \sum_{k=1}^{n} \lambda_k^\ell$.

The spectrum of a graph, $Sp(G)$, with respect to its adjacency matrix consists of the eigenvalues of its adjacency matrix with their multiplicity. We use the notation: $Sp(G) = \{\lambda_1^{m_1}, \lambda_2^{m_2}, \ldots, \lambda_s^{m_s}\}$, where $m_i$ is the multiplicity of $\lambda_i$ for $i = 1, \ldots, s$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_s$. For brevity, we refer to the eigenvalues of the adjacency matrix of the graph as the eigenvalues of the graph. Two graphs $G_1$ and $G_2$ are said to be co-spectral if $Sp(G_1) = Sp(G_2)$.  

7
The sum, \( w_\ell = \sum_{j=1}^{n} \lambda_j^\ell \), of the \( \ell \)-th powers of the eigenvalues is known as the \( \ell \)-th spectral moment. It is folklore that \( w_0 = n \), \( w_1 = 0 \), \( w_2 \) is two times the number of edges in \( G \), and \( w_3 \) is six times the number of triangles in \( G \). This is notable because the number of vertices, edges, and triangles are completely determined by the spectrum of \( G \).

Let \( C_n \) denote the \( n \)-cycle and \( K_n \) denote the complete graph on \( n \) vertices. The complete \( s \)-partite graph \( K_{p_1,p_2,...,p_s} \) is a graph with vertex set \( V = V_1 \cup V_2 \cup \cdots \cup V_s \) where the \( V_i \) are nonempty disjoint sets and \( |V_i| = p_i \) for \( 1 \leq i \leq s \), with \( x \) in \( V_i \) adjacent to \( y \) in \( V_j \) if and only if \( i \neq j \). An \( s \)-partite graph has vertex set the vertex set of \( K_{p_1,p_2,...,p_s} \) and edge set a subset of the edge set of \( K_{p_1,p_2,...,p_s} \). Let \( G \cup H \) denote the disjoint union of graphs \( G \) and \( H \). We mention that \( K_{1,4} \) and \( K_1 \cup C_4 \) have the same spectrum but not the same number of 4-cycles. For this reason, equations for spectral moments for \( w_\ell \) for \( \ell > 3 \) cannot be as simple as the equations mentioned above for \( \ell \leq 3 \). Nevertheless, it is possible to find such equations, as we show in Chapter 3.

Since the spectrum of a disconnected graph is the union of the spectra of its connected components, from this point on we only consider connected graphs.

We mention that there are other types of graph spectra that are studied in spectral graph theory [94]: the Laplacian spectrum, the signless Laplacian spectrum, the normalized Laplacian spectrum, and the Seidel spectrum. In this document we restrict ourselves to the adjacency spectrum.

The following result is a special case of a famous theorem of Perron [148] and Frobenius [80] for non-negative matrices:

**Theorem 1** (Perron-Frobenius). If \( G \) is a connected graph with at least 2 vertices, then

- its largest eigenvalue \( \lambda_1 \) has multiplicity one;
- for the eigenvalue \( \lambda_1 \), the corresponding eigenvector has all positive coordinates;
- any other eigenvalue \( \lambda \) satisfies \(-\lambda_1 \leq \lambda < \lambda_1\);
- the deletion of any edge of \( G \) decreases the largest eigenvalue.

The largest eigenvalue \( \lambda_1 \) is called the spectral radius of \( G \).

An \( r \)-regular graph is a graph for which each vertex is adjacent to exactly \( r \) other vertices. If we specifically consider \( r \)-regular graphs \( G \), we have that \( r \) is the spectral radius of \( G \). This result is quickly seen since each row of the adjacency matrix \( A \) of an \( r \)-regular graph has exactly \( r \) non-zero entries, each with value 1, and so \( (1,1,...,1)^T \) is an eigenvector with eigenvalue \( r \). This implies that every other eigenvector is orthogonal to \( (1,1,...,1)^T \) and thus contains a negative coordinate. By Theorem 1, we have that \( |\lambda| \leq r \) for all eigenvalues \( \lambda \) of \( G \) and the multiplicity of the spectral radius \( r \) is one. We note that more generally, when the graph is not necessarily connected, the multiplicity of the spectral radius \( r \) is equal to the number of connected components of \( G \) (see [23] for example). Since, consider \( A \) in block diagonal form with blocks consisting of the connected components with zero everywhere else. Each block will be similar on its own to a diagonal matrix of its eigenvalues.

A bipartite graph is a 2-partite graph. If we consider bipartite graphs, we have that the spectrum is symmetric about 0: if \( \lambda \) is an eigenvalue of the bipartite graph, \( G \), then \(-\lambda \) is also an eigenvalue of \( G \) with the same multiplicity. This result was originally shown by Coulson and Rushbrooke [47] as they investigated the structure of molecules.
2.2 Spectral moment equations

In this subsection we review some past papers where spectral moments are related to the counts of subgraphs of a graph. We also mention some results related to counting walks in regular graphs that we will need later.

Consider a walk $W = v_0 v_1 \cdots v_\ell$. If $v_{i-1} = v_{i+1}$ for some $i$ then $W$ is reducible, otherwise it is irreducible. In the reducible case, $W$ can be reduced at index $i$ by omitting $v_i$ and $v_{i+1}$. We let $\text{red}(W)$ be the irreducible result of repeatedly omitting $v_i$ and $v_{i+1}$ for $i = \min\{j \mid v_{j-1} = v_{j+1}\}$ until no such $j$ exists. By a result of Godsil [83], if reductions are iteratively applied to a walk until it is irreducible, then the result is independent of the reductions chosen.

**Lemma 2.** If $W$ reduces to $W'$ by any sequence of reductions, then $\text{red}(W) = \text{red}(W').$

**Proof.** Case A: $W$ reduces to $W'$ in at most one step. Suppose $W$ has length $l$. If $l \leq 2$ then either $W$ is of the form $v_0 v_1 v_0$ and $W' = v_0$ or $W' = W$. Thus $\text{red}(W') = \text{red}(W)$.

Now, we use induction on the length $l \geq 2$ and assume that $\text{red}(W') = \text{red}(W)$ for $l \leq n$. We consider $l = n + 1$ where $W$ is reduced to $W'$ at index $i$. Let $j$ be the smallest index where $W$ can be reduced. Then $j \leq i$.

If $j = i$ then $\text{red}(W') = \text{red}(W)$ by definition.

If $j < i - 1$ then we reduce both $W'$ and $W$ at index $j$ to give walks $W''$ and $W^*$ respectively.

$\text{red}(W') = \text{red}(W'')$ by the inductive hypothesis since $W'$ reduces to $W''$ at index $j$

$= \text{red}(W^*)$ by the hypothesis since $W^*$ reduces to $W''$ at index $i - 2$

$= \text{red}(W)$ by reducing $W$ at index $j$ followed by the sequence of reductions by which $W^*$ reduces to $\text{red}(W^*)$

So suppose $j = i - 1$. Then, $W$ has $v_{i-2} v_{i-1} = v_i v_{i+1}$. Thus $W$ reduces to $W'$ at both $i$ and $j$, which gives the desired result.

Case B: $W$ reduces to $W'$ in more than one step. We use induction on the number of steps $k \geq 1$. The base case follows from Case A. Assume that $\text{red}(W) = \text{red}(W')$ when $W$ reduces to $W'$ in $k \leq n$ steps. Let $k = n + 1$ and suppose that the first reduction step was at index $i$. Then, $W'$ reduces to some walk $W^*$ at index $i$ which further reduces to $W'$. By the inductive hypothesis, $\text{red}(W') = \text{red}(W^*)$. Now reduce $W$ at index $i$ followed by the sequence of reductions for which $W^*$ reduces to $\text{red}(W^*)$. We have that $\text{red}(W^*) = \text{red}(W)$ and the desired result follows. \qed

Let $v_0, v_1, \ldots, v_\ell$ denote the vertices of a graph $G$. A primitive circuit is a closed irreducible walk $W$ of length $\geq 3$ such that $v_1 \neq v_{\ell-1}$. Each primitive circuit $W$ is taken to represent all cyclic permutations of the sequence of edges of $W$ and also the reverse walk $v_\ell v_{\ell-1} \cdots v_0$. McKay [141] determines a generating function for the number of closed walks of length $\ell$ in an $r$-regular graph $G$ in terms of $n, r$, and the number of primitive circuits of length $\ell$ in $G$. McKay’s result is as follows. Let $G$ be an $r$-regular graph of order $n$. Let $p_\ell$ be the number of primitive circuits of length $\ell \geq 0$ in
G. For \( w(x) = \sum_{\ell=0}^{\infty} w_{\ell} x^{\ell} \) and \( p(x) = \sum_{\ell=3}^{\infty} \frac{\ell^{2} x^{\ell}}{1-x^2} p_{\ell} \),

\[
w(x) = \frac{r-2 - r(1-4(r-1)x^2)^{1/2}}{2(r^2x^2-1)} + \frac{2}{1-(4(r-1)x^2)^{1/2}} \left( \frac{1 - (1-4(r-1)x^2)^{1/2}}{2(r-1)x} \right) .
\]

In [178], van Dam investigates \( r \)-regular graphs \( G \) with four distinct eigenvalues. It is shown that in such graphs, the number of closed walks of length \( \ell \) from a given starting vertex is independent of the choice of starting vertex for all \( \ell \). Therefore, \( \frac{1}{n} w_{\ell} \) is a non-negative integer for all \( \ell \). This work is extended in [180] where van Dam and Spence determine the feasible spectra \( \{r^1, \lambda'^1, \lambda'^2, \lambda'^3\} \). With the aid of a computer, they were able to answer the existence question for 214 of the 244 feasible spectra. For each realizable spectrum, all graphs with that spectrum were found.

Jiang et al. [112] present a procedure for enumerating the spectral moments of molecules (connected graphs with maximum vertex degree three) in terms of subgraph counts. They give a procedure for finding the counts of subgraphs containing no cycles and find equations for \( w_{\ell} \) for values of \( \ell \) up to 14. This allows them to approximate a formula for the total \( \Pi \) electron energy. They consider how moments are dependent on the connectivities of molecules with both bonds and multi-atom subgraphs contributing to the equations. Jiang and Tang [110] continue to investigate moment equations for acyclic carbon chains. The moment equations for molecules consisting entirely of hexagonal rings are investigated in [96]. Jiang and Zhang [111] seek to further illuminate the relationship between molecular behaviour and structure using spectral moments. They illustrate the procedure for obtaining some of the subgraphs with cycles whose counts appear in moment equations; giving \( w_{\ell} \) up to \( \ell = 12 \) for this set of subgraphs. Jiang and Zhang claim that moments allow them to directly determine molecular information. They give a scheme based on moment analysis and energy partitioning for determining the stability and reactivity of conjugated hydrocarbons and mention that moments enabled them to bypass the usual molecular orbital theory methods. There are many more similar and related results in both chemistry and mathematics [24, 71, 72, 73, 91, 92, 97, 113].

A fullerene graph or simply fullerene is a 3-regular, planar graph with all faces 5-cycles or 6-cycles including the external face. Zhang and Balasubramanian [201] consider fullerenes containing isolated 5-cycles and derive analytical formulae for the first 14 moments in terms of counts of subgraphs. They observe that the first 11 moments depend only on the counts of cycles. Cvetković et al. [52] define a measure of fullerenes, called the width, related to the distance from vertices in the fullerene to the nearest 5-cycle. Cvetković and Stevanović [60] prove that for small values of \( \ell \), \( w_{\ell} \) depends only on \( \ell \), the width, and the number of vertices of the fullerene graph.

The coalescence of two graphs, \( G_1 \) with a distinguished vertex \( v_1 \) and \( G_2 \) with a distinguished vertex \( v_2 \), is the graph obtained by identifying vertices \( v_1 \) and \( v_2 \). A lollipop \( L(m,k) \) is the coalescence of \( C_m \) with \( m \geq 3 \) vertices and a path \( P_{k+1} \) with \( k+1 \geq 2 \) vertices and distinguished vertex one of the vertices of degree one. Boulet and Jouve [29] give a way to count closed walks of a graph \( G \) in terms of its subgraph counts. Here, the graph \( G \) and the subgraphs described are not necessarily regular. They use this information to show that there are no co-spectral non-isomorphic lollipops and that each lollipop graph is determined by its spectrum. An example of an equation from their paper is as follows: for a graph \( G \) with no triangles or 3-cycles,

\[
\]
where \( [H] \) denotes the counts of subgraphs \( H \) in \( G \) as before. We note that Haemers et al. [95] previously showed that co-spectral lollipops are isomorphic and that lollipops \( L(m,k) \) with \( m \) odd are determined by their spectrum but they did not consider lollipops with \( m \) even and the methods used were different.

A walk of length 0 is trivial. If \( \text{red}(W) \) is trivial then we say that \( W \) is totally-reducible. Let \( u_\ell \) be the number of totally-reducible walks of length \( \ell \) in \( G \). A totally-reducible walk must have even length, so \( u_\ell = 0 \) for all odd \( \ell \). For even \( \ell \), McKay [140] showed that

\[
    u_\ell = n \sum_{i=0}^{\ell/2} \binom{\ell}{i} \frac{\ell - 2i + 1}{\ell - i + 1} r^i.
\]  

(2.2)

A matching of size \( i \) or an \( i \)-matching in a graph \( G \) is a set of \( i \) edges of \( G \) in which no two edges share a vertex. Both Friedland et al. [78] and Wanless [195] use generating functions to find equations relating the number of matchings to the counts of certain small subgraphs.

In [78] a generating polynomial for the number of matchings in a graph is found. The matching polynomial of a graph \( G_1 \cup G_2 \) is the product of the matching polynomial of the graph \( G_1 \) and the matching polynomial of the graph \( G_2 \). Friedland et al. find the matching polynomials for various paths, cycles, and their unions. They determine some graphs with the same matching polynomial. Finally, an expression for the number of \( \ell \)-matchings of an \( r \)-regular graph is given in terms of \( r \), \( n \) and the counts of various paths, cycles, the unions of paths and cycles, and a few other small graphs for \( \ell \) up to 4.

When the vertices of a walk \( W = v_0v_1\cdots v_\ell \) are all distinct then \( W \) is a path. For \( 0 \leq i \leq \ell \), we call the walk \( W_i = v_0v_1\cdots v_i \) a prefix of \( W \). If \( \text{red}(W_i) \) is a path for each \( i \) then \( W \) is called a tree-like walk. A tree-like walk is closed if and only if it is totally-reducible. Wanless [195] expresses the number, \( \epsilon_\ell \), of totally-reducible walks of length \( \ell \) which are not tree-like in an \( r \)-regular graph \( G \) in terms of \( n \), \( r \), and the number of certain subgraphs of \( G \). Wanless derives the polynomials \( \epsilon_\ell \) using generating functions. A rooted tree is a tree with a distinguished vertex, called the root. A rooted walk in a rooted tree is a walk that starts at the root. Counting closed walks in a regular graph can be equated to counting closed walks in an infinite, nearly regular tree with a distinguished root vertex. The following result is from [195]:

**Lemma 3.** Let \( T \) be an infinite rooted tree in which the root has degree \( k_1 \) and every other vertex has degree \( k_2 + 1 \). The generating function for the number of closed rooted walks in \( T \) is

\[
    T_{k_1} = \frac{2k_2}{2k_2 - k_1 + k_1\sqrt{1 - 4x^2k_2}}.
\]

**Proof.** We generate closed rooted walks in \( T \) as follows:

1. We perform a trivial walk in 1 way or

2. We perform a non-trivial closed walk:
   - there are \( k_1 \) choices for the next vertex after the root,
   - there are \( T_{k_2} \) closed walks not involving the root from the chosen vertex,
   - there are \( T_{k_1} \) ways to make a closed walk upon returning to the root.
CHAPTER 2. LITERATURE REVIEW

Since walking away from the root and returning involves a factor $x^2$, we have

$$T_{k_1} = 1 + k_1T_{k_2}T_{k_1}x^2.$$  

We take $k_1 = k_2$ to count walks that do not include the root and get

$$k_2x^2T_{k_2}^2 - T_{k_2} + 1 = 0.$$  

Upon solving the quadratic, we take the correct root (after considering the limit as $x \to 0$) to give

$$T_{k_2} = 1 - \frac{\sqrt{1 - 4x^2k_2^2}}{2x^2k_2}.$$  

Finally, substituting this value for $T_{k_2}$ into the above expression for $T_{k_1}$ gives the desired result.

Wanless seeks to count all the desired walks by extending the most basic walks. A walk $W'$ is formed by adding a diversion to $W = v_0v_1 \cdots v_{\ell}$ if $W' = v_0 \cdots v_id_{v_{i+1}} \cdots v_{\ell}$ where $d$ is a walk such that $v_id$ is totally-reducible (of length $\geq 0$), and if $i < \ell$ then no prefix of $v_id$ reduces to $v_iv_{i+1}$. This generating function is given as

**Lemma 4.** Let $W = v_0v_1 \cdots v_{\ell}$ be a walk of length $\ell$ in an $(r+1)$-regular graph, $G$. The generating function for walks in $G$ formed by adding a diversion to $W$ is $x^\ell T_r^\ell T_{r+1}$.

**Proof.** Adding diversions results in a walk $W' = v_0d_0v_1d_1 \cdots v_id_\ell$. The generating function for $v_id_i$ is $T_r$ for $0 \leq i < \ell$ and is $T_{r+1}$ for $i = \ell$. The $\ell$ edges, $d_iv_{i+1}$ for $0 \leq i < \ell$, result in the factor $x^\ell$. \qed

Wanless also shows that the process of reducing a walk is unique.

**Lemma 5.** If $W'$ reduces to $W$ then there is a unique way to write $W' = v_0d_0v_1d_1 \cdots v_id_\ell$ where $W = v_0v_1 \cdots v_{\ell}$ and each $v_id_i$ is a diversion.

A walk $W'$ has been formed by adding a tail of length $t$ to a closed walk $W$ if

$$W' = u_1u_2 \cdots u_tWu_t \cdots u_1$$

where $u_1 \cdots u_tv_0$ is an irreducible walk of length $t$ and neither $u_tv_0$ nor $v_0u_t$ occurs in $W$. Diversions and tails are called extensions. Adding diversions and tails are combined in the following way:

**Lemma 6.** Suppose $W = v_0v_1 \cdots v_{\ell}$ is a closed walk of length $2\ell$ in an $(r+1)$-regular graph $G$ and let $H$ be the subgraph induced by the edges of $W$. Suppose the first vertex of $W$ has degree $\delta$ in $H$. The generating function for walks in $G$ that are extensions of $W$ is

$$\psi(\ell, \delta) = x^{2\ell} T_r^{2\ell} T_{k+1} \left( \frac{1 - (\delta - 1)x^2T_k^2}{1 - r x^2 T_k^2} \right),$$

where each walk is counted the number of times that it can be formed by adding tails and/or diversions to $W$.

An algorithm is designed to find the equations for $\epsilon_\ell$: the counts of all closed non-tree-like walks in terms of the counts of subgraphs. This algorithm determines subgraphs $H$ that are essential to
the equation, eliminates over-counting errors based on inclusion-exclusion principles, and gives the counts of all closed non-tree-like walks as coefficients of the counts of subgraphs based upon the above generating functions. We give an example of the algorithm output. Let $C_{i_1,i_2,\ldots,i_h}$ denote cycles of length $i_1,i_2,\ldots,i_h$ sharing a single vertex and $\Theta_{i_1,i_2,\ldots,i_h}$ two vertices joined by internally disjoint paths of lengths $i_1,i_2,\ldots,i_h$. These are depicted in Figure 3.1. This equation is for an $(r+1)$-regular graph $G$:

$$
\epsilon_{12} = (1320r^3 - 6)[C_3] + 528r^2[C_4] + 120r[C_5] + 192[K_4] - (480r + 12)[\Theta_{2,2,1}] \\
- 48([\Theta_{3,2,1}] + [\Theta_{2,2,2}] + [C_{3,3}]).
$$

The values of $\epsilon_{\ell}$ for $\ell$ up to 16 and up to 20 in the bipartite case are explicitly reproduced in [195]. The number of matchings of size $\ell$ in an $r$-regular graph can be determined given $u_\ell - \epsilon_\ell$. Wanless presents expressions for the number of matchings of size $\ell$ up to 12.

### 2.3 Integral graphs

An integral graph is a graph for which all eigenvalues of its adjacency matrix are integers.

Integral graphs have applications such as modeling the quantum spin networks that permit perfect state transfer [3, 6, 18, 19, 20, 43, 44, 59, 74, 85, 86, 87, 149, 159, 165, 174], and modeling multiprocessor interconnection networks especially in connection to the load balancing problem [6, 57, 59, 64].

Harary and Schwenk [98] first introduced the problem of finding integral graphs in 1974. In general, this problem is a difficult one. For this reason, results in the area pertain to investigating particular classes of integral graphs and/or using graph operations or transformations for the construction of integral graphs. There have been many interesting discoveries, a lot of which we will briefly mention here. We also refer the reader to the three survey papers on this topic: Integral trees – a survey by Li and Wang [131], A survey on integral graphs by Balińska et al. [12], and A survey of results on integral trees and integral graphs by Wang [181].

Some examples of integral graphs are the complete graphs, $K_n$, with spectra $\{(n-1)^1, -1^{n-1}\}$; the cocktail party graphs, $CP(n)$, with spectra $\{(2n-2)^1, 0^n, -2^{n-1}\}$; balanced complete multipartite graphs, $K_{n/k,n/k,\ldots,n/k}$, with spectra $\{(n-n/k)^1, (n-1)^{n-k}, -(n/k)^{k-1}\}$; and complete bipartite graphs, $K_{m,n}$, for which $mn$ a perfect square, with spectra $\{\sqrt{mn}^1, 0^{m+n-2}, -\sqrt{mn}^1\}$ (which includes the stars $K_{1,m^2}$). The cycles, $C_n$, have eigenvalues $\{2\cos(2k\pi/n) \mid k = 1, \ldots, n\}$ and thus $C_n$ is only integral for $n = 3, 4, 6$.

We call the set of vertices connected to $v$ by an edge, the neighbours of $v$. A strongly regular graph is an $r$-regular graph with exactly $e$ common neighbours between every pair of adjacent vertices and $f$ common neighbours between every pair of non-adjacent vertices for some $r, e,$ and $f$. A strongly regular graph $G$ is integral if $(e-f)^2 - 4(f-r) = s^2$ for some positive integer $s$.

In the original paper on integral graphs [98], Harary and Schwenk make mention of some graph operations which preserve integrality: the product (or conjunction), the sum (or Cartesian product), and the strong sum (or strong product). Let $G_1, G_2$ be graphs with vertex sets $V(G_1), V(G_2)$. The product of $G_1$ and $G_2$, $G_1 \times G_2$, is the graph with vertex set the Cartesian product $V(G_1) \times V(G_2)$ with $(x, a)$ adjacent to $(y, b)$ if and only if $x$ is adjacent to $y$ in $G_1$ and $a$ is adjacent to $b$ in $G_2$. The
sum of \( G_1 \) and \( G_2 \), \( G_1 + G_2 \), is the graph with vertex set the Cartesian product \( V(G_1) \times V(G_2) \) with \((x, a)\) adjacent to \((y, b)\) if and only if \( x = y \) and \( a \) is adjacent to \( b \) in \( G_2 \) or \( x \) is adjacent to \( y \) in \( G_1 \) and \( a = b \). The strong sum of \( G_1 \) and \( G_2 \), \( G_1 \oplus G_2 \), is the graph with vertex set the Cartesian product \( V(G_1) \times V(G_2) \) with \((x, a)\) adjacent to \((y, b)\) if and only if \( x \) is equal or adjacent to \( y \) in \( G_1 \), and \( a \) is equal or adjacent to \( b \) in \( G_2 \). We summarize their conclusions which were based on Schwenk’s work on the characteristic polynomial in [160]. Let \( \lambda_1, \lambda_2, \ldots, \lambda_n \) be the eigenvalues of \( G_1 \) and \( \mu_1, \mu_2, \ldots, \mu_p \) be the eigenvalues of \( G_2 \); they found that \( G_1 \times G_2 \) has eigenvalues \( \lambda_i \mu_j \), \( G_1 + G_2 \) has eigenvalues \( \lambda_i + \mu_j \), and \( G_1 \oplus G_2 \) has eigenvalues \( \lambda_i \mu_j + \lambda_i + \mu_j \) for \( i = 1, \ldots, m \) and \( j = 1, \ldots, p \).

The complete product of vertex disjoint graphs \( G_1 \) and \( G_2 \), \( G_1 \vee G_2 \), is the graph obtained by joining each vertex of \( G_1 \) to all vertices of \( G_2 \). The complete product of an \( r_1 \)-regular graph with \( n_1 \) vertices and an \( r_2 \)-regular graph with \( n_2 \) vertices is integral if and only if both graphs are integral and \((r_1 - r_2)^2 + 4n_1n_2 \) is a perfect square [98].

The complement of a graph \( G \), \( \overline{G} \), is the graph with \( V(\overline{G}) = V(G) \) and \( x \) adjacent to \( y \) if and only if \( x \) is not adjacent to \( y \) in \( G \). The line graph of a graph \( G \), \( L(G) \), is the graph with \( V(L(G)) = E(G) \) and \( x \) adjacent to \( y \) if and only if edges \( x \) and \( y \) in \( G \) have a common endpoint. Harary and Schwenk [98] show that given a regular integral graph \( G \), the complement \( \overline{G} \) and the line graph \( L(G) \) are both integral.

The subdivision graph of a graph \( G \), \( S(G) \), is the graph obtained by inserting a single new vertex into each edge of \( G \). In [55], it was shown that \( L_2(G_1) = L(S(G_1)) \) is integral if and only if \( G_1 \) is the union of complete graphs all having the same number of vertices.

A more general operation, the non-complete extended \( p \)-sum of graphs (NEPS), first defined in [53], preserves integrality. Let \( B \) be a set of binary \( m \)-tuples which does not contain the \( m \)-tuple \((0, \ldots, 0)\). The NEPS of graphs \( G_1, \ldots, G_m \) with basis \( B \), \( G = \text{NEPS}(G_1, \ldots, G_m; B) \), is the graph with vertex set the Cartesian product \( V(G_1) \times \cdots \times V(G_m) \), with \((u_1, \ldots, u_m)\) adjacent to \((v_1, \ldots, v_m)\) if and only if there exists \((\beta_1, \ldots, \beta_m) \in B \) such that \( u_i \) is adjacent to \( v_i \) in \( G_i \) whenever \( \beta_i = 1 \) and \( u_i = v_i \) whenever \( \beta_i = 0 \). The graphs \( G_1, \ldots, G_m \) are the factors of \( G \). Thus, \( G_1 \times G_2 \) is equivalent to \( \text{NEPS}(G_1, G_2; \{(1, 1)\}) \), \( G_1 + G_2 \) to \( \text{NEPS}(G_1, G_2; \{(0, 1), (1, 0)\}) \), and \( G_1 \oplus G_2 \) to \( \text{NEPS}(G_1, G_2; \{(0, 1), (1, 0), (1, 1)\}) \). Let graph \( G_i \) have order \( n_i \) and eigenvalues \( \lambda_{i_1}, \ldots, \lambda_{i_{n_i}} \) for \( i = 1, \ldots, m \), the graph \( \text{NEPS}(G_1, \ldots, G_m; B) \), where \( B \) is the basis of NEPS of graphs \( G_1, \ldots, G_m \), has eigenvalues

\[
\sum_{\beta \in B} \lambda_{1_{j_1}}^{\beta_{1_{j_1}}} \cdots \lambda_{m_{j_m}}^{\beta_{m_{j_m}}}
\]

for each \( j_k \in \{1, \ldots, n_k\} \) for \( k = 1, \ldots, m \) where \( 0^0 = 1 \). If the graphs \( G_1, \ldots, G_m \) are all integral then \( \text{NEPS}(G_1, \ldots, G_m; B) \), where \( B \) is the basis of NEPS of graphs \( G_1, \ldots, G_m \), is integral. If the graphs \( G_1, \ldots, G_m \) are all regular and integral then \( \text{NEPS}(G_1, \ldots, G_m; B) \), where \( B \) is the basis of NEPS of graphs \( G_1, \ldots, G_m \), is regular and integral [172].

Many infinite families of integral complete \( s \)-partite graphs for \( s \geq 3 \) have been constructed [102, 154, 185, 186, 191].

Let \( pG \) denote the disjoint union of \( p \) copies of \( G \). Lepovic presents results on integral graphs which belong to several graph classes: \( pK_{a,b} \) [123], \( pK_{a} \cup \beta K_{b} \) [124], \( pK_{a} \cup \beta pK_{b} \) [125], \( pK_{a} \cup \beta K_{a} \) [126], \( pK_{a,a} \cup \beta K_{b,b} \) [127], \( pK_{a,a,b,b,...,b} \) [128].
The corona of $G_1$ and $G_2$, $G_1 \circ G_2$, is the graph obtained by taking one copy of $G_1$ and $|V(G_1)|$ copies of $G_2$ and then joining the $i$th vertex of $G_1$ to every vertex in the $i$th copy of $G_2$. The following results can be found in [188]: $K_p \circ K_{p^2}$ is integral when $p = m(m + 1)$ for a positive integer $m$, $K_{a,b} \circ K_{p^2}$ is integral when there exist positive integers $a, b, c$ such that $\sqrt{a}b$ is a positive integer and $p^2 = c(c + \sqrt{a}b)$, and $K_{a,a,\ldots,a} \circ K_{p^2}$ is integral where $K_{a,a,\ldots,a}$ is a complete $s$-partite graph if there exist positive integers $b, c$ such that $p^2 = b(b + as - a) = c(c + a)$.

The $(n + 1)$-regular graph $K_{n,n+1} \equiv K_{n+1,n}$ on $4n + 2$ vertices is obtained from two disjoint copies of $K_{n,n+1}$ with vertex classes $V_1 = \{x_i \mid i = 1, 2, \ldots, n\}$, $V_2 = \{y_i \mid i = 1, 2, \ldots, n + 1\}$ and $U_1 = \{a_i \mid i = 1, 2, \ldots, n\}$, $U_2 = \{b_i \mid i = 1, 2, \ldots, n + 1\}$, respectively, by adding the edges $\{y_i b_i \mid i = 1, 2, \ldots, n + 1\}$. Let $K_{1,p}$ be a graph with vertex classes $V_1 = \{x_1\}$ and $V_2 = \{y_i \mid i = 1, 2, \ldots, p\}$. Let the $i$-th graph $K_p$ of $(p - 1)K_p$ have the vertex set $\{w_{ij} \mid j = 1, 2, \ldots, p\}$, where $i = 1, 2, \ldots, p - 1$. Then the $p$-regular graph $K_{1,p}[(p - 1)K_p]$ on $p^2 + 1$ vertices is obtained by adding the edges $\{w_{ij} w_{ji} \mid j = 1, 2, \ldots, p - 1\}$ for $i = 1, 2, \ldots, p$ between the graph $K_{1,p}$ and the graph $(p - 1)K_p$. Wang et al. [193] determine that the class of graphs $K_{p,p+1} \equiv K_{p+1,p}$ is integral and that the class of graphs $K_{1,p}[(p - 1)K_p]$ is integral when $p = m^2 + m + 1$ for some positive integer $m$.

Let $k \ast_G H$ denote the graph obtained by taking $k$ copies of a rooted graph $H$ and joining the root of each copy to all vertices of a graph $G$. Let $k \bullet H$ denote the graph obtained by identifying the roots of $k$ copies of a rooted graph $H$. We list a few of the constructions of Indulal and Vijayakumar [108]. Given an $r$-regular graph $G$, the graph $k \ast_G K_{r+2}$ is integral if and only if $(r + 2)^2 + 4pk$ is a perfect square. When $k = \frac{n^2 - (r+2)^2}{4p}$ and $m > r + 2$, this gives an infinite family of integral graphs. The graph $k \bullet K_{m,n}$, with any vertex in the class of $n$ vertices taken as the root, is integral if and only if both $m(n-1)$ and $m(n-1) + mk$ are perfect squares.


Integral trees are another family that has been investigated extensively [31, 32, 36, 41, 42, 98, 99, 100, 101, 129, 130, 133, 183, 184, 187, 188, 189, 190, 192, 194, 196, 197, 200]. Still, much is unknown. These results have only covered trees with diameter less than 11. More recently, Csikvári [48] proved the existence of integral trees with arbitrarily large even diameter. Ghorbani et al. [82] showed the existence of infinitely many integral trees of arbitrarily large odd diameter.

In [61], Cvetković et al. produce the only 7 molecular graphs with integral spectra. In so doing, they find the only connected integral graphs which are not 3-regular with maximum vertex degree three: $K_1$, $K_2$, $K_3$, $C_4$, $C_6$, $K_2 \circ 2K_1$, $S(K_{1,3})$.

Radosavljević and Šimić [153, 166] show that there are 13 connected non-bipartite integral graphs that are not regular with maximum vertex degree four. In [45], it was shown that up to isomorphism there are 22 non-bipartite integral graphs with spectral radius three. Partial results on non-regular bipartite integral graphs with maximum vertex degree four appear in [14, 15, 16].

Ahmadi et al. [4] consider the possibilities for adjacency matrices of integral graphs. They are able to get a loose upper bound on the number of labeled integral graphs with $n$ vertices as less than or equal to $2^{n(n-1)/2}n^{n/3}$ for sufficiently large $n$. We mention that the number of labeled graphs on $n$ vertices is $2^{n(n-1)/2}n!$.

Table 2.1 gives the total number of connected integral graphs of order $n$ for $1 \leq n \leq 12$ as summarized by Balijńska et al. [12]. We note that this table contains the correct value for $n = 11$ from the original source [10] (see also [51]).
The aid of a computer was essential to generating the graphs for \( n = 8, \ldots, 12 \). The methods and details are documented in the following papers: [8, 9, 10, 11].

![Table 2.1: Connected integral graphs with \( n \) vertices](image)

Table 2.2 gives the number of all connected regular integral graphs up to 12 vertices. These graphs were presented with pictures in [10, 11]. The connected regular integral graphs up to \( n = 10 \) were also documented by Abdollahi and Vatandoost [2]. In error, they missed a graph on 7 vertices by failing to take into account the complement of the disconnected integral graph \( C_4 \cup C_3 \).

![Table 2.2: Connected regular integral graphs with \( n \) vertices](image)

The finiteness of the number of integral graphs was considered in [54]. One of the results there is that the set of connected integral graphs with a given maximum vertex degree is finite.

We reproduce the results of Cvetković et al. [63] in Theorem 7 and Cvetković [53] in Theorem 8 and then deduce that there are only finitely many connected \( r \)-regular integral graphs.

The *eccentricity* of a vertex \( v \) in a graph \( G \) is the maximum distance from \( v \) to any vertex in \( G \). The *radius* of a graph \( G \) is the minimum eccentricity of any vertex in \( G \). The maximum distance over all pairs of vertices in a graph \( G \) is called the *diameter* of \( G \).

**Theorem 7.** Let \( r \geq 3 \). Given a connected \( r \)-regular bipartite graph, \( G \), with \( n \) vertices and radius \( R \),

\[
n \leq \frac{2(r-1)^R - 2}{r - 2}.
\]

**Proof.** Choose a vertex, \( u \), with eccentricity \( R \). Let \( N_k(u) = \{ v \in V \mid d(u, v) = k \} \) and \( d_k(u) = |N_k(u)| \). So \( d_0(u) = 1 \), \( d_1(u) = r \), and \( d_k(u) \leq (r-1)d_{k-1}(u) \) for \( k = 2, \ldots, R - 1 \) since each \( v \in N_{k-1}(u) \) has at most \( r-1 \) neighbours in \( N_k(u) \). By induction on \( k \), it follows that \( d_k(u) \leq r(r-1)^{k-1} \) for \( k = 2, \ldots, R - 1 \).

Next we consider \( d_R(u) \): Since \( G \) is bipartite, each vertex in \( N_R(u) \) has all of its \( r \) neighbours in \( N_{R-1}(u) \). For this reason, the previous argument overcounts each such vertex \( r \) times and we get
2.3. INTEGRAL GRAPHS

\[ d_R(u) \leq \frac{(r-1)d_{R-1}(u)}{r} \leq \frac{(r-1)r(r-1)^{R-2}}{r} = (r-1)^{R-1}. \]

Now

\[ n = \sum_{k=0}^{R} d_k(u) \leq 1 + \sum_{k=1}^{R-1} r(r-1)^{k-1} + (r-1)^{R-1} \]

\[ = 1 + \frac{r(1-(r-1)^{R-1})}{1-(r-1)} + (r-1)^{R-1} \]

\[ = \frac{(r-2) + r((r-1)^{R-1} - 1) + (r-2)(r-1)^{R-1}}{r-2} \]

\[ = \frac{2(r-1)^R - 2}{r-2}. \]

\[ \square \]

Note: This bound does not work for \( r = 2 \) since we find the sum of the geometric series which does not evaluate at this point.

**Theorem 8.** For a connected graph \( G \) with diameter \( D \), we have \( D \leq s - 1 \), where \( s \) is the number of distinct eigenvalues of \( G \).

**Proof.** (By contradiction) Suppose \( D \geq s \). Let \( a_{xy}^{(k)} \) denote the \( (x,y) \) entry of matrix \( A^k \). Then for some vertices \( u, v \in G \), the smallest \( k \) such that \( a_{uv}^{(k)} \neq 0 \) is \( k = D \). Now, consider the minimal polynomial of \( A \): \( \phi(\lambda) = (\lambda - \lambda_1) \cdots (\lambda - \lambda_s) = \lambda^s + b_1 \lambda^{s-1} + \cdots + b_s \lambda^0 \). Now \( A \) is a root of this polynomial, so \( \phi(A) = 0 \) and thus \( A^k \phi(A) = A^{k+s} + b_1 A^{k+s-1} + \cdots + b_s A^k = 0 \) for all \( k \geq 0 \). Letting \( k = D - s \), we get \( A^D + b_1 A^{D-1} + \cdots + b_s A^{D-s} = 0 \). This implies that \( a_{uv}^{(D)} = 0 \) which is a contradiction. \( \square \)

The **bipartite double cover** of \( G \) is the bipartite graph \( G \times K_2 \). Similar to the result by Schwenk [161] used in [173] and [176], we have

**Lemma 9.** If \( G \) is an \( r \)-regular integral graph, then the bipartite double cover of \( G \) is a bipartite \( r \)-regular integral graph.

If \( G \) is bipartite then the bipartite double cover consists of two disjoint copies of \( G \). We note that from this definition, with the observation that \( Sp(K_2) = \{-1,1\} \), it can be easily seen that the spectrum of a bipartite graph is symmetric about the eigenvalue 0.

An \( r \)-regular integral graph, has at most \( 2r+1 \) distinct eigenvalues. Thus \( R \leq D \leq 2r+1-1 = 2r \) by Theorem 8. Given Lemma 9, we have that the maximum order of an \( r \)-regular non-bipartite integral graph is bounded above by the maximum order of the \( r \)-regular bipartite integral graphs. It follows from Theorem 7 that for a connected \( r \)-regular integral graph, \( n \leq \frac{2(r-1)^{2r-1}}{r-2} \).

Thus, connected 3-regular integral graphs have at most 126 vertices and connected 4-regular integral graphs have at most 6560 vertices.

All the cubic integral graphs have been determined. They were found independently by two sets of authors [37, 54, 161] using different methods. In [54] and [37], the authors considered all possible sets.
of distinct eigenvalues, a polynomial by Hoffman [104], and the spectral moments $w_1$, $w_2$, and $w_3$; and then used theoretical reasoning and a computer search to find the graphs that exist. Schwenk [161] proved Lemma 9 for 3-regular $G = H \times K_2$ and determined the graphs $G$ by theoretical reasoning before deriving the graphs $H$. The girth of a graph is the length of the shortest cycle contained in the graph. There are exactly 13 connected cubic integral graphs: $K_4$, $K_3^3$, $C_3 + K_2$, $C_4 + K_2$, the Petersen graph, the graph obtained from $K_3^3$ by specifying a pair of non-adjacent vertices and replacing each of them by a triangle, the graph obtained by taking two copies of $K_2^3$ and adding the edges of a matching of size 3 between the pairs of size 3 in each copy, $L_2(K_4)$, $C_4 + K_2$, the Desargues’ graph and its co-spectral mate, a bipartite graph on 24 vertices with girth 6, and the Tutte 8-cage. These graphs have largest order 30 which is significantly less than 126.

The quartic integral graphs have not yet been fully determined.

### 2.4 Quartic integral graphs

We use the acronym QIG as shorthand for connected quartic integral graph. Since all non-bipartite QIGs have a bipartite double cover that is a 4-regular integral graph, finding all QIGs involves first considering only bipartite graphs.

For connected 4-regular bipartite integral graphs, $Sp(G) = \{4, 3^x, 2^y, 1^z, 0^2w, -1^x, -2^y, -3^z, -4\}$; which we abbreviate by simply specifying the quadruple $[x, y, z, w]$. Cvetković et al. [63] found quadruples $[x, y, z, w]$ that are candidates for the spectrum of a bipartite QIG. They called these possible spectra. For 4-regular bipartite $G$, they first produced equations for $w_\ell$ up to $\ell = 6$:

\[
\begin{align*}
\quad w_0 &= n, & w_4 &= 28n + 8[C_4], \\
\quad w_2 &= 4n, & w_6 &= 232n + 144[C_4] + 12[C_6].
\end{align*}
\]

Cvetković et al. then added the following result of Hoffman [104]. Let $J$ denote the square matrix of order $n$ with every entry 1 and $I$ denote the identity matrix of order $n$. Given a connected $r$-regular graph $G$ of order $n$ with adjacency matrix $A$ and distinct eigenvalues $\mu_1 = r, \mu_2, \ldots, \mu_j$,

\[
J \prod_{i=2}^{j} (r - \mu_i) = n \prod_{i=2}^{j} (A - \mu_i I).
\]

Cvetković et al. divide the possible spectra into various cases depending on whether values $x, y, z, w$ are zero or non-zero and find all solutions to equations (2.3) that satisfy the Hoffman identity for $n \leq 6560$. Using these methods, they lower the upper bound on the number of vertices of a QIG to 5040.

The last section of [63] details the constructions used to derive several QIGs. For each of the cubic integral graphs, $G_i$ for $i = 1, \ldots, 13$, the graphs $L(G_i)$, $L(G_i) \times K_2$, $G_i + K_2$, and $G_i \oplus K_2$ are QIGs.

Let $q_v$ denote the number of 4-cycles to which the vertex $v$ belongs. Stevanović [171] considers $Q_v = \sum_{i=1} v_i q_v$, where $v_1, \ldots, v_4$ are the neighbours of $v$, for all vertices $v$ of a graph $G$. He uses the set of 4 equations for each $v$ arising from graph angles [49, 50, 56] - a generalization of (2.3) - to further eliminate spectra from the list of possible spectra. Stevanović is able to lower the upper
bound on the number of vertices of a QIG to 1260 with the exception of 5 spectra (one of which has 5040 vertices).

Let $H$ with eigenvalues $\mu_1, \mu_2, \ldots, \mu_m$ be a (vertex) induced subgraph of a graph $G$ with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Cauchy’s interlacing theorem (see [62], p.19) states that $\lambda_i \geq \mu_i \geq \lambda_{n-m-i}$ for $i = 1, \ldots, m$. Stevanović [173] considers a vertex $v$ of a desired unknown graph $G$. He then uses $q_v$, an equation determined by counting the walks of length 2 starting at $v$ in two ways, and a derivation of the Hoffman identity for 5 (and then 6) distinct eigenvalues to find limited possibilities for the number of common neighbors between $v$ and other vertices of $G$. From this information a large part of a hypothetical QIG is constructed and tested against Cauchy’s interlacing theorem for certain subgraphs. Stevanović determines that there are 16 bipartite QIGs avoiding $\pm 3$ in their spectrum. It follows that there are 8 non-bipartite QIGs avoiding $\pm 3$ in their spectrum. These 24 graphs are depicted in [173]. The complete proof is given in [172].

As a result of the efforts published in [63, 171, 173], there are 1888 possible spectra printed in [63] as well as a list of 65 known QIGs. These graphs are given a label $I_{n, index}$ and are accompanied by the spectrum and sometimes a short description.

Stevanović et al. [176] extend the equations for the $\ell$th spectral moment by adding an inequality for $\ell = 8$:

$$w_8 \geq 2092n + 2024[C_4] + 288[C_6].$$

(2.4)

They make use of a correspondence between closed walks in an $r$-regular graph and walks in an infinite $r$-regular tree and find recurrence relations for the number of closed walks. Using this new inequality for $w_8$, Stevanović et al. are able to reduce the list of possible spectra to a list with 828 entries. The upper bound for $n$ is improved to give $8 \leq n \leq 560$. All QIGs with $n \leq 24$ that realize one of the possible spectra are found and listed with drawings in [176]. Included in these graphs, are 14 non-bipartite QIGs but these were all previously known by the results in [9, 10, 11].

Stevanović [172] showed that NEPS($G_1, \ldots, G_m; B$), where $G_1, \ldots, G_m$ are each themselves representable as a NEPS of graphs is isomorphic to NEPS of their factors; $G_1, \ldots, G_m$; for some basis. This result enables Stevanović to simply consider NEPS($G_1, \ldots, G_m; B$) where $G_1, \ldots, G_m$ are connected regular integral graphs which are not representable as a NEPS of graphs to determine the QIGs that are a NEPS of graphs. A NEPS of graphs that is isomorphic to a QIG satisfies the following conditions:

- If the degree of the largest factor of a NEPS is at most 2 then all factors of the NEPS are $C_3$ or $K_2$. There are 14 non-isomorphic QIGs that are a NEPS of this type.

- If the degree, $\Delta$, of the largest factor is strictly greater than 2 then one of the factors is a $\Delta$-regular integral graph and the remaining factors are isomorphic to $K_2$. Graphs of this type are of the form $H \times K_2$ where $H$ is a non-bipartite QIG, $H + K_2$ where $H$ is a 3-regular integral graph, or $H \oplus K_2$ where $H$ is a non-bipartite 3-regular graph.

Balińska et al. [13] present exact and randomized algorithms for generating QIGs. Using two of their algorithms, they are able to produce all QIGs for $12 < n < 20$ by a computer search. These graphs are given as the spectrum with the number of co-spectral graphs for that spectrum. A third algorithm is suggested for enumerating the graphs with at least 20 vertices but not attempted.
Stevanović [175] considers finding r-regular bipartite integral graphs with 5 distinct eigenvalues: the graphs with spectrum of the form \( \{r, x^y, 0^{2z}, -x^y, -r\} \). Of course, this covers more than the case: \( r = 4 \) but we mention it here. The following result is proved by Stevanović.

**Lemma 10.** The number of closed walks of length 6 in an r-regular bipartite graph \( G \) with \( n \) vertices is equal to \( nr(5r^2 - 6r + 2) + 48(r - 1)[C_4] + 12[C_6] \).

The equations \( w_0, w_2, \) and \( w_4 \) for r-regular graphs (similar to Equations (2.3) but \( w_4 \) requires a counting argument) and \( w_6 \) given by Lemma 10 are used to find an extensive list of candidate spectra for graphs with 5 distinct eigenvalues. Construction and non-existence results follow for some of these spectra which can be found in [175].

### 2.5 Cayley integral graphs

A Cayley graph \( \text{Cay}(\Gamma, S) \) for a group \( \Gamma \) and connection set \( S \subseteq \Gamma \) is the graph with vertex set \( \Gamma \) and with \( a \) connected to \( b \) if and only if \( ba^{-1} \in S \). Abdollahi and Vatandoost [2] determine which of the known regular integral graphs up to \( n = 11 \) are Cayley integral graphs. We summarize their results in Table 2.3.

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**Table 2.3:** Connected Cayley integral graphs with \( n \) vertices

Given groups \( \Gamma_1 \) and \( \Gamma_2 \), a group homomorphism is a function \( \alpha : \Gamma_1 \to \Gamma_2 \) such that \( \alpha(gh) = \alpha(g)\alpha(h) \). Let \( \mathbb{C}^* \) denote the group of non-zero complex numbers with multiplication. A character of a finite group \( \Gamma \) is a homomorphism \( \chi : \Gamma \to \mathbb{C}^* \). The eigenvalues of a Cayley graph \( \text{Cay}(\Gamma, S) \) are equal to \( \sum_{\chi \in S} \chi(s) \) for each character \( \chi \) of \( \Gamma \) [134]. Lovász also shows that if \( \Gamma \) is an Abelian group, then there are precisely \( n \) distinct characters of \( \Gamma \).

So [169] characterizes integral Cayley graphs over cyclic groups with the following result: for a cyclic group \( \Gamma \), \( G = \text{Cay}(\Gamma, S) \) is integral if and only if \( \sum_{k \in S} \omega^k \) is an integer where \( \omega \) is a primitive \( n \)-th root of unity of \( \Gamma \). A circulant graph on \( n \) vertices is a Cayley graph \( \text{Cay}(\mathbb{Z}_n, S) \) for some \( S \subseteq \mathbb{Z}_n \setminus \{0\} \). Known examples of circulant graphs include complete graphs and cycles. So was able to prove that there are precisely \( 2^{\tau(n)} - 1 \) non-isomorphic integral circulant graphs on \( n \) vertices, where there are \( \tau(n) \) divisors of \( n \).

4. We define the greatest common divisor of \( k \)-tuples of non-negative integers \( x = (x_1, \ldots, x_k) \) and \( m = (m_1, \ldots, m_k) \) to be \( \gcd(x, m) = (d_1, \ldots, d_k) = d \) where \( d_i = \gcd(x_i, m_i) \) for \( i = 1, \ldots, k \). Let \( \Gamma \) be an abelian group represented as a direct product of cyclic groups, \( \Gamma = \mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_k}, m_i \geq 1 \) for \( i = 1, \ldots, k \). For the divisor tuple \( d = (d_1, \ldots, d_k) \) of \( m = (m_1, \ldots, m_k) \) where \( d_i \) is a divisor of \( m_i \) for \( i = 1, \ldots, k \); we define \( S_r(D) = \{ x \in \Gamma \mid \gcd(x, m) = d \} \). We define the set \( S_r(D) = \bigcup_{i=1}^j S_r(d^{(i)}) \) where \( D = \{d^{(1)}, \ldots, d^{(j)}\} \) is a set of distinct divisor tuples of \( m \). A gcd-graph over a finite Abelian group \( \Gamma \), is a Cayley graph \( \text{Cay}(\Gamma, S_r(D)) \). It follows from a result of So [169] that the graphs \( \text{Cay}(\mathbb{Z}_n, S_{\mathbb{Z}_n}(\{1\})) \) (the unitary Cayley graphs) are integral (see also [116, 158]) and more generally that the graphs \( \text{Cay}(\mathbb{Z}_n, S_{\mathbb{Z}_n}(D)) \) are integral. Klotz and Sander [118] prove that every gcd-graph
Cay(Γ, S Publication Date) is integral. In \[119\], Klotz and Sander prove that if \(G\) is a NEPS of complete graphs \(NEPS(K_{m_1}, \ldots, K_{m_k}; B)\), then \(G\) is isomorphic to a gcd-graph over \(\Gamma = \mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_k}\).

An \(r\)-regular graph \(G\) with eigenvalues \(\lambda_1, \ldots, \lambda_n\) is a Ramanujan graph if for \(n > 3\), \(\lambda(G) = \max_{|\lambda| < r} |\lambda|\) is defined, and \(\lambda(G) \leq 2\sqrt{r-1}\). Droll [70] obtains a complete characterization of the cases in which the unitary Cayley graph \(\text{Cay}(\mathbb{Z}_n, S_{\mathbb{Z}_n}, \{(1)\})\) is a Ramanujan graph. Let \(p\) be a prime number and \(s\) be a positive integer such that \(p^s \geq 3\). Le and Sander [122] extended the result of Droll by classifying the graphs \(\text{Cay}(\mathbb{Z}_{p^s}, S_{\mathbb{Z}_{p^s}}, D))\) that are Ramanujan for each prime power \(p^s\) and arbitrary divisor tuple \(D\).

Let the Boolean algebra \(B(\Gamma)\) generated by the subgroups of \(\Gamma\) denote the lattice of subsets obtained by arbitrary finite intersections, unions, and complements of the subgroups. For an Abelian group \(\Gamma\), \(S\) belongs to the Boolean algebra \(B(\Gamma)\) generated by the subgroups of \(\Gamma\) if and only if Cay(\(\Gamma, S\)) is integral. This result was proven by various authors over time. In [117], Klotz and Sander prove that for an Abelian group \(\Gamma\), if \(S\) belongs to the Boolean algebra \(B(\Gamma)\) generated by the subgroups of \(\Gamma\), then Cay(\(\Gamma, S\)) is integral. If \(\Gamma\) is a cyclic group then So [169] shows that the converse is true. Finally it is proved in [5, 118] that the converse is true when \(\Gamma\) is an Abelian group.

The following are some examples of graphs that satisfy the previous result. An \(m\)-Sudoku puzzle, is an \(m \times m\) arrangement of \(m \times m\) blocks of cells such that each cell is either empty or contains a number from 1 to \(m^2\) and there is a unique way to fill in the empty cells so that every row, column, and block of the puzzle contains all of the numbers 1 to \(m^2\) exactly once. A Sudoku graph, Sud(\(m\)), is a graph with vertices corresponding to the \(m^4\) cells of the puzzle and \(x\) adjacent to \(y\) if and only if \(x\) and \(y\) correspond to cells from the same row, column, or block. Sander [157] shows that for \(m \in \mathbb{Z}^+\), Sud(\(m\)) is isomorphic to the NEPS of graphs \(NEPS(K_m, K_m, K_m, K_m; \{(0, 1, 0, 1), (1, 1, 0, 0), (0, 0, 1, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)\})\). Thus \(m\)-Sudoku puzzles are integral graphs. In [117], Klotz and Sander show that for an Abelian group \(\Gamma\) and \(S \in B(\Gamma)\) that Sud(\(n\)) is an integral Cayley graph Cay(\(\Gamma, S\)). Similarly they show that a variation of Sudoku, positional Sudoku, has graph SudP(\(m\)) that is an integral Cayley graph. In the same paper we see that the Hamming graph is another example of an integral Cayley graph on an Abelian group \(\Gamma\) with \(S \in B(\Gamma)\) (see also [158]).

A multiset is a set \(S\) with multiplicity function \(\mu_S: S \rightarrow \mathbb{N}\) where we say \(\mu_S(a) = 0\) for all \(a \notin S\). A Cayley multigraph MCay(\(\Gamma, S\)) is a Cayley graph with multiple edges permitted such that the number of edges joining \(a\) to \(b\) in MCay(\(\Gamma, S\)) is \(\mu_S(ba^{-1})\). Let the collection \(C(\Gamma)\), or integral cone over \(B(\Gamma)\), be the Boolean algebra of all multisets that can be expressed as non-negative integer combinations of the subgroups that generate \(B(\Gamma)\). DeVos et al. [69] prove that for every finite group \(\Gamma\) and every \(S\) in \(C(\Gamma)\), where \(B(\Gamma)\) is generated by the normal subgroups of \(\Gamma\), the Cayley multigraph MCay(\(\Gamma, S\)) is integral.

Klotz and Sander [117] show that if every Cayley graph Cay(\(\Gamma, S\)) over \(\Gamma\) is integral then \(\Gamma \in \{\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4^t, \mathbb{Z}_5^t \times \mathbb{Z}_3^t, \mathbb{Z}_2^t \times \mathbb{Z}_4^t\}, s \geq 1, t \geq 1\).

Abdollahi and Vatandoost [1] showed that there are exactly 7 connected cubic integral Cayley graphs. They determine that Cay(\(\Gamma, S\)) with \(|S| = 3\) is integral if and only if \(\Gamma\) is isomorphic to one of the following groups: \(\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_4, \mathbb{Z}_6, \mathbb{Z}_3^t, \mathbb{Z}_2^t \times \mathbb{Z}_4, D_6, \mathbb{Z}_2 \times \mathbb{Z}_6, D_{12}, A_4, S_4, D_8 \times \mathbb{Z}_3, D_6 \times \mathbb{Z}_4,\) or \(A_4 \times \mathbb{Z}_2\).
CHAPTER 2. LITERATURE REVIEW

The orders possible for Cayley QIGs on finite Abelian groups have been determined by Abdollahi and Vatandoost [2]. They showed that for an Abelian group, $\Gamma$, if $\text{Cay}(\Gamma, S)$ is a Cayley QIG then $|\Gamma| \in \{5, 6, 8, 9, 10, 12, 16, 18, 20, 24, 25, 32, 36, 40, 48, 50, 60, 64, 72, 80, 96, 100, 120, 144\}$, but they did not establish whether Cayley QIGs of these orders exist. We settle this in Section 4.4.

2.6 Strongly regular graphs

We denote a strongly regular graph with $n$ vertices of degree $r$, $e$ common neighbours between adjacent vertices, and $f$ common neighbours between non-adjacent vertices by $\text{SR}(n, r, e, f)$. Strongly regular graphs conceptually arose as an association scheme with two classes in connection with the theory of partially balanced designs [28]. They were introduced in this way by Bose [26]. In the same paper, he relates strongly regular graphs to partial geometries. A partial geometry $\text{PG}(K, R, T)$ is a partial linear space $(X, \mathcal{L})$ with constant line size $K$ such that each point is on $R$ lines and given a line $L \in \mathcal{L}$ and a point $x \notin L$ there are exactly $T$ lines through $x$ that meet $L$. The point graph of a partial geometry has the points as vertices and $x$ adjacent to $y$ if and only if $x \in L$ and $y \in L$ for some $L \in \mathcal{L}$.

The point graph of a $\text{PG}(K, R, T)$ is a $\text{SR}(K(1 + (K-1)(R-1)), R(K-1), (K-2) + (R-1)(T-1), RT)$ (and possibly the complement of a disconnected graph). For this reason, the necessary conditions for a strongly regular graph give necessary conditions for the existence of a partial geometry. A strongly regular graph is called pseudo-geometric if it has a parameter set such that it could be the point graph of a partial geometry and geometric if it is the point graph of a partial geometry. Bose [26] proved that if a pseudo-geometric strongly regular graph, corresponding to a $\text{PG}(K, R, T)$, satisfies

$$2K > R(R-1) + T(R+1)(R^2 - 2R + 2)$$

then the graph is geometric. Bose goes on to consider pseudo-geometric strongly regular graphs as: mutually orthogonal latin squares, Lattice designs, triangular graphs, balanced incomplete block designs (specifically singly linked block schemes), and elliptic non-degenerate quadrics.

We refer the reader to the survey papers on the topic: Brouwer and van Lint [35], Hubaut [106], Seidel [164].

The complement of an $\text{SR}(n, r, e, f)$ is an $\text{SR}(n, n-r-1, n-2r+e-2, n-2r+f)$. When $f = 0$, the strongly regular graphs consist of a disjoint union of complete subgraphs of equal size. From this point on we will only consider the strongly regular graphs for which $\text{SR}(n, r, e, f)$ and its complement are connected. For this reason, we will have $f > 0$.

Some examples of strongly regular graphs are the Moore graphs (see Section 2.7) and the triangular graphs. A triangular graph, $T(v)$, is the graph with vertices the unordered pairs from a set of $v \geq 5$ elements with $x$ adjacent to $y$ whenever they have an element in common.

There are some necessary conditions for the existence of a strongly regular graph with parameter set $(n, r, e, f)$ (see for example [39]):

- $0 < f < r < n - 1$;
- from the parameters of the complementary graph it follows that $n - 2r + f - 2 \geq 0$;
and most significantly, the spectrum is \( \text{Sp}(G) = \{ r^1, \lambda_1^{m_1}, \lambda_2^{m_2} \} \) where \( \lambda_1 \) is strictly greater than \( \lambda_2 \), \( m_1 = \frac{-r(\lambda_1+1)(r-\lambda_1)}{(r+\lambda_1\lambda_2)(\lambda_1-\lambda_2)} \), and \( m_2 = \frac{r(\lambda_2+1)(r-\lambda_2)}{(r+\lambda_1\lambda_2)(\lambda_1-\lambda_2)} \).

Clearly, \( m_1 \) and \( m_2 \) must be non-negative integers.

Based on these conditions and since \( f \neq 0 \), the eigenvalues \( r, \lambda_1, \) and \( \lambda_2 \) must all be distinct. It also follows that a regular connected graph and its complement are strongly regular with \( f \neq 0 \) if its adjacency matrix has three eigenvalues.

Where these necessary conditions are met, a parameter set \((n, r, e, f)\) is considered feasible. For a feasible parameter set, it is of interest to determine firstly whether it is realizable by a graph and secondly to find all graphs \( \text{SR}(n, r, e, f) \). A list of feasible parameter sets and whether or not they are realizable, including references, can be found in [30]. Currently, the smallest feasible parameter set for which the existence question remains unanswered is \((65, 32, 15, 16)\).

Let \( E_j \) for \( 0 \leq j \leq 2 \) be a basis of minimal idempotents of the vector space \( \langle I, A, \overline{A} \rangle \) with respect to matrix multiplication \( \circ \). We define the Krein parameters \( q_{ij}^{k} \) from the equation

\[
E_i \circ E_j = \sum_{k=0}^{2} q_{ij}^{k} E_k.
\]

A \( k \)-clique of a graph \( G \) is a set of \( k \) pairwise adjacent vertices of \( G \). A \( co \)-clique is a set of pairwise non-adjacent vertices.

We mention the results that have been used to determine that no graph with a given feasible parameter set is possible. Each of the following have lead to new non-existence results (see Table 2.4):

- **The Half-Case:** If \( m_1 = m_2 \) then \((n, r, e, f) = (4f + 1, 2f, f - 1, f)\). A \( \text{SR}(4f + 1, 2f, f - 1, f) \) exists if and only if a conference matrix \( C \) of order \( n + 1 \) exists. Such a \( C \) can only exist if \( n \) is the sum of two squares (see for example [22]).

- **The Krein Conditions:** The Krein parameters can be shown to be non-negative and by computing values for the parameters the four Krein conditions were found [67, 103, 121, 162]. The two non-trivial conditions, \( q_{11}^{1} \) and \( q_{22}^{2} \), can be written as Krein1: \( (\lambda_1 + 1)(r + \lambda_1 + 2\lambda_1\lambda_2) \leq (r + \lambda_1)(\lambda_2 + 1)^2 \), and Krein2: \( (\lambda_2 + 1)(r + \lambda_2 + 2\lambda_1\lambda_2) \leq (r + \lambda_2)(\lambda_1 + 1)^2 \).

- **The Absolute Bound:** There is a result that bounds the number of vectors which make only two distinct angles with each other (see for example [68, 120]). The result is used to form a spherical 2-distance set with vectors of the \( i \)th eigenspace of the adjacency matrix of a strongly regular graph together with a matrix \( E_i \) for \( i = 1, 2 \) (as in the basis matrices in the Krein parameters). The following bound is derived as a consequence (see for example [164]): \( n \leq \frac{1}{4} m_1(m_1 + 3) \) and similarly \( n \leq \frac{1}{4} m_2(m_2 + 3) \). This bound was later improved by Neumaier [145] to \( n \leq \frac{1}{2} m_1(m_1 + 1) \) if \( q_{11}^{1} \neq 0 \).

- **The Claw Bound:** Neumaier [144] took bound (2.5), for when a strongly regular graph is geometric, and furthered the ideas. In a form suggested by Brouwer and van Lint (see [164] for example), it reads: if \( f \neq \lambda_2^2, f \neq \lambda_2(\lambda_2 + 1) \) then \( 2(\lambda_1 + 1) \leq \lambda_2(\lambda_2 + 1)(f + 1) \).
• **The Case** $f = 1$: If $f = 1$ then the graph induced by the neighbours of a vertex is a union of complete graphs on $(e + 1)$ vertices and thus $(e + 1)$ divides $r$. Furthermore, $\frac{nr}{(e+1)(e+2)}$ is the total number of complete graphs on $(e + 2)$ vertices (remarked by [27]).

• **The Case** $f = 2$: If $f = 2$ then the neighbours of a vertex structurally form a partial linear space with girth at least five. Thus $(e + 1)$ divides $r$ if $r < \frac{1}{2}e(e + 3)$ [34].

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Table 2.4: Strongly regular graph non-existence results for $n \leq 100$
2.7. MOORE GRAPHS

- (49, 16, 3, 6): Bussemaker et al. [38] use a combination of counting techniques, enumeration, linear algebra, and computer computation to eliminate this feasible parameter set.

- (57, 14, 1, 4): Wilbrink and Brouwer [199] prove that a strongly regular graph with this parameter set cannot exist using counting arguments for subgraphs. More specifically they prove and use the following result: Let $C$ be a co-clique in an $SR(n, r, e, f)$ with smallest eigenvalue $\lambda_2$. Then, $|C| \leq \frac{n(-\lambda_2)}{r-\lambda_2}$ and equality holds if and only if each vertex not in $C$ is adjacent to exactly $\frac{r|C|}{n-|C|}$ points of $C$.

- (76, 21, 2, 7): The PG(3, 6, 1) with point graph an $SR(76, 21, 2, 7)$, was proven not to exist (see for example [147]). In [93], Haemers showed that the pseudo-geometric $SR(76, 21, 2, 7)$ would have to be geometric and therefore cannot exist.

More recently, the possible prime divisors of the order of automorphism groups of unknown graphs have been examined. Makhnev and Minakov [137] show that if a $SR(99, 14, 1, 2)$ exists then the order of the automorphism group is divisible by a prime $p = 2, 3, 7, 11$. Paduchikh [146] shows that if a $SR(85, 14, 3, 2)$ exists then the order of the automorphism group is divisible by a prime $p = 2, 3, 5, 7, 17$. Behbahani and Lam [21] are able to determine the possible prime divisors of the orders for all 16 unknown strongly regular graphs of Table 2.4. This includes an improvement for $SR(99, 14, 1, 2)$ to $p = 2$ or 3 and for $SR(85, 14, 3, 2)$ to $p = 2$.

Other more recent non-existence results for $n > 100$ include the following.

- (486, 165, 36, 66): In [136], the subgraphs induced by the neighbours of any given vertex are considered. Using this information, they are able to prove that a pseudo-geometric graph $SR(486, 165, 36, 66)$ for a PG(6, 33, 2) does not to exist.

- (324, 57, 0, 12): It is shown by exhaustive computer search that the known list of 2-(k(k - 1)/2 + 1, k, 2) symmetric designs for $k = 11$ is complete. This result implies that there exists no $SR(324, 57, 0, 12)$ [115].

### 2.7 Moore graphs

A Moore graph is a strongly regular graph with $e = 0$ and $f = 1$. Equivalently, it is a graph with diameter $D$ and girth $2D + 1$. Two trivial infinite families of Moore graphs are the odd cycles $C_{2k+1}$ and the complete graphs $K_n$.

The problem of describing such graphs was originally posed by E. F. Moore in 1960 (see [105] for example) as one of finding regular graphs with degree $\Delta$ and diameter $D$ for which $n = 1 + \Delta \sum_{i=1}^{D} (\Delta - 1)^{i-1}$. Thus, this value for $n$ is called the Moore bound. This is an upper bound on the number of vertices in the greater problem of finding large graphs with specified degree and diameter. This bound is easily derived. Designate one vertex as the root of the desired graph with degree $\Delta$ and diameter $D$. Then, sum up the maximum number of vertices possible at each distance from the root:

$$1 + \Delta + \Delta(\Delta - 1) + \cdots + \Delta(\Delta - 1)^{D-1}.$$  

Moore used number theory to show that some pairings of $\Delta$ and $D$ are not possible.
Hoffman and Singleton tackled this problem in [105]. They define an important recurrence from the Moore bound: for degree $\Delta \geq 0$, $F_{i+1}(x) = xF_i(x) - (\Delta - 1)F_{i-1}(x)$ where $F_1(x) = x + 1$ and $F_0(x) = 1$. They then prove that the adjacency matrix $A$ of a Moore graph satisfies the equation $F_D(A) = J$. From considering this equation when $D = 2$ or $3$, they are able to determine the possible eigenvalues and multiplicities of a Moore graph and thus the (order, degree, spectrum) triplets possible for each diameter. Their methods use various properties of the matrices and some results on the irreducibility of the polynomial $F_D(x)$ over the rationals. The techniques used for their uniqueness results involved determining certain permutation matrices corresponding to the relationships between vertices at distance $D$ and examining their properties. We summarize their contributions.

- If a Moore graph of diameter 2 exists then $\Delta = 2, 3, 7$, or 57.

- For these first three values of $\Delta$ unique graphs exist; the pentagon, the Petersen graph, and the Hoffman-Singleton graph.

- For diameter 3, the unique Moore graph is the heptagon.

Later, two different combinatorial proofs that the Hoffman-Singleton graph is unique were published by James [109] and by Fan and Schwenk [75].

It is still not known whether a graph $G$, with degree $\Delta = 57$ and diameter 2 exists although its parameter set is feasible. If it exists, $n = 3250$, $Sp(G) = \{57^1, 7^{1729}, -8^{1520}\}$, $\text{Aut}(G)$ is not a rank 3 group [7], and $G$ is not vertex-transitive (see for example [40]). More recent work on the missing Moore graph has been done in investigating the symmetries and the automorphism group of $G$. Makhnev and Paduchikh [138] show that $|\text{Aut}(G)| \leq 550$ if $\text{Aut}(G)$ has even order. The same authors further investigate $\text{Aut}(G)$ and give possibilities for its order and what must divide the order if it is not even [139]. Mačaj and Šiřáň [135] prove that if $|\text{Aut}(G)|$ is odd then $|\text{Aut}(G)|$ divides $19 \times 3^2, 13 \times 3, 5^2 \times 11, 7^2 \times 3, 7 \times 5, 5^3 \times 3$, or $3^3 \times 5$; and if $|\text{Aut}(G)|$ is even then $|\text{Aut}(G)|$ divides $11 \times 5 \times 2, 5^3 \times 2, 2^3 \times 2$, or $2p, p \in \{7, 11, 19\}$. Thus, $|\text{Aut}(G)| \leq 375$ if it is odd and $|\text{Aut}(G)| \leq 110$ if it is even.

For $\Delta \geq 3$ and $D \geq 3$, no Moore graphs exist. Independently, Damerell [65] and Bannai and Ito [17] were able to prove this result. Many partial non-existence results were also obtained [25, 79, 150].

A $(\Delta, g)$-cage is a regular graph of degree $\Delta$ and girth $g$ with minimum possible order. The Moore bound is a lower bound on the number of vertices for a $(\Delta, 2D + 1)$-cage (see for example [23]). Thus a $(\Delta, g)$-cage of order $n = 1 + \Delta \sum_{i=1}^{D} (\Delta - 1)^{i-1}$ is a Moore graph if $g = 2D + 1$. Using this connection between Moore graphs and cages, Singleton [167] showed that a graph with diameter $D$ and girth $2D + 1$ must be regular.

Given a fixed integer $s \geq 3$, conditions called uniqueness and exchange are defined in [66] that relate the universal cover $U$ of a graph to a family $A$ of connected 2-regular subgraphs of $U$. These conditions involve considering arcs of length $s$ contained in $A$. Delgado and Stellmacher are able to prove that if these uniqueness and exchange conditions are satisfied, an equivalence relation on the universal cover of a graph whose quotient graph is a bipartite graph of diameter $s - 1$ and girth $2(s - 1)$ can be defined. In [177], van Bon determines that if $U$ and $A$ satisfy uniqueness and shifted exchange conditions, then the relation $x \equiv y$ if and only if $d(x, y) = 2s - 1$ and there exists an element
of $A$ containing $x$ and $y$, extends to an equivalence relation on the universal cover of a graph such that the quotient graph is a Moore graph of diameter $s - 1$. The results of this paper are used to characterize the Moore graphs. Then, van Bon is able to reconstruct the Hoffman-Singleton graph based on these conditions.

Other research on Moore graphs which we will not cover includes topics such as generalized Moore graphs, radial Moore graphs, directed Moore graphs, and mixed Moore graphs. There is a survey paper on Moore graphs and the degree/diameter problem which covers Moore graphs and these more general topics in greater detail [143].
Chapter 3

Spectral Moments of Regular Graphs in Terms of Subgraph Counts

3.1 Spectral moments equations

Our aim is to provide equations that give the spectral moments of a regular graph as a linear combination of the number of copies of certain small subgraphs of the graph. By using generating functions to count a certain type of closed walk, we extend the list of equations to beyond what appears in the current literature. We give a general method for finding equations such as (2.3) for spectral moments of regular graphs, whether or not they are bipartite. The subgraphs that appear in our equations will be called \textit{contributors} (in (2.3) all of the contributors are cycles, but this is not true in general). Our methods build on those of Friedland et al. [78] and Wanless [195] mentioned in Section 2.2. Using a special class of closed walks, similar to the “primitive circuits” used by McKay [141], we will derive generating functions that give expressions for all closed walks in terms of the counts of contributors.

Our notation for contributors is as follows. We let $C_{i_1 \cdots i_q}$ be an $i_1$-cycle joined to an $i_2$-cycle by an edge. The graphs $C_{i_1 \cdots i_q}$ and $\Theta_{i_1, i_2, \ldots, i_q}$ were described in Section 2.2. Examples of this notation appear in Figure 3.1. If at any point we encounter contributors that cannot be described by our notation, we draw a picture of the subgraph like those in Figure 3.1.

![Figure 3.1: Contributor Notation](image)

Formally, we define a contributor to be a connected graph with minimum degree at least 2. As we will show in Theorem 15, a contributor affects the spectral moment, $w_i$, if and only if its edges can be covered by a closed walk of length $i$. Any given $w_i$ is only affected by finitely many contributors because a contributor with $j$ edges can only affect $w_i$ if $j \leq i$. We note that although our contributors are the same as in [195], they are used differently because we are counting slightly different types of walks.
3.2 Base walks and extensions

Let \( z_\ell = w_\ell - u_\ell \) where we reiterate the fact that \( w_\ell \) is the number of closed walks of length \( \ell \) in \( G \) and \( u_\ell \) is the number of totally-reducible walks of length \( \ell \) in \( G \).

Combining \( u_\ell \), as given by (2.2), with our methods to find \( z_\ell \), which we develop, we will achieve our goal of finding \( w_\ell \).

We view a closed but not totally-reducible walk as consisting of a “base walk”, which is the part that prevents it from being totally-reducible, together with some “extensions” which do not change its basic structure. For the sake of efficiency, walks that are extensions of the same base walk will be counted together. The primitive circuits in \([141]\) played an analogous role to our base walks.

For a closed walk \( W = v_0v_1 \cdots v_\ell \) we allow the following extensions:

1. A tail is an irreducible walk \( x_0x_1 \cdots x_\ell \) of length \( \ell \geq 0 \) with \( x_{\ell-1} \neq \{v_1, v_{\ell-1}\} \) and \( x_\ell = v_0 \). We say that \( W' = x_0x_1 \cdots x_{\ell-1}v_0 \cdots v_\ell x_{\ell-1} \cdots x_1x_0 \) is formed by adding a tail of length \( \ell \) to \( W \).

2. A diversion is a totally-reducible walk \( v_id \) of length \( \ell \geq 0 \) where if \( i < \ell \) then no prefix of \( v_id \) reduces to \( v_iv_{i+1} \). We say that \( W' = v_0v_1 \cdots v_idv_{i+1} \cdots v_\ell \) is formed by adding a diversion to \( W \).

We say that \( W \) extends to \( W' \) if \( W' \) is formed from \( W \) by adding a single tail and then any number of diversions. We stress that this includes the possibility that \( W' = W \), and also the option of adding diversions to the tail. Now that we know what extensions are allowed, we can define the base walks to be closed but not totally-reducible walks that are not an extension of any other walk. This means that \( W = v_0v_1 \cdots v_\ell \) is a base walk if and only if

1. \( W \) is non-trivial: \( \ell > 0 \),
2. \( W \) is closed: \( v_0 = v_\ell \),
3. \( W \) is irreducible: \( v_{i-1} \neq v_{i+1} \) for \( 0 < i < \ell \),
4. \( W \) has no non-trivial tail: \( v_1 \neq v_{\ell-1} \).

We mention that extensions are not the same in \([195]\). Diversions are the same but tails are different. Thus we can adopt Lemma 5 from \([195]\) as given in Section 2.2.

In Section 3.3 we will find generating functions that count all possible extensions of a walk. Our goal is to express all closed walks that are not totally-reducible in a unique way as the extension of some base walk. We now show the uniqueness of extensions.

**Theorem 11.** Given a closed and not totally-reducible walk \( W' \), there exists a unique base walk \( W \) such that \( W \) extends to \( W' \). Furthermore, there is a unique way to write

\[
W' = x_0d_0 \cdots x_{\ell-1}d_{\ell-1}v_0d_1v_1d_{\ell+1} \cdots v_\ell d_{\ell+\ell}x_{\ell-1}d_{\ell+\ell+1} \cdots x_0d_{2\ell+\ell}
\]

where \( W = v_0v_1 \cdots v_\ell \), each \( x_id_i \) and \( v_id_i \) is a diversion, and \( x_0x_1 \cdots x_{\ell-1}v_0 \) is a tail.

**Proof.** Given \( W' \), we can find red(\( W' \)) = \( w_0v_1 \cdots w_\ell \). Note that \( \ell \geq 3 \) since \( W' \) is closed and not totally-reducible. Find the minimum \( j \) such that \( w_j \neq w_{\ell-j} \) for \( 1 \leq j < \ell/2 \). A value for \( j \) satisfying
this condition exists since \( \text{red}(W') \) is irreducible. Let \( W = w_{j-1}w_j \cdots w_{\ell-j+1} \). Since \( j < \frac{\ell}{2} \), we know that \( W \) is non-trivial. Also, \( W \) is closed since \( w_{j-1} = w_{\ell-j+1} \). By the definition of \( \text{red}(W') \), \( W \) is irreducible. Finally, \( W \) has a trivial tail since \( w_j \neq w_{\ell-j} \). Therefore, \( W \) is a base walk as desired and \( \text{red}(W') \) is formed by adding a tail to \( W \).

Suppose \( V \) is a base walk such that adding a tail yields \( W'' \) and then adding diversions gives \( W' \). By definition of base walks and tails, \( V \) and \( W'' \) are irreducible. Also, \( W' \) reduces to \( W'' \) so \( W'' = \text{red}(W') \), by Lemma 2. It now follows that \( V = W \), since \( V \) cannot be longer than \( W \) by part 4 of the definition of base walks and \( V \) cannot be shorter than \( W \) given how we defined tails.

Thus, given \( W' \) there is a unique base walk \( W \) that extends to \( W' \), and a unique way to add the tail in that extension process. The uniqueness of (3.1) is shown by Lemma 5.

3.3 Generating walks around subgraphs

In this section we give a number of single variable generating functions for walks. In all cases the coefficient of \( x^i \) is the number of walks of length \( i \) in the class being counted. Now, we combine the functions of Lemma 3 and Lemma 4 given with proofs in Section 2.2. We note that the proof of the following is similar to the proof of Lemma 6 which can be found in [195].

**Lemma 12.** Suppose \( W = v_0v_1 \cdots v_\ell \) is a closed walk of length \( \ell \) in an \((r+1)\)-regular graph, \( G \). The generating function for walks in \( G \) that are extensions of \( W \) is

\[
\psi(\ell) = x^{\ell}T_r^\ell T_{r+1}^\ell \left( \frac{1 - x^2T_r^2}{1 - rx^2T_r^2} \right).
\]

**Proof.** By Lemma 4, the number of walks formed by adding diversions to a walk is generated by \( x^{\ell}T_r^\ell T_{r+1}^\ell \). Now if we add a tail, \( x_1 \cdots x_t v_0 \), of length \( t > 0 \); there are: \( r+1-2 \) choices for \( x_t \) since \( x_t \neq v_1 \) and \( x_t \neq v_{\ell-1} \), and \( r \) choices for \( x_i \) for \( 1 \leq i < t \); so \( (r-1)r^{t-1} \) choices of tail. Thus,

\[
\psi(\ell) = x^{\ell}T_r^\ell T_{r+1}^\ell + \sum_{t \geq 1} (r-1)r^{t-1}x^{\ell+2t}T_r^{\ell+2t}T_{r+1}^{\ell+2t}.
\]

\[
= x^{\ell}T_r^\ell T_{r+1}^\ell (1 + \frac{r-1}{r} \sum_{t \geq 1} r^t x^{2t}T_r^{2t})
\]

\[
= x^{\ell}T_r^\ell T_{r+1}^\ell \left( 1 + \frac{r-1}{r} \left( \frac{rx^2T_r^2}{1 - rx^2T_r^2} \right) \right)
\]

which simplifies to the required expression. 

Base walks are closed, so we can use \( \psi(\ell) \) to extend base walks of length \( \ell \). Note that \( \psi(\ell) \) only depends on the length of the walk we start with and the regularity of \( G \). Given the number \( b_i \) of base walks of length \( i \) in \( G \), the generating function for all closed walks that are extensions of base walks of length at most \( \ell \) is

\[
\sum_{i=1}^{\ell} b_i \psi(i).
\]
3.3. GENERATING WALKS AROUND SUBGRAPHS

Let $\mathcal{H}_\ell$ denote the set of all (isomorphism classes of) subgraphs induced by the base walks of length at most $\ell$. For our purposes the subgraph induced by a walk is the (simple) subgraph induced by the edges of the walk. To find $\mathcal{H}_\ell$ we need to obtain the base walks of length $i$, for $3 \leq i \leq \ell$. To do this, we generate all reduced walks of a given length, removing any such walks with a tail. For each base walk we find the subgraph it induces and keep a catalogue of such subgraphs up to isomorphism (using nauty [142]).

Algorithm 1

\begin{verbatim}
\eta(H) := 0
for i from #edges in H to \ell do
  b^H_i := 0
  for each vertex j of H do
    findBW(j, j, b^H_i, H, i)
  end for
  Add b^H_i \cdot \psi(i) to \eta(H)
end for

function findBW(W, v, b, S, len)
  if W has length \texttt{len} then
    if W is closed, has a trivial tail, and induces S then
      Increment b
    end if
  else
    for all neighbours u of vertex v in S do
      if Wu is irreducible then
        findBW(Wu, u, b, S, len)
      end if
    end for
  end if
end function
\end{verbatim}

From the set $\mathcal{H}_\ell$, we count the number of base walks of each length up to $\ell$ that occur. The function findBW(W, v, b, S, len) of Algorithm 1 recursively finds the number, $b = b^S_\text{len}$, of base walks of length \texttt{len} that induce the contributor $S$. The parameter $W$ is an irreducible walk with end vertex the parameter $v$. Each call to findBW extends the length of $W$ by one. The values of $b^H_i$ calculated in Algorithm 1 are related to the $b_i$ from (3.2) by $b_i = \sum_{H \in \mathcal{H}_\ell} b^H_i$.

For each contributor $H \in \mathcal{H}_\ell$ we use Algorithm 1 to find $\eta(H)$, a power series in $x$ for the contribution of contributor $H$. For example, at the completion of Algorithm 1 when $\ell = 9$ and $H = C_3$,

$$\eta(H) = 6x^3 + 30rx^5 + 6x^6 + 126r^2x^7 + 48rx^8 + (6 + 504r^3)x^9 + O(x^{10}).$$

It is important to note in general that $\eta(H)$ is only useful for the powers of $x$ up to $\ell$. The coefficients of terms beyond $x^\ell$ are missing the contribution of base walks of length greater than $\ell$.

We next show how to find $z_i$ from the calculated values of $\eta(H)$ for $H \in \mathcal{H}_\ell$. 


\textbf{Theorem 13.} The generating function for closed and not totally-reducible walks of length at most $\ell$ in $G$ is
\[ \sum_{i \leq \ell} z^i x^i = \sum_{H \in \mathcal{H}_\ell} \eta(H)[H] + O(x^{\ell+1}). \tag{3.3} \]

\textbf{Proof.} The base walks counted by $b_i^H$ in Algorithm 1 are closed and not totally-reducible by definition. Extending these walks cannot destroy either of these properties, so every walk counted by $\eta(H)[H]$ is closed and not totally-reducible. Let $W'$ be any closed and not totally-reducible walk of length at most $\ell$. By Theorem 11, $W'$ is obtained by extension of some base walk $W$. Let $H_W$ be the subgraph induced by $W$. Since the length of $W$ is no more than $\ell$, we know that $H_W \in \mathcal{H}_\ell$. The uniqueness clauses in Theorem 11 guarantee that $W'$ is counted exactly once by $\eta(H_W)[H_W]$ and is not counted by $\eta(H)[H]$ for any $H \neq H_W$. Hence $W'$ is counted exactly once on both sides of (3.3). The theorem follows.

In (3.3), we see that $H \in \mathcal{H}_\ell$ appears as a variable in $w_i$ if a base walk that induces $H$ extends to a walk of length $i$. We say in this case that $H$ affects $w_i$. For the remainder of this section, we characterize the circumstances under which $H$ affects $w_i$.

An \textit{Eulerian tour} of a multigraph $M$ is a closed walk containing every edge of $M$ exactly once. A multigraph with an Eulerian tour is said to be \textit{Eulerian}. We make use of the well known result that a multigraph is Eulerian if and only if every vertex has even degree.

A base walk that induces $H$ is an Eulerian tour in a multigraph with underlying graph $H$. We next prove a partial converse of this statement.

\textbf{Lemma 14.} Let $M$ be an Eulerian multigraph with underlying graph the contributor $H$. If $M$ has $j$ edges and at most two edges between any pair of vertices, then there exists a base walk of length $j$ that is an Eulerian tour of $M$.

\textbf{Proof.} The sequence of edges followed by any closed walk may be cyclically permuted to obtain another closed walk, in which case we say that the walk itself has been \textit{cyclically permuted}. By cyclically permuting any closed walk that has a non-trivial tail, the tail can be moved to a place where it can be reduced. Conversely, if a closed walk is reducible it can be cyclically permuted so that the point of reduction becomes a tail instead. In this proof we count a non-trivial tail or a point at which a walk may be reduced as a \textit{flaw}.

Consider an Eulerian tour $W$ of $M$. Suppose $W$ has a flaw. Then by cyclically permuting $W$ we can obtain a walk $W' = v_0v_1\cdots v_j$ in which $v_0 = v_2$. Since $M$ has at most two edges between any pair of vertices, $W'$ cannot walk directly between $v_0$ and $v_1$ after the first two edges. However, $v_i = v_1$ for some $i \in \{4, 5, \ldots, j-2\}$ because $H$ has minimum degree at least two. We reverse the subsequence $v_1v_2\cdots v_i$ of $W'$ to give $W'' = v_0v_i\cdots v_2v_1v_{i+1}\cdots v_j$. Since $W''$ includes the same edges as $W$, it is an Eulerian tour of $M$. Moreover, $W''$ has fewer flaws than $W$ because $v_{i-1} \neq v_0$ and $v_{i+1} \neq v_2$ (otherwise there are more than two edges between $v_0$ and $v_1$ in $M$).

By repeating the above process we must eventually reach a flawless Eulerian tour of $M$, which provides the desired base walk.

Lemma 14 does not generalize to multigraphs with more than two edges between some pair of vertices. Consider replacing one edge of any cycle $C_n$ by three parallel edges. Although the resulting multigraph is Eulerian, there is no base walk that is an Eulerian tour on its edges.
3.3. GENERATING WALKS AROUND SUBGRAPHS

Another general issue is that an Eulerian tour that induces a graph \( H \) may be reducible to a walk that induces a proper subgraph of \( H \). For an extreme example, consider the totally-reducible walk \( W = v_0v_1v_0v_1v_0v_2v_1v_2v_0 \), which is an Eulerian tour of a multigraph with underlying graph \( C_3 \). However, there exists an Eulerian tour \( W' = v_0v_1v_0v_1v_0v_2v_1v_2v_0 \) of the same multigraph. We see that \( W' \) is an extension of a base walk, even though \( W \) is not. We show in the next proof that this is always the case; if there is any Eulerian tour of a multigraph then there will be one which is an extension of a base walk.

**Theorem 15.** A contributor \( H \) affects \( w_i \) if and only if there exists a closed walk of length \( i \) that induces \( H \).

**Proof.** (\( \Rightarrow \)) Let \( W \) be a base walk that induces \( H \) and \( W' \) be an extension of \( W \). Note that \( W' \) is not necessarily restricted to the edges of \( H \) in \( G \). If \( W \) has length \( j \) and \( W' \) has length \( i \) then by the definition of extensions, \( i - j \) is even. We form a new walk \( W'' \) by following the edges of \( W \), then tracing back and forth across the final edge \((i - j)/2 \) times in each direction. The result is a closed walk of length \( i \) that induces \( H \), as desired.

(\( \Leftarrow \)) Any closed walk that induces \( H \) is an Eulerian tour of some multigraph \( M \) with underlying graph \( H \). We claim that there is an Eulerian tour of \( M \) that cannot be reduced to a walk that induces a graph with fewer edges than \( H \). Construct a graph \( M' \) from \( M \) by removing pairs of parallel edges until there is at least one and at most two edges between each pair of vertices. Note that \( M' \) still has underlying graph \( H \), and since the degree at each vertex is still even, \( M' \) is Eulerian. Thus by Lemma 14 there exists a base walk \( W \) that is an Eulerian tour of \( M' \). Now for each pair of parallel edges previously removed between a pair of vertices, say \( u \) and \( v \), we add a diversion \( uvu \) to \( W \). This extension of the base walk \( W \) is an Eulerian tour of \( M \). The result follows.

**Corollary 16.** Let \( H \) be a contributor with \( e \) edges. Let \( m \) be the minimum number of edges in an Eulerian multigraph \( M \) with underlying graph \( H \).

1. \( m \) is the minimum value for which \( H \) affects \( w_m \) and \( e \leq m < 2e \).

2. \( H \) affects \( w_{m+2s} \) for all integers \( s \geq 0 \).

3. If \( H \) is bipartite then \( H \) only affects \( w_i \) for even integers \( i \geq m \).

4. If \( H \) is not bipartite then \( H \) affects \( w_i \) for all \( i \geq 2e - 1 \).

**Proof.** When \( H \) is Eulerian, \( m = e \). We can always double each edge of \( H \) to obtain an Eulerian \( M \) but this is not the best possible. Since \( H \) has minimum degree at least 2, it contains a cycle. Constructing \( M \) by doubling all edges of \( H \) except those in the cycle, we prove (a). Since adding a pair of parallel edges preserves the Eulerian property, we have (b). Part (c) is true since bipartite graphs have no closed walks of odd length. If \( H \) is not bipartite then we can double all the edges or double all the edges except for one odd cycle, which implies (d).
3.4 Equations relating moments to subgraph counts

By combining (2.2) with (3.3) we obtain the following result for the spectral moments of an \((r+1)\)-regular graph with \(n\) vertices:

\[
\begin{align*}
w_0 &= n, \\
w_1 &= 0, \\
w_2 &= (1 + r)n, \\
w_3 &= 6[C_3], \\
w_4 &= (1 + 3r + 2r^2)n + 8[C_4], \\
w_5 &= 30r[C_3] + 10[C_5], \\
w_6 &= (1 + 5r + 9r^2 + 5r^3)n + 6[C_3] + 48r[C_4] + 12[C_6] + 24[C_{3,3}] + 12[\Theta_{2,2,1}], \\
w_7 &= 126r^2[C_3] + 70r[C_5] + 14[C_7] + 28[C_{4,3}] + 28[\Theta_{2,2,1}] + 14[\Theta_{3,2,1}] + 84[\Theta_{2,2,2,1}], \\
w_8 &= (1 + 7r + 20r^2 + 28r^3 + 14r^4)n + 48r[C_3] + (8 + 224r^2)[C_4] + 96r[C_6] \\
&\quad + 16[C_8] + 192r[C_{3,3}] + 32[C_{4,4}] + 32[C_{5,3}] + 32[C_{3,3}] + 144[K_4] \\
&\quad + 96r[\Theta_{2,2,1}] + 48[\Theta_{2,2,2}] + 16[\Theta_{3,2,1}] + 16[\Theta_{4,3,1}] + 16[\Theta_{4,2,1}] + 96[\Theta_{2,2,2,2}] \\
&\quad + 96[\Theta_{3,2,2,1}] + 64[\Theta_{4,3,2,1}], \\
w_9 &= (6 + 504r^3)[C_3] + 360r^2[C_5] + 126r[C_7] + 18[C_9] + 72[C_{3,3}] + 252r[C_{4,3}] \\
&\quad + 36[C_{5,4}] + 36[C_{6,3}] + 144[C_{3,3,3}] + 36[C_{4,3,3}] + 288[K_4] + (36 + 252r)[\Theta_{2,2,1}] \\
&\quad + (18 + 126r)[\Theta_{3,2,1}] + (36 + 756r)[\Theta_{2,2,2,1}] + 36[\Theta_{3,2,2}] + 18[\Theta_{4,2,1}] + 18[\Theta_{4,3,1}] \\
&\quad + 18[\Theta_{5,2,1}] + 108[\Theta_{3,3,2,1}] + 108[\Theta_{4,3,2,1}] + 108[\Theta_{4,2,2,1}] \\
&\quad + 90[\begin{array}{c} \includegraphics[width=0.5cm]{triangle}
\end{array}] + 108[\begin{array}{c} \includegraphics[width=0.5cm]{square}
\end{array}] + 36[\begin{array}{c} \includegraphics[width=0.5cm]{pentagon}
\end{array}] + 72[\begin{array}{c} \includegraphics[width=0.5cm]{hexagon}
\end{array}] + 72[\begin{array}{c} \includegraphics[width=0.5cm]{heptagon}
\end{array}] \\
&\quad + 108[\begin{array}{c} \includegraphics[width=0.5cm]{octagon}
\end{array}] + 180[\begin{array}{c} \includegraphics[width=0.5cm]{nonagon}
\end{array}] + 72[\begin{array}{c} \includegraphics[width=0.5cm]{decagon}
\end{array}] + 288[\begin{array}{c} \includegraphics[width=0.5cm]{undecagon}
\end{array}].
\end{align*}
\] (3.4)

Knowing any information about \(G\) that limits the possibilities for subgraphs can lead to equations with significantly fewer contributors. For example, a lower bound on the girth would eliminate any contributors containing short cycles. We present the equations for graphs with girth 5.

\[
\begin{align*}
w_0 &= n, \\
w_1 &= 0, \\
w_2 &= (1 + r)n, \\
w_3 &= 0, \\
w_4 &= (1 + 3r + 2r^2)n, \\
w_5 &= 10[C_5], \\
w_6 &= (1 + 5r + 9r^2 + 5r^3)n + 12[C_6], \\
w_7 &= 70r[C_5] + 14[C_7], \\
w_8 &= (1 + 7r + 20r^2 + 28r^3 + 14r^4)n + 96r[C_6] + 16[C_8], \\
w_9 &= 360r^2[C_5] + 126r[C_7] + 18[C_9],
\end{align*}
\]
3.4. EQUATIONS RELATING MOMENTS TO SUBGRAPH COUNTS

\[ w_{10} = (1 + 9r + 35r^2 + 75r^3 + 90r^4 + 42r^5)n + 10|C_5| + 540r^2|C_6| + 160r|C_8| + 20|C_{10}|
+ 40|C_{5,5}| + 20|\Theta_{3,3,2}| + 20|\Theta_{4,4,1}|, \]
\[ w_{11} = 1650r^3|C_5| + 770r^2|C_7| + 198r|C_9| + 22|C_{11}| + 44|C_{6,5}| + 44|\Theta_{3,3,2}| + 22|\Theta_{4,3,2}|
+ 22|\Theta_{5,4,1}| + 132|\Theta_{3,3,3,2}|, \]
\[ w_{12} = (1 + 11r + 54r^2 + 154r^3 + 275r^4 + 297r^5 + 132r^6)n + 120r|C_5| + (12 + 2640r^3)|C_6|
+ 1056r^2|C_8| + 240r|C_{10}| + 24|C_{12}| + 480r|C_{5,5}| + 48|C_{6,6}| + 48|C_{7,5}| + 48|C_{5,5-5}|
+ 24|\Theta_{6,4,1}| + 24|\Theta_{4,3,2}| + 24|\Theta_{4,4,2}| + 24|\Theta_{5,3,2}| + 24|\Theta_{5,5,1}| + 144|\Theta_{3,3,3,3}|
+ 144|\Theta_{4,3,3,2}| + 240r|\Theta_{3,3,2}| + 240r|\Theta_{4,4,1}| + 72|\Theta_{3,3,3,3}| + 72|\Theta_{4,4,1,1}| + 96|\Theta_{3,3,3,4}|, \]
\[ w_{13} = 7150r^4|C_5| + 4004r^3|C_7| + 1404r^2|C_9| + 286r|C_{11}| + 26|C_{13}| + 572r|C_{6,5}|
+ 52|C_{8,5}|
+ 52|C_{6,5-5}| + 572r|\Theta_{3,3,2}|
+ 26|\Theta_{4,3,2}| + 52|\Theta_{4,4,1}|
+ 52|\Theta_{4,3,3}|
+ 26|\Theta_{5,3,2}|
+ 286r|\Theta_{5,4,1}|
+ 1716r|\Theta_{3,3,3,2}|
+ 26|\Theta_{5,4,2}|
+ 26|\Theta_{6,3,2}|
+ 26|\Theta_{6,5,1}|
+ 26|\Theta_{7,4,1}|
+ 156|\Theta_{4,3,3,3}|
+ 156|\Theta_{4,4,4,2}|
+ 156|\Theta_{4,4,4,1}|
+ 156|\Theta_{5,3,3,2}|
+ 78|\Theta_{3,3,3,4}|
+ 78|\Theta_{3,3,3,4}|
+ 104|\Theta_{4,4,1,1}|
+ 104|\Theta_{4,4,1,1}|
+ 104|\Theta_{4,4,1,1}|
. \]

Another example is to consider bipartite graphs, where contributors that contain odd length cycles cannot occur. For bipartite \((r + 1)\)-regular graphs on \(n\) vertices we have \(w_i = 0\) for all odd \(i\), and

\[ w_0 = n, \]
\[ w_2 = (1 + r)n, \]
\[ w_4 = (1 + 3r + 2r^2)n + 8|C_4|, \]
\[ w_6 = (1 + 5r + 9r^2 + 5r^3)n + 48r|C_4| + 12|C_6|, \]
\[ w_8 = (1 + 7r + 20r^2 + 28r^3 + 14r^4)n + (8 + 224r^2)|C_4| + 96r|C_6| + 16|C_8| + 32|C_{4,4}|
+ 48|\Theta_{2,2,2}|
+ 16|\Theta_{3,3,1}|
+ 96|\Theta_{2,2,2,2}, \]
\[ w_{10} = (1 + 9r + 35r^2 + 75r^3 + 90r^4 + 42r^5)n + (80r + 960r^3)|C_4|
+ 540r^2|C_6|
+ 160r|C_8|
+ 20|C_{10}|
+ 320r|C_{4,4}|
+ 40|C_{6,4}|
+ 40|C_{4,4-4}|
+ 480r|\Theta_{2,2,2}|
+ (40 + 160r)|\Theta_{3,3,1}|
+ 960r|\Theta_{2,2,2,2}|
+ 40|\Theta_{4,4,2}|
+ 20|\Theta_{5,3,1}|
+ 120|\Theta_{3,3,3,1}|
+ 120|\Theta_{4,2,2,2}|
+ 120|\Theta_{3,3,3,1}|
+ 80|\Theta_{3,3,3,4}|
+ 80|\Theta_{3,3,3,4}|
+ 32|\Theta_{4,4,1,1}|
+ 32|\Theta_{4,4,1,1}|
+ 32|\Theta_{4,4,1,1}|
. \]
\[ w_{12} = (1 + 11r + 54r^2 + 154r^3 + 275r^4 + 297r^5 + 132r^6)n + (8 + 528r^2 + 3960r^4)|C_4|
+ (12 + 2640r^3)|C_6|
+ 1056r^2|C_8|
+ 240r|C_{10}|
+ 24|C_{12}|
+ (96 + 2112r^2)|C_{4,4}|
+ 480r|C_{6,4}|
+ 192|C_{4,4-4}|
+ 48|C_{6,6}|
+ 48|C_{8,4}|
+ 480r|C_{4,-4}|
+ 48|C_{6,-4}|
+ (240 + 3168r^2)|\Theta_{2,2,2}|
+ (48 + 480r + 1056r^2)|\Theta_{3,3,1}|
+ (1920 + 6336r^2)|\Theta_{2,2,2,2}|
+ (24 + 480r)|\Theta_{4,2,2}|
+ 72|\Theta_{3,3,3}|
+ (24 + 240r)|\Theta_{5,3,1}|
+ 4320|\Theta_{2,2,2,2}|
+ (48 + 1440r)|\Theta_{3,3,3,1}|
+ 1440r|\Theta_{4,2,2,2}|
+ 24|\Theta_{4,4,2,2}|
+ 48|\Theta_{6,2,2}|
+ 24|\Theta_{5,5,1}|
+ 24|\Theta_{7,3,1}|
+ 144|\Theta_{3,3,3,3}|
+ 144|\Theta_{4,4,4,2}|
+ 144|\Theta_{5,3,3,1}|
+ 144|\Theta_{6,2,2,2}|
+ 2880|\Theta_{2,2,2,2,2}|
+ (648 + 1440r)|\Theta_{3,3,3,4}|
+ (288 + 960r)|\Theta_{3,3,3,4}|
+ 312|\Theta_{3,3,3,4}|
+ 312|\Theta_{3,3,3,4}|
+ 48|\Theta_{3,3,3,4}|
+ 48|\Theta_{3,3,3,4}|
+ 72|\Theta_{3,3,3,4}|
+ 72|\Theta_{3,3,3,4}|
. \]
We note that more equations are easily determined but the increasing number of contributors makes them unsuitable to present in this forum.

From these equations, we were able to reproduce the reduced possible spectra list of [176] including the values of $[C_4]$ and $[C_6]$ for all entries. In addition, we were able to determine $[C_8]$ in many cases. One example of this can be found in Section 4.2.

### 3.5 Remarks

The equations we have presented provide much more information than (2.4). However, if preferred they can also be used to derive inequalities like (2.4) simply by dropping some of the terms. Our equations are well suited to this use, since all terms are positive, unlike the equations in [195] which employ the principle of inclusion-exclusion and hence have many terms of opposite signs.

Other uses of our equations have yet to be explored, but we are confident that they will prove useful in a variety of contexts.
Chapter 4

Quartic Integral Cayley Graphs

4.1 Vertex-transitive integral graphs

We give exhaustive lists of connected 4-regular integral Cayley graphs and connected 4-regular integral arc-transitive graphs. We first restrict our attention to the bipartite case because of Lemma 9.

We are able to add to the result of [2] mentioned in Section 2.5 and find that the precise set of orders of Cayley QIGs on Abelian groups is \{5, 6, 8, 9, 10, 12, 16, 18, 24, 36\}. More generally, we show that for any group \(\Gamma\), if Cay(\(\Gamma\), \(S\)) is a Cayley QIG then

\[|\Gamma| \in \{5, 6, 8, 9, 10, 12, 16, 18, 20, 24, 30, 32, 36, 40, 48, 60, 72, 120\}.\]

Furthermore, for each of these orders we give the Cayley QIGs that exist. We note that the orders: 25, 50, 64, and 100 could be almost immediately eliminated based on the possible spectra list orders of [176].

For a given Cayley graph \(G\), there may exist many different pairs (\(\Gamma\), \(S\)) of groups \(\Gamma\) and connection sets \(S\) such that \(G \cong\) Cay(\(\Gamma\), \(S\)). We call isomorphic Cayley graphs on the same group \(\Gamma\) equivalent if their connection sets are from the same orbit of the automorphism group of \(\Gamma\):

**Definition 17.** Let \(\Gamma\) be a group and \(\text{Aut}(\Gamma)\) be the automorphism group of \(\Gamma\). If Cay(\(\Gamma\), \(S\)) \(\cong\) Cay(\(\Gamma\), \(T\)) and \(S^\sigma = T\) for some \(\sigma \in \text{Aut}(\Gamma)\) then Cay(\(\Gamma\), \(S\)) and Cay(\(\Gamma\), \(T\)) are equivalent.

Any other connection sets give non-equivalent Cayley Graphs. Cayley graphs from different groups are non-equivalent. There are, up to isomorphism, only 32 connected quartic integral Cayley graphs; but each graph is realized in up to 18 non-equivalent ways. Of the 32 graphs, 17 are bipartite graphs realizing a possible spectrum.

A graph is arc-transitive if its automorphism group acts transitively upon ordered pairs of adjacent vertices. There are, up to isomorphism, only 27 connected quartic integral graphs that are arc-transitive. Of the 27 graphs, 16 are bipartite graphs realizing a possible spectrum, 5 of which are not Cayley graphs.

Section 4.2 details the methods used for refining the set of possible spectra from [176] by removing ones that cannot be realized by vertex-transitive QIGs. Section 4.3 summarises the algorithm used for finding all of the bipartite Cayley QIGs. Section 4.4 gives our results. It includes tables giving...
the details of the Cayley QIGs and the bipartite arc-transitive QIGs, some drawings, and some non-bipartite QIGs that result from finding quotients of our bipartite graphs.

4.2 Vertex-transitive cases

A graph is vertex-transitive if its automorphism group acts transitively upon its vertices. In this section, our aim is to compile a set \( \Xi \) that includes all possible spectra that might be realized by a vertex-transitive QIG, but is otherwise as small as we can make it. Initially we take \( \Xi \) to be all possible spectra from [176], and candidates will be progressively removed from the set as we work through this section.

We use the same notation for subgraphs as was described in Section 3.1 and depicted in Figure 3.1.

In [63], Equations (4.1) and (4.2) are used to determine \([C_4]\) and \([C_6]\) for a given \([x, y, z, w]\).

\[
2(4^4 + 3^4 x + 2^4 y + z) = 28n + 8[C_4], \tag{4.1}
\]
\[
2(4^6 + 3^6 x + 2^6 y + z) = 232n + 144[C_4] + 12[C_6]. \tag{4.2}
\]

In Chapter 3, these equations were extended to higher spectral moments of general regular graphs. By specialising to 4-regular bipartite graphs, we obtain the following equations:

\[
2(4^8 + 3^8 x + 2^8 y + z) = 2092n + 2024[C_4] + 288[C_6] + 16[C_8] + 32[C_{4,4}]
+ 96[\Theta_{2,2,2,2}] + 48[\Theta_{2,2,2}] + 16[\Theta_{3,3,1}],
\]
\[
2(4^{10} + 3^{10} x + 2^{10} y + z) = 19864n + 26160[C_4] + 4860[C_6] + 480[C_8] + 20[C_{10}]
+ 960[C_{4,4}] + 40[C_{4-4}] + 40[C_{6,4}] + 1440[\Theta_{2,2,2}]
+ 520[\Theta_{3,3,1}] + 2880[\Theta_{2,2,2,2}] + 40[\Theta_{4,2,2,2}] + 20[\Theta_{5,3,1}]
+ 120[\Theta_{3,3,3,1}] + 120[\Theta_{4,2,2,2}] + 120\begin{tikzpicture}
\end{tikzpicture} + 80\begin{tikzpicture}
\end{tikzpicture}.
\tag{4.3}
\]

We use Equations (4.3) to determine the girth where \([C_4] = [C_6] = 0\) for a given \([x, y, z, w]\) and also to determine the values for \([C_8]\) and \([C_{10}]\) where possible. Vertex-transitive graphs have the same number of \(i\)-cycles incident with each vertex, so the number of vertices divides \(i[C_i]\). We apply this observation for \(i \in \{4, 6, 8, 10\}\) to the possible spectra for which the value of \([C_i]\) can be deduced. We eliminate those quadruples that cannot be realized by a vertex-transitive QIG from \(\Xi\).

For example, if we consider \([5, 6, 11, 1]\) with \(n = 48\), \([C_4] = 24\), and \([C_6] = 140\) then

\[
\frac{4[C_4]}{n} = \frac{4(24)}{48} = \frac{2}{3} \in \mathbb{N} \quad \text{but} \quad \frac{6[C_6]}{n} = \frac{6(140)}{48} = \frac{35}{2} \notin \mathbb{N},
\]

where \(\mathbb{N}\) denotes the set of non-negative integers. Thus \([5, 6, 11, 1]\) is eliminated from \(\Xi\). In contrast, for \([12, 12, 20, 3]\) with \(n = 96\), \([C_4] = 24\), \([C_6] = 128\), and \([C_8] = 528\),

\[
\frac{4[C_4]}{n} = \frac{4(24)}{96} = \frac{1}{6} \in \mathbb{N}, \quad \frac{6[C_6]}{n} = \frac{6(128)}{96} = \frac{8}{3} \in \mathbb{N}, \quad \text{and} \quad \frac{8[C_8]}{n} = \frac{8(528)}{96} = \frac{44}{3} \notin \mathbb{N}.
\]

In this case, \([C_{10}] = 6240 - [\Theta_{5,3,1}] - 2[C_{6,4}]\) and so we consider it unknown. Thus \([12, 12, 20, 3]\) remains in \(\Xi\).
It is also plausible to eliminate quadruples from $\Xi$ using arguments specific to particular cases. We give one example to demonstrate the possibility. Consider $[24, 4, 40, 3]$ with $[C_4] = 72$ and $[C_6] = 0$. There are $4(72)/144 = 2$ copies of $C_4$ incident at each vertex. Since $[C_6] = 0$, we know $[\Theta_{3,3,1}] = 0$. Also, with only two 4-cycles at each vertex, $[\Theta_{2,2,2,2}] = 0$. Since two 4-cycles meet at exactly one vertex of a $C_4 \cdot 4$, $[C_4 \cdot 4] = 144$. From Equation (4.3) we get that,

$$2(4^8 + 3^8(24) + 2^8(4) + 40) = 2092(144) + 2024(72) + 16[C_8] + 32(144),$$

which gives the contradiction $[C_8] = -216$. Thus we remove $[24, 4, 40, 3]$ from $\Xi$. This entry is underlined in Table 4.1.

We eliminate two quadruples from $\Xi$ using the following Lemma [23, Prop. 16.6]:

**Lemma 18.** Let $G$ be a vertex-transitive graph which has degree $r$ and an even number of vertices. If $\lambda$ is a simple eigenvalue of $G$, then $\lambda$ is one of the integers $2\alpha - r$ for $0 \leq \alpha \leq r$.

The orders associated with the eliminated quadruples are 36 and 72. Both entries have 1 as a simple eigenvalue. These entries are underlined and highlighted in bold in Table 4.1.

Using the above methods, we reduced the set $\Xi$ from the initial 828 possible spectra to 59 quadruples in the final version. Henceforth $\Xi$ will refer to this final set of 59 quadruples (see Appendix A).

Table 4.1 summarizes the process of finding $\Xi$. For every order, we consider each $[x, y, z, w]$ and check whether we get integer counts at each vertex for each $C_i$ where $[C_i]$ is known and $i \in \{4, 6, 8, 10\}$. A ‘q’ in the table denotes that all possible spectra for a given $n$ have $4[C_4]/n \notin \mathbb{N}$. An ‘h’ denotes that some possible spectra for a given $n$ satisfy $4[C_4]/n \in \mathbb{N}$ but all possible spectra for that $n$ have $6[C_6]/n \notin \mathbb{N}$. If $i[C_i]/n \in \mathbb{N}$ for all $i$ where $[C_i]$ is known for a specific possible spectra, then the girth is recorded. Thus an entry of 4, 4, 6 would indicate that there are three possible spectra in $\Xi$ associated with that order, and if the quadruples are all realized by graphs then two graphs have girth 4 and one has girth 6.

<table>
<thead>
<tr>
<th>$n$</th>
<th>Girth</th>
<th>$n$</th>
<th>Girth</th>
<th>$n$</th>
<th>Girth</th>
<th>$n$</th>
<th>Girth</th>
</tr>
</thead>
<tbody>
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<td>4</td>
<td>36</td>
<td>4,4,4</td>
<td>96</td>
<td>4</td>
<td>240</td>
<td>6,8</td>
</tr>
<tr>
<td>10</td>
<td>4</td>
<td>40</td>
<td>4</td>
<td>112</td>
<td>h</td>
<td>252</td>
<td>h</td>
</tr>
<tr>
<td>12</td>
<td>4,4</td>
<td>42</td>
<td>4</td>
<td>120</td>
<td>4,4,6,6,6</td>
<td>280</td>
<td>8</td>
</tr>
<tr>
<td>14</td>
<td>q</td>
<td>48</td>
<td>4</td>
<td>126</td>
<td>4,6</td>
<td>288</td>
<td>6</td>
</tr>
<tr>
<td>16</td>
<td>4</td>
<td>56</td>
<td>h</td>
<td>140</td>
<td>h</td>
<td>336</td>
<td>h</td>
</tr>
<tr>
<td>18</td>
<td>4</td>
<td>60</td>
<td>4,4,4,4,6</td>
<td>144</td>
<td>4,4,6</td>
<td>360</td>
<td>6,6,8</td>
</tr>
<tr>
<td>20</td>
<td>4</td>
<td>70</td>
<td>6</td>
<td>160</td>
<td>q</td>
<td>420</td>
<td>8</td>
</tr>
<tr>
<td>24</td>
<td>4,4,4</td>
<td>72</td>
<td>4,4,4,6,6</td>
<td>168</td>
<td>6</td>
<td>480</td>
<td>8</td>
</tr>
<tr>
<td>28</td>
<td>q</td>
<td>80</td>
<td>h</td>
<td>180</td>
<td>4,4,6,6,6</td>
<td>504</td>
<td>h</td>
</tr>
<tr>
<td>30</td>
<td>4,4,6</td>
<td>84</td>
<td>h</td>
<td>210</td>
<td>6</td>
<td>560</td>
<td>10</td>
</tr>
<tr>
<td>32</td>
<td>6</td>
<td>90</td>
<td>4,4,6,6</td>
<td>224</td>
<td>h</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 4.1: Finding the set of possible spectra for vertex transitive graphs
4.3 The algorithm

In this section we outline our method for finding bipartite Cayley QIGs, using the set $\Xi$ compiled in Section 4.2.

Define $\Omega$ to be the set of orders associated with the spectra in $\Xi$. Cayley graphs are vertex-transitive, so we only consider groups $\Gamma$ of order $n \in \Omega$. To reduce the number of groups to be considered, we use a result similar to one in [152]. Let $\mathbb{Z}_a$ denote the cyclic group of order $a$ and let $\Gamma'$ denote the commutator subgroup of a group $\Gamma$.

Lemma 19. Let $\Gamma$ be a finite group and let $\text{Cay}(\Gamma, S)$ be a connected Cayley graph of degree at most 4. Then $\Gamma/\Gamma'$ is isomorphic to one of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_a$ with $a \geq 2$; $\mathbb{Z}_a \times \mathbb{Z}_b$ with $a, b \geq 2$; or $\mathbb{Z}_a$ with $a \geq 1$.

Proof. Since $\text{Cay}(\Gamma, S)$ is connected and has degree at most 4, $\Gamma$ is generated by an inverse-closed set of at most 4 elements. This must also be true of the quotient group $\Gamma/\Gamma'$. Now since $\Gamma/\Gamma'$ is Abelian, the result follows. $\square$

By Lemma 19, we need only consider groups $\Gamma$ with $\Gamma/\Gamma'$ isomorphic to one of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_a$, $\mathbb{Z}_a \times \mathbb{Z}_b$, or $\mathbb{Z}_a$. We denote the set of groups that satisfy this property by $\Phi$.

To construct connected simple undirected 4-regular Cayley graphs $\text{Cay}(\Gamma, S)$, we considered inverse-closed sets $S$ of four non-identity elements of $\Gamma$ that generate $\Gamma$. The search was pruned by placing additional restrictions on $S$. Let $g$ denote the girth of the graph $\text{Cay}(\Gamma, S)$.

1. Since $\text{Cay}(\Gamma, S)$ is bipartite, the order of $s$ is even for each $s \in S$.

2. If $s_1, s_2 \in S$ and $s_1 \neq s_2^{-1}$, then the order of $s_1 s_2$ is at least $g/2$ (in particular non-involutions have order no smaller than the girth).

3. For any set of connection sets that result in equivalent Cayley graphs (in the sense of Definition 17), only one representative is chosen.

We note that the minimum girth possible for $\text{Cay}(\Gamma, S)$ is given by Table 4.1.

We briefly expand upon the occurrence of a restriction on connection sets with no involutions. Consider a pair of connection sets $S$ and $T$ with $\sigma \in \text{Aut}(\Gamma)$ such that $S^\sigma = T$. The algorithm chooses sets by successively stabilizing elements point-wise. Two equivalent connection sets occur when some element $s$ in $S$ is also in $T$ but is not a fixed point of $\sigma$. In this situation $s = \sigma(s') \in T$ for some $s \neq s' \in S$. For example, the following $(S, T, \sigma)$ triple occurs for the group $\Gamma = \mathbb{Z}_{16}$: $\{(1, 15, 3, 13), (1, 15, 5, 11), (1, 5, 9, 13)(2, 10)(3, 15, 11, 7)(6, 14)\}$. Similar situations can occur when there are 2 or 4 involutions.

In our computations we divided up the generation of connection sets $S$ based upon whether they would contain 0, 2, or 4 involutions. Algorithm 2 demonstrates our methods for generating connection sets containing exactly four involutions. The other two cases are similar. The orbit of an element $u$ in $P$ is the set $Pu = \{p(u) \mid p \in P\}$. The set $P_S = \{p \in P \mid p(s) = s \text{ for all } s \in S\}$ is the point-wise stabilizer of the set of elements $S$. 


Algorithm 2

for all $\Gamma \in \Phi$ do
    $\text{Inv} := \text{involutions of } \Gamma \text{ satisfying restrictions } 1 - 2$
    $\text{CS}(\Gamma) := \{\}$
    $cs := \{\}$
    $\text{FINDCS}(\text{CS}(\Gamma), \text{Inv}, cs, \Gamma)$
end for

function $\text{FINDCS}(C, I, c, gp)$
    if $|c|$ equals 4 and $c$ generates $gp$ then
        if $C \cup \{c\}$ satisfies restriction 3 then
            Add $c$ to $C$
        end if
    else
        for all orbits of $\text{Aut}(gp)_c$ acting on $I$ do
            Consider first element $o$ of current orbit
            if $o \notin c$ and $c \cup \{o\}$ satisfies restriction 2 then
                $I^* := \text{elements of } I \text{ in later orbits}$
                $\text{FINDCS}(C, I^*, c \cup \{o\}, gp)$
            end if
        end for
    end if
end function

We summarize the results of our computations in Table 4.2. The values for $n \in \Omega$ appear as the first column and in the second column the number of groups of order $n$ is given. (We reiterate that $\Omega$ does not include orders eliminated by the vertex-transitive tests of Section 4.2). The number of groups in $\Phi$ of order $n$ are listed in column three. Column 4 contains the number of connection sets $S$ among the groups counted by column 3, subject to the restrictions on $S$ given above. The graphs $\text{Cay}(\Gamma, S)$ that are bipartite are counted in column 5. The number of isomorphism classes of these graphs appears in column 6. The number of isomorphism classes of integral graphs is recorded in column 7. The last column gives the number of isomorphism classes of arc-transitive integral graphs.

4.4 Quartic integral graphs

In this section we present the graphs that our computations discovered, starting with the bipartite Cayley case.

4.4.1 Bipartite Cayley integral graphs

As a result of the computation described in Section 4.3, we have:

Theorem 20. There are precisely 17 isomorphism classes of connected 4-regular bipartite integral Cayley graphs, as detailed in Table 4.3.

For each bipartite Cayley QIG in Table 4.3 we give $n$ and the spectrum $[x, y, z, w]$. Graphs appearing in the paper by Cvetković et al. [63] are labelled $I_{n,\text{index}}$ as in that paper. If the graph is
### Table 4.2: Results at each step of our computations for finding Cayley graphs

<table>
<thead>
<tr>
<th>$n$</th>
<th>#Groups $\Gamma$</th>
<th># $\Gamma \in \Phi$</th>
<th>#Sets $S$</th>
<th>#Bipartite Cay($\Gamma, S$)</th>
<th>#Isomorphism Classes</th>
<th>#Integral</th>
<th>#Arc-Transitive</th>
</tr>
</thead>
<tbody>
<tr>
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<td>5</td>
<td>5</td>
<td>13</td>
<td>7</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>10</td>
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<td>2</td>
<td>2</td>
<td>2</td>
<td>1</td>
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<td>5</td>
<td>19</td>
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<td>2</td>
<td>1</td>
</tr>
<tr>
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<td>1</td>
</tr>
<tr>
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<td>1</td>
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</tr>
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<td>1</td>
<td>1</td>
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<td>14</td>
<td>149</td>
<td>105</td>
<td>48</td>
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<td>1</td>
</tr>
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<td>55</td>
<td>55</td>
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<td>51</td>
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<td>96</td>
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<td>49</td>
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<td>236</td>
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<td>3545</td>
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<td>6563</td>
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<td>168</td>
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<tr>
<td>180</td>
<td>37</td>
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<td>1017</td>
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<td>40</td>
<td>4080</td>
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<td>3223</td>
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<td>-</td>
</tr>
<tr>
<td>288</td>
<td>1045</td>
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<td>24815</td>
<td>15695</td>
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<td>162</td>
<td>160</td>
<td>15928</td>
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<td>11524</td>
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</tr>
<tr>
<td>420</td>
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<td>9271</td>
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<td>-</td>
</tr>
<tr>
<td>480</td>
<td>1213</td>
<td>1148</td>
<td>68179</td>
<td>63804</td>
<td>48322</td>
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<td>-</td>
</tr>
<tr>
<td>560</td>
<td>180</td>
<td>177</td>
<td>21764</td>
<td>21433</td>
<td>18704</td>
<td>0</td>
<td>-</td>
</tr>
</tbody>
</table>
in the census of Potočnik et al. [151, 152] then we give the index in their notation: AT4Val[n][index].

In two columns, we give the groups and connection sets that give rise to each Cayley graph. The first column contains the group, $\Gamma$, with a presentation of that group. We stick as close as possible to the convention of using generators in $\{a, b, c, d, e\}$ for cyclic groups, $\{s, t, u, v\}$ for symmetric or alternating groups, and $\{r, f\}$ for the quaternion group, the dihedral group, or the quasidihedral group. The last column contains the number of involutions in the connection set, $S$, followed by the connection set itself in terms of the generators from the previous column.

<table>
<thead>
<tr>
<th>Group</th>
<th>Connection Sets</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G_1 : n = 8 \quad {0, 0, 0, 3} \quad I_{8,1} \quad AT4Val[8][1]$</td>
<td></td>
</tr>
<tr>
<td>$\mathbb{Z}_8$</td>
<td>$\langle a \mid a^8 \rangle$</td>
</tr>
<tr>
<td>$\mathbb{Z}_4 \times \mathbb{Z}_2$</td>
<td>$\langle a \mid a^4 \rangle \times \langle b \mid b^2 \rangle$</td>
</tr>
<tr>
<td>$\mathbb{D}_8$</td>
<td>$\langle r, f \mid r^4, f^2, (rf)^2 \rangle$</td>
</tr>
<tr>
<td>$\mathbb{Q}_8$</td>
<td>$\langle r, f \mid r^4, f^2, r^2f^2, rf^{-1} \rangle$</td>
</tr>
<tr>
<td>$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$</td>
<td>$\langle a \mid a^2 \rangle \times \langle b \mid b^2 \rangle \times \langle c \mid c^2 \rangle$</td>
</tr>
</tbody>
</table>

| $G_2 : n = 10 \quad \{0, 0, 4, 0\} \quad I_{10,1} \quad AT4Val[10][2]$ | |
| $\mathbb{D}_{10}$ | $\langle r, f \mid r^4, f^2, (rf)^2 \rangle$ | $4 \{f, fr, fr^2, r^2f\}$ |
| $\mathbb{Z}_{10}$ | $\langle a \mid a^{10} \rangle$ | $0 \{a, a^3, a^7, a^9\}$ |

| $G_3 : n = 12 \quad \{0, 2, 0, 3\} \quad I_{12,4} \quad AT4Val[12][2]$ | |
| $\mathbb{Z}_3 \times \mathbb{Z}_4$ | $\langle a, b \mid a^4, b^4, abab^{-1} \rangle$ | $0 \{b, b^3, ba, b^3a\}$ |
| $\mathbb{Z}_{12}$ | $\langle a \mid a^{12} \rangle$ | $0 \{a, a^3, a^7, a^{11}\}$ |
| $\mathbb{D}_{12}$ | $\langle r, f \mid r^6, f^2, (rf)^2 \rangle$ | $2 \{r^2f, f, r^5, r\}$ |
| $\mathbb{Z}_6 \times \mathbb{Z}_2$ | $\langle a \mid a^6 \rangle \times \langle b \mid b^2 \rangle$ | $0 \{a^5, a^2b, a, a^4b\}$ |

| $G_4 : n = 12 \quad \{0, 1, 4, 0\} \quad I_{12,2}$ | |
| $\mathbb{D}_{12}$ | $\langle r, f \mid r^6, f^2, (rf)^2 \rangle$ | $2 \{rf, r^3, r, r^5\}$ |
| $\mathbb{D}_{12}$ | $\langle r, f \mid r^6, f^2, (rf)^2 \rangle$ | $4 \{rf, r^3, r^2f, r^3f\}$ |
| $\mathbb{D}_{12}$ | $\langle r, f \mid r^6, f^2, (rf)^2 \rangle$ | $4 \{rf, r^4f, r^3f, r^5f\}$ |
### Chapter 4. Quartic Integral Cayley Graphs

#### $G_6$: $n = 16$  \  $[0, 4, 0, 3]$  \  $I_{16,1}$  \  AT4Val$[16][1]$  

<table>
<thead>
<tr>
<th>Group</th>
<th>Presentation</th>
<th>2 {a^3b, a^5, a}</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Z_6 \times Z_2$</td>
<td>$&lt; a \mid a^6 &gt; \times &lt; b \mid b^2 &gt;$</td>
<td></td>
</tr>
<tr>
<td>$Z_4 \times Z_4$</td>
<td>$&lt; a \mid a^4 &gt; \times &lt; b \mid b^4 &gt;$</td>
<td>0 {a, b, a^3, b^3}</td>
</tr>
<tr>
<td>$(Z_4 \times Z_2) \times Z_2$</td>
<td>$&lt; a, b, c \mid a^4, b^2, c^2, aba^{-1}b^{-1}, (aac)^2, (bc)^2, bacac^{-1}c &gt;$</td>
<td>2 {ac, a^2bc, a^3bc, a^2c}</td>
</tr>
<tr>
<td>$Z_4 \times Z_4$</td>
<td>$&lt; a, b \mid a^3b, a^3c, ab$</td>
<td>0 {a, a^2c, a^3, abc}</td>
</tr>
<tr>
<td>$Z_8 \times Z_2$</td>
<td>$&lt; a, b \mid a^8, b, aba^{-1}b &gt;$</td>
<td>0 {a, ab, a^3b, a^7}</td>
</tr>
<tr>
<td>$QD_{16}$</td>
<td>$&lt; r, f \mid r^8, f^2, rf^5f &gt;$</td>
<td>2 {r, r^4f, r^6f, r^7}</td>
</tr>
<tr>
<td>$Z_4 \times Z_2 \times Z_2$</td>
<td>$&lt; a \mid a^4 &gt; \times &lt; b \mid b^2 &gt; \times &lt; c \mid c^2 &gt;$</td>
<td>2 {a, b, c, a^3}</td>
</tr>
<tr>
<td>$Z_2 \times D_8$</td>
<td>$&lt; a \mid a^2 &gt; \times &lt; r, f \mid r^4, f^2, (rf)^2 &gt;$</td>
<td>4 {a, f, r^3f, r^2f}</td>
</tr>
<tr>
<td>$Z_2 \times Z_2 \times Z_2 \times Z_2$</td>
<td>$&lt; a \mid a^2 &gt; \times &lt; b \mid b^2 &gt; \times &lt; c \mid c^2 &gt; \times &lt; d \mid d^2 &gt;$</td>
<td>4 {a, b, c, d}</td>
</tr>
</tbody>
</table>

#### $G_6$: $n = 18$  \  $[0, 4, 0, 4]$  \  $I_{18,1}$  \  AT4Val$[18][2]$  

<table>
<thead>
<tr>
<th>Group</th>
<th>Presentation</th>
<th>2 {s, st, ats, a^2ts}</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Z_3 \times S_3$</td>
<td>$&lt; a \mid a^3 &gt; \times &lt; s, t \mid s^2, t^3, (st)^2 &gt;$</td>
<td>0 {sa, sa^2, sat, sa^2t}</td>
</tr>
<tr>
<td>$(Z_3 \times Z_3) \times Z_2$</td>
<td>$&lt; a, b, c \mid a^3, b^3, c^2, aba^{-1}b^{-1}, (ac)^2, (bc)^2 &gt;$</td>
<td>4 {c, ca, cb, cab}</td>
</tr>
<tr>
<td>$Z_6 \times Z_3$</td>
<td>$&lt; a \mid a^6 &gt; \times &lt; b \mid b^3 &gt;$</td>
<td>0 {a, a^5, a^3b, a^3b^2}</td>
</tr>
</tbody>
</table>

#### $G_7$: $n = 24$  \  $[0, 8, 0, 3]$  \  $I_{24,2}$  \  AT4Val$[24][1]$  

<table>
<thead>
<tr>
<th>Group</th>
<th>Presentation</th>
<th>2 {s, st, at, a^3sts}</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Z_4 \times S_3$</td>
<td>$&lt; a \mid a^4 &gt; \times &lt; s, t \mid s^2, t^3, (st)^2 &gt;$</td>
<td>0 {sa, sa^2, sat, sa^2t}</td>
</tr>
<tr>
<td>$(Z_6 \times Z_2) \times Z_2$</td>
<td>$&lt; a, b, c \mid a^6, b^2, c^2, aba^{-1}b^{-1}, (aac)^2, a^3(ch)^2 &gt;$</td>
<td>2 {a^3c, a^2c, ab, a^5b}</td>
</tr>
<tr>
<td>$Z_3 \times D_4$</td>
<td>$&lt; a \mid a^3 &gt; \times &lt; r, f \mid r^4, f^2, (rf)^2 &gt;$</td>
<td>0 {ar^3f, a^2r^3f, ar^3, a^2r}</td>
</tr>
<tr>
<td>$S_4$</td>
<td>$&lt; s, t \mid s^2, t^3, (st)^4 &gt;$</td>
<td>4 {st^2sts, t^2st, stst^2s, tsts^2}</td>
</tr>
<tr>
<td>$Z_2 \times A_4$</td>
<td>$&lt; a \mid a^2 &gt; \times &lt; s, t \mid s^2, t^3, (st)^3 &gt;$</td>
<td>0 {ast, astst, astst, atst}</td>
</tr>
</tbody>
</table>
### 4.4. QUARTIC INTEGRAL GRAPHS

**G₈: n = 24** \[2, 2, 6, 1\] \[I₄₄,₃\]

<table>
<thead>
<tr>
<th>[\mathbb{Z}_2 \times \mathbb{Z}_2 \times S_3]</th>
<th>[&lt; a \mid a^2 &gt; &lt; b \mid b^2 &gt; &lt; s, t \mid s^2, t^3, (st)^2 &gt;]</th>
<th>4 {s, bs, st, ast}</th>
</tr>
</thead>
<tbody>
<tr>
<td>[\mathbb{Z}_4 \times S_3]</td>
<td>[&lt; a \mid a^4 &gt; &lt; s, t \mid s^2, t^3, (st)^2 &gt;]</td>
<td>2 {a, a^2, s, st}</td>
</tr>
<tr>
<td>&amp;</td>
<td>2 {s, st, at, a^2ts}</td>
<td></td>
</tr>
<tr>
<td>[D_{24}]</td>
<td>[&lt; r, f \mid r^{12}, f^2, (rf)^2 &gt;]</td>
<td>4 {f, rf, r^5f, r^9f}</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2 {f, r^3f, r^9f, r^{11}f}</td>
</tr>
<tr>
<td>[\mathbb{Z}_2 \times (\mathbb{Z}_3 \times \mathbb{Z}_4)]</td>
<td>[&lt; a \mid a^2 &gt; &lt; b, c \mid b^3, c^4, bebc^{-1} &gt;]</td>
<td>0 {c, c^5, ab, ac^3bc}</td>
</tr>
<tr>
<td>[(Z_6 \times Z_2) \times Z_2]</td>
<td>[&lt; a, b, c \mid a^3, b^2, c^2, aba^{-1}b^{-1}, (ac)^2, (bc)^4 &gt;]</td>
<td>4 {c, b, ca, cbc}</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2 {c, bc, ba, bca}</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2 {c, ca, cbcac, bab}</td>
</tr>
<tr>
<td>[\mathbb{Z}_{12} \times \mathbb{Z}_2]</td>
<td>[&lt; a \mid a^{12} &gt; &lt; b \mid b^2 &gt;]</td>
<td>0 {a^3, a^9, a^4b, a^8b}</td>
</tr>
<tr>
<td>[\mathbb{Z}_3 \times D_8]</td>
<td>[&lt; a \mid a^3 &gt; &lt; r, f \mid r^4, f^2, (rf)^2 &gt;]</td>
<td>2 {r^{12}f, rf, r^5f, r^9f}</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0 {r, r^3, a^3f, a^2r^3f}</td>
</tr>
<tr>
<td>[\mathbb{Z}_2 \times \mathbb{Z}_2 \times S_3]</td>
<td>[&lt; a \mid a^2 &gt; &lt; b \mid b^2 &gt; &lt; s, t \mid s^2, t^3, (st)^2 &gt;]</td>
<td>4 {s, b, h, a, st}</td>
</tr>
<tr>
<td></td>
<td></td>
<td>4 {s, b, h, at, ast}</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2 {s, sb, ast, at}</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2 {s, sb, at, ast}</td>
</tr>
<tr>
<td>[\mathbb{Z}_6 \times \mathbb{Z}_2 \times \mathbb{Z}_2]</td>
<td>[&lt; a \mid a^6 &gt; &lt; b \mid b^2 &gt; &lt; c \mid c^2 &gt;]</td>
<td>2 {a^3, b, a^2c, a^4c}</td>
</tr>
</tbody>
</table>

**G₉: n = 24** \[3, 0, 5, 3\] \[I₄₄,₄\]

<table>
<thead>
<tr>
<th>[S_4]</th>
<th>[&lt; s, t \mid s^2, t^3, (st)^4 &gt;]</th>
<th>4 {s, t^2st, st^2st, stst^2st}</th>
</tr>
</thead>
<tbody>
<tr>
<td>[\mathbb{Z}_2 \times A_4]</td>
<td>[&lt; a \mid a^2 &gt; &lt; s, t \mid s^2, t^3, (st)^3 &gt;]</td>
<td>2 {a, as, at^2s, ast}</td>
</tr>
</tbody>
</table>

**G₁₀: n = 30** \[0, 10, 4, 0\] \[I₃₀,₁\] \[AT₄Val[30][4]\]

<table>
<thead>
<tr>
<th>[\mathbb{Z}_5 \times S_3]</th>
<th>[&lt; a \mid a^5 &gt; &lt; s, t \mid s^2, t^3, (st)^2 &gt;]</th>
<th>0 {as, a^2st, a^4s, a^3st}</th>
</tr>
</thead>
<tbody>
<tr>
<td>[D_{30}]</td>
<td>[&lt; r, f \mid r^{15}, f^2, (rf)^2 &gt;]</td>
<td>4 {f, r^2f, r^3f, r^{11}f}</td>
</tr>
</tbody>
</table>

**G₁₁: n = 32** \[0, 12, 0, 3\] \[I₃₂,₁\] \[AT₄Val[32][4]\]

<table>
<thead>
<tr>
<th>[\mathbb{Z}_8 \times \mathbb{Z}_4]</th>
<th>[&lt; a, b \mid a^3, b^4, ab^2a^{-1}b^{-2}, aba^{-1}b^{-1} &gt;]</th>
<th>0 {a, a^7, ab, a^3b^3}</th>
</tr>
</thead>
<tbody>
<tr>
<td>[(Z_8 \times Z_2) \times Z_2]</td>
<td>[&lt; a, b, c \mid a^3, b^2, c^2, a^2ba^6b, (aac)^2, (bc)^2, ba^{-1}cac &gt;]</td>
<td>2 {a^4c, a^2c, a^7bc, a^5c}</td>
</tr>
<tr>
<td>[\mathbb{Z}_2,((Z_4 \times Z_2) \times Z_2),(Z_4 \times Z_2)]</td>
<td>[&lt; a, b \mid a^4, b^4, ab^2a^{-1}b^{-2}, a^4b^2, aba^{-1}b^{-1}ab^{-1}a^{-1}b, aba^6ba &gt;]</td>
<td>0 {ba, a^3b, a^4, a^5}</td>
</tr>
</tbody>
</table>
$\langle a, b, c \rangle = \langle a^b, b^a, c^a, abc, acb \rangle$
### 4.4. QUARTIC INTEGRAL GRAPHS

<table>
<thead>
<tr>
<th>$\mathbb{Z}_6 \times D_8$</th>
<th>$\mathbb{Z}_2 \times ((\mathbb{Z}_6 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$&lt; a</td>
<td>a^6 &gt; \times &lt; r, f</td>
</tr>
<tr>
<td>$\mathbb{Z}_6 \times D_8$</td>
<td>$\mathbb{Z}_2 \times ((\mathbb{Z}_6 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2)$</td>
</tr>
<tr>
<td>$&lt; a</td>
<td>a^6 &gt; \times &lt; r, f</td>
</tr>
<tr>
<td>$\mathbb{Z}_6 \times D_8$</td>
<td>$\mathbb{Z}_2 \times ((\mathbb{Z}_6 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2)$</td>
</tr>
<tr>
<td>$&lt; a</td>
<td>a^6 &gt; \times &lt; r, f</td>
</tr>
</tbody>
</table>

### G_{15} : n = 72

<table>
<thead>
<tr>
<th>$\mathbb{Z}_3 \times S_4$</th>
<th>$\mathbb{Z}_2 \times (\mathbb{Z}_3 \times \mathbb{Z}_4)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$&lt; a</td>
<td>a^3 &gt; \times &lt; s, t</td>
</tr>
<tr>
<td>$\mathbb{Z}_3 \times (\mathbb{Z}_3 \times \mathbb{Z}_4)$</td>
<td>$\mathbb{Z}_2 \times (\mathbb{Z}_3 \times \mathbb{Z}_4)$</td>
</tr>
<tr>
<td>$(\mathbb{Z}_3 \times A_4) \rtimes \mathbb{Z}_2$</td>
<td>$2 {ath, ab, tsh, tbs}$</td>
</tr>
<tr>
<td>$&lt; a, s, t, b</td>
<td>a^3, s^2, t^3, b^2, asa^{-1} s^{-1}, ata^{-1} t^{-1}, stbsbt^{-1}, (ab)^2, (b)^2, (st)^3 &gt;$</td>
</tr>
<tr>
<td>$A_4 \times S_3$</td>
<td>$\mathbb{Z}_2 \times (\mathbb{Z}_3 \times \mathbb{Z}_4)$</td>
</tr>
<tr>
<td>$&lt; s, t</td>
<td>s^2, t^3, (st)^3 &gt; \times &lt; u, v</td>
</tr>
<tr>
<td>$\mathbb{Z}_6 \times A_4$</td>
<td>$\mathbb{Z}_2 \times (\mathbb{Z}_3 \times \mathbb{Z}_4)$</td>
</tr>
<tr>
<td>$&lt; a</td>
<td>a^6 &gt; \times &lt; s, t</td>
</tr>
</tbody>
</table>

### G_{16} : n = 72

<table>
<thead>
<tr>
<th>$\mathbb{Z}_2 \times (\mathbb{Z}_3 \times \mathbb{Z}_4)$</th>
<th>$\mathbb{Z}_2 \times (\mathbb{Z}_3 \times \mathbb{Z}_4)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbb{Z}_3 \times (\mathbb{Z}_3 \times \mathbb{Z}_4)$</td>
<td>$\mathbb{Z}_3 \times (\mathbb{Z}_3 \times \mathbb{Z}_4)$</td>
</tr>
<tr>
<td>$&lt; a, b, c, d</td>
<td>a^3, b^3, c^3, d^3, ab^3 a^{-1} b^{-1}, ac^3 a^{-1} c^{-1}, b^3 b^{-1} d^{-1}, adda^{-1}, bca^{-1} c^{-1}, c^2 d^2 &gt;$</td>
</tr>
<tr>
<td>$\mathbb{Z}_3 \times (\mathbb{Z}_3 \times \mathbb{Z}_4)$</td>
<td>$\mathbb{Z}_3 \times (\mathbb{Z}_3 \times \mathbb{Z}_4)$</td>
</tr>
<tr>
<td>$&lt; a, b, c, d</td>
<td>a^3, b^4, c^3, d^3, (ab)^{-1} 2, acac^{-1}, (ad)^2, cbcb^{-1}, bdb^{-1} d^{-1}, cdc^{-1} d^{-1} &gt;$</td>
</tr>
<tr>
<td>$\mathbb{Z}_3 \times (\mathbb{Z}_3 \times \mathbb{Z}_4)$</td>
<td>$\mathbb{Z}_3 \times (\mathbb{Z}_3 \times \mathbb{Z}_4)$</td>
</tr>
<tr>
<td>$&lt; a</td>
<td>a^6 &gt; \times &lt; b, c</td>
</tr>
<tr>
<td>$\mathbb{Z}_3 \times (\mathbb{Z}_3 \times \mathbb{Z}_4)$</td>
<td>$\mathbb{Z}_3 \times (\mathbb{Z}_3 \times \mathbb{Z}_4)$</td>
</tr>
<tr>
<td>$&lt; a</td>
<td>a^6 &gt; \times &lt; b, c</td>
</tr>
<tr>
<td>$\mathbb{Z}_3 \times (\mathbb{Z}_3 \times \mathbb{Z}_4)$</td>
<td>$\mathbb{Z}_3 \times (\mathbb{Z}_3 \times \mathbb{Z}_4)$</td>
</tr>
<tr>
<td>$&lt; a</td>
<td>a^3 &gt; \times &lt; b, c, d</td>
</tr>
<tr>
<td>$\mathbb{Z}_3 \times (\mathbb{Z}_3 \times \mathbb{Z}_4)$</td>
<td>$\mathbb{Z}_3 \times (\mathbb{Z}_3 \times \mathbb{Z}_4)$</td>
</tr>
<tr>
<td>$&lt; a</td>
<td>a^3 &gt; \times &lt; b, c, d</td>
</tr>
<tr>
<td>$\mathbb{Z}_2 \times (\mathbb{Z}_3 \times \mathbb{Z}_4)$</td>
<td>$\mathbb{Z}_2 \times (\mathbb{Z}_3 \times \mathbb{Z}_4)$</td>
</tr>
<tr>
<td>$(S_4 \times S_4) \times \mathbb{Z}_2$</td>
<td>$\mathbb{Z}_2 \times (\mathbb{Z}_3 \times \mathbb{Z}_4)$</td>
</tr>
<tr>
<td>$&lt; s, t, u, v, a</td>
<td>s^2, t^3, u^2, v^3, a^2, tve^{-1} v^{-1}, (uv)^2, (av)^2, svst^{-1}, vasasu &gt;$</td>
</tr>
<tr>
<td>$\mathbb{Z}_2 \times (\mathbb{Z}_3 \times \mathbb{Z}_4)$</td>
<td>$\mathbb{Z}_2 \times (\mathbb{Z}_3 \times \mathbb{Z}_4)$</td>
</tr>
<tr>
<td>$&lt; a</td>
<td>a^2 &gt; \times &lt; b, c, d</td>
</tr>
<tr>
<td>$\mathbb{Z}_2 \times (\mathbb{Z}_3 \times \mathbb{Z}_4)$</td>
<td>$\mathbb{Z}_2 \times (\mathbb{Z}_3 \times \mathbb{Z}_4)$</td>
</tr>
<tr>
<td>$&lt; a</td>
<td>a^2 &gt; \times &lt; b, c, d</td>
</tr>
<tr>
<td>Group</td>
<td>Description</td>
</tr>
<tr>
<td>-------</td>
<td>-------------</td>
</tr>
<tr>
<td>$\mathbb{Z}_2 \times S_3 \times S_3$</td>
<td>$&lt; a \mid a^2 &gt; \times &lt; s, t \mid s^2, t^3, (st)^2 &gt; \times &lt; u, v \mid u^2, v^3, (uv)^2 &gt;$</td>
</tr>
<tr>
<td>$\mathbb{Z}_2 \times \mathbb{Z}_6 \times S_3$</td>
<td>$&lt; a \mid a^2 &gt; \times &lt; b \mid b^6 &gt; \times &lt; s, t \mid s^2, t^3, (st)^2 &gt;$</td>
</tr>
<tr>
<td>$G_{17} : n = 120 \quad [12, 28, 4, 15] \quad AT4Val[120][4]$</td>
<td></td>
</tr>
<tr>
<td>$S_5$</td>
<td>$&lt; s, t \mid s^2, t^5, (st)^4, (st^2st^3)^2 &gt;$</td>
</tr>
<tr>
<td>$Z_2 \times A_5$</td>
<td>$&lt; a \mid a^3 &gt; \times &lt; s, t \mid s^2, t^3, (st)^5 &gt;$</td>
</tr>
<tr>
<td>$S_3 \times (\mathbb{Z}_5 \rtimes \mathbb{Z}_4)$</td>
<td>$&lt; s, t \mid s^2, t^3, (st)^2 &gt; \times &lt; a, b \mid a^5, b^4, ab^{-1}a^2b, a^2b^{-1}a^{-1}b &gt;$</td>
</tr>
<tr>
<td>$\mathbb{Z}_5 \times S_4$</td>
<td>$&lt; a \mid a^5 &gt; \times &lt; s, t \mid s^2, t^3, (st)^4 &gt;$</td>
</tr>
<tr>
<td>$(\mathbb{Z}_5 \rtimes A_4) \rtimes \mathbb{Z}_2$</td>
<td>$&lt; a, s, t, b \mid a^5, s^2, t^3, b^2, asa^{-1}s^{-1}, ata^{-1}t^{-1}, bsb^{-1}s, (st)^5, (tb)^3, (ab)^5 &gt;$</td>
</tr>
</tbody>
</table>

Table 4.3: Quartic bipartite integral Cayley graphs

Drawings for all but the three largest bipartite Cayley QIGs appear in Figure 4.4. With over 70 vertices, it is difficult to present $G_{15}, G_{16},$ and $G_{17}$ clearly.
4.4. QUARTIC INTEGRAL GRAPHS

4.4.2 Bipartite arc-transitive integral graphs

We considered all arc-transitive 4-regular graphs from the census of Potočnik et al. [151, 152] and tested them for integrality. The only arc-transitive bipartite QIGs that are not Cayley and thus not accounted for in Table 4.3 are the five that appear in Table 4.5. We let \([\Gamma : H] = \{Ha | a \in \Gamma\}\) denote the set of right cosets of \(H \leq \Gamma\). A Schreier coset graph \(\text{Sch}(\Gamma, H, HSH)\) for a group \(\Gamma\), subgroup \(H \leq \Gamma\), and connection set \(S \subseteq \Gamma\) is the graph with vertex set \([\Gamma : H]\) and with \(Ha\) connected to \(Hb\) if and only if \(ba^{-1} \in HSH\). We represent these 5 graphs as Schreier coset graphs. We give the order \(n\) and the spectrum \([x, y, z, w]\) followed by the graph index from [151, 152]. Graphs appearing in the paper by Cvetković et al. are labelled with the notation of [63]: \(I_{n, \text{index}}\). The first line consists of the group \(\Gamma\), with a presentation of that group. The second line consists of the subgroup \(H\) and its generators in terms of the generators of \(\Gamma\) followed by the connection set \(S\) in terms of the generators of \(\Gamma\).
Let \( V(G) \) denote the vertices of a graph \( G \), and \( E(G) \) the unordered pairs of vertices which are edges of \( G \). A homomorphism from a graph \( G \) to a graph \( H \) is a map \( V(G) \to V(H) \) which preserves adjacency. Each homomorphism induces an edge map \( E(G) \to E(H) \). If the vertex and edge maps of the homomorphism are both surjective then we say that \( H \) is a quotient of \( G \). In this section we find new integral graphs that are quotients of the integral graphs found in Table 4.3 and Table 4.5. However, first we describe our method for finding all Cayley QIGs.

### 4.4.3 Integral graphs as quotients

Let \( V(G) \) denote the vertices of a graph \( G \), and \( E(G) \) the unordered pairs of vertices which are edges of \( G \). A homomorphism from a graph \( G \) to a graph \( H \) is a map \( V(G) \to V(H) \) which preserves adjacency. Each homomorphism induces an edge map \( E(G) \to E(H) \). If the vertex and edge maps of the homomorphism are both surjective then we say that \( H \) is a quotient of \( G \). In this section we find new integral graphs that are quotients of the integral graphs found in Table 4.3 and Table 4.5. However, first we describe our method for finding all Cayley QIGs.

The census \([151, 152]\) of arc-transitive graphs contains all arc-transitive graphs with at most 640 vertices. Thus, the upper bound of 560 given in \([176]\) for the order of a bipartite QIG, ensures that Table 4.3 and Table 4.5 contain all bipartite arc-transitive QIGs. The non-bipartite arc-transitive QIGs will be given in Sections 4.4.4 and 4.4.5. However, first we describe our method for finding all Cayley QIGs.

### Table 4.5: Quartic bipartite arc-transitive non-Cayley integral graphs

<table>
<thead>
<tr>
<th>( F_1 )</th>
<th>( n = 60 )</th>
<th>4, 16, 4, 5</th>
<th>( I_{60,1} )</th>
<th>AT4Val[60][4]</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathbb{Z}_2 \times \mathbb{Z}_2 \times S_5 ) :</td>
<td>(&lt; a \mid a^2 &gt; x \prec b \mid b^2 &gt; x \prec s, t \mid s^2, t^2, (st)^4, (st^3)^2 &gt;)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( D_3 : )</td>
<td>(&lt; bstst^2st^{-1}, abstst &gt;, {s, bl^2, s^t, bl^{-2}})</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( F_2 )</th>
<th>( n = 70 )</th>
<th>6, 14, 14, 0</th>
<th>( I_{70,1} )</th>
<th>AT4Val[70][4]</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathbb{Z}_2 \times S_7 ) :</td>
<td>(&lt; a \mid a^2 &gt; x \prec s, t \mid s^2, t^7, (st)^9, (st^2st^5)^2, (stst^{-1})^3 &gt;)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( S_3 \times S_1 : )</td>
<td>(&lt; t^2st^{-2}, t^{-2}st^{-1}(st)^2, t^2(st)^2(ts)t^{-1}, t(st)^2(ts)^2, stst^{-1}s &gt;, {ast^4, atstst^{-1}, at, )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( at^{-1}} )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( F_3 )</th>
<th>( n = 90 )</th>
<th>9, 16, 19, 0</th>
<th>( I_{90,1} )</th>
<th>AT4Val[90][1]</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathbb{Z}_2 \times \text{PGL}(2, 9) : )</td>
<td>(&lt; a \mid a^2 &gt; x \prec s, t \mid x, y, z \mid x^8, y^3, z^2, xz^5z, yz^2z^{-1}, x, y^4xy^{-1}y, (xyx^2)^2, xyx^{-2}y^{-1}x^4yx^{-1}y^{-1} &gt;)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( {z \times D_3 \times \mathbb{Z}_2 } : )</td>
<td>(&lt; yzxy^{-1}x, x^{-1}y^2xy^{-1}y, x^2z^2y^{-1}x^{-1}y &gt;, {ayx^{-1}y^{-1}x, az, ayxy^{-1}x, )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( ax^{-1}y^{-1}x } )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( F_4 )</th>
<th>( n = 180 )</th>
<th>22, 28, 34, 5</th>
<th>AT4Val[180][12]</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathbb{Z}_2 \times S_4 \times S_5 ) :</td>
<td>(&lt; a \mid a^2 &gt; x \prec s, t \mid s^2, t^3, (st)^2 &gt; \prec u, v \mid u^2, v^5, (uv)^4, (uv^2uv^3)^2 &gt;)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( D_9 : )</td>
<td>(&lt; v^{-2}uv^2, vuv^2u, astuvuv^2uv^{-1} &gt;, {at^{-1}v^{-2}, atuv^2, su, atv^2})</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( F_5 )</th>
<th>( n = 210 )</th>
<th>27, 28, 49, 0</th>
<th>AT4Val[210][10]</th>
</tr>
</thead>
<tbody>
<tr>
<td>( S_7 : )</td>
<td>(&lt; s, t \mid s^2, t^7, (st)^9, (st^2st^5)^2, (stst^{-1})^3 &gt;)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( S_4 : )</td>
<td>(&lt; tst^3st^3, tst^{-3}st^3, (st^3t^3)^2ts, (ts)^3st &gt;, {ts^3, st^4, (st)^3, (ts)^3})</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 4.5: Quartic bipartite arc-transitive non-Cayley integral graphs
4.4. QUARTIC INTEGRAL GRAPHS

We start by considering special classes of possible homomorphisms. Suppose $G$ is a graph and $\alpha$ is a function from $E(G)$ to a group $\Gamma$. A $\Gamma$-voltage graph $\text{Vol}(G, \alpha)$ for $G$ is the graph with vertex set $V(G)$ and edge set $E(G)$ with edges labeled by an element of $\Gamma$ as given by the function $\alpha$. The derived graph of a $\Gamma$-voltage graph $\text{Vol}(G, \alpha)$ is the graph with vertex set $V(G)$ and edge set $E(G)$ with edges labeled by an element of $\Gamma$ as given by the function $\alpha$. The derived graph of $\text{Vol}(G, \alpha)$ is the graph with vertex set the Cartesian product $V(G) \times \Gamma$ with $(a, x)$ connected to $(b, y)$ whenever both $\{a, b\} \in E(G)$ and $y = x * \alpha(\{a, b\})$ where $*$ is the group operation of $\Gamma$. Projection onto the first coordinate, by definition, maps the derived graph of $\text{Vol}(G, \alpha)$ onto $G$, and this map is a surjective homomorphism. Hence $G$ is a quotient of the derived graph.

As an interesting example for quartic integral graphs, we found a $\mathbb{Z}_3$-voltage graph $\text{Vol}(F_1, \alpha)$, with derived graph $F_4$. Thus, $F_1$ is a quotient of $F_4$.

A bipartite double cover $G \times K_2$ is the derived graph of the $\mathbb{Z}_2$-voltage graph $\text{Vol}(G, \alpha)$ where $\alpha$ is the constant function assigning 1 to every edge.

We give an example for quartic integral graphs that was also noted in [63]. An odd graph $O_i$ is the graph with one vertex for each of the $(i − 1)$-element subsets of a $(2i − 1)$-element set and with edges joining disjoint subsets. The graph $F_2$ is the bipartite double cover of the integral graph $O_4$.

We want to find all graphs which have their bipartite double cover among the bipartite graphs that we have discovered. This requires us to find quotients of our bipartite graphs. Since it is computationally easy to do, we will actually consider a more general class of homomorphisms than what is required for the task just described. This will increase the number of quartic integral graphs that we find. However, we make no effort to be exhaustive in finding all possible quotients.

Let $\vartheta$ be an automorphism of some graph $G$. For convenience, we will refer to the orbits in the action on $G$ of the group generated by $\vartheta$ as simply “orbits of $\vartheta$”. We say that $\vartheta$ is $k$-semiregular if all its orbits have the same size, $k$. Note that if $G = H \times K_2$ then the natural homomorphism from $G$ onto $H$ maps orbits of a 2-semiregular automorphism of $G$ to single vertices of $H$. With this as motivation, the class of homomorphisms that we consider is the following. We identify any $k$-semiregular automorphism, $\vartheta$ of a target graph $G$. Our homomorphism is to collapse each orbit of $\vartheta$ to a single point.

We wrote a routine in Magma [168] to find such quotients of a target graph $G$, as follows. For one representative, $\vartheta$, of each conjugacy class of (nontrivial) semiregular automorphisms of $G$, we collapsed the orbits of $\vartheta$ to single vertices to obtain a quotient $H$. If $H$ was a 4-regular graph we checked to see if it was integral. If it was, then we printed it out and called the routine recursively on $H$.

In some cases we were only interested in finding those $H$ for which $G$ is a bipartite double cover. In such instances, it suffices to only consider 2-semiregular automorphisms and we do not need to make recursive calls to the routine.

We applied our Magma routine to all target graphs $G_i$ for $i \in 1, \ldots, 17$ and to most of the arc-transitive graphs from the census of Potočnik et al. [151, 152]. There are graphs in the census with extremely large automorphism groups, and they were impractical for our simple routine. So we decided to only include target graphs from the census if their automorphism group had order no more than $2^{20}$.

The results of this Magma routine will be given in the following subsections.
4.4.4 Non-bipartite Cayley integral graphs

In this section we report all quartic Cayley integral graphs that are not bipartite. We rely on this Lemma:

Lemma 21. If $G$ is a 4-regular Cayley graph then $G \times K_2$, the bipartite double cover of $G$, is isomorphic to a 4-regular Cayley graph.

Proof. If $G = \text{Cay}(\Gamma, S)$ then we define $G' = \text{Cay}(\Gamma \times \mathbb{Z}_2, \{(s, 1) \mid s \in S\})$. This graph $G'$ is an undirected Cayley graph. Consider the following: $(g, 0)$ is adjacent to $(g, 0)(s, 1) = (gs, 1)$ and $(g, 1)$ is adjacent to $(g, 1)(s, 1) = (gs, 0)$ for $g \in \Gamma$. We have verified that $G'$ is isomorphic to $G \times K_2$ which gives the desired result.  

Hence we can find all the graphs we seek by applying the Magma routine of Section 4.4.3 to our graphs $G_i$ where $i = 1, \ldots, 17$. We use the following result by Sabidussi [156] to decide which of the graphs that we find are Cayley graphs:

Lemma 22. A graph $G$ is a Cayley graph if and only if $\text{Aut}(G)$ contains a regular subgroup.

Table 4.6 summarizes our results using the Magma routine of Section 4.4.3 when restricted to the 2-semiregular automorphisms for each given $G_i$. We give the number of non-bipartite graphs found, followed by the numbers of those that are Cayley, vertex-transitive, and arc-transitive.

<table>
<thead>
<tr>
<th>Initial Graph</th>
<th>#Non-bipartite</th>
<th>#Cayley</th>
<th>#Vertex-transitive</th>
<th>#Arc-transitive</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G_1$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$G_2$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$G_3$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$G_4$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$G_5$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$G_6$</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$G_7$</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$G_8$</td>
<td>4</td>
<td>2</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>$G_9$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$G_{10}$</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$G_{11}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$G_{12}$</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$G_{13}$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$G_{14}$</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>$G_{15}$</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$G_{16}$</td>
<td>13</td>
<td>2</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>$G_{17}$</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 4.6: Non-bipartite graphs found for $G_i$
4.4. QUARTIC INTEGRAL GRAPHS

The non-bipartite graphs counted in column 2 up to row 8 of Table 4.6 were previously found by Stevanović et al. [176]. All graphs counted by column 2 from rows 9 to 17 were previously unknown with the exception of the graph with bipartite double cover $G_{10}$ and one of the two graphs with bipartite double cover $G_{14}$. In Table 4.7, we expand upon the counts of non-bipartite Cayley graphs in column three of Table 4.6 by producing a breakdown of the groups and the connection sets of the underlying graphs. We follow the same conventions as in Table 4.3 except that we use different notation for the spectrum, since there is no longer symmetry about the origin.

<table>
<thead>
<tr>
<th>Group</th>
<th>Connection Sets</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H_1$ : $n = 5$ $- 1^4, 4^4$ $I_{5,1}$ AT4Val$[5][1]$</td>
<td>$\langle a</td>
</tr>
<tr>
<td>$Z_5$</td>
<td></td>
</tr>
<tr>
<td>$H_2$ : $n = 6$ $- 2^2, 0^3, 4^1$ $I_{6,1}$ AT4Val$[6][1]$</td>
<td>$\langle s, t</td>
</tr>
<tr>
<td>$S_3$</td>
<td></td>
</tr>
<tr>
<td>$Z_6$</td>
<td>$0 {a^3, a^2, a, a^4}$</td>
</tr>
<tr>
<td>$H_3$ : $n = 8$ $- 2^3, 0^2, 2^1, 4^1$ $I_{8,2}$</td>
<td>$\langle a</td>
</tr>
<tr>
<td>$Z_4 \times Z_2$</td>
<td></td>
</tr>
<tr>
<td>$D_8$</td>
<td>$2 {r, r^2, r^3, fr^2}$ $4 {f, r^2, rf, fr}$</td>
</tr>
<tr>
<td>$\langle r, f</td>
<td>r^4, f^2, (rf)^2 \rangle$</td>
</tr>
<tr>
<td>$Z_2 \times Z_2 \times Z_2$</td>
<td>$4 {b, a, abc, ac}$</td>
</tr>
<tr>
<td>$\langle a</td>
<td>a^2 \rangle \times &lt; b</td>
</tr>
<tr>
<td>$H_4$ : $n = 9$ $- 2^4, 1^4, 4^1$ $I_{9,2}$ AT4Val$[9][1]$</td>
<td>$\langle a</td>
</tr>
<tr>
<td>$Z_3 \times Z_3$</td>
<td></td>
</tr>
<tr>
<td>$H_5$ : $n = 12$ $- 2^5, 0^3, 2^3, 4^1$ $I_{12,7}$ AT4Val$[12][1]$</td>
<td>$\langle s, t</td>
</tr>
<tr>
<td>$A_4$</td>
<td></td>
</tr>
<tr>
<td>$H_6$ : $n = 12$ $- 3^2, -1^4, 0^1, 1^2, 2^2, 4^1$ $I_{12,5}$</td>
<td>$\langle a, b</td>
</tr>
<tr>
<td>$Z_3 \times Z_4$</td>
<td></td>
</tr>
<tr>
<td>$Z_{12}$</td>
<td>$0 {a^3, a^4, a^8, a^9}$</td>
</tr>
<tr>
<td>$\langle a</td>
<td>a^{12} \rangle$</td>
</tr>
<tr>
<td>$D_{12}$</td>
<td>$&lt; r, f \mid r^6, f^2, (r f)^2 &gt;$</td>
</tr>
<tr>
<td>$\mathbb{Z}_6 \times \mathbb{Z}_2$</td>
<td>$&lt; a \mid a^6 &gt; \times &lt; b \mid b^2 &gt;$</td>
</tr>
</tbody>
</table>

**H**$_7$: $n = 12$ $- 3^2, -2^2, 0^1, 1^6, 4^1$ $I_{12.1}$

| $\mathbb{Z}_3 \times \mathbb{Z}_4$ | $< a, b \mid a^3, b^4, abab^{-1}, ab^2a^2b >$ | $0 \{ b^3, b^2a, b^2a^2 \}$ |
| $\mathbb{Z}_{12}$ | $< a \mid a^{12} >$ | $0 \{ a^3, a^{10}, a^2, a^9 \}$ |
| $D_{12}$ | $< r, f \mid r^6, f^2, (r f)^2 >$ | $2 \{ f, fr^3, r, f r^3 \}$ | $4 \{ r^3, f, fr, fr^3 \}$ |
| $\mathbb{Z}_6 \times \mathbb{Z}_2$ | $< a \mid a^6 > \times < b \mid b^2 >$ | $2 \{ a^3b, a^3, a^3b, a^4b \}$ |

**H**$_9$: $n = 18$ $- 3^2, -2^4, 0^6, 1^4, 3^2, 4^1$ $I_{18.4}$

| $\mathbb{Z}_3 \times S_3$ | $< a \mid a^3 > \times < s, t \mid s^2, t^3, (st)^2 >$ | $2 \{ a, s, a^2, st^2 \}$ | $0 \{ t, t^2, s t^2, s a^2 t^2 \}$ |
| $(\mathbb{Z}_3 \times \mathbb{Z}_3) \times \mathbb{Z}_2$ | $< a, b, c \mid a^3, b^3, c^2, cac^{-1}a^{-2}, bc^{-1}b^{-2}c, aba^{-1}b^{-1} >$ | $2 \{ a, c, a^2, cb^2 \}$ |
| $\mathbb{Z}_6 \times \mathbb{Z}_3$ | $< a \mid a^6 > \times < b \mid b^3 >$ | $0 \{ a^2b, a^5b^2, ab, a^4b^2 \}$ |

**H**$_{10}$: $n = 20$ $- 2^6, -1^4, 0^3, 3^4, 4^1$

| $\mathbb{Z}_5 \times \mathbb{Z}_4$ | $< a, b \mid a^5, b^4, ab^3a^3b, ab^2ab^2 >$ | $2 \{ a^2b^2, ab^2, a^2b, a^4b^3 \}$ |

**H**$_{10}$: $n = 24$ $- 3^3, -2^3, -1^5, 0^3, 1^5, 2^1, 3^3, 4^1$ $I_{24.5}$

| $S_4$ | $< s, t \mid s^2, t^3, (st)^4 >$ | $2 \{ s, st^2s, (tst)^2, t(t s)^2 \}$ |
| $\mathbb{Z}_2 \times A_4$ | $< a \mid a^2 > \times < s, t \mid s^2, t^3, (st)^3 >$ | $2 \{ s, st^2, a, ts \}$ |

**H**$_{11}$: $n = 24$ $- 3^4, -2^3, -1^2, 0^3, 1^8, 2^1, 3^2, 4^1$

| $\mathbb{Z}_4 \times S_3$ | $< a \mid a^4 > \times < s, t \mid s^2, t^3, (st)^2 >$ | $2 \{ a^3t, at^2, sa^2, a^2 \}$ |
| $(\mathbb{Z}_6 \times \mathbb{Z}_2) \times \mathbb{Z}_2$ | $< a, b, c \mid a^6, b^2, c^2, aba^{-1}b^{-1}, (a^3c)^2b, cbc^{-1}b^{-1}, a^2ca^2c >$ | $4 \{ ca^2, cb, b, a^3 \}$ |
| $\mathbb{Z}_3 \times D_8$ | $< a \mid a^3 > \times < r, f \mid r^4, f^2, (r f)^2 >$ | $2 \{ f, r^2, ar, a^2r^{-1} \}$ |
| $\mathbb{Z}_2 \times \mathbb{Z}_2 \times S_3$ | $< a \mid a^2 > \times < b \mid b^2 > \times < s, t \mid s^2, t^3, (st)^2 >$ | $4 \{ s a b t, s b t^2, a b, s a \}$ |
4.4. QUARTIC INTEGRAL GRAPHS

<table>
<thead>
<tr>
<th>Table 4.7: Quartic non-bipartite integral Cayley graphs</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H_{12}: n = 36$</td>
</tr>
<tr>
<td>$\mathbb{Z}_3 \times A_4$</td>
</tr>
<tr>
<td>$H_{13}: n = 36$</td>
</tr>
<tr>
<td>$\mathbb{Z}_3 \times (\mathbb{Z}_3 \rtimes \mathbb{Z}_4)$</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td>$S_3 \times S_3$</td>
</tr>
<tr>
<td>$Z_6 \times S_3$</td>
</tr>
<tr>
<td>$H_{14}: n = 36$</td>
</tr>
<tr>
<td>$\mathbb{Z}_3 \times (\mathbb{Z}_3 \rtimes \mathbb{Z}_4)$</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td>$S_3 \times S_3$</td>
</tr>
<tr>
<td>$Z_6 \times S_3$</td>
</tr>
<tr>
<td>$H_{15}: n = 60$</td>
</tr>
<tr>
<td>$A_5$</td>
</tr>
</tbody>
</table>

Thus, by Theorem 20 and Lemma 21 we have that \( \{G_i \mid 1 \leq i \leq 17\} \cup \{H_j \mid 1 \leq j \leq 15\} \) is the complete set of Cayley QIGs.

4.4.5 Non-bipartite arc-transitive integral graphs

In Section 4.4.2, we listed all bipartite arc-transitive QIGs from the census of Potočnik et al. [151, 152]. When searching this census for integral graphs, we also found arc-transitive QIGs that are not bipartite. There are 46 such graphs that are not Cayley and thus not already accounted for in Table 4.7. By [176], the bipartite double cover of any QIG has order at most 560, so we can be sure that the census contains all the arc-transitive QIGs. In fact, the following folklore result tells us more:
Lemma 23. The bipartite double cover of an arc-transitive graph is arc-transitive.

Proof. Let $G$ be an arc-transitive graph. Then $H = G \times K_2$ has vertices $(a,x)$ for all $a \in G$ and $x \in \mathbb{Z}_2$ and arcs $((a,x), (b,x + 1))$ whenever $a$ is adjacent to $b$ in $G$.

Claim 1: For all $\sigma \in \text{Aut}(G)$, $\sigma' : V(H) \to V(H)$ defined by $\sigma'((a,x)) = (\sigma(a), x + 1)$ is an automorphism of $H$. Consider any element $(a, x) \in H$, then since $\sigma$ is a permutation of the vertices of $G$, we have that there is an element $(\sigma^{-1}(a), x + 1) \in H$ such that $\sigma'((\sigma^{-1}(a), x + 1)) = (a, x)$. Thus $\sigma'$ is a permutation of the vertices of $H$. Consider any arc $((a,x), (b,x + 1))$ in $E(H)$. By applying $\sigma'$, $((a,x), (b,x + 1)) \to ((\sigma(a), x + 1), (\sigma(b), x))$. Since $a$ was adjacent to $b$ in $G$, $\sigma(a)$ is adjacent to $\sigma(b)$ in $G$. Now since the parities of the second coordinates are still different, $((\sigma(a), x + 1), (\sigma(b), x))$ is an arc.

Claim 2: The map $\sigma^* : V(H) \to V(H)$ defined by $\sigma^*((a,x)) = (a, x + 1)$ is an automorphism of $H$. It is quite clear that $\sigma^*$ is an order 2 permutation that swaps the vertices with the same first coordinate. We consider $((a,x), (b,x + 1))$ in $E(H)$. By $\sigma^*$, $((a,x), (b,x + 1)) \to ((a, x + 1), (b, x))$. Since $a$ was adjacent to $b$ in $G$ and the parities of the second coordinates are still different, $((a, x + 1), (b, x))$ is an arc.

Claim 3: For all pairs of arcs $e_1 = ((a,x), (b,x + 1))$ and $e_2 = ((c,y), (d, y + 1))$ in $E(H)$, there exists $\alpha \in \text{Aut}(H)$ such that $\alpha(e_1) = e_2$. If $c = a$ and $d = b$, then $\alpha = \sigma^*$ or the identity automorphism. Otherwise, since $G$ is arc-transitive, there exists $\sigma_1 \in \text{Aut}(G)$ such that $(a,b) \mapsto (c,d)$ where $(a,b)$ and $(c,d)$ are arcs of $G$. If $y = x + 1$ then $\alpha = \sigma_1'$. Otherwise $\alpha = \sigma^* \circ \sigma_1'$.

Thus, $H$ is arc-transitive.

This last result provides a useful cross-check of our results and of the Magma routine from Section 4.4.3. It tells us that by applying the routine (restricted to 2-semiregular automorphisms) to the bipartite arc-transitive QIGs from Tables 4.3 and 4.5, we should find all the arc-transitive integral non-bipartite graphs. This list should tally with the list obtained by directly screening the census for integral graphs, which is what happened in practice.

We now list the spectrum of the non-bipartite arc-transitive QIGs that are not Cayley and whose bipartite double cover is one of the $G_i$ for $i = 1, \ldots, 17$ or $F_i$ for $i = 1, \ldots, 5$. We denote these graphs by $J_i$ for $i = 1, \ldots, 6$. Graphs appearing in the paper by Cvetković et al. are included using the notation of [63]: $I_{n, \text{index}}$. We give the graph index from the census of Potočnik et al. [151, 152] in their notation: AT4Val[n][index].

- From $G_{10}$, $J_1 \cong I_{15,2} \cong \text{AT4Val}[15][1] : [-2^5, -1^4, 2^5, 4^1]$,
- From $F_1$, $J_2 \cong \text{AT4Val}[30][2] : [-3^4, -2^5, -1^4, 0^5, 2^{11}, 4^1]$,
  $J_3 \cong I_{30,4} \cong \text{AT4Val}[30][3] : [-2^{11}, -1^4, 0^5, 2^5, 3^4, 4^1]$,
- From $F_2$, $J_4 \cong I_{35,1} \cong \text{AT4Val}[35][2] \cong O_4 : [-3^6, -1^{14}, 2^{14}, 4^1]$,
- From $F_3$, $J_5 \cong I_{45,1} \cong \text{AT4Val}[45][1] : [-2^{16}, -1^9, 1^{10}, 3^9, 4^1]$,
- From $F_4$, $J_6 \cong \text{AT4Val}[90][8] : [-3^{14}, -2^{27}, -1^{24}, 0^8, 1^{10}, 2^{21}, 3^8, 4^1]$.

Of the arc-transitive non-bipartite non-Cayley graphs, only $J_2$ and $J_6$ were not previously known to be integral. Thus, the arc-transitive QIGs from the census are as follows: $G_1$, $G_2$, $G_3$, $G_5$, $G_6$, $G_7$, $J_1$, $J_2$, $J_3$, $J_4$, $J_5$, $J_6$. 

CHAPTER 4. QUARTIC INTEGRAL CAYLEY GRAPHS

56
4.4. QUARTIC INTEGRAL GRAPHS

$G_{10}$, $G_{11}$, $G_{12}$, $G_{15}$, $G_{17}$, $F_1$, $F_2$, $F_3$, $F_4$, $F_5$, $H_1$, $H_2$, $H_4$, $H_5$, $H_{12}$, $J_1$, $J_2$, $J_3$, $J_4$, $J_5$, and $J_6$. We summarize these results by the following Lemma:

**Lemma 24.** There are exactly 27 quartic integral graphs that are arc-transitive; 16 of which are bipartite.

4.4.6 The only vertex-transitive graph on 32 vertices

On considering the only spectrum with 32 vertices in Appendix A, we are able to prove that $G_{11}$ is the unique vertex-transitive QIG with this spectrum.

We consider a vertex-transitive graph $G$ with 32 vertices, girth 6, and spectrum the quadruple $[0, 12, 0, 3]$. Since $G$ is vertex-transitive, the orbit-stabilizer theorem (see [89] for example) gives us that $|G| = 2^5$ divides $|\text{Aut}(G)|$.

A $p$-subgroup $H$ of a group $\Gamma$ is a subgroup of $\Gamma$ where every element $h \in H$ has order a power of the prime $p$. A Sylow $p$-subgroup $P$ of a group $\Gamma$ is a $p$-subgroup that is contained in no larger $p$-subgroup of $\Gamma$. Let $A$ be a Sylow 2-subgroup of $\text{Aut}(G)$ [77]. The center $Z(\Gamma)$ is defined as $Z(\Gamma) = \{z \in \Gamma \mid z\alpha = \alpha z \text{ for all } \alpha \in \Gamma\}$. If $Z(\Gamma) = \{e\}$ where $e$ is the identity element of $\Gamma$ then we say that the center of $\Gamma$ is trivial.

The following theorem appears in [198, Thm 3.4].

**Theorem 25.** Let $p$ be a prime number and let $A$ be a Sylow $p$-subgroup of $\text{Aut}(G)$. Suppose $p^s$ divides $|\text{Aut}(G)v|$ for a vertex $v \in V(G)$. Then $p^s$ also divides $|Av|$.

Now since $G$ is transitive, $|\text{Aut}(G)v| = 2^5$ for all $v \in V(G)$. By Theorem 25, $2^5$ divides $|Av|$ for all $v \in V(G)$ and thus $Av = V(G)$. Therefore, $A$ is transitive on the vertices of $G$.

Since $A$ is finite and non-trivial, it has a non-trivial center [77]. Let $\sigma$ be a central element in $A$ of order 2. Since $\sigma$ is a central element, the group $\langle \sigma \rangle$ is a normal subgroup of $A$.

Given a graph $H$, a partition $\Pi$ of $V(H)$ is a set of disjoint non-empty subsets of $V(H)$ whose union is $V(H)$. We will refer to these subsets as cells. A partition, $\Pi = (C_1, \ldots, C_k)$ is equitable if for every choice of $i$ and $j$, each vertex in $C_i$ has the same number of neighbours in $C_j$. Given an equitable partition $\Pi$ of a graph $H$, the quotient graph $H/\Pi$ of $H$ with respect to $\Pi$ is the graph with the cells of $\Pi$ as its vertices and with edge $(C_i, C_j)$ for every edge $(x, y) \in E(H)$ where $x \in C_i$ and $y \in C_j$. Thus, a quotient graph may have multiple edges and loops.

The partition, $\Pi = (C_1, \ldots, C_k)$, of $V(G)$ into orbits of $\sigma$ is an equitable partition of the graph $G$ [155, p. 76]. Since the group $\langle \sigma \rangle$ is normal, each cell contains two vertices [198, Prop 7.1]. Now since $A$ is transitive on $V(G)$, $A$ acts transitively on these cells [23, p. 173]. Thus, the quotient graph $G/\Pi$ of $G$ with respect to $\Pi$ is a transitive multigraph with 16 vertices.

Every eigenvalue of $G/\Pi$ is an eigenvalue of the graph $G$ [94]. For this reason, we only consider graphs with integer eigenvalues. For any integral graph that satisfies the known conditions of $G/\Pi$, we check that it lifts to $G$.

The pair of vertices in one cell is mapped to the pair of vertices in another cell and so these pairs of edges in $G/\Pi$ can be thought of as a single edge in the ‘frame’ of $G/\Pi$. Each pair of vertices in a cell is either adjacent or non-adjacent. This gives 2 cases:

- The frame of $G/\Pi$ is a cubic transitive graph. It can be checked that there is no such cubic graph that is integral.
• The frame of $G/II$ is a quartic transitive graph. It can be checked that a unique quartic graph candidate exists that satisfies integrality: $C_4 \times C_4$.

In lifting from $C_4 \times C_4$ to $G$ each edge between cells is replaced by two edges. These edges can be thought of as ‘parallel’ or ‘crossed’, depending on whether they join vertices in the same copy of $C_4 \times C_4$ or not. Since $G$ has no 4-cycles, each cycle must have an odd number of ‘parallel’ edges to be a suitable candidate for $G$. Using this observation, it can be checked by exhaustion that there is a unique way to lift $G/II$ to $G$. This unique completion is isomorphic to $G_{11}$. There is no bipartite QIG on 64 vertices. Therefore, by Theorem 21, there are no non-bipartite QIGs with 32 vertices. Thus, $G_{11}$ is the unique QIG on 32 vertices that is vertex-transitive.

### 4.4.7 Other quartic integral graphs

Finally, we list the spectra of the remaining QIGs which we found using the Magma routine of Section 4.4.3 in its full generality. These are graphs that are neither Cayley nor arc-transitive, but are quotients of the graphs $G_i$ for $i = 1, \ldots, 17$ and/or of the graphs AT4Val[$n$][index] for $n \leq 640$ with automorphism groups of order less than $2^{20}$. We note that many of these graphs were obtained from multiple starting graphs, but we only list each graph once.

We list the spectrum of the bipartite QIGs first. We denote these graphs by $M_i$ for $i = 1, \ldots, 9$ and follow the same conventions as in the list for $J_i$ where $i = 1, \ldots, 6$ except that we use the quadruple form for the spectrum of a bipartite graph.

- From AT4Val[60][4] we have $M_1 \cong I_{30, 3} : [1, 8, 3, 2]$,
- From $G_{15} \cong AT4Val[72][12]$ we have $M_2 \cong I_{36, 1} : [2, 8, 6, 1]$, $M_3 \cong I_{36, 2} : [3, 6, 5, 3]$,
- From $G_{17} \cong AT4Val[120][4]$ we have $M_4 : [3, 4, 1, 6]$, $M_5 : [6, 12, 2, 9]$,
- From AT4Val[180][12] we have $M_6 : [9, 16, 19, 0]$, $M_7 : [10, 14, 18, 2]$,
- From AT4Val[216][12] we have $M_8 : [3, 5, 9, 0]$,
- From AT4Val[546][48] we have $M_9 : [5, 4, 7, 4]$.

We do not list graphs with at most 24 vertices since all bipartite QIGs on 24 or fewer vertices are known [176]. The 6 graphs $M_4, \ldots, M_9$ were not previously known to be bipartite QIGs. We find that $M_6$ is co-spectral to $F_3$, but 5 of the above spectra were not previously known to be realized by any graph.

We give an example of how a quotient graph can arise from multiple graphs. In addition to AT4Val[180][12], $M_7$ is a quotient of AT4Val[360][10] and AT4Val[540][17].

Next, we list the spectrum of the non-bipartite QIGs. We denote these graphs by $L_i$ where $i \in 1, \ldots, 44$.

- From AT4Val[30][3], $L_1 \cong I_{15, 4} : [-2^5, -1^3, 0^2, 2^3, 3^1, 4^1]$.
- From $G_{12} \cong AT4Val[36][3]$, $L_2 : [-3^3, -2^2, -1^1, 0^5, 1^3, 2^2, 3^1, 4^1]$.
- From AT4Val[36][6], $L_3 \cong I_{18, 5} : [-2^7, -1^2, 0^1, 1^4, 2^1, 3^2, 4^1]$, $L_4 \cong I_{18, 6} : [-2^6, -1^3, 0^3, 1^2, 3^1, 4^1]$.


4.5. REMARKS

- From AT4Val[60][4], \( L_5 : [-3^3, -2^7, -1^3, 0^6, 1^1, 2^9, 3^1, 4^1] \), and \( L_6 : [-3^2, -2^3, -1^2, 0^6, 2^7, 3^2, 4^1] \).

- From AT4Val[70][4], \( L_7 : [-3^5, -2^4, -1^9, 1^5, 2^10, 3^4, 4^1] \), and \( L_8 : [-3^4, -2^6, -1^8, 1^6, 2^8, 3^2, 4^1] \).

- From \( G_{15} \cong AT4Val[72][12] \), \( L_9 : [-3^1, -2^5, -1^3, 0^4, 1^3, 3^1, 4^1] \), \( L_{10} : [-3^2, -2^4, -1^4, 0^1, 1^2, 2^4, 4^1] \), \( L_{11} : [-3^2, -2^5, 0^1, 1^6, 2^3, 4^1] \), \( L_{12} : [-3^3, -2^2, -1^2, 0^1, 1^1, 2^3, 4^1] \), \( L_{13} : [-3^3, -2^5, -1^3, 0^1, 1^3, 2^3, 3^1, 4^1] \), \( L_{14} : [-3^3, -2^4, -1^3, 0^1, 1^4, 2^2, 3^1, 4^1] \), \( L_{15} : [-3^3, -2^9, -1^5, 0^3, 1^5, 2^7, 3^3, 4^1] \), \( L_{16} : [-3^4, 2^9, -1^2, 0^3, 1^3, 2^7, 3^2, 4^1] \), \( L_{17} : [-3^2, -2^11, -1^4, 0^3, 1^6, 2^9, 3^4, 4^1] \), and \( L_{18} : [-3^3, -2^9, -1^5, 0^3, 1^5, 2^7, 3^3, 4^1] \).

- From \( G_{16} \), \( L_{19} : [-3^4, -2^7, -1^6, 0^1, 1^{10}, 2^3, 3^4, 4^1] \), \( L_{20} : [-3^4, -2^9, -1^2, 0^1, 1^{14}, 2^1, 3^4, 4^1] \), \( L_{21} : [-3^4, -2^7, -1^{10}, 0^1, 1^6, 2^5, 3^4, 4^1] \), \( L_{22} : [-3^4, -2^6, -1^8, 0^1, 1^8, 2^4, 3^4, 4^1] \), \( L_{23} : [-3^4, -2^8, -1^4, 0^1, 1^{12}, 2^2, 3^4, 4^1] \), \( L_{24} : [-3^4, -2^6, -1^8, 0^1, 1^8, 2^4, 3^4, 4^1] \), \( L_{25} : [-3^3, -2^8, -1^4, 0^1, 1^{12}, 2^2, 3^4, 4^1] \), \( L_{26} : [-3^3, -2^7, -1^9, 0^1, 1^7, 2^3, 3^5, 4^1] \), \( L_{27} : [-3^3, -2^8, -1^7, 0^1, 1^9, 2^2, 3^3, 4^1] \), \( L_{28} : [-3^4, -2^7, -1^6, 0^1, 1^{10}, 2^3, 3^4, 4^1] \), and \( L_{29} : [-3^4, -2^6, -1^8, 0^1, 1^8, 2^4, 3^4, 4^1] \).

- From AT4Val[90][1], \( L_{30} : [-3^4, -2^{10}, -1^9, 1^{10}, 2^6, 3^5, 4^1] \).

- From AT4Val[90][8], \( L_{31} : [-3^5, -2^6, -1^{14}, 1^5, 2^{10}, 3^4, 4^1] \).

- From \( G_{17} \cong AT4Val[120][4] \), \( L_{32} : [-3^3, -2^7, -1^1, 0^9, 1^1, 2^5, 3^3, 4^1] \), \( L_{33} : [-3^3, -2^7, -1^1, 0^9, 1^1, 2^5, 3^3, 4^1] \), \( L_{34} : [-3^4, -2^5, -1^2, 0^9, 2^3, 3^2, 4^1] \), \( L_{35} : [-3^7, -2^{13}, -1^3, 0^{15}, 1^1, 2^{15}, 3^5, 4^1] \).

- From AT4Val[180][12], \( L_{36} : [-3^4, -2^8, -1^{12}, 0^2, 1^6, 2^6, 3^6, 4^1] \), \( L_{37} : [-3^9, -2^{17}, -1^9, 0^6, 1^{15}, 2^{11}, 3^{13}, 4^1] \), \( L_{38} : [-3^{11}, -2^{13}, -1^{21}, 0^5, 1^{13}, 2^{15}, 3^{11}, 4^1] \), and \( L_{39} : [-3^{12}, -2^{15}, -1^{14}, 0^5, 1^{20}, 2^{13}, 3^{10}, 4^1] \).

- From AT4Val[210][10], \( L_{40} : [-3^{16}, -2^9, -1^{29}, 1^{20}, 2^{19}, 3^{11}, 4^1] \).

- From AT4Val[273][4], \( L_{41} : [-3^4, -2^4, -1^6, 0^4, 1^1, 3^4, 4^1] \).

- From AT4Val[546][48], \( L_{42} : [-3^2, -2^3, -1^5, 0^4, 1^2, 2^1, 3^3, 4^1] \), \( L_{43} : [-3^3, -2^2, -1^4, 0^4, 1^3, 2^2, 3^2, 4^1] \), \( L_{44} : [-3^3, -2^2, -1^4, 0^4, 1^3, 2^2, 3^2, 4^1] \).

We do not list graphs with at most 12 vertices since all non-bipartite QIGs on 12 or fewer vertices are known [11, 176]. Of the 44 graphs given above, only \( L_1 \), \( L_3 \) and \( L_4 \) previously appear in the literature about integral graphs. The remaining 41 non-bipartite QIGs are new.

4.5 Remarks

There are precisely 32 connected 4-regular integral Cayley graphs up to isomorphism. Table 4.3 lists the 17 graphs of the 32 which are bipartite and Table 4.7 gives the details of the 15 non-bipartite graphs.

There are exactly 27 quartic integral graphs that are arc-transitive. We found that 16 of the 27 graphs are bipartite; these appear in Table 4.3 and Table 4.5. We found that 16 of the 27 graphs are Cayley graphs; these appear in Table 4.3 and Table 4.7.
There are integral Cayley bipartite graphs that can be decomposed into $H \times K_2$ where $H$ is Cayley and arc-transitive, Cayley but not arc-transitive, or arc-transitive but not Cayley. The graph $G_{10}$ is our only example of this last possibility; refer to Table 4.6.

The new 4-regular integral graphs that we found that are co-spectral to other graphs are as follows: $G_{13}$ co-spectral to $I_{40,1}$ and $I_{10,2}$, $G_{15}$ to $I_{72,1}$, $H_9$ to $I_{20,8}$, $H_{12}$ to $I_{36,4}$, and $F_3$ to $M_6$. We also mention the co-spectral graphs among the known integral graphs: $G_5$ is co-spectral to $I_{16,2}$ and another graph appearing in [176], $G_9$ to $I_{20,8}$, $H_9$ to $I_{20,8}$, and $F_3$ to $M_6$. We also mention the co-spectral graphs among the known integral graphs: $G_5$ is co-spectral to $I_{16,2}$, and another graph appearing in [176], $G_9$ to $I_{20,8}$, $H_9$ to $I_{20,8}$, and $F_3$ to $M_6$.

We find that some integral Cayley graphs are co-spectral to integral non-Cayley graphs and that some integral arc-transitive graphs are co-spectral to integral graphs that are not arc-transitive. For example, the arc-transitive Cayley graph $H_5$ has a co-spectral mate $I_{12,6}$, that is neither arc-transitive nor Cayley.

As can also be seen in Table 4.3, there are isomorphic integral graphs that are non-equivalent Cayley graphs $\text{Cay}(\Gamma, S)$ and $\text{Cay}(\Gamma^*, S^*)$ in the sense of Definition 17. This can occur for $\Gamma \neq \Gamma^*$ as well as $\Gamma = \Gamma^*$ with $S \neq S^*$. Consider $G_{12}$, which has 12 non-equivalent Cayley Graphs on 6 different groups. For $\Gamma = S_3 \times S_3$, there are 4 non-equivalent Cayley graphs with connection sets occurring for each of the three possible numbers of involutions. There is only one Cayley graph of order 40 up to equivalence. For all other orders the bipartite integral Cayley graphs are not unique up to equivalence. In the non-bipartite case; $H_1, H_4, H_5, H_9, H_{12},$ and $H_{15}$ are all unique up to equivalence.

There are non-isomorphic integral Cayley graphs with the same number of vertices. As can be seen in Table 4.3 for the bipartite case, there are two graphs on 12 vertices, three graphs on 24 vertices, and two graphs on 72 vertices up to isomorphism. For all other orders there is at most one graph up to isomorphism. There are many more examples in the non-bipartite case (refer to Table 4.7).

There exist non-isomorphic integral Cayley graphs for the same group $\Gamma$. Consider $G_i$ for $i = 7, 8, 9$ in Table 4.3. The following 6 groups are examples of this: $Z_2 \times A_4$, $Z_3 \times D_8$, $Z_2 \times Z_2 \times S_3$, $S_4$, $(Z_6 \times Z_2) \times Z_2$, and $Z_4 \times S_3$.

We began with the 828 possible spectra from [176], and then narrowed our focus to a set $\Xi$ of 59 candidates for vertex transitive graphs; refer to Table 4.1. Of these, we found 22 which are realised by Cayley graphs or arc-transitive graphs. In Section 4.4.7, by taking quotients, we found an additional 6 bipartite integral graphs that are neither arc-transitive nor Cayley, but realize a possible spectrum.

In Section 4.4.6, we prove that $G_{11}$ is the unique QIG on 32 vertices that is vertex-transitive.

Overall, we found 9 bipartite quartic integral graphs (namely, $G_{16}$, $G_{17}$, $F_4$, $F_5$, $M_4$, $M_5$, $M_7$, $M_8$, $M_9$) that realize spectra not previously known to be achieved. It is open whether the remaining possible spectra are realized by any 4-regular bipartite integral graphs.

Acknowledgements

The results in this chapter were found using a mixture of the computer algebra packages Magma [168] and GAP [81] (including the GRAPE package [170] for GAP), as well as nauty [142].
Chapter 5

Counting Subgraphs of Strongly Regular Graphs

5.1 Counting subgraphs in terms of smaller subgraphs

In this section we develop our method for counting subgraphs of a strongly regular graph in terms of the parameters of the graph and the counts of some subgraphs with fewer vertices.

Consider a strongly regular graph \( G = \text{SR}(n, r, e, f) \). Let \( H \) denote a connected subgraph of \( G \). Our method enables us to derive expressions for the counts of connected subgraphs \( H \) of \( G \) that have minimum degree one or two. We will give our results for counting these two categories of subgraphs separately.

For \( S \subset V(H) \), let \( H \setminus S \) denote the graph induced by \( V(H) \setminus S \). For ease of notation, we denote an edge between vertex \( x \) and vertex \( y \) by \( xy \). Given a graph \( H \) and an edge \( xy \), let \( H + xy \) denote the graph \( H \) with vertex set \( V(H) \) and edge set \( E(H) \cup \{xy\} \). In particular, \( H + xy = H \) if \( xy \) is already an edge of \( H \). For any graph \( H \), let \([H, v_1, v_2, \ldots, v_i]\) for \( i = 0, 1, \ldots, 3 \), denote the number of subgraphs isomorphic to \( H \) in \( G \). In all other cases, we have

\[
[H, v_1, v_2, \ldots, v_i] = [H] |\text{Aut}(H)_{v_1}| \cdots |\text{Aut}(H)_{\{v_1, \ldots, v_{i-1}\}}v_i|.
\]

If \( v \) is a cut-vertex of the graph \( H \) then \( H \setminus \{v\} \) is a disconnected graph.

**Theorem 26.** Consider \( G = \text{SR}(n, r, e, f) \) and a graph \( H \) containing a vertex \( s \) of degree one. Let \( u \) denote the neighbour of \( s \) in \( H \) and let \( N \) denote the neighbours of \( u \) in \( H \setminus \{s\} \). Let \( X \) denote a set of vertices of \( H \setminus \{(u, s) \cup N\} \) that are orbit representatives under the action of \( \text{Aut}(H \setminus \{s\})_{\{u\}} \). Then,

\[
[H] = \frac{(r - |N|)\{H \setminus \{s\}, u\} - \sum_{x \in X} [H \setminus \{s\} + ux, u, x]}{|\text{Aut}(H)u||\text{Aut}(H)_{\{u\}}s|}.
\]

**Proof.** We count \([H, u, s]\), the number of graphs \( H \) with distinguished vertices \( u \) and \( s \), in two ways. By definition,

\[
[H, u, s] = [H] |\text{Aut}(H)u||\text{Aut}(H)_{\{u\}}s|.
\]
The second way is as follows. We choose a copy of \( H \{ s \} \) with distinguished vertex \( u \). Then, we choose a neighbour \( s \) of \( u \). There are \([H \{ s \}, u]\) such graphs in \( G \). Since \( G \) is regular, there are \( r - |N| \) choices for \( s \) in \( G \). However, these choices for \( s \) include the undesired possibility that a vertex of \( H \) is chosen. We eliminate the counts of graphs for these choices by subtracting \([H \{ s \} + u x, u, x]\) for each vertex \( x \) in the set \( V(H) \) that is not \( u \), any of the neighbours of \( u \), nor a second vertex in the same orbit under the action of \( \text{Aut}(H \{ s \})_{(u,v)} \) (since this would lead to the subtraction of the counts of a graph multiple times). These vertices \( x \) are precisely the vertices of \( X \).

In solving for \([H]\), we arrive at the desired expression. \( \square \)

**Theorem 27.** Consider \( G = \text{SR}(n, r, e, f) \) and a graph \( H \) containing a vertex \( s \) of degree two, such that \( s \) is not a cut-vertex of \( H \). Let \( u \) and \( v \) denote the two neighbours of \( s \) in \( H \) and let \( C \) denote the common neighbours of \( u \) and \( v \) in \( H \{ s \} \). Let \( X \) denote a set of vertices of \( H \{ \{ u, v, s \} \cup C \} \) that are orbit representatives under the action of \( \text{Aut}(H \{ \{ s \} \})_{(u,v)} \). If \( u \) is adjacent to \( v \) in \( H \) then,

\[
[H] = \frac{(e - |C|)[H \{ s \}, u, v] - \sum_{x \in X}[H \{ s \} + u x + x v, u, v, x]}{|\text{Aut}(H)u||\text{Aut}(H)_{(u,v)}||\text{Aut}(H)_{(u,v)} s|}.
\]

Otherwise,

\[
[H] = \frac{(f - |C|)[H \{ s \}, u, v] + (e - f)[H \{ s \} + u v, u, v] - \sum_{x \in X}[H \{ s \} + u x + x v, u, v, x]}{|\text{Aut}(H)u||\text{Aut}(H)_{(u,v)}||\text{Aut}(H)_{(u,v)} s|}.
\]

**Proof.** For the time being, suppose that \( f \geq |C| \) and \( e \geq |C| \). We count \([H, u, v, s]\) the number of graphs \( H \) with distinguished vertices \( u, v, \) and \( s \); in two ways. By definition,

\[
[H, u, v, s] = [H]|\text{Aut}(H)u||\text{Aut}(H)_{(u,v)}||\text{Aut}(H)_{(u,v)} s|.
\]

The second way has two cases.

- **Case 1 - vertex \( u \) is adjacent to vertex \( v \) in \( H \):** We choose a copy of \( H \{ s \} \) with distinguished vertices \( u \) and \( v \). Then, we choose a neighbour \( s \) of \( u \). There are \([H \{ s \}, u, v]\) such graphs in \( G \). Since \( G \) is strongly regular, there are \( e - |C| \) choices for \( s \) in \( G \). However, these choices for \( s \) include the undesired possibility that a vertex of \( H \) is chosen. We eliminate the counts of graphs for these choices by subtracting \([H \{ s \} + u x + x v, u, v, x]\) for each vertex \( x \) in the set \( V(H) \) that is not \( u, v \), any of the common neighbours of \( u \) and \( v \), nor a second vertex in the same orbit under the action of \( \text{Aut}(H \{ s \})_{(u,v)} \). These vertices \( x \) are precisely the vertices of \( X \).

- **Case 2 - vertex \( u \) is not adjacent to vertex \( v \) in \( H \):**

  1. Either, we choose a copy of \( H \{ s \} \) with distinguished vertices \( u \) and \( v \) and strictly no edge \( uv \). Then, we choose a neighbour \( s \) of \( u \).

     a. There are \([H \{ s \}, u, v] - [H \{ s \} + u v, u, v]\) such graphs in \( G \). Since \( G \) is strongly regular, there are \( f - |C| \) choices for \( s \) in \( G \). Similar to the last case, we eliminate the counts of graphs for choices for \( s \) where \( s \in V(H) \) by subtracting \([H \{ s \} + u x + x v, u, v, x]\) for each vertex \( x \in X \).
2. Or we choose a copy of $H\backslash\{s\}$ with distinguished vertices $u$ and $v$ and the edge $uv$ added. Then, we choose a neighbour $s$ of $u$.

- There are $[H\backslash\{s\} + uv, u, v]$ such graphs in $G$. Since $G$ is strongly regular, there are $e - |C|$ choices for $s$ in $G$. Once again, we eliminate the counts of graphs for choices for $s$ where $s \in V(H)$ by subtracting $[H\backslash\{s\} + uv + ux + xv, u, v, x]$ for each vertex $x \in X$.

By collecting terms and then solving for $[H]$, we arrive at the desired expression in each case.

If $f < |C|$, then no graphs $H$ with strictly no edge $uv$ occur in $G$ and all the counts of graphs in Case 2.1 are zero. Thus, $[H\backslash\{s\}, u, v] = [H\backslash\{s\} + uv, u, v]$ and the expression for Case 2 simplifies to that of Case 1.

If $e < |C|$, no graphs $H$ with the edge $uv$ occur in $G$ and thus the terms containing counts of graphs with the added edge $uv$ are zero. The expression for Case 2 simplifies to $(f - |C|)[H\backslash\{s\}, u, v] - \sum_{x \in X}[H\backslash\{s\} + ux + xv, u, v, x]$.

Thus, under these conditions, these equations are still correct. \hfill \Box

We note that if the addition of an edge results in a subgraph of $G$ that does not satisfy the values $e$ and $f$ then the counts of that subgraph will be zero and thus the term corresponding to that subgraph will contribute zero to the equation.

5.2 Subgraph counts in terms of SR parameters

We give our algorithm which recursively finds an expression for the counts of subgraphs of a strongly regular graph. Where possible, the method \textsc{findEx} in Algorithm 3 takes a subgraph $H$ and outputs $[H]$ in terms of the variables $n, r, e,$ and $f$ in the following way:

- If $H = K_1$ then output $n$ since $[H] = n$.

- If $H$ has a vertex of degree one, express $[H]$ using Theorem 26. Call \textsc{findEx} on all graphs that appear in the expression.

- If $H$ has a vertex $s$ of degree two that is not a cut-vertex of $H$, express $[H]$ using Theorem 27. Call \textsc{findEx} on all graphs that appear in the expression.

- If no suitable vertex of $H$ is found then output the variable $[H]$.

This algorithm terminates since the graphs at each step of the recursion have one less vertex (and in the worst case, two more edges). We note that there may be many possible choices for a vertex of degree one or two. In our implementation we prefer to first select degree one vertices if they are present in $H$, but this is not necessary. This decision reduces the number of intermediate graphs possible by never introducing graphs with more than one vertex of degree one (with the exception of the path graphs). For the same reason, we insist that the chosen $s$ of degree 2 has a neighbour of degree at least 3 where possible (this is not possible when $H$ is a cycle).
Algorithm 3

for all graphs $GG$ in File do
    Print [ name of $GG$] = findEx($GG$)
end for

description Returns a list of vertices $oList$ where $oList[v]$ contains the canonical representative of the orbit of $v$ under the action of the point-wise stabilizer of $set$ in $\Gamma$.

function FINDORB($\Gamma$, $set$)
    $oList := []$
    for each orbit $p$ of the vertices in the stabilizer of $set$ in $\Gamma$ do
        for each vertex $v$ in $p$ do
            $oList[v] :=$ the first vertex occurring in $p$
        end for
    end for
    Return $oList$
end function

description Returns the value of $|\Gamma v_1| \cdot \cdots \cdot |\Gamma_{\{v_1,\ldots,v_{i-1}\}} v_i|$ where $set = \{v_1,\ldots,v_i\}$.

function ORBITSz($\Gamma$, $set$)
    if $v$ is the only vertex in $set$ then
        Find the orbit $orb$ of vertex $v$ in $\Gamma$
        Return (size of $orb$)
    else
        Remove the first vertex $v$ from $set$
        Find the orbit $orb$ of vertex $v$ in $\Gamma$
        $\Gamma :=$ the stabilizer of $v$ in $\Gamma$
        Return (size of $orb$)$\times$ORBITSz($\Gamma$, $set$)
    end if
end function

function FINDEx($G$)
    if $G$ is $K_1$ then
        Print $n$
    else
        if $\exists$ a vertex $s$ of $G$ with degree 1 then
            $u :=$ neighbour of $s$ in $G$
            Print $1/\text{ORBITSz}$(Aut($G$), $\{u,s\}$) $\times$ $(r - \text{(degree of } u) - 1)$
            Remove edge $us$ from $G$
            Remove vertex $s$ from $G$
            $orb :=$ FINDORB(Aut($G$), $\{u\}$)
            Print ORBITSz(Aut($G$), $\{u\}$)
            FINDEx($G$)
            $added := \{\}$
        end if
        for each vertex $x$ in $G$ do
            if $x \neq u$ and $ux$ is not an edge of $G$ then
                if $orb[x]$ not in $added$ then
                    Print $\text{ORBITSz}$(Aut($G$), $\{u\}$)
                    FindEx($G$)
                    $added := \{orb[x]\}$
                end if
            end if
        end for
    end if
end function
5.2. SUBGRAPH COUNTS IN TERMS OF SR PARAMETERS

Add \( \text{orb}[x] \) to \( \text{added} \)
\[
G' := G + ux
\]
Print \(-\text{ORBITSZ}(\text{Aut}(G'), \{u, x\})\)
FIND\text{EX}(G')
end if
end if
end for

else
if \( \exists \) a vertex \( s \) of \( G \) with degree 2 that is not a cut vertex then
Let \( u \) and \( v \) be the neighbours of \( s \) in \( G \)
Print \( 1/\text{ORBITSZ}(\text{Aut}(G), \{u, v, s\}) \)
Remove edges \( us \) and \( sv \) from \( G \)
Remove vertex \( s \) from \( G \)
\( \text{CNs} := \) number of common neighbours of \( u \) and \( v \) in \( G \)
\( \text{orb} := \text{FINDORB}(\text{Aut}(G), \{u, v\}) \)
if \( uv \) is an edge of \( G \) then
Print \((e - \text{CNs}) * \text{ORBITSZ}(\text{Aut}(G), \{u, v\})\)
else
Print \((f - \text{CNs}) * \text{ORBITSZ}(\text{Aut}(G), \{u, v\})\)
end if
FIND\text{EX}(G)
if \( uv \) is not an edge of \( G \) then
\[
G' := G + uv
\]
Print \(+ (e - f) * \text{ORBITSZ}(\text{Aut}(G'), \{u, v\})\)
FIND\text{EX}(G')
end if
\( \text{added} := \{\} \)
for each vertex \( x \) in \( V(G) \setminus \{u, v\} \) do
if \( ux \) is not an edge of \( G \) or \( xv \) is not an edge of \( G \) then
if \( \text{orb}[x] \) not in \( \text{added} \) then
Add \( \text{orb}[x] \) to \( \text{added} \)
\[
G' := G + ux + xv
\]
Print \(-\text{ORBITSZ}(\text{Aut}(G'), \{u, v, x\})\)
FIND\text{EX}(G')
end if
end if
end for
else
Print [name of \( G \)]
end if
end if
end function

As an example of the different possible non-isomorphic choices for a start vertex, we give the equations after just one step of the algorithm for the subgraph \( \Theta_{4,3,2} \):

\[
[\Theta_{4,3,2}] = e[L(5, 2)] + 2(f - e - 1)[\Theta_{3,3,2}] - [\begin{tikzpicture}[baseline=-0.5ex]
\draw (-0.5,0) -- (0,0) -- (0.5,0);
\begin{scope}[xshift=-1cm]
\draw (0,0) -- (0.5,0.5) -- (0,1) -- (-0.5,0.5) -- cycle;
\end{scope}
\end{tikzpicture}] - [\begin{tikzpicture}[baseline=-0.5ex]
\draw (0,0) -- (0.5,1) -- (1,0) -- cycle;
\end{tikzpicture}] - \frac{1}{2}[\begin{tikzpicture}[baseline=-0.5ex]
\draw (0,0) -- (0.5,0.5) -- (1,0);
\end{tikzpicture}];
\]
\[
[\Theta_{4,3,2}] = e[L(6, 1)] + 2(f - e)[\Theta_{4,2,2}] - 2[\Theta_{3,3,2}] - [\Theta_{5,2,1}] - \frac{1}{2}[\begin{tikzpicture}[baseline=-0.5ex]
\draw (0,0) -- (0.5,1) -- (1,0) -- cycle;
\end{tikzpicture}] - \frac{1}{2}[\begin{tikzpicture}[baseline=-0.5ex]
\draw (0,0) -- (0.5,0.5) -- (1,0);
\end{tikzpicture}] - [\begin{tikzpicture}[baseline=-0.5ex]
\draw (0,0) -- (0.5,1) -- (1,0) -- cycle;
\begin{scope}[xshift=-1cm]
\draw (0,0) -- (0.5,0.5) -- (1,0) -- cycle;
\end{scope}
\end{tikzpicture}];
\]
\[
[\Theta_{4,3,2}] = 7e[C_7] + (f - e - 2)[\Theta_{4,3,1}] - 2[\Theta_{5,2,1}] - [\begin{tikzpicture}[baseline=-0.5ex]
\draw (0,0) -- (0.5,1) -- (1,0) -- cycle;
\end{tikzpicture}].
\]
The final algorithm result is the same equation for all three choices of start vertex:

\[
[\Theta_{4,3,2}] = \frac{1}{2} r (-148 f + 216 f^2 + 103 e f - 8 r^2 e - 49 f^2 r - 10 f r^2 + 27 f^3 r + 3 f^2 r^2 - 4 f^4 r + 2 e r^3 \\
+ 14 r^2 e^2 - 2 e r^2 - 7 e r^3 + 9 r^2 e^3 - 17 r e^3 - 2 e^3 r^3 + e^2 r^4 + 2 e r^2 e^4 - 3 r e^4 - r e^5 + 6 e r + 67 r f \\
+ e^4 - 6 e^2 - 165 f^3 + 10 e^3 + 53 f^4 - 114 f r e + 33 f r^2 e + 32 f^2 r e - 9 f^2 r^2 e - 13 f^3 r e \\
- 9 f r^2 e^2 + 22 f r^2 e^2 + 2 f^4 r e - 5 f^3 r e^2 + 2 f e^2 r^3 + f r e^4 + f^3 r e^2 \\
- 3 f r e^3 - 11 f^5 + f^6 + 10 f e^2 - 17 f^3 e - 19 f e^2 + 2 f^4 e^2 + 3 r f^3 e^3 + f e^5 \\
- 34 f^2 e^2 + 21 f^4 e - 9 f^3 e^2 + 8 f e^3 - 2 f^2 e^3 - 3 r^2 e^3 - 3 f^5 e + 2 f^3 e^3 - 3 f^2 e^4) n \\
+ 12(28 + 5 e^2 - 26 f + 15 e + 6 f^2 - r - 11 e f) [K_4] - 6 e^2 f^3] .
\]

### 5.3 Equations for subgraphs of a strongly regular graph

We give the derived equations for the counts of subgraphs \( H \) of a graph \( \text{SR}(n, r, e, f) \) for which \( H \) has at most 8 edges and all vertices of degree at least two.

\[
[C_3] = \frac{1}{6} e r n , \\
[C_4] = \frac{1}{8} r (f r - r + e^2 - f + 1 - e f) n , \\
[C_5] = \frac{1}{10} r (r^2 f - f r - 2 e f + e r f - 3 e r + 5 e + e^3 - 2 f e^2 - 2 f^2 + e f^2) n , \\
[\Theta_{2,1,1}] = \frac{1}{4} (e - 1) e r n , \\
[C_6] = \frac{1}{12} r (-r^2 f^2 + 4 - 6 e^2 r + 10 f r - 5 r^2 f - 6 e f - 6 r - 4 e - 6 f - 2 e r f^2 + 4 e f^2 + 2 e r f + 15 e^2 \\
+ 2 r^2 + f^2 + e r^2 - f e^2 - 3 e f^2 + 3 r e f^3 + r e f^3 + e r^3 + 4 f^3 - 3 f e^3 + 3 f^2 e^2) n , \\
[C_{3,3}] = \frac{1}{8} e r (e r - 4 e + 2) n , \\
[\Theta_{3,2,1}] = \frac{1}{2} e r (f r - r + e^2 - f - e f + 3 - 2 e) n , \\
[\Theta_{2,2,2}] = \frac{1}{12} r (r^2 f^2 - f^2 - 3 f r + 3 f - e f^2 + 2 r - 2 - 3 e^2 + 3 e f + e^3) n , \\
[C_7] = \frac{1}{12} r (4 r^2 f^2 - 10 r^2 f - 10 e^3 r - 42 e r - 7 f r + 12 r^2 f - 35 e f + 56 e - 3 e r f^2 + 10 e f^2 + 43 e r f \\
- 28 e^2 + 35 e^3 + 7 f^2 - 12 e r^2 f + 12 e r^2 e - 35 e f^2 + 10 e r^2 - 6 f r^3 + 3 e r^3 + 6 e f^3 + f e r^3 \\
+ f e^2 r^2 + r^2 f^3 + r f^3 - 2 f^3 - 2 e r^2 f^2 - 3 f^2 e r^2 + 4 f e^3 + 9 f^2 e^2 - f^2 r^3 + f r^4 + e f^4 \\
+ e^5 + f^4 - r f^4 + 6 f^2 e^3 - 4 f^3 e^2 - 4 f e^4) n , \\
[\Theta_{2,2,2,1}] = \frac{1}{12} (e - 2)(e - 1) e r n , \\
[C_{4,3}] = \frac{1}{4} e r (r^2 f^2 - r^2 - 5 f r - e r f + 5 r + e^2 r + 4 f + 4 e f - 10 + 8 e - 6 e^2) n , \\
[\bigwedge_{3,3}] = \frac{1}{2} (e - 1)^2 e r n - 12[K_4] , \\
[\Theta_{3,3,1}] = \frac{1}{4} r (2 f e r^2 + 4 - 5 r - 2 e^3 - 2 f e^2 + 4 e f^2 + 2 e r f - 7 e f - 2 e - 4 r f^2 + 3 f^2 - 4 e^3 + 11 e^2 \\
+ 9 f r - 7 f - 2 e^2 r - 2 r f + r^2 + e^4 - 2 e f^2 + f^2 e^2 + r^2 f^3) n + 6[K_4] .
\]
\[ [C_{3-3}] = \frac{1}{5}er(2er^2 - 4er - 2e^2 - 4 + 12e)n + 6[K_4], \]

\[ [\Theta_{4,2,1}] = \frac{1}{2}r(-2f^2 + r^2 f + 2f + f^2 - 3fr + erf - 7 - 3e^2 - 11e - 3er + 2r + e^3 - rf^2 + ef^2)n + 12[K_4], \]

\[ [\Theta_{3,2,2}] = \frac{1}{4}r(e f^3 - rf^3 + f^3 r + 6e f + 5fe^2 - 2ef^2 - 5erf - 11e - f^2 - 5e^3 - rf^3 + 7e^2 + fr - e^2 r + 5er - r^2 f + e^4 + f^3 - f^2 e^2 + r^2 f^2)n, \]

\[ [\Theta_{3,3,2}] = \frac{1}{4}r(-3r^2 f^2 + rf^2 - 2e^3 r - 13er + 12e^2 r - fr + rf^2 - 17ef + 32e + 21erf - 46e^2 + 23e^3 + f^2 - 6er^2 f - 6fe^2 r - 3fe^2 + er^2 - 2r^2 f^3 + 4rf^3 - 8e^4 - 2f^3 + 2er^2 f^2 - 3f^2 e^2 + 2fe^3 r + 10f e^3 - f^2 e^2 + f^2 r^3 - e^4 + e^5 - f^4 + rf^4 + 2f^3 e^2 - 2fe^4)n + 6(e - f - 4)[K_4], \]

\[ [C_8] = \frac{1}{16}r(13r^2 f^2 - 49r f^2 + 36 + 32er - 168e^2 r + 129fr - 79r^2 f - 41ef - 59r - 112e - 73f - 94erf + 96e f^2 - 16er f + 345e^2 + 28r^2 - 5e^3 - 112e^3 + 36f^2 - 2f^2 er^3 + 35er^2 f + 116f c^2 r - 88fe^3 + 30e^2 r^2 + 29fr s^5 - 6er f^3 - 19e^3 - 14fr^3 - 21fe^3 r^2 - 3r^2 f^3 + 13r^3 - 15e^4 r + 70e^4 - 11f^3 + 24er^2 f - 21f^2 e r^2 + 32fe^3 r - 120fe^3 + 66f c^2 e^2 + f^2 r^3 - 7f r^4 - 4er f^4 + 8e f^4 - 3f^2 e^2 r^2 + 3f^4 - r^2 f^4 - 2rf^4 + f e^4 + f c^3 r^2 - 4f^2 e^3 r + 3er^3 + 6f^3 e^2 r + f c^2 r^3 - f^2 r^4 + 16f^2 e^3 + f^3 r^3 - 18f^3 e^3 - 5f e^4 + e f^4 + e^6 + 10f^2 e^4 - 5f^4 e^2 + 5f^4 c^2 - 4f^3 e^3 - f^5 + 5f^5 r - f^5 c^3 e + 33)[K_4], \]

\[ [\Theta_{2,2,2,1}] = \frac{1}{48}r(r f^3 - f^3 - 6r f^2 + 6f^2 - ef^3 + 11fr - 11f + 6ef^2 - 6r + 6 + 11e^2 - 11ef - 6e^3 + e^4)n, \]

\[ [\Theta_{3,2,2,1}] = \frac{1}{4}r(e - 1)(e^2 - ef - 4e + fr - r + 5 - f)n + 12[K_4], \]

\[ [C_{4,4}] = \frac{1}{8}r(-6r^2 f^2 + 17r^2 f - 12 - 4er + 14e^2 r - 33fr + 12r^2 f + 19ef + 17r + 16 + 23f + 10er f^2 - 16e f^2 - 6er f - 51e^2 + 6v^2 + r^3 + 12f^2 + 2er f^2 - 14fe r^2 + 8fe^2 - 2e^2 r^2 - 2fr^3 + ef^3 + 2fe r^2 - r f^3 + e^4 r - 9e^4 + 3e^2 - 2er f^2 + f^2 e^2 r - 2fe^3 r + 12f e^3 - 4f c^2 e^2 + f^2 r^3)n + 12[K_4], \]

\[ [C_{5,3}] = \frac{1}{4}r(12fe r^2 - 5r^2 f - 10er f + 6ef + 4f r^2 + 10fr - 6f + 24 + 20e^2 - 56e + 21er - 6r - 8e^3 + 5r f^2 - 4e f^2 - 3er f^2 - r^2 f^2 - 2fe^2 r + er f^2 + e^3 r + er f^2 + fr^3)n - 36[K_4], \]

\[ [\Theta_{4,3,1}] = \frac{1}{4}r(-3r^2 f^2 - rf f^2 - 4e^2 r - 21er + 4e^2 r - 3fr + 4r f^2 - 28ef + 40e - 5er f^2 + 9e f^2 + 26er f - 32e^3 + 27e^3 + 3e^3 - 4er f^2 - 16fe^2 + 3er^2 - fr^3 + 2er f^3 - 4ef^3 + fe^2 r^2 - r^2 f^3 + 4rf^3 - 6e^4 - 3f^3 - 4f^2 e^2 r + 2fe^3 r + f c^3 + 9f^2 e^2 + f^2 r^3 + e^5 + 3f^2 e^3 - f^3 e^2 - 3e f^4)n + 12(5e - 4f - 2)[K_4], \]

\[ [\Theta_{4,2,2}] = \frac{1}{4}r(-6r^2 f^2 + 18er f^2 - 3e^2 r - 12 + 13e^2 r - 34fr + 11r^2 f + 22ef + 16r + 12e + 24f + 11er f^2 - 20ef^2 - 8er f - 53e^2 - 24r^2 + 23e^3 - 13f^2 - er^2 f - 12f^2 e r + 15fe^2 - fr^3 - er f^3 + 4ef^3 + f e^2 r^2 - r^2 f^3 - r f^3 - 7e^4 + 2f^3 - f^2 e^2 r + f e^3 r + 11fe^3 - 7f^2 e^2 + f^2 r^3 - ef^4 + e^5 - f^4 + rf f^2 + 2f^3 e^2 - 2fe^4)n + 12(e - f - 2)[K_4], \]

\[ [\Theta_{5,2,1}] = \frac{1}{4}r(-r^2 f^2 + 2r f^2 + 28 + 8er - 6e^2 r + 18fr - 7r^2 f - 4ef - 12r - 40e - 14f \]
CHAPTER 5. COUNTING SUBGRAPHS OF STRONGLY REGULAR GRAPHS

\[
- 2erf^2 + 2ef^2 - 2erf + 27e^2 + 2r^2 - 4e^3 - f^2 + erf^2 + 3fe^2 + fr^3 - ef^3 \\
+ rfr^3 + e^4 - r^3 - 3fe^3 + 3f^2e^2)n + 24(2e - 3 - f)[K_4],
\]
\[
\mathbf{[\star \star \star \star]} = \frac{1}{2}er(-er + erf + 9e + e^3 - 2ef - fe^2 + f - 4 - 4e^2)n + 12(2 - e + f)[K_4],
\]
\[
\mathbf{[\star \star \star \star \star]} = e(e - 1)(e^2 - 2e - ef + fr - r - 2f + 5)n + 24(2 - 2e + f)[K_4],
\]
\[
\mathbf{[\star \star \star \star \star \star]} = \frac{1}{2}er(-ef - 2fr - 4e^2 + 8e - 6 + rf^2 - 2ef^2 + r + 2ef^2 + 3f)n \\
+ 12(1 - e + f^2 + 2e - 2ef)[K_4],
\]
\[
\mathbf{[\star \star \star \star \star \star \star]} = \frac{1}{3}er(e - 1)(ef - 4e + fr + 2 - f^2)n + 6(5 - r - f - 2ef + e^2 + f^2)[K_4],
\]
\[
\mathbf{[\star \star \star \star \star \star \star \star]} = \frac{1}{4}er(e - 1)(er - 6e + 4)n + 12[K_4],
\]
\[
\mathbf{[\star \star \star \star \star \star \star \star \star]} = \frac{1}{4}er(e - 1)(er - 4e + 2)n + 12(2 - e)[K_4],
\]
\[
\mathbf{[\star \star \star \star \star \star \star \star \star \star]} = \frac{1}{2}er(-f^2 + 5f + rf^2 + 2r - 3fr - 4e^2 - ef^2 + e^3 - 8 + ef + 7e)n + 24(f - e)[K_4],
\]
\[
\mathbf{[\star \star \star \star \star \star \star \star \star \star \star]} = \frac{1}{4}er(4ef^2 - 6ef + 2f^2 + 18fr - 18f + 32 + 26e^2 - 36e - r^3 + 5er - 16r - 5e^2r \\
- 5r^2f - 66e^3 + 5r^2 - 2r^2f + 2ef)^2 + e^2r^2 + fr^3)n + 12(3e - 2f - 2)[K_4],
\]
\[
\mathbf{[\star \star \star \star \star \star \star \star \star \star \star \star]} = \frac{1}{8}er(3e^3 - 5er^2 + 16er - 4e^2r - 4erf - 2f + 6ef + 4ef^2 + 16 + 28e^2 - 52e - 4e^3)n \\
+ 6(e - f - 4)[K_4],
\]
\[
\mathbf{[\star \star \star \star \star \star \star \star \star \star \star \star \star]} = \frac{1}{4}er(-er + erf - fe^2 + 7e + e^3 - ef - 2 - 4e^2)n + 12[K_4] - 2\mathbf{[\star \star \star \star \star]},
\]
\[
\mathbf{[\star \star \star \star \star \star \star \star \star \star \star \star \star \star]} = 6(e - 2)[K_4],
\]
\[
\mathbf{[\star \star \star \star \star \star \star \star \star \star \star \star \star \star \star]} = \frac{1}{8}r(f - 2)(rf^2 - f^2 - erf^2 + 4ef - 3fr + 3f - fe^2 + 2e^3 + 2r - 2 - 4e^2)n + 3(e - f)^2[K_4] \\
- \mathbf{[\star \star \star \star \star \star]},
\]

(5.1)

Next, we give the derived equations for subgraphs \( H \) of a graph \( SR(n, r, 0, f) \) for which \( H \) has at most 9 edges and all vertices of degree at least two. The variables \([K_4], \mathbf{[\star \star \star \star]}, \text{and } \mathbf{[\star \star \star \star \star]}\) are given the value of zero since their corresponding graphs do not appear in a graph with \( e = 0 \).

\[\mathbf{[C_4]} = \frac{1}{8}(f - 1)(r - 1)rn,\]
\[\mathbf{[C_5]} = \frac{1}{10}fr(r - 1)(r - f)n,\]
\[\mathbf{[C_6]} = \frac{1}{12}r(r - 1)(r^2f + 2r - 4fr - f^2r - 4 + 6f + f^3 - f^2)n,\]
\[\mathbf{[\Theta_{2,2,2}]} = \frac{1}{12}(f - 2)(f - 1)(r - 1)rn,\]
\[\mathbf{[C_7]} = \frac{1}{12}fr(r - 1)(r - f)(f^2 - 2f + r^2 - 5r + 7)n,\]
\[\mathbf{[\Theta_{3,3,1}]} = \frac{1}{4}r(r - 1)(f - 1)(fr - 3f - r + 4)n,\]
\[\mathbf{[\Theta_{3,2,2}]} = \frac{1}{4}fr(r - 1)(f - 1)(r - f)n,\]
\[\mathbf{[\Theta_{3,3,3}]} = \frac{1}{4}fr(r - 1)(f - r)(-fr + f^2 + 2f - 1)n,\]
\[\mathbf{[C_8]} = \frac{1}{16}(r - 1)(fr^4 - 6r^3f - r^3f^2 + 23r^2f + r^2f^3 - 5r^2 - 56fr + 23r - f^4r + 13f^2r - 2f^3r \\
+ 73f - 36f^2 - 3f^4 - 36 + 11f^3 + f^5)n,\]
\[\mathbf{[\Theta_{2,2,2,2}]} = \frac{1}{48}(f - 3)(f - 2)(f - 1)(r - 1)rn,\]
5.3. EQUATIONS FOR SUBGRAPHS OF A STRONGLY REGULAR GRAPH

\[ [C_{4,4}] = \frac{1}{5} r(r - 1)(f - 1)(r^2 f - f^2 - 5fr + 11f - r^2 + 5r - 12)n, \]
\[ [\Theta_{4,3,1}] = \frac{1}{2} fr(r - 1)(r - 3)(f - 1)(r - f)n, \]
\[ [\Theta_{4,2,2}] = \frac{1}{4} r(r - 1)(f - 1)(f^3 - f^2 r - f^2 + r^2 f - 6fr + 12f + 4r - 12)n, \]
\[ [\Theta_{4,4,1}] = \frac{1}{4} fr(r - 1)(f - r)(f^2 r - 2f^2 + 3fr - 3f - r^2 f + 1)n, \]
\[ [\Theta_{4,3,2}] = \frac{1}{4} fr(r - 1)(r - f)(f^3 - f^2 r + f^2 + r^2 f - 6fr + 10f + 2r - 6)n, \]
\[ [\Theta_{3,3,3}] = \frac{1}{12} (r - 1)(r^3 f^2 - r^2 + 6r^2 f - 9r^2 f^2 - 2r^2 f^3 - 42fr + 31f^2 r + 3f^4 r + 4f^3 r + 15r \]
\[ - 6f^4 - 36 - 54f^2 - f^3 + 81f + 10f^3)n, \]
\[ [C_9] = \frac{1}{18} fr(r - 1)(r - f)(f^4 - 4f^3 + r^2 f^2 - 5f^2 r + 21f^2 - 11r^2 f + 48fr - 81f + r^4 - 7r^3 \]
\[ + 30r^2 - 72r + 81)n, \]
\[ [C_{5,4}] = \frac{1}{4} fr(r - 1)(f - 1)(f - r)(2f - r^2 + 5r - 10)n, \]
\[ [\Theta_{5,3,1}] = \frac{1}{2} r(r - 1)(f - 1)(f^3 r - 3f^3 - r^2 f^2 + 10f^2 + r^3 f - 7r^2 f + 25fr - 45f + 2r^2 - 15r + 36)n, \]
\[ [\Theta_{5,2,2}] = \frac{1}{4} fr(r - 1)(f - 1)(r - f)(f^2 - 2f + r^2 - 7r + 13)n. \]

(5.2)

Note that if a graph \( H \) has a vertex of degree 4, the corresponding equation may not contain the factor \((r - 3)\). However, if such a subgraph \( H \) in a graph \( G \) with \( r = 3 \) is considered then another parameter of \( G \), \( e \) or \( f \), will always give a zero factor in the equation for \( H \). For example, consider the equation for the graph \( \Theta_{2,2,2,2} \) in the equations for a \( \text{SR}(n, r, 0, f) \):

\[ [\Theta_{2,2,2,2}] = \frac{1}{15}(f - 3)(f - 2)(f - 1)(r - 1)n. \]

The factor \((r - 3)\) does not occur but \((f - 3)\), \((f - 2)\), and \((f - 1)\) do occur. Such a value for \( f \) is only possible if \( r \) is at least 4.

We further simplify and give the derived equations for subgraphs \( H \) of a Moore graph \( \text{SR}(n, r, 0, 1) \), for which \( H \) has at most 11 edges and all vertices of degree at least two. The variables \([K_4],[K_5],[K_6],[K_7],[K_8],[K_9],[K_{10}],[K_{11}],[K_{12}],[K_{13}],[K_{14}],[K_{15}],[K_{16}],[K_{17}]\), and \([K_{3,3}]\) are given the value of zero since their corresponding graphs do not appear in a graph with \( e = 0 \).

\[ [C_5] = \frac{1}{10}(r - 1)^2rn, \]
\[ [C_6] = \frac{1}{12} r(r - 2)(r - 1)^2n, \]
\[ [C_7] = \frac{1}{11} r(r - 2)(r - 3)(r - 1)^2n, \]
\[ [\Theta_{3,3,2}] = \frac{1}{4} r(r - 2)(r - 1)^2n, \]
\[ [C_8] = \frac{1}{16} r(r - 2)(r^2 - 4r + 5)(r - 1)^2n, \]
\[ [\Theta_{4,4,1}] = \frac{1}{4} r(r - 1)^2(r - 2)^2n, \]
\[ [\Theta_{4,3,2}] = \frac{1}{3} r(r - 2)(r - 3)(r - 1)^2n, \]
\[ [\Theta_{3,3,3}] = \frac{1}{12} r(r - 2)(r - 3)(r - 1)^2n, \]
\[ [C_9] = \frac{1}{18} r(r - 2)(r^3 - 5r^2 + 10r - 9)(r - 1)^2 n, \]
\[ [\Theta] = \frac{1}{8} r(r - 2)(r - 1)^2 n, \]
\[ [C_{5,5}] = \frac{1}{5} r(r - 2)(r - 3)(r - 1)^3 n, \]
\[ [\Theta_{5,4,1}] = \frac{1}{2} r(r - 1)^2 (r - 2)^3 n, \]
\[ [\Theta_{5,3,2}] = \frac{1}{2} r(r - 2)(r - 1)^2 (r - 3)^2 n, \]
\[ [\Theta_{4,4,2}] = \frac{1}{4} r(r - 2)(r^2 - 5r + 7)(r - 1)^2 n, \]
\[ [\Theta_{4,3,3}] = \frac{1}{4} r(r - 2)(r - 3)(r - 4)(r - 1)^2 n, \]
\[ [C_10] = \frac{1}{20} r(r - 2)(r - 3)(r^3 - 3r^2 + r - 1)(r - 1)^2 n, \]
\[ [\Theta_{3,3,3,2}] = \frac{1}{12} r(r - 2)(r - 3)(r - 1)^2 n, \]
\[ [\Theta_{6,4,1}] = \frac{1}{2} r(r - 2)(r - 3)^2 (r - 1)^3 n, \]
\[ [\Theta_{5,4,2}] = \frac{1}{2} r(r - 2)(r^3 - 7r^2 + 18r - 17)(r - 1)^2 n, \]
\[ [C_{5-5}] = \frac{1}{8} r(r - 2)(r^3 - 4r^2 + r + 8)(r - 1)^2 n, \]
\[ [\Theta_{5,5,1}] = \frac{1}{2} r(r - 2)(r - 3)(r^2 - 3r + 1)(r - 1)^2 n, \]
\[ [\Theta_{5,4,2}] = \frac{1}{2} r(r - 2)(r - 3)(r^2 - 5r + 7)(r - 1)^2 n, \]
\[ [\Theta_{5,3,3}] = \frac{1}{4} r(r - 2)(r - 3)(r^2 - 5r + 8)(r - 1)^2 n, \]
\[ [\Theta_{4,4,3}] = \frac{1}{4} r(r - 2)(r - 3)(r^2 - 7r + 13)(r - 1)^2 n, \]
\[ [C_{11}] = \frac{1}{22} r(r - 2)(r - 3)(r^4 - 4r^3 - 2r^2 + 16r + 2)(r - 1)^2 n. \quad (5.3) \]

### 5.4 Subgraph counts for some specific graph parameter sets

We give the non-zero counts of subgraphs up to 11 edges and all cycles up to 17 edges, for a SR(3250, 57, 0, 1), the only Moore graph where existence is still in question.

\[ [C_5] = 58094400, \quad [\Theta] = 862701840000, \]
\[ [C_6] = 26626660000, \quad [\Theta] = 862701840000, \]
\[ [C_7] = 123243120000, \quad [\Theta] = 431350920000, \]
\[ [\Theta_{3,3,2}] = 7987980000, \quad [C_{6-5}] = 1327698131760000, \]
5.4. SUBGRAPH COUNTS FOR SOME SPECIFIC GRAPH PARAMETER SETS

\[
\begin{align*}
\Theta_{4,4,1} &= 60429068700000, \\
\Theta_{5,4,2} &= 43933890000000, \\
\Theta_{5,5} &= 826701840000, \\
\Theta_{3,3,3} &= 1437836400000, \\
\Theta_{5,3,3} &= 300895559600000, \\
\Theta_{5,5} &= 43933890000000, \\
\Theta_{4,3,2} &= 1298129482680000, \\
\Theta_{4,3,3} &= 862701840000, \\
\Theta_{5,4,2} &= 1234957683960000, \\
C_8 &= 3993990000, \\
C_9 &= 12077825760000, \\
C_{10} &= 23732288580000, \\
C_{11} &= 862701840000, \\
C_{12} &= 22861598760000, \\
C_{13} &= 1328129482680000, \\
C_{14} &= 2608810364160000, \\
C_{15} &= 2611414445640000, \\
C_{16} &= 13599903240, \\
C_{17} &= 470572695936.
\end{align*}
\]

We give the non-zero counts of subgraphs up to 9 edges and some graphs on 10 edges, for a
SR(162, 21, 0, 3), where existence is still in question.

\[
\begin{align*}
\Theta_{4,3,2} &= 48327279000000, \\
\Theta_{5,3,2} &= 46585899360000, \\
\Theta_{4,4,2} &= 23732288580000, \\
\Theta_{4,3,3} &= 22861598760000, \\
\Theta_{5,5} &= 15140589832368000, \\
\Theta_{3,3,3} &= 1437836400000, \\
\Theta_{5,4,1} &= 862701840000, \\
\Theta_{4,4,3} &= 878677800000, \\
C_4 &= 17010, \\
C_5 &= 367416, \\
C_6 &= 5420520, \\
\Theta_{2,2,2} &= 11340, \\
\Theta_{3,3,3} &= 90804240, \\
\Theta_{3,3,1} &= 1258740, \\
\Theta_{3,2,2} &= 15092198505674986055199840000, \\
\Theta_{3,3,2} &= 891902340, \\
\Theta_{4,4,3} &= 7348320,
\end{align*}
\]

\[
\begin{align*}
\Theta_{3,3,2} &= 27120812040, \\
\Theta_{3,3,3} &= 624607200, \\
\Theta_{3,3,1} &= 569494800, \\
\Theta_{3,2,2} &= 62001450, \\
\Theta_{3,3,2} &= 8298090360, \\
\Theta_{4,4,2} &= 31085230680, \\
\Theta_{4,4,3} &= 31085230680, \\
\Theta_{4,4,2} &= 30715977600, \\
\Theta_{4,4,3} &= 13881895020, \\
\Theta_{4,4,3} &= 13599903240, \\
\Theta_{4,4,3} &= 470572695936.
\end{align*}
\]
We give the non-zero counts of subgraphs up to 9 edges and some graphs on 10 edges, for a SR(176, 25, 0, 4), where existence is still in question.

\[
\begin{align*}
[C_4] & = 39600, & [\Theta_{4,3,2}] & = 7380172800, \\
[C_5] & = 887040, & [\Theta_{3,3,3}] & = 1109275200, \\
[C_6] & = 15998400, & [C_9] & = 139754630400, \\
[\Theta_{2,2,2}] & = 52800, & & \\
[C_7] & = 326304000, & [C_{5,4}] & = 3339705600, \\
[\Theta_{3,3,1}] & = 5306400, & [\Theta_{5,3,1}] & = 6304161600, \\
[\Theta_{3,2,2}] & = 6652800, & [\Theta_{5,2,2}] & = 3133468800, \\
[\Theta_{3,3,2}] & = 170755200, & [\Theta_{5,3,2}] & = 301593600, \\
[C_8] & = 6752394000, & [\Theta_{5,5}] & = 46083945600, \\
[\Theta_{2,2,2,2}] & = 13200, & [\Theta_{5,4,1}] & = 173979590400, \\
[C_{4,4}] & = 60033600, & [\Theta_{5,3,2}] & = 174511814400, \\
[\Theta_{4,3,1}] & = 29273200, & [\Theta_{4,4,2}] & = 7908183600, \\
[\Theta_{4,2,2}] & = 133372800, & [\Theta_{4,3,3}] & = 79095139200, \\
[\Theta_{4,4,1}] & = 158400, & [C_{10}] & = 2919185660160. \\
\end{align*}
\]

Non-existence would follow if a count came out to be something other than a non-negative integer. We checked the not-known-to-exist cases appearing in Table 2.4 (strongly regular graph parameters with $n \leq 100$) for the counts of subgraphs up to 9 edges, but did not find anything. The counts of subgraphs for other strongly regular graphs not-known-to-exist could also be listed here but we simply leave these few as examples of the value of this result.

### 5.5 Remarks

Equations for any subgraph with a vertex of degree 1 or 2 (where the latter is not a cut-vertex) can be produced using the procedure presented in this thesis. For example, we give the equations for the counts of some graphs with a degree one vertex (or vertices) in a SR($n, r, e, f$):

\[
\begin{align*}
[P_2] & = \frac{1}{2}rn, \\
[P_3] & = \frac{1}{2}(r-1)rn, \\
[P_4] & = \frac{1}{2}r(r^2-2r+1-e)n, \\
[P_5] & = \frac{1}{2}r(r^3-3r^2+4r-fr-2er-2+3e+f-e^2+ef)n, \\
[L(3, 1)] & = \frac{1}{2}(r-2)ern, \\
[L(3, 2)] & = \frac{1}{2}er(r^2-3r+4-2e)n.
\end{align*}
\]
If it is so desired, the algorithm presented need not be used recursively and the counts of subgraphs can be found in terms of the counts of subgraphs with one less vertex. For example,

\[
[\Theta_{4,3,2}] = 7e[C_7] + (f - e - 2)[\Theta_{4,3,1}] - 2[\Theta_{5,2,1}] - \[
\]

This can lead to results that relate the counts of two target subgraphs to each other.

We give the counts of subgraphs in a strongly regular graph up to 9, 10, or 11 edges depending on the restrictions specified. Subgraph counts for graphs of these sizes could be computed in a short amount of time. Unfortunately, the number of graphs in intermediate steps of the recursion grows rapidly as the size of the subgraph increases. There are many options for producing results for subgraphs with greater than 11 edges. The algorithm could easily be adapted to store the resulting expressions of smaller graphs as counts are being computed. This way, instead of making a recursive call for every graph with fewer vertices as it is encountered, it simply looks up the expression for those graphs. There are also options to parallelize the algorithm and make use of a multiple processor grid system.

It seems that the equations given in this chapter for counts of subgraphs in strongly regular graphs contain the same information as the spectral moment equations of Chapter 3 when applied to the same graphs. This seems plausible since both types of equations count subgraphs in terms of the eigenvalues of the parent graph. However, it is not yet clear exactly how these equations are related. It would be of interest to prove that they contain the same information, if indeed that is the case. Otherwise, they could potentially lead to a contradiction that shows non-existence of some strongly regular graphs.

It also seems possible that looking at relations between subgraphs (combinations of \(\Theta_{2,...,2}\) graphs for example) for specific strongly regular graph parameter sets could be used to establish non-existence results. These uses for our equations have yet to be fully explored.
Chapter 6

Conclusion

6.1 Results summary

In this thesis, we made progress towards a goal of spectral graph theory: to relate the eigenvalues of a graph to the structural properties of the graph.

6.1.1 Spectral moment equations

We were able to extend the known spectral moment equations to walk lengths greater than 6. These equations relate the eigenvalues of a regular graph to the counts of small subgraphs. This information can be used in a variety of ways, but especially as a tool for studying regular graphs with a selective set of eigenvalues.

The most general version of this result is the spectral moments $w_\ell$ of an $(r+1)$-regular graph with $n$ vertices. The expressions for $w_\ell$ for $\ell = 0, 1, \ldots, 9$ are given as Equations (3.4) in Section 3.4. We gave examples for the equations when applied to graphs with girth 5 as Equations (3.5) and bipartite graphs as Equations (3.6), where certain contributors are known not to occur. Our equations for bipartite graphs verify the results of [63] and [174] by reproducing Equations (1.1) and Lemma 10 respectively. We reiterate that knowing any information about the graphs being studied that limits the possibilities for subgraphs can lead to equations with significantly fewer contributors and thus there are a lot more specializations that these equations can be tailored to.

The equations we have presented provide much more information than inequalities presented in [176]. However, if preferred they can also be used to derive inequalities simply by dropping some of the terms. Our equations are well suited to this use, since all terms are positive, unlike the equations in [195] which employ the principle of inclusion-exclusion and hence have many terms of opposite signs.

In Chapter 4, these equations are employed to eliminate feasible spectra for graphs that are vertex-transitive. This lead to our result of finding all connected 4-regular bipartite integral Cayley graphs.

Other uses of our equations have yet to be explored, but we are confident that they will prove useful in a variety of contexts. For example, they may help in constructing co-spectral graphs by identifying subgraphs for which the counts may differ.
6.1. RESULTS SUMMARY

6.1.2 Quartic integral graphs

As integral graphs are being recognized for their uses in modelling multiprocessor interconnection networks, quantum spin networks, and the like; it is timely for us to contribute a wealth of new 4-regular integral graphs.

Our results show that there are only 32 connected 4-regular integral Cayley graphs up to isomorphism. Table 4.3 lists the 17 graphs of the 32 which are bipartite and Table 4.7 gives the details of the 15 non-bipartite graphs. Most of the bipartite graphs are depicted in Figure 4.4. There are exactly 27 quartic integral graphs that are arc-transitive. We determined that 16 of the 27 graphs are bipartite; these appear in Table 4.3 and Table 4.5. Of these 27 graphs, 16 are Cayley graphs; these appear in Table 4.3 and Table 4.7.

We began with the 828 possible spectra from [176], and then narrowed our focus to a set \( \Xi \) of 59 candidates for vertex-transitive graphs; refer to Table 4.1 and Appendix A. Of these, we found 22 which are realized by Cayley graphs or arc-transitive graphs. In Section 4.4.7, by taking quotients, we found an additional 6 bipartite integral graphs that are neither arc-transitive nor Cayley, but realize a possible spectrum.

In Section 4.4.6, we show that \( G_{11} \) is the only 4-regular integral graph with 32 vertices that is vertex-transitive. The proof relies on the theory of quotient graphs and permutation group actions.

Overall, we found 9 bipartite quartic integral graphs (namely, \( G_{16}, G_{17}, F_4, F_5, M_4, M_5, M_7, M_8, M_9 \)) that realise spectra not previously known to be achieved. It is still an open problem, to determine whether the remaining possible spectra are realized by any 4-regular bipartite integral graphs.

6.1.3 Subgraph counts of strongly regular graphs

Our final contribution was to count the connected subgraphs \( H \) of a strongly regular graph in terms of the parameters \( n, r, e, \) and \( f \). For some graphs \( H \), we are able to express the number of copies of \( H \) in terms of \( n, r, e, \) and \( f \) alone. Counts of other graphs are in terms of smaller subgraphs with minimum degree 3, in addition to \( n, r, e, \) and \( f \).

We first proved results for determining the number of copies of a graph that there are in a strongly regular graph in terms of the counts of graphs with fewer vertices. Next, we designed an algorithm which applies these results recursively to graphs until a single vertex or a graph of minimum degree 3 remains. This method allowed us to derive equations for the counts of subgraphs \( H \) of a graph \( \text{SR}(n, r, e, f) \) for which \( H \) has all vertices of degree at least two. The equations for the counts of subgraphs \( H \) up to 8 edges in a strongly regular graph \( \text{SR}(n, r, e, f) \) are given as Equations (5.1) in Section 5.3. In the same section, the equations for the counts of subgraphs \( H \) up to 9 edges in a strongly regular graph \( \text{SR}(n, r, 0, f) \) are given as Equations (5.2). Finally, the equations for the counts of subgraphs \( H \) up to 11 edges in a strongly regular graph \( \text{SR}(n, r, 0, 1) \) are given as Equations (5.3).

The counts of subgraphs \( H \) with minimum degree 2 and up to 11 edges are given for the ‘missing’ Moore graph \( \text{SR}(3250, 57, 0, 1) \) as Equations (5.4) in Section 5.4. In the same section, the counts of subgraphs \( H \) with minimum degree 2 and up to 9 edges are given for the not-known-to-exist strongly regular graphs \( \text{SR}(162, 21, 0, 3) \) as Equations (5.5) and \( \text{SR}(176, 25, 0, 4) \) as Equations (5.6).
Again there is a lot of flexibility for the use of this result. Equations for any subgraph with a vertex of degree 1 or 2 (where the latter is not a cut-vertex) can be produced using the described procedure. If it is so desired, the algorithm presented need not be used recursively and the counts of subgraphs can be found in terms of the counts of subgraphs with one less vertex. For example,

$$[\Theta_{4,3,2}] = 7e[C_7] + (f - e - 2)[\Theta_{4,3,1}] - 2[\Theta_{5,2,1}] - \left[\begin{array}{c}
\end{array}\right].$$

This can lead to results that relate the counts of two target subgraphs to each other.

These and other uses for our equations have yet to be fully explored.

6.2 Future work

There are many unanswered questions arising from the topics discussed in this thesis that have yet to be addressed. We present a few of these problems here.

It is of interest to further investigate the 4-regular bipartite integral possible spectra of [176]. To our knowledge, the smallest quadruple for which the existence question is not completely solved is $[1, 7, 3, 2]$ with 28 vertices. In the more restricted vertex-transitive case (see Appendix A), all vertex-transitive graphs on at most 31 vertices can be found at the following location: [155] (the graphs with $n \leq 26$ have been verified at this stage). Given our result that there is a unique graph on 32 vertices, the smallest quadruple for which the existence question is not completely solved is $[4, 4, 4, 5]$ with 36 vertices. We know that there exists a graph $G_{12}$ satisfying these parameters, but it is not known whether or not this graph is the unique vertex-transitive graph with that spectrum. Our methods in Section 4.4.3 for finding the quotient of an existing graph can be broadened. We placed an upper bound of $2^{20}$ on the order of the automorphism group of a graph. This allowed our computation to complete in a reasonable time. However, it may have resulted in us missing some integral graphs that could be found by a more sophisticated analysis of the examples with large automorphism group. It might also be worth considering more general quotients than the ones we took. We only considered quotients by orbits of a cyclic group of automorphisms.

There is opportunity to apply the techniques of Section 4.4.6 to possible spectra of graphs with order a power of a prime $p$ or order $2pm$ where $m$ is a positive integer greater than 2. For example, the spectrum $[6, 0, 14, 0]$ in Appendix A has $42 = 2 \times 7 \times 3$ vertices. Using nauty’s genbg [142] to generate the set of all 4-regular bipartite multigraphs with 3 vertices in each part, we determined that there are only four integral graph candidates. The only thing left is to find the possible lifts from these candidates to a graph with the desired spectrum. Beyond these techniques, there is more information about quotient graphs that we could attempt to employ. In addition to the eigenvalues of the quotient graph $G/\Pi$ being a subset of the eigenvalues of the graph $G$, other results include that:

- $G$ and $G/\Pi$ have the same spectral radius [84, Cor 2.3] and
- the eigenvectors of $G/\Pi$ can be lifted to provide the eigenvectors of $G$ [84, p. 77].

There are also interlacing techniques, graph angle methods in relation to eigenspaces, and possibilities for vertex partitioning (in relation to quotient matrices), which we plan to exhaust in the near future to find other existing graphs.
6.2. FUTURE WORK

The new 4-regular integral graphs that we found that are co-spectral to other graphs are as follows: $G_{13}$ co-spectral to $I_{40,1}$ and $I_{40,2}$, $G_{15}$ to $I_{72,1}$, $H_{9}$ to $I_{20,8}$, $H_{12}$ to $I_{36,4}$, and $F_{3}$ to $M_{6}$. We also mention the co-spectral graphs among the known integral graphs: $G_{5}$ is co-spectral to $I_{16,2}$ and another graph appearing in [176], $G_{6}$ to $I_{18,2}$ and $I_{18,3}$, $G_{7}$ to $I_{24,1}$, $F_{1}$ to $I_{60,2}$, $H_{5}$ to $I_{12,6}$, and $J_{3}$ to $I_{30,5}$. There are many options to explore for determining co-spectral mates for graphs that we have found. For example, there are eigenvalue preserving switching techniques that we have yet to attempt: Seidel switching [163] and Godsil-McKay switching [88].

Given a graph $G$ with diameter $D$ and spectral radius $\lambda$, the second type mixed tightness $t_{2}(G)$ is defined as $(D + 1)\lambda$. It has been argued that graphs with small tightness $t_{2}$ are well suited for multiprocessor interconnection networks [64]. It would be of interest to extend the classification of integral graphs with small tightness presented in [64] by adding the graphs of this thesis that are well suited.

We mentioned that from the equations of Chapter 3, we were able to reproduce the reduced possible spectra list for 4-regular bipartite integral graphs including the values of $[C_{4}]$ and $[C_{6}]$ for all entries. In addition, we are able to determine $[C_{8}]$ in many cases. It would be worth attempting to produce a possible spectra list for 5-regular bipartite integral graphs using inequalities where the number of subgraphs in the equations become too numerous.

Spectral moments of regular graphs are used to study properties of fullerenes in [60]. They define a measure of width, $w$, of a fullerene. Several unanswered questions remain that we would like to investigate using our spectral moment equations of Chapter 3. For example, we would consider $w_{\ell}$ for even $\ell$ between $2w + 8$ and $4w + 10$.

We mentioned that given our equations for strongly regular graphs, we checked the not-known-to-exist cases appearing in Table 2.4 (strongly regular graph parameters with $n \leq 100$) for the counts of subgraphs up to 9 edges, but did not find any negative or non-integer valued counts. We plan to check the parameter sets of strongly regular graphs where $n > 100$. It may also prove fruitful to look at the equations for subgraphs with at least 10 edges or even those subgraphs with vertices of degree one. In the event that none of these attempts gives an example of a parameter set that can be shown not feasible, then it would be of interest to understand why the equations we found do not have the power to provide a non-existence result.

In the event that graphs with some known parameters are being investigated, it is possible to derive congruence results for the remaining parameters since each equation must evaluate to a non-negative integer. As an example, consider the first equation from the presented equations for a Moore graph $SR(n, r, 0, 1)$: $[C_{5}] = \frac{1}{10}(r - 1)^{2}rn$. Any parameter set with these values will only have a solution if 5 divides $n$ or $r$ is congruent to 0 or 1 mod 5. Then one could consider the equation $[C_{7}] = \frac{1}{18}r(r - 2)(r - 3)(r - 1)^{2}n$ and deduce that simultaneously 7 divides $n$ or $r$ is congruent to 0, 1, 2, or 3 mod 7. Unfortunately, 5 divides 3250 and $r = 7$ so this does not eliminate the ‘missing’ Moore graph $SR(3250, 57, 0, 1)$. This example was inconclusive. However, it is very possible that considering a better equation (or combination of equations) would lead to non-existence results.

Our methods do not give an equation for a subgraph with minimum degree 3. However, minimum degree 3 graphs do appear on the right side of our equations; for example, consider $K_{4}$ in this equation for a graph $SR(n, r, c, f)$:

$$[\Delta] = \frac{1}{2}(e - 1)^{2}ern - 12[K_{4}]$$.
For this reason, we hoped that counts for these graphs could be deduced. In particular, we wanted to determine the counts of the Petersen graph $P$, in the ‘missing’ Moore graph $\text{SR}(3250, 57, 0, 1)$. This is interesting because $P$ is the smallest minimum degree 3 graph that can occur in $\text{SR}(3250, 57, 0, 1)$. It occurs in the Hoffman-Singleton graph 525 times. However, in any attempt that we have made to solve for $[P]$, by forming a system of linear equations that include the variable $[P]$, the result is inconclusive because $[P]$ is a free variable. It would be interesting to investigate this further. There are more combinations of equations to consider. Do other subgraphs with minimum degree 3 behave the same way? If so, does it indicate that the counts of these subgraphs are not determined by the spectrum?

It seems that the equations given in Chapter 5 for counts of subgraphs in strongly regular graphs contribute no new information to the spectral moment equations of Chapter 3. This seems plausible since both equations depend heavily on the eigenvalues of the parent graph. However, it is not yet clear exactly how these equations are related. It would be interesting to investigate this relationship and if possible deduce one set of equations from the other.

There is opportunity to attempt ad-hoc methods between subgraphs similar to the technique of eliminating the spectrum $[24, 4, 40, 3]$ in Section 4.2. We give the counts of subgraphs for strongly regular graphs which are not known to exist. This includes the 57-regular Moore graph on 3250 vertices. We also have lists of possible spectra like that of Appendix A where given a spectrum it is not known whether or not it is realizable. If we take a specific strongly regular parameter set or a specific spectrum and consider the relevant spectral moment equations and/or strongly regular graph equations in conjunction with relations between subgraphs, then non-existence results seem very attainable. The equations of this thesis afford a wealth of counts of the various subgraphs and their relations to each other creating many graph specific opportunities for elimination.

In $[33]$, it is conjectured that the Petersen graph is the only connected strongly regular graph that is not Hamiltonian. The authors give the result that for a connected graph $G = \text{SR}(n, r, e, f)$ with smallest eigenvalue $\lambda$: if $\lambda \notin \mathbb{Z}$ or if $-\lambda \leq f + 1$, then $G$ is Hamiltonian. The expression for $[C_{10}]$ given by our algorithm for Moore graph parameters is $\frac{1}{20}r(r - 2)(r - 3)(r^3 - 3r^2 + r - 1)(r - 1)^2n$ and this evaluates to 0 for the Petersen graph $\text{SR}(10, 3, 0, 1)$. The first graph for which Hamiltonicity is still unknown is for a graph that is not known to exist: $\text{SR}(99, 14, 1, 2)$ with $\lambda = -4$. In practice, our current algorithm would create too many terms for a large graph like $C_{99}$ in the middle steps of recursion. However, it might be possible to appropriately discard most of the terms and in keeping some, derive a positive lower bound on the number of Hamiltonian cycles.

There are many more interesting problems to discover and explore in the area of spectral graph theory. The equations we presented and the graphs we found as a result leave more questions than answers regarding what can be said about the relationship between the eigenvalues of a regular graph and the counts of its various subgraphs.
Appendix A

Feasible Vertex-Transitive Spectra

The following is a set of possible spectra that might be realized by a connected 4-regular bipartite integral graph $G$ that is vertex-transitive. This set was determined in Section 4.2. The entries are given as $n \ x \ y \ z \ w\ [C_4] [C_6]$ where $|V(G)| = n$ and $Sp(G) = \{4, 3^x, 2^y, 1^z, 0^ww, -1^z, -2^y, -3^x, -4\}$.

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References


REFERENCES


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