# Counting Subgraphs of Regular Graphs using Spectral Moments

by

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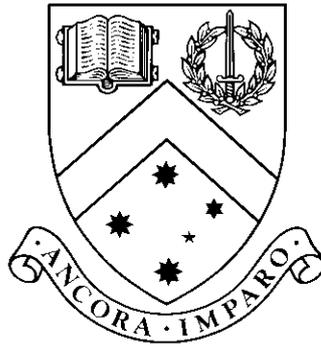
Thesis

Submitted by Marsha Elizabeth Minchenko

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And there is nothing new under the sun.

Ecclesiastes 1:9b

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Abstract

This thesis introduces a novel approach to counting subgraphs of regular graphs. This is useful for relating the algebraic properties of a graph G to its structural properties; namely the eigenvalues of the adjacency matrix of G to the counts of various subgraphs of G.

It was previously known that the ℓth spectral moment of G is equal to the number of closed walks of length ℓ in G. The first four spectral moments were found to relate the eigenvalues of G to the number of vertices, edges, triangles, and quadrilaterals in G. For bipartite regular graphs, the 6th spectral moment was found to relate the eigenvalues of G to the degree and the number of vertices, quadrilaterals, and hexagons in G.

We present a general method for counting all closed walks of a given length in connected regular graphs. We describe a set of base walks and their possible extensions to the set of all closed walks of G. All the extensions are handled using generating functions, resulting in a substantial reduction in the amount of direct enumeration required. Consequently, we are able to derive equations relating the ℓth spectral moment to the degree, the number of vertices, and the counts of various subgraphs in a regular graph.

Such equations have been used to help in the search for members of interesting families of graphs. One such graph family is the integral graphs. These have only integers as eigenvalues. We build on previous results that give a list of spectra (the eigenvalues with their multiplicities) that are possible for a graph that is connected, 4-regular, bipartite, and integral. From this list, we consider the subgraph configurations necessary to eliminate spectra that cannot be realized by a vertex-transitive graph. Using this refined list, we are able to find all connected, 4-regular, integral Cayley graphs.

We present a method for counting connected subgraphs H of a strongly regular graph. Strongly regular graphs are r-regular with e common neighbours between adjacent vertices, and f common neighbours between non-adjacent vertices. For some subgraphs H, we are able to express the number of copies of H in terms of r, e, f, and the number of vertices. Counts of other subgraphs are calculated in terms of the counts of smaller subgraphs. We use basic counting arguments to prove these results and outline the algorithm which applies these results recursively on subgraphs. We give examples of the equations obtained for several cases of parameter values where it is yet to be established whether strongly regular graphs exist.

We look at how both sets of equations derived in this thesis can be used to investigate the 'missing' Moore graph. Moore Graphs are strongly regular graphs with $e = 0$ and $f = 1$. Some well known examples are the Petersen graph and the Hoffman-Singleton graph. It is not known if a 57-regular Moore graph exists. If it does exist then it is integral with spectrum $\{57, 7^{1729}, -8^{1520}\}$. Our equations give further insights into the structure of the 'missing' Moore graph such as the numbers of various cycles and theta graphs.

It is our hope that our equations for counting subgraphs in regular graphs and for counting subgraphs in strongly regular graphs will be useful in various ways. These include finding other specified families of graphs, finding the numbers of subgraphs of interest, investigating co-spectral pairs of graphs, and proving the non-existence of graphs.

Counting Subgraphs of Regular Graphs using Spectral Moments

Declaration

I declare that this thesis is my own work and has not been submitted in any form for another degree or diploma at any university or other institute of tertiary education. Information derived from the published and unpublished work of others has been acknowledged in the text and a list of references is given.

Marsha Elizabeth Minchenko

Counting Subgraphs of Regular Graphs using Spectral Moments

Declaration (Co-Authored Papers)

Material from the following joint papers, with the relative author contributions indicated, has been incorporated into the thesis.

Title	Authors	Location in thesis
Spectral moments of regular graphs in terms of subgraph counts	M. Minchenko (80%) I. Wanless (20%)	Chapter 3
Connected quartic bipartite integral Cayley graphs	M. Minchenko (80%) I. Wanless (20%)	Sections $4.1 - 4.4.5$, $4.4.7 - 4.5$
Counting subgraphs of strongly regular graphs	M. Minchenko (80%) I. Wanless (20%)	Chapter 5

Marsha Elizabeth Minchenko

Prof. Ian M. Wanless

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<div align="right">Marsha Elizabeth Minchenko</div>

Monash University
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Chapter 1

Introduction

1.1 Problem and motivation

Graphs are used to model and investigate various problems and systems, where objects and the connections between the objects are described by vertices and edges respectively. Atoms and their bonds, cities and the roads between them, computers and information that is being sent and received are all examples of situations in which graphs can encapsulate the underlying relationships.

Matrices can be used to represent graphs. The eigenvalues of a matrix are a special set of scalars associated with the matrix. The set of all eigenvalues corresponding to the matrix representation of a graph is known as its spectrum. Spectral graph theory seeks to relate the spectrum of a graph to the structural properties of the graph.

The origins of spectral graph theory can be traced to the papers of Günthard and Primas [90] and Cvetković and Gutman [58]. In these papers it was recognized that Hückel's model in quantum chemistry [107] was also a mathematical theory of graph spectra relating the underlying graph to the eigenvalues, which in this case represented energy levels of certain electrons. Another early problem was posed as "can one hear the shape of a drum?" [114] and this was translated to a problem in spectral graph theory [76] with the graph modelling the shape of a drum while the eigenvalues corresponded to the sound of that drum. Eigenvalues associated with graphs have since arisen in many applications. Many mathematicians worked in this area in the 1950s and 1960s and by the year 1970 there were over 80 papers published on eigenvalues in graphs. In 1980, the book, *Spectra of Graphs* by Cvetković et al. [62], covered most of the results in the area at the time. It is still one of the best resources as an introduction to this growing field of mathematics.

The paper by Günthard and Primas [90], also raised the question, "which graphs are determined by their spectrum?". It was thought that the spectrum uniquely determined the graph until a pair of graphs with the same set of eigenvalues was presented in [46]. The answer given instinctively by a survey paper on this area [179] was "almost all graphs", but as they state more factually "for almost all graphs the answer to this question is still unknown".

The eigenvalues of a graph are always real numbers. The more restricted class of graphs where the eigenvalues are all integer valued are the *integral graphs*. The task of determining the graphs with this property was initiated in the early seventies [98] but the problem of finding all integral graphs has been called intractable [98]. At the beginning of the 21st century a use for integral graphs

as a certain quantum spin network in quantum information processing was determined [43] and this lead to a new emphasis on this topic. There are situations in which integral graphs are preferable to other graphs because they have the advantage of involving only integer arithmetic [64].

1.2   Research contributions

This thesis makes contributions to the area of spectral graph theory. We prove results that give information about the subgraphs of a graph based on the set of eigenvalues of the graph. We are able to use these results to find previously unknown integral graphs. Other discoveries in this thesis are useful for investigating strongly regular graphs.

Our main research contributions in the thesis can be summarized as follows.

In Chapter 3, we find equations that relate the eigenvalues of a regular graph to the number of isomorphic copies of certain small graphs which appear as subgraphs of the graph.

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ denote the eigenvalues of the adjacency matrix of G. The sum, $w_\ell = \sum_{j=1}^{n} \lambda_j^\ell$, of the ℓ-th powers of the eigenvalues is known as the ℓ-th spectral moment and is equal to the number of closed walks of length ℓ in G. It is folklore that $w_1 = 0$, w_2 is two times the number of edges of G, and w_3 is six times the number of triangles in G. By using generating functions to count a certain type of closed walk, we extend this list of equations to beyond what appears in the current literature.

Let C_i denote the i-cycle. For any graph H, let $[H]$ denote the number of (not necessarily induced) subgraphs in G that are isomorphic to H, where the parent graph G will be implicitly specified by the context. For 4-regular bipartite G, equations for w_ℓ up to $\ell = 6$ were given by Cvetković *et al.* [63]:

$$\begin{aligned} w_0 &= n, & w_4 &= 28n + 8[C_4], \\ w_2 &= 4n, & w_6 &= 232n + 144[C_4] + 12[C_6]. \end{aligned} \tag{1.1}$$

Stevanović *et al.* [176] added an inequality for $\ell = 8$:

$$w_8 \geqslant 2092n + 2024[C_4] + 288[C_6]. \tag{1.2}$$

We give a general method for finding equations such as (1.1) for spectral moments of regular graphs, whether or not they are bipartite.

The subgraphs that appear in our equations will be called *contributors* (in (1.1) all of the contributors are cycles, but this is not true in general). Our methods build on those of Friedland *et al.* [78] and Wanless [195] who relate the numbers of matchings to the counts of contributors. We use a special class of closed walks, called the not totally-reducible walks, similar to the "primitive circuits" used by McKay [141] to count spanning trees.

There is a result in the literature that can be used to count the totally-reducible walks in a graph [140]. We count closed walks as the sum of totally-reducible walks and closed not totally-reducible walks. The ideas of base walks and their extensions are introduced. We prove that given a closed and not totally-reducible walk W', there exists a unique base walk W such that W extends to W'.

A suitable generating function is derived to count all not totally-reducible walks that extend a base walk of length ℓ. Suppose $W = v_0 v_1 \cdots v_\ell$ is a closed walk of length ℓ in an $(r + 1)$-regular

graph, G. It has been shown for an infinite rooted tree T where the root has degree k_1 and every other vertex has degree $k_2 + 1$, that the generating function for closed rooted walks in T is

$$T_{k_1} = \frac{2k_2}{2k_2 - k_1 + k_1\sqrt{1 - 4x^2k_2}}.$$

We prove that the generating function for walks in an r-regular graph G that are extensions of W is

$$\psi(\ell) = x^\ell T_r^\ell T_{r+1} \left(\frac{1 - x^2 T_r^2}{1 - rx^2 T_r^2} \right).$$

Algorithms are used to take a contributor H and find b_i, the number of base walks of length i up to ℓ that induce H; to calculate $\eta(H) = \sum_{i=1}^{\ell} b_i \psi(i)$; as well as to determine $\mathcal{H}_\ell$, the set of all contributors that can be induced by the base walks of length up to ℓ. Finally, it is shown that the generating function for closed and not totally-reducible walks of length at most ℓ in a graph G is

$$\sum_{H \in \mathcal{H}_\ell} \eta(H)[H] + O(x^{\ell+1}).$$

In this way we derive generating functions that give expressions for the numbers of closed walks in terms of the counts of contributors.

The chapter ends by characterising the contributors that can occur in the resulting equations.

In Chapter 4, we use a variety of methods to find new quartic integral graphs. We produce exhaustive lists of the connected 4-regular integral Cayley graphs and the connected 4-regular integral arc-transitive graphs.

The spectrum of a connected 4-regular bipartite integral graph has the form $\{4, 3^x, 2^y, 1^z, 0^{2w}, -1^z, -2^y, -3^x, -4\}$; which can be abbreviated by the quadruple $[x, y, z, w]$. Cvetković et al. [63] showed that the number of vertices in a connected r-regular bipartite graph with radius R is bounded above by $(2(r - 1)^R - 2)/(r - 2)$. Thus, it can be shown that connected 4-regular bipartite integral graphs have at most 6560 vertices.

Cvetković *et al.* [63] found quadruples $[x, y, z, w]$ that are candidates for the spectrum of a bipartite quartic integral graph. They called these *possible spectra*. Research activities regarding the set of possible spectra fall into two streams: eliminate possible spectra based on new information and/or techniques, or find graphs that realize a possible spectrum. Useful tools include an identity by Hoffman [104], spectral moment equations like (1.1), and inequality (1.2). All quartic integral graphs (QIGs) that avoid eigenvalues of ± 3 and realize a possible spectrum are found in [173]. Stevanović [171] eliminates spectra using equations arising from graph angles. In the same paper he determines that the possible values for the order n of the graph are between 8 and 1260, except for 5 identified spectra. Stevanović *et al.* [176] use their inequality for w_8 to determine that $8 \leqslant n \leqslant 560$. All of the bipartite QIGs with $n \leqslant 24$ that realize one of the possible spectra were found and are listed with drawings in [176].

The equations for the spectral moments from Chapter 3, when specialized to bipartite 4-regular graphs, are sufficient to reproduce the list of possible spectra presented in [176]. In Section 4.2, we show that some spectra on this list cannot be achieved by any vertex-transitive graph.

The Cayley QIGs on finite Abelian groups have been studied by Abdollahi *et al.* [2]. They showed that for an Abelian group, Γ, if $\mathrm{Cay}(\Gamma, S)$ is a Cayley QIG then

$$|\Gamma| \in \{5, 6, 8, 9, 10, 12, 16, 18, 20, 24, 25, 32, 36, 40, 48, 50, 60, 64, 72, 80, 96, 100, 120, 144\},$$

but they did not establish whether Cayley QIGs of these orders exist. We find that the set of orders of Cayley QIGs on Abelian groups is precisely $\{5, 6, 8, 9, 10, 12, 16, 18, 24, 36\}$. More generally, we consider all groups and find that many Cayley QIGs are on non-Abelian groups. We show that for any group Γ, if $\mathrm{Cay}(\Gamma, S)$ is a Cayley QIG then

$$|\Gamma| \in \{5, 6, 8, 9, 10, 12, 16, 18, 20, 24, 30, 32, 36, 40, 48, 60, 72, 120\}.$$

Furthermore, for each of these orders we give the Cayley QIGs that exist.

Our findings are that up to isomorphism, there are exactly

- 32 connected 4-regular integral Cayley graphs,

- 17 connected 4-regular bipartite integral Cayley graphs,

- 27 connected 4-regular integral graphs that are arc-transitive, and

- 16 connected 4-regular bipartite integral graphs that are arc-transitive.

Upon investigating the relationship between two of the bipartite arc-transitive non-Cayley integral graphs F_1 and F_4, we uncovered the method for constructing new integral graphs that are quotients of existing integral graphs. This leads to our finding additional connected 4-regular integral graphs that are not Cayley and not arc-transitive.

The graphs with only 3 distinct eigenvalues are the strongly regular graphs. In Chapter 5, we present equations for the counts of subgraphs of a strongly regular graph in terms of its parameters and the counts of a few small subgraphs. In this case, the parameters are the standard ones: the number of vertices n, the regularity r, the number of common neighbours between adjacent vertices e, and the number of common neighbours between non-adjacent vertices f. We denote a strongly regular graph by $\mathrm{SR}(n, r, e, f)$.

Necessary conditions have been established for the existence of a strongly regular graph with parameter set (n, r, e, f), and where these conditions are met, a parameter set is considered feasible. For a feasible parameter set, it is of interest to determine firstly whether it is realizable by a graph and secondly to find all graphs $\mathrm{SR}(n, r, e, f)$. A list of feasible parameter sets and whether or not they are realizable, including references, can be found in [30]. Currently, the smallest feasible parameter set for which the existence question remains unanswered is $(65, 32, 15, 16)$. For $n \leqslant 100$, the existence question is still unanswered for 16 sets of feasible parameters (and their complements).

We present results for determining the number of copies of a subgraph H in a strongly regular graph in terms of the counts of subgraphs with strictly fewer vertices. When the existence question for a feasible parameter set is unanswered, our methods give subgraph counts that a graph with that parameter set would have to have. Thus, this tool is useful for both studying subgraphs of existing graphs as well as considering the existence question for feasible parameter sets.

We summarize our results here but the details and an explanation of the notation used can be found in Sections 5.1 and 5.2. Consider a strongly regular graph $\mathrm{SR}(n, r, e, f)$ and a graph H containing a vertex s of degree one. Let u denote the neighbour of s in H and let N denote the neighbours of u in $H\backslash\{s\}$. Let X denote a set of vertices of $H\backslash(\{u, s\} \cup N)$ that are orbit representatives under the action of $\mathrm{Aut}(H\backslash\{s\})_{\{u\}}$. Then,

$$[H] = \frac{(r - |N|)[H\backslash\{s\}, u] - \sum_{x \in X}[H\backslash\{s\} + ux, u, x]}{|\mathrm{Aut}(H)u||\mathrm{Aut}(H)_{\{u\}}s|}.$$

Consider a strongly regular graph $\mathrm{SR}(n, r, e, f)$ and a graph H containing a vertex s of degree two, such that s is not a cut-vertex of H. Let u and v denote the two neighbours of s in H and let C denote the common neighbours of u and v in $H\backslash\{s\}$. Let X denote a set of vertices of $H\backslash(\{u, v, s\} \cup C)$ that are orbit representatives under the action of $\mathrm{Aut}(H\backslash\{s\})_{\{u,v\}}$. If u is adjacent to v in H then,

$$[H] = \frac{(e - |C|)[H\backslash\{s\}, u, v] - \sum_{x \in X}[H\backslash\{s\} + ux + xv, u, v, x]}{|\mathrm{Aut}(H)u||\mathrm{Aut}(H)_{\{u\}}v||\mathrm{Aut}(H)_{\{u,v\}}s|}.$$

Otherwise,

$$[H] = \frac{(f - |C|)[H\backslash\{s\}, u, v] + (e - f)[H\backslash\{s\} + uv, u, v] - \sum_{x \in X}[H\backslash\{s\} + ux + xv, u, v, x]}{|\mathrm{Aut}(H)u||\mathrm{Aut}(H)_{\{u\}}v||\mathrm{Aut}(H)_{\{u,v\}}s|}.$$

An algorithm is used to recursively call these theorems for successively smaller graphs until the graphs consist of a single vertex or a graph with minimum degree 3. We thereby express the number of copies of some subgraphs in terms of the standard parameters of strongly regular graphs. Counts of other subgraphs are calculated in terms of the counts of smaller subgraphs.

One of the most famous open problems in algebraic graph theory is whether or not a 57-regular Moore graph exists. A *Moore* graph is a regular graph with degree Δ and diameter D for which $n = 1 + \Delta \sum_{i=1}^{D}(\Delta - 1)^{i-1}$. A Moore graph of diameter 2 is a strongly regular graph with no common neighbours between adjacent vertices and exactly 1 common neighbour between non-adjacent vertices. Hoffman and Singleton [105] prove that any Moore graph with diameter 2 must have degree 2, 3, 7, or 57. They also show that unique graphs exist for the first 3 values: the pentagon, the Petersen graph, and the Hoffman-Singleton graph. The 4th value corresponds to the strongly regular graph $G = \mathrm{SR}(3250, 57, 0, 1)$. If such a graph exists, $|V(G)| = 3250$, $Sp(G) = \{57^1, 7^{1729}, -8^{1520}\}$, $\mathrm{Aut}(G)$ is not a rank 3 group [7], and G is not vertex-transitive (see for example [40]). More recent work on the missing Moore graph has been done in investigating the symmetries and the automorphism group of G [135, 138, 139].

At the end of the chapter we present a few case studies. We implement our algorithm to count small subgraphs of various strongly regular graphs which are not known to exist. Among these case studies, we present the counts of small subgraphs of the famous "missing" Moore graph.

1.3   Publications

Publications arising from this thesis include:

Minchenko, M. (2011), Connected quartic bipartite integral Cayley graphs. International Workshop On Combinatorial Algorithms 2011. Accepted poster.

Minchenko, M. and Wanless, I. M. (2013), Spectral moments of regular graphs in terms of subgraph counts. Accepted to Linear Algebra and its Applications.

Minchenko, M. and Wanless, I. M. (2013), Connected quartic bipartite integral Cayley graphs. Submitted.

Minchenko, M. and Wanless, I. M. Counting subgraphs of strongly regular graphs. In preparation.

Chapter 2

Literature Review

2.1 Background

All graphs, $G = \{V(G), E(G)\}$, in this document are simple and undirected with edge set $E(G)$ and vertex set $V(G)$ of size n.

The *adjacency matrix*, $A = [a_{ij}]$, of G is the $n \times n$ matrix defined as

$$a_{ij} = \begin{cases} 1 & \text{if } i \text{ is adjacent to } j, \\ 0 & \text{otherwise.} \end{cases}$$

The eigenvalues of A are the values λ satisfying the equation $Av = \lambda v$ for non-zero vectors v. The adjacency matrix is real and symmetric. Hence, all its eigenvalues are real and A is similar to a diagonal matrix with diagonal consisting of the eigenvalues of A [132]. The *trace* of A, $\mathrm{Tr}(A)$ is the sum of the diagonal entries. Now since similar matrices have the same trace;

$$\mathrm{Tr}(A) = \sum_{k=1}^{n} \lambda_k. \tag{2.1}$$

A *walk* W of length ℓ in G is a sequence of vertices $v_0 v_1 \cdots v_\ell$, where v_i is adjacent to v_{i-1} for each $i = 1, 2, \ldots, \ell$. When $v_0 = v_\ell$ we say that W is a *closed walk*. If we consider the matrix A^2 and look at one entry, we have

$$A_{i,j}^2 = a_{i,1} a_{1,j} + a_{i,2} a_{2,j} + \ldots + a_{i,n} a_{n,j}$$

which is the number of walks of length 2 from vertex i to vertex j. Similarly $A_{i,j}^\ell$ gives the number of walks of length ℓ from vertex i to vertex j.

Thus the entries along the diagonal in A^ℓ give the number of walks of length ℓ from a given vertex to itself; and the trace gives the total number of closed walks of length ℓ in G. By (2.1), $\mathrm{Tr}(A^\ell) = \sum_{k=1}^{n} \lambda_k^\ell$.

The *spectrum* of a graph, $Sp(G)$, with respect to its adjacency matrix consists of the eigenvalues of its adjacency matrix with their multiplicity. We use the notation: $Sp(G) = \{\lambda_1^{m_1}, \lambda_2^{m_2}, \ldots, \lambda_s^{m_s}\}$, where m_i is the multiplicity of λ_i for $i = 1, \ldots, s$ and $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_s$. For brevity, we refer to the eigenvalues of the adjacency matrix of the graph as the eigenvalues of the graph. Two graphs G_1 and G_2 are said to be *co-spectral* if $Sp(G_1) = Sp(G_2)$.

The sum, $w_\ell = \sum_{j=1}^n \lambda_j^\ell$, of the ℓ-th powers of the eigenvalues is known as the ℓ-th spectral moment. It is folklore that $w_0 = n$, $w_1 = 0$, w_2 is two times the number of edges in G, and w_3 is six times the number of triangles in G. This is notable because the number of vertices, edges, and triangles are completely determined by the spectrum of G.

Let C_n denote the n-cycle and K_n denote the complete graph on n vertices. The complete s-partite graph $K_{p_1, p_2, \ldots, p_s}$ is a graph with vertex set $V = V_1 \cup V_2 \cup \cdots \cup V_s$ where the V_i are nonempty disjoint sets and $|V_i| = p_i$ for $1 \leqslant i \leqslant s$, with x in V_i adjacent to y in V_j if and only if $i \neq j$. An s-partite graph has vertex set the vertex set of $K_{p_1, p_2, \ldots, p_s}$ and edge set a subset of the edge set of $K_{p_1, p_2, \ldots, p_s}$. Let $G \cup H$ denote the disjoint union of graphs G and H. We mention that $K_{1,4}$ and $K_1 \cup C_4$ have the same spectrum but not the same number of 4-cycles. For this reason, equations for spectral moments for w_ℓ for $\ell > 3$ cannot be as simple as the equations mentioned above for $\ell \leqslant 3$. Nevertheless, it is possible to find such equations, as we show in Chapter 3.

Since the spectrum of a disconnected graph is the union of the spectra of its connected components, from this point on we only consider connected graphs.

We mention that there are other types of graph spectra that are studied in spectral graph theory [94]; the Laplacian spectrum, the signless Laplacian spectrum, the normalized Laplacian spectrum, and the Seidel spectrum. In this document we restrict ourselves to the adjacency spectrum.

The following result is a special case of a famous theorem of Perron [148] and Frobenius [80] for non-negative matrices:

Theorem 1 (Perron-Frobenius)**.** *If G is a connected graph with at least 2 vertices, then*

- *its largest eigenvalue λ_1 has multiplicity one;*

- *for the eigenvalue λ_1, the corresponding eigenvector has all positive coordinates;*

- *any other eigenvalue λ satisfies $-\lambda_1 \leqslant \lambda < \lambda_1$;*

- *the deletion of any edge of G decreases the largest eigenvalue.*

The largest eigenvalue λ_1 is called the *spectral radius* of G.

An *r-regular* graph is a graph for which each vertex is adjacent to exactly r other vertices. If we specifically consider r-regular graphs G, we have that r is the spectral radius of G. This result is quickly seen since each row of the adjacency matrix A of an r-regular graph has exactly r non-zero entries, each with value 1, and so $(1, 1, .., 1)^T$ is an eigenvector with eigenvalue r. This implies that every other eigenvector is orthogonal to $(1, 1, .., 1)^T$ and thus contains a negative coordinate. By Theorem 1, we have that $|\lambda| \leqslant r$ for all eigenvalues λ of G and the multiplicity of the spectral radius r is one. We note that more generally, when the graph is not necessarily connected, the multiplicity of the spectral radius r is equal to the number of connected components of G (see [23] for example). Since, consider A in block diagonal form with blocks consisting of the connected components with zero everywhere else. Each block will be similar on its own to a diagonal matrix of its eigenvalues.

A *bipartite* graph is a 2-partite graph. If we consider bipartite graphs, we have that the spectrum is symmetric about 0: if λ is an eigenvalue of the bipartite graph, G, then $-\lambda$ is also an eigenvalue of G with the same multiplicity. This result was originally shown by Coulson and Rushbrooke [47] as they investigated the structure of molecules.

2.2 Spectral moment equations

In this subsection we review some past papers where spectral moments are related to the counts of subgraphs of a graph. We also mention some results related to counting walks in regular graphs that we will need later.

Consider a walk $W = v_0 v_1 \cdots v_\ell$. If $v_{i-1} = v_{i+1}$ for some i then W is *reducible*, otherwise it is *irreducible*. In the reducible case, W can be *reduced* at index i by omitting v_i and v_{i+1}. We let $\mathrm{red}(W)$ be the irreducible result of repeatedly omitting v_i and v_{i+1} for $i = \min\{j \mid v_{j-1} = v_{j+1}\}$ until no such j exists. By a result of Godsil [83], if reductions are iteratively applied to a walk until it is irreducible, then the result is independent of the reductions chosen.

Lemma 2. *If W reduces to W' by any sequence of reductions, then $\mathrm{red}(W) = \mathrm{red}(W')$.*

Proof. Case A: W reduces to W' in at most one step. Suppose W has length l. If $l \leqslant 2$ then either W is of the form $v_0 v_1 v_0$ and $W' = v_0$ or $W' = W$. Thus $\mathrm{red}(W') = \mathrm{red}(W)$.

Now, we use induction on the length $l \geqslant 2$ and assume that $\mathrm{red}(W') = \mathrm{red}(W)$ for $l \leqslant n$. We consider $l = n + 1$ where W is reduced to W' at index i. Let j be the smallest index where W can be reduced. Then $j \leqslant i$.

If $j = i$ then $\mathrm{red}(W') = \mathrm{red}(W)$ by definition.

If $j < i - 1$ then we reduce both W' and W at index j to give walks W'^* and W^* respectively.

$$
\begin{aligned}
\mathrm{red}(W') &= \mathrm{red}(W'^*) \text{ by the inductive hypothesis since } W' \text{ reduces to } W'^* \text{ at index } j \\
&= \mathrm{red}(W^*) \text{ by the hypothesis since } W^* \text{ reduces to } W'^* \text{ at index } i - 2 \\
&= \mathrm{red}(W) \text{ by reducing } W \text{ at index } j \text{ followed by the sequence of reductions by which } W^* \\
&\quad \text{reduces to } \mathrm{red}(W^*)
\end{aligned}
$$

So suppose $j = i - 1$. Then, W has $v_{i-2} v_{i-1} = v_i v_{i+1}$. Thus W reduces to W' at both i and j, which gives the desired result.

Case B: W reduces to W' in more than one step. We use induction on the number of steps $k \geqslant 1$. The base case follows from Case A. Assume that $\mathrm{red}(W) = \mathrm{red}(W')$ when W reduces to W' in $k \leqslant n$ steps. Let $k = n + 1$ and suppose that the first reduction step was at index i. Then, W reduces to some walk W^* at index i which further reduces to W'. By the inductive hypothesis, $\mathrm{red}(W') = \mathrm{red}(W^*)$. Now reduce W at index i followed by the sequence of reductions for which W^* reduces to $\mathrm{red}(W^*)$. We have that $\mathrm{red}(W^*) = \mathrm{red}(W)$ and the desired result follows. $\square$

Let $v_0, v_1, \ldots, v_\ell$ denote the vertices of a graph G. A *primitive circuit* is a closed irreducible walk W of length $\geqslant 3$ such that $v_1 \neq v_{\ell-1}$. Each primitive circuit W is taken to represent all cyclic permutations of the sequence of edges of W and also the reverse walk $v_\ell v_{\ell-1} \cdots v_0$. McKay [141] determines a generating function for the number of closed walks of length ℓ in an r-regular graph G in terms of n, r, and the number of primitive circuits of length ℓ in G. McKay's result is as follows. Let G be an r-regular graph of order n. Let p_ℓ be the number of primitive circuits of length $\ell \geqslant 0$ in

G. For $w(x) = \sum_{\ell=0}^{\infty} w_\ell x^\ell$ and $p(x) = \sum_{\ell=3}^{\infty} \frac{\ell x^\ell}{1-x^\ell} p_\ell$,

$$w(x) = \frac{r - 2 - r(1 - 4(r-1)x^2)^{1/2}}{2(r^2 x^2 - 1)} n + \frac{2}{(1 - 4(r-1)x^2)^{1/2}} p\left(\frac{1 - (1 - 4(r-1)x^2)^{1/2}}{2(r-1)x}\right).$$

In [178], van Dam investigates r-regular graphs G with four distinct eigenvalues. It is shown that in such graphs, the number of closed walks of length ℓ from a given starting vertex is independent of the choice of starting vertex for all ℓ. Therefore, $\frac{1}{n} w^\ell$ is a non-negative integer for all ℓ. This work is extended in [180] where van Dam and Spence determine the feasible spectra $\{r^1, \lambda_1^{m_1}, \lambda_2^{m_2}, \lambda_3^{m_3}\}$. With the aid of a computer, they were able to answer the existence question for 214 of the 244 feasible spectra. For each realizable spectrum, all graphs with that spectrum were found.

Jiang et al. [112] present a procedure for enumerating the spectral moments of molecules (connected graphs with maximum vertex degree three) in terms of subgraph counts. They give a procedure for finding the counts of subgraphs containing no cycles and find equations for w_ℓ for values of ℓ up to 14. This allows them to approximate a formula for the total Π electron energy. They consider how moments are dependent on the connectivities of molecules with both bonds and multi-atom subgraphs contributing to the equations. Jiang and Tang [110] continue to investigate moment equations for acyclic carbon chains. The moment equations for molecules consisting entirely of hexagonal rings are investigated in [96]. Jiang and Zhang [111] seek to further illuminate the relationship between molecular behaviour and structure using spectral moments. They illustrate the procedure for obtaining some of the subgraphs with cycles whose counts appear in moment equations; giving w_ℓ up to $\ell = 12$ for this set of subgraphs. Jiang and Zhang claim that moments allow them to directly determine molecular information. They give a scheme based on moment analysis and energy partitioning for determining the stability and reactivity of conjugated hydrocarbons and mention that moments enabled them to bypass the usual molecular orbital theory methods. There are many more similar and related results in both chemistry and mathematics [24, 71, 72, 73, 91, 92, 97, 113].

A fullerene graph or simply *fullerene* is a 3-regular, planar graph with all faces 5-cycles or 6-cycles including the external face. Zhang and Balasubramanian [201] consider fullerenes containing isolated 5-cycles and derive analytical formulae for the first 14 moments in terms of counts of subgraphs. They observe that the first 11 moments depend only on the counts of cycles. Cvetković et al. [52] define a measure of fullerenes, called the width, related to the distance from vertices in the fullerene to the nearest 5-cycle. Cvetković and Stevanović [60] prove that for small values of ℓ, w_ℓ depends only on ℓ, the width, and the number of vertices of the fullerene graph.

The *coalescence* of two graphs, G_1 with a distinguished vertex v_1 and G_2 with a distinguished vertex v_2, is the graph obtained by identifying vertices v_1 and v_2. A *lollipop* $L(m, k)$ is the coalescence of C_m with $m \geqslant 3$ vertices and a path P_{k+1} with $k + 1 \geqslant 2$ vertices and distinguished vertex one of the vertices of degree one. Boulet and Jouve [29] give a way to count closed walks of a graph G in terms of its subgraph counts. Here, the graph G and the subgraphs described are not necessarily regular. They use this information to show that there are no co-spectral non-isomorphic lollipops and that each lollipop graph is determined by its spectrum. An example of an equation from their paper is as follows: for a graph G with no triangles or 3-cycles,

$$w_6 = 12[C_6] + 2[P_2] + 12[P_3] + 6[P_4] + 12[K_{1,3}] + 48[C_4] + 12[L(4, 1)],$$

where $[H]$ denotes the counts of subgraphs H in G as before. We note that Haemers et al. [95] previously showed that co-spectral lollipops are isomorphic and that lollipops $L(m,k)$ with m odd are determined by their spectrum but they did not consider lollipops with m even and the methods used were different.

A walk of length 0 is *trivial*. If $\text{red}(W)$ is trivial then we say that W is *totally-reducible*. Let u_ℓ be the number of totally-reducible walks of length ℓ in G. A totally-reducible walk must have even length, so $u_\ell = 0$ for all odd ℓ. For even ℓ, McKay [140] showed that

$$u_\ell = n \sum_{i=0}^{\ell/2} \binom{\ell}{i} \frac{\ell - 2i + 1}{\ell - i + 1} r^i. \tag{2.2}$$

A *matching* of size i or an i-matching in a graph G is a set of i edges of G in which no two edges share a vertex. Both Friedland et al. [78] and Wanless [195] use generating functions to find equations relating the number of matchings to the counts of certain small subgraphs.

In [78] a generating polynomial for the number of matchings in a graph is found. The matching polynomial of a graph $G_1 \cup G_2$ is the product of the matching polynomial of the graph G_1 and the matching polynomial of the graph G_2. Friedland et al. find the matching polynomials for various paths, cycles, and their unions. They determine some graphs with the same matching polynomial. Finally, an expression for the number of ℓ-matchings of an r-regular graph is given in terms of r, n and the counts of various paths, cycles, the unions of paths and cycles, and a few other small graphs for ℓ up to 4.

When the vertices of a walk $W = v_0 v_1 \cdots v_\ell$ are all distinct then W is a *path*. For $0 \leqslant i \leqslant \ell$, we call the walk $W_i = v_0 v_1 \cdots v_i$ a *prefix* of W. If $\text{red}(W_i)$ is a path for each i then W is called a *tree-like* walk. A tree-like walk is closed if and only if it is totally-reducible. Wanless [195] expresses the number, ϵ_ℓ, of totally-reducible walks of length ℓ which are not tree-like in an r-regular graph G in terms of n, r, and the number of certain subgraphs of G. Wanless derives the polynomials ϵ_ℓ using generating functions. A *rooted tree* is a tree with a distinguished vertex, called the *root*. A *rooted walk* in a rooted tree is a walk that starts at the root. Counting closed walks in a regular graph can be equated to counting closed walks in an infinite, nearly regular tree with a distinguished root vertex. The following result is from [195]:

Lemma 3. *Let T be an infinite rooted tree in which the root has degree k_1 and every other vertex has degree $k_2 + 1$. The generating function for the number of closed rooted walks in T is*

$$T_{k_1} = \frac{2k_2}{2k_2 - k_1 + k_1 \sqrt{1 - 4x^2 k_2}}.$$

Proof. We generate closed rooted walks in T as follows:

1. We perform a trivial walk in 1 way or

2. We perform a non-trivial closed walk:

   - there are k_1 choices for the next vertex after the root,
   - there are T_{k_2} closed walks not involving the root from the chosen vertex,
   - there are T_{k_1} ways to make a closed walk upon returning to the root.

Since walking away from the root and returning involves a factor x^2, we have

$$T_{k_1} = 1 + k_1 T_{k_2} T_{k_1} x^2.$$

We take $k_1 = k_2$ to count walks that do not include the root and get

$$k_2 x^2 T_{k_2}^2 - T_{k_2} + 1 = 0.$$

Upon solving the quadratic, we take the correct root (after considering the limit as $x \to 0$) to give

$$T_{k_2} = \frac{1 - \sqrt{1 - 4x^2 k_2}}{2x^2 k_2}.$$

Finally, substituting this value for T_{k_2} into the above expression for T_{k_1} gives the desired result. $\square$

Wanless seeks to count all the desired walks by extending the most basic walks. A walk W' is formed by adding a *diversion* to $W = v_0 v_1 \cdots v_\ell$ if $W' = v_0 \cdots v_i d v_{i+1} \cdots v_\ell$ where d is a walk such that $v_i d$ is totally-reducible (of length $\geqslant 0$), and if $i < \ell$ then no prefix of $v_i d$ reduces to $v_i v_{i+1}$. This generating function is given as

Lemma 4. *Let $W = v_0 v_1 \cdots v_\ell$ be a walk of length ℓ in an $(r+1)$-regular graph, G. The generating function for walks in G formed by adding a diversion to W is $x^\ell T_r^\ell T_{r+1}$.*

Proof. Adding diversions results in a walk $W' = v_0 d_0 v_1 d_1 \ldots v_\ell d_\ell$. The generating function for $v_i d_i$ is T_r for $0 \leqslant i < \ell$ and is T_{r+1} for $i = \ell$. The ℓ edges, $d_i v_{i+1}$ for $0 \leqslant i < \ell$, result in the factor x^ℓ. $\square$

Wanless also shows that the process of reducing a walk is unique.

Lemma 5. *If W' reduces to W then there is a unique way to write $W' = v_0 d_0 v_1 d_1 \ldots v_\ell d_\ell$ where $W = v_0 v_1 \cdots v_\ell$ and each $v_i d_i$ is a diversion.*

A walk W' has been formed by adding a *tail* of length t to a closed walk W if

$$W' = u_1 u_2 \cdots u_t W u_t \cdots u_1$$

where $u_1 \cdots u_t v_0$ is an irreducible walk of length t and neither $u_t v_0$ nor $v_0 u_t$ occurs in W. Diversions and tails are called *extensions*. Adding diversions and tails are combined in the following way:

Lemma 6. *Suppose $W = v_0 v_1 \ldots v_\ell$ is a closed walk of length 2ℓ in an $(r+1)$-regular graph G and let H be the subgraph induced by the edges of W. Suppose the first vertex of W has degree δ in H. The generating function for walks in G that are extensions of W is*

$$\psi(\ell, \delta) = x^{2\ell} T_k^{2\ell} T_{k+1} \left(\frac{1 - (\delta - 1) x^2 T_k^2}{1 - r x^2 T_k^2} \right),$$

where each walk is counted the number of times that it can be formed by adding tails and/or diversions to W.

An algorithm is designed to find the equations for ϵ_ℓ: the counts of all closed non-tree-like walks in terms of the counts of subgraphs. This algorithm determines subgraphs H that are essential to

the equation, eliminates over-counting errors based on inclusion-exclusion principles, and gives the counts of all closed non-tree-like walks as coefficients of the counts of subgraphs based upon the above generating functions. We give an example of the algorithm output. Let $C_{i_1 \cdot i_2 \cdots i_h}$ denote cycles of length $i_1, i_2, \ldots, i_h$ sharing a single vertex and $\Theta_{i_1, i_2, \ldots, i_h}$ two vertices joined by internally disjoint paths of lengths $i_1, i_2, \ldots, i_h$. These are depicted in Figure 3.1. This equation is for an $(r+1)$-regular graph G:

$$\epsilon_{12} = (1320r^3 - 6)[C_3] + 528r^2[C_4] + 120r[C_5] + 192[K_4] - (480r + 12)[\Theta_{2,2,1}]$$
$$- 48([\Theta_{3,2,1}] + [\Theta_{2,2,2}] + [C_{3 \cdot 3}]).$$

The values of ϵ_ℓ for ℓ up to 16 and up to 20 in the bipartite case are explicitly reproduced in [195]. The number of matchings of size ℓ in an r-regular graph can be determined given $u_\ell - \epsilon_\ell$. Wanless presents expressions for the number of matchings of size ℓ up to 12.

2.3 Integral graphs

An *integral* graph is a graph for which all eigenvalues of its adjacency matrix are integers.

Integral graphs have applications such as modeling the quantum spin networks that permit perfect state transfer [3, 6, 18, 19, 20, 43, 44, 59, 74, 85, 86, 87, 149, 159, 165, 174], and modeling multiprocessor interconnection networks especially in connection to the load balancing problem [6, 57, 59, 64].

Harary and Schwenk [98] first introduced the problem of finding integral graphs in 1974. In general, this problem is a difficult one. For this reason, results in the area pertain to investigating particular classes of integral graphs and/or using graph operations or transformations for the construction of integral graphs. There have been many interesting discoveries, a lot of which we will briefly mention here. We also refer the reader to the three survey papers on this topic: *Integral trees – a survey* by Li and Wang [131], *A survey on integral graphs* by Balińska et al. [12], and *A survey of results on integral trees and integral graphs* by Wang [181].

Some examples of integral graphs are the complete graphs, K_n, with spectra $\{(n-1)^1, -1^{n-1}\}$; the cocktail party graphs, $CP(n)$, with spectra $\{(2n-2)^1, 0^n, -2^{n-1}\}$; balanced complete multipartite graphs, $K_{n/k, n/k, \ldots, n/k}$, with spectra $\{(n - n/k)^1, 0^{n-k}, -(n/k)^{k-1}\}$; and complete bipartite graphs, $K_{m,n}$, for which mn a perfect square, with spectra $\{\sqrt{mn}^1, 0^{m+n-2}, -\sqrt{mn}^1\}$ (which includes the stars K_{1,p^2}). The cycles, C_n, have eigenvalues $\{2\cos(2k\pi/n) \mid k = 1, \ldots, n\}$ and thus C_n is only integral for $n = 3, 4, 6$.

We call the set of vertices connected to v by an edge, the *neighbours* of v. A *strongly regular* graph is an r-regular graph with exactly e common neighbours between every pair of adjacent vertices and f common neighbours between every pair of non-adjacent vertices for some r, e, and f. A strongly regular graph G is integral if $(e - f)^2 - 4(f - r) = s^2$ for some positive integer s.

In the original paper on integral graphs [98], Harary and Schwenk make mention of some graph operations which preserve integrality: the product (or conjunction), the sum (or Cartesian product), and the strong sum (or strong product). Let G_1, G_2 be graphs with vertex sets $V(G_1), V(G_2)$. The *product* of G_1 and G_2, $G_1 \times G_2$, is the graph with vertex set the Cartesian product $V(G_1) \times V(G_2)$ with (x, a) adjacent to (y, b) if and only if x is adjacent to y in G_1 and a is adjacent to b in G_2. The

sum of G_1 and G_2, $G_1 + G_2$, is the graph with vertex set the Cartesian product $V(G_1) \times V(G_2)$ with (x, a) adjacent to (y, b) if and only if $x = y$ and a is adjacent to b in G_2 or x is adjacent to y in G_1 and $a = b$. The *strong sum* of G_1 and G_2, $G_1 \oplus G_2$, is the graph with vertex set the Cartesian product $V(G_1) \times V(G_2)$ with (x, a) adjacent to (y, b) if and only if x is equal or adjacent to y in G_1, and a is equal or adjacent to b in G_2. We summarize their conclusions which were based on Schwenk's work on the characteristic polynomial in [160]. Let $\lambda_1, \lambda_2, \ldots, \lambda_m$ be the eigenvalues of G_1 and $\mu_1, \mu_2, \ldots, \mu_p$ be the eigenvalues of G_2; they found that $G_1 \times G_2$ has eigenvalues $\lambda_i \mu_j$, $G_1 + G_2$ has eigenvalues $\lambda_i + \mu_j$, and $G_1 \oplus G_2$ has eigenvalues $\lambda_i \mu_j + \lambda_i + \mu_j$ for $i = 1, \ldots, m$ and $j = 1, \ldots, p$.

The *complete product* of vertex disjoint graphs G_1 and G_2, $G_1 \nabla G_2$, is the graph obtained by joining each vertex of G_1 to all vertices of G_2. The complete product of an r_1-regular graph with n_1 vertices and an r_2-regular graph with n_2 vertices is integral if and only if both graphs are integral and $(r_1 - r_2)^2 + 4n_1 n_2$ is a perfect square [98].

The *complement* of a graph G_1, $\overline{G_1}$, is the graph with $V(\overline{G_1}) = V(G_1)$ and x adjacent to y if and only if x is not adjacent to y in G_1. The *line graph* of a graph G_1, $L(G_1)$, is the graph with $V(L(G_1)) = E(G_1)$ and x adjacent to y if and only if edges x and y in G_1 have a common endpoint. Harary and Schwenk [98] show that given a regular integral graph G_1, the complement $\overline{G_1}$ and the line graph $L(G_1)$ are both integral.

The *subdivision graph* of a graph G_1, $S(G_1)$, is the graph obtained by inserting a single new vertex into each edge of G_1. In [55], it was shown that $L_2(G_1) = L(S(G_1))$ is integral if and only if G_1 is the union of complete graphs all having the same number of vertices.

A more general operation, the *non-complete extended p-sum of graphs* (NEPS), first defined in [53], preserves integrality. Let B be a set of binary m-tuples which does not contain the m-tuple $(0, \ldots, 0)$. The NEPS of graphs $G_1, \ldots, G_m$ with basis B, $G = \mathrm{NEPS}(G_1, \ldots, G_m; B)$, is the graph with vertex set the Cartesian product $V(G_1) \times \cdots \times V(G_m)$, with $(u_1, \ldots, u_m)$ adjacent to $(v_1, \ldots, v_m)$ if and only if there exists $(\beta_1, \ldots, \beta_m) \in B$ such that u_i is adjacent to v_i in G_i whenever $\beta_i = 1$ and $u_i = v_i$ whenever $\beta_i = 0$. The graphs $G_1, \ldots, G_m$ are the factors of G. Thus, $G_1 \times G_2$ is equivalent to $\mathrm{NEPS}(G_1, G_2; \{(1, 1)\})$, $G_1 + G_2$ to $\mathrm{NEPS}(G_1, G_2; \{(0, 1), (1, 0)\})$, and $G_1 \oplus G_2$ to $\mathrm{NEPS}(G_1, G_2; \{(0, 1), (1, 0), (1, 1)\})$. Let graph G_i have order n_i and eigenvalues $\lambda_{i_1}, \ldots, \lambda_{i_{n_i}}$ for $i \in 1, \ldots, m$, the graph $\mathrm{NEPS}(G_1, \ldots, G_m; B)$, where B is the basis of NEPS of graphs $G_1, \ldots, G_m$, has eigenvalues

$$\sum_{\beta \in B} \lambda_{1_{j_1}}^{\beta_1} \cdots \lambda_{m_{j_m}}^{\beta_m}$$

for each $j_k \in \{1, \ldots, n_k\}$ for $k = 1, \ldots, m$ where $0^0 = 1$. If the graphs $G_1, \ldots, G_m$ are all integral then $\mathrm{NEPS}(G_1, \ldots, G_m; B)$, where B is the basis of NEPS of graphs $G_1, \ldots, G_m$, is integral. If the graphs $G_1, \ldots, G_m$ are all regular and integral then $\mathrm{NEPS}(G_1, \ldots, G_m; B)$, where B is the basis of NEPS of graphs $G_1, \ldots, G_m$, is regular and integral [172].

Many infinite families of integral complete s-partite graphs for $s \geqslant 3$ have been constructed [102, 154, 185, 186, 191].

Let pG denote the disjoint union of p copies of G. Lepović presents results on integral graphs which belong to several graph classes: $\overline{\alpha K_{a,b}}$ [123], $\overline{\alpha K_a \cup \beta K_{b,b}}$ [124], $\overline{\alpha K_a \cup \beta K_b}$ [125], $\overline{\alpha K_a \cup \beta K_{b,b}}$ [126], $\overline{\alpha K_{a,a} \cup \beta K_{b,b}}$ [127], $\overline{\alpha K_{a,a,\ldots,a,b,b,\ldots,b}}$ [128].

The *corona* of G_1 and G_2, $G_1 \circ G_2$, is the graph obtained by taking one copy of G_1 and $|V(G_1)|$ copies of G_2 and then joining the ith vertex of G_1 to every vertex in the ith copy of G_2. The following results can be found in [188]: $K_p \circ \overline{K_p}$ is integral when $p = m(m+1)$ for a positive integer m, $K_{a,b} \circ \overline{K_{p^2}}$ is integral when there exist positive integers a, b, c such that $\sqrt{ab}$ is a positive integer and $p^2 = c(c + \sqrt{ab})$, and $K_{a,a,\dots,a} \circ \overline{K_{p^2}}$ is integral where $K_{a,a,\dots,a}$ is a complete s-partite graph if there exist positive integers b, c such that $p^2 = b(b + as - a) = c(c + a)$.

The $(n+1)$-regular graph $K_{n,n+1} \equiv K_{n+1,n}$ on $4n+2$ vertices is obtained from two disjoint copies of $K_{n,n+1}$ with vertex classes $V_1 = \{x_i \mid i = 1, 2, \dots, n\}$, $V_2 = \{y_i \mid i = 1, 2, \dots, n+1\}$ and $U_1 = \{a_i \mid i = 1, 2, \dots, n\}$, $U_2 = \{b_i \mid i = 1, 2, \dots, n+1\}$, respectively, by adding the edges $\{y_i b_i \mid i = 1, 2, \dots, n+1\}$. Let $K_{1,p}$ be a graph with vertex classes $V_1 = \{x_1\}$ and $V_2 = \{y_i \mid i = 1, 2, \dots, p\}$. Let the i-th graph K_p of $(p-1)K_p$ have the vertex set $\{w_{i_j} \mid j = 1, 2, \dots, p\}$, where $i = 1, 2, \dots, p-1$. Then the p-regular graph $K_{1,p}[(p-1)K_p]$ on $p^2 + 1$ vertices is obtained by adding the edges $\{y_i w_{j_i} \mid j = 1, 2, \dots, p-1\}$ for $i = 1, 2, \dots, p$ between the graph $K_{1,p}$ and the graph $(p-1)K_p$. Wang et al. [193] determine that the class of graphs $K_{p,p+1} \equiv K_{p+1,p}$ is integral and that the class of graphs $K_{1,p}[(p-1)K_p]$ is integral when $p = m^2 + m + 1$ for some positive integer m.

Let $k *_G H$ denote the graph obtained by taking k copies of a rooted graph H and joining the root of each copy to all vertices of a graph G. Let $k \bullet H$ denote the graph obtained by identifying the roots of k copies of a rooted graph H. We list a few of the constructions of Indulal and Vijayakumar [108]. Given an r-regular graph G, the graph $k *_G K_{r+2}$ is integral if and only if $(r+2)^2 + 4pk$ is a perfect square. When $k = \frac{m^2 - (r+2)^2}{4p}$ and $m > r + 2$, this gives an infinite family of integral graphs. The graph $k \bullet K_{m,n}$, with any vertex in the class of n vertices taken as the root, is integral if and only if both $m(n-1)$ and $m(n-1) + mk$ are perfect squares.

Wang and Hoede [182] construct 15 infinite classes of non-regular bipartite integral graphs.

Integral trees are another family that has been investigated extensively [31, 32, 36, 41, 42, 98, 99, 100, 101, 129, 130, 133, 183, 184, 187, 188, 189, 190, 192, 194, 196, 197, 200]. Still, much is unknown. These results have only covered trees with diameter less than 11. More recently, Csikvári [48] proved the existence of integral trees with arbitrarily large even diameter. Ghorbani et al. [82] showed the existence of infinitely many integral trees of arbitrarily large odd diameter.

In [61], Cvetković et al. produce the only 7 molecular graphs with integral spectra. In so doing, they find the only connected integral graphs which are not 3-regular with maximum vertex degree three: K_1, K_2, K_3, C_4, C_6, $K_2 \circ 2K_1$, $S(K_{1,3})$.

Radosavljević and Simić [153, 166] show that there are 13 connected non-bipartite integral graphs that are not regular with maximum vertex degree four. In [45], it was shown that up to isomorphism there are 22 non-bipartite integral graphs with spectral radius three. Partial results on non-regular bipartite integral graphs with maximum vertex degree four appear in [14, 15, 16].

Ahmadi et al. [4] consider the possibilities for adjacency matrices of integral graphs. They are able to get a loose upper bound on the number of labeled integral graphs with n vertices as less than or equal to $2^{\frac{n(n-1)}{2} - \frac{n}{400}}$ for sufficiently large n. We mention that the number of labeled graphs on n vertices is $2^{\frac{n(n-1)}{2}}$.

Table 2.1 gives the total number of connected integral graphs of order n for $1 \leqslant n \leqslant 12$ as summarized by Balińska et al. [12]. We note that this table contains the correct value for $n = 11$ from the original source [10] (see also [51]).

The aid of a computer was essential to generating the graphs for $n = 8, \ldots, 12$. The methods and details are documented in the following papers: [8, 9, 10, 11].

n	1	2	3	4	5	6	7	8	9	10	11	12
#	1	1	1	2	3	6	7	22	24	83	113	325

Table 2.1: Connected integral graphs with n vertices

Table 2.2 gives the number of all connected regular integral graphs up to 12 vertices. These graphs were presented with pictures in [10, 11]. The connected regular integral graphs up to $n = 10$ were also documented by Abdollahi and Vatandoost [2]. In error, they missed a graph on 7 vertices by failing to take into account the complement of the disconnected integral graph $C_4 \cup C_3$.

n	1	2	3	4	5	6	7	8	9	10	11	12
#	1	1	1	2	1	5	2	6	7	15	3	60

Table 2.2: Connected regular integral graphs with n vertices

The finiteness of the number of integral graphs was considered in [54]. One of the results there is that the set of connected integral graphs with a given maximum vertex degree is finite.

We reproduce the results of Cvetković et al. [63] in Theorem 7 and Cvetković [53] in Theorem 8 and then deduce that there are only finitely many connected r-regular integral graphs.

The *eccentricity* of a vertex v in a graph G is the maximum distance from v to any vertex in G. The *radius* of a graph G is the minimum eccentricity of any vertex in G. The maximum distance over all pairs of vertices in a graph G is called the *diameter* of G.

Theorem 7. *Let $r \geqslant 3$. Given a connected r-regular bipartite graph, G, with n vertices and radius R,*

$$n \leqslant \frac{2(r-1)^R - 2}{r - 2}.$$

Proof. Choose a vertex, u, with eccentricity R. Let $N_k(u) = \{v \in V \mid d(u, v) = k\}$ and $d_k(u) = |N_k(u)|$. So $d_0(u) = 1$, $d_1(u) = r$, and $d_k(u) \leqslant (r-1)d_{k-1}(u)$ for $k = 2, ..., R-1$ since each $v \in N_{k-1}(u)$ has at most $r-1$ neighbours in $N_k(u)$. By induction on k, it follows that $d_k(u) \leqslant r(r-1)^{k-1}$ for $k = 2, ..., R-1$.

Next we consider $d_R(u)$: Since G is bipartite, each vertex in $N_R(u)$ has all of its r neighbours in $N_{R-1}(u)$. For this reason, the previous argument overcounts each such vertex r times and we get

$d_R(u) \leqslant \frac{(r-1)d_{R-1}(u)}{r} \leqslant \frac{(r-1)r(r-1)^{R-2}}{r} = (r-1)^{R-1}$. Now

$$n = \sum_{k=0}^{R} d_k(u) \leqslant 1 + \sum_{k=1}^{R-1} r(r-1)^{k-1} + (r-1)^{R-1}$$

$$= 1 + \frac{r(1-(r-1)^{R-1})}{1-(r-1)} + (r-1)^{R-1}$$

$$= \frac{(r-2) + r((r-1)^{R-1}-1) + (r-2)(r-1)^{R-1}}{r-2}$$

$$= \frac{((r-2)+r)(r-1)^{R-1} + ((r-2)-r)}{r-2}$$

$$= \frac{2(r-1)^R - 2}{r-2}.$$

$\square$

Note: This bound does not work for $r = 2$ since we find the sum of the geometric series which does not evaluate at this point.

Theorem 8. *For a connected graph G with diameter D, we have $D \leqslant s - 1$, where s is the number of distinct eigenvalues of G.*

Proof. (By contradiction) Suppose $D \geqslant s$. Let $a_{xy}^{(z)}$ denote the (x, y) entry of matrix A^z. Then for some vertices $u, v \in G$, the smallest k such that $a_{uv}^{(k)} \neq 0$ is $k = D$. Now, consider the minimal polynomial of A: $\phi(\lambda) = (\lambda - \lambda_1) \cdots (\lambda - \lambda_s) = \lambda^s + b_1 \lambda^{s-1} + \cdots + b_s \lambda^0$. Now A is a root of this polynomial, so $\phi(A) = 0$ and thus $A^k \phi(A) = A^{k+s} + b_1 A^{k+s-1} + \cdots + b_s A^k = 0$ for all $k \geqslant 0$. Letting $k = D - s$, we get $A^D + b_1 A^{D-1} + \cdots + b_s A^{D-s} = 0$. This implies that $a_{uv}^{(D)} = 0$ which is a contradiction. $\square$

The *bipartite double cover* of G is the bipartite graph $G \times K_2$. Similar to the result by Schwenk [161] used in [173] and [176], we have

Lemma 9. *If G is an r-regular integral graph, then the bipartite double cover of G is a bipartite r-regular integral graph.*

If G is bipartite then the bipartite double cover consists of two disjoint copies of G. We note that from this definition, with the observation that $Sp(K_2) = \{-1, 1\}$, it can be easily seen that the spectrum of a bipartite graph is symmetric about the eigenvalue 0.

An r-regular integral graph, has at most $2r+1$ distinct eigenvalues. Thus $R \leqslant D \leqslant 2r+1-1 = 2r$ by Theorem 8. Given Lemma 9, we have that the maximum order of an r-regular non-bipartite integral graph is bounded above by the maximum order of the r-regular bipartite integral graphs. It follows from Theorem 7 that for a connected r-regular integral graph, $n \leqslant \frac{2(r-1)^{2r}-2}{r-2}$.

Thus, connected 3-regular integral graphs have at most 126 vertices and connected 4-regular integral graphs have at most 6560 vertices.

All the cubic integral graphs have been determined. They were found independently by two sets of authors [37, 54, 161] using different methods. In [54] and [37], the authors considered all possible sets

of distinct eigenvalues, a polynomial by Hoffman [104], and the spectral moments w_1, w_2, and w_3; and
then used theoretical reasoning and a computer search to find the graphs that exist. Schwenk [161]
proved Lemma 9 for 3-regular $G = H \times K_2$ and determined the graphs G by theoretical reasoning
before deriving the graphs H. The *girth* of a graph is the length of the shortest cycle contained
in the graph. There are exactly 13 connected cubic integral graphs: K_4, $K_{3,3}$, $C_3 + K_2$, $C_4 + K_2$,
the Petersen graph, the graph obtained from $K_{3,3}$ by specifying a pair of non-adjacent vertices and
replacing each of them by a triangle, the graph obtained by taking two copies of $K_{2,3}$ and adding
the edges of a matching of size 3 between the pairs of size 3 in each copy, $L_2(K_4)$, $C_6 + K_2$, the
Desargues' graph and its co-spectral mate, a bipartite graph on 24 vertices with girth 6, and the
Tutte 8-cage. These graphs have largest order 30 which is significantly less than 126.

The quartic integral graphs have not yet been fully determined.

2.4   Quartic integral graphs

We use the acronym QIG as shorthand for connected quartic integral graph. Since all non-bipartite
QIGs have a bipartite double cover that is a 4-regular integral graph, finding all QIGs involves first
considering only bipartite graphs.

For connected 4-regular bipartite integral graphs, $Sp(G) = \{4, 3^x, 2^y, 1^z, 0^{2w}, -1^z, -2^y, -3^x, -4\}$;
which we abbreviate by simply specifying the quadruple $[x, y, z, w]$.

Cvetković et al. [63] found quadruples $[x, y, z, w]$ that are candidates for the spectrum of a bipartite
QIG. They called these *possible spectra*. For 4-regular bipartite G, they first produced equations for
w_ℓ up to $\ell = 6$:

$$
\begin{aligned}
w_0 &= n, & w_4 &= 28n + 8[C_4], \\
w_2 &= 4n, & w_6 &= 232n + 144[C_4] + 12[C_6].
\end{aligned}
\tag{2.3}
$$

Cvetković et al. then added the following result of Hoffman [104]. Let J denote the square matrix of
order n with every entry 1 and I denote the identity matrix of order n. Given a connected r-regular
graph G of order n with adjacency matrix A and distinct eigenvalues $\mu_1 = r, \mu_2, \ldots, \mu_j$,

$$
J \prod_{i=2}^{j} (r - \mu_i) = n \prod_{i=2}^{j} (A - \mu_i I).
$$

Cvetković et al. divide the possible spectra into various cases depending on whether values x, y, z, w
are zero or non-zero and find all solutions to equations (2.3) that satisfy the Hoffman identity for
$n \leqslant 6560$. Using these methods, they lower the upper bound on the number of vertices of a QIG to
5040.

The last section of [63] details the constructions used to derive several QIGs. For each of the
cubic integral graphs, G_i for $i = 1, \ldots, 13$, the graphs $L(G_i)$, $L(G_i) \times K_2$, $G_i + K_2$, and $G_i \oplus K_2$ are
QIGs.

Let q_v denote the number of 4-cycles to which the vertex v belongs. Stevanović [171] considers
$Q_v = \sum_{i=1}^{4} q_{v_i}$, where $v_1, \ldots, v_4$ are the neighbours of v, for all vertices v of a graph G. He uses
the set of 4 equations for each v arising from graph angles [49, 50, 56] - a generalization of (2.3) -
to further eliminate spectra from the list of possible spectra. Stevanović is able to lower the upper

bound on the number of vertices of a QIG to 1260 with the exception of 5 spectra (one of which has 5040 vertices).

Let H with eigenvalues $\mu_1, \mu_2, \ldots, \mu_m$ be a (vertex) induced subgraph of a graph G with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Cauchy's interlacing theorem (see [62], p.19) states that $\lambda_i \geqslant \mu_i \geqslant \lambda_{n-m+i}$ for $i = 1, \ldots, m$. Stevanović [173] considers a vertex v of a desired unknown graph G. He then uses q_v, an equation determined by counting the walks of length 2 starting at v in two ways, and a derivation of the Hoffman identity for 5 (and then 6) distinct eigenvalues to find limited possibilities for the number of common neighbours between v and other vertices of G. From this information a large part of a hypothetical QIG is constructed and tested against Cauchy's interlacing theorem for certain subgraphs. Stevanović determines that there are 16 bipartite QIGs avoiding ± 3 in their spectrum. It follows that there are 8 non-bipartite QIGs avoiding ± 3 in their spectrum. These 24 graphs are depicted in [173]. The complete proof is given in [172].

As a result of the efforts published in [63, 171, 173], there are 1888 possible spectra printed in [63] as well as a list of 65 known QIGs. These graphs are given a label $I_{n,index}$ and are accompanied by the spectrum and sometimes a short description.

Stevanović et al. [176] extend the equations for the ℓth spectral moment by adding an inequality for $\ell = 8$:

$$w_8 \geqslant 2092n + 2024[C_4] + 288[C_6]. \tag{2.4}$$

They make use of a correspondence between closed walks in an r-regular graph and walks in an infinite r-regular tree and find recurrence relations for the number of closed walks. Using this new inequality for w_8, Stevanović et al. are able to reduce the list of possible spectra to a list with 828 entries. The upper bound for n is improved to give $8 \leqslant n \leqslant 560$. All QIGs with $n \leqslant 24$ that realize one of the possible spectra are found and listed with drawings in [176]. Included in these graphs, are 14 non-bipartite QIGs but these were all previously known by the results in [9, 10, 11].

Stevanović [172] showed that NEPS$(G_1, \ldots, G_m; B)$, where $G_1, \ldots, G_m$ are each themselves representable as a NEPS of graphs is isomorphic to NEPS of their factors; $G_1, \ldots, G_m$; for some basis. This result enables Stevanović to simply consider NEPS$(G_1, \ldots, G_m; B)$ where $G_1, \ldots, G_m$ are connected regular integral graphs which are not representable as a NEPS of graphs to determine the QIGs that are a NEPS of graphs. A NEPS of graphs that is isomorphic to a QIG satisfies the following conditions:

- If the degree of the largest factor of a NEPS is at most 2 then all factors of the NEPS are C_3 or K_2. There are 14 non-isomorphic QIGs that are a NEPS of this type.

- If the degree, Δ, of the largest factor is strictly greater than 2 then one of the factors is a Δ-regular integral graph and the remaining factors are isomorphic to K_2. Graphs of this type are of the form $H \times K_2$ where H is a non-bipartite QIG, $H + K_2$ where H is a 3-regular integral graph, or $H \oplus K_2$ where H is a non-bipartite 3-regular graph.

Balińska et al. [13] present exact and randomized algorithms for generating QIGs. Using two of their algorithms, they are able to produce all QIGs for $12 < n < 20$ by a computer search. These graphs are given as the spectrum with the number of co-spectral graphs for that spectrum. A third algorithm is suggested for enumerating the graphs with at least 20 vertices but not attempted.

Stevanović [175] considers finding r-regular bipartite integral graphs with 5 distinct eigenvalues: the graphs with spectrum of the form $\{r, x^y, 0^{2z}, -x^y, -r\}$. Of course, this covers more than the case: $r = 4$ but we mention it here. The following result is proved by Stevanović.

Lemma 10. *The number of closed walks of length* 6 *in an* r-*regular bipartite graph* G *with* n *vertices is equal to* $nr(5r^2 - 6r + 2) + 48(r - 1)[C_4] + 12[C_6]$.

The equations w_0, w_2, and w_4 for r-regular graphs (similar to Equations (2.3) but w_4 requires a counting argument) and w_6 given by Lemma 10 are used to find an extensive list of candidate spectra for graphs with 5 distinct eigenvalues. Construction and non-existence results follow for some of these spectra which can be found in [175].

2.5   Cayley integral graphs

A *Cayley graph* $\mathrm{Cay}(\Gamma, S)$ for a group Γ and *connection set* $S \subset \Gamma$ is the graph with vertex set Γ and with a connected to b if and only if $ba^{-1} \in S$. Abdollahi and Vatandoost [2] determine which of the known regular integral graphs up to $n = 11$ are Cayley integral graphs. We summarize their results in Table 2.3.

n	1	2	3	4	5	6	7	8	9	10	11
#	1	1	1	2	1	5	1	6	3	5	1

Table 2.3: Connected Cayley integral graphs with n vertices

Given groups Γ_1 and Γ_2, a *group homomorphism* is a function $\alpha : \Gamma_1 \to \Gamma_2$ such that $\alpha(gh) = \alpha(g)\alpha(h)$. Let $\mathbb{C}^*$ denote the group of non-zero complex numbers with multiplication. A character of a finite group Γ is a homomorphism $\chi : \Gamma \to \mathbb{C}^*$. The eigenvalues of a Cayley graph $\mathrm{Cay}(\Gamma, S)$ are equal to $\sum_{s \in S} \chi(s)$ for each character χ of Γ [134]. Lovász also shows that if Γ is an Abelian group, then there are precisely n distinct characters of Γ.

So [169] characterizes integral Cayley graphs over cyclic groups with the following result: for a cyclic group Γ, $G = \mathrm{Cay}(\Gamma, S)$ is integral if and only if $\sum_{k \in S} \omega^k$ is an integer where ω is a primitive n-th root of unity of Γ. A *circulant* graph on n vertices is a Cayley graph $\mathrm{Cay}(\mathbb{Z}_n, S)$ for some $S \subseteq \mathbb{Z}_n \backslash \{0\}$. Known examples of circulant graphs include complete graphs and cycles. So was able to prove that there are precisely $2^{\tau(n)-1}$ non-isomorphic integral circulant graphs on n vertices, where there are $\tau(n)$ divisors of n.

We define the greatest common divisor of k-tuples of non-negative integers $x = (x_1, \ldots, x_k)$ and $m = (m_1, \ldots, m_k)$ to be $\gcd(x, m) = (d_1, \ldots, d_k) = d$ where $d_i = \gcd(x_i, m_i)$ for $i = 1, \ldots, k$. Let Γ be an abelian group represented as a direct product of cyclic groups, $\Gamma = \mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_k}, m_i \geqslant 1$ for $i = 1, \ldots, k$. For the divisor tuple $d = (d_1, \ldots, d_k)$ of $m = (m_1, \ldots, m_k)$ where d_i is a divisor of m_i for $i = 1, \ldots, k$; we define $S_\Gamma(d) = \{x \in \Gamma \mid \gcd(x, m) = d\}$. We define the set $S_\Gamma(D) = \bigcup_{i=1}^{j} S_\Gamma(d^{(i)})$ where $D = \{d^{(1)}, \ldots, d^{(j)}\}$ is a set of distinct divisor tuples of m. A *gcd-graph* over a finite Abelian group Γ, is a Cayley graph $\mathrm{Cay}(\Gamma, S_\Gamma(D))$. It follows from a result of So [169] that the graphs $\mathrm{Cay}(\mathbb{Z}_n, S_{\mathbb{Z}_n}(\{1\}))$ (the unitary Cayley graphs) are integral (see also [116, 158]) and more generally that the graphs $\mathrm{Cay}(\mathbb{Z}_n, S_{\mathbb{Z}_n}(D))$ are integral. Klotz and Sander [118] prove that every gcd-graph

$\mathrm{Cay}(\Gamma, S_\Gamma(D))$ is integral. In [119], Klotz and Sander prove that if G is a NEPS of complete graphs $\mathrm{NEPS}(K_{m_1}, \ldots, K_{m_k}; B)$, then G is isomorphic to a gcd-graph over $\Gamma = \mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_k}$.

An r-regular graph G with eigenvalues $\lambda_1, \ldots, \lambda_n$ is a *Ramanujan* graph if for $n > 3$, $\lambda(G) = \max_{|\lambda_i| < r} |\lambda_i|$ is defined, and $\lambda(G) \leqslant 2\sqrt{r-1}$. Droll [70] obtains a complete characterization of the cases in which the unitary Cayley graph $\mathrm{Cay}(\mathbb{Z}_n, S_{\mathbb{Z}_n}(\{1\}))$ is a Ramanujan graph. Let p be a prime number and s be a positive integer such that $p^s \geqslant 3$. Le and Sander [122] extended the result of Droll by classifying the graphs $\mathrm{Cay}(\mathbb{Z}_{p^s}, S_{\mathbb{Z}_{p^s}}(D))$ that are Ramanujan for each prime power p^s and arbitrary divisor tuple D.

Let the Boolean algebra $\mathbb{B}(\Gamma)$ generated by the subgroups of Γ denote the lattice of subsets obtained by arbitrary finite intersections, unions, and complements of the subgroups. For an Abelian group Γ, S belongs to the Boolean algebra $\mathbb{B}(\Gamma)$ generated by the subgroups of Γ if and only if $\mathrm{Cay}(\Gamma, S)$ is integral. This result was proven by various authors over time. In [117], Klotz and Sander prove that for an Abelian group Γ, if S belongs to the Boolean algebra $\mathbb{B}(\Gamma)$ generated by the subgroups of Γ, then $\mathrm{Cay}(\Gamma, S)$ is integral. If Γ is a cyclic group then So [169] shows that the converse is true. Finally it is proved in [5, 118] that the converse is true when Γ is an Abelian group.

The following are some examples of graphs that satisfy the previous result. An m-Sudoku puzzle, is an $m \times m$ arrangement of $m \times m$ blocks of cells such that each cell is either empty or contains a number from 1 to m^2 and there is a unique way to fill in the empty cells so that every row, column, and block of the puzzle contains all of the numbers 1 to m^2 exactly once. A Sudoku graph, $\mathrm{Sud}(m)$, is a graph with vertices corresponding to the m^4 cells of the puzzle and x adjacent to y if and only if x and y correspond to cells from the same row, column, or block. Sander [157] shows that for $m \in \mathbb{Z}^+$, $\mathrm{Sud}(m)$ is isomorphic to the NEPS of graphs $\mathrm{NEPS}(K_m, K_m, K_m, K_m; \{(0,1,0,1),(1,1,0,0),(0,0,1,1),(1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1)\})$. Thus m-Sudoku puzzles are integral graphs. In [117], Klotz and Sander show that for an Abelian group Γ and $S \in \mathbb{B}(\Gamma)$ that $Sud(n)$ is an integral Cayley graph $\mathrm{Cay}(\Gamma, S)$. Similarly they show that a variation of Sudoku, *positional Sudoku*, has graph $\mathrm{SudP}(m)$ that is an integral Cayley graph. In the same paper we see that the Hamming graph is another example of an integral Cayley graph on an Abelian group Γ with $S \in \mathbb{B}(\Gamma)$ (see also [158]).

A *multiset* is a set S with multiplicity function $\mu_S : S \to \mathbb{N}$ where we say $\mu_S(a) = 0$ for all $a \notin S$. A *Cayley multigraph* $\mathrm{MCay}(\Gamma, S)$ is a Cayley graph with multiple edges permitted such that the number of edges joining a to b in $\mathrm{MCay}(\Gamma, S)$ is $\mu_S(ba^{-1})$. Let the collection $\mathbb{C}(\Gamma)$, or *integral cone* over $\mathbb{B}(\Gamma)$, be the Boolean algebra of all multisets that can be expressed as non-negative integer combinations of the subgroups that generate $\mathbb{B}(\Gamma)$. DeVos et al. [69] prove that for every finite group Γ and every S in $\mathbb{C}(\Gamma)$, where $\mathbb{B}(\Gamma)$ is generated by the normal subgroups of Γ, the Cayley multigraph $\mathrm{MCay}(\Gamma, S)$ is integral.

Klotz and Sander [117] show that if every Cayley graph $\mathrm{Cay}(\Gamma, S)$ over Γ is integral then $\Gamma \in \{\mathbb{Z}_2^s, \mathbb{Z}_3^s, \mathbb{Z}_4^s, \mathbb{Z}_2^s \times \mathbb{Z}_3^t, \mathbb{Z}_2^s \times \mathbb{Z}_4^t\}, s \geqslant 1, t \geqslant 1$.

Abdollahi and Vatandoost [1] showed that there are exactly 7 connected cubic integral Cayley graphs. They determine that $\mathrm{Cay}(\Gamma, S)$ with $|S| = 3$ is integral if and only if Γ is isomorphic to one of the following groups: $\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_4, \mathbb{Z}_6, S_3, \mathbb{Z}_2^3, \mathbb{Z}_2 \times \mathbb{Z}_4, D_8, \mathbb{Z}_2 \times \mathbb{Z}_6, D_{12}, A_4, S_4, D_8 \times \mathbb{Z}_3, D_6 \times \mathbb{Z}_4$, or $A_4 \times \mathbb{Z}_2$.

The orders possible for Cayley QIGs on finite Abelian groups have been determined by Abdollahi and Vatandoost [2]. They showed that for an Abelian group, Γ, if $\mathrm{Cay}(\Gamma, S)$ is a Cayley QIG then

$$|\Gamma| \in \{5, 6, 8, 9, 10, 12, 16, 18, 20, 24, 25, 32, 36, 40, 48, 50, 60, 64, 72, 80, 96, 100, 120, 144\},$$

but they did not establish whether Cayley QIGs of these orders exist. We settle this in Section 4.4.

2.6 Strongly regular graphs

We denote a strongly regular graph with n vertices of degree r, e common neighbours between adjacent vertices, and f common neighbours between non-adjacent vertices by $\mathrm{SR}(n, r, e, f)$. Strongly regular graphs conceptually arose as an association scheme with two classes in connection with the theory of partially balanced designs [28]. They were introduced in this way by Bose [26]. In the same paper, he relates strongly regular graphs to partial geometries. A *partial geometry* $\mathrm{PG}(K, R, T)$ is a partial linear space $(X, \mathscr{L})$ with constant line size K such that each point is on R lines and given a line $L \in \mathscr{L}$ and a point $x \notin L$ there are exactly T lines through x that meet L. The *point graph* of a partial geometry has the points as vertices and x adjacent to y if and only if $x \in L$ and $y \in L$ for some $L \in \mathscr{L}$. The point graph of a $\mathrm{PG}(K, R, T)$ is an $\mathrm{SR}(K(1 + \frac{(K-1)(R-1)}{T}), R(K-1), (K-2) + (R-1)(T-1), RT)$ (and possibly the complement of a disconnected graph). For this reason, the necessary conditions for a strongly regular graph give necessary conditions for the existence of a partial geometry. A strongly regular graph is called *pseudo-geometric* if it has a parameter set such that it could be the point graph of a partial geometry and *geometric* if it is the point graph of a partial geometry. Bose [26] proved that if a pseudo-geometric strongly regular graph, corresponding to a $\mathrm{PG}(K, R, T)$, satisfies

$$2K > R(R - 1) + T(R + 1)(R^2 - 2R + 2) \tag{2.5}$$

then the graph is geometric. Bose goes on to consider pseudo-geometric strongly regular graphs as: mutually orthogonal latin squares, Lattice designs, triangular graphs, balanced incomplete block designs (specifically singly linked block schemes), and elliptic non-degenerate quadrics.

We refer the reader to the survey papers on the topic: Brouwer and van Lint [35], Hubaut [106], Seidel [164].

The complement of an $\mathrm{SR}(n, r, e, f)$ is an $\mathrm{SR}(n, n - r - 1, n - 2r + e - 2, n - 2r + f)$. When $f = 0$, the strongly regular graphs consist of a disjoint union of complete subgraphs of equal size. From this point on we will only consider the strongly regular graphs for which $\mathrm{SR}(n, r, e, f)$ and its complement are connected. For this reason, we will have $f > 0$.

Some examples of strongly regular graphs are the Moore graphs (see Section 2.7) and the triangular graphs. A *triangular graph*, $T(v)$, is the graph with vertices the unordered pairs from a set of $v \geqslant 5$ elements with x adjacent to y whenever they have an element in common.

There are some necessary conditions for the existence of a strongly regular graph with parameter set (n, r, e, f) (see for example [39]):

- $0 < f < r < n - 1$;

- from the parameters of the complementary graph it follows that $n - 2r + f - 2 \geqslant 0$;

- and most significantly, the spectrum is $Sp(G) = \{r^1, \lambda_1^{m_1}, \lambda_2^{m_2}\}$ where λ_1 is strictly greater than λ_2, $m_1 = \frac{-r(\lambda_1+1)(r-\lambda_1)}{(r+\lambda_1\lambda_2)(\lambda_1-\lambda_2)}$, and $m_2 = \frac{r(\lambda_2+1)(r-\lambda_2)}{(r+\lambda_1\lambda_2)(\lambda_1-\lambda_2)}$.

Clearly, m_1 and m_2 must be non-negative integers.

Based on these conditions and since $f \neq 0$, the eigenvalues r, λ_1, and λ_2 must all be distinct. It also follows that a regular connected graph and its complement are strongly regular with $f \neq 0$ if its adjacency matrix has three eigenvalues.

Where these necessary conditions are met, a parameter set (n, r, e, f) is considered *feasible*. For a feasible parameter set, it is of interest to determine firstly whether it is realizable by a graph and secondly to find all graphs $SR(n, r, e, f)$. A list of feasible parameter sets and whether or not they are realizable, including references, can be found in [30]. Currently, the smallest feasible parameter set for which the existence question remains unanswered is $(65, 32, 15, 16)$.

Let E_j for $0 \leqslant j \leqslant 2$ be a basis of minimal idempotents of the vector space $< I, A, \overline{A} >$ with respect to matrix multiplication $\circ$. We define the *Krein parameters* q_{ij}^k from the equation

$$E_i \circ E_j = \sum_{k=0}^{2} q_{ij}^k E_k.$$

A *k-clique* of a graph G is a set of k pairwise adjacent vertices of G. A *co-clique* is a set of pairwise non-adjacent vertices.

We mention the results that have been used to determine that no graph with a given feasible parameter set is possible. Each of the following have lead to new non-existence results (see Table 2.4):

- **The Half-Case:** If $m_1 = m_2$ then $(n, r, e, f) = (4f+1, 2f, f-1, f)$. A $SR(4f+1, 2f, f-1, f)$ exists if and only if a conference matrix C of order $n + 1$ exists. Such a C can only exist if n is the sum of two squares (see for example [22]).

- **The Krein Conditions:** The Krein parameters can be shown to be non-negative and by computing values for the parameters the four Krein conditions were found [67, 103, 121, 162]. The two non-trivial conditions, q_{11}^1 and q_{22}^2, can be written as Krein1: $(\lambda_1+1)(r+\lambda_1+2\lambda_1\lambda_2) \leqslant (r + \lambda_1)(\lambda_2 + 1)^2$, and Krein2: $(\lambda_2 + 1)(r + \lambda_2 + 2\lambda_1\lambda_2) \leqslant (r + \lambda_2)(\lambda_1 + 1)^2$.

- **The Absolute Bound:** There is a result that bounds the number of vectors which make only two distinct angles with each other (see for example [68, 120]). The result is used to form a spherical 2-distance set with vectors of the ith eigenspace of the adjacency matrix of a strongly regular graph together with a matrix E_i for $i = 1, 2$ (as in the basis matrices in the Krein parameters). The following bound is derived as a consequence (see for example [164]): $n \leqslant \frac{1}{2}m_1(m_1+3)$ and similarly $n \leqslant \frac{1}{2}m_2(m_2+3)$. This bound was later improved by Neumaier [145] to $n \leqslant \frac{1}{2}m_1(m_1 + 1)$ if $q_{11}^1 \neq 0$.

- **The Claw Bound:** Neumaier [144] took bound (2.5), for when a strongly regular graph is geometric, and furthered the ideas. In a form suggested by Brouwer and van Lint (see [164] for example), it reads: if $f \neq \lambda_2^2$, $f \neq \lambda_2(\lambda_2 + 1)$ then $2(\lambda_1 + 1) \leqslant \lambda_2(\lambda_2 + 1)(f + 1)$.

- **The Case $f = 1$:** If $f = 1$ then the graph induced by the neighbours of a vertex is a union of complete graphs on $(e+1)$ vertices and thus $(e+1)$ divides r. Furthermore, $\frac{nr}{(e+1)(e+2)}$ is the total number of complete graphs on $(e+2)$ vertices (remarked by [27]).

- **The Case $f = 2$:** If $f = 2$ then the neighbours of a vertex structurally form a partial linear space with girth at least five. Thus $(e+1)$ divides r if $r < \frac{1}{2}e(e+3)$ [34].

	n	r	e	f	Comments
-	21	10	4	5	Half-case
-	28	9	0	4	Krein2; absolute bound
		18	12	10	Krein1; absolute bound
-	33	16	7	8	Half-case
-	49	16	3	6	Bussemaker et al. [38]
		32	21	20	
-	50	21	4	12	Absolute bound
		28	18	12	Absolute bound
-	56	22	3	12	Krein2; absolute bound
		33	22	15	Krein1; absolute bound
-	57	14	1	4	Wilbrink and Brouwer [199]
		42	31	30	
-	57	28	13	14	Half-case
-	63	22	1	11	Krein2; absolute bound
		40	28	20	
-	64	21	0	10	Krein2; absolute bound
		42	30	22	Krein1; absolute bound
-	64	30	18	10	Absolute bound
		33	12	22	Absolute bound
?	65	32	15	16	
?	69	20	7	5	
		48	32	36	
-	69	34	16	17	Half-case
?	75	32	10	16	PG(5, 8, 2)?
		42	25	21	
-	76	21	2	7	Haemers [93]
		54	39	36	
?	76	30	8	14	
		45	28	24	
?	76	35	18	14	
		40	18	24	
-	77	38	18	19	Half-case
-	81	40	13	26	Absolute bound
		40	25	14	Absolute bound
?	85	14	3	2	
		70	57	60	
?	85	30	11	10	
		54	33	36	
?	85	42	20	21	
?	88	27	6	9	
		60	41	40	
-	93	46	22	23	Half-case
?	95	40	12	20	
		54	33	27	
?	96	35	10	14	
		60	38	36	
?	96	38	10	18	
		57	36	30	
?	96	45	24	18	
		50	22	30	
?	99	14	1	2	
		84	71	72	
?	99	42	21	15	
		56	28	36	
?	100	33	8	12	
		66	44	42	
-	100	33	18	7	Absolute bound
		66	39	52	Absolute bound

Table 2.4: Strongly regular graph non-existence results for $n \leqslant 100$

- $(49, 16, 3, 6)$: Bussemaker et al. [38] use a combination of counting techniques, enumeration, linear algebra, and computer computation to eliminate this feasible parameter set.

- $(57, 14, 1, 4)$: Wilbrink and Brouwer [199] prove that a strongly regular graph with this parameter set cannot exist using counting arguments for subgraphs. More specifically they prove and use the following result: Let C be a co-clique in an $\mathrm{SR}(n, r, e, f)$ with smallest eigenvalue λ_2. Then, $|C| \leqslant \frac{n(-\lambda_2)}{r - \lambda_2}$ and equality holds if and only if each vertex not in C is adjacent to exactly $\frac{r|C|}{n - |C|}$ points of C.

- $(76, 21, 2, 7)$: The $\mathrm{PG}(3, 6, 1)$ with point graph an $\mathrm{SR}(76, 21, 2, 7)$, was proven not to exist (see for example [147]). In [93], Haemers showed that the pseudo-geometric $\mathrm{SR}(76, 21, 2, 7)$ would have to be geometric and therefore cannot exist.

More recently, the possible prime divisors of the order of automorphism groups of unknown graphs have been examined. Makhnev and Minakov [137] show that if a $\mathrm{SR}(99, 14, 1, 2)$ exists then the order of the automorphism group is divisible by a prime $p = 2, 3, 7$, or 11. Paduchikh [146] shows that if a $\mathrm{SR}(85, 14, 3, 2)$ exists then the order of the automorphism group is divisible by a prime $p = 2, 3, 5, 7$, or 17. Behbahani and Lam [21] are able to determine the possible prime divisors of the orders for all 16 unknown strongly regular graphs of Table 2.4. This includes an improvement for $\mathrm{SR}(99, 14, 1, 2)$ to $p = 2$ or 3 and for $\mathrm{SR}(85, 14, 3, 2)$ to $p = 2$.

Other more recent non-existence results for $n > 100$ include the following.

- $(486, 165, 36, 66)$: In [136], the subgraphs induced by the neighbours of any given vertex are considered. Using this information, they are able to prove that a pseudo-geometric graph $\mathrm{SR}(486, 165, 36, 66)$ for a $\mathrm{PG}(6, 33, 2)$ does not to exist.

- $(324, 57, 0, 12)$: It is shown by exhaustive computer search that the known list of $2\text{-}(k(k - 1)/2 + 1, k, 2)$ symmetric designs for $k = 11$ is complete. This result implies that there exists no $\mathrm{SR}(324, 57, 0, 12)$ [115].

2.7 Moore graphs

A *Moore* graph is a strongly regular graph with $e = 0$ and $f = 1$. Equivalently, it is a graph with diameter D and girth $2D + 1$. Two trivial infinite families of Moore graphs are the odd cycles C_{2k+1} and the complete graphs K_n.

The problem of describing such graphs was originally posed by E. F. Moore in 1960 (see [105] for example) as one of finding regular graphs with degree Δ and diameter D for which $n = 1 + \Delta \sum_{i=1}^{D} (\Delta - 1)^{i-1}$. Thus, this value for n is called the *Moore bound*. This is an upper bound on the number of vertices in the greater problem of finding large graphs with specified degree and diameter. This bound is easily derived. Designate one vertex as the root of the desired graph with degree Δ and diameter D. Then, sum up the maximum number of vertices possible at each distance from the root:

$$1 + \Delta + \Delta(\Delta - 1) + \cdots + \Delta(\Delta - 1)^{D-1}.$$

Moore used number theory to show that some pairings of Δ and D are not possible.

Hoffman and Singleton tackled this problem in [105]. They define an important recurrence from the Moore bound: for degree $\Delta \geqslant 0$, $F_{i+1}(x) = xF_i(x) - (\Delta - 1)F_{i-1}(x)$ where $F_1(x) = x + 1$ and $F_0(x) = 1$. They then prove that the adjacency matrix A of a Moore graph satisfies the equation $F_D(A) = J$. From considering this equation when $D = 2$ or 3, they are able to determine the possible eigenvalues and multiplicities of a Moore graph and thus the (order, degree, spectrum) triplets possible for each diameter. Their methods use various properties of the matrices and some results on the irreducibility of the polynomial $F_D(x)$ over the rationals. The techniques used for their uniqueness results involved determining certain permutation matrices corresponding to the relationships between vertices at distance D and examining their properties. We summarize their contributions.

- If a Moore graph of diameter 2 exists then $\Delta = 2, 3, 7$, or 57.

- For these first three values of Δ unique graphs exist; the pentagon, the Petersen graph, and the Hoffman-Singleton graph.

- For diameter 3, the unique Moore graph is the heptagon.

Later, two different combinatorial proofs that the Hoffman-Singleton graph is unique were published by James [109] and by Fan and Schwenk [75].

It is still not known whether a graph G, with degree $\Delta = 57$ and diameter 2 exists although its parameter set is feasible. If it exists, $n = 3250$, $Sp(G) = \{57^1, 7^{1729}, -8^{1520}\}$, $\mathrm{Aut}(G)$ is not a rank 3 group [7], and G is not vertex-transitive (see for example [40]). More recent work on the missing Moore graph has been done in investigating the symmetries and the automorphism group of G. Makhnev and Paduchikh [138] show that $|\mathrm{Aut}(G)| \leqslant 550$ if $\mathrm{Aut}(G)$ has even order. The same authors further investigate $\mathrm{Aut}(G)$ and give possibilities for its order and what must divide the order if it is not even [139]. Mačaj and Širáň [135] prove that if $|\mathrm{Aut}(G)|$ is odd then $|\mathrm{Aut}(G)|$ divides $19 \times 3^2, 13 \times 3, 5^2 \times 11, 7^2 \times 3, 7 \times 5, 5^3 \times 3$, or $3^3 \times 5$; and if $|\mathrm{Aut}(G)|$ is even then $|\mathrm{Aut}(G)|$ divides $11 \times 5 \times 2, 5^2 \times 2, 3^3 \times 2$, or $2p, p \in \{7, 11, 19\}$. Thus, $|\mathrm{Aut}(G)| \leqslant 375$ if it is odd and $|\mathrm{Aut}(G)| \leqslant 110$ if it is even.

For $\Delta \geqslant 3$ and $D \geqslant 3$, no Moore graphs exist. Independently, Damerell [65] and Bannai and Ito [17] were able to prove this result. Many partial non-existence results were also obtained [25, 79, 150].

A (Δ, g)-*cage* is a regular graph of degree Δ and girth g with minimum possible order. The Moore bound is a lower bound on the number of vertices for a $(\Delta, 2D + 1)$-cage (see for example [23]). Thus a (Δ, g)-cage of order $n = 1 + \Delta \sum_{i=1}^{D} (\Delta - 1)^{i-1}$ is a Moore graph if $g = 2D + 1$. Using this connection between Moore graphs and cages, Singleton [167] showed that a graph with diameter D and girth $2D + 1$ must be regular.

Given a fixed integer $s \geqslant 3$, conditions called uniqueness and exchange are defined in [66] that relate the universal cover U of a graph to a family A of connected 2-regular subgraphs of U. These conditions involve considering arcs of length s contained in A. Delgado and Stellmacher are able to prove that if these uniqueness and exchange conditions are satisfied, an equivalence relation on the universal cover of a graph whose quotient graph is a bipartite graph of diameter $s - 1$ and girth $2(s - 1)$ can be defined. In [177], van Bon determines that if U and A satisfy uniqueness and shifted exchange conditions, then the relation $x \equiv y$ if and only if $d(x, y) = 2s - 1$ and there exists an element

of A containing x and y, extends to an equivalence relation on the universal cover of a graph such that the quotient graph is a Moore graph of diameter $s - 1$. The results of this paper are used to characterize the Moore graphs. Then, van Bon is able to reconstruct the Hoffman-Singleton graph based on these conditions.

Other research on Moore graphs which we will not cover includes topics such as generalized Moore graphs, radial Moore graphs, directed Moore graphs, and mixed Moore graphs. There is a survey paper on Moore graphs and the degree/diameter problem which covers Moore graphs and these more general topics in greater detail [143].

Chapter 3

Spectral Moments of Regular Graphs in Terms of Subgraph Counts

3.1 Spectral moments equations

Our aim is to provide equations that give the spectral moments of a regular graph as a linear combination of the number of copies of certain small subgraphs of the graph. By using generating functions to count a certain type of closed walk, we extend the list of equations to beyond what appears in the current literature. We give a general method for finding equations such as (2.3) for spectral moments of regular graphs, whether or not they are bipartite. The subgraphs that appear in our equations will be called *contributors* (in (2.3) all of the contributors are cycles, but this is not true in general). Our methods build on those of Friedland et al. [78] and Wanless [195] mentioned in Section 2.2. Using a special class of closed walks, similar to the "primitive circuits" used by McKay [141], we will derive generating functions that give expressions for all closed walks in terms of the counts of contributors.

Our notation for contributors is as follows. We let $C_{i_1 - i_2}$ be an i_1-cycle joined to an i_2-cycle by an edge. The graphs $C_{i_1 \cdot i_2 \cdots i_h}$ and $\Theta_{i_1, i_2, \ldots, i_h}$ were described in Section 2.2. Examples of this notation appear in Figure 3.1. If at any point we encounter contributors that cannot be described by our notation, we draw a picture of the subgraph like those in Figure 3.1.

(a) $C_{4 \cdot 3 \cdot 3}$ (b) C_{4-3} (c) $\Theta_{2,2,1}$ (d) K_4
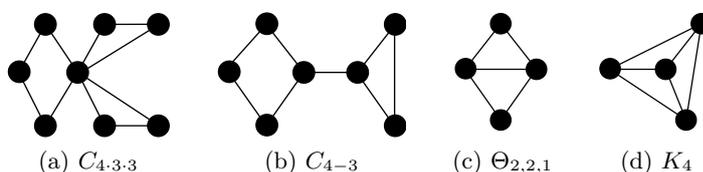
Figure 3.1: Contributor Notation

Formally, we define a contributor to be a connected graph with minimum degree at least 2. As we will show in Theorem 15, a contributor affects the spectral moment, w_i, if and only if its edges can be covered by a closed walk of length i. Any given w_i is only affected by finitely many contributors because a contributor with j edges can only affect w_i if $j \leqslant i$. We note that although our contributors are the same as in [195], they are used differently because we are counting slightly different types of walks.

3.2 Base walks and extensions

Let $z_\ell = w_\ell - u_\ell$ where we reiterate the fact that w_ℓ is the number of closed walks of length ℓ in G and u_ℓ is the number of totally-reducible walks of length ℓ in G.

Combining u_ℓ, as given by (2.2), with our methods to find z_ℓ, which we develop, we will achieve our goal of finding w_ℓ.

We view a closed but not totally-reducible walk as consisting of a "base walk", which is the part that prevents it from being totally-reducible, together with some "extensions" which do not change its basic structure. For the sake of efficiency, walks that are extensions of the same base walk will be counted together. The primitive circuits in [141] played an analogous role to our base walks.

For a closed walk $W = v_0 v_1 \cdots v_\ell$ we allow the following extensions:

1. A *tail* is an irreducible walk $x_0 x_1 \cdots x_t$ of length $t \geqslant 0$ with $x_{t-1} \notin \{v_1, v_{\ell-1}\}$ and $x_t = v_0$. We say that $W' = x_0 x_1 \cdots x_{t-1} v_0 \cdots v_\ell x_{t-1} \cdots x_1 x_0$ is formed by adding a tail of length t to W.

2. A *diversion* is a totally-reducible walk $v_i d$ of length $\ell \geqslant 0$ where if $i < \ell$ then no prefix of $v_i d$ reduces to $v_i v_{i+1}$. We say that $W' = v_0 v_1 \cdots v_i d v_{i+1} \cdots v_\ell$ is formed by adding a diversion to W.

We say that W *extends* to W' if W' is formed from W by adding a single tail and then any number of diversions. We stress that this includes the possibility that $W' = W$, and also the option of adding diversions to the tail. Now that we know what extensions are allowed, we can define the base walks to be closed but not totally-reducible walks that are not an extension of any other walk. This means that $W = v_0 v_1 \cdots v_\ell$ is a *base walk* if and only if

1. W is non-trivial: $\ell > 0$,

2. W is closed: $v_0 = v_\ell$,

3. W is irreducible: $v_{i-1} \neq v_{i+1}$ for $0 < i < \ell$,

4. W has no non-trivial tail: $v_1 \neq v_{\ell-1}$.

We mention that extensions are not the same in [195]. Diversions are the same but tails are different. Thus we can adopt Lemma 5 from [195] as given in Section 2.2.

In Section 3.3 we will find generating functions that count all possible extensions of a walk. Our goal is to express all closed walks that are not totally-reducible in a unique way as the extension of some base walk. We now show the uniqueness of extensions.

Theorem 11. *Given a closed and not totally-reducible walk W', there exists a unique base walk W such that W extends to W'. Furthermore, there is a unique way to write*

$$W' = x_0 d_0 \cdots x_{t-1} d_{t-1} v_0 d_t v_1 d_{t+1} \cdots v_\ell d_{t+\ell} x_{t-1} d_{t+\ell+1} \cdots x_0 d_{2t+\ell} \tag{3.1}$$

where $W = v_0 v_1 \cdots v_\ell$, each $x_i d_i$ and $v_i d_i$ is a diversion, and $x_0 x_1 \cdots x_{t-1} v_0$ is a tail.

Proof. Given W', we can find $\mathrm{red}(W') = w_0 w_1 \cdots w_\ell$. Note that $\ell \geqslant 3$ since W' is closed and not totally-reducible. Find the minimum j such that $w_j \neq w_{\ell-j}$ for $1 \leqslant j < \ell/2$. A value for j satisfying

this condition exists since red(W') is irreducible. Let $W = w_{j-1}w_j \cdots w_{\ell-j+1}$. Since $j < \ell/2$, we know that W is non-trivial. Also, W is closed since $w_{j-1} = w_{\ell-j+1}$. By the definition of red(W'), W is irreducible. Finally, W has a trivial tail since $w_j \neq w_{\ell-j}$. Therefore, W is a base walk as desired and red(W') is formed by adding a tail to W.

Suppose V is a base walk such that adding a tail yields W'' and then adding diversions gives W'. By definition of base walks and tails, V and W'' are irreducible. Also, W' reduces to W'' so $W'' = $ red(W'), by Lemma 2. It now follows that $V = W$, since V cannot be longer than W by part 4 of the definition of base walks and V cannot be shorter than W given how we defined tails.

Thus, given W' there is a unique base walk W that extends to W', and a unique way to add the tail in that extension process. The uniqueness of (3.1) is shown by Lemma 5.   □

3.3 Generating walks around subgraphs

In this section we give a number of single variable generating functions for walks. In all cases the coefficient of x^i is the number of walks of length i in the class being counted. Now, we combine the functions of Lemma 3 and Lemma 4 given with proofs in Section 2.2. We note that the proof of the following is similar to the proof of Lemma 6 which can be found in [195].

Lemma 12. *Suppose $W = v_0 v_1 \cdots v_\ell$ is a closed walk of length ℓ in an $(r+1)$-regular graph, G. The generating function for walks in G that are extensions of W is*

$$\psi(\ell) = x^\ell T_r^\ell T_{r+1} \left(\frac{1 - x^2 T_r^2}{1 - rx^2 T_r^2} \right).$$

Proof. By Lemma 4, the number of walks formed by adding diversions to a walk is generated by $x^\ell T_r^\ell T_{r+1}$. Now if we add a tail, $x_1 \ldots x_t v_0$, of length $t > 0$; there are: $r + 1 - 2$ choices for x_t since $x_t \neq v_1$ and $x_t \neq v_{\ell-1}$, and r choices for x_i for $1 \leqslant i < t$; so $(r-1)r^{t-1}$ choices of tail. Thus,

$$\psi(\ell) = x^\ell T_r^\ell T_{r+1} + \sum_{t \geqslant 1} (r-1)r^{t-1} x^{\ell+2t} T_r^{\ell+2t} T_{r+1}$$

$$= x^\ell T_r^\ell T_{r+1} (1 + \frac{r-1}{r} \sum_{t \geqslant 1} r^t x^{2t} T_r^{2t})$$

$$= x^\ell T_r^\ell T_{r+1} \left(1 + \frac{r-1}{r} \left(\frac{rx^2 T_r^2}{1 - rx^2 T_r^2} \right) \right)$$

which simplifies to the required expression.   □

Base walks are closed, so we can use $\psi(\ell)$ to extend base walks of length ℓ. Note that $\psi(\ell)$ only depends on the length of the walk we start with and the regularity of G. Given the number b_i of base walks of length i in G, the generating function for all closed walks that are extensions of base walks of length at most ℓ is

$$\sum_{i=1}^{\ell} b_i \psi(i). \tag{3.2}$$

Let $\mathcal{H}_\ell$ denote the set of all (isomorphism classes of) subgraphs induced by the base walks of length at most ℓ. For our purposes the subgraph induced by a walk is the (simple) subgraph induced by the *edges* of the walk. To find $\mathcal{H}_\ell$ we need to obtain the base walks of length i, for $3 \leqslant i \leqslant \ell$. To do this, we generate all reduced walks of a given length, removing any such walks with a tail. For each base walk we find the subgraph it induces and keep a catalogue of such subgraphs up to isomorphism (using nauty [142]).

Algorithm 1

  $\eta(H) := 0$
  for i from #edges in H to ℓ **do**
    $b_i^H := 0$
    for each vertex j of H **do**
      FINDBW(j, j, b_i^H, H, i)
    end for
    Add $b_i^H \cdot \psi(i)$ to $\eta(H)$
  end for

  function FINDBW(W,v,b,S,len)
    if W has length len **then**
      if W is closed, has a trivial tail, and induces S **then**
        Increment b
      end if
    else
      for all neighbours u of vertex v in S **do**
        if Wu is irreducible **then**
          FINDBW(Wu, u, b, S, len)
        end if
      end for
    end if
  end function

From the set $\mathcal{H}_\ell$, we count the number of base walks of each length up to ℓ that occur. The function FINDBW(W, v, b, S, len) of Algorithm 1 recursively finds the number, $b = b_{len}^S$, of base walks of length len that induce the contributor S. The parameter W is an irreducible walk with end vertex the parameter v. Each call to FINDBW extends the length of W by one. The values of b_i^H calculated in Algorithm 1 are related to the b_i from (3.2) by $b_i = \sum_{H \in \mathcal{H}_\ell} b_i^H$.

For each contributor $H \in \mathcal{H}_\ell$ we use Algorithm 1 to find $\eta(H)$, a power series in x for the contribution of contributor H. For example, at the completion of Algorithm 1 when $\ell = 9$ and $H = C_3$,

$$\eta(H) = 6x^3 + 30rx^5 + 6x^6 + 126r^2x^7 + 48rx^8 + (6 + 504r^3)x^9 + O(x^{10}).$$

It is important to note in general that $\eta(H)$ is only useful for the powers of x up to ℓ. The coefficients of terms beyond x^ℓ are missing the contribution of base walks of length greater than ℓ.

We next show how to find z_i from the calculated values of $\eta(H)$ for $H \in \mathcal{H}_\ell$.

Theorem 13. *The generating function for closed and not totally-reducible walks of length at most ℓ in G is*

$$\sum_{i \leqslant \ell} z_i x^i = \sum_{H \in \mathcal{H}_\ell} \eta(H)[H] + O(x^{\ell+1}). \tag{3.3}$$

Proof. The base walks counted by b_i^H in Algorithm 1 are closed and not totally-reducible by definition. Extending these walks cannot destroy either of these properties, so every walk counted by $\eta(H)[H]$ is closed and not totally-reducible. Let W' be any closed and not totally-reducible walk of length at most ℓ. By Theorem 11, W' is obtained by extension of some base walk W. Let H_W be the subgraph induced by W. Since the length of W is no more than ℓ, we know that $H_W \in \mathcal{H}_\ell$. The uniqueness clauses in Theorem 11 guarantee that W' is counted exactly once by $\eta(H_W)[H_W]$ and is not counted by $\eta(H)[H]$ for any $H \neq H_W$. Hence W' is counted exactly once on both sides of (3.3). The theorem follows. $\qquad\square$

In (3.3), we see that $H \in \mathcal{H}_\ell$ appears as a variable in w_i if a base walk that induces H extends to a walk of length i. We say in this case that H *affects* w_i. For the remainder of this section, we characterize the circumstances under which H affects w_i.

An *Eulerian tour* of a multigraph M is a closed walk containing every edge of M exactly once. A multigraph with an Eulerian tour is said to be *Eulerian*. We make use of the well known result that a multigraph is Eulerian if and only if every vertex has even degree.

A base walk that induces H is an Eulerian tour in a multigraph with underlying graph H. We next prove a partial converse of this statement.

Lemma 14. *Let M be an Eulerian multigraph with underlying graph the contributor H. If M has j edges and at most two edges between any pair of vertices, then there exists a base walk of length j that is an Eulerian tour of M.*

Proof. The sequence of edges followed by any closed walk may be cyclically permuted to obtain another closed walk, in which case we say that the walk itself has been *cyclically permuted*. By cyclically permuting any closed walk that has a non-trivial tail, the tail can be moved to a place where it can be reduced. Conversely, if a closed walk is reducible it can be cyclically permuted so that the point of reduction becomes a tail instead. In this proof we count a non-trivial tail or a point at which a walk may be reduced as a *flaw*.

Consider an Eulerian tour W of M. Suppose W has a flaw. Then by cyclically permuting W we can obtain a walk $W' = v_0 v_1 \cdots v_j$ in which $v_0 = v_2$. Since M has at most two edges between any pair of vertices, W' cannot walk directly between v_0 and v_1 after the first two edges. However, $v_i = v_1$ for some $i \in \{4, 5, \ldots, j-2\}$ because H has minimum degree at least two. We reverse the subsequence $v_1 v_2 \cdots v_i$ of W' to give $W'' = v_0 v_i v_{i-1} \cdots v_2 v_1 v_{i+1} \cdots v_j$. Since W'' includes the same edges as W, it is an Eulerian tour of M. Moreover, W'' has fewer flaws than W because $v_{i-1} \neq v_0$ and $v_{i+1} \neq v_2$ (otherwise there are more than two edges between v_0 and v_1 in M).

By repeating the above process we must eventually reach a flawless Eulerian tour of M, which provides the desired base walk. $\qquad\square$

Lemma 14 does not generalize to multigraphs with more than two edges between some pair of vertices. Consider replacing one edge of any cycle C_n by three parallel edges. Although the resulting multigraph is Eulerian, there is no base walk that is an Eulerian tour on its edges.

Another general issue is that an Eulerian tour that induces a graph H may be reducible to a walk that induces a proper subgraph of H. For an extreme example, consider the totally-reducible walk $W = v_0v_1v_0v_1v_0v_2v_1v_2v_0$, which is an Eulerian tour of a multigraph with underlying graph C_3. However, there exists an Eulerian tour $W' = v_0v_1v_0v_1v_2v_0v_1v_2v_0$ of the same multigraph. We see that W' is an extension of a base walk, even though W is not. We show in the next proof that this is always the case; if there is any Eulerian tour of a multigraph then there will be one which is an extension of a base walk.

Theorem 15. *A contributor H affects w_i if and only if there exists a closed walk of length i that induces H.*

Proof. ($\Rightarrow$) Let W be a base walk that induces H and W' be an extension of W. Note that W' is not necessarily restricted to the edges of H in G. If W has length j and W' has length i then by the definition of extensions, $i - j$ is even. We form a new walk W'' by following the edges of W, then tracing back and forth across the final edge $(i - j)/2$ times in each direction. The result is a closed walk of length i that induces H, as desired.

($\Leftarrow$) Any closed walk that induces H is an Eulerian tour of some multigraph M with underlying graph H. We claim that there is an Eulerian tour of M that cannot be reduced to a walk that induces a graph with fewer edges than H. Construct a graph M' from M by removing pairs of parallel edges until there is at least one and at most two edges between each pair of vertices. Note that M' still has underlying graph H, and since the degree at each vertex is still even, M' is Eulerian. Thus by Lemma 14 there exists a base walk W that is an Eulerian tour of M'. Now for each pair of parallel edges previously removed between a pair of vertices, say u and v, we add a diversion uvu to W. This extension of the base walk W is an Eulerian tour of M. The result follows. □

Corollary 16. *Let H be a contributor with e edges. Let m be the minimum number of edges in an Eulerian multigraph M with underlying graph H.*

1. *m is the minimum value for which H affects w_m and $e \leqslant m < 2e$.*

2. *H affects w_{m+2s} for all integers $s \geqslant 0$.*

3. *If H is bipartite then H only affects w_i for even integers $i \geqslant m$.*

4. *If H is not bipartite then H affects w_i for all $i \geqslant 2e - 1$.*

Proof. When H is Eulerian, $m = e$. We can always double each edge of H to obtain an Eulerian M but this is not the best possible. Since H has minimum degree at least 2, it contains a cycle. Constructing M by doubling all edges of H except those in the cycle, we prove (a). Since adding a pair of parallel edges preserves the Eulerian property, we have (b). Part (c) is true since bipartite graphs have no closed walks of odd length. If H is not bipartite then we can double all the edges or double all the edges except for one odd cycle, which implies (d). □

3.4 Equations relating moments to subgraph counts

By combining (2.2) with (3.3) we obtain the following result for the spectral moments of an $(r+1)$-regular graph with n vertices:

$$w_0 = n,$$
$$w_1 = 0,$$
$$w_2 = (1+r)n,$$
$$w_3 = 6[C_3],$$
$$w_4 = (1+3r+2r^2)n + 8[C_4],$$
$$w_5 = 30r[C_3] + 10[C_5],$$
$$w_6 = (1+5r+9r^2+5r^3)n + 6[C_3] + 48r[C_4] + 12[C_6] + 24[C_{3\cdot3}] + 12[\Theta_{2,2,1}],$$
$$w_7 = 126r^2[C_3] + 70r[C_5] + 14[C_7] + 28[C_{4\cdot3}] + 28[\Theta_{2,2,1}] + 14[\Theta_{3,2,1}] + 84[\Theta_{2,2,2,1}],$$
$$w_8 = (1+7r+20r^2+28r^3+14r^4)n + 48r[C_3] + (8+224r^2)[C_4] + 96r[C_6]$$
$$\qquad + 16[C_8] + 192r[C_{3\cdot3}] + 32[C_{4\cdot4}] + 32[C_{5\cdot3}] + 32[C_{3-3}] + 144[K_4]$$
$$\qquad + 96r[\Theta_{2,2,1}] + 48[\Theta_{2,2,2}] + 16[\Theta_{3,2,1}] + 16[\Theta_{3,3,1}] + 16[\Theta_{4,2,1}] + 96[\Theta_{2,2,2,2}]$$
$$\qquad + 96[\Theta_{3,2,2,1}] + 64[\text{⟨graph⟩}],$$
$$w_9 = (6+504r^3)[C_3] + 360r^2[C_5] + 126r[C_7] + 18[C_9] + 72[C_{3\cdot3}] + 252r[C_{4\cdot3}]$$
$$\qquad + 36[C_{5\cdot4}] + 36[C_{6\cdot3}] + 144[C_{3\cdot3\cdot3}] + 36[C_{4-3}] + 288[K_4] + (36+252r)[\Theta_{2,2,1}]$$
$$\qquad + (18+126r)[\Theta_{3,2,1}] + (36+756r)[\Theta_{2,2,2,1}] + 36[\Theta_{3,2,2}] + 18[\Theta_{4,2,1}] + 18[\Theta_{4,3,1}]$$
$$\qquad + 18[\Theta_{5,2,1}] + 108[\Theta_{3,2,2,2}] + 108[\Theta_{3,3,2,1}] + 108[\Theta_{4,2,2,1}]$$
$$\qquad + 90[\text{⟨graph⟩}] + 108[\text{⟨graph⟩}] + 36[\text{⟨graph⟩}] + 72[\text{⟨graph⟩}] + 72[\text{⟨graph⟩}]$$
$$\qquad + 108[\text{⟨graph⟩}] + 180[\text{⟨graph⟩}] + 72[\text{⟨graph⟩}] + 288[\text{⟨graph⟩}]. \tag{3.4}$$

Knowing any information about G that limits the possibilities for subgraphs can lead to equations with significantly fewer contributors. For example, a lower bound on the girth would eliminate any contributors containing short cycles. We present the equations for graphs with girth 5.

$$w_0 = n,$$
$$w_1 = 0,$$
$$w_2 = (1+r)n,$$
$$w_3 = 0,$$
$$w_4 = (1+3r+2r^2)n,$$
$$w_5 = 10[C_5],$$
$$w_6 = (1+5r+9r^2+5r^3)n + 12[C_6],$$
$$w_7 = 70r[C_5] + 14[C_7],$$
$$w_8 = (1+7r+20r^2+28r^3+14r^4)n + 96r[C_6] + 16[C_8],$$
$$w_9 = 360r^2[C_5] + 126r[C_7] + 18[C_9],$$

$$w_{10} = (1 + 9r + 35r^2 + 75r^3 + 90r^4 + 42r^5)n + 10[C_5] + 540r^2[C_6] + 160r[C_8] + 20[C_{10}]$$
$$+ 40[C_{5\cdot5}] + 20[\Theta_{3,3,2}] + 20[\Theta_{4,4,1}],$$

$$w_{11} = 1650r^3[C_5] + 770r^2[C_7] + 198r[C_9] + 22[C_{11}] + 44[C_{6\cdot5}] + 44[\Theta_{3,3,2}] + 22[\Theta_{4,3,2}]$$
$$+ 22[\Theta_{5,4,1}] + 132[\Theta_{3,3,3,2}],$$

$$w_{12} = (1 + 11r + 54r^2 + 154r^3 + 275r^4 + 297r^5 + 132r^6)n + 120r[C_5] + (12 + 2640r^3)[C_6]$$
$$+ 1056r^2[C_8] + 240r[C_{10}] + 24[C_{12}] + 480r[C_{5\cdot5}] + 48[C_{6\cdot6}] + 48[C_{7\cdot5}] + 48[C_{5-5}]$$
$$+ 24[\Theta_{6,4,1}] + 24[\Theta_{4,3,2}] + 24[\Theta_{4,4,2}] + 24[\Theta_{5,3,2}] + 24[\Theta_{5,5,1}] + 144[\Theta_{3,3,3,3}]$$
$$+ 144[\Theta_{4,3,3,2}] + 240r[\Theta_{3,3,2}] + 240r[\Theta_{4,4,1}] + 72[\Theta_{3,3,3}] + 72[\raisebox{-0.3em}{\includegraphics{}}] + 96[\raisebox{-0.3em}{\includegraphics{}}],$$

$$w_{13} = 7150r^4[C_5] + 4004r^3[C_7] + 1404r^2[C_9] + 286r[C_{11}] + 26[C_{13}] + 572r[C_{6\cdot5}] + 52[C_{7\cdot6}]$$
$$+ 52[C_{8\cdot5}] + 52[C_{6-5}] + 572r[\Theta_{3,3,2}] + (26 + 286r)[\Theta_{4,3,2}] + 52[\Theta_{4,4,1}] + 52[\Theta_{4,3,3}]$$
$$+ 26[\Theta_{5,3,2}] + 286r[\Theta_{5,4,1}] + 1716r[\Theta_{3,3,3,2}] + 26[\Theta_{5,4,2}] + 26[\Theta_{6,3,2}] + 26[\Theta_{6,5,1}]$$
$$+ 26[\Theta_{7,4,1}] + 156[\Theta_{4,3,3,3}] + 156[\Theta_{4,4,3,2}] + 156[\Theta_{4,4,4,1}] + 156[\Theta_{5,3,3,2}] + 78[\raisebox{-0.3em}{\includegraphics{}}]$$
$$+ 78[\raisebox{-0.3em}{\includegraphics{}}] + 104[\raisebox{-0.3em}{\includegraphics{}}] + 104[\raisebox{-0.3em}{\includegraphics{}}] + 104[\raisebox{-0.3em}{\includegraphics{}}]. \tag{3.5}$$

Another example is to consider bipartite graphs, where contributors that contain odd length cycles cannot occur. For bipartite $(r + 1)$-regular graphs on n vertices we have $w_i = 0$ for all odd i, and

$$w_0 = n,$$
$$w_2 = (1 + r)n,$$
$$w_4 = (1 + 3r + 2r^2)n + 8[C_4],$$
$$w_6 = (1 + 5r + 9r^2 + 5r^3)n + 48r[C_4] + 12[C_6],$$
$$w_8 = (1 + 7r + 20r^2 + 28r^3 + 14r^4)n + (8 + 224r^2)[C_4] + 96r[C_6] + 16[C_8] + 32[C_{4\cdot4}]$$
$$+ 48[\Theta_{2,2,2}] + 16[\Theta_{3,3,1}] + 96[\Theta_{2,2,2,2}],$$
$$w_{10} = (1 + 9r + 35r^2 + 75r^3 + 90r^4 + 42r^5)n + (80r + 960r^3)[C_4] + 540r^2[C_6] + 160r[C_8]$$
$$+ 20[C_{10}] + 320r[C_{4\cdot4}] + 40[C_{6\cdot4}] + 40[C_{4-4}] + 480r[\Theta_{2,2,2}] + (40 + 160r)[\Theta_{3,3,1}]$$
$$+ 960r[\Theta_{2,2,2,2}] + 40[\Theta_{4,2,2}] + 20[\Theta_{5,3,1}] + 120[\Theta_{3,3,3,1}] + 120[\Theta_{4,2,2,2}]$$
$$+ 120[\raisebox{-0.3em}{\includegraphics{}}] + 80[\raisebox{-0.3em}{\includegraphics{}}],$$
$$w_{12} = (1 + 11r + 54r^2 + 154r^3 + 275r^4 + 297r^5 + 132r^6)n + (8 + 528r^2 + 3960r^4)[C_4]$$
$$+ (12 + 2640r^3)[C_6] + 1056r^2[C_8] + 240r[C_{10}] + 24[C_{12}] + (96 + 2112r^2)[C_{4\cdot4}]$$
$$+ 480r[C_{6\cdot4}] + 192[C_{4\cdot4\cdot4}] + 48[C_{6\cdot6}] + 48[C_{8\cdot4}] + 480r[C_{4-4}] + 48[C_{6-4}]$$
$$+ (240 + 3168r^2)[\Theta_{2,2,2}] + (48 + 480r + 1056r^2)[\Theta_{3,3,1}] + (1920 + 6336r^2)[\Theta_{2,2,2,2}]$$
$$+ (24 + 480r)[\Theta_{4,2,2}] + 72[\Theta_{3,3,3}] + (24 + 240r)[\Theta_{5,3,1}] + 4320[\Theta_{2,2,2,2,2}]$$
$$+ (48 + 1440r)[\Theta_{3,3,3,1}] + 1440r[\Theta_{4,2,2,2}] + 24[\Theta_{4,4,2}] + 48[\Theta_{6,2,2}] + 24[\Theta_{5,5,1}]$$
$$+ 24[\Theta_{7,3,1}] + 144[\Theta_{3,3,3,3}] + 144[\Theta_{4,4,2,2}] + 144[\Theta_{5,3,3,1}] + 144[\Theta_{6,2,2,2}]$$
$$+ 2880[\Theta_{2,2,2,2,2,2}] + (648 + 1440r)[\raisebox{-0.3em}{\includegraphics{}}] + (288 + 960r)[\raisebox{-0.3em}{\includegraphics{}}] + 312[\raisebox{-0.3em}{\includegraphics{}}]$$
$$+ 432[\raisebox{-0.3em}{\includegraphics{}}] + 48[\raisebox{-0.3em}{\includegraphics{}}] + 48[\raisebox{-0.3em}{\includegraphics{}}] + 72[\raisebox{-0.3em}{\includegraphics{}}] + 72[\raisebox{-0.3em}{\includegraphics{}}]$$

$$+ 96[\ \text{⬚}\ ] + 144[\ \text{⬚}\ ] + 192[\ \text{⬚}\ ] + 216[\ \text{⬚}\ ] + 288[\ \text{⬚}\ ] + 432[\ \text{⬚}\ ]$$

$$+ 912[\ \text{⬚}\ ] + 48[\ \text{⬚}\ ] + 96[\ \text{⬚}\ ] + 96[\ \text{⬚}\ ] + 96[\ \text{⬚}\ ] + 144[\ \text{⬚}\ ]$$

$$+ 240[\ \text{⬚}\ ] + 240[\ \text{⬚}\ ] + 432[\ \text{⬚}\ ] + 96[\ \text{⬚}\ ] + 96[\ \text{⬚}\ ] + 288[\ \text{⬚}\ ]$$

$$+ 384[\ \text{⬚}\ ] + 384[\ \text{⬚}\ ] + 576[\ \text{⬚}\ ] + 1056[\ \text{⬚}\ ]. \tag{3.6}$$

We note that more equations are easily determined but the increasing number of contributors makes them unsuitable to present in this forum.

From these equations, we were able to reproduce the reduced possible spectra list of [176] including the values of $[C_4]$ and $[C_6]$ for all entries. In addition, we were able to determine $[C_8]$ in many cases. One example of this can be found in Section 4.2.

3.5 Remarks

The equations we have presented provide much more information than (2.4). However, if preferred they can also be used to derive inequalities like (2.4) simply by dropping some of the terms. Our equations are well suited to this use, since all terms are positive, unlike the equations in [195] which employ the principle of inclusion-exclusion and hence have many terms of opposite signs.

Other uses of our equations have yet to be explored, but we are confident that they will prove useful in a variety of contexts.

Chapter 4

Quartic Integral Cayley Graphs

4.1 Vertex-transitive integral graphs

We give exhaustive lists of connected 4-regular integral Cayley graphs and connected 4-regular integral arc-transitive graphs. We first restrict our attention to the bipartite case because of Lemma 9.

We are able to add to the result of [2] mentioned in Section 2.5 and find that the precise set of orders of Cayley QIGs on Abelian groups is $\{5, 6, 8, 9, 10, 12, 16, 18, 24, 36\}$. More generally, we show that for any group Γ, if $\mathrm{Cay}(\Gamma, S)$ is a Cayley QIG then

$$|\Gamma| \in \{5, 6, 8, 9, 10, 12, 16, 18, 20, 24, 30, 32, 36, 40, 48, 60, 72, 120\}.$$

Furthermore, for each of these orders we give the Cayley QIGs that exist. We note that the orders: 25, 50, 64, and 100 could be almost immediately eliminated based on the possible spectra list orders of [176].

For a given Cayley graph G, there may exist many different pairs (Γ, S) of groups Γ and connection sets S such that $G \cong \mathrm{Cay}(\Gamma, S)$. We call isomorphic Cayley graphs on the same group Γ *equivalent* if their connection sets are from the same orbit of the automorphism group of Γ:

Definition 17. *Let Γ be a group and $\mathrm{Aut}(\Gamma)$ be the automorphism group of Γ. If $\mathrm{Cay}(\Gamma, S) \cong \mathrm{Cay}(\Gamma, T)$ and $S^\sigma = T$ for some $\sigma \in \mathrm{Aut}(\Gamma)$ then $\mathrm{Cay}(\Gamma, S)$ and $\mathrm{Cay}(\Gamma, T)$ are equivalent.*

Any other connection sets give non-equivalent Cayley Graphs. Cayley graphs from different groups are non-equivalent. There are, up to isomorphism, only 32 connected quartic integral Cayley graphs; but each graph is realized in up to 18 non-equivalent ways. Of the 32 graphs, 17 are bipartite graphs realizing a possible spectrum.

A graph is *arc-transitive* if its automorphism group acts transitively upon ordered pairs of adjacent vertices. There are, up to isomorphism, only 27 connected quartic integral graphs that are arc-transitive. Of the 27 graphs, 16 are bipartite graphs realizing a possible spectrum, 5 of which are not Cayley graphs.

Section 4.2 details the methods used for refining the set of possible spectra from [176] by removing ones that cannot be realized by vertex-transitive QIGs. Section 4.3 summarises the algorithm used for finding all of the bipartite Cayley QIGs. Section 4.4 gives our results. It includes tables giving

the details of the Cayley QIGs and the bipartite arc-transitive QIGs, some drawings, and some non-bipartite QIGs that result from finding quotients of our bipartite graphs.

4.2   Vertex-transitive cases

A graph is *vertex-transitive* if its automorphism group acts transitively upon its vertices. In this section, our aim is to compile a set Ξ that includes all possible spectra that might be realized by a vertex-transitive QIG, but is otherwise as small as we can make it. Initially we take Ξ to be all possible spectra from [176], and candidates will be progressively removed from the set as we work through this section.

We use the same notation for subgraphs as was described in Section 3.1 and depicted in Figure 3.1.

In [63], Equations (4.1) and (4.2) are used to determine $[C_4]$ and $[C_6]$ for a given $[x, y, z, w]$.

$$2(4^4 + 3^4 x + 2^4 y + z) = 28n + 8[C_4], \tag{4.1}$$

$$2(4^6 + 3^6 x + 2^6 y + z) = 232n + 144[C_4] + 12[C_6]. \tag{4.2}$$

In Chapter 3, these equations were extended to higher spectral moments of general regular graphs. By specialising to 4-regular bipartite graphs, we obtain the following equations:

$$2(4^8 + 3^8 x + 2^8 y + z) = 2092n + 2024[C_4] + 288[C_6] + 16[C_8] + 32[C_{4\cdot 4}]$$
$$+ 96[\Theta_{2,2,2}] + 48[\Theta_{2,2,2}] + 16[\Theta_{3,3,1}],$$

$$2(4^{10} + 3^{10} x + 2^{10} y + z) = 19864n + 26160[C_4] + 4860[C_6] + 480[C_8] + 20[C_{10}]$$
$$+ 960[C_{4\cdot 4}] + 40[C_{4-4}] + 40[C_{6\cdot 4}] + 1440[\Theta_{2,2,2}]$$
$$+ 520[\Theta_{3,3,1}] + 2880[\Theta_{2,2,2,2}] + 40[\Theta_{4,2,2}] + 20[\Theta_{5,3,1}]$$
$$+ 120[\Theta_{3,3,3,1}] + 120[\Theta_{4,2,2,2}] + 120[\text{⬦}] + 80[\text{▭}]. \tag{4.3}$$

We use Equations (4.3) to determine the girth where $[C_4] = [C_6] = 0$ for a given $[x, y, z, w]$ and also to determine the values for $[C_8]$ and $[C_{10}]$ where possible. Vertex-transitive graphs have the same number of i-cycles incident with each vertex, so the number of vertices divides $i[C_i]$. We apply this observation for $i \in \{4, 6, 8, 10\}$ to the possible spectra for which the value of $[C_i]$ can be deduced. We eliminate those quadruples that cannot be realized by a vertex-transitive QIG from Ξ.

For example, if we consider $[5, 6, 11, 1]$ with $n = 48$, $[C_4] = 24$, and $[C_6] = 140$ then

$$\frac{4[C_4]}{n} = \frac{4(24)}{48} = 2 \in \mathbb{N} \quad \text{but} \quad \frac{6[C_6]}{n} = \frac{6(140)}{48} = \frac{35}{2} \notin \mathbb{N},$$

where $\mathbb{N}$ denotes the set of non-negative integers. Thus $[5, 6, 11, 1]$ is eliminated from Ξ. In contrast, for $[12, 12, 20, 3]$ with $n = 96$, $[C_4] = 24$, $[C_6] = 128$, and $[C_8] = 528$,

$$\frac{4[C_4]}{n} = \frac{4(24)}{96} = 1 \in \mathbb{N}, \quad \frac{6[C_6]}{n} = \frac{6(128)}{96} = 8 \in \mathbb{N}, \quad \text{and} \quad \frac{8[C_8]}{n} = \frac{8(528)}{96} = 44 \in \mathbb{N}.$$

In this case, $[C_{10}] = 6240 - [\Theta_{5,3,1}] - 2[C_{6\cdot 4}]$ and so we consider it unknown. Thus $[12, 12, 20, 3]$ remains in Ξ.

It is also plausible to eliminate quadruples from Ξ using arguments specific to particular cases. We give one example to demonstrate the possibility. Consider $[24, 4, 40, 3]$ with $[C_4] = 72$ and $[C_6] = 0$. There are $4(72)/144 = 2$ copies of C_4 incident at each vertex. Since $[C_6] = 0$, we know $[\Theta_{3,3,1}] = 0$. Also, with only two 4-cycles at each vertex, $[\Theta_{2,2,2,2}] = [\Theta_{2,2,2}] = 0$. Since two 4-cycles meet at exactly one vertex of a $C_{4.4}$, $[C_{4.4}] = 144$. From Equation (4.3) we get that,

$$2(4^8 + 3^8(24) + 2^8(4) + 40) = 2092(144) + 2024(72) + 16[C_8] + 32(144),$$

which gives the contradiction $[C_8] = -216$. Thus we remove $[24, 4, 40, 3]$ from Ξ. This entry is underlined in Table 4.1.

We eliminate two quadruples from Ξ using the following Lemma [23, Prop. 16.6]:

Lemma 18. *Let G be a vertex-transitive graph which has degree r and an even number of vertices. If λ is a simple eigenvalue of G, then λ is one of the integers $2\alpha - r$ for $0 \leqslant \alpha \leqslant r$.*

The orders associated with the eliminated quadruples are 36 and 72. Both entries have 1 as a simple eigenvalue. These entries are underlined and highlighted in bold in Table 4.1.

Using the above methods, we reduced the set Ξ from the initial 828 possible spectra to 59 quadruples in the final version. Henceforth Ξ will refer to this final set of 59 quadruples (see Appendix A).

Table 4.1 summarizes the process of finding Ξ. For every order, we consider each $[x, y, z, w]$ and check whether we get integer counts at each vertex for each C_i where $[C_i]$ is known and $i \in \{4, 6, 8, 10\}$. A 'q' in the table denotes that all possible spectra for a given n have $4[C_4]/n \notin \mathbb{N}$. An 'h' denotes that some possible spectra for a given n satisfy $4[C_4]/n \in \mathbb{N}$ but all possible spectra for that n have $6[C_6]/n \notin \mathbb{N}$. If $i[C_i]/n \in \mathbb{N}$ for all i where $[C_i]$ is known for a specific possible spectra, then the girth is recorded. Thus an entry of $4, 4, 6$ would indicate that there are three possible spectra in Ξ associated with that order, and if the quadruples are all realized by graphs then two graphs have girth 4 and one has girth 6.

n	Girth	n	Girth	n	Girth	n	Girth
8	4	36	**4**,4,4	96	4	240	6,8
10	4	40	4	112	h	252	h
12	4,4	42	4	120	4,4,6,6,6	280	8
14	q	48	4	126	4,6	288	6
16	4	56	h	140	h	336	h
18	4	60	4,4,4,4,6	144	**4**,4,6	360	6,6,8
20	4	70	6	160	q	420	8
24	4,4,4	72	**4**,4,4,6,6	168	6	480	8
28	q	80	h	180	4,6,6,6	504	h
30	4,4,6	84	h	210	6	560	10
32	6	90	4,4,6,6	224	h		

Table 4.1: Finding the set of possible spectra for vertex transitive graphs

4.3   The algorithm

In this section we outline our method for finding bipartite Cayley QIGs, using the set Ξ compiled in Section 4.2.

Define Ω to be the set of orders associated with the spectra in Ξ. Cayley graphs are vertex-transitive, so we only consider groups Γ of order $n \in \Omega$. To reduce the number of groups to be considered, we use a result similar to one in [152]. Let $\mathbb{Z}_a$ denote the cyclic group of order a and let Γ' denote the commutator subgroup of a group Γ.

Lemma 19. *Let Γ be a finite group and let $\mathrm{Cay}(\Gamma, S)$ be a connected Cayley graph of degree at most 4. Then Γ/Γ' is isomorphic to one of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$; $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_a$ with $a \geqslant 2$; $\mathbb{Z}_a \times \mathbb{Z}_b$ with $a, b \geqslant 2$; or $\mathbb{Z}_a$ with $a \geqslant 1$.*

Proof. Since $\mathrm{Cay}(\Gamma, S)$ is connected and has degree at most 4, Γ is generated by an inverse-closed set of at most 4 elements. This must also be true of the quotient group Γ/Γ'. Now since Γ/Γ' is Abelian, the result follows. $\square$

By Lemma 19, we need only consider groups Γ with Γ/Γ' isomorphic to one of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_a$, $\mathbb{Z}_a \times \mathbb{Z}_b$, or $\mathbb{Z}_a$. We denote the set of groups that satisfy this property by Φ.

To construct connected simple undirected 4-regular Cayley graphs $\mathrm{Cay}(\Gamma, S)$, we considered inverse-closed sets S of four non-identity elements of Γ that generate Γ. The search was pruned by placing additional restrictions on S. Let g denote the girth of the graph $\mathrm{Cay}(\Gamma, S)$.

1. Since $\mathrm{Cay}(\Gamma, S)$ is bipartite, the order of s is even for each $s \in S$.

2. If $s_1, s_2 \in S$ and $s_1 \neq s_2^{-1}$, then the order of $s_1 s_2$ is at least $g/2$ (in particular non-involutions have order no smaller than the girth).

3. For any set of connection sets that result in equivalent Cayley graphs (in the sense of Definition 17), only one representative is chosen.

We note that the minimum girth possible for $\mathrm{Cay}(\Gamma, S)$ is given by Table 4.1.

We briefly expand upon the occurrence of a restriction on connection sets with no involutions. Consider a pair of connection sets S and T with $\sigma \in \mathrm{Aut}(\Gamma)$ such that $S^\sigma = T$. The algorithm chooses sets by successively stabilizing elements point-wise. Two equivalent connection sets occur when some element s in S is also in T but is not a fixed point of σ. In this situation $s = \sigma(s') \in T$ for some $s \neq s' \in S$. For example, the following (S, T, σ) triple occurs for the group $\Gamma = \mathbb{Z}_{16}$: $(\{1, 15, 3, 13\}, \{1, 15, 5, 11\}, (1, 5, 9, 13)(2, 10)(3, 15, 11, 7)(6, 14))$. Similar situations can occur when there are 2 or 4 involutions.

In our computations we divided up the generation of connection sets S based upon whether they would contain 0, 2, or 4 involutions. Algorithm 2 demonstrates our methods for generating connection sets containing exactly four involutions. The other two cases are similar. The *orbit* of an element u in P is the set $Pu = \{p(u) \mid p \in P\}$. The set $P_S = \{p \in P \mid p(s) = s \text{ for all } s \in S\}$ is the *point-wise stabilizer* of the set of elements S.

Algorithm 2

---

for all $\Gamma \in \Phi$ **do**
    $Inv :=$ involutions of Γ satisfying restrictions $1 - 2$
    $\mathrm{CS}(\Gamma) := \{\}$
    $cs := \{\}$
    FINDCS$(\mathrm{CS}(\Gamma), Inv, cs, \Gamma)$
end for

function FINDCS$(\mathcal{C},I,c,gp)$
    if $|c|$ equals 4 and c generates gp **then**
        if $\mathcal{C} \cup \{c\}$ satisfies restriction 3 **then**
            Add c to $\mathcal{C}$
        end if
    else
        for all orbits of $\mathrm{Aut}(gp)_c$ acting on I **do**
            Consider first element o of current orbit
            if $o \notin c$ and $c \cup \{o\}$ satisfies restriction 2 **then**
                $I^* :=$ elements of I in later orbits
                FINDCS$(\mathcal{C}, I^*, c \cup \{o\}, gp)$
            end if
        end for
    end if
end function

We summarize the results of our computations in Table 4.2. The values for $n \in \Omega$ appear as the first column and in the second column the number of groups of order n is given. (We reiterate that Ω does not include orders eliminated by the vertex-transitive tests of Section 4.2). The number of groups in Φ of order n are listed in column three. Column 4 contains the number of connection sets S among the groups counted by column 3, subject to the restrictions on S given above. The graphs $\mathrm{Cay}(\Gamma, S)$ that are bipartite are counted in column 5. The number of isomorphism classes of these graphs appears in column 6. The number of isomorphism classes of integral graphs is recorded in column 7. The last column gives the number of isomorphism classes of arc-transitive integral graphs.

4.4 Quartic integral graphs

In this section we present the graphs that our computations discovered, starting with the bipartite Cayley case.

4.4.1 Bipartite Cayley integral graphs

As a result of the computation described in Section 4.3, we have:

Theorem 20. *There are precisely* 17 *isomorphism classes of connected* 4-*regular bipartite integral Cayley graphs, as detailed in Table* 4.3.

For each bipartite Cayley QIG in Table 4.3 we give n and the spectrum $[x, y, z, w]$. Graphs appearing in the paper by Cvetković et al. [63] are labelled $I_{n,index}$ as in that paper. If the graph is

n	#Groups Γ	#$\Gamma \in \Phi$	#Sets S	#Bipartite $\mathrm{Cay}(\Gamma, S)$	#Isomorphism Classes	#Integral	#Arc-Transitive
8	5	5	13	7	1	1	1
10	2	2	2	2	1	1	1
12	5	5	19	11	3	2	1
16	14	14	66	44	5	1	1
18	5	5	12	12	5	1	1
20	5	5	34	20	8	0	-
24	15	15	151	98	23	3	1
30	4	4	31	31	17	1	1
32	51	48	58	51	16	1	1
36	14	14	149	105	48	1	1
40	14	14	201	146	54	1	0
42	6	6	55	55	36	0	-
48	52	51	840	616	177	1	0
60	13	13	385	281	161	0	-
70	4	4	96	96	73	0	-
72	50	49	1014	765	338	2	1
90	10	10	236	236	175	0	-
96	231	218	4434	3545	1292	0	-
120	47	47	2833	1968	1123	1	1
126	16	16	427	427	346	0	-
144	197	190	6563	5350	2722	0	-
168	57	57	2388	2212	1601	0	-
180	37	37	2927	2497	1883	0	-
210	12	12	1172	1172	1017	0	-
240	208	205	10884	9885	6791	0	-
280	40	40	4080	3929	3223	0	-
288	1045	968	26391	24815	15695	0	-
360	162	160	15928	14703	11524	0	-
420	41	41	10558	10204	9271	0	-
480	1213	1148	68179	63804	48322	0	-
560	180	177	21764	21433	18704	0	-

Table 4.2: Results at each step of our computations for finding Cayley graphs

in the census of Potočnik et al. [151, 152] then we give the index in their notation: AT4Val[n][index]. In two columns, we give the groups and connection sets that give rise to each Cayley graph. The first column contains the group, Γ, with a presentation of that group. We stick as close as possible to the convention of using generators in $\{a, b, c, d, e\}$ for cyclic groups, $\{s, t, u, v\}$ for symmetric or alternating groups, and $\{r, f\}$ for the quaternion group, the dihedral group, or the quasidihedral group. The last column contains the number of involutions in the connection set, S, followed by the connection set itself in terms of the generators from the previous column.

Group	Connection Sets (#involutions S)
G_1 : n = 8 $[0, 0, 0, 3]$ $I_{8,1}$ AT4Val[8][1]	
$\mathbb{Z}_8$ $< a \mid a^8 >$	$0\ \{a, a^3, a^5, a^7\}$
$\mathbb{Z}_4 \times \mathbb{Z}_2$ $< a \mid a^4 > \times < b \mid b^2 >$	$2\ \{a, b, a^3, a^2b\}$ $0\ \{a, ab, a^3, a^3b\}$
D_8 $< r, f \mid r^4, f^2, (rf)^2 >$	$4\ \{f, fr, fr^2, rf\}$ $2\ \{f, r, fr^2, frf\}$
Q_8 $< r, f \mid r^4, f^4, r^2f^2, rfrf^{-1} >$	$0\ \{r, f, r^3, r^2f\}$
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ $< a \mid a^2 > \times < b \mid b^2 > \times < c \mid c^2 >$	$4\ \{a, b, c, abc\}$
G_2 : n = 10 $[0, 0, 4, 0]$ $I_{10,1}$ AT4Val[10][2]	
D_{10} $< r, f \mid r^4, f^2, (rf)^2 >$	$4\ \{f, fr, fr^2, r^2f\}$
$\mathbb{Z}_{10}$ $< a \mid a^{10} >$	$0\ \{a, a^3, a^7, a^9\}$
G_3 : n = 12 $[0, 2, 0, 3]$ $I_{12,4}$ AT4Val[12][2]	
$\mathbb{Z}_3 \rtimes \mathbb{Z}_4$ $< a, b \mid a^3, b^4, abab^{-1} >$	$0\ \{b, b^3, ba, b^3a\}$
$\mathbb{Z}_{12}$ $< a \mid a^{12} >$	$0\ \{a, a^5, a^7, a^{11}\}$
D_{12} $< r, f \mid r^6, f^2, (rf)^2 >$	$2\ \{r^2f, f, r^5, r\}$ $4\ \{r^4f, rf, r^2f, r^5f\}$
$\mathbb{Z}_6 \times \mathbb{Z}_2$ $< a \mid a^6 > \times < b \mid b^2 >$	$0\ \{a^5, a^2b, a, a^4b\}$
G_4 : n = 12 $[0, 1, 4, 0]$ $I_{12,2}$	
D_{12} $< r, f \mid r^6, f^2, (rf)^2 >$	$2\ \{rf, r^3, r, r^5\}$ $4\ \{rf, r^3, r^5f, r^3f\}$ $4\ \{rf, r^4f, r^5f, r^3f\}$

$\mathbb{Z}_6 \times \mathbb{Z}_2$ $< a \mid a^6 > \times < b \mid b^2 >$	$2 \ \{a^3, b, a^5, a\}$

G_5 : n = 16   [0, 4, 0, 3] $I_{16,1}$ AT4Val[16][1]

$\mathbb{Z}_4 \times \mathbb{Z}_4$ $< a \mid a^4 > \times < b \mid b^4 >$	$0 \ \{a, b, a^3, b^3\}$
$(\mathbb{Z}_4 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$ $< a, b, c \mid a^4, b^2, c^2, aba^{-1}b^{-1}, (aac)^2, (bc)^2, baca^{-1}c >$	$2 \ \{ac, a^2bc, a^3bc, a^2c\}$ $2 \ \{bc, a^3b, a^2c, ab\}$ $0 \ \{a, a^3c, a^3, abc\}$
$\mathbb{Z}_4 \rtimes \mathbb{Z}_4$ $< a, b \mid a^4, b^4, aba^{-1}b >$	$0 \ \{a, a^3ba, a^3, b\}$
$\mathbb{Z}_8 \rtimes \mathbb{Z}_2$ $< a, b \mid a^8, b^2, aba^3b >$	$0 \ \{a, ab, a^3b, a^7\}$
QD_{16} $< r, f \mid r^8, f^2, rfr^5f >$	$2 \ \{r, r^4f, r^6f, r^7\}$
$\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ $< a \mid a^4 > \times < b \mid b^2 > \times < c \mid c^2 >$	$2 \ \{a, b, c, a^3\}$
$\mathbb{Z}_2 \times D_8$ $< a \mid a^2 > \times < r, f \mid r^4, f^2, (rf)^2 >$	$4 \ \{a, f, r^3f, r^2f\}$ $4 \ \{f, r^3f, af, rf\}$ $4 \ \{f, r^3f, af, arf\}$ $2 \ \{a, r, f, r^3\}$ $2 \ \{a, r, af, r^3\}$
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ $< a \mid a^2 > \times < b \mid b^2 > \times < c \mid c^2 > \times < d \mid d^2 >$	$4 \ \{a, b, c, d\}$

G_6 : n = 18   [0, 4, 4, 0] $I_{18,1}$ AT4Val[18][2]

$\mathbb{Z}_3 \times S_3$ $< a \mid a^3 > \times < s, t \mid s^2, t^3, (st)^2 >$	$2 \ \{s, st, ats, a^2ts\}$ $0 \ \{sa, sa^2, sat, sa^2t\}$
$(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2$ $< a, b, c \mid a^3, b^3, c^2, aba^{-1}b^{-1}, (ac)^2, (bc)^2 >$	$4 \ \{c, ca, cb, cab\}$
$\mathbb{Z}_6 \times \mathbb{Z}_3$ $< a \mid a^6 > \times < b \mid b^3 >$	$0 \ \{a, a^5, a^3b, a^3b^2\}$

G_7 : n = 24   [0, 8, 0, 3] $I_{24,2}$ AT4Val[24][1]

$\mathbb{Z}_4 \times S_3$ $< a \mid a^4 > \times < s, t \mid s^2, t^3, (st)^2 >$	$2 \ \{s, st, at, a^3sts\}$
$(\mathbb{Z}_6 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$ $< a, b, c \mid a^6, b^2, c^2, aba^{-1}b^{-1}, (aac)^2, a^3(cb)^2 >$	$2 \ \{a^3c, a^2c, ab, a^5b\}$
$\mathbb{Z}_3 \times D_8$ $< a \mid a^3 > \times < r, f \mid r^4, f^2, (rf)^2 >$	$0 \ \{ar^3f, a^2r^3f, ar^3, a^2r\}$
S_4 $< s, t \mid s^2, t^3, (st)^4 >$	$4 \ \{st^2sts, t^2st, stst^2s, tst^2\}$ $0 \ \{ts, st^2, ststs, tst\}$ $2 \ \{st^2sts, tst^2, ts, st^2\}$
$\mathbb{Z}_2 \times A_4$ $< a \mid a^2 > \times < s, t \mid s^2, t^3, (st)^3 >$	$0 \ \{ast, astst, asts, atst\}$

$\mathbb{Z}_2 \times \mathbb{Z}_2 \times S_3$ $< a \mid a^2 > \times < b \mid b^2 > \times < s, t \mid s^2, t^3, (st)^2 >$	$4 \ \{s, bs, st, ast\}$

G$_8$: n $= 24$ $[2, 2, 6, 1]$ I$_{24,3}$

$\mathbb{Z}_4 \times S_3$ $< a \mid a^4 > \times < s, t \mid s^2, t^3, (st)^2 >$	$2 \ \{a, a^3, s, st\}$ $2 \ \{s, st, ats, a^3ts\}$
D_{24} $< r, f \mid r^{12}, f^2, (rf)^2 >$	$4 \ \{f, rf, r^5f, r^6f\}$ $2 \ \{f, r^3, r^9, r^8f\}$
$\mathbb{Z}_2 \times (\mathbb{Z}_3 \rtimes \mathbb{Z}_4)$ $< a \mid a^2 > \times < b, c \mid b^3, c^4, bcbc^{-1} >$	$0 \ \{c, c^3, ab, ac^3bc\}$
$(\mathbb{Z}_6 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$ $< a, b, c \mid a^3, b^2, c^2, aba^{-1}b^{-1}, (ac)^2, (bc)^4 >$	$4 \ \{c, b, ca, cbc\}$ $2 \ \{c, bcb, ba, bcac\}$ $2 \ \{c, ca, cbcac, bac\}$ $0 \ \{cb, bc, ba, bcac\}$
$\mathbb{Z}_{12} \times \mathbb{Z}_2$ $< a \mid a^{12} > \times < b \mid b^2 >$	$0 \ \{a^3, a^9, a^4b, a^8b\}$
$\mathbb{Z}_3 \times D_8$ $< a \mid a^3 > \times < r, f \mid r^4, f^2, (rf)^2 >$	$2 \ \{r^3f, rf, a^2f, af\}$ $0 \ \{r, r^3, ar^3f, a^2r^3f\}$
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times S_3$ $< a \mid a^2 > \times < b \mid b^2 > \times < s, t \mid s^2, t^3, (st)^2 >$	$4 \ \{s, b, a, st\}$ $4 \ \{s, b, st, ats\}$ $4 \ \{s, sb, ast, ats\}$ $2 \ \{s, b, at, asts\}$ $2 \ \{s, sb, at, asts\}$
$\mathbb{Z}_6 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ $< a \mid a^6 > \times < b \mid b^2 > \times < c \mid c^2 >$	$2 \ \{a^3, b, a^2c, a^4c\}$

G$_9$: n $= 24$ $[3, 0, 5, 3]$ I$_{24,4}$

S_4 $< s, t \mid s^2, t^3, (st)^4 >$	$4 \ \{s, t^2st, st^2sts, stst^2st\}$
$\mathbb{Z}_2 \times A_4$ $< a \mid a^2 > \times < s, t \mid s^2, t^3, (st)^3 >$	$2 \ \{a, as, at^2s, ast\}$

G$_{10}$: n $= 30$ $[0, 10, 4, 0]$ I$_{30,1}$ AT4Val[30][4]

$\mathbb{Z}_5 \times S_3$ $< a \mid a^5 > \times < s, t \mid s^2, t^3, (st)^2 >$	$0 \ \{as, a^2st, a^4s, a^3st\}$
D_{30} $< r, f \mid r^{15}, f^2, (rf)^2 >$	$4 \ \{f, r^2f, r^3f, r^{11}f\}$

G$_{11}$: n $= 32$ $[0, 12, 0, 3]$ I$_{32,1}$ AT4Val[32][4]

$\mathbb{Z}_8 \rtimes \mathbb{Z}_4$ $< a, b \mid a^8, b^4, ab^2a^{-1}b^2, aba^3b^{-1} >$	$0 \ \{a, a^7, ab, a^3b^3\}$
$(\mathbb{Z}_8 \rtimes \mathbb{Z}_2) \rtimes \mathbb{Z}_2$ $< a, b, c \mid a^8, b^2, c^2, a^2ba^6b, (aac)^2, (bc)^2, ba^{-1}cac >$	$2 \ \{a^4c, a^2c, a^7bc, a^5c\}$
$\mathbb{Z}_2.((\mathbb{Z}_4 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2) = (\mathbb{Z}_2 \times \mathbb{Z}_2).(\mathbb{Z}_4 \times \mathbb{Z}_2)$ $< a, b \mid a^8, b^4, ab^2a^{-1}b^2, a^4b^2, aba^{-1}b^{-1}ab^{-1}a^{-1}b, aba^6ba >$	$0 \ \{ba, a^3b, a^3, a^5\}$

$(\mathbb{Z}_4 \times \mathbb{Z}_4) \rtimes \mathbb{Z}_2$ $< a, b, c \mid a^4, b^4, c^2, aba^{-1}b^{-1}, aca^3c, (bbc)^2(bc)^4, a^3(bc)^2 >$	$2 \; \{ab^2c, c, bc, a^3bc\}$
$\mathbb{Z}_4.D_8 = \mathbb{Z}_4.(\mathbb{Z}_4 \times \mathbb{Z}_2)$ $< a, b \mid a^8, b^8, aba^3b, ab^{-1}a^3b^{-1}, ab^{-1}a^{-1}b^3 >$	$0 \; \{a, a^7, a^7ba, a^4b\}$
$(\mathbb{Z}_4 \times \mathbb{Z}_4) \rtimes \mathbb{Z}_2$ $< a, b, c \mid a^4, b^4, c^2, aba^{-1}b^{-1}, (ac)^2, (bc)^2 >$	$4 \; \{c, cb, ca, abc\}$
$(\mathbb{Z}_8 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$ $< a, b, c \mid a^8, b^2, c^2, aba^{-1}b, aca^{-1}c, a^4bcbc, (bc)^4 >$	$2 \; \{b, c, abc, a^3bc\}$
$\mathbb{Z}_2 \times QD_{16}$ $< a \mid a^2 > \times < r, f \mid r^8, f^2, rfr^5f >$	$2 \; \{ar, r^3, r^5, r^2f\}$
$(\mathbb{Z}_8 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$ $< a, b, c \mid a^8, b^2, c^2, aba^{-1}b^{-1}, (ac)^2, a^4(bc)^2 >$	$4 \; \{a^7c, a^2c, ac, a^4b\}$
$(\mathbb{Z}_2 \times D_8) \rtimes \mathbb{Z}_2$ $< a, r, f, b \mid a^2, r^4, f^2, b^2, aba^{-1}b^{-1}, r(fa)^2, r^2(bf)^2 >$	$4 \; \{r^2f, ar^2, rf, rab\}$
$(\mathbb{Z}_2 \times Q_8) \rtimes \mathbb{Z}_2$ $< a, r, f, b \mid a^2, r^4, f^4, b^2, ara^{-1}r^{-1}, afa^{-1}f^{-1}, r^2f^2, rfrf^{-1},$ 　$(rrb)^2, r^2(ab)^2, brbr^{-1}f >$	$2 \; \{r^2b, a, ar^3fb, ar^3b\}$
$(\mathbb{Z}_2 \times Q_8) \rtimes \mathbb{Z}_2$ $< a, r, f, b \mid a^2, r^4, f^4, b^2, ara^{-1}r^{-1}, afa^{-1}f^{-1}, r^2f^2, rfrf^{-1},$ 　$(rb)^2, fbf^{-1}b^{-1}, arbar^{-1}b >$	$4 \; \{b, a, br, brf\}$

G$_{12}$: n = 36　　[4, 4, 4, 5]　　I$_{36,3}$　　AT4Val[36][3]

$\mathbb{Z}_3 \times (\mathbb{Z}_3 \rtimes \mathbb{Z}_4)$ $< a \mid a^3 > \times < b, c \mid b^3, c^4, bcbc^{-1} >$	$0 \; \{ac, a^2c^3, a^2cb, ac^3b\}$
$(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_4$ $< a, b, c \mid a^3, b^3, c^4, aba^{-1}b^{-1}, (acc)^2, acac^{-1}b^{-1} >$	$0 \; \{a^2b^2c^3, b^2c, a^2bc^3, ac\}$
$S_3 \times S_3$ $< s, t \mid s^2, t^3, (st)^2 > \times < u, v \mid u^2, v^3, (uv)^2 >$	$4 \; \{u, s, uv, st\}$ $4 \; \{u, uv, svu, stvu\}$ $2 \; \{su, stu, tsv, tsuvu\}$ $0 \; \{tu, stsu, sv, suvu\}$
$\mathbb{Z}_6 \times S_3$ $< a \mid a^6 > \times < s, t \mid s^2, t^3, (st)^2 >$	$2 \; \{a, a^5, a^3s, a^3st\}$ $2 \; \{a^3s, a^3st, a^2ts, a^4ts\}$ $0 \; \{as, a^3t, a^5s, a^3sts\}$
$\mathbb{Z}_2 \times ((\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2)$ $< a \mid a^2 > \times < b, c, d \mid b^3, c^3, d^2, bcb^{-1}c^{-1}, (bd)^2, (cd)^2 >$	$4 \; \{d, dc, adb, adcdbd\}$ $2 \; \{d, dc, ab, adbd\}$
$\mathbb{Z}_6 \times \mathbb{Z}_6$ $< a \mid a^6 > \times < b \mid b^6 >$	$0 \; \{a, a^5, b, b^5\}$

G$_{13}$: n = 40　　[4, 6, 4, 5]

$\mathbb{Z}_2 \times (\mathbb{Z}_5 \rtimes \mathbb{Z}_4)$ $< a \mid a^2 > \times < b, c \mid b^5, c^4, cbc^{-1}b^2, cb^2c^{-1}b^{-1} >$	$2 \; \{ac^2, ac^2b, c, c^3\}$

G$_{14}$: n = 48　　[6, 4, 10, 3]　　I$_{48,1}$

$\mathbb{Z}_2 \times \mathbb{Z}_4 \times S_3$ $< a \mid a^2 > \times < b \mid b^4 > \times < s, t \mid s^2, t^3, (st)^2 >$	$2 \; \{s, a, bt, b^3sts\}$

$D_8 \times S_3$ $< r, f \mid r^4, f^2, (rf)^2 > \times < s, t \mid s^2, t^3, (st)^2 >$	$4 \ \{s, rfs, fts, r^2fst\}$ $4 \ \{rf, rfs, fts, r^2fst\}$ $2 \ \{s, rfs, r^3t, rsts\}$ $2 \ \{rf, rfs, r^3t, rsts\}$
$\mathbb{Z}_2 \times ((\mathbb{Z}_6 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2)$ $< a \mid a^2 > \times < b, c, d \mid b^6, c^2, d^2, bcb^{-1}c^{-1}, (bbd)^2, b^3(dc)^2 >$	$4 \ \{a, c, b^4d, b^3d\}$
$\mathbb{Z}_6 \times D_8$ $< a \mid a^6 > \times < r, f \mid r^4, f^2, (rf)^2 >$	$2 \ \{a^3, a^3r^3f, ar, a^5r^3\}$
$\mathbb{Z}_2 \times S_4$ $< a \mid a^2 > \times < s, t \mid s^2, t^3, (st)^4 >$	$4 \ \{a, s, stst^2s, st^2sts\}$ $4 \ \{as, at^2st, atst^2, stst^2st\}$ $2 \ \{s, astst^2, atst^2s, atst^2st\}$
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times A_4$ $< a \mid a^2 > \times < b \mid b^2 > \times < s, t \mid s^2, t^3, (st)^3 >$	$2 \ \{a, abtst^2, abtst, abt^2st^2\}$
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times S_3$ $< a \mid a^2 > \times < b \mid b^2 > \times < c \mid c^2 > \times < s, t \mid s^2, t^3, (st)^2 >$	$4 \ \{s, b, cst, ats\}$

G_{15} : n = 72   [6, 16, 10, 3]   AT4Val[72][12]

$\mathbb{Z}_3 \times S_4$ $< a \mid a^3 > \times < s, t \mid s^2, t^3, (st)^4 >$	$0 \ \{ast, a^2tst, a^2t^2s, aststs\}$
$(\mathbb{Z}_3 \times A_4) \rtimes \mathbb{Z}_2$ $< a, s, t, b \mid a^3, s^2, t^3, b^2, asa^{-1}s^{-1}, ata^{-1}t^{-1}, stbsbt^{-1}, (ab)^2,$ $(tb)^2, (st)^3 >$	$4 \ \{atb, ab, tsbt, tbs\}$
$A_4 \times S_3$ $< s, t \mid s^2, t^3, (st)^3 > \times < u, v \mid u^2, v^3, (uv)^2 >$	$0 \ \{tu, t^2u, tsuv, st^2uv\}$
$\mathbb{Z}_6 \times A_4$ $< a \mid a^6 > \times < s, t \mid s^2, t^3, (st)^3 >$	$0 \ \{ast, a^3t, a^3t^2, a^5stst\}$

G_{16} : n = 72   [8, 10, 16, 1]

|	
$(\mathbb{Z}_3 \times (\mathbb{Z}_3 \rtimes \mathbb{Z}_4)) \rtimes \mathbb{Z}_2$ $< a, b, c, d \mid a^3, b^3, c^4, d^2, aba^{-1}b^{-1}, aca^{-1}c^{-1}, bdb^{-1}d^{-1},$ $adad^{-1}, bcbc^{-1}, c^2d^2 >$	$2 \ \{dc, dacb, ad, d^2ad\}$
$(\mathbb{Z}_6 \times S_3) \rtimes \mathbb{Z}_2$ $< a, b, c, d \mid a^2, b^4, c^3, d^3, (ab^{-1})^2, acac^{-1}, (ad)^2, cbcb^{-1},$ $bdb^{-1}d^{-1}, cdc^{-1}d^{-1} >$	$4 \ \{a, ab^2, abd, abacda\}$ $2 \ \{ab, abcd, cb, b^2cb\}$
$\mathbb{Z}_6 \times (\mathbb{Z}_3 \rtimes \mathbb{Z}_4)$ $< a \mid a^6 > \times < b, c \mid b^3, c^4, bcbc^{-1} >$	$0 \ \{ab^2, a^5b, a^3b^2c, a^3b^2c^3\}$
$\mathbb{Z}_3 \times ((\mathbb{Z}_6 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2)$ $< a \mid a^3 > \times < b, c, d \mid b^6, c^2, d^2, bcb^{-1}c^{-1}, (bbd)^2, b^3(dc)^2 >$	$2 \ \{b^5d, b^2d, a^2b^4c, ab^2c\}$ $0 \ \{b^2cd, b^5cd, a^2b^4c, ab^2c\}$
$(S_3 \times S_3) \rtimes \mathbb{Z}_2$ $< s, t, u, v, a \mid s^2, t^3, u^2, v^3, a^2, tvt^{-1}v^{-1}, (uv)^2, (av)^2,$ $svst^{-1}, asasu >$	$4 \ \{a, sastsat, s, sast\}$ $2 \ \{sas, stsa, atsa, asat^2\}$ $2 \ \{asa, atsat^2, sastst, asastsat\}$ $0 \ \{sa, as, atsat, asastst^2\}$
$\mathbb{Z}_2 \times ((\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_4)$ $< a \mid a^2 > \times < b, c, d \mid b^3, c^3, d^4, bcb^{-1}c^{-1}, (bdd)^2, bdbd^{-1}c^{-1} >$	$2 \ \{ad^2, ab^2c^2d^2, ab^2d^3, ab^2cd\}$ $0 \ \{ab^2cd, ab^2d^3, abc^2, ab^2c\}$

$\mathbb{Z}_2 \times S_3 \times S_3$ $< a \mid a^2 > \times < s,t \mid s^2,t^3,(st)^2 > \times < u,v \mid u^2,v^3,(uv)^2 >$	4 $\{u,s,auvst,atsvu\}$ 4 $\{u,au,suv,stvu\}$ 2 $\{u,s,atv,astsuvu\}$
$\mathbb{Z}_2 \times \mathbb{Z}_6 \times S_3$ $< a \mid a^2 > \times < b \mid b^6 > \times < s,t \mid s^2,t^3,(st)^2 >$	2 $\{ts,ab^3ts,bsts,b^5t\}$

G_{17} : n = 120   [12, 28, 4, 15]   AT4Val[120][4]

S_5 $< s,t \mid s^2,t^5,(st)^4,(st^2st^3)^2 >$	0 $\{t^2st^3, tst^2st^2st, st^2stst, tst^4\}$ 4 $\{t(st)^2tst^4, st^2(st)^2t, (t^2s)^2ts, (st^2)^2st\}$
$\mathbb{Z}_2 \times A_5$ $< a \mid a^2 > \times < s,t \mid s^2,t^3,(st)^5 >$	2 $\{a(tst^2s)^2t, ast(ts)^2, ast^2(st)^2, a(st)^3ts\}$
$S_3 \times (\mathbb{Z}_5 \rtimes \mathbb{Z}_4)$ $< s,t \mid s^2,t^3,(st)^2 > \times < a,b \mid a^5,b^4,ab^{-1}a^2b,a^2b^{-1}a^{-1}b >$	2 $\{sb^2, stb^2a, tb^3, stsb\}$
$\mathbb{Z}_5 \times S_4$ $< a \mid a^5 > \times < s,t \mid s^2,t^3,(st)^4 >$	0 $\{a^2st, a^3t^2s, aststs, a^4tst\}$
$(\mathbb{Z}_5 \times A_4) \rtimes \mathbb{Z}_2$ $< a,s,t,b \mid a^5,s^2,t^3,b^2,asa^{-1}s^{-1},ata^{-1}t^{-1},bsbt^{-1}st,$ $(st)^3,(tb)^2,(ab)^2 >$	4 $\{tba^2, bta, btbsb, tabs\}$

Table 4.3: Quartic bipartite integral Cayley graphs

Drawings for all but the three largest bipartite Cayley QIGs appear in Figure 4.4. With over 70 vertices, it is difficult to present G_{15}, G_{16}, and G_{17} clearly.

G_1

G_2

G_3

G_4

G_5

G_6

G₇   **G₈**   **G₉**

**G₁₀**   **G₁₁**   **G₁₂**

G₁₃   **G₁₄**
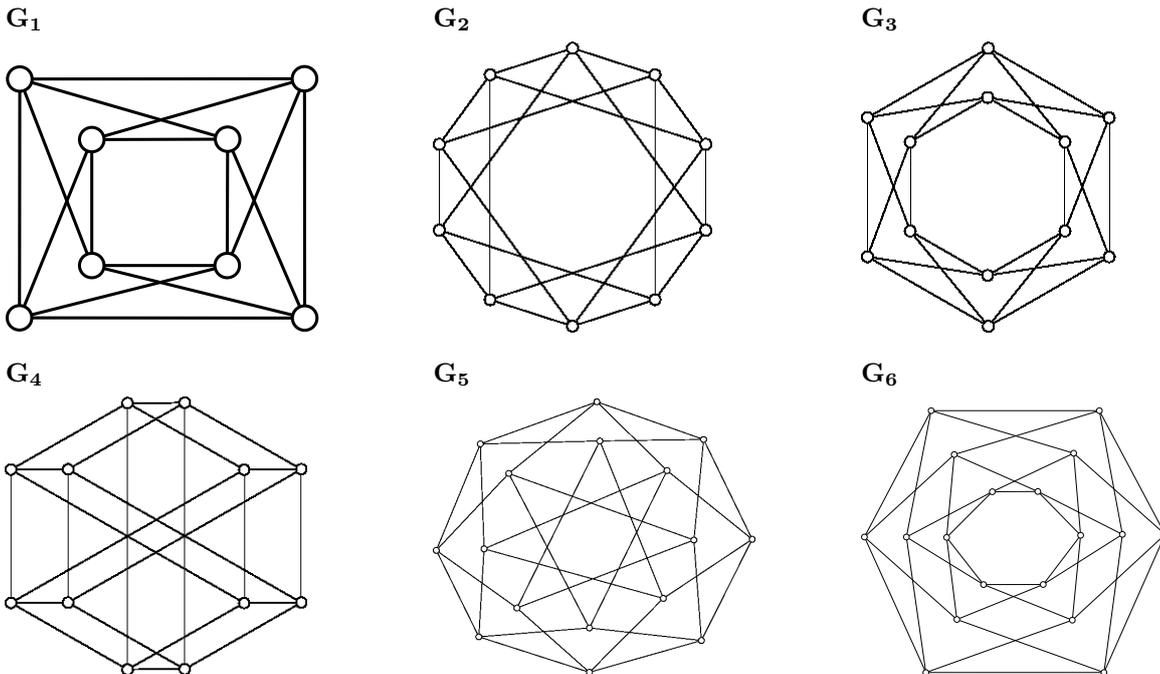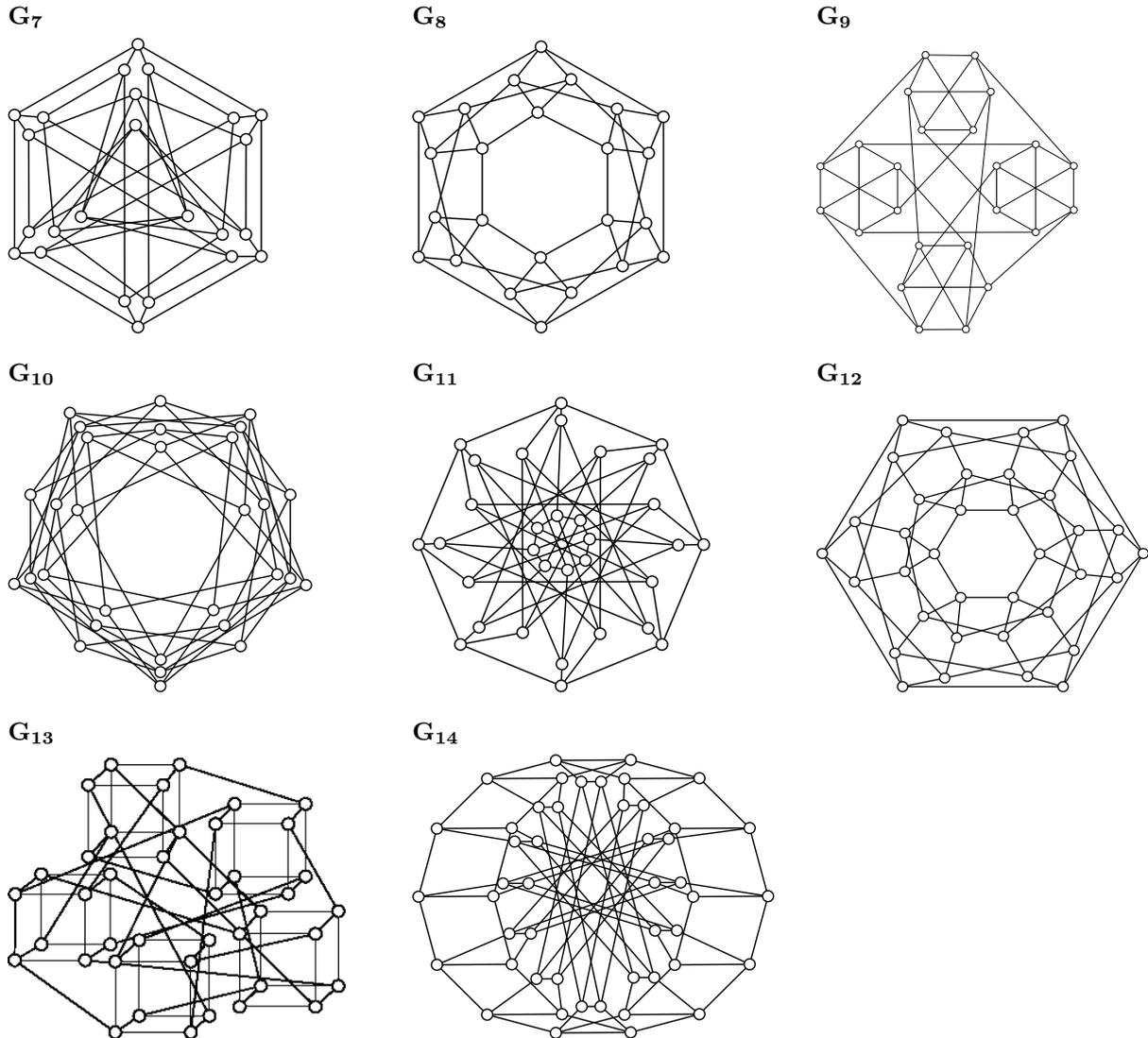
Table 4.4: Drawings of quartic bipartite integral Cayley graphs G_1 to G_{14}

4.4.2 Bipartite arc-transitive integral graphs

We considered all arc-transitive 4-regular graphs from the census of Potočnik et al. [151, 152] and tested them for integrality. The only arc-transitive bipartite QIGs that are not Cayley and thus not accounted for in Table 4.3 are the five that appear in Table 4.5. We let $[\Gamma : H] = \{Ha \mid a \in \Gamma\}$ denote the set of right cosets of $H \in \Gamma$. A *Schreier coset graph* $\mathrm{Sch}(\Gamma, H, HSH)$ for a group Γ, subgroup $H \leqslant \Gamma$, and connection set $S \subset \Gamma$ is the graph with vertex set $[\Gamma : H]$ and with Ha connected to Hb if and only if $ba^{-1} \in HSH$. We represent these 5 graphs as Schreier coset graphs. We give the order n and the spectrum $[x, y, z, w]$ followed by the graph index from [151, 152]. Graphs appearing in the paper by Cvetković *et al.* are labelled with the notation of [63]: $I_{n,index}$. The first line consists of the group Γ, with a presentation of that group. The second line consists of the subgroup H and its generators in terms of the generators of Γ followed by the connection set S in terms of the generators of Γ.

Group
Subgroup, Subset

F_1 : n = 60   [4, 16, 4, 5] $I_{60,1}$ AT4Val[60][4]

$\mathbb{Z}_2 \times \mathbb{Z}_2 \times S_5$: $< a \mid a^2 > \times < b \mid b^2 > \times < s, t \mid s^2, t^5, (st)^4, (st^2st^3)^2 >$
D_8 : $< bstst^2st^{-1}, abstst >$, $\{s, bt^2, s^t, bt^{-2}\}$

F_2 : n = 70   [6, 14, 14, 0] $I_{70,1}$ AT4Val[70][4]

$\mathbb{Z}_2 \times S_7$: $< a \mid a^2 > \times < s, t \mid s^2, t^7, (st)^6, (st^2st^5)^2, (stst^{-1})^3 >$
$S_3 \times S_4$: $< t^2st^{-2}, t^{-2}st^{-1}(st)^2t, t^2(st)^2(ts)^3t^{-1}, t(st)^2(ts)^3, stst^{-1}s >$, $\{ast^4, atstst^{-1}, at,$
   $at^{-1}\}$

F_3 : n = 90   [9, 16, 19, 0] $I_{90,1}$ AT4Val[90][1]

$\mathbb{Z}_2 \times P\Gamma L(2,9)$: $< a \mid a^2 > \times < x, y, z \mid x^8, y^3, z^2, xzx^5z, yzy^{-1}z^{-1}, xyxy^{-1}x^6yx^6y^{-1},$
   $(xyx^2y)^2, xyx^{-2}y^{-1}x^4yx^{-1}y^{-1} >$
$(\mathbb{Z}_2 \times D_8) \rtimes \mathbb{Z}_2$: $< yzxy^{-1}x, x^{-1}yxzx^{-1}y, x^2zy^{-1}x^{-1}y^{-1}xy >$, $\{ayx^{-1}y^{-1}x, az, ayxy^{-1}x,$
   $axy^{-1}x^{-1}y\}$

F_4 : n = 180   [22, 28, 34, 5]   AT4Val[180][12]

$\mathbb{Z}_2 \times S_3 \times S_5$: $< a \mid a^2 > \times < s, t \mid s^2, t^3, (st)^2 > \times < u, v \mid u^2, v^5, (uv)^4, (uv^2uv^3)^2 >$
D_8 : $< v^{-2}uv^2, vuv^2uv^2, astuvuv^2uv^{-1} >$, $\{at^{-1}v^{-2}, atuv^2, su, atv^2\}$

F_5 : n = 210   [27, 28, 49, 0]   AT4Val[210][10]

S_7 : $< s, t \mid s^2, t^7, (st)^6, (st^2st^5)^2, (stst^{-1})^3 >$
S_4 : $< tst^3st^3, tst^{-2}st, (st)^2t^2st^{-1}(st)^2tst >$, $\{t^3s, st^4, (st)^3, (ts)^3\}$

Table 4.5: Quartic bipartite arc-transitive non-Cayley integral graphs

The census [151, 152] of arc-transitive graphs contains all arc-transitive graphs with at most 640 vertices. Thus, the upper bound of 560 given in [176] for the order of a bipartite QIG, ensures that Table 4.3 and Table 4.5 contain all bipartite arc-transitive QIGs. The non-bipartite arc-transitive QIGs will be given in Sections 4.4.4 and 4.4.5. However, first we describe our method for finding all Cayley QIGs.

4.4.3 Integral graphs as quotients

Let $V(G)$ denote the vertices of a graph G, and $E(G)$ the unordered pairs of vertices which are edges of G. A homomorphism from a graph G to a graph H is a map $V(G) \to V(H)$ which preserves adjacency. Each homomorphism induces an edge map $E(G) \to E(H)$. If the vertex and edge maps of the homomorphism are both surjective then we say that H is a *quotient* of G. In this section we find new integral graphs that are quotients of the integral graphs found in Table 4.3 and Table 4.5. To specify a quotient of a graph G it suffices to know G and the vertex map (the edges of the quotient are implied by the surjectivity of the edge map).

We start by considering special classes of possible homomorphisms. Suppose G is a graph and α is a function from $E(G)$ to a group Γ. A Γ-*voltage graph* $\text{Vol}(G, \alpha)$ for G is the graph with vertex set $V(G)$ and edge set $E(G)$ with edges labeled by an element of Γ as given by the function α. The *derived graph* of a Γ-voltage graph $\text{Vol}(G, \alpha)$ is the graph with vertex set the Cartesian product $V(G) \times \Gamma$ with (a, x) connected to (b, y) whenever both $\{a, b\} \in E(G)$ and $y = x * \alpha(\{a, b\})$ where $*$ is the group operation of Γ. Projection onto the first coordinate, by definition, maps the derived graph of $\text{Vol}(G, \alpha)$ onto G, and this map is a surjective homomorphism. Hence G is a quotient of the derived graph.

As an interesting example for quartic integral graphs, we found a $\mathbb{Z}_3$-voltage graph $\text{Vol}(F_1, \alpha)$, with derived graph F_4. Thus, F_1 is a quotient of F_4.

A bipartite double cover $G \times K_2$ is the derived graph of the $\mathbb{Z}_2$-voltage graph $\text{Vol}(G, \alpha)$ where α is the constant function assigning 1 to every edge.

We give an example for quartic integral graphs that was also noted in [63]. An *odd graph* O_i is the graph with one vertex for each of the $(i-1)$-element subsets of a $(2i-1)$-element set and with edges joining disjoint subsets. The graph F_2 is the bipartite double cover of the integral graph O_4.

We want to find all graphs which have their bipartite double cover among the bipartite graphs that we have discovered. This requires us to find quotients of our bipartite graphs. Since it is computationally easy to do, we will actually consider a more general class of homomorphisms than what is required for the task just described. This will increase the number of quartic integral graphs that we find. However, we make no effort to be exhaustive in finding all possible quotients.

Let ϑ be an automorphism of some graph G. For convenience, we will refer to the orbits in the action on G of the group generated by ϑ as simply "orbits of ϑ". We say that ϑ is *k-semiregular* if all its orbits have the same size, k. Note that if $G = H \times K_2$ then the natural homomorphism from G onto H maps orbits of a 2-semiregular automorphism of G to single vertices of H. With this as motivation, the class of homomorphisms that we consider is the following. We identify any k-semiregular automorphism, ϑ of a target graph G. Our homomorphism is to collapse each orbit of ϑ to a single point.

We wrote a routine in Magma [168] to find such quotients of a target graph G, as follows. For one representative, ϑ, of each conjugacy class of (nontrivial) semiregular automorphisms of G, we collapsed the orbits of ϑ to single vertices to obtain a quotient H. If H was a 4-regular graph we checked to see if it was integral. If it was, then we printed it out and called the routine recursively on H.

In some cases we were only interested in finding those H for which G is a bipartite double cover. In such instances, it suffices to only consider 2-semiregular automorphisms and we do not need to make recursive calls to the routine.

We applied our Magma routine to all target graphs G_i for $i \in 1, \ldots, 17$ and to most of the arc-transitive graphs from the census of Potočnik et al. [151, 152]. There are graphs in the census with extremely large automorphism groups, and they were impractical for our simple routine. So we decided to only include target graphs from the census if their automorphism group had order no more than 2^{20}.

The results of this Magma routine will be given in the following subsections.

4.4.4   Non-bipartite Cayley integral graphs

In this section we report all quartic Cayley integral graphs that are not bipartite. We rely on this Lemma:

Lemma 21. *If G is a 4-regular Cayley graph then $G \times K_2$, the bipartite double cover of G, is isomorphic to a 4-regular Cayley graph.*

Proof. If $G = \text{Cay}(\Gamma, S)$ then we define $G' = \text{Cay}(\Gamma \times \mathbb{Z}_2, \{(s, 1) \mid s \in S\})$. This graph G' is an undirected Cayley graph. Consider the following: $(g, 0)$ is adjacent to $(g, 0)(s, 1) = (gs, 1)$ and $(g, 1)$ is adjacent to $(g, 1)(s, 1) = (gs, 0)$ for $g \in \Gamma$. We have verified that G' is isomorphic to $G \times K_2$ which gives the desired result. $\square$

Hence we can find all the graphs we seek by applying the Magma routine of Section 4.4.3 to our graphs G_i where $i = 1, \ldots, 17$. We use the following result by Sabidussi [156] to decide which of the graphs that we find are Cayley graphs:

Lemma 22. *A graph G is a Cayley graph if and only if $\text{Aut}(G)$ contains a regular subgroup.*

Table 4.6 summarizes our results using the Magma routine of Section 4.4.3 when restricted to the 2-semiregular automorphisms for each given G_i. We give the number of non-bipartite graphs found, followed by the numbers of those that are Cayley, vertex-transitive, and arc-transitive.

Initial Graph	#Non-bipartite	#Cayley	#Vertex-transitive	#Arc-transitive
G_1	0	0	0	0
G_2	1	1	1	1
G_3	1	1	1	1
G_4	0	0	0	0
G_5	1	1	1	0
G_6	2	1	1	1
G_7	2	1	1	1
G_8	4	2	2	0
G_9	0	0	0	0
G_{10}	1	0	1	1
G_{11}	0	0	0	0
G_{12}	2	1	1	0
G_{13}	1	1	1	0
G_{14}	2	2	2	0
G_{15}	5	1	1	1
G_{16}	13	2	2	0
G_{17}	2	1	1	0

Table 4.6: Non-bipartite graphs found for G_i

The non-bipartite graphs counted in column 2 up to row 8 of Table 4.6 were previously found by Stevanović et al. [176]. All graphs counted by column 2 from rows 9 to 17 were previously unknown with the exception of the graph with bipartite double cover G_{10} and one of the two graphs with bipartite double cover G_{14}. In Table 4.7, we expand upon the counts of non-bipartite Cayley graphs in column three of Table 4.6 by producing a breakdown of the groups and the connection sets of the underlying graphs. We follow the same conventions as in Table 4.3 except that we use different notation for the spectrum, since there is no longer symmetry about the origin.

Group	Connection Sets (#involutions S)
$\mathbf{H_1}$: $\mathbf{n = 5}$ $\mathbf{-1^4, 4^1}$ $\mathbf{I_{5,1}}$ $\mathbf{AT4Val[5][1]}$	
$\mathbb{Z}_5$ $< a \mid a^5 >$	$0\ \{a^3, a^2, a^4, a\}$
$\mathbf{H_2}$: $\mathbf{n = 6}$ $\mathbf{-2^2, 0^3, 4^1}$ $\mathbf{I_{6,1}}$ $\mathbf{AT4Val[6][1]}$	
S_3 $< s, t \mid s^2, t^3, (st)^2 >$	$2\ \{st, t, t^2, s\}$
$\mathbb{Z}_6$ $< a \mid a^6 >$	$0\ \{a^5, a^2, a, a^4\}$
$\mathbf{H_3}$: $\mathbf{n = 8}$ $\mathbf{-2^3, 0^3, 2^1, 4^1}$ $\mathbf{I_{8,2}}$	
$\mathbb{Z}_4 \times \mathbb{Z}_2$ $< a \mid a^4 > \times < b \mid b^2 >$	$2\ \{a, a^2, a^3, b\}$
D_8 $< r, f \mid r^4, f^2, (rf)^2 >$	$2\ \{r, r^2, r^3, fr^2\}$ $4\ \{f, r^2, rf, fr\}$
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ $< a \mid a^2 > \times < b \mid b^2 > \times < c \mid c^2 >$	$4\ \{b, a, abc, ac\}$
$\mathbf{H_4}$: $\mathbf{n = 9}$ $\mathbf{-2^4, 1^4, 4^1}$ $\mathbf{I_{9,2}}$ $\mathbf{AT4Val[9][1]}$	
$\mathbb{Z}_3 \times \mathbb{Z}_3$ $< a \mid a^3 > \times < b \mid b^3 >$	$0\ \{a^2b, ab^2, a^2, a\}$
$\mathbf{H_5}$: $\mathbf{n = 12}$ $\mathbf{-2^5, 0^3, 2^3, 4^1}$ $\mathbf{I_{12,7}}$ $\mathbf{AT4Val[12][1]}$	
A_4 $< s, t \mid s^2, t^3, (st)^3 >$	$0\ \{t^2s, ts, st^2, st\}$
$\mathbf{H_6}$: $\mathbf{n = 12}$ $\mathbf{-3^2, -1^4, 0^1, 1^2, 2^2, 4^1}$ $\mathbf{I_{12,5}}$	
$\mathbb{Z}_3 \rtimes \mathbb{Z}_4$ $< a, b \mid a^3, b^4, abab^{-1}, ab^2a^2b^2 >$	$0\ \{a, ba^2, b^3a^2, a^2\}$
$\mathbb{Z}_{12}$ $< a \mid a^{12} >$	$0\ \{a^3, a^4, a^8, a^9\}$

D_{12} $< r, f \mid r^6, f^2, (rf)^2 >$	$2\ \{r^4, fr^4, r^2, fr\}$ $2\ \{r^3, f, r^4, r^2\}$
$\mathbb{Z}_6 \times \mathbb{Z}_2$ $< a \mid a^6 > \times < b \mid b^2 >$	$2\ \{a^2, b, a^3, a^4\}$

H_7 : n = 12 $-\ 3^2, -2^2, 0^1, 1^6, 4^1$ $I_{12,1}$

$\mathbb{Z}_3 \rtimes \mathbb{Z}_4$ $< a, b \mid a^3, b^4, abab^{-1}, ab^2a^2b^2 >$	$0\ \{b^3, b, b^2a, b^2a^2\}$
$\mathbb{Z}_{12}$ $< a \mid a^{12} >$	$0\ \{a^3, a^{10}, a^2, a^9\}$
D_{12} $< r, f \mid r^6, f^2, (rf)^2 >$	$2\ \{f, fr^3, r, r^5\}$ $4\ \{r^3, f, fr, fr^5\}$
$\mathbb{Z}_6 \times \mathbb{Z}_2$ $< a \mid a^6 > \times < b \mid b^2 >$	$2\ \{a^3b, a^3, a^2b, a^4b\}$

H_8 : n = 18 $-\ 3^2, -2^4, 0^5, 1^4, 3^2, 4^1$ $I_{18,4}$

$\mathbb{Z}_3 \times S_3$ $< a \mid a^3 > \times < s, t \mid s^2, t^3, (st)^2 >$	$2\ \{a, s, a^2, st^2\}$ $0\ \{t, t^2, sat^2, sa^2t^2\}$
$(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2$ $< a, b, c \mid a^3, b^3, c^2, cac^{-1}a^{-2}, bc^{-1}b^{-2}c, aba^{-1}b^{-1} >$	$2\ \{a, c, a^2, cb^2\}$
$\mathbb{Z}_6 \times \mathbb{Z}_3$ $< a \mid a^6 > \times < b \mid b^3 >$	$0\ \{a^2b, a^5b^2, ab, a^4b^2\}$

H_9 : n = 20 $-\ 2^6, -1^4, 0^5, 3^4, 4^1$

$\mathbb{Z}_5 \rtimes \mathbb{Z}_4$ $< a, b \mid a^5, b^4, ab^3a^3b, ab^2ab^2 >$	$2\ \{a^2b^2, ab^2, a^2b, a^4b^3\}$

H_{10} : n = 24 $-\ 3^3, -2^3, -1^5, 0^3, 1^5, 2^1, 3^3, 4^1$ $I_{24,5}$

S_4 $< s, t \mid s^2, t^3, (st)^4 >$	$2\ \{s, st^2st, (tst)^2, t(ts)^2\}$
$\mathbb{Z}_2 \times A_4$ $< a \mid a^2 > \times < s, t \mid s^2, t^3, (st)^3 >$	$2\ \{s, st^2, a, ts\}$

H_{11} : n = 24 $-\ 3^4, -2^3, -1^2, 0^3, 1^8, 2^1, 3^2, 4^1$

$\mathbb{Z}_4 \times S_3$ $< a \mid a^4 > \times < s, t \mid s^2, t^3, (st)^2 >$	$2\ \{a^3t, at^2, sa^2, a^2\}$
$(\mathbb{Z}_6 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$ $< a, b, c \mid a^6, b^2, c^2, aba^{-1}b^{-1}, (a^3c)^2b, cbc^{-1}b^{-1}, a^2ca^2c >$	$4\ \{ca^2, cb, b, a^3\}$
$\mathbb{Z}_3 \times D_8$ $< a \mid a^3 > \times < r, f \mid r^4, f^2, (rf)^2 >$	$2\ \{f, r^2, ar, a^2r^{-1}\}$
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times S_3$ $< a \mid a^2 > \times < b \mid b^2 > \times < s, t \mid s^2, t^3, (st)^2 >$	$4\ \{sabt, sbt^2, ab, sa\}$

H_{12} : n = 36 $-2^{13}, -1^6, 0^3, 1^4, 2^3, 3^6, 4^1$ AT4Val[36][6]

$\mathbb{Z}_3 \times A_4$ $< a \mid a^3 > \times < s, t \mid s^2, t^3, (st)^3 >$	$0 \ \{ta, tst, t^2a^2, t^2st^2\}$

H_{13} : n = 36 $-3^4, -2^{10}, 0^1, 1^{16}, 3^4, 4^1$

$\mathbb{Z}_3 \times (\mathbb{Z}_3 \rtimes \mathbb{Z}_4)$ $< a, b, c \mid a^3, b^3, c^4, aba^{-1}b^{-1}, aca^{-1}c^{-1}, bc^{-1}b^{-2}c >$	$0 \ \{ac^2b, cb^2, a^2c^2b^2, c^3b^2\}$
$(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_4$ $< a, b, c \mid a^3, b^3, c^4, aba^{-1}b^{-1}, ac^{-1}a^{-1}bc, ac^2ac^2, acbc^{-1}b, c^2bc^2b >$	$2 \ \{a^2bc^3, a^2c, ac^2, ab^2c^2\}$
$S_3 \times S_3$ $< s, t \mid s^2, t^3, (st)^2 > \times < u, v \mid u^2, v^3, (uv)^2 >$	$4 \ \{suvt, s, sut^2, uv^2\}$
$\mathbb{Z}_6 \times S_3$ $< a \mid a^6 > \times < s, t \mid s^2, t^3, (st)^2 >$	$2 \ \{a^5t^2, at, sa^3t, st\}$

H_{14} : n = 36 $-3^4, -2^4, -1^{12}, 0^1, 1^4, 2^6, 3^4, 4^1$

$\mathbb{Z}_3 \times (\mathbb{Z}_3 \rtimes \mathbb{Z}_4)$ $< a, b, c \mid a^3, b^3, c^4, aba^{-1}b^{-1}, aca^{-1}c^{-1}, bc^{-1}b^{-2}c >$	$0 \ \{c^3b, cb, a^2b, ab^2\}$
$(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_4$ $< a, b, c \mid a^3, b^3, c^4, aba^{-1}b^{-1}, ac^{-1}a^{-1}bc, ac^2ac^2, acbc^{-1}b, c^2bc^2b >$	$0 \ \{a^2bc^3, a^2c, b^2, b\}$
$S_3 \times S_3$ $< s, t \mid s^2, t^3, (st)^2 > \times < u, v \mid u^2, v^3, (uv)^2 >$	$2 \ \{v^2t, s, uv, vt^2\}$
$\mathbb{Z}_6 \times S_3$ $< a \mid a^6 > \times < s, t \mid s^2, t^3, (st)^2 >$	$2 \ \{a^2t, sa^3t, a^4t^2, st\}$

H_{15} : n = 60 $-3^4, -2^{17}, -1^4, 0^{15}, 2^{11}, 3^8, 4^1$

A_5 $< s, t \mid s^2, t^3, (st)^5 >$	$2 \ \{st^2(st)^2, tst^2(st)^2t, (st)^3ts,$ $((st)^2t)^2st\}$

Table 4.7: Quartic non-bipartite integral Cayley graphs

Thus, by Theorem 20 and Lemma 21 we have that $\{G_i \mid 1 \leqslant i \leqslant 17\} \cup \{H_j \mid 1 \leqslant j \leqslant 15\}$ is the complete set of Cayley QIGs.

4.4.5   Non-bipartite arc-transitive integral graphs

In Section 4.4.2, we listed all bipartite arc-transitive QIGs from the census of Potočnik et al. [151, 152]. When searching this census for integral graphs, we also found arc-transitive QIGs that are not bipartite. There are 6 such graphs that are not Cayley and thus not already accounted for in Table 4.7. By [176], the bipartite double cover of any QIG has order at most 560, so we can be sure that the census contains all the arc-transitive QIGs. In fact, the following folklore result tells us more:

Lemma 23. *The bipartite double cover of an arc-transitive graph is arc-transitive.*

Proof. Let G be an arc-transitive graph. Then $H = G \times K_2$ has vertices (a, x) for all $a \in G$ and $x \in \mathbb{Z}_2$ and arcs $((a, x), (b, x + 1))$ and $((b, x), (a, x + 1))$ whenever a is adjacent to b in G.

Claim 1: For all $\sigma \in \text{Aut}(G)$, $\sigma' : V(H) \to V(H)$ defined by $\sigma'((a, x)) = (\sigma(a), x + 1)$ is an automorphism of H. Consider any element $(a, x) \in H$, then since σ is a permutation of the vertices of G, we have that there is an element $(\sigma^{-1}(a), x+1) \in H$ such that $\sigma'((\sigma^{-1}(a), x+1)) = (a, x)$. Thus σ' is a permutation of the vertices of H. Consider any arc $((a, x), (b, x + 1))$ in $E(H)$. By applying σ', $((a, x), (b, x + 1)) \to ((\sigma(a), x + 1), (\sigma(b), x))$. Since a was adjacent to b in G, $\sigma(a)$ is adjacent to $\sigma(b)$ in G. Now since the parities of the second coordinates are still different, $((\sigma(a), x+1), (\sigma(b), x))$ is an arc.

Claim 2: The map $\sigma^* : V(H) \to V(H)$ defined by $\sigma^*((a, x)) = (a, x + 1)$ is an automorphism of H. It is quite clear that σ^* is an order 2 permutation that swaps the vertices with the same first coordinate. We consider $((a, x), (b, x + 1))$ in $E(H)$. By σ^*, $((a, x), (b, x + 1)) \to ((a, x + 1), (b, x))$. Since a was adjacent to b in G and the parities of the second coordinates are still different, $((a, x + 1), (b, x))$ is an arc.

Claim 3: For all pairs of arcs $e_1 = ((a, x), (b, x+1))$ and $e_2 = ((c, y), (d, y+1))$ in $E(H)$, there exists $\alpha \in \text{Aut}(H)$ such that $\alpha(e_1) = e_2$. If $c = a$ and $d = b$, then $\alpha = \sigma^*$ or the identity automorphism. Otherwise, since G is arc-transitive, there exists $\sigma_1 \in \text{Aut}(G)$ such that $(a, b) \to (c, d)$ where (a, b) and (c, d) are arcs of G. If $y = x + 1$ then $\alpha = \sigma_1'$. Otherwise $\alpha = \sigma^* \circ \sigma_1'$.

Thus, H is arc-transitive. $\square$

This last result provides a useful cross-check of our results and of the Magma routine from Section 4.4.3. It tells us that by applying the routine (restricted to 2-semiregular automorphisms) to the bipartite arc-transitive QIGs from Tables 4.3 and 4.5, we should find all the arc-transitive integral non-bipartite graphs. This list should tally with the list obtained by directly screening the census for integral graphs, which is what happened in practice.

We now list the spectrum of the non-bipartite arc-transitive QIGs that are not Cayley and whose bipartite double cover is one of the G_i for $i = 1, \ldots, 17$ or F_i for $i = 1, \ldots, 5$. We denote these graphs by J_i for $i = 1, \ldots, 6$. Graphs appearing in the paper by Cvetković *et al.* are included using the notation of [63]: $I_{n,index}$. We give the graph index from the census of Potočnik et al. [151, 152] in their notation: AT4Val[n][index].

- From G_{10}, $J_1 \cong I_{15,2} \cong$ AT4Val[15][1] : $[-2^5, -1^4, 2^5, 4^1]$,

- From F_1, $J_2 \cong$ AT4Val[30][2] : $[-3^4, -2^5, -1^4, 0^5, 2^{11}, 4^1]$,
  $J_3 \cong I_{30,4} \cong$ AT4Val[30][3] : $[-2^{11}, -1^4, 0^5, 2^5, 3^4, 4^1]$,

- From F_2, $J_4 \cong I_{35,1} \cong$ AT4Val[35][2] $\cong O_4$: $[-3^6, -1^{14}, 2^{14}, 4^1]$,

- From F_3, $J_5 \cong I_{45,1} \cong$ AT4Val[45][1] : $[-2^{16}, -1^9, 1^{10}, 3^9, 4^1]$,

- From F_4, $J_6 \cong$ AT4Val[90][8] : $[-3^{14}, -2^7, -1^{24}, 0^5, 1^{10}, 2^{21}, 3^8, 4^1]$.

Of the arc-transitive non-bipartite non-Cayley graphs, only J_2 and J_6 were not previously known to be integral. Thus, the arc-transitive QIGs from the census are as follows: G_1, G_2, G_3, G_5, G_6, G_7,

G_{10}, G_{11}, G_{12}, G_{15}, G_{17}, F_1, F_2, F_3, F_4, F_5, H_1, H_2, H_4, H_5, H_{12}, J_1, J_2, J_3, J_4, J_5, and J_6. We summarize these results by the following Lemma:

Lemma 24. *There are exactly* 27 *quartic integral graphs that are arc-transitive;* 16 *of which are bipartite.*

4.4.6 The only vertex-transitive graph on 32 vertices

On considering the only spectrum with 32 vertices in Appendix A, we are able to prove that G_{11} is the unique vertex-transitive QIG with this spectrum.

We consider a vertex-transitive graph G with 32 vertices, girth 6, and spectrum the quadruple $[0, 12, 0, 3]$. Since G is vertex-transitive, the orbit-stabilizer theorem (see [89] for example) gives us that $|G| = 2^5$ divides $|\operatorname{Aut}(G)|$.

A *p-subgroup H* of a group Γ is a subgroup of Γ where every element $h \in H$ has order a power of the prime p. A *Sylow p-subgroup P* of a group Γ is a p-subgroup that is contained in no larger p-subgroup of Γ. Let A be a Sylow 2-subgroup of $\operatorname{Aut}(G)$ [77]. The *center* $Z(\Gamma)$ is defined as $Z(\Gamma) = \{z \in \Gamma \mid z\alpha = \alpha z \text{ for all } \alpha \in \Gamma\}$. If $Z(\Gamma) = \{e\}$ where e is the identity element of Γ then we say that the center of Γ is *trivial*.

The following theorem appears in [198, Thm 3.4].

Theorem 25. *Let p be a prime number and let A be a Sylow p-subgroup of $\operatorname{Aut}(G)$. Suppose p^s divides $|\operatorname{Aut}(G)v|$ for a vertex $v \in V(G)$. Then p^s also divides $|Av|$.*

Now since G is transitive, $|\operatorname{Aut}(G)v| = 2^5$ for all $v \in V(G)$. By Theorem 25, 2^5 divides $|Av|$ for all $v \in V(G)$ and thus $Av = V(G)$. Therefore, A is transitive on the vertices of G.

Since A is finite and non-trivial, it has a non-trivial center [77]. Let σ be a central element in A of order 2. Since σ is a central element, the group $\langle \sigma \rangle$ is a normal subgroup of A.

Given a graph H, a *partition* Π of $V(H)$ is a set of disjoint non-empty subsets of $V(H)$ whose union is $V(H)$. We will refer to these subsets as cells. A partition, $\Pi = (C_1, \dots, C_k)$ is *equitable* if for every choice of i and j, each vertex in C_i has the same number of neighbours in C_j. Given an equitable partition Π of a graph H, the *quotient graph H/Π* of H with respect to Π is the graph with the cells of Π as its vertices and with edge (C_i, C_j) for every edge $(x, y) \in E(H)$ where $x \in C_i$ and $y \in C_j$. Thus, a quotient graph may have multiple edges and loops.

The partition, $\Pi = (C_1, \dots, C_k)$, of $V(G)$ into orbits of σ is an equitable partition of the graph G [155, p. 76]. Since the group $\langle \sigma \rangle$ is normal, each cell contains two vertices [198, Prop 7.1]. Now since A is transitive on $V(G)$, A acts transitively on these cells [23, p. 173]. Thus, the quotient graph G/Π of G with respect to Π is a transitive multigraph with 16 vertices.

Every eigenvalue of G/Π is an eigenvalue of the graph G [94]. For this reason, we only consider graphs with integer eigenvalues. For any integral graph that satisfies the known conditions of G/Π, we check that it lifts to G.

The pair of vertices in one cell is mapped to the pair of vertices in another cell and so these pairs of edges in G/Π can be thought of as a single edge in the 'frame' of G/Π. Each pair of vertices in a cell is either adjacent or non-adjacent. This gives 2 cases:

- The frame of G/Π is a cubic transitive graph. It can be checked that there is no such cubic graph that is integral.

- The frame of G/Π is a quartic transitive graph. It can be checked that a unique quartic graph candidate exists that satisfies integrality: $C_4 \times C_4$.

In lifting from $C_4 \times C_4$ to G each edge between cells is replaced by two edges. These edges can be thought of as 'parallel' or 'crossed', depending on whether they join vertices in the same copy of $C_4 \times C_4$ or not. Since G has no 4-cycles, each cycle must have an odd number of 'parallel' edges to be a suitable candidate for G. Using this observation, it can be checked by exhaustion that there is a unique way to lift G/Π to G. This unique completion is isomorphic to G_{11}. There is no bipartite QIG on 64 vertices. Therefore, by Theorem 21, there are no non-bipartite QIGs with 32 vertices. Thus, G_{11} is the unique QIG on 32 vertices that is vertex-transitive.

4.4.7   Other quartic integral graphs

Finally, we list the spectra of the remaining QIGs which we found using the Magma routine of Section 4.4.3 in its full generality. These are graphs that are neither Cayley nor arc-transitive, but are quotients of the graphs G_i for $i = 1, \ldots, 17$ and/or of the graphs AT4Val[n][index] for $n \leqslant 640$ with automorphism groups of order less than 2^{20}. We note that many of these graphs were obtained from multiple starting graphs, but we only list each graph once.

We list the spectrum of the bipartite QIGs first. We denote these graphs by M_i for $i = 1, \ldots, 9$ and follow the same conventions as in the list for J_i where $i = 1, \ldots, 6$ except that we use the quadruple form for the spectrum of a bipartite graph.

- From AT4Val[60][4] we have $M_1 \cong I_{30,3} : [1, 8, 3, 2]$,

- From $G_{15} \cong$ AT4Val[72][12] we have $M_2 \cong I_{36,1} : [2, 8, 6, 1]$, $M_3 \cong I_{36,2} : [3, 6, 5, 3]$,

- From $G_{17} \cong$ AT4Val[120][4] we have $M_4 : [3, 4, 1, 6]$, $M_5 : [6, 12, 2, 9]$,

- From AT4Val[180][12] we have $M_6 : [9, 16, 19, 0]$, $M_7 : [10, 14, 18, 2]$,

- From AT4Val[216][12] we have $M_8 : [3, 5, 9, 0]$,

- From AT4Val[546][48] we have $M_9 : [5, 4, 7, 4]$.

We do not list graphs with at most 24 vertices since all bipartite QIGs on 24 or fewer vertices are known [176]. The 6 graphs $M_4, \ldots, M_9$ were not previously known to be bipartite QIGs. We find that M_6 is co-spectral to F_3, but 5 of the above spectra were not previously known to be realized by any graph.

We give an example of how a quotient graph can arise from multiple graphs. In addition to AT4Val[180][12], M_7 is a quotient of AT4Val[360][10] and AT4Val[540][17].

Next, we list the spectrum of the non-bipartite QIGs. We denote these graphs by L_i where $i \in 1, \ldots, 44$.

- From AT4Val[30][3], $L_1 \cong I_{15,4} : [-2^5, -1^3, 0^2, 2^3, 3^1, 4^1]$.

- From $G_{12} \cong$ AT4Val[36][3], $L_2 : [-3^3, -2^2, -1^1, 0^5, 1^3, 2^2, 3^1, 4^1]$.

- From AT4Val[36][6], $L_3 \cong I_{18,5} : [-2^7, -1^2, 0^1, 1^4, 2^1, 3^2, 4^1]$,
  $L_4 \cong I_{18,6} : [-2^6, -1^3, 0^3, 1^2, 3^3, 4^1]$.

- From AT4Val[60][4], $L_5 : [-3^3, -2^7, -1^3, 0^5, 1^1, 2^9, 3^1, 4^1]$, and
  $L_6 : [-3^2, -2^9, -1^2, 0^5, 1^2, 2^7, 3^2, 4^1]$.

- From AT4Val[70][4], $L_7 : [-3^5, -2^4, -1^9, 1^5, 2^{10}, 3^1, 4^1]$, and
  $L_8 : [-3^4, -2^6, -1^8, 1^6, 2^8, 3^2, 4^1]$.

- From $G_{15} \cong$ AT4Val[72][12], $L_9 : [-3^1, -2^5, -1^3, 0^1, 1^3, 2^3, 3^1, 4^1]$,
  $L_{10} : [-3^2, -2^3, -1^4, 0^1, 1^2, 2^5, 4^1]$, $L_{11} : [-3^2, -2^5, 0^1, 1^6, 2^3, 4^1]$,
  $L_{12} : [-3^2, -2^4, -1^2, 0^1, 1^4, 2^4, 4^1]$, $L_{13} : [-3^1, -2^5, -1^3, 0^1, 1^3, 2^3, 3^1, 4^1]$,
  $L_{14} : [-3^2, -2^4, -1^1, 0^3, 1^4, 2^2, 3^1, 4^1]$, $L_{15} : [-3^3, -2^9, -1^5, 0^3, 1^5, 2^7, 3^3, 4^1]$,
  $L_{16} : [-3^4, 2^9, -1^2, 0^3, 1^8, 2^7, 3^2, 4^1]$, $L_{17} : [-3^2, -2^{11}, -1^4, 0^3, 1^6, 2^5, 3^4, 4^1]$, and
  $L_{18} : [-3^3, -2^9, -1^5, 0^3, 1^5, 2^7, 3^3, 4^1]$.

- From G_{16}, $L_{19} : [-3^4, -2^7, -1^6, 0^1, 1^{10}, 2^3, 3^4, 4^1]$, $L_{20} : [-3^4, -2^9, -1^2, 0^1, 1^{14}, 2^1, 3^4, 4^1]$,
  $L_{21} : [-3^4, -2^5, -1^{10}, 0^1, 1^6, 2^5, 3^4, 4^1]$, $L_{22} : [-3^4, -2^6, -1^8, 0^1, 1^8, 2^4, 3^4, 4^1]$,
  $L_{23} : [-3^4, -2^8, -1^4, 0^1, 1^{12}, 2^2, 3^4, 4^1]$, $L_{24} : [-3^4, -2^6, -1^8, 0^1, 1^8, 2^4, 3^4, 4^1]$,
  $L_{25} : [-3^4, -2^8, -1^4, 0^1, 1^{12}, 2^2, 3^4, 4^1]$, $L_{26} : [-3^3, -2^7, -1^9, 0^1, 1^7, 2^3, 3^5, 4^1]$,
  $L_{27} : [-3^3, -2^8, -1^7, 0^1, 1^9, 2^2, 3^5, 4^1]$, $L_{28} : [-3^4, -2^7, -1^6, 0^1, 1^{10}, 2^3, 3^4, 4^1]$, and
  $L_{29} : [-3^4, -2^6, -1^8, 0^1, 1^8, 2^4, 3^4, 4^1]$.

- From AT4Val[90][1], $L_{30} : [-3^4, -2^{10}, -1^9, 1^{10}, 2^6, 3^5, 4^1]$.

- From AT4Val[90][8], $L_{31} : [-3^5, -2^6, -1^{14}, 1^5, 2^{10}, 3^4, 4^1]$.

- From $G_{17} \cong$ AT4Val[120][4], $L_{32} : [-3^3, -2^7, -1^1, 0^9, 1^1, 2^5, 3^3, 4^1]$,
  $L_{33} : [-3^3, -2^7, -1^1, 0^9, 1^1, 2^5, 3^3, 4^1]$, $L_{34} : [-3^4, -2^5, -1^2, 0^9, 2^7, 3^2, 4^1]$,
  $L_{35} : [-3^7, -2^{13}, -1^3, 0^{15}, 1^1, 2^{15}, 3^5, 4^1]$.

- From AT4Val[180][12], $L_{36} : [-3^4, -2^8, -1^{12}, 0^2, 1^6, 2^6, 3^6, 4^1]$,
  $L_{37} : [-3^9, -2^{17}, -1^{19}, 0^5, 1^{15}, 2^{11}, 3^{13}, 4^1]$, $L_{38} : [-3^{11}, -2^{13}, -1^{21}, 0^5, 1^{13}, 2^{15}, 3^{11}, 4^1]$, and
  $L_{39} : [-3^{12}, -2^{15}, -1^{14}, 0^5, 1^{20}, 2^{13}, 3^{10}, 4^1]$.

- From AT4Val[210][10], $L_{40} : [-3^{16}, -2^9, -1^{29}, 1^{20}, 2^{19}, 3^{11}, 4^1]$.

- From AT4Val[273][4], $L_{41} : [-3^1, -2^4, -1^6, 0^4, 1^1, 3^4, 4^1]$.

- From AT4Val[546][48], $L_{42} : [-3^2, -2^3, -1^5, 0^4, 1^2, 2^1, 3^3, 4^1]$,
  $L_{43} : [-3^3, -2^2, -1^4, 0^4, 1^3, 2^2, 3^2, 4^1]$, $L_{44} : [-3^3, -2^2, -1^4, 0^4, 1^3, 2^2, 3^2, 4^1]$.

We do not list graphs with at most 12 vertices since all non-bipartite QIGs on 12 or fewer vertices are known [11, 176]. Of the 44 graphs given above, only L_1, L_3 and L_4 previously appear in the literature about integral graphs. The remaining 41 non-bipartite QIGs are new.

4.5 Remarks

There are precisely 32 connected 4-regular integral Cayley graphs up to isomorphism. Table 4.3 lists the 17 graphs of the 32 which are bipartite and Table 4.7 gives the details of the 15 non-bipartite graphs.

There are exactly 27 quartic integral graphs that are arc-transitive. We found that 16 of the 27 graphs are bipartite; these appear in Table 4.3 and Table 4.5. We found that 16 of the 27 graphs are Cayley graphs; these appear in Table 4.3 and Table 4.7.

There are integral Cayley bipartite graphs that can be decomposed into $H \times K_2$ where H is Cayley and arc-transitive, Cayley but not arc-transitive, or arc-transitive but not Cayley. The graph G_{10} is our only example of this last possibility; refer to Table 4.6.

The new 4-regular integral graphs that we found that are co-spectral to other graphs are as follows: G_{13} co-spectral to $I_{40,1}$ and $I_{40,2}$, G_{15} to $I_{72,1}$, H_9 to $I_{20,8}$, H_{12} to $I_{36,4}$, and F_3 to M_6. We also mention the co-spectral graphs among the known integral graphs: G_5 is co-spectral to $I_{16,2}$ and another graph appearing in [176], G_6 to $I_{18,2}$ and $I_{18,3}$, G_7 to $I_{24,1}$, F_1 to $I_{60,2}$, H_5 to $I_{12,6}$, and J_3 to $I_{30,5}$.

We find that some integral Cayley graphs are co-spectral to integral non-Cayley graphs and that some integral arc-transitive graphs are co-spectral to integral graphs that are not arc-transitive. For example, the arc-transitive Cayley graph H_5 has a co-spectral mate $I_{12,6}$, that is neither arc-transitive nor Cayley.

As can also be seen in Table 4.3, there are isomorphic integral graphs that are non-equivalent Cayley graphs $\mathrm{Cay}(\Gamma, S)$ and $\mathrm{Cay}(\Gamma^*, S^*)$ in the sense of Definition 17. This can occur for $\Gamma \neq \Gamma^*$ as well as $\Gamma = \Gamma^*$ with $S \neq S^*$. Consider G_{12}, which has 12 non-equivalent Cayley Graphs on 6 different groups. For $\Gamma = S_3 \times S_3$, there are 4 non-equivalent Cayley graphs with connection sets occurring for each of the three possible numbers of involutions. There is only one Cayley graph of order 40 up to equivalence. For all other orders the bipartite integral Cayley graphs are not unique up to equivalence. In the non-bipartite case; $H_1, H_4, H_5, H_9, H_{12}$, and H_{15} are all unique up to equivalence.

There are non-isomorphic integral Cayley graphs with the same number of vertices. As can be seen in Table 4.3 for the bipartite case, there are two graphs on 12 vertices, three graphs on 24 vertices, and two graphs on 72 vertices up to isomorphism. For all other orders there is at most one graph up to isomorphism. There are many more examples in the non-bipartite case (refer to Table 4.7).

There exist non-isomorphic integral Cayley graphs for the same group Γ. Consider G_i for $i = 7, 8, 9$ in Table 4.3. The following 6 groups are examples of this: $\mathbb{Z}_2 \times A_4$, $\mathbb{Z}_3 \times D_8$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times S_3$, S_4, $(\mathbb{Z}_6 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$, and $\mathbb{Z}_4 \times S_3$.

We began with the 828 possible spectra from [176], and then narrowed our focus to a set Ξ of 59 candidates for vertex transitive graphs; refer to Table 4.1. Of these, we found 22 which are realised by Cayley graphs or arc-transitive graphs. In Section 4.4.7, by taking quotients, we found an additional 6 bipartite integral graphs that are neither arc-transitive nor Cayley, but realize a possible spectrum.

In Section 4.4.6, we prove that G_{11} is the unique QIG on 32 vertices that is vertex-transitive.

Overall, we found 9 bipartite quartic integral graphs (namely, G_{16}, G_{17}, F_4, F_5, M_4, M_5, M_7, M_8, M_9) that realize spectra not previously known to be achieved. It is open whether the remaining possible spectra are realized by any 4-regular bipartite integral graphs.

Acknowledgements

Chapter 5

Counting Subgraphs of Strongly Regular Graphs

5.1   Counting subgraphs in terms of smaller subgraphs

In this section we develop our method for counting subgraphs of a strongly regular graph in terms of the parameters of the graph and the counts of some subgraphs with fewer vertices.

Consider a strongly regular graph $G = \mathrm{SR}(n, r, e, f)$. Let H denote a connected subgraph of G. Our method enables us to derive expressions for the counts of connected subgraphs H of G that have minimum degree one or two. We will give our results for counting these two categories of subgraphs separately.

For $S \subset V(H)$, let $H \backslash S$ denote the graph induced by $V(H) \backslash S$. For ease of notation, we denote an edge between vertex x and vertex y by xy. Given a graph H and an edge xy, let $H + xy$ denote the graph H with vertex set $V(H)$ and edge set $E(H) \cup \{xy\}$. In particular, $H + xy = H$ if xy is already an edge of H. For any graph H, let $[H, v_1, v_2, \ldots, v_i]$ for $i = 0, 1, \ldots, 3$, denote the number of subgraphs in G with distinguished vertices $v_1, v_2, \ldots, v_i$ of H. Thus, when $i = 0$ we have $[H]$ simply denoting the number of subgraphs isomorphic to H in G. In all other cases, we have

$$[H, v_1, v_2, \ldots, v_i] = [H] |\operatorname{Aut}(H)v_1| \cdots |\operatorname{Aut}(H)_{\{v_1, \ldots, v_{i-1}\}} v_i|.$$

If v is a *cut-vertex* of the graph H then $H \backslash \{v\}$ is a disconnected graph.

Theorem 26. *Consider $G = \mathrm{SR}(n, r, e, f)$ and a graph H containing a vertex s of degree one. Let u denote the neighbour of s in H and let N denote the neighbours of u in $H \backslash \{s\}$. Let X denote a set of vertices of $H \backslash (\{u, s\} \cup N)$ that are orbit representatives under the action of $\operatorname{Aut}(H \backslash \{s\})_{\{u\}}$. Then,*

$$[H] = \frac{(r - |N|)[H \backslash \{s\}, u] - \sum_{x \in X} [H \backslash \{s\} + ux, u, x]}{|\operatorname{Aut}(H)u| |\operatorname{Aut}(H)_{\{u\}} s|}.$$

Proof. We count $[H, u, s]$, the number of graphs H with distinguished vertices u and s, in two ways. By definition,

$$[H, u, s] = [H] |\operatorname{Aut}(H)u| |\operatorname{Aut}(H)_{\{u\}} s|.$$

The second way is as follows. We choose a copy of $H\backslash\{s\}$ with distinguished vertex u. Then, we choose a neighbour s of u. There are $[H\backslash\{s\}, u]$ such graphs in G. Since G is regular, there are $r - |N|$ choices for s in G. However, these choices for s include the undesired possibility that a vertex of H is chosen. We eliminate the counts of graphs for these choices by subtracting $[H\backslash\{s\} + ux, u, x]$ for each vertex x in the set $V(H)$ that is not u, any of the neighbours of u, nor a second vertex in the same orbit under the action of $\mathrm{Aut}(H\backslash\{s\})_{\{u\}}$ (since this would lead to the subtraction of the counts of a graph multiple times). These vertices x are precisely the vertices of X.

In solving for $[H]$, we arrive at the desired expression. □

Theorem 27. *Consider $G = \mathrm{SR}(n, r, e, f)$ and a graph H containing a vertex s of degree two, such that s is not a cut-vertex of H. Let u and v denote the two neighbours of s in H and let C denote the common neighbours of u and v in $H\backslash\{s\}$. Let X denote a set of vertices of $H\backslash(\{u, v, s\} \cup C)$ that are orbit representatives under the action of $\mathrm{Aut}(H\backslash\{s\})_{\{u,v\}}$. If u is adjacent to v in H then,*

$$[H] = \frac{(e - |C|)[H\backslash\{s\}, u, v] - \sum_{x \in X}[H\backslash\{s\} + ux + xv, u, v, x]}{|\mathrm{Aut}(H)u||\mathrm{Aut}(H)_{\{u\}}v||\mathrm{Aut}(H)_{\{u,v\}}s|}.$$

Otherwise,

$$[H] = \frac{(f - |C|)[H\backslash\{s\}, u, v] + (e - f)[H\backslash\{s\} + uv, u, v] - \sum_{x \in X}[H\backslash\{s\} + ux + xv, u, v, x]}{|\mathrm{Aut}(H)u||\mathrm{Aut}(H)_{\{u\}}v||\mathrm{Aut}(H)_{\{u,v\}}s|}.$$

Proof. For the time being, suppose that $f \geqslant |C|$ and $e \geqslant |C|$. We count $[H, u, v, s]$; the number of graphs H with distinguished vertices u, v, and s; in two ways. By definition,

$$[H, u, v, s] = [H]|\mathrm{Aut}(H)u||\mathrm{Aut}(H)_{\{u\}}v||\mathrm{Aut}(H)_{\{u,v\}}s|.$$

The second way has two cases.

- **Case 1 - vertex u is adjacent to vertex v in H:** We choose a copy of $H\backslash\{s\}$ with distinguished vertices u and v. Then, we choose a neighbour s of u. There are $[H\backslash\{s\}, u, v]$ such graphs in G. Since G is strongly regular, there are $e - |C|$ choices for s in G. However, these choices for s include the undesired possibility that a vertex of H is chosen. We eliminate the counts of graphs for these choices by subtracting $[H\backslash\{s\} + ux + xv, u, v, x]$ for each vertex x in the set $V(H)$ that is not u, v, any of the common neighbours of u and v, nor a second vertex in the same orbit under the action of $\mathrm{Aut}(H\backslash\{s\})_{\{u,v\}}$. These vertices x are precisely the vertices of X.

- **Case 2 - vertex u is not adjacent to vertex v in H:**

  1. Either, we choose a copy of $H\backslash\{s\}$ with distinguished vertices u and v and strictly no edge uv. Then, we choose a neighbour s of u.

     - There are $[H\backslash\{s\}, u, v] - [H\backslash\{s\} + uv, u, v]$ such graphs in G. Since G is strongly regular, there are $f - |C|$ choices for s in G. Similar to the last case, we eliminate the counts of graphs for choices for s where $s \in V(H)$ by subtracting $([H\backslash\{s\} + ux + xv, u, v, x] - [H\backslash\{s\} + uv + ux + xv, u, v, x])$ for each vertex $x \in X$.

2. Or we choose a copy of $H\backslash\{s\}$ with distinguished vertices u and v and the edge uv added. Then, we choose a neighbour s of u.

    – There are $[H\backslash\{s\} + uv, u, v]$ such graphs in G. Since G is strongly regular, there are $e - |C|$ choices for s in G. Once again, we eliminate the counts of graphs for choices for s where $s \in V(H)$ by subtracting $[H\backslash\{s\} + uv + ux + xv, u, v, x]$ for each vertex $x \in X$.

By collecting terms and then solving for $[H]$, we arrive at the desired expression in each case.

If $f < |C|$, then no graphs H with strictly no edge uv occur in G and all the counts of graphs in Case 2.1 are zero. Thus, $[H\backslash\{s\}, u, v] = [H\backslash\{s\} + uv, u, v]$ and the expression for Case 2 simplifies to that of Case 1.

If $e < |C|$, no graphs H with the edge uv occur in G and thus the terms containing counts of graphs with the added edge uv are zero. The expression for Case 2 simplifies to $(f - |C|)[H\backslash\{s\}, u, v] - \sum_{x \in X}[H\backslash\{s\} + ux + xv, u, v, x]$.

Thus, under these conditions, these equations are still correct. $\qquad\square$

We note that if the addition of an edge results in a subgraph of G that does not satisfy the values e and f then the counts of that subgraph will be zero and thus the term corresponding to that subgraph will contribute zero to the equation.

5.2 Subgraph counts in terms of SR parameters

We give our algorithm which recursively finds an expression for the counts of subgraphs of a strongly regular graph. Where possible, the method FINDEX in Algorithm 3 takes a subgraph H and outputs $[H]$ in terms of the variables n, r, e, and f in the following way:

- If $H = K_1$ then output n since $[H] = n$.

- If H has a vertex of degree one, express $[H]$ using Theorem 26. Call FINDEX on all graphs that appear in the expression.

- If H has a vertex s of degree two that is not a cut-vertex of H, express $[H]$ using Theorem 27. Call FINDEX on all graphs that appear in the expression.

- If no suitable vertex of H is found then output the variable $[H]$.

This algorithm terminates since the graphs at each step of the recursion have one less vertex (and in the worst case, two more edges). We note that there may be many possible choices for a vertex of degree one or two. In our implementation we prefer to first select degree one vertices if they are present in H, but this is not necessary. This decision reduces the number of intermediate graphs possible by never introducing graphs with more than one vertex of degree one (with the exception of the path graphs). For the same reason, we insist that the chosen s of degree 2 has a neighbour of degree at least 3 where possible (this is not possible when H is a cycle).

Algorithm 3

---

for all graphs GG in *File* **do**
    Print [name of GG] $=$
    FINDEX(GG)
end for

description Returns a list of vertices *oList* where *oList*[v] contains the canonical representative of the orbit of v under the action of the point-wise stabilizer of *set* in Γ.
function FINDORB(Γ,*set*)
    oList := []
    for each orbit p of the vertices in the stabilizer of *set* in Γ **do**
        for each vertex v in p **do**
            oList[v] := the first vertex occurring in p
        end for
    end for
    Return *oList*
end function

description Returns the value of $|\Gamma v_1| \cdots |\Gamma_{\{v_1,\ldots,v_{i-1}\}} v_i|$ where $set = \{v_1, \ldots, v_{i-1}, v_i\}$.
function ORBITSZ(Γ,*set*)
    if v is the only vertex in *set* **then**
        Find the orbit *orb* of vertex v in Γ
        Return (size of *orb*)
    else
        Remove the first vertex v from *set*
        Find the orbit *orb* of vertex v in Γ
        Γ := the stabilizer of v in Γ
        Return (size of *orb*)∗ORBITSZ(Γ, *set*)
    end if
end function

function FINDEX(G)
    if G is K_1 **then**
        Print n
    else
        if $\exists$ a vertex s of G with degree 1 **then**
            u := neighbour of s in G
            Print 1/ORBITSZ(Aut(G), $\{u, s\}$) $\ast$ ($r -$ (degree of u) $- 1)\ast$
            Remove edge us from G
            Remove vertex s from G
            orb := FINDORB(Aut(G), $\{u\}$)
            Print ORBITSZ(Aut(G), $\{u\}$)
            FINDEX(G)
            added := {}
            for each vertex x in G **do**
                if $x \neq u$ and ux is not an edge of G **then**
                    if *orb*[x] not in *added* **then**

Add $orb[x]$ to $added$
$G' := G + ux$
Print $-\textsc{orbitSz}(\text{Aut}(G'), \{u, x\})$
$\textsc{findEx}(G')$
    end if
    end if
  end for
else
  if $\exists$ a vertex s of G with degree 2 that is not a cut vertex **then**
    Let u and v be the neighbours of s in G
    Print $1/\textsc{orbitSz}(\text{Aut}(G), \{u, v, s\})$
    Remove edges us and sv from G
    Remove vertex s from G
    $CNs :=$ number of common neighbours of u and v in G
    $orb := \textsc{findOrb}(\text{Aut}(G), \{u, v\})$
    if uv is an edge of G **then**
      Print $(e - CNs) * \textsc{orbitSz}(\text{Aut}(G), \{u, v\})$
    else
      Print $(f - CNs) * \textsc{orbitSz}(\text{Aut}(G), \{u, v\})$
    end if
    $\textsc{findEx}(G)$
    if uv is not an edge of G **then**
      $G' := G + uv$
      Print $+(e - f) * \textsc{orbitSz}(\text{Aut}(G'), \{u, v\})$
      $\textsc{findEx}(G')$
    end if
    $added := \{\}$
    for each vertex x in $V(G)\backslash\{u, v\}$ **do**
      if ux is not an edge of G or xv is not an edge of G **then**
        if $orb[x]$ not in $added$ **then**
          Add $orb[x]$ to $added$
          $G' := G + ux + xv$
          Print $-\textsc{orbitSz}(\text{Aut}(G'), \{u, v, x\})$
          $\textsc{findEx}(G')$
        end if
      end if
    end for
  else
    Print [name of G]
  end if
end if
end if
end if
end function

As an example of the different possible non-isomorphic choices for a start vertex, we give the equations after just one step of the algorithm for the subgraph $\Theta_{4,3,2}$:

$$[\Theta_{4,3,2}] = e[L(5,2)] + 2(f - e - 1)[\Theta_{3,3,2}] - [\,\blacktriangleleft\!\triangleright\,] - [\Theta_{4,3,1}] - [\,\triangledown\!\!\!\triangleleft\,] - \tfrac{1}{2}[\,\triangle\!\!\triangle\,],$$

$$[\Theta_{4,3,2}] = e[L(6,1)] + 2(f - e)[\Theta_{4,2,2}] - 2[\Theta_{3,3,2}] - [\Theta_{5,2,1}] - \tfrac{1}{2}[\,\hexagon\,] - \tfrac{1}{2}[\,\triangle\!\!\triangle\,] - [\,\triangle\!\!\triangle\,],$$

$$[\Theta_{4,3,2}] = 7e[C_7] + (f - e - 2)[\Theta_{4,3,1}] - 2[\Theta_{5,2,1}] - [\,\triangle\!\!\triangle\,].$$

The final algorithm result is the same equation for all three choices of start vertex:

$$[\Theta_{4,3,2}] = \tfrac{1}{2}r(-148f + 216f^2 + 103ef - 8r^2e - 49f^2r - 10fr^2 + 27f^3r + 3f^2r^2 - 4f^4r + 2er^3$$
$$+ 14r^2e^2 - 2re^2 - 7e^2r^3 + 9r^2e^3 - 17re^3 - 2e^3r^3 + e^2r^4 + 2r^2e^4 - 3re^4 - re^5 + 6er + 67rf$$
$$+ e^4 - 6e^2 - 165f^3 + 10e^3 + 53f^4 - 114fre + 33fr^2e + 32f^2re - 9f^2r^2e - 13f^3re$$
$$- 9fr^2e^2 + 22fre^2 + 20f^2re^2 + 2f^4re - 5f^3re^2 + 2fe^2r^3 + fre^4 + f^3r^2e$$
$$- 3fer^3 - 11f^5 + f^6 + e^5 + 10f^2e - 17f^3e - 19fe^2 + 2f^4e^2 + 3rf^2e^3 + fe^5$$
$$- 34f^2e^2 + 21f^4e - 9f^3e^2 + 8fe^3 - 2fe^4 - 3r^2fe^3 - 3f^5e + 2f^3e^3 - 3f^2e^4)n$$
$$+ 12(28 + 5e^2 - 26f + 15e + 6f^2 - r - 11ef)[K_4] - 6[\text{⬦}].$$

5.3 Equations for subgraphs of a strongly regular graph

We give the derived equations for the counts of subgraphs H of a graph $\mathrm{SR}(n, r, e, f)$ for which H has at most 8 edges and all vertices of degree at least two.

$$[C_3] = \tfrac{1}{6}ern,$$
$$[C_4] = \tfrac{1}{8}r(fr - r + e^2 - f + 1 - ef)n,$$
$$[C_5] = \tfrac{1}{10}r(r^2f - fr - 2ef + erf - 3er + 5e + e^3 - 2fe^2 - rf^2 + f^2 + ef^2)n,$$
$$[\Theta_{2,2,1}] = \tfrac{1}{4}(e - 1)ern,$$
$$[C_6] = \tfrac{1}{12}r(-r^2f^2 + 4 - 6e^2r + 10fr - 5r^2f - 6ef - 6r - 4e - 6f - 2erf^2 + 4ef^2 + 2erf + 15e^2$$
$$+ 2r^2 + f^2 + er^2f + fe^2r - 3fe^2 + fr^3 - ef^3 + rf^3 + e^4 - f^3 - 3fe^3 + 3f^2e^2)n,$$
$$[C_{3\cdot3}] = \tfrac{1}{8}er(er - 4e + 2)n,$$
$$[\Theta_{3,2,1}] = \tfrac{1}{2}er(fr - r + e^2 - f - ef + 3 - 2e)n,$$
$$[\Theta_{2,2,2}] = \tfrac{1}{12}r(rf^2 - f^2 - 3fr + 3f - ef^2 + 2r - 2 - 3e^2 + 3ef + e^3)n,$$
$$[C_7] = \tfrac{1}{14}r(4r^2f^2 - 10rf^2 - 10e^3r - 42er - 7fr + 12r^2f - 35ef + 56e - 3erf^2 + 10ef^2 + 43erf$$
$$- 28e^2 + 35e^3 + 7f^2 - 12er^2f + 12fe^2r - 35fe^2 + 10er^2 - 6fr^3 + 3erf^3 - 6ef^3 + fer^3$$
$$+ fe^2r^2 + r^2f^3 + rf^3 - 2f^3 - 2er^2f^2 - 3f^2e^2r + fe^3r - 4fe^3 + 9f^2e^2 - f^2r^3 + fr^4 + ef^4$$
$$+ e^5 + f^4 - rf^4 + 6f^2e^3 - 4f^3e^2 - 4fe^4)n,$$
$$[\Theta_{2,2,2,1}] = \tfrac{1}{12}(e - 2)(e - 1)ern,$$
$$[C_{4\cdot3}] = \tfrac{1}{4}er(r^2f - r^2 - 5fr - erf + 5r + e^2r + 4f + 4ef - 10 + 8e - 6e^2)n,$$
$$[\text{⬦}] = \tfrac{1}{2}(e - 1)^2ern - 12[K_4],$$
$$[\Theta_{3,3,1}] = \tfrac{1}{4}r(2fe^2r + 4 - 5r - 2fe^3 - 2fe^2 + 4ef^2 + 2erf - 7ef - 2e - 4rf^2 + 3f^2 - 4e^3 + 11e^2$$
$$+ 9fr - 7f - 2e^2r - 2r^2f + r^2 + e^4 - 2erf^2 + f^2e^2 + r^2f^2)n + 6[K_4],$$

$$[C_{3-3}] = \tfrac{1}{8}er(er^2 - 4er - 4e^2 - 4 + 12e)n + 6[K_4],$$

$$[\Theta_{4,2,1}] = \tfrac{1}{2}er(-2fe^2 + r^2f + 2f + f^2 - 3fr + erf - 7 - 3e^2 + 11e - 3er + 2r + e^3 - rf^2 + ef^2)n$$
$$+ 12[K_4],$$

$$[\boxtimes] = \tfrac{1}{4}(f - 2)(e - 1)ern + 6(e - f)[K_4],$$

$$[\Theta_{3,2,2}] = \tfrac{1}{4}r(ef^3 - fe^3 + fe^2r + 6ef + 5fe^2 - 2ef^2 - 5erf - 11e - f^2 - 5e^3 - rf^3 + 7e^2 + fr - e^2r$$
$$+ 5er - r^2f + e^4 + f^3 - f^2e^2 + r^2f^2)n,$$

$$[\Theta_{3,3,2}] = \tfrac{1}{4}r(-3r^2f^2 + rf^2 - 2e^3r - 13er + 12e^2r - fr + r^2f - 17ef + 32e + 21erf - 46e^2 + 23e^3$$
$$+ f^2 - 6er^2f - 6fe^2r - 3fe^2 + er^2 - 2r^2f^3 + 4rf^3 - 8e^4 - 2f^3 + 2er^2f^2 - 3f^2e^2r + 2fe^3r$$
$$+ 10fe^3 - f^2e^2 + f^2r^3 - ef^4 + e^5 - f^4 + rf^4 + 2f^3e^2 - 2fe^4)n + 6(e - f - 4)[K_4],$$

$$[C_8] = \tfrac{1}{16}r(13r^2f^2 - 49rf^2 + 36 + 32er - 168e^2r + 129fr - 79r^2f - 41ef - 59r - 112e - 73f$$
$$- 94erf^2 + 96ef^2 - 16erf + 345e^2 + 28r^2 - 5r^3 - 112e^3 + 36f^2 - 2f^2er^3 + 35er^2f$$
$$+ 116fe^2r - 88fe^2 + 30e^2r^2 + 29fr^3 + 6erf^3 - 19ef^3 - 14fer^3 - 21fe^2r^2 - 3r^2f^3$$
$$+ 13rf^3 - 15e^4r + 70e^4 - 11f^3 + 24er^2f^2 - 21f^2e^2r + 32fe^3r - 120fe^3 + 66f^2e^2 + f^2r^3$$
$$- 7fr^4 - 4erf^4 + 8ef^4 - 3f^2e^2r^2 + 3f^4 - r^2f^4 - 2rf^4 + fe^4r + fe^3r^2 - 4f^2e^3r + 3er^2f^3$$
$$+ 6f^3e^2r + fe^2r^3 - f^2r^4 + 16f^2e^3 + f^3r^3 - 18f^3e^2 - 5fe^4 + efr^4 + e^6 + 10f^2e^4 - 5fe^5$$
$$+ 5f^4e^2 - 10f^3e^3 - f^5 + f^5r - f^5e + fr^5)n + 33[K_4],$$

$$[\Theta_{2,2,2,2}] = \tfrac{1}{48}r(rf^3 - f^3 - 6rf^2 + 6f^2 - ef^3 + 11fr - 11f + 6ef^2 - 6r + 6 + 11e^2 - 11ef - 6e^3 + e^4)n,$$

$$[\Theta_{3,2,2,1}] = \tfrac{1}{4}e(e - 1)r(e^2 - ef - 4e + fr - r + 5 - f)n + 12[K_4],$$

$$[C_{4\cdot4}] = \tfrac{1}{8}r(-6r^2f^2 + 17rf^2 - 12 - 4er + 14e^2r - 33fr + 12r^2f + 19ef + 17r + 16e + 23f$$
$$+ 10erf^2 - 16ef^2 - 6erf - 51e^2 - 6r^2 + r^3 + 24e^3 - 12f^2 + 2er^2f - 14fe^2r + 8fe^2$$
$$- 2e^2r^2 - 2fr^3 + ef^3 + 2fe^2r^2 - rf^3 + e^4r - 9e^4 + f^3 - 2er^2f^2 + f^2e^2r - 2fe^3r$$
$$+ 12fe^3 - 4f^2e^2 + f^2r^3)n - 12[K_4],$$

$$[C_{5\cdot3}] = \tfrac{1}{4}er(12fe^2 - 5r^2f - 10erf + 6ef - 4f^2 + 10fr - 6f + 24 + 20e^2 - 56e + 21er - 6r - 8e^3$$
$$+ 5rf^2 - 4ef^2 - 3er^2 - r^2f^2 - 2fe^2r + erf^2 + e^3r + er^2f + fr^3)n - 36[K_4],$$

$$[\Theta_{4,3,1}] = \tfrac{1}{2}r(-3r^2f^2 - rf^2 - 4e^3r - 21er + 4e^2r - 3fr + 4r^2f - 28ef + 40e - 5erf^2 + 9ef^2$$
$$+ 26erf - 32e^2 + 27e^3 + 3f^2 - 4er^2f - 16fe^2 + 3er^2 - fr^3 + 2erf^3 - 4ef^3 + fe^2r^2$$
$$- r^2f^3 + 4rf^3 - 6e^4 - 3f^3 - 4f^2e^2r + 2fe^3r + fe^3 + 9f^2e^2 + f^2r^3 + e^5 + 3f^2e^3 - f^3e^2$$
$$- 3fe^4)n + 12(5e - 4f - 2)[K_4],$$

$$[\Theta_{4,2,2}] = \tfrac{1}{4}r(-6r^2f^2 + 18rf^2 - 3e^3r - 12 + 13e^2r - 34fr + 11r^2f + 22ef + 16r + 12e + 24f$$
$$+ 11erf^2 - 20ef^2 - 8erf - 53e^2 - 4r^2 + 23e^3 - 13f^2 - er^2f - 12fe^2r + 15fe^2 - fr^3$$
$$- erf^3 + 4ef^3 + fe^2r^2 - r^2f^3 - rf^3 - 7e^4 + 2f^3 - f^2e^2r + fe^3r + 11fe^3 - 7f^2e^2$$
$$+ f^2r^3 - ef^4 + e^5 - f^4 + rf^4 + 2f^3e^2 - 2fe^4)n + 12(e - f - 2)[K_4],$$

$$[\Theta_{5,2,1}] = \tfrac{1}{2}er(-r^2f^2 + 2rf^2 + 28 + 8er - 6e^2r + 18fr - 7r^2f - 4ef - 12r - 40e - 14f$$

$$-2erf^2 + 2ef^2 - 2erf + 27e^2 + 2r^2 - 4e^3 - f^2 + er^2f + fe^2r + 3fe^2 + fr^3 - ef^3$$
$$+ rf^3 + e^4 - f^3 - 3fe^3 + 3f^2e^2)n + 24(2e - 3 - f)[K_4],$$

$[\triangle\!\!\triangle] = \frac{1}{2}er(-er + erf + 9e + e^3 - 2ef - fe^2 + f - 4 - 4e^2)n + 12(2 - e + f)[K_4],$

$[\boxslash] = e(e-1)r(e^2 - 2e - ef + fr - r - 2f + 5)n + 24(2 - 2e + f)[K_4],$

$[\triangle\!\!] = \frac{1}{2}er(-ef - 2fr - 4e^2 + 8e - 6 + rf^2 - 2ef^2 + r + 2fe^2 + 3f)n$
$$+ 12(1 - e + f^2 + e^2 + f - 2ef)[K_4],$$

$[\diamond] = \frac{1}{4}er(e-1)(ef - 4e + fr + 2 - f^2)n + 6(5 - r - f - 2ef + e^2 + f^2)[K_4],$

$[\diamond\!\!\cdot] = \frac{1}{4}er(e-1)(er - 6e + 4)n + 12[K_4],$

$[\bowtie] = \frac{1}{4}er(e-1)(er - 4e + 2)n + 12(2 - e)[K_4],$

$[\diamond] = \frac{1}{2}er(-f^2 + 5f + rf^2 + 2r - 3fr - 4e^2 - ef^2 + e^3 - 8 + ef + 7e)n + 24(f - e)[K_4],$

$[C_{4-3}] = \frac{1}{4}er(4fe^2 - 6ef + 2f^2 + 18fr - 18f + 32 + 26e^2 - 36e - r^3 + 5er - 16r - 5e^2r$
$$- 5r^2f - 6e^3 + 5r^2 - 2rf^2 + 2ef^2 - er^2f + e^2r^2 + fr^3)n + 12(3e - 2f - 2)[K_4],$$

$[\cdot\!\!\succ\!\!\prec\!\!\cdot] = \frac{1}{8}er(er^3 - 5er^2 + 16er - 4e^2r - 4erf - 2f + 6ef + 4fe^2 + 16 + 28e^2 - 52e - 4e^3)n$
$$+ 6(e - f - 4)[K_4],$$

$[\langle\!\!\!\;\rangle] = \frac{1}{4}er(-er + erf - fe^2 + 7e + e^3 - ef - 2 - 4e^2)n + 12[K_4] - 2[\diamond],$

$[\diamond\!\!\!\triangleleft] = 6(e - 2)[K_4],$

$[\cdot\!\!\diamond\!\!\cdot] = \frac{1}{8}r(f-2)(rf^2 - f^2 - ef^2 + 4ef - 3fr + 3f - fe^2 + 2e^3 + 2r - 2 - 4e^2)n + 3(e - f)^2[K_4]$
$$- [\diamond]. \tag{5.1}$$

Next, we give the derived equations for subgraphs H of a graph $\mathrm{SR}(n, r, 0, f)$ for which H has at most 9 edges and all vertices of degree at least two. The variables $[K_4]$, $[\diamond]$, and $[\diamond\!\!\triangleleft]$ are given the value of zero since their corresponding graphs do not appear in a graph with $e = 0$.

$[C_4] = \frac{1}{8}(f-1)(r-1)rn,$

$[C_5] = \frac{1}{10}fr(r-1)(r-f)n,$

$[C_6] = \frac{1}{12}r(r-1)(r^2f + 2r - 4fr - f^2r - 4 + 6f + f^3 - f^2)n,$

$[\Theta_{2,2,2}] = \frac{1}{12}(f-2)(f-1)(r-1)rn,$

$[C_7] = \frac{1}{14}fr(r-1)(r-f)(f^2 - 2f + r^2 - 5r + 7)n,$

$[\Theta_{3,3,1}] = \frac{1}{4}r(r-1)(f-1)(fr - 3f - r + 4)n,$

$[\Theta_{3,2,2}] = \frac{1}{4}fr(r-1)(f-1)(r-f)n,$

$[\Theta_{3,3,2}] = \frac{1}{4}fr(r-1)(f-r)(-fr + f^2 + 2f - 1)n,$

$[C_8] = \frac{1}{16}r(r-1)(fr^4 - 6r^3f - r^3f^2 + 23r^2f + r^2f^3 - 5r^2 - 56fr + 23r - f^4r + 13f^2r - 2f^3r$
$$+ 73f - 36f^2 - 3f^4 - 36 + 11f^3 + f^5)n,$$

$[\Theta_{2,2,2,2}] = \frac{1}{48}(f-3)(f-2)(f-1)(r-1)rn,$

$$[C_{4\cdot4}] = \tfrac{1}{8}r(r-1)(f-1)(r^2f - f^2 - 5fr + 11f - r^2 + 5r - 12)n,$$

$$[\Theta_{4,3,1}] = \tfrac{1}{2}fr(r-1)(r-3)(f-1)(r-f)n,$$

$$[\Theta_{4,2,2}] = \tfrac{1}{4}r(r-1)(f-1)(f^3 - f^2r - f^2 + r^2f - 6fr + 12f + 4r - 12)n,$$

$$[\text{}] = \tfrac{1}{8}(f-2)^2(f-1)(r-1)rn,$$

$$[\Theta_{4,4,1}] = \tfrac{1}{4}fr(r-1)(f-r)(f^2r - 2f^2 + 3fr - 3f - r^2f + 1)n,$$

$$[\Theta_{4,3,2}] = \tfrac{1}{2}fr(r-1)(r-f)(f^3 - f^2r + f^2 + r^2f - 6fr + 10f + 2r - 6)n,$$

$$[\Theta_{3,3,3}] = \tfrac{1}{12}r(r-1)(r^3f^2 - r^2 + 6r^2f - 9r^2f^2 - 2r^2f^3 - 42fr + 31f^2r + 3f^4r + 4f^3r + 15r$$
$$- 6f^4 - 36 - 54f^2 - f^5 + 81f + 10f^3)n,$$

$$[C_9] = \tfrac{1}{18}fr(r-1)(r-f)(f^4 - 4f^3 + r^2f^2 - 5f^2r + 21f^2 - 11r^2f + 48fr - 81f + r^4 - 7r^3$$
$$+ 30r^2 - 72r + 81)n,$$

$$[\text{}] = \tfrac{1}{2}fr(r-1)(f-1)^2(r-f)n,$$

$$[C_{5\cdot4}] = \tfrac{1}{4}fr(r-1)(f-1)(f-r)(2f - r^2 + 5r - 10)n,$$

$$[\Theta_{5,3,1}] = \tfrac{1}{2}r(r-1)(f-1)(f^3r - 3f^3 - r^2f^2 + 10f^2 + r^3f - 7r^2f + 25fr - 45f + 2r^2 - 15r + 36)n,$$

$$[\Theta_{5,2,2}] = \tfrac{1}{4}fr(r-1)(f-1)(r-f)(f^2 - 2f + r^2 - 7r + 13)n. \tag{5.2}$$

Note that if a graph H has a vertex of degree 4, the corresponding equation may not contain the factor $(r-3)$. However, if such a subgraph H in a graph G with $r = 3$ is considered then another parameter of G, e or f, will always give a zero factor in the equation for H. For example, consider the equation for the graph $\Theta_{2,2,2,2}$ in the equations for a $\mathrm{SR}(n, r, 0, f)$:

$$[\Theta_{2,2,2,2}] = \tfrac{1}{48}(f-3)(f-2)(f-1)(r-1)rn.$$

The factor $(r-3)$ does not occur but $(f-3)$, $(f-2)$, and $(f-1)$ do occur. Such a value for f is only possible if r is at least 4.

We further simplify and give the derived equations for subgraphs H of a Moore graph $\mathrm{SR}(n, r, 0, 1)$, for which H has at most 11 edges and all vertices of degree at least two. The variables $[K_4]$, [], [], [], $[K_5]$, [], [], [], [], [], [], [], [], and $[K_{3,3}]$ are given the value of zero since their corresponding graphs do not appear in a graph with $e = 0$.

$$[C_5] = \tfrac{1}{10}(r-1)^2rn,$$

$$[C_6] = \tfrac{1}{12}r(r-2)(r-1)^2n,$$

$$[C_7] = \tfrac{1}{14}r(r-2)(r-3)(r-1)^2n,$$

$$[\Theta_{3,3,2}] = \tfrac{1}{4}r(r-2)(r-1)^2n,$$

$$[C_8] = \tfrac{1}{16}r(r-2)(r^2 - 4r + 5)(r-1)^2n,$$

$$[\Theta_{4,4,1}] = \tfrac{1}{4}r(r-1)^2(r-2)^2n,$$

$$[\Theta_{4,3,2}] = \tfrac{1}{2}r(r-2)(r-3)(r-1)^2n,$$

$$[\Theta_{3,3,3}] = \tfrac{1}{12}r(r-2)(r-3)(r-1)^2n,$$

$$[C_9] = \tfrac{1}{18}r(r-2)(r^3 - 5r^2 + 10r - 9)(r-1)^2 n,$$

$$[\text{}] = \tfrac{1}{8}r(r-2)(r-1)^2 n,$$

$$[C_{5\cdot5}] = \tfrac{1}{8}r(r-2)(r-3)(r-1)^3 n,$$

$$[\Theta_{5,4,1}] = \tfrac{1}{2}r(r-1)^2(r-2)^3 n,$$

$$[\Theta_{5,3,2}] = \tfrac{1}{2}r(r-2)(r-1)^2(r-3)^2 n,$$

$$[\Theta_{4,4,2}] = \tfrac{1}{4}r(r-2)(r^2 - 5r + 7)(r-1)^2 n,$$

$$[\Theta_{4,3,3}] = \tfrac{1}{4}r(r-2)(r-3)(r-4)(r-1)^2 n,$$

$$[C_{10}] = \tfrac{1}{20}r(r-2)(r-3)(r^3 - 3r^2 + r - 1)(r-1)^2 n,$$

$$[\Theta_{3,3,3,2}] = \tfrac{1}{12}r(r-2)(r-3)(r-1)^2 n,$$

$$[\text{}] = \tfrac{1}{2}r(r-2)(r-3)(r-1)^2 n,$$

$$[\text{}] = \tfrac{1}{2}r(r-1)^2(r-2)^2 n,$$

$$[\text{}] = \tfrac{1}{2}r(r-2)(r-3)(r-1)^2 n,$$

$$[\text{}] = \tfrac{1}{2}r(r-2)(r-3)(r-1)^2 n,$$

$$[\text{}] = \tfrac{1}{4}r(r-2)(r-3)(r-1)^2 n,$$

$$[C_{6\cdot5}] = \tfrac{1}{4}r^2(r-2)(r-1)^2(r-3)^2 n,$$

$$[\Theta_{6,4,1}] = \tfrac{1}{2}r(r-2)(r-3)^2(r-1)^3 n,$$

$$[\Theta_{5,4,2}] = \tfrac{1}{2}r(r-2)(r^3 - 7r^2 + 18r - 17)(r-1)^2 n,$$

$$[C_{5-5}] = \tfrac{1}{8}r(r-2)(r^3 - 4r^2 + r + 8)(r-1)^2 n,$$

$$[\Theta_{5,5,1}] = \tfrac{1}{4}r(r-2)(r-3)(r^2 - 3r + 1)(r-1)^2 n,$$

$$[\Theta_{5,4,2}] = \tfrac{1}{2}r(r-2)(r-3)(r^2 - 5r + 7)(r-1)^2 n,$$

$$[\Theta_{5,3,3}] = \tfrac{1}{4}r(r-2)(r-3)(r^2 - 5r + 8)(r-1)^2 n,$$

$$[\Theta_{4,4,3}] = \tfrac{1}{4}r(r-2)(r-3)(r^2 - 7r + 13)(r-1)^2 n,$$

$$[C_{11}] = \tfrac{1}{22}r(r-2)(r-3)(r^4 - 4r^3 - 2r^2 + 16r + 2)(r-1)^2 n. \qquad (5.3)$$

5.4   Subgraph counts for some specific graph parameter sets

We give the non-zero counts of subgraphs up to 11 edges and all cycles up to 17 edges, for a $SR(3250, 57, 0, 1)$, the only Moore graph where existence is still in question.

$$[C_5] = 58094400,$$
$$[\text{}] = 862701840000,$$
$$[C_6] = 2662660000,$$
$$[\text{}] = 862701840000,$$
$$[C_7] = 123243120000,$$
$$[\text{}] = 431350920000,$$
$$[\Theta_{3,3,2}] = 7987980000,$$
$$[C_{6\cdot5}] = 1327698131760000,$$

$$[C_8] = 6042906870000,$$
$$[\Theta_{4,4,1}] = 439338900000,$$
$$[\Theta_{4,3,2}] = 862701840000,$$
$$[\Theta_{3,3,3}] = 143783640000,$$
$$[C_9] = 300896555960000,$$
$$[\text{}] = 3993990000,$$
$$[C_{5\cdot5}] = 12077825760000,$$
$$[\Theta_{5,4,1}] = 48327279000000,$$
$$[\Theta_{5,3,2}] = 46585899360000,$$
$$[\Theta_{4,4,2}] = 23732288580000,$$
$$[\Theta_{4,3,3}] = 22861598760000,$$
$$[C_{10}] = 15140589832368000,$$
$$[\Theta_{3,3,3,2}] = 143783640000,$$
$$[\text{}] = 862701840000,$$
$$[\text{}] = 878677800000,$$

$$[\Theta_{6,4,1}] = 2608810364160000,$$
$$[\Theta_{5,4,2}] = 2611414445640000,$$
$$[C_{5-5}] = 688012705380000,$$
$$[\Theta_{5,5,1}] = 1328129482680000,$$
$$[\Theta_{5,4,2}] = 2563087166640000,$$
$$[\Theta_{5,3,3}] = 1281974934240000,$$
$$[\Theta_{4,4,3}] = 1234957683960000,$$
$$[C_{11}] = 769345344658800000,$$
$$[C_{12}] = 39410133308205660000,$$
$$[C_{13}] = 2032307051020979040000,$$
$$[C_{14}] = 105394174390609944840000,$$
$$[C_{15}] = 5492063712719478454560000,$$
$$[C_{16}] = 287381727957179467832220000,$$
$$[C_{17}] = 15092198505674986055199840000.$$

$$(5.4)$$

We give the non-zero counts of subgraphs up to 9 edges and some graphs on 10 edges, for a $SR(162, 21, 0, 3)$, where existence is still in question.

$$[C_4] = 17010,$$
$$[C_5] = 367416,$$
$$[C_6] = 5420520,$$
$$[\Theta_{2,2,2}] = 11340,$$
$$[C_7] = 90804240,$$
$$[\Theta_{3,3,1}] = 1258740,$$
$$[\Theta_{3,2,2}] = 1837080,$$
$$[\Theta_{3,3,2}] = 45008460,$$
$$[C_8] = 1574530650,$$
$$[C_{4\cdot4}] = 11634840,$$
$$[\Theta_{4,3,1}] = 66134880,$$
$$[\Theta_{4,2,2}] = 30005640,$$
$$[\text{}] = 17010,$$
$$[\Theta_{4,4,1}] = 891902340,$$

$$[\Theta_{4,3,2}] = 1576214640,$$
$$[\Theta_{3,3,3}] = 235146240,$$
$$[C_9] = 27120812040,$$
$$[\text{}] = 7348320,$$
$$[C_{5\cdot4}] = 624607200,$$
$$[\Theta_{5,3,1}] = 1168382880,$$
$$[\Theta_{5,2,2}] = 569494800,$$
$$[\text{}] = 62001450,$$
$$[C_{5\cdot5}] = 8298090360,$$
$$[\Theta_{5,4,1}] = 31085230680,$$
$$[\Theta_{5,3,2}] = 30715977600,$$
$$[\Theta_{4,4,2}] = 13881895020,$$
$$[\Theta_{4,3,3}] = 13599903240,$$
$$[C_{10}] = 470572695936.$$

$$(5.5)$$

We give the non-zero counts of subgraphs up to 9 edges and some graphs on 10 edges, for a SR$(176, 25, 0, 4)$, where existence is still in question.

$$
\begin{aligned}
[C_4] &= 39600, \\
[C_5] &= 887040, \\
[C_6] &= 15998400, \\
[\Theta_{2,2,2}] &= 52800, \\
[C_7] &= 326304000, \\
[\Theta_{3,3,1}] &= 5306400, \\
[\Theta_{3,2,2}] &= 6652800, \\
[\Theta_{3,3,2}] &= 170755200, \\
[C_8] &= 6752394000, \\
[\Theta_{2,2,2,2}] &= 13200, \\
[C_{4\cdot4}] &= 60033600, \\
[\Theta_{4,3,1}] &= 292723200, \\
[\Theta_{4,2,2}] &= 133372800, \\
[\text{图}] &= 158400, \\
[\Theta_{4,4,1}] &= 4087036800,
\end{aligned}
\qquad
\begin{aligned}
[\Theta_{4,3,2}] &= 7380172800, \\
[\Theta_{3,3,3}] &= 1109275200, \\
[C_9] &= 139754630400, \\
[\text{图}] &= 39916800, \\
[C_{5\cdot4}] &= 3339705600, \\
[\Theta_{5,3,1}] &= 6304161600, \\
[\Theta_{5,2,2}] &= 3133468800, \\
[\text{图}] &= 301593600, \\
[C_{5\cdot5}] &= 46083945600, \\
[\Theta_{5,4,1}] &= 173979590400, \\
[\Theta_{5,3,2}] &= 174511814400, \\
[\Theta_{4,4,2}] &= 79081833600, \\
[\Theta_{4,3,3}] &= 79095139200, \\
[C_{10}] &= 2919185660160.
\end{aligned}
\tag{5.6}
$$

Non-existence would follow if a count came out to be something other than a non-negative integer. We checked the not-known-to-exist cases appearing in Table 2.4 (strongly regular graph parameters with $n \leqslant 100$) for the counts of subgraphs up to 9 edges, but did not find anything. The counts of subgraphs for other strongly regular graphs not-known-to-exist could also be listed here but we simply leave these few as examples of the value of this result.

5.5   Remarks

Equations for any subgraph with a vertex of degree 1 or 2 (where the latter is not a cut-vertex) can be produced using the procedure presented in this thesis. For example, we give the equations for the counts of some graphs with a degree one vertex (or vertices) in a SR(n, r, e, f):

$$
\begin{aligned}
[P_2] &= \tfrac{1}{2}rn, \\
[P_3] &= \tfrac{1}{2}(r-1)rn, \\
[P_4] &= \tfrac{1}{2}r(r^2 - 2r + 1 - e)n, \\
[P_5] &= \tfrac{1}{2}r(r^3 - 3r^2 + 4r - fr - 2er - 2 + 3e + f - e^2 + ef)n, \\
[L(3,1)] &= \tfrac{1}{2}(r-2)ern, \\
[L(3,2)] &= \tfrac{1}{2}er(r^2 - 3r + 4 - 2e)n.
\end{aligned}
$$

If it is so desired, the algorithm presented need not be used recursively and the counts of subgraphs can be found in terms of the counts of subgraphs with one less vertex. For example,

$$[\Theta_{4,3,2}] = 7e[C_7] + (f - e - 2)[\Theta_{4,3,1}] - 2[\Theta_{5,2,1}] - [\;\raisebox{-0.4ex}{\includegraphics[height=2.2ex]{subgraph}}\;].$$

This can lead to results that relate the counts of two target subgraphs to each other.

We give the counts of subgraphs in a strongly regular graph up to 9, 10, or 11 edges depending on the restrictions specified. Subgraph counts for graphs of these sizes could be computed in a short amount of time. Unfortunately, the number of graphs in intermediate steps of the recursion grows rapidly as the size of the subgraph increases. There are many options for producing results for subgraphs with greater than 11 edges. The algorithm could easily be adapted to store the resulting expressions of smaller graphs as counts are being computed. This way, instead of making a recursive call for every graph with fewer vertices as it is encountered, it simply looks up the expression for those graphs. There are also options to parallelize the algorithm and make use of a multiple processor grid system.

It seems that the equations given in this chapter for counts of subgraphs in strongly regular graphs contain the same information as the spectral moment equations of Chapter 3 when applied to the same graphs. This seems plausible since both types of equations count subgraphs in terms of the eigenvalues of the parent graph. However, it is not yet clear exactly how these equations are related. It would be of interest to prove that they contain the same information, if indeed that is the case. Otherwise, they could potentially lead to a contradiction that shows non-existence of some strongly regular graphs.

It also seems possible that looking at relations between subgraphs (combinations of $\Theta_{2,\ldots,2}$ graphs for example) for specific strongly regular graph parameter sets could be used to establish non-existence results. These uses for our equations have yet to be fully explored.

Chapter 6

Conclusion

6.1 Results summary

In this thesis, we made progress towards a goal of spectral graph theory: to relate the eigenvalues of a graph to the structural properties of the graph.

6.1.1 Spectral moment equations

We were able to extend the known spectral moment equations to walk lengths greater than 6. These equations relate the eigenvalues of a regular graph to the counts of small subgraphs. This information can be used in a variety of ways, but especially as a tool for studying regular graphs with a selective set of eigenvalues.

The most general version of this result is the spectral moments w_ℓ of an $(r+1)$-regular graph with n vertices. The expressions for w_ℓ for $\ell = 0, 1, \ldots, 9$ are given as Equations (3.4) in Section 3.4. We gave examples for the equations when applied to graphs with girth 5 as Equations (3.5) and bipartite graphs as Equations (3.6), where certain contributors are known not to occur. Our equations for bipartite graphs verify the results of [63] and [174] by reproducing Equations (1.1) and Lemma 10 respectively. We reiterate that knowing any information about the graphs being studied that limits the possibilities for subgraphs can lead to equations with significantly fewer contributors and thus there are a lot more specializations that these equations can be tailored to.

The equations we have presented provide much more information than inequalities presented in [176]. However, if preferred they can also be used to derive inequalities simply by dropping some of the terms. Our equations are well suited to this use, since all terms are positive, unlike the equations in [195] which employ the principle of inclusion-exclusion and hence have many terms of opposite signs.

In Chapter 4, these equations are employed to eliminate feasible spectra for graphs that are vertex-transitive. This lead to our result of finding all connected 4-regular bipartite integral Cayley graphs.

Other uses of our equations have yet to be explored, but we are confident that they will prove useful in a variety of contexts. For example, they may help in constructing co-spectral graphs by identifying subgraphs for which the counts may differ.

6.1.2 Quartic integral graphs

As integral graphs are being recognized for their uses in modelling multiprocessor interconnection networks, quantum spin networks, and the like; it is timely for us to contribute a wealth of new 4-regular integral graphs.

Our results show that there are only 32 connected 4-regular integral Cayley graphs up to isomorphism. Table 4.3 lists the 17 graphs of the 32 which are bipartite and Table 4.7 gives the details of the 15 non-bipartite graphs. Most of the bipartite graphs are depicted in Figure 4.4. There are exactly 27 quartic integral graphs that are arc-transitive. We determined that 16 of the 27 graphs are bipartite; these appear in Table 4.3 and Table 4.5. Of these 27 graphs, 16 are Cayley graphs; these appear in Table 4.3 and Table 4.7.

We began with the 828 possible spectra from [176], and then narrowed our focus to a set Ξ of 59 candidates for vertex-transitive graphs; refer to Table 4.1 and Appendix A. Of these, we found 22 which are realized by Cayley graphs or arc-transitive graphs. In Section 4.4.7, by taking quotients, we found an additional 6 bipartite integral graphs that are neither arc-transitive nor Cayley, but realize a possible spectrum.

In Section 4.4.6, we show that G_{11} is the only 4-regular integral graph with 32 vertices that is vertex-transitive. The proof relies on the theory of quotient graphs and permutation group actions.

Overall, we found 9 bipartite quartic integral graphs (namely, G_{16}, G_{17}, F_4, F_5, M_4, M_5, M_7, M_8, M_9) that realise spectra not previously known to be achieved. It is still an open problem, to determine whether the remaining possible spectra are realized by any 4-regular bipartite integral graphs.

6.1.3 Subgraph counts of strongly regular graphs

Our final contribution was to count the connected subgraphs H of a strongly regular graph in terms of the parameters n, r, e, and f. For some graphs H, we are able to express the number of copies of H in terms of n, r, e, and f alone. Counts of other graphs are in terms of smaller subgraphs with minimum degree 3, in addition to n, r, e, and f.

We first proved results for determining the number of copies of a graph that there are in a strongly regular graph in terms of the counts of graphs with fewer vertices. Next, we designed an algorithm which applies these results recursively to graphs until a single vertex or a graph of minimum degree 3 remains. This method allowed us to derive equations for the counts of subgraphs H of a graph $\mathrm{SR}(n, r, e, f)$ for which H has all vertices of degree at least two. The equations for the counts of subgraphs H up to 8 edges in a strongly regular graph $\mathrm{SR}(n, r, e, f)$ are given as Equations (5.1) in Section 5.3. In the same section, the equations for the counts of subgraphs H up to 9 edges in a strongly regular graph $\mathrm{SR}(n, r, 0, f)$ are given as Equations (5.2). Finally, the equations for the counts of subgraphs H up to 11 edges in a strongly regular graph $\mathrm{SR}(n, r, 0, 1)$ are given as Equations (5.3).

The counts of subgraphs H with minimum degree 2 and up to 11 edges are given for the 'missing' Moore graph $\mathrm{SR}(3250, 57, 0, 1)$ as Equations (5.4) in Section 5.4. In the same section, the counts of subgraphs H with minimum degree 2 and up to 9 edges are given for the not-known-to-exist strongly regular graphs $\mathrm{SR}(162, 21, 0, 3)$ as Equations (5.5) and $\mathrm{SR}(176, 25, 0, 4)$ as Equations (5.6).

Again there is a lot of flexibility for the use of this result. Equations for any subgraph with a vertex of degree 1 or 2 (where the latter is not a cut-vertex) can be produced using the described procedure. If it is so desired, the algorithm presented need not be used recursively and the counts of subgraphs can be found in terms of the counts of subgraphs with one less vertex. For example,

$$[\Theta_{4,3,2}] = 7e[C_7] + (f - e - 2)[\Theta_{4,3,1}] - 2[\Theta_{5,2,1}] - [\text{}].$$

This can lead to results that relate the counts of two target subgraphs to each other.

These and other uses for our equations have yet to be fully explored.

6.2 Future work

There are many unanswered questions arising from the topics discussed in this thesis that have yet to be addressed. We present a few of these problems here.

It is of interest to further investigate the 4-regular bipartite integral possible spectra of [176]. To our knowledge, the smallest quadruple for which the existence question is not completely solved is $[1, 7, 3, 2]$ with 28 vertices. In the more restricted vertex-transitive case (see Appendix A), all vertex-transitive graphs on at most 31 vertices can be found at the following location: [155] (the graphs with $n \leqslant 26$ have been verified at this stage). Given our result that there is a unique graph on 32 vertices, the smallest quadruple for which the existence question is not completely solved is $[4, 4, 4, 5]$ with 36 vertices. We know that there exists a graph G_{12} satisfying these parameters, but it is not known whether or not this graph is the unique vertex-transitive graph with that spectrum. Our methods in Section 4.4.3 for finding the quotient of an existing graph can be broadened. We placed an upper bound of 2^{20} on the order of the automorphism group of a graph. This allowed our computation to complete in a reasonable time. However, it may have resulted in us missing some integral graphs that could be found by a more sophisticated analysis of the examples with large automorphism group. It might also be worth considering more general quotients than the ones we took. We only considered quotients by orbits of a cyclic group of automorphisms.

There is opportunity to apply the techniques of Section 4.4.6 to possible spectra of graphs with order a power of a prime p or order $2pm$ where m is a positive integer greater than 2. For example, the spectrum $[6, 0, 14, 0]$ in Appendix A has $42 = 2 * 7 * 3$ vertices. Using nauty's genbg [142] to generate the set of all 4-regular bipartite multigraphs with 3 vertices in each part, we determined that there are only four integral graph candidates. The only thing left is to find the possible lifts from these candidates to a graph with the desired spectrum. Beyond these techniques, there is more information about quotient graphs that we could attempt to employ. In addition to the eigenvalues of the quotient graph G/Π being a subset of the eigenvalues of the graph G, other results include that:

- G and G/Π have the same spectral radius [84, Cor 2.3] and

- the eigenvectors of G/Π can be lifted to provide the eigenvectors of G [84, p. 77].

There are also interlacing techniques, graph angle methods in relation to eigenspaces, and possibilities for vertex partitioning (in relation to quotient matrices), which we plan to exhaust in the near future to find other existing graphs.

The new 4-regular integral graphs that we found that are co-spectral to other graphs are as follows: G_{13} co-spectral to $I_{40,1}$ and $I_{40,2}$, G_{15} to $I_{72,1}$, H_9 to $I_{20,8}$, H_{12} to $I_{36,4}$, and F_3 to M_6. We also mention the co-spectral graphs among the known integral graphs: G_5 is co-spectral to $I_{16,2}$ and another graph appearing in [176], G_6 to $I_{18,2}$ and $I_{18,3}$, G_7 to $I_{24,1}$, F_1 to $I_{60,2}$, H_5 to $I_{12,6}$, and J_3 to $I_{30,5}$. There are many options to explore for determining co-spectral mates for graphs that we have found. For example, there are eigenvalue preserving switching techniques that we have yet to attempt: Seidel switching [163] and Godsil-McKay switching [88].

Given a graph G with diameter D and spectral radius λ, the *second type mixed tightness* $t_2(G)$ is defined as $(D+1)\lambda$. It has been argued that graphs with small tightness t_2 are well suited for multiprocessor interconnection networks [64]. It would be of interest to extend the classification of integral graphs with small tightness presented in [64] by adding the graphs of this thesis that are well suited.

We mentioned that from the equations of Chapter 3, we were able to reproduce the reduced possible spectra list for 4-regular bipartite integral graphs including the values of $[C_4]$ and $[C_6]$ for all entries. In addition, we are able to determine $[C_8]$ in many cases. It would be worth attempting to produce a possible spectra list for 5-regular bipartite integral graphs using inequalities where the number of subgraphs in the equations become too numerous.

Spectral moments of regular graphs are used to study properties of fullerenes in [60]. They define a measure of width, w, of a fullerene. Several unanswered questions remain that we would like to investigate using our spectral moment equations of Chapter 3. For example, we would consider w_ℓ for even ℓ between $2w+8$ and $4w+10$.

We mentioned that given our equations for strongly regular graphs, we checked the not-known-to-exist cases appearing in Table 2.4 (strongly regular graph parameters with $n \leqslant 100$) for the counts of subgraphs up to 9 edges, but did not find any negative or non-integer valued counts. We plan to check the parameter sets of strongly regular graphs where $n > 100$. It may also prove fruitful to look at the equations for subgraphs with at least 10 edges or even those subgraphs with vertices of degree one. In the event that none of these attempts gives an example of a parameter set that can be shown not feasible, then it would be of interest to understand why the equations we found do not have the power to provide a non-existence result.

In the event that graphs with some known parameters are being investigated, it is possible to derive congruence results for the remaining parameters since each equation must evaluate to a non-negative integer. As an example, consider the first equation from the presented equations for a Moore graph $\mathrm{SR}(n,r,0,1)$: $[C_5] = \frac{1}{10}(r-1)^2 rn$. Any parameter set with these values will only have a solution if 5 divides n or r is congruent to 0 or 1 mod 5. Then one could consider the equation $[C_7] = \frac{1}{14}r(r-2)(r-3)(r-1)^2 n$ and deduce that simultaneously 7 divides n or r is congruent to 0, 1, 2, or 3 mod 7. Unfortunately, 5 divides 3250 and $r = 7$ so this does not eliminate the 'missing' Moore graph $\mathrm{SR}(3250, 57, 0, 1)$. This example was inconclusive. However, it is very possible that considering a better equation (or combination of equations) would lead to non-existence results.

Our methods do not give an equation for a subgraph with minimum degree 3. However, minimum degree 3 graphs do appear on the right side of our equations; for example, consider K_4 in this equation for a graph $\mathrm{SR}(n,r,e,f)$:

$$[\triangle\!\triangle] = \tfrac{1}{2}(e-1)^2 ern - 12[K_4].$$

For this reason, we hoped that counts for these graphs could be deduced. In particular, we wanted to determine the counts of the Petersen graph P, in the 'missing' Moore graph $\mathrm{SR}(3250, 57, 0, 1)$. This is interesting because P is the smallest minimum degree 3 graph that can occur in $\mathrm{SR}(3250, 57, 0, 1)$. It occurs in the Hoffman-Singleton graph 525 times. However, in any attempt that we have made to solve for $[P]$, by forming a system of linear equations that include the variable $[P]$, the result is inconclusive because $[P]$ is a free variable. It would be interesting to investigate this further. There are more combinations of equations to consider. Do other subgraphs with minimum degree 3 behave the same way? If so, does it indicate that the counts of these subgraphs are not determined by the spectrum?

It seems that the equations given in Chapter 5 for counts of subgraphs in strongly regular graphs contribute no new information to the spectral moment equations of Chapter 3. This seems plausible since both equations depend heavily on the eigenvalues of the parent graph. However, it is not yet clear exactly how these equations are related. It would be interesting to investigate this relationship and if possible deduce one set of equations from the other.

There is opportunity to attempt ad-hoc methods between subgraphs similar to the technique of eliminating the spectrum [24, 4, 40, 3] in Section 4.2. We give the counts of subgraphs for strongly regular graphs which are not known to exist. This includes the 57-regular Moore graph on 3250 vertices. We also have lists of possible spectra like that of Appendix A where given a spectrum it is not known whether or not it is realizable. If we take a specific strongly regular parameter set or a specific spectrum and consider the relevant spectral moment equations and/or strongly regular graph equations in conjunction with relations between subgraphs, then non-existence results seem very attainable. The equations of this thesis afford a wealth of counts of the various subgraphs and their relations to each other creating many graph specific opportunities for elimination.

In [33], it is conjectured that the Petersen graph is the only connected strongly regular graph that is not Hamiltonian. The authors give the result that for a connected graph $G = \mathrm{SR}(n, r, e, f)$ with smallest eigenvalue λ: if $\lambda \notin \mathbb{Z}$ or if $-\lambda \leqslant f + 1$, then G is Hamiltonian. The expression for $[C_{10}]$ given by our algorithm for Moore graph parameters is $\frac{1}{20} r(r-2)(r-3)(r^3 - 3r^2 + r - 1)(r-1)^2 n$ and this evaluates to 0 for the Petersen graph $\mathrm{SR}(10, 3, 0, 1)$. The first graph for which Hamiltonicity is still unknown is for a graph that is not known to exist: $\mathrm{SR}(99, 14, 1, 2)$ with $\lambda = -4$. In practice, our current algorithm would create too many terms for a large graph like C_{99} in the middle steps of recursion. However, it might be possible to appropriately discard most of the terms and in keeping some, derive a positive lower bound on the number of Hamiltonian cycles.

There are many more interesting problems to discover and explore in the area of spectral graph theory. The equations we presented and the graphs we found as a result leave more questions than answers regarding what can be said about the relationship between the eigenvalues of a regular graph and the counts of its various subgraphs.

Appendix A

Feasible Vertex-Transitive Spectra

The following is a set of possible spectra that might be realized by a connected 4-regular bipartite integral graph G that is vertex-transitive. This set was determined in Section 4.2. The entries are given as n x y z w $[C_4]$ $[C_6]$ where $|V(G)| = n$ and $Sp(G) = \{4, 3^x, 2^y, 1^z, 0^{2w}, -1^z, -2^y, -3^x, -4\}$.

n	x	y	z	w	$[C_4]$	$[C_6]$
8	0	0	0	3	36	96
10	0	0	4	0	30	130
12	0	1	4	0	27	138
12	0	2	0	3	30	112
16	0	4	0	3	24	128
18	0	4	4	0	18	162
20	0	5	4	0	15	170
24	0	8	0	3	12	160
24	2	2	6	1	30	124
24	3	0	5	3	42	80
30	0	10	4	0	0	210
30	3	2	9	0	30	130
30	4	1	4	5	45	60
32	0	12	0	3	0	192
36	4	4	4	5	36	84
36	5	1	7	4	45	66
40	4	6	4	5	30	100
42	6	0	14	0	42	98
48	6	4	10	3	36	96
60	4	16	4	5	0	180
60	6	9	14	0	15	170
60	7	8	9	5	30	100
60	8	7	4	10	45	30
60	9	1	19	0	45	90
70	6	14	14	0	0	210
72	11	4	13	7	54	24
72	6	16	10	3	0	192
72	8	10	16	1	18	156
72	9	10	7	9	36	60
90	12	13	4	15	45	0
90	13	7	19	5	45	60
90	8	22	4	10	0	150
90	9	16	19	0	0	210
96	12	12	20	3	24	128
120	12	28	4	15	0	120
120	13	22	19	5	0	180
120	15	20	9	15	30	40
120	16	14	24	5	30	100
120	19	6	29	5	60	20
126	13	28	7	14	0	126
126	20	7	28	7	63	0
144	16	28	16	11	0	144
144	20	16	28	7	36	72
168	20	28	28	7	0	168
180	20	40	4	25	0	60
180	21	34	19	15	0	120
180	22	28	34	5	0	180
180	26	19	34	10	45	30
210	27	28	49	0	0	210
240	28	52	4	35	0	0
240	30	40	34	15	0	120
280	34	56	14	35	0	0
288	36	52	28	27	0	48
360	46	64	34	35	0	0
360	47	58	49	25	0	60
360	48	52	64	15	0	120
420	55	70	49	35	0	0
480	64	76	64	35	0	0
560	76	84	84	35	0	0

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