CHINESE MIDDLE SCHOOL MATHEMATICS TEACHERS’ PRACTICES AND PERSPECTIVES VIEWED THROUGH A WESTERN LENS

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Summary

This study was designed to investigate Chinese middle school mathematics teachers’ practices and perspectives on knowing and learning that may underlie their observed practices. In particular, this study addressed the following research questions: (a) What pedagogical activities and strategies can be identified in Chinese middle school mathematics teachers’ practices? (b) What teacher perspectives of mathematics knowing and learning may underlie these teachers’ practices?

To address these questions, the researcher, a native Chinese who moved to Australia and became an Australian citizen, used a constructivist (western) conceptual framework as a lens. This framework combined two lines of previous work, Ernest’s (1989b) characterisation of mathematics teaching and Simon and Tzur et al.’s (Heinz et al., 2000b; Simon et al., 2000; Tzur et al., 2001) distinction of perspectives that seem to underlie reform-oriented teaching practices.

Eleven year-7 middle school mathematics teachers from two different schools located in a Southeast province in Mainland China volunteered to participate in this study. For each of these 11 teachers, a set of data was collected, consisting of two video recorded classroom observations (consecutive lessons) interspersed with three video recorded interviews with the teacher about those observed lessons (before, between, and after each classroom observations). Transcripts of the interviews and classroom observations were translated into English. Six teachers were then selected as case studies and their practices and perspectives were analysed using the constant comparison method proposed by Glaser and Strauss (1967) as part of the Grounded Theory methodology.

Findings of this study indicate that, typically, the participating Chinese teachers used a five-component cyclic teaching practice—Reviewing, Bridging, Variation, Summarising, and Reflection/Planning. The first four components constituted their implemented lesson structure. Data are presented to depict the nature of these first four components along with the participating teachers’ rationale for using each component the way they did.

The significance of these findings is discussed in terms of a perspective about knowing and learning that seem to underlie the participating teachers’ five-component practice. This perspective appears to differ from perspectives that were identified previously. Coupled with the four-component lesson structure this perspective is discussed in terms of how it may inform mathematics teachers’ development of effective pedagogical practices, as well as future research and theory development.
Statement

This thesis contains no material which has been submitted for examination in any other course or accepted for the award of any other degree or diploma in any university and, to the best of my knowledge and belief, contains no material previously published by another person except when due reference is made in the text.

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Dedicated to Father and Mother,

My life-long love and friendship.

xiexie (thank you)!

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Chapter 1:

Introduction

1.1 Background to this Thesis
This thesis seeks to describe practices that characterise the pedagogy of Chinese middle school mathematics teachers who participated in this study and to identify the perspectives on knowing and learning that seem to underlie their practices. This focus contributes to the growing interest on mathematics education in China among international and Chinese researchers (An, 2006; Cai, 2007; Correa, Perry, Sims, Miller, & Fang, 2008; Fan, Chen, Zhu, Qiu, & Hu, 2006; Gu, Huang, & Marton, 2006; Ma, 1999b; Wong, 2006; Xie & Carspecken, 2008). This chapter introduces the study and its educational logic, discusses the significance of the study, and presents the research questions.

1.2 Educational Logic
The educational logic of this study begins with the ways of thinking, rooted in one’s culture, that underlie mathematics teachers’ pedagogical activities. Ernest (1989a) and Bruner (1996) noted that mathematics teachers’ beliefs have a powerful impact on their teaching practices. Cooney (2001) suggested that what teachers do in their classrooms is ultimately a product of their beliefs. Teachers draw upon their cultural beliefs as a normative framework that guides their teaching practices (Cai, Perry, Wong, & Wang, 2009; Wilson & Cooney, 2002). Bruner (1996) emphasised that teachers’ practices and behaviours are influenced by their cultural conceptions of effective teaching. That is, mathematics teaching and learning is a cultural, context-dependent activity (Bishop, 1994; Cai, Lin, & Fan, 2006; Stigler & Hiebert, 1999). Different cultures hold different beliefs, philosophies regarding teaching and learning in general and approaches to mathematics education in particular (Xie & Carspecken, 2008). Similarly, Correa, Perry, Sims, Miller, and Fang (2008) asserted that, to a large extent,
teachers’ beliefs about instruction and learning are shaped by culturally shared experiences and values. These researchers noted that in order to improve educational experiences for students we need to acknowledge and consider how teacher beliefs and practices are linked and embedded within cultural contexts. This current study drew on the link between culture, teaching perspectives, and teachers’ practices by focusing on both what participating teachers did in their classrooms to promote students’ learning and the rationale they gave for their ways of teaching.

According to Robitaille and Travers (1992), studies that examine different culturally-oriented beliefs and approaches to teaching and learning can provide opportunities for sharing, discussing, and debating important issues in mathematics education. Such studies can also promote understanding of how to teach and learn mathematics differently and effectively under a common goal (An et al., 2004). Stigler and Perry (1988) pointed out that international studies can lead researchers and educators to a deeper understanding of various aspects of mathematics education. Such studies can also lead mathematics teachers to reflect on their own teaching practices and bring about better choices and improved teaching practices. With these comparisons, teachers are more likely to also question their own teaching practices and become aware of alternatives to their current practices.

This study used a Western lens to examine, and link, middle schools mathematics teachers’ practices and perspectives in China. This examination follows a logical sequence of how mathematics teaching and learning takes place. Mathematics teachers’ perspectives of mathematics and its teaching and learning impact teachers’ practices, and vice versa. Cobb, Wood, and Yackel (1990) contended that beliefs and practices influence one another, co-develop, and are not related in a linearly causal way in either direction. Researchers (Richardson, Anders, Tidwell, & Lloyd, 1991; Staub & Stern, 2002; Trigerll, Prosser, &
Waterhouse, 1999; Watkins & Biggs, 2001) demonstrated a close relationship between teacher beliefs, instructional practices, and student learning. Figure 1.1 illustrates these reflexive relationships among cultural contexts, teachers’ perspectives, and teachers’ practices.

Figure 1.1: Relationship among teachers’ culture, perspectives, and practices.

The larger, rectangular frame depicts the overarching context—cultural milieu—in which teaching-learning processes occur. This context gives rise to teachers’ interrelated perspectives about mathematics, mathematics learning, and mathematics teaching—perspectives that both shape and are shaped by teaching practices.

It should be noted that this thesis is a cross-cultural study, but not in the obvious sense of comparing mathematics teaching and learning in two different countries. Rather, it crosses cultural boundaries by using a Western (constructivist) lens to investigate the phenomenon of mathematics teaching and learning in China. Of course, the constructivist lens is only one way of interpreting the data. For example, it may have been possible to choose a behaviourist approach (Gagne, 1965; Guthrie, 1942; Thorndike, 1932) or a critical theory perspective (McLaren & Kincheloe, 1995; Kincheloe, 2008), but the constructivist theory seemed a more
appropriate framework for the interpretive approach used in the thesis. Furthermore, it is stressed that the thesis does not seek to compare the approaches to teaching between Chinese and Western teachers. Rather, it seeks to view approaches to teaching mathematics by these teachers in China through a Western constructivist framework.

The importance of this thesis lies in interpreting and linking how the participating Chinese teachers think about learning and how they go about teaching mathematical concepts not yet known to their students. Two main reasons support this importance. First, teaching practices and learning activities are context dependent (Biggs, 1992; Samuelowicz & Bain, 1992; Watkins & Regmi, 1992). Different cultures have different traditions, values, and beliefs about the functions of education, purposes of schooling, different expectations and understandings of the role of teachers and students, and different standards and requirements for judging students’ learning outcomes. Studying particular aspects of mathematics education in a particular cultural context, in this case China, can contribute to understanding interrelationships among these factors.

Second, classrooms are intentionally designed for students’ learning. Teachers play the central role in the classroom for teaching and learning to take place. In other words, teaching is an important factor in fostering effective students’ learning. Teachers’ conceptions of the subject matter, and its teaching and learning play an important role in teachers’ practices, which in turn affect students’ learning outcomes. Cultural norms and practices frame teachers’ beliefs and perspectives on teaching and learning, as well as their understanding of the subject matter they teach. Teachers’ perspectives on content, pedagogy, and epistemology—what does knowing mean and how it evolves through learning—drive what and how they plan for a lesson, the instructional strategies they choose, the learning activities and tasks they design,
and the assignments they give. It is these aspects of the educational logic that are the focus of this study.

1.3 Aims/Objectives of the Study
This study aimed to:

(a) Identify typical ways in which the participating Chinese mathematics teachers carried out their teaching (e.g., a lesson structure); and

(b) Examine how these teachers’ perspectives are linked to their observed practices.

1.4 Significance of this Study
This sub-section elaborates six reasons why using a constructivist framework to examine Chinese mathematics teachers’ practices at the middle grades can make a significant contribution to the knowledge base in mathematics education. The first reason is the apparent interest in the consistently high level of Chinese students’ mathematical outcomes. Evidence from large-scale and small-scale comparative studies show that, on average, Chinese students typically outperform their Western counterparts in mathematics and science (Chen & Stevenson, 1995b; Fan & Zhu, 2006; OECD, 2009; Stevenson, Chen, & Lee, 1993; Stevenson & Stigler, 1992). Studying how Chinese teachers think about mathematics and its learning, and how their thinking is translated into organisation of classroom teaching-learning activities, can help to better understand the contribution of teaching to these outcomes. It is also recognized that Chinese students are generally positively motivated and actively encouraged by their parents, but such cultural attributes seem insufficient to explain the differences in outcomes, especially given that Western schools seem better resourced than schools in China.

Secondly, numerous cross-cultural studies have been conducted to explain the Chinese mathematics education and student outcomes (Chen & Stevenson, 1995; Perry, 2000; Stigler
& Stevenson, 1990; Stigler & Perry, 1988). Most of these studies were conducted by Western scholars or educators (described here as “outsiders”). Recently, a growing number of studies approached Chinese mathematics education from an insider’s perspectives (Cai, 1998; Cai, 2000; Ma, 1999a; Paine & Ma, 1993; Paine, 1997; Wang & Paine, 2003). In Chapter 2, this study is situated in relation to these works. Here, the advantages in conducting this study by this researcher are pointed out. The researcher was born and grew up in Mainland China, and experienced its educational system as a student up to the completion of an undergraduate degree. Subsequently she moved to Australia and experienced its educational system as a master’s and a doctoral student, as well as a teacher in Australian schools. That is, in this qualitative study, the researcher was part of the instruments for cross-cultural comparison (Patton, 2002). In particular, the researcher was positioned as both an insider to the Chinese education system, and as a conceptual outsider, that is, an educator and researcher trained in and utilising a Western perspective. Such a position can help to shed light on how and why Chinese teaching promotes first-rate student learning of mathematics. As an insider equipped with a Western constructivist lens, the researcher had an advantage in investigating Chinese classrooms through observing teachers’ lessons and analysing their practices. If, as an international community of mathematics educators, we can understand how mathematics teachers in China are effective, we can also better understand the link between teaching and learning, a link that has become a focus of much research in recent years (Cai, 2003; Mullis, Martin, & Foy, 2008; Mullis et al., 2000; Tzur, 2008).

Thirdly, this thesis can contribute to the development of accounts that characterise teachers’ perspectives in order to better understand their rationale for their teaching methods. For example, Heinz, Kinzel, Simon, and Tzur, (2000), Simon, Tzur, Heinz, Kinzel, and Smith, (2000), and Tzur, Simon, Heinz, and Kinzel, (2001) distinguished three major perspectives that mathematics teachers in the US seem to have about mathematics and its learning:
traditional, perception-based, and conception-based perspectives (these are discussed in detail in Chapter 2). However, those perspectives were identified in a Western context. Conducting this thesis in China could enable examining the applicability of their strategy of inquiry to the Eastern context, and possibly extending their work. By identifying perspectives that underlie Chinese mathematics teachers’ practices, this thesis may contribute to identifying goals and/or processes conducive to teachers’ professional development elsewhere.

Fourthly, this thesis is significant in its use of a Western strategy of inquiry in an Eastern context. Different methodologies have been used in recent years to study teaching practices in classrooms and how teachers think about those practices (Stevenson & Stigler, 1992; Stigler & Perry, 1988). However, teachers’ professed beliefs can be inconsistent with their teaching practices (Thompson, 1992). Moreover, Leatham (2006) pointed out that researchers often report inconsistency between teachers’ beliefs and actions because (a) teachers may not be able to articulate their beliefs and (b) there is no one-to-one correspondence between what teachers state and what researchers think that those statements mean. In order to make inferences about teachers’ perspectives, researchers need to collect data that specifically link and corroborate what teachers do and what they say (their rationale) about what they do.

To this end, large-scale questionnaires for measuring teachers’ views and beliefs, let alone for finding out their practices, seem insufficient, because teachers adhere to different cultural values and different understandings of educational phenomena. Furthermore, Pajares (1992) argued that beliefs must be inferred from what one says, intends, and does; they cannot be observed or measured directly. For example, teachers from different cultures are likely to have different interpretations of the ‘same question’ in a survey due to assimilating it into a different frame of reference. Teachers may also have a different, culturally oriented tendency to respond in a way that pleases those who authored a survey, which might have an impact on
the type of response they give. Leung (1995) argued that no matter how well questionnaires concerning instructional practices are designed, the data collected could not substitute for the rich information one obtains via classroom observations. On the other hand, direct observation alone has its own limitations. As classroom teaching and learning are complex processes, direct observations alone cannot capture the teachers’ reasoning for teaching the ways they do. Asking teachers about their teaching, how and why they organise classroom activities, what they look for in students’ work and what the teachers do in response to it, and how teachers conceive of teacher–student interactions in service of mathematics learning all seem needed to better explain what teachers do.

To avoid the aforementioned limitations of studying teaching via using only surveys and interviews, or only observations, this thesis adopted Simon and Tzur’s (1999) strategy of inquiry of *Account of Practice* (AoP, elaborated in Chapter 3). This strategy of inquiry was designed for studying teaching practices in classrooms and characterising teachers’ underlying perspectives. The AoP strategy of inquiry differs from studies that rely on teachers’ self-reports. In an AoP, the researcher generates rich data sets obtained from videotaping at least two consecutive lessons in which the teacher teaches a single, new (to students) topic, along with taped interviews before, between, and after each classroom observation. The AoP strategy of inquiry was developed and employed in the U.S., and to the researcher’s knowledge has not been applied in a Chinese context. Thus, this thesis contributes to examining the suitability of AoP for studying non-Western teachers’ practices and characterising their perspectives. This thesis may also contribute to improving that strategy of inquiry due to applying it in a different cultural context.

Fifthly, this thesis uses a new approach to comparing teachers’ practices and perspectives. It differs from recent studies that focused, for example, on Chinese (and Eastern) teachers’
mathematical content knowledge (Ball & Bass, 2001; Ma, 1999a; Shulman, 1987). These studies were based on the assumption that teachers’ mathematics knowledge is central to effective teaching and student learning. Other studies of Chinese teachers’ practices have been conducted from a social-cultural perspective (Gu, Huang, & Marton, 2006). However, an extensive search of the research literature indicated that there has been no substantial articulation of Chinese teaching in terms of identifying teachers’ epistemological and psychological stances and how these stances impact their teaching practices. It is in this latter sense that this thesis can contribute to the growing body of literature on mathematics education in China and elsewhere.

Finally, this thesis is likely to make an important contribution due to its focus on mathematics teachers’ pedagogical activities in the middle school, a grade-band in which algebra is being taught and gaps among student outcomes seem to widen. Studies like the International Association for the Evaluation of Educational Achievement [IEA] (1988), Beaton et al. (1996), Stevenson and Stigler (1992), Stigler, Lee, and Stevenson (1987), and Stigler and Stevenson (1991), suggested that Eastern students have consistently outperformed their Western counterparts in mathematics. Specifically, the TIMSS (Mullis et al., 2000; Mullis et al., 2008) study showed that the gap in mathematics achievements between the East and the West increases between the fourth and the eighth grades. By studying successful teaching (in terms of student outcomes) during this period, a better understanding might be gained about factors that contribute to this gap. Explaining these factors is particularly important for students learning to reason algebraically. Algebra has become a central focus of reform in the mathematics curriculum and has been considered by many to be an important gatekeeper to higher mathematics and science courses and careers (Moses & Cobb, 2001; Oakes, Muir, & Joseph, 2000; Phillips & Lappan, 1998). Students cannot continue to higher levels of
mathematics, science, engineering, and technology (STEM) without a solid foundation in algebra. Thus, focusing on middle school can be beneficial for research in this key area.

This thesis did not set out to make judgments about whether Chinese teachers’ beliefs, epistemologies, pedagogies, teaching practices, or instructional strategies are better than those of their Western counterparts. Rather, it aimed to foster a better understanding of how and why the participating Chinese mathematics teachers organise their practices the way they do. In turn, this may help to promote teacher educators’ and policy makers’ considerations of ways to improve practising and prospective teachers’ development.

1.5 Research Questions
This thesis reports an investigation of Chinese middle-school mathematics teachers’ practices and conceptions of mathematics, and the teaching and learning that seemed to inform and be informed by these practices. It was designed to provide insights into links between these practices and perspectives by probing into teachers’ explanations of particular activities observed in their classrooms. In particular, this thesis addressed the following questions:

(1) What pedagogical activities and strategies can be identified in Chinese middle school mathematics teachers’ practices?

(2) What teacher perspectives of mathematics knowing and learning may underlie these teachers’ practices?

1.6 Outline of the Thesis
In Chapter 2, the relevant literature is reviewed, including previous works that constituted the conceptual framework of this thesis.

In Chapter 3, the methodology utilised for addressing the research questions is described, including the theoretical underpinning of the Account of Practice qualitative data collection
and analysis strategy of inquiry, sampling criteria for participating schools and teachers, and considerations of findings credibility.

In Chapter 4, findings about the participating Chinese teachers’ ways of organizing their lessons (practice) and reasons for such organization in terms of how a teacher thinks about the process of learning mathematics (perspective) are presented. These findings depict a four-component Chinese lesson structure.

In Chapter 5, these findings are discussed in terms of relation to previous works in the field and contribution of this thesis beyond those works. The discussion focuses on the effectiveness of the four-component lesson practice and its compatibility with Western (constructivist) models of teaching, and on distinguishing a pedagogical approach that seems to underlie this practice.
Chapter 2:

Researching Mathematics Teaching in China

The purpose of this chapter is twofold: to situate this study within the broader research literature about mathematics teaching in China and to present the conceptual framework that guided this study. The chapter begins with an introduction of literature about the key role that mathematics teachers serve in student learning and the importance that teachers’ perspectives have for understanding their practice. The introductory part also delineates the reasons for conducting this study in China. Next, key aspects of mathematics education in China are discussed, including students’ outstanding outcomes, Chinese cultural approaches to learning and teaching (compared to Western approaches), the need to study teachers’ thinking about their mathematics teaching, and a review of major studies of this latter issue. Then, a constructivist conceptual framework that guided the researcher’s work is explicated, including background about this theory of learning, approaches to teaching that draw on this theory, and characterisations of teaching and teacher perspectives postulated by mathematics educators. Combined, the discussion of Chinese approaches and the constructivist (Western) framework help to depict how the researcher was positioned to conduct this cross-cultures qualitative study.

2.1 Introduction to Literature Review

This sub-section introduces a review of the literature about mathematics teaching and the role of teachers in promoting students’ learning. It highlights the need to focus not only on how teachers perform their practice but also how they think about it. Then, the reasons that led the researcher to conduct this study in China are discussed.
In the past few decades, more emphasis has been placed on the role that mathematics teachers and the tasks/activities/problems they use play in their students’ learning processes (Hill, Rowan, & Ball, 2005; Jaworski, 2005, 2007a; Sullivan, 2008; Watson & Sullivan, 2008; Zaslavsky, 2007). Teachers play an important role in organising and shaping the learning context and therefore influence what is being taught and how it is taught (Cross, 2009). Bybee (1993) pointed out that a teacher is thought to be the centre of the educational change process, so it is critical to identify and understand teachers’ thinking about their practice.

In recent years, researchers have paid a growing attention not only to how teachers carry out their practices but also to how they think about what they do (da Ponte & Chapman, 2006; Phillip, 2007; Thompson, 1992). The reason seems quite clear: teachers continually face the challenges of deciding for themselves, ahead of lessons and in real-time, how to transform their academic knowledge and personal beliefs into classroom practices that foster students’ learning. In particular, numerous studies attempted to better understand how teachers conceive of the nature of knowledge and learning, that is, teacher epistemological stances, and how such stances may affect teachers’ curriculum implementation and instructional approaches (Cobb, Wood, & Yackel, 1991; Ernest, 1989a; Pajares, 1992; Philipp, 2007; Prawat, 1992; Raymond, 1997; Simon et al., 2000; Torff, 2005; Wilson & Cooney, 2002). Wilson and Cooney (2002) asserted that a considerable amount of research on mathematics teachers’ beliefs was based on the assumption that “what teachers believe is a significant determiner of what gets taught, how it gets taught, and what gets learned in the classroom” (p. 128).

In reference to the work of Furinghetti and Pehkonen (2002) and Pajares (1992), Correa et al. (2008) noted that the term “teacher beliefs”, also known as “implicit theories”, “orientations”, and “teacher perspectives”, has been used to mean many different things. In spite of those
different meanings, they indicated that teachers’ thinking is influential in determining how they frame problem situations and tasks for teaching mathematics. Thus, teacher thinking is thought to be a strong predictor of teacher behaviours (Thompson, 1992; Torff & Warburton, 2005). Researchers (Brown & Cooney, 1982; Cooney, 1985; Richardson, 1996) asserted that teachers’ beliefs about mathematics and mathematics teaching underlie their practices—the methods they choose to use for promoting their students’ learning in the classroom. Ball (1991) and Ernest (1989a) suggested that, to a large extent, differences found in teachers’ work depend on their views of teaching and learning of mathematics. Similarly, Cross (2009) noted that how a teacher conceptualises the nature of mathematics has direct impact on his or her teaching. To better understand why and how mathematics teaching in the classroom fosters (or not) students’ learning, and to explain how mathematics teacher education can promote pedagogical improvements, teachers’ views must be addressed. Following the works of Ernest (1989a, 1989b, 1989c), Gu, et al., (2006), Steinbring (1998), Thompson and Saldanha (2000), Simon et al. (2000), and Tzur (2008), this study focused in part on teachers’ epistemological stances about what constitutes mathematical knowing and how one might learn mathematics as a root for how they create learning opportunities for their students.

Including a focus on mathematics teachers’ epistemological stances seems a strategic choice because, as Philipp (2007) noted, such stances are a central factor for understanding students’ experiences with and achievements in school mathematics. This view resonates with Shulman’s (1986) distinction of teachers’ ‘pedagogical content knowledge’, which refers to the combination of content and pedagogy they use to make the mathematics in a curriculum accessible to their students. It is also consistent with the emphasis placed by the third International Mathematics and Science Study’s (Mullis et al., 2000; Mullis et al., 2008) on getting detailed information on how teachers teach mathematics. Stipek, Givvin, Salmon, and MacGyvers (2001) noted that teachers’ thinking guides their decisions in the classroom,
including the degree of students’ autonomy they promote and forms of assessment they use in and outside of classroom. Thus, it is important to study not only teachers’ content knowledge, but also their underlying views of the nature of mathematics and its teaching and learning, which impact their teaching practices and, in turn, students’ learning (Thompson, 1992).

A challenging problem in studying teachers’ conceptions and their instructional practices is that only some elements of this endeavour, such as teachers’ actions and language, are observable. The views that underlie teachers’ overt behaviours are implicit and may be hard to access and infer. Many teachers may never give explicit, systematic thought to the philosophies of teaching and learning they adhere to, even if they had many years of teaching experience. Moreover, as Fang (1996) and Thompson (1992) cautioned, there may be wide inconsistencies between professed beliefs (what teachers say) and classroom practices (what teachers do). In this regard, Thompson (1992) raised a conceptual and methodological issue: How may one examine teachers’ thinking in relation to their actual work? She argued that to get a coherent picture of both (and of possible gaps between them), researchers should employ methods that coordinate observation data from teachers’ real work in classrooms and interview data in which teachers explain these classroom data.

When a study includes a focus on teacher epistemological stances, one should take into account that those stances “may be shaped largely by culturally shared experiences and values” (Correa et al., 2008, p. 140). To understand the complex links between teachers’ culturally embedded epistemological stances and practices, the two can be studied in conjunction (da Ponte & Chapman, 2006). This view is consistent with Leung’s (1995) assertion that researchers in cross-cultural curriculum studies are becoming increasingly aware of the need to look at instructional practices in the classroom as an instance of the cultural milieu in which these practices are situated. Yet, many past studies of instructional practices relied on
teacher self-reports or on student reports of teaching-learning processes. Fewer studies adopted the observation scheme. As example of the latter, in which classroom practices in Asian countries and the US were compared, Stevenson and Stigler (1992) and Stigler and Perry (1988) used substantial classroom observations. These observations led to identification of remarkable differences among Chinese, Japanese, and American teacher practices. These researchers suggested that many of the differences seem to reflect differences in the teachers’ cultures. As part of this study, the researcher investigated Chinese teachers’ conceptions of mathematics and of mathematics learning as a means to interpret their practices through a constructivist lens. In line with the arguments above, differences between Chinese and Western cultures are further identified in this chapter.

Two main reasons underlie the choice to conduct this study in China. First, China has a unified national curriculum for mathematics, which is used by mathematics teachers across the most populated country in the world. This curriculum and the way Chinese teachers use it have been the focus of growing interest and research (Cai, 2000; Fan et al., 2006; Geary, Bow-Thomas, Liu, & Siegler, 1996; Jiang & Eggleton, 1995; Wang & Cai, 2007; Watkins & Biggs, 1996). Two main reasons for this interest are the first-rate Chinese student outcomes in mathematics and the contrast with the non-centralised, reform-oriented educational systems in the Western world (Xie & Carspecken, 2008). It should be noted that only two provinces in China (Shanghai and Zhejiang) have the flexibility to produce and use their own curricula. Yet, these two also have to tightly follow the national curriculum when selecting textbooks and other teaching resources, due to the end-of-school national exam, which echoes the Chinese high regard for mastery of basic skills and knowledge (Zhang, Li, & Tang, 2006).

In 2001, the Basic Education Curriculum Material Development Centre in China issued the Curriculum Standards for 9-year compulsory education. These standards represent a new
wave of curriculum reform in China, which seems highly influenced by recent reforms in mathematics education in the west (Wong, Han, & Lee, 2006; Zhao, 2009). This implies that in China the selection of teaching content, teacher education, and teaching and learning of the mathematics in most provinces would be quite similar. In particular, the two schools in which this study was conducted are in Zhejiang (province); in both, the Chinese national curriculum serves as the main teacher resource and guide.

Second, the researcher was born and grew up in Mainland China, and experienced its educational system as a student through to completion of her first university degree. Then, she moved to Australia and experienced its educational system as a master’s and a doctoral student. In this Western context, she also served as a teacher in Australian schools. Such a combination is an asset for a qualitative study that involves the use of a Western perspective for inferring Chinese teacher perspectives through interviews and classroom observations. In such a study, the researcher’s experience and perspective constitute part of the research instruments (Adler & Adler, 1994; Bogdan & Biklen, 1992; Guba, 1990). This researcher was therefore positioned as both an insider familiar with Chinese education and culture and as an outsider who observes Chinese teachers’ work through her experience of Australian education.

In the rest of this chapter, the discussion focuses on the importance of studying teachers’ conceptions of teaching in the way these conceptions manifest in the classroom. Then, the conceptual framework for this study is described, proceeding from a review of the Western, constructivist theory of learning and its implications for teaching into the specific constructivist lens used to address the research questions, including perspectives identified by Western, constructivist researchers (Ernest, 1989a; 1989b; Simon et al., 2000) to make sense of mathematics teachers’ practices.
2.2 Significance of a Study on Chinese Mathematics Teachers’ Work

This sub-section discusses aspects of mathematics education in China. It begins with a brief presentation of a key reason for the growing interest in this issue, namely, students’ outstanding outcomes, which are thought of as partly reflecting Chinese teachers’ pedagogical approaches. Then, common ways in which mathematics instruction proceeds in Chinese schools and classrooms are addressed, based on studies conducted by Chinese (insiders) and Western (outsiders) researchers. To better understand those common ways, the sub-section moves on to explaining cultural features of mathematics education in the East and in the West, including the Confucian-Heritage Cultural features of the former. Finally, the importance of studying teachers’ thinking about their teaching and the need to link their thinking with specific ways they implement their practice is explained.

2.2.1 East Asian Students’ Outstanding Mathematical Achievements

Over the past few decades, a number of large-scale international comparative studies have been conducted to examine students’ mathematical achievements in different countries, (Mullis et al., 2008; OECD, 2009). Students from East Asian countries, like Hong Kong (SAR), Japan, Mainland China, Singapore, Korea, and Taiwan, participated in those studies. Students from these countries have (a) consistently excelled in international comparisons of mathematics achievements and (b) typically outperformed their Western counterparts (Mullis et al., 2000; OECD, 2001; Robitaille & Garden, 1989; Stevenson, 1993; Stevenson et al., 1990). For example, the TIMSS (2003) results showed that East Asian students (grade 4 and 8) occupied the first five places in mathematics in the following order: Singapore, Korea, Hong Kong, Chinese Taipei, and Japan. In the Second International Assessment of Educational Progress (1990-1991, see PISA, 2000), more than a dozen countries and regions participated in a study that aimed to collect and report data on what students know and can do, on educational and cultural factors associated with student achievements, and on students’
attitudes. In that study, students at two age levels (9-year-old, and 13-year-old) were assessed. Students from Mainland China ranked first in the 13-year age group; 15-years old Hong Kong students took first place in the Programme for International Student Assessment (PISA). The most recent PISA study (OECD, 2009) included students from Mainland China (Shanghai), who were ranked first in mathematics among students from all 67 participating countries. In addition, in the International Mathematics Olympiads (1994-2003), students from Mainland China were ranked first in 1995, 1997, and 1999-2002 (Fan & Zhu, 2000). It can therefore be concluded that there are important national or cultural factors contributing to this success.

Some cross-cultural studies that compared US and East Asian students’ mathematical achievements provided similar findings: East Asian students consistently outperformed American students in almost every area of mathematical knowledge (Stevenson et al., 1993; Stevenson & Stigler, 1992; Zhou, Peverly, & Lin, 2006). Chinese students’ outstanding academic performance in mathematics was also manifested in a number of small-scale, cross-national comparative studies conducted by individual researchers over the past two decades, most notably by Stigler, Stevenson, and their colleagues (Chen, Chuangsheng, Lee, & Stevenson, 1995; Stevenson et al., 1993; Stevenson & Stigler, 1992). Stevenson et al. (1992) used the term ‘learning gap’ in order to report their findings of wide differences in student achievements between the US on one hand and Japan and China on the other hand. Several factors have been hypothesised and explored to explain this “learning gap.” For example, some researchers (Wang, 2002; Yang & Cobb, 1995) studied culturally-related differences such as classroom practices, parents’ expectations, time spent on learning mathematics in school, and students’ motivation, beliefs, and effort. Others (Lee, Ichikawa, & Stevenson, 1987; Sutter, 2000) studied the content and organisation of mathematics curricula. It should be noted that when more challenging, open-ended tasks were used, Chinese students did not necessarily show superior performance (Cai, 1998; Cai & Cifarelli, 2006). A few studies (An,
Kulm, & Wu, 2006; Ma, 1999) focused on how different cultural traditions impact mathematics teaching and, particularly, how Chinese teachers’ thinking impact their ways of teaching. This study contributed to the aforementioned literature through collecting and analysing data about Chinese teachers’ work with students on solving quite challenging problems in algebra while making their lessons conducive to students’ learning.

2.2.2 Mathematics Teaching in China
To understand the way mathematics teachers perform their work in China, it is important to consider some of the conditions of their work. Chinese class-sizes range from 40 to 70 students (Ding, Li, Li, & Kulm, 2008), which is commonly viewed as not conducive to effective instruction and student learning (Rice, 1999; Zurawsky, 2003). Due to this class-size, and consistent with the examination-driven national curriculum, mathematics pedagogy in China appears, at least to Western observers, to be teacher-centred (Leung, 2001). Chinese teachers seem to have little flexibility in selecting the content they teach. Thus, they seem to strictly follow the textbook at the expense of paying attention to individual students’ learning needs and interests. Accordingly, Chinese students were depicted as docile, obedient, uncritical, inclined to learn by rote, lacking intrinsic motivation, and aimed only for the exam (Gardner, 1989; Ginsberg, 1992). All these aspects do not match the Western view of classroom environment conducive to empowering student learning. In this study, the class-size was typically about 45 students, and observable teacher practices included activities that can be considered traditional from a Western perspective. However, the data analysis shows that teachers focused on promoting students’ understanding by deliberately reactivating what a teacher seemed to consider as their available knowledge.
2.2.3 Interpreting Chinese Ways of Teaching Mathematics

Chinese practices for teaching mathematics have attracted the interest of educators and researchers in mathematics education and cross-cultural studies (Biggs, 1994; Biggs & Watkins, 1996, 2001; Huang & Leung, 2006; Leung, 1995, 2001; Ma, 1999; Paine, 1990; Stevenson & Stigler, 1992; Wang & Lin, 2005). Some researchers (Fan, Wong, Cai, & Li, 2006; Ma, 1999) suggested that factors such as cultural beliefs, rigorous curriculum, and teacher competence play important roles in those practices. Li and Li (2009) identified essential aspects of the ‘Chinese teaching culture’, such as the ‘teaching research group’. In China, mathematics teachers are required and organised for (a) studying the curriculum and planning lessons together, (b) observing and critiquing each other’s teaching, and (c) collaboratively analysing their student learning (Lewis, 2000; Paine, 1997; Paine & Ma, 1993). To support their work outside the classroom, Chinese teachers typically teach about two lessons per day; the rest of their time is devoted to checking students’ class work and homework, and to planning their lessons. In such a system, apprenticeship practices and professional ranking and promotion methods might have been a major factor in promoting excellent mathematics classroom instruction in China (Li & Huang, 2008). It should be noted that in this study the researcher did not include the aspect of group planning in data collection, but it is discussed as part of a pattern that was identified in their teaching to better understand how their work proceeds from one day to another.

Recently, an entire book, How Chinese Learn Mathematics: Perspectives from Insiders (Fan et al., 2006), was devoted to addressing various aspects of mathematics education in China. An (2006) asserted that “most Chinese teachers would use one or two hours for daily planning” (p. 121). She pointed out that Chinese teachers’ lesson plans are not simple outlines, but a detailed set of teaching notes that specify the mathematical goals for student learning, the materials and teaching methods to be used, the types of questions to be asked and examples
given (including deliberate errors), alternative ways for solving problems, and how a lesson should be summarised. An (2006, 2008) suggested further that Chinese teachers focus on different cognitive levels of their students as they usually design a multi-layered plan to help students achieve proficiency in mathematics.

Another lens on Chinese mathematics teaching was offered by Paine (1990), who described it as a ‘Virtuoso model’. In the Virtuoso model, lessons are dominated by teacher-talk in an expository and explanatory format, with students constituting an *actively participating audience*. According to Paine, teaching in Chinese classrooms involves gradual transmission of knowledge to students in a precise and elegant language. Lesson unfolding resembles an artistic performance. To explain how teachers prepare for such a performance, Paine (2002) conducted a qualitative study on teachers’ instruction in Shanghai, Mainland China. She found that teachers’ collaborative study of the mathematical ‘focal points’, ‘difficult points’, and ‘hinges’ of teaching materials could have contributed to what Ma (1999) depicted as Chinese teachers’ profound understanding of fundamental mathematics (PUFM). This study focused on specific ways in which participating teachers planned and orchestrated their lessons, reflecting the virtuoso model and detailed sets of teaching notes they used, as well as the rationale for using such sets.

Other researchers (Stevenson & Lee, 1997; Stigler & Stevenson, 1991) also pointed out that East Asian mathematics lessons are coherent and polished. They emphasised how such lessons revolve around solving and discussing mathematically challenging problems. They further argued that in such an environment students are actively involved in the learning tasks and enjoy ample opportunities to think mathematically and develop their conceptual understanding of mathematics (Lee, 1998; Stevenson & Lee, 1995). That is, the apparent teacher-dominated, lecture style of teaching in China does not necessarily mean that teachers
make less effort to actively engage their students (Li, Kulm, Huang, & Ding, 2009; Mok, 2006). Rather, as Xie and Carspecken (2008) stressed, the curriculum and expected teachers’ work attempt to promote mental activity via abstract, general mathematical methods applied to abstract objects and governed by abstract, general principles. This study focused specifically on the thinking processes that teachers seemed to foster as they engaged their students in solving problems, some of which were quite challenging.

Ma’s (1999) seminal study, which drew on Shulman’s (1986) key notions of content knowledge and pedagogical content knowledge, addressed the relationship between those two types of knowledge in US and Chinese teachers. She found that Chinese teachers had more coherent mathematics knowledge and deeper mathematical understandings than US teachers (PUFM), which seems related to the fact that they are specialists (i.e., teachers at all grade levels, primary included, teach only mathematics). Most importantly, she also provided insights into the relationship between teachers’ content knowledge and their beliefs and pedagogy. She described the Chinese teachers’ thinking about mathematics as “connected” and marked by “longitudinal coherence”. She pointed out that the Chinese teachers were more likely to demonstrate their PUFM via flexibly representing mathematical ideas in their classrooms.

Moreover, Ma (1999) contended that Chinese teachers emphasise the understanding of abstract concepts from multiple perspectives, while American teachers paid great attention to procedural knowledge and tangible, hands-on experiences. She found that, quite often, Chinese teachers might devote an entire lesson (40-50 minutes) to the solution of a single mathematical problem, while approaching a mathematics topic/concept from many different angles. This crucial point has been observed in this study and is elaborated in Chapter 4. Ma further reported that in Chinese primary classrooms, students are often encouraged to solve
one mathematical problem in several ways. These findings were consistent with Paine’s (1990) ‘Virtuoso model’ of teaching, and were corroborated by Gu et al.’s (2006) study of the Chinese ‘teaching with variation’ method and with Zheng’s (2001) conclusion that Chinese student outcomes are rooted in solving problems in multiple methods. Similarly, Fan and Zhu (2006) and Wang (2002) argued that Chinese mathematics teachers engage their students in finding *multiple solutions and justifications* to non-routine mathematical problems. This is contrary to common perceptions of Chinese teaching.

A related line of research on teachers’ pedagogical content knowledge, by An et al. (2006), compared Chinese and US teachers’ knowledge of student thinking. They suggested that Chinese mathematics teachers are particularly aware of the importance of students’ prior knowledge. The ideas that Chinese views of mathematics teaching are based on students’ prior knowledge, and that they focus on assisting students in connecting between the prior knowledge and new mathematical concepts, were suggested by a few researchers (An et al., 2006; Correa et al., 2008; Ma, 1999). Later in this chapter, and in the analysis (Chapter 4), the constructivist core notion of assimilation is used to shed light on the importance of this aspect of mathematics teaching in China.

Lim (2007) characterised Chinese mathematics teaching through a Malaysian lens. He distinguished six features that make it effective: (1) teaching with variation of problems and solutions; (2) emphasis on students’ use of precise mathematical language coupled with teacher asking of high level questions such as ‘why?’, ‘how?’, and ‘what if?’; (3) strong emphasis on logical, mathematical thinking, and thus on justifying one’s answers mathematically; (4) creation of a highly ordered and serious, learning-driven classroom culture; (5) close and affirming teacher-student rapport; and (6) strong collaborative culture amongst mathematics teachers.
The studies reviewed above indicated that a critical examination of mathematics teachers’ pedagogy in China should penetrate deeper than observable features, such as large class size, whole classroom teaching, examination-driven curriculum, teacher-led practices, authoritarian teacher, content-oriented rather than process-oriented teaching, and/or emphasis on memorisation (Gu et al., 2006). In particular, it seems essential to examine how teachers organise their lessons and why, how students are involved in the process of learning, and how teachers foster students’ active engagement in learning via problem solving (e.g., teaching with variation on the basis of students’ prior knowledge). This study attempted to accomplish such articulation of observable teachers’ behaviours to better address reasons for carrying out teaching in these ways. The following discussion reviews literature that addressed cultural features of mathematics education in Eastern and Western countries.

2.2.4 Cultural Features of Mathematics Education in the East and West

To examine teachers’ practices and their rationale for using them, it is important to recognize that school teaching and learning are culture- and context-dependent (Biggs, 1992; Samuelowicz & Bain, 1992; Watkins & Regmi, 1992). Teachers and students from different cultural backgrounds hold different views about teaching, learning, and what is important in education, which might cause differences in their motives, approaches, performance, and learning outcomes (Biggs, 1996). Watkins (2000) and Zhu, Valeke, and Schellens (2008) proposed that cultural variables such as philosophical perspectives, value orientations, and motivation greatly impact students’ learning and how they and their teachers conceive of learning.

By-and-large, East Asian countries share the Confucian culture despite differences in political systems, economy, and educational structures. In particular, Asian countries share much of the Confucian values in education (Leung, 2001). Ho (1991) referred to these East Asian
countries as *Confucian-Heritage Cultures* (CHC; cited in Leung, 2001). To situate the study in the Chinese culture, the following sub-section elaborates the characteristics of CHC.

### 2.2.4.1 Characterising Confucian-Heritage Cultures (CHC)

Chinese culture is regarded as one of the Confucian-heritage cultures (CHC) (Watkins & Biggs, 2001). Confucianism, which was also named a *civil religion* (Acton, 2003), has prevailed in China for thousands of years. The core ideas of CHC are to maintain wisdom, harmony, and peace throughout the nation. In the Confucian tradition, education and schooling are regarded as significant not only for personal improvement but also for social improvement (Lee, 1996). Gao (1998) cited Confucius from the book *Da Xue* (The Great Learning, 2000 years ago):

> Through the investigation of things, knowledge is perfected. With the perfection of knowledge, thoughts become sincere. With sincerity in thoughts, the heart is rectified. Through rightness in heart and mind, the self is cultivated and disciplined. When the self is disciplined, the family can be rightly regulated. When the family is rightly regulated, the state can be well-governed. (p. 2)

This ancient quote points out that Chinese society is slanted toward a collectivistic rather than individualistic culture. Collectivism is an important characteristic of CHC and education plays a central role in promoting the collective good. CHC begins from and centers on loyalty to the family (Salili, 1996). Thus, students are expected to work hard not only to satisfy their own goals but also and chiefly to meet goals set by their families (Stevenson & Lee, 1990). Students’ success in life is defined by and relies on significant others like family, peers, or even the society as a whole (Holloway, 1988; Salili, 1996; Watkins, 2000). In CHC, the pressure to succeed academically for the social good is omnipresent for children regardless of parents’ socio-economic status or education level, and it is a matter of *family face* (Ho, 1993). The Chinese culture encourages students to demonstrate humble assent to authority, persistence, obligation, and conformity to social norms. These virtues stand in sharp contrast to Western emphases on individual’s self-reliance, responsibility, and creativity (Triandis,
McCusker, & Hui, 1990; Zhu et al., 2008). These culturally oriented traits seem, in part, responsible for the high level of discipline observed in the classrooms of teachers who participated in this study.

Confucius viewed learning as hard work and as a change in students’ attitudes to allow sustained mental engagement and thus deep, internal transformation of the learner. Learning was not regarded just or mainly for its own sake, but for seeking a civil service job and thus contribution to societal improvement. Confucius’ philosophy asserted that learning should be effortful, pragmatic, respectful, and enable students to acquire essential knowledge (Tweed & Lehman, 2002). In this view, **effort determines one’s learning and progress** more than abilities. Watkins and Biggs (1996) stated that for many Chinese—students, teachers, and parents alike—intelligence is not innate or relatively fixed, but rather something which can be improved through persistent, hard work. This stance seems compatible with Gardner’s (1983) work on multiple intelligences and with recent views on talent as a developing trait (Jiannong & Zha, 2000; Wieczerkowski, Cropley, & Prado, 2000). An ancient Chinese idiom captures this core idea: ‘Although studying anonymously for 10 years, once you are successful, you will become well-known worldwide’ (Gao, 1998).

Directly related to the notions of persistence and hard work for success is the Chinese core idea that failure is a necessary element for successful development. This is captured in another Chinese idiom: ‘failure is the mother of success’. Chinese teachers and students tend to believe that we learn and grow from our own as well as others’ mistakes. Wang and Murphy (2006) pointed out that Chinese teachers view errors made by individual students as a valuable learning opportunity for the particular student and the rest of the class. This view on the role of errors is also evident in the Chinese dialectical view of knowing, which asserts that to know ‘what something is’ requires continual engagement with ‘what it is not’ (Xie &
Carspecken, 2008), which errors help to accentuate. Therefore, teachers often discuss students’ mistakes publicly to benefit students in the class. With this kind of classroom norm, students feel it is natural to make mistakes, and they rarely feel embarrassed when their mistakes are exposed in public. Rather, they conceive of their own and others’ mistakes as a good learning opportunity, and believe that teachers are exposing their mistakes in order to help them. The data collected in the participating teachers’ lessons (and interviews) of this study indicated such a view, as well as the humble acceptance of teachers’ continual request of low-achieving students’ to share (and expose) their mistakes.

CHC expects students to respect and obey teachers—the masters of subject knowledge. Correspondingly, Chinese teachers adhere to the belief that they have a responsibility for cultivating not only students’ cognitive capacity but also their moral development (Gao, 1998). They believe that promoting students’ positive attitudes towards society and sensible moral behaviours is part of their responsibility (Gao & Watkins, 2001). This is in line with the Chinese cultural value of *Jiao Shu Yu Ren* (教书育人), which means teachers’ work entails developing a good person (Watkins, 2000). Teachers should always act as a good moral and behavioural model for students to follow, both inside and outside the classroom, the so called *Wei Ren Shi Biao* (为人师表) (Gao, 1998).

The core ideas of CHC explained above imply that, in a Chinese classroom, a high degree of teacher authority does not necessarily mean that students are learning passively. Rather, in this study students were observed being actively involved while submitting to the authority of the “master”, that is, to knowing and learning (Mok et al., 2001). This is possible because CHC students have a predisposition to learn (Hess & Azuma, 1991) and a deep sense of respect towards teachers (Biggs, 1996). Gao and Watkins (2001) suggested that teacher-centred and student-centred pedagogies occur simultaneously in CHC classrooms, which is
beneficial to student learning. Biggs (1998) pointed out that in a Chinese classroom, featured with collectivist and Confucian values, there are few management problems and the teacher can focus on meaningful learning. This may allow using whole-class methods in which the teacher is working with students individually, in pairs, or in groups whilst maintaining dialogue with the whole class. For both teachers and students, a good student is not just or mainly one who excels academically; rather, it is one who always pays attention to the teacher as the main road for academic success. Watkins (2000) suggested that student outcomes are nurtured in CHC classrooms because students think that listening attentively is their duty, and they are learning through actively listening and comparing their own solutions to others. A typical classroom practice that continually capitalises on this fundamental tendency is when the entire class or most of the students respond to a teacher’s questions as if they are members in a ‘speaking chorus’. This study provides further data and analysis of how teachers’ authority, exerted within the CHC context, could ‘live together’ with students’ high levels of mental engagement. The following discussion compares and contrasts features of mathematics pedagogy in Eastern and Western countries.

2.2.4.2 Features of Mathematics Pedagogy Identified in the East and West
To characterise East Asian mathematics classrooms on the basis of underlying values of CHC and contrast them with Western classrooms, Leung (2001) presented six dichotomies (see below). However, he clarified that these dichotomies do not mean that all East Asian countries are on one side and all Western counties are on the other side. Rather, the dichotomies can help to characterise the relative positions of the two cultures.

(1) **Product (content) versus process:** The CHC traditional view of mathematics is of a body of knowledge with a distinct structure (Xie & Carspecken, 2008). In such a view, mathematics pedagogy focuses on acquiring the body of knowledge through teacher guidance.
The emphasis in East Asian mathematics classrooms has been on the mathematics content and the basic skills and methods needed to deal with and understand the content (Zhang et al., 2006). The underlying belief is that the content (mathematical structure) is the core. In contrast, contemporary Western reforms seem to be focused on the process of doing mathematics. For example, advocated classroom activities (NCTM, 2000) such as problem solving and open-ended investigations are highly promoted and preferred over lecture, memory drills, and rote exercises. The underlying belief is that the process of learning is the core. This study reports how participating Chinese teachers’ interpreted the body of knowledge (e.g., algebraic fractions and equations) through problems they selected to instantiate abstract principles and methods.

(2) Rote versus meaningful learning: East Asian classrooms stress the importance of reflection and understanding in learning, for which memorisation has always been an accepted means, even when trying to memorise things that one does not yet fully understand. Leung (2001) defined rote learning as committing things to memory without understanding. Thus East Asian students have been often criticised for learning by rote, but the literature reviewed above suggests that memorisation in the Chinese teaching context included mental activity and understanding. In contrast, contemporary Western mathematics educators advocate meaningful learning. They emphasise that students should understand before they commit to memory, if there is a need to memorise at all. The Western conception of learning holds that memorisation without thorough understanding is rote-learning; true learning cannot be accomplished by rote, but through meaningful construction of the mathematics. For example, Marton et al. (1996) found that Western educators generally believed that memorisation is synonymous with rote-learning and does not lead to understanding; Chinese teachers believed that memorisation can be used as a means to deepen students’ understanding. This study corroborated the previous findings, while highlighting the link
between particular ways in which rote memorisation was used in support of students’ grasping general, abstract ideas and applying them to particular instances.

(3) Studying hard versus pleasurable learning: The traditional view in East Asian countries, especially in China, asserts that students should put in their studies substantial effort and perseverance in order to succeed. Learning is a serious endeavour that necessitates hard work. Pleasure and satisfaction are the end result of the hard work. In contrast, Western educators in general hold the view that learning should be an enjoyable experience for students. The core idea is that students learn more effectively when they enjoy what they do, so learning should take place in a pleasurable way. This study did not collect student data, but it does discuss teachers’ thinking about the role students’ interests may play in their learning.

(4) Extrinsic versus intrinsic motivations: According to Leung (2001), East Asian countries seem to focus on extrinsic motivation, such as encouraging students to learn so they get higher marks in national examinations in order to go to a better university, for family face, and for getting a higher social status. Examination in China has long been a source of motivation for students’ learning. It is deeply rooted in the Chinese culture, where the selection of high offices in the government is through competitive examinations. In addition to examinations, East Asian teachers and parents ascribe great importance to education. Their expectations are explicitly communicated in schools, classrooms, and at home. Those ‘external’ motivators hold that even if students are not interested in learning mathematics, they have to work hard and do their best to excel academically. Watkins and Biggs (2000, 1996) and Watkins (2000) pointed out that in the Chinese collectivist culture students are expected to adopt substantial strategies for learning so that others’ motives, like family face, job ambitions, material rewards, and peer pressures will be met. As Correa et al. (2008) noted, students’ interest in what they learn is of great importance to Chinese teachers; however, such
an interest may not reflect only or mainly the individual’s intrinsic needs. In contrast to such a collective view of motivation and coupled with individualistic core values, Western societies cherish intrinsic motivation in learning mathematics. They believe that the best way to motivate a student to learn mathematics is to get him or her personally engaged. In this regard, Niles (1995) stated that the capitalist core notion of competition is a major motivator for Western students—to be the best individual in a group, which is an ego-enhancing value. In other words, individual competitiveness emphasises internal need for success as a means for overcoming fear of failure, where winning is its own reward (Atkinson, 1964). This study shows a mix of teachers’ praise to individual students’ good work and use of mistakes to promote learning. Whereas the background knowledge of the national examination might have been a variable in teachers’ and students’ work, it was not mentioned by teachers in this study as a driving force in their organisation and teaching of the lessons, perhaps because at Year 7 this is not (yet) a major consideration.

(5) **Whole class teaching versus individualised learning:** The East Asian collectivistic philosophy stresses social integration and harmony. Consequently, learning (and teaching) together is highly treasured. It is important for individuals to fit into the social structure, and the teacher’s role is to both promote and be a moral model of this process. In East Asian countries teaching usually takes place in a large group setting, which results in the typical mode of whole class direct teaching. In contrast, Western societies stress individual independence and creativity. Accordingly, teachers are expected to tailor their teaching to every student’s personal needs and abilities. This Western view (NCTM 2000) underlies the strong emphasis on small group work, prevalent tutoring programs, and the grouping of students into ‘tracks’ on the basis of their ability and academic aspirations. Some of the specific activities described and examined in Chapter 4 (e.g., the entire class speaking like a chorus, or following a student’s mistaken solution) provide a window on how teaching large
classes helped to create a sense of collective in the students while also promoting understanding of individual students.

(6) Teacher competence—subject matter versus pedagogy: In the East Asian tradition, the CHC deeply-rooted image of a high quality mathematics teacher is of one who is a master of the mathematical knowledge. A popular Chinese metaphor suggests that ‘a teacher needs to have a bucket of water before she is able to give (i.e. teach) students a bowl of water’. If a mathematics teacher does not have very strong subject knowledge he or she will not be respected. For effective mathematics teaching, a solid grasp of subject knowledge is considered more important than a sound knowledge of pedagogy. Such a view is reflected in Ma’s (1999) findings that without PUFM it was impossible for elementary mathematics teachers to invoke appropriate pedagogical interventions. That is, in China being a master of the subject matter seems to be considered a necessary condition to being a facilitator of students’ learning. Thus, in China mathematics at all grade levels, including primary schools, is taught by designated mathematics teachers (specialists). In contrast, in Western countries effective teaching is equated with being a good facilitator of the student's learning processes, including learning of things teachers may not know themselves. In the information age, it seems impossible for teachers to truly master the immense, fast-changing body of knowledge. Furthermore, the Internet provides students with easy access to knowledge that no teacher could ever master. Thus, in the Western world being competent in pedagogy (teaching students how to learn on their own) seems more important than being competent in one’s subject area. This study did not examine teachers’ content knowledge directly. However, through a case of a novice teacher with a master’s degree in mathematics, Chapter 4 provides data that pertain to the issue of how this knowledge may be insufficient for effective teaching.
This summary of six features of mathematics teaching and learning in the East and West indicates culturally rooted differences in educational traditions, values, and philosophies. Being aware of the differences can help understanding the different teaching practices. In particular, the literature reviewed above establishes the basis for this study in terms of pointing to the role that culturally sensitive use of a Western perspective may play in better explaining Chinese mathematics teaching practices. The following discussion elaborates the importance of studying not only how mathematics teachers carry out their teaching but also the rationale for doing it in these ways.

2.3 Importance of Studying Teachers’ Thinking about their Mathematics Teaching

It can be assumed that the way a teacher thinks about the nature of learning influences the way he or she teaches and subsequently what and how students learn (Kember, 1998; Trigwell, Prosser, & Waterhouse, 1999; Watkins & Biggs, 2001). Kember (1998) argued for the following educational logic: A teacher’s culture impacts her or his thinking about teaching, which drives the teacher’s practice; in turn, a teachers’ practice impacts student approaches to and achievements in learning. For example, Kember and Kwan (2000) found that teachers who conceived of teaching as merely transmitting knowledge would tend to use a content-based pedagogy, whereas those who conceived of teaching as facilitative tend to adopt a student-centred pedagogy. This study was designed to further examine such a link between what Chinese mathematics teachers do and say about what they do in order to promote their students’ learning, and figure out in what ways these links may be similar or different than findings about teachers in the US.

Numerous studies of teachers’ conceptions of teaching have been conducted (Boulton-Lewis, Smith, McCrindle, Burnett, & Campbell, 2001; Gao & Watkins, 2001; Kember, 1998; Kember & Kwan, 2000; Martin & Balla, 1991; Pratt, 1992; Prosser, Trigwell, & Taylor, 1994;
Samuelowicz & Bain, 1992; Tang, 1999). Table 2.1 presents a comparison of some of these studies. A recent summary of those studies, which provides the basis for Table 2.1, was conducted by Eley (2006), who categorised conceptions of teaching on a continuum ranging from teaching as transmission of information through teaching as facilitating student construction of mathematics. This continuum is consistent with most of the studies summarised in Table 2.1.

2.4 Importance of Linking Teachers’ Thinking to their Classroom Practice

In mathematics classrooms, teachers and students are the key participants in teaching-learning processes. Ball, Lubiensky, and Mewborn (2001) noted that “what teachers and students are able (to) do together with mathematics in classrooms is at the heart of mathematics education” (p. 433). The classroom is the place intentionally designed to facilitate students’ learning. Brophy and Good (1986) and Hiebert (1999) pointed out that the relationships between classroom teaching and learning are quite complicated, but that teaching makes a difference in students’ learning.

Research on teaching can stimulate discussion of ways to improve students’ classroom learning opportunities. Lim (2007) noted that before a lesson commences the mathematics teacher needs to plan it—select and decide what content to teach, and choose a teaching strategy that suits the students’ level of understanding. This is echoed in Tzur’s (2008) assertion that any mathematics teacher faces two fundamental questions—what and how to teach next, so students are motivated to persist in their pursuit of mathematics. Such decisions seem to be afforded and constrained by the mathematics teachers’ content knowledge (Ma, 1999), their teaching skills, and their philosophy, values and beliefs about mathematics and its teaching (Bishop, 1991; Ernest, 1989b; Thompson, 1992).
Table 2.1: A comparison of studies about conceptions of teaching

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<tbody>
<tr>
<td>Country</td>
<td>Australia</td>
<td>UK, Australia</td>
<td>Cross-culture</td>
<td>Australia</td>
<td>Australia</td>
<td>Hong Kong</td>
<td>Hong Kong</td>
<td>Australia</td>
<td>Mainland China</td>
</tr>
<tr>
<td>Participants</td>
<td>20 university teachers</td>
<td>13 academic teachers</td>
<td>Adult educators</td>
<td>24 university teachers</td>
<td>13 tertiary teachers</td>
<td>28 university lectures</td>
<td>17 university lecturers</td>
<td>16 secondary school teachers</td>
<td>450 Secondary school teachers</td>
</tr>
<tr>
<td>Presenting information</td>
<td>Imparting information</td>
<td>Engineering: delivering content</td>
<td>Transferring concepts in syllabus</td>
<td>Transmitting knowledge</td>
<td>Transmitting knowledge</td>
<td>Passing information</td>
<td>Content/skills transmission</td>
<td>Knowledge delivery</td>
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<tr>
<td>Transmitting information</td>
<td>Transmission of knowledge and attitudes</td>
<td>Help students acquire concepts of the syllabus</td>
<td>Transferring teacher’s knowledge</td>
<td>Transferring structured knowledge</td>
<td>Preparing students for assessment</td>
<td>Exam preparation</td>
<td></td>
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<tr>
<td>Teacher-Centred (content-oriented)</td>
<td>Apprenticeship: modeling</td>
<td>Illustrating application of theory to practice</td>
<td>Demonstrating application</td>
<td>Skills/understanding</td>
<td></td>
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<tr>
<td>Student-teacher interaction (apprenticeship)</td>
<td>Developing concepts and principles through interaction with students</td>
<td>Facilitate student understanding</td>
<td>Student-teacher interaction</td>
<td>Facilitation of understanding</td>
<td></td>
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<tr>
<td>Teaching</td>
<td>Developing the capacity in students to be experts</td>
<td>Developmental: cultivating the intellect</td>
<td>Help students develop conceptions</td>
<td>Intellectual development</td>
<td>Helping learners’ understanding</td>
<td>Facilitating students to become independent learner</td>
<td>Ability development</td>
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<tr>
<td>Student-Centred (learning oriented)</td>
<td>Bring about conceptual change in students</td>
<td>Change students’ conceptions of the world</td>
<td>Help students change conceptions</td>
<td>Conceptual change</td>
<td>Facilitate change in perspectives or attitudes</td>
<td>Transformation of students</td>
<td>Attitude promotion</td>
<td></td>
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</table>

Nurture personal agency
Social reform
The TIMSS (1995) and TIMSS-R (1999) showed that teachers and their teaching practices were important factors correlated with students’ achievement. Within reform-oriented mathematics teaching and learning environments advocated worldwide, some countries have put forward curriculum reform documents, or ‘Standards’, to set norms for mathematics teachers’ desired classroom practices. The mathematics curricula of various educational regions in the world were reformed at the turn of the Millennium (Wong, Han, & Lee, 2006). For example, in Australia the National Statement on Mathematics for Australian Schools was published in 1990. In the US, the first set of the National Council of Teachers of Mathematics (NCTM) Standards was published in 1989. It was followed by the Teaching Standards (NCTM, 1991), Assessment Standards (1995), and a recently revised, comprehensive document of Principles and Standards (NCTM, 2000), which integrated all three documents. Mainland China has also undergone a curriculum reform toward the end of the 20th century (BECMDC, 2001, see below).

The Australian Association of Mathematics Teachers (AAMT, 2006) Standards for excellence in teaching mathematics in Australian schools are organised into three domains: professional knowledge, professional attributes, and professional practice. The AAMT standards constituted a national consensus for the profession by articulating the knowledge, skills, and attributes required for good teaching of mathematics. Considering, for example, the professional practice domain, the AAMT Standards stated that a positive impact on learning outcomes can be “reflected in the learning environments they [teachers] establish, the lessons they plan, their uses of technologies and other resources, their teaching practices, and the ways in which they assess and report on student learning” (p. 1). Professional knowledge was further depicted as “knowledge of students; knowledge of mathematics; and knowledge of students’ learning mathematics” (p. 2). The AAMT (2006) also noted that “effective schools are only effective to the extent that they have effective teachers” (p. 4). Knowing about reform documents and recommendations in the West helped to guide the researcher in framing the problem and questions of this study, and most importantly in noticing aspects of participating teachers’ work.
(e.g., do they use real-world problems) and formulating interview questions about specific ways
they taught new mathematical ideas to their students and/or interpreted students’ understanding
(e.g., asking a teacher to explain the sequencing of problems and the role that engaging students’
in small group work had supposedly served).

In China, the National Curriculum in mathematics, developed by the Basic Education
Curriculum Material Development Centre (BECMDC, 2001), was revised to include six new
national mathematics curriculum standards. These underlying principles in the development of
mathematics curricular materials are: Curriculum, Content, Learning, Student Activities,
Assessment, and Information Technology. The overall objectives of the mathematics curriculum
were spelled out in the dimensions of knowledge and skills, mathematical thinking, problem
solving, and affect and attitude (BECMDC, 2001, pp. 6-7, cited in Wong, Han, & Lee, 2006).

Whereas professional standards and curriculum principles are important, they do not, in
themselves, determine how teaching actually proceeds (Ball, 1996; Lampert & Ball, 1999).
Rather, many factors affect curriculum implementation. The most important factor seems to be
the mathematics teacher, particularly the role he or she plays in the classroom (Clarke, Clarke, &
Sullivan, 1996; Ma, Zhao, & Tuo, 2006). Ma et al. (2006) and McCaffrey et al. (2001) further
argued that teachers seldom implement a curriculum exactly as stated in curriculum documents;
rather they would decide what to teach and how to teach based on what transpires in their
classrooms, and those decisions are affected by their knowledge, beliefs, and school cultures, as
well as constraints teachers face or perceive (Herbst, 2008).

One aspect of how teachers drive classroom learning is the tasks (problem situations) they plan
and implement to nurture student learning (Ainley, Pratt, & Hansen, 2006; Liljedahl, Chernoff,
& Zazkis, 2007; Simon & Tzur, 2004; Watson & Sullivan, 2008; Zaslavsky, 2008). At issue is
that teachers’ implementation of tasks may change the intellectual demands required for solving
those tasks (Stein, Grover, & Henningsen, 1996; Stein & Lane, 1996). Based on his collaboration with Sullivan and Zaslavsky, Tzur (2008) further pointed out three characteristic ways in which teachers modify tasks generated and provided by mathematics teacher educators. First, during planning, a teacher might conclude that a designed task is not likely to accomplish the intended goal for student learning, and before the lesson she or he would adjust it to fit one’s own expectation of how student learning would unfold. Second, a teacher might begin a lesson with the intended task, but as the lesson continues she or he interprets students’ work on the task as lack of progress and changes the task for what the teacher considers a more suitable one. Third, a teacher might plan and implement a task assuming that one’s teaching corresponds with the mathematics educator’s expectations, whereas actually it does not. Accordingly, Tzur noted three plausible sources that affect teacher task modifications: (a) teachers’ conceptions of the mathematics learning goal that the tasks are designed to promote; (b) teachers’ facility with using those tasks as a pedagogical tool; and (c) teachers’ implicit and explicit epistemological stance about how students come to understand a mathematical concept they are yet to learn, and the role the task may play in this learning process. In this study, by interviewing teachers before the lesson to understand their plan and then observing their class, data collection and analysis allowed a focus on changes in tasks, which were addressed by the teacher in responses to the researcher’s questions during the follow-up interview (a few examples of such changes and teachers’ rationale for them are presented in Chapter 4).

In summary, the literature reviewed above suggests that teachers’ practices are rooted in their conceptions and beliefs, which are context- and cultural-dependent (Bishop, 1994). Recent mathematics education efforts have focused much attention on explaining the links between teachers’ ways of thinking about and implementing mathematics teaching in their classroom practice (Even & Ball, 2009; Philipp, 2007; Sullivan & Wood, 2008). This dissertation study has both theoretical and practical significance in that it can further examine those links. A unique feature of this examination is that it does not intend to compare teaching practices in two
different countries. Rather the comparison across cultures is accomplished via the use of a Western conceptual lens applied to the collection and analysis of data about the participating Chinese teachers’ practices and perspectives. The following discussion explains this framework.

2.5 Conceptual framework
This study was designed to investigate the participating Chinese teachers’ practices and views of the nature of mathematics, its teaching, and learning as manifested in their practice. The researcher adopted a constructivist view of knowing and learning, which along with her Western educational experience (both as student and teacher) provided the cross-cultural lens for this study. The research problem and questions of this study focused on ways in which Chinese teachers carry out and think about their mathematics teaching. This sub-section first provides background about constructivist stances on learning and implications for teaching. Then, constructivist accounts of teaching that informed this study are discussed (Ernest, 1989a; 1989b, 1989c; Heinz et al., 2000; Simon et al., 2000; Tzur et al., 2001).

2.5.1 Constructivism and its Implications for Teaching
This sub-section describes the constructivist conceptual framework that guided this study. It begins with background about this theory of learning, including discussion of its basic principles. Then, this sub-section depicts constructivist-informed teaching approaches. Finally, constructivist characterisations of teaching and teacher perspectives, which could serve as backdrop for studying the participating Chinese teachers’ perspectives, are described. This conceptual framework was important to this study in that it helped to identify aspects of the participating teachers’ practices and thinking that could help to explain the impact on students’ success (e.g., reactivating students’ available knowledge early in a lesson).

2.5.1.1 Background on Constructivism
Constructivism is a theory of learning or meaning making (Confrey & Kazak, 2006; von Glasersfeld, 1995b). It builds on the pioneering work of Piaget (1971, 1985, 1970, 1980) and to a
lesser extent also on Dewey (Dewey, 1902, 1933; Dewey & Bentley, 1949). It addresses how an individual constructs her or his understandings as a transformation in what is already known into novel ideas (Beswick, 2007). Constructivism does not address teaching directly and it is conceived of as a general perspective on learning rather than a pedagogical technique (Simon, 1995a; Steffe, 1990a, 1990b). Furthermore, constructivism provides a framework that supports a range of pedagogical practices consistent with its principles (Beswick, 2007; Cobb, 1994). According to von Glasersfeld (1990), learning is dependent on extant knowledge that has been constructed in the context of prior experiences.

Proponents of the constructivist theory find it valuable not simply because it makes learning engaging for students, but because it can facilitate teachers in fostering learning and retention better than other theories (Abbott & Fouts, 2003; Conley, 1993). Constructivism has evolved to include quite a range of views, including social constructivism and radical constructivism (Kelowna, 2006). The metaphor of construction aptly summarises the position that knowledge is built in the minds of learners through transformation of what they bring to the learning situation—their own personal backgrounds, experiences and aptitudes (Tobin, 1990). From a constructivist point of view, because knowing is considered to be filtered and structured through one’s experiences, it tells us not about the world as it is but about our organisation of our experiences in the world. Learning is thus an adaptive process that organises an individual’s experiential world (von Glasersfeld, 1995b); it is not a process of discovering an independent, pre-existing world outside the mind of the knower (Matthews, 1992).

Opponents of constructivism, such as Richardson (2003), argued that there are still many unanswered questions in the constructivist theory literature. For example, the questions of how knowledge is created in the mind, what constitutes ‘the social experience’, or how does subject matter affect the way in which it is constructed in practice remain open. He further argued that if pedagogical understandings rooted in constructivism are to develop further, issues like student
learning, the elements of effective constructivist teaching, teachers’ subject-matter knowledge, and cultural differences require considerable attention. Likewise, Keeves (2002) claimed that “constructivism is both incomplete and inadequate for the effective learning and teaching of mathematics and science at the upper secondary school level” (p. 114). A common criticism of constructivism is that it seems to imply that anyone’s construction of the world is as viable as another’s and therefore the world only exists as constructed in the mind of the knower—a view known as solipsism (Duit, 1995). In contrast, von Glasersfeld (1995b) asserted that knowledge must not only be viable personally, but also inherent in the social and physical contexts in which actions may occur, and is constantly held in check against reality constraints, including the views of other people in one’s social-cultural group. This important feature of constructivism was helpful in making sense of how teachers who participated in this study continually engaged students in comparing one’s and others’ solutions to problems. Such comparison seemed to be at the heart of the teachers’ practice (see Section 4.3 about Teaching with Variation), and to play a prominent role in teachers’ explanations of how students abstract a method that remains the same across these comparisons.

The very notion of ‘constructivist teaching’ is a controversial issue. Various types of constructivism, and pedagogical approaches that they orient, have emerged, including radical, social, physical, evolutionary, post-modern, and information-processing and cybernetic systems views of constructivism (Prawat, 1996; Steffe & Gale, 1995). Derry (1992) pointed out that constructivism has been claimed by “various epistemological camps” that do not consider each other “theoretical comrades” (p. 415). Despite of the disagreements, many educators and researchers appear to have come to an agreement about how a constructivist epistemology should affect educational practice and learning, and there seem to be general agreement on the roles of teachers and learners. The following sub-section reviews basic ideas of constructivism about learning and its implications for teaching.
2.5.1.2 Basic Principles of Constructivism

Constructivist theory stresses that knowledge cannot be separated from the way one came to know it (von Glasersfeld, 1995b). In his view, this theory of learning consists of two basic principles, one psychological and the other epistemological (Treagust, Duit, & Fraser, 1996; Tzur, 2010). von Glasersfeld’s (1990) stated the first principle as follows: “knowledge is not passively received either through the senses or by way of communication. Knowledge is actively built up by the cognising subject” (p. 22). This principle implies that a teacher cannot directly put ideas into students’ heads; rather, while being engaged in teaching-learning processes each student actively constructs her or his own meanings for the mathematics being taught. The second (‘radical’) principle states that the function of cognition is adaptive; it serves the organisation of the experiential world—not the discovery of an ontological (‘objective’) reality (von Glasersfeld, 1995b). In this sense, absolute, objective truth cannot be found; humans can only construct viable explanations of their experiences (von Glasersfeld, 1990).

Accordingly, von Glasersfeld (1996a) asserted that there are two main educational implications of constructivism. First, learning is a process of knowledge construction rather than of absorption. Duffy and Jonassen (1992) and Fosnot (1996) agreed with this assertion and emphasised that because humans’ knowledge is constructed based on their own perceptions and conceptions of the world, each person may construct slightly different meanings for mathematical concepts. von Glasersfeld (1995a) further argued that “from a constructivist’s view, learning is not a stimulus-response phenomenon. It requires self-regulation and the building of conceptual structures through reflection and abstraction” (p. 14). Learning occurs only when learners are actively involved in the (mental) construction and reorganisation of concepts. The implication seems straightforward—concepts cannot be transmitted directly from a teacher to a learner by means of words or other observable actions (von Glasersfeld, 1995a, 1995b). For educators, the challenge is to be able to build a hypothetical model of the conceptual world of students because that world
This leads to the second implication for teaching, which draws on the epistemological (assimilation) principle—what students can perceive and conceive of the teacher’s intended learning is afforded and constrained by the student’s available (assimilatory) conceptions. It requires teachers to understand that the mathematics students ‘see’ in the world may be qualitatively different from the teachers’ understanding (Tzur, 2010). Steffe (1995) referred to this distinction as first order model (one’s own mathematics, be it student or teacher) and second-order model (e.g., a teacher’s model of students’ mathematical models). The implication is also straightforward, but much harder to accomplish with teachers: a teacher who adheres to the constructivist epistemological principle of assimilation should teach mathematics in such a way that always reactivates and fosters transformation in students’ assimilatory conceptions. That is, a student’s learning, and therefore teaching, can only start and proceed from where students are conceptually (Simon, 1995a; Tzur, 2008). This principle served as a major guiding lens in the researcher’s noticing of specific features of the participating teachers’ practices (e.g., engaging students in solving problems that they learned in the primary school) and inquiring into those practices (e.g., asking a teacher why such problems were used in the order they were used).

The second educational implication of constructivism is that knowledge is highly related to the contexts in which a learner experiences and constructs it (Duffy & Jonassen, 1992; von Glasersfeld, 1996b). This implies that cognitive experience and change better take place in authentic activities for the students, which brings constructivist theory close to the situated learning perspective (Boaler, 2000; Lave & Wenger, 1991; Noss & Hoyles, 1996). Greeno (1997) stated that situated learning “takes the theory of social and ecological interaction as its basis [and emphasises the] information structures in the contents of people’s interactions” (p. 5). Whereas as a social-cultural perspective that follows Vygotsky’s (1978) work situated learning
emphasises social interaction, constructivist theory emphasises interaction within one’s physical and social environments, which is necessarily ‘filtered’ through her or his assimilatory conceptions (Confrey & Kazak, 2006).

In a constructivist view, building understanding requires that learners have the opportunities to articulate their ideas, to test those ideas through experimentation and conversation, and to consider connections between the phenomena that they are examining and other applications of the concepts (Dykstra, 1996; Julyan & Duckworth, 1996, cited in Chen (2003)). The opportunity for learners to discuss and clarify their experiences is essential, because it encourages self-organisation and reflective abstraction. Fosnot (1996), Simon et al. (2004), Steffe (1990a, 1990b), and Thompson (1985) suggested that reflective abstraction is the driving force of meaningful learning of mathematics. Accordingly, a constructivist-informed classroom is seen as a community of cognising members engaged in (inter)activity, reflection, and constant conversation. The researcher applied this stance in this study in the sense that teachers’ work was not just described in terms of observable behaviours but also interpreted in terms of possible learning opportunities such behaviours might have created for students’ abstraction of the concepts taught.

2.5.1.3 Constructivist-Informed Teaching
In von Glasersfeld’s (1995b) radical constructivist conception of learning, the teacher plays the role of “midwife in the birth of understanding [as opposed to a] mechanics of knowledge transfer” (p. 15). The teacher’s role is not to dispense knowledge, but to provide students with opportunities and incentives to build it on the basis of, and transformation in, what they already came to know (von Glasersfeld, 1996b). In particular, a teacher should perceive of students’ errors not as a misconception, but as genuine indication of the learner’s conceptions at a given time, what at that moment makes sense to the learner (Fosnot, 1996; von Glasersfeld, 1995a). To modify what, from the teacher’s first order model of mathematics, appears as students’
misconceptions the teacher needs to infer an explanation of how the student might arrive at her or his answers. Using such inferences (a teacher’s second order model of students’ mathematics), the teacher asks questions or provides a different presentation to allow the students to notice their errors and construct a concept that’s more compatible with the teacher’s (von Glasersfeld, 1995a, 1995b).

Because teaching oriented by a constructivist theory begins from what students already know, students and their experiences are placed at the centre of the learning process. Further, students are not expected to simply memorise the content and repeat it on tests and assignments, but must be actively involved and take responsibility for their own learning (Perkins, 1992). Pirie and Kieren (1992) proposed a set of principles for mathematics teachers who want to create classroom environments consistent with constructivist principles: (1) focusing on students’ knowledge and experiences; (2) facilitating dialogues where shared understandings of the relevant mathematics can be negotiated, including a classroom norm that expects students to justify their ideas; and (3) purposely using tasks, raising questions, or presenting information to stimulate and foster students’ reflection on and construction of new understandings. Similarly, Brooks and Brooks (1999) proposed five guiding principles of constructivism for teaching. The first, in line with the Dutch Realistic Mathematics Education approach (Freudenthal, 1973; Gravemeijer, 1994), is to use realistic problems that are of relevance to students. The second is to structure learning around key concepts, which is consistent with the NCTM (2000) PSSM and with Simon’s (2006a) notion of Key Developmental Understandings. The third is seeking and valuing students’ points of view, which is consistent with Steffe’s (1995) notion of Second Order Model. The fourth is adapting curriculum to address students’ suppositions, which seems like a different way to say ‘assimilation’. The fifth is continually assessing student learning in the context of teaching (as opposed to by external tests outside that context), which is consistent with Tzur’s (2007) recent articulation of three levels of assessment. Again, it should be noted that, consistent with the core constructivist notion of assimilation, in this study the researcher
focused much of her attention on ways in which participating teachers seemed to identify and capitalise on students’ available knowledge. The following discussion elaborates and links two lines of constructivists’ works on teaching and teacher perspectives.

2.5.2 Constructivist Characterisations of Teaching and Teacher Perspectives
As part of the conceptual framework that guided this study about Chinese mathematics teaching, two lines of constructivists’ work, Ernest’s (1989b) and Simon and Tzur et al. (Heinz et al., 2000b; Simon et al., 2000; Tzur et al., 2001) were coordinated. The discussion below elaborates each of these two lines.

2.5.2.1 Ernest’s Characterisation of Mathematics Teaching
Ernest (1989a; 1989b, 1989c) proposed that teaching reforms cannot take place unless teachers change their deeply held beliefs about mathematics, its teaching and learning. He identified three main components of the knowledge systems of teachers. These include teachers’ views of the (a) nature of mathematics (subject), (b) features of mathematics teaching (pedagogy), and (c) process of learning mathematics (epistemology).

Knowledge Component 1: Subject Matter. According to Ernest, teachers’ views of the nature of mathematics are important because they heavily impact the other two components. Teachers’ conceptions of the nature of mathematics may be held implicitly or explicitly. Ernest distinguished three philosophies based on the observed occurrence in the teaching of mathematics, as well as in the philosophy of mathematics and science.

(a) Instrumentalist view: mathematics is like a bag of tools, an accumulation of facts, rules, and skills to be used in the pursuance of some external end.

(b) Platonist view: mathematics is a static but unified body of certain knowledge. Mathematics exists outside the knower and can thus be discovered (but is not a mental creation of the cognising individual).
Problem solving view: mathematics is a dynamic, continually expanding field of human creation and invention, a cultural product. It exists within and among humans as a process of enquiry and coming to know, not as a finished product as its results remain open to revision.

These distinctions were used in this study as a lens for comprehending the participating teachers’ view of mathematics and how they seem to inform their pedagogy (e.g., while all teachers were asked questions that could elicit such views, only one teacher explicitly stated a view that seemed consistent with the third type.

Knowledge Component 2: Pedagogy. According to different roles he identified teachers play in classrooms, Ernest (1989a; 1989b, 1989c) proposed three models of mathematics pedagogy that correspond to the three views of mathematics knowing: instructor, explainer, and facilitator. He specified these models through the teacher’s view of her role in accomplishing the intended outcomes of instruction. The instructor model focuses on students’ skill mastery and correct, fast performance. The explainer model focuses on students’ attainment of conceptual understandings of the unified body of mathematical knowledge. The facilitator model focuses on students’ confidence and competence in using their understandings for solving and posing problems. In this study, the researcher used this typology to examine participating teachers’ roles in the observed classrooms, and revealed a diversity that spanned all three (with individual deviations).

Knowledge Component 3: Epistemology. Ernest proposed two models of teachers’ views on how students’ learning of mathematics takes place:

(a) Learning as passive reception of knowledge by submissive and compliant learners;

(b) Learning as active construction of understanding through exploration, development of students’ autonomy, and pursuit of own interests.

In this study, the researcher paid particular attention to gaining insights into teachers’ epistemological views. As the data and analysis in Chapter 4 show, none of the participating
Chinese teachers seemed to consider learning as passive reception. However, their view of the active process by which students come to grasp new concepts did not fit within the notion of construction as transformation in available concepts. Using these two key views as backdrop, the researcher had to interpret the participating teachers’ views in a way that differs from both (which is discussed in Chapter 5). The three components (subject, pedagogy, and epistemology) of teachers’ knowledge identified by Ernest (1989a, 1989b, 1989c) were corroborated and extended in recent empirical studies that used a strategy of inquiry designed to link teaching practices with teachers’ rationales. The following discussion presents these studies and links them to Ernest’s work as the backbone component of the conceptual framework of this study when examining teachers’ perspectives.

**2.5.2.2 Perspectives that Underlie Teaching Practices**

In recent years, a few of studies (Heinz, et al., 2000b; Simon, et al., 2000; Tzur, et al., 2001) focused on how perspectives that underlie teachers’ practices may develop from traditional conceptions of mathematics, learning, and teaching, toward practices more consistent with constructivist principles and current reform efforts. The three characteristic perspectives they identified extend the work of Ernest by working from the bottom up, that is, from analysing empirically grounded data about teachers’ actual work in classrooms and how they think about their work—into a theoretical account of the perspective that may underlie the teachers’ practices.

These researchers investigated teachers’ development of reform-oriented practices by using a ‘teacher development experiment’ strategy of inquiry (Simon, 2000). This strategy of inquiry is rooted in a constructivist epistemological stance. It combines the use of whole-group teaching experiments in teacher education courses with case studies of individual mathematics teachers. In particular, these researchers elaborated on and used a qualitative strategy of inquiry they termed the ‘account of practice’ (Simon & Tzur, 1999). Through this strategy of inquiry (see next chapter), those researchers developed and postulated three distinct pedagogical perspectives.
that underlie mathematics teachers’ practices: traditional, perception-based, and conception-based perspectives.

A ‘traditional’ perspective of mathematics learning is marked by a teacher’s: (a) view of mathematics as independent of human experience; (b) a platonic view of coming to know; and (c) a passive view of learners (e.g., teacher lecturing, transmission of ideas to students, reading textbooks, and exercising many examples following teacher demonstrations). A ‘perception-based’ perspective (PBP) is marked by a teacher’s: (a) view of mathematics as part of the external world independent of human activity, in an external reality that is accessible to all through perception; (b) a discovery view of coming to know—mathematics is a meaningful, interconnected body that can be attained through first-hand experiences (i.e., perception); and (c) active (‘hands-on’) view of learning—students learn mathematics via actively perceiving of mathematical objects, principles, and relationships among them. A ‘conception-based’ perspective (constructivist, CBP) is marked by a teacher’s: (a) view of mathematics as being created through human activity and reflection, where mathematics knowing is inseparable from the knower; (b) a strong emphasis on the constraining and affording roles that an individual’s current knowledge (assimilatory conceptions) play in what they come to see and understand; and (c) an active view of the learner—mathematics learning is a building-up mental process of continually transforming one’s available (assimilatory) conceptions, that is, accommodation in one’s mental structures (Heinz, et al., 2000b; Simon, et al., 2000; Tzur, et al., 2001). Table 2.2 below summarises the three perspectives.
Table 2.2: Three perspectives identified by Simon et al.

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<th>Perspective</th>
<th>Psychology</th>
<th>Epistemology</th>
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<tr>
<td>Traditional</td>
<td>Passive learning</td>
<td>Mathematics exists independent of the learner</td>
</tr>
<tr>
<td>PBP</td>
<td>Active learning; no attention to existing knowledge</td>
<td>Mathematics exists independent of the learner</td>
</tr>
<tr>
<td>CBP</td>
<td>Active learning; Transforming existing structures/operations into new ones</td>
<td>Mathematics a learner knows depends on her/his assimilatory conceptions; Grasping the new entails activating and then transforming the known</td>
</tr>
</tbody>
</table>

A key point in the identification of the three perspectives was that common to both traditional show-and-tell perspectives and perception-based perspectives seems to be a view of mathematical knowledge as ‘out there’, independent of the knower. Not surprisingly, teachers whose practice is rooted in traditional approaches to mathematics teaching seem to attempt to transmit mathematical ideas to students. Teachers whose practice is rooted in perception-based perspectives seem to emphasise students’ active, first-hand experience of mathematical ideas and links among them. Teachers who adhere to a perception-based perspective do not seem to see their primary role as directly transmitting the intended ideas to students. Rather, their role is to create an environment conducive to students’ perception (‘discovery’) of those ideas via engaging them in hands-on activities that allow ‘seeing’ and connecting the intended ideas, and to explore or discover the already existing mathematics (Tzur et al., 2001).

In this sense, a perception-based perspective marks a noticeable transformation of a teacher’s traditional practice—from a passive to an active view of learning (e.g., using manipulatives, engendering students’ participation in small-group work, sharing and discussing solutions to meaningful problems, etc.). However, this change does not seem to be accompanied by an epistemological shift in a teacher’s view on the nature of mathematical knowing. That is, a perception-based teacher overlooks the issue of how, if mathematics does not exist independent
of the knower, may a student come to know it, which characterises conception-based perspectives (Tzur, 2008). Tzur et al. (2001) stressed that a conception-based perspective is rooted in such a shift, which centres on the principle of assimilation—human beings’ knowing has no access to a reality independent of how they experience it (von Glasersfeld, 1995b).

To formulate pedagogy that is rooted in a conception-based perspective, Tzur et al. (2001) and Heinz et al. (2000) elaborated on Simon’s (1995b) cyclical model of teaching. In such a pedagogy, to design lessons and activities that can promote students’ learning a teacher uses his or her knowledge of or about mathematics, teaching, and learning, and his or her hypotheses of students’ current (assimilatory) conceptions, including their possible responses to some particular activities. Simon (1995b) refers to the three components of the “teacher’s learning goal, teacher’s plan for learning activities, and teachers’ hypotheses of how students’ conceptions might develop” as a hypothetical learning trajectory (HLT)—“the teacher’s prediction as to the path by which learning might proceed” (p. 135). Simon’s teaching model is cyclic because the teacher implements the planned HLT while continually focusing on the students’ interpretations of the planned activities and collecting evidence of how they understand the mathematics. Through the continuous interaction with students, the teacher gains insights into students’ evolving conceptions. These insights evoke the teacher to modify the goals and activities for the next lesson, that is, for an adjustment of the HLT, and so on. Figure 2.1 illustrates Simon’s (1995b) mathematics teaching cycle.
In this study, the mathematics teaching cycle was used as a lens to guide, particularly, aspects of teachers’ plans and how they were maintained, or changed, during the observed lessons. When interviewed about an upcoming lesson, teachers were asked to specify (with examples) what was their goal for students’ learning, what teaching activities they planned to accomplish these goals, and how they anticipated the learning process to evolve for students at different levels. Based on teachers’ specific responses to those questions, further probing targeted teachers’ mathematical and pedagogical knowledge that supported their plans. And, during a post-observation interview, the researcher focused much of her questions on obtaining teachers’ understanding of their students’ understanding before and after the lesson. Simply put, interweaving the observations and interviews provided rich data about the entire cycle.

In regards to this cycle, Tzur et al. (2001) identified a tension in the role the teacher’s own mathematics may play. On one hand, teachers must use their own mathematics knowledge (first order model) as an apparatus to set learning goals for their students. On the other hand, to make
sense of students’ mathematical ideas a teacher also has to learn to ‘bracket’ her or his own mathematics in order to infer into students’ mathematics on the basis of their observable actions and language. Tzur et al. (2001) contended that “the perspective of teaching as a reflection-interaction cycle implies that teachers’ knowledge—conceptions of mathematics, perspectives on mathematical activity and representations, and teaching-learning processes of particular mathematical content—is constantly changing” (p. 231). Such changes in teachers’ knowledge were frequently noticed, documented, and analysed in this study (e.g., a novice teacher with a master’s degree in mathematics seemed to learn about the differences between his and students’ understanding as he was reflecting on the lesson during a follow-up interview).

Along with a focus on teacher’s mathematical understandings, in a constructivist-informed model of teaching, a great emphasis is placed on a teacher’s understanding of students’ assimilatory conceptions, because these conceptions afford and constrain what students experience, see, and learn (Simon, 1995b). This is consistent with von Glasersfeld’s (1996b) and Steffe’s (1995) notion of building a hypothetical (second order) model of the conceptual world of students. Within such a perspective, the teacher assumes that students’ worlds, mathematics included, may be different from what he or she intends for them to learn, because they construct knowledge based on their perceptions and conceptions of the world. A teacher’s goal is to foster students’ knowing of the mathematics she or he knows, while being acutely aware that accomplishing such a goal requires starting from students’ current conceptions.

To complement a constructivist framework for explaining learning with a corresponding pedagogical approach, Tzur (2008) proposed a 7-step cycle (see Figure 2.2) that elaborated on Simon’s (1995a) and Simon and Tzur’s (2004) pedagogical approaches. As one would expect of a pedagogy rooted in the constructivist principle of assimilation, the 7-step cycle begins with specifying students’ available conceptions. A teacher identifies in students’ conceptions a goal they may set, a mental activity they can carry out, and outcomes they may notice and relate anew
to the activity. A similar analysis is conducted to the intended mathematics. Through relating the analysis of students’ conceptions and the intended mathematics (goal for students’ learning), the teacher then selects, designs and implements tasks that (a) reactivate students’ available conceptions and (b) orient their noticing of and reflection on new, intended outcomes. As students work on the tasks and respond to follow-up prompts/questions, the teacher actively monitors their progress and renegotiates the tasks to orient their reflection. This monitoring provides the basis of a new cycle for the next lesson, and so on. This 7-step cycle served the researcher in comparing and contrasting the participating teachers’ practices and rationale with each of the steps (e.g., how and why teachers selected a few problems from primary school to engage students in and reactivate their current conceptions as a means to grasp the new concepts). This 7-step cycle is presented here to illustrate the aforementioned steps; the discussion in Chapter 5 returns to this cycle and links it with the results of this study.

Figure 2.2: Tzur’s (2008) 7-step cycle.
2.5.2.3 Coordinating the Two Lines of Work

The two constructivist-informed lines of scholarly work about teaching, Ernest’s (1989a, 1989b, 1989c), and Simon and Tzur et al.’s (2000), seemed consistent with one another (see Table 2.3). Both lines seemed to similarly characterise teachers’ views in terms of the teacher’s stance on mathematical knowledge as it links to the process of learning and corresponding teacher roles. Ernest’s (1989a, 1989b) two categories of Instrumentalist and Platonist views of knowledge seem to be rooted in an epistemological stance that is common to both the traditional and perception-based perspectives (mathematics exists ‘out there’ independent of the knower). Ernest’s category of Problem Solving, with its emphasis on knowledge as dynamic, inseparable from the knower, and created through human activity, seems consistent with the principle of assimilation and learning as transformation in existing conceptions. Accordingly, both lines of work distinguished similar teachers’ views of learners (passive vs. active) and of the ways one comes to know something that she or he did not yet know (absorption of information, exercising hands-on examples of concepts, and changing one’s understanding via the process of problem solving). Ernest’s three teacher types (instructor, explainer, and facilitator) seemed consistent with the implied traditional, perception-based, and conception-based perspectives (respectively).

Table 2.3: A coordinated approach to mathematics teaching.

<table>
<thead>
<tr>
<th>View of knowledge</th>
<th>View of learning</th>
<th>View of teaching</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Traditional perspective</strong></td>
<td><strong>Perception-based perspective</strong></td>
<td><strong>Conception-based perspective</strong></td>
</tr>
<tr>
<td>Independent of the knower; out there; unrelated facts and rules.</td>
<td>Independent of the knower; out there; unified; static; discovered;</td>
<td>Dynamic; not a product; Depends on the knower; assimilation</td>
</tr>
<tr>
<td>Passive recipient</td>
<td>Active discovery and understanding of math</td>
<td>Active construction of mathematical understandings</td>
</tr>
<tr>
<td>Transmission; lecturing; ‘instructor’</td>
<td>Teacher as ‘pointer’ or ‘explainer’</td>
<td>Engage in tasks + orient reflection; ‘facilitator’</td>
</tr>
</tbody>
</table>

An advantage that this integrated lens has if used as a conceptual framework in this study is the organisation of an overarching system that coordinates practices and perspectives. In particular, it synthesises the three components proposed by Ernest (seen in the column titles) into a characteristic perspective consisting of all three components (seen across horizontal lines). The
study capitalised on this advantage by helping the researcher to stay focused and comprehensive in noticing specific behaviours during the lessons and elaborating on them so that all three components and links among them were addressed by the teacher, in specific details, during the interviews. For example, she would ask a teacher who used problems from the primary school what was the purpose of such a pedagogical move, and probe further into how reactivating this ‘old’ knowledge (the teacher’s rationale) could help students learn the ‘new’ knowledge during the observed lesson.

A disadvantage of this integrated lens is that it may convey a mutually exclusive, complete set of practices/perspectives. However, boundaries among the different perspectives may blur and not be easy to determine from the data, and additional perspectives may be underlying mathematics teachers’ practices. In fact, from the data about the participating Chinese teachers’ work in this study the researcher distinguished such an additional perspective, which did not seem to fit within any of the three in this framework (it is discussed in Chapter 5). That is, while this framework provided a starting point for this study, the researcher did not attempt to ‘force’ the data into these categories. Rather, she proceeded in the analysis from the empirically grounded data into making sense of these data while focusing on key features of the teachers’ apparent rationale and finally organising these features into a coherent perspective.

2.6 Summary of Literature Review
The discussion above depicted the background literature about plausible relationships between mathematics teachers’ practices and the perspectives that may underlie those practices. This discussion emphasised the key role that cultural beliefs about learning play in teachers’ perspectives, which in turn impact their work. In particular, it highlighted important differences between CHC and Western cultures, both of which constitute the researcher’s experience and perspective. The perspective about learning and teaching that the researcher adopted—constructivism—was described and explained in terms of how learning proceeds and teaching
practices it seems to imply. In particular, two lines of work (Ernest, Simon et al.) about teaching, informed by constructivist views, were considered and coordinated to provide the backbone for this study. The next chapter discusses the research design and methods of this study.
Chapter 3:

Research Design and Methods

This study was designed to examine Chinese middle school mathematics teachers’ pedagogical practices and gain insight into their underlying perspectives by probing into teachers’ reasons and meanings for what they do. As explained in the previous two chapters, it is assumed that how a teacher goes about teaching a particular piece of mathematics makes sense to the teacher—and this study was set out to articulate that sense. Because the focus was on linking descriptions of teaching processes with inferences into teachers’ rationale, qualitative research design was chosen (Bogdan & Biklen, 1992; Creswell, 2005).

The particular qualitative strategy of inquiry (Denzin & Lincoln, 1994) used for this study was Simon and Tzur’s (1999) *Account of Practice* (AoP). This strategy of inquiry is described later in this chapter. Two key reasons underlie choosing the AoP strategy of inquiry. First, it explicitly focuses on linking a teacher’s practice with her or his perspective on (a) the nature of mathematical knowledge and (b) the process of learning. The researcher assumed that using the AoP could enable identification of practices, perspectives, and links among them in Chinese middle school mathematics teachers’ work. Second, the AoP strategy of inquiry grew out of collaboration between two mathematics educators by combining their expertise in research on students’ conceptual learning (Tzur, 1999, 2000, 2004) with research on mathematics teaching and teacher development (Heinz et al., 2000; Simon, 1995; Simon & Blume, 1994; Tzur et al., 2001). Thus, the AoP is a qualitative strategy of inquiry developed particularly for studying teaching that assumes the content—mathematics—matters. It centres on the unfolding of mathematical thinking of teacher and students as they engage in interactive work on specific tasks during mathematics lessons, and on the teacher’s rationale for planning and implementing (and changing) those specific tasks.
To situate and explain the AoP strategy of inquiry and its suitability for this study, this chapter begins with a description of qualitative research. Then, the theoretical underpinnings of AoP are discussed, by situating this strategy of inquiry in the overlap between phenomenology-phenomenography (Bogdan & Biklen, 1992; Groenewald, 2004) and grounded theory (Glaser & Strauss, 1967), as illustrated in Figure 3.1. Then, the AoP strategy of inquiry is described, followed by presentation of the study participants (including sampling criteria for schools and teachers), data collection procedures (video recorded interviews and classroom observations; field notes) and the reliability, validity, and ethical issues associated with those procedures. Data analysis processes are described next, including issues of translation accuracy, followed by a more general discussion of the reliability and validity of qualitative research and their implications for this study. The following sub-section introduces key aspects of qualitative research.

![Figure 3.1: Theoretical underpinnings of the Account of Practice strategy of inquiry.](image)

### 3.1 Qualitative Research

Qualitative research differs from quantitative research in both goals and means. Quantitative studies attempt to identify and quantify relationships among variables that explain and predict quantifiable aspects of a phenomenon under study. To this end, researchers measure and analyse...
variables, and relationships among them, by using mathematical and statistical models. Qualitative research studies, on the other hand, focus on processes that are not or cannot be measured, particularly on meanings that people have for aspects of their lived experiences. As Straus and Corbin (1990) asserted, qualitative research is "any kind of research that produces findings not arrived at by means of statistical procedures or other means of quantification" (p. 17).

Denzin and Lincoln (2000) defined qualitative research as systematic inquiries “involving an interpretive, naturalistic approach to its subject matter” (p. 2). Qualitative researchers observe, listen to, and document a phenomenon in natural settings while attempting to understand and interpret the meanings and relevance it has for people in the research settings. Qualitative researchers therefore employ multiple methods of data collection, such as “case study, personal experience, introspective, life story, interview, observational, historical, interactional and visual texts” (p. 2) as a means to provide rich descriptions of how research subjects perceive the phenomenon under study (Denzin & Lincoln, 1994). Using and coordinating multiple methods and data sources, known as triangulation, provides an in-depth understanding of the phenomenon while increasing the study’s “rigor, breadth, and depth” (p. 2).

Bogdan and Biklen (1992) proposed a similar definition of qualitative research, which focuses on the naturalistic approach to humans’ life based on contexts and situations in which the examined phenomenon takes place. Similarly, Patton (2002) emphasised that qualitative research seeks to understand a phenomenon in the “real world setting [where] the researcher does not attempt to manipulate the phenomenon of interest” (p. 39). Instead, findings are arrived at through collecting and analysing data in real-world settings in which the phenomenon of interest unfolds. Bogdan and Biklen (1992) added that such data collection and analysis is “soft,” in that it is “rich in description of people, places, and conversations, and not easily handled by statistical procedures” (p. 2).
Bogdan and Biklen (1992) emphasised two additional features of qualitative research—the researcher’s role and the process of analysis. Unlike quantitative studies, in which objectivity is assumed and sought, qualitative studies assume that description and interpretation of a phenomenon are framed by the researcher’s perspective. Thus, in a qualitative study “the researcher is the key instrument” (p. 29) and the process of data analysis is inductive, that is, “things are open at the beginning and more directed and specific at the bottom” (p. 31). In addition to the techniques mentioned by Denzin and Lincoln (2000), Bogdan and Biklen (1992) emphasised the use of observation, in-depth interviews, and field notes as three major types of qualitative methods particularly suitable for qualitative research in education. These methods constitute the bulk of data collection in the AoP strategy of inquiry.

Several major approaches have been developed under the wide ‘umbrella’ of qualitative research designs. These approaches include ethnography, narrative research and hermeneutics, grounded theory, phenomenology, phenomenography and action research (Creswell, 2005). Bogdan and Biklen (1992) noted that these approaches are rooted in and reflect different theoretical underpinnings. They suggested that although the use of the term phenomenology is itself debated, “most qualitative researchers reflect some sort of a phenomenological perspective” (p. 33). This is also the case of this study, as the AoP strategy of inquiry draws on (but is not limited to) phenomenology and phenomenography. The next sub-section presents the theoretical underpinnings of this qualitative study of Chinese mathematics teachers’ practices and perspectives (meanings, rationale) interpreted to underlie those teachers’ practices.

3.2 Theoretical Underpinnings of this Study

Two major qualitative approaches that underpin the particular (AoP) strategy of inquiry used for this study are phenomenology-phenomenography and grounded theory. As the following discussion of each approach reveals, phenomenology focuses on providing rich descriptions of the phenomenon under study, whereas grounded theory focuses on creating accounts that explain...
the phenomenon by conceptualising categories and themes out of empirical data. The purpose of this study was to accomplish both—describe participating Chinese middle school mathematics teachers’ practices and theorise about their teaching in terms of inferred teacher perspectives that underlie it. In this study, aspects of the teachers’ practices, such as teacher and student activities, lesson plans (problem sequences) and teacher-students interactions when plans were implemented, and components that teachers used to organise their lessons—were linked with insights into how participating teachers explained why those plans were created and what the implementation intended to accomplish. The AoP strategy of inquiry combines these two purposes and thus seems to lie in the overlap between phenomenology-phenomenography and grounded theory (as depicted in Figure 3.1 above). Accordingly, the following sub-sections introduce key aspects of phenomenology and phenomenography and then of grounded theory.

3.2.1 Phenomenology and Phenomenography
This section compares and contrasts Phenomenology and Phenomenography as methodologies.

3.2.1.1 Phenomenology
Phenomenology is a qualitative research approach designed to study the experience of people from the first-person point of view (Bogdan & Biklen, 1992; Groenewald, 2004; Stanford Encyclopedia of Philosophy, 2008). Its roots are traced to the work of Edmund Husserl in his logical investigations in the early 1900s, which were developed into a movement by Heidegger, Sartre, Merleau-Ponty, Gadamer and Schütz (Bengtsson, 1993, cited in Hasselgren & Beach, 1997). According to Groenewald (2004), the aim of the phenomenologist is to describe the phenomenon as accurately as possible while remaining true to observable facts. What phenomenologists emphasise and attempt to describe are subjective aspects of people’s behaviour (Bodgan & Biklen, 1992). They further argued that the goal of a phenomenologist is to understand the research subjects from their own point of view. Thus, a phenomenologist
conceives of the phenomenon under study as being socially constructed by both the research subjects and the researcher (Berger & Luckmann, 1967).

Researchers who adopt phenomenological approaches set out to study the subjective appearance of the phenomenon as opposed to the objective reality (Stanford Encyclopedia of Philosophy, 2008). Phenomenologists do not assume they know what specific things mean to the people they are studying. Rather, they approach these people’s lives from a silent stance in order to grasp the meanings to those they are studying (Bogdan & Biklen, 1992). To capture and provide rich description of the phenomenon within a given research setting, a phenomenological study needs to be organised so that data emerge and support descriptions of participants’ subjective realities (Kensit, 2000). In this study, the researcher attempted to approach aspects of the participating teachers’ professional lives in a silent mode, via observing their lessons without interacting with the teacher or the students. A major way of systematically obtaining such participants’ subjective descriptions has been provided by phenomenography, which is discussed next.

**3.2.1.2 Phenomenography**

According to Hasselgren and Beach (1997), the term phenomenography has its etymological roots in the Greek words phainomenon (appearance) and graphein (description). Combined, the two words mean a description of appearances. Richardson (1999) noted that the earliest research considered as phenomenographic was an investigation carried out by Marton and his colleagues at the University of Göteborg in Sweden. Their work was concerned with qualitative differences among individual students’ perceptions of the outcome and processes of learning. These researchers’ purpose was to try to see the world from the student’s perspective, and they conceptualised mental processes such as perceiving, apprehending, and understanding as referring to a person’s experience (Ashworth & Lucas, 1998).
Marton (1976, 1981, 1986, 1988, 1994) defined phenomenography as an empirically based approach, an inquiry orientation aiming at description, analysis, and understanding of various aspects of different people’s experiences. That is, phenomenography is directed toward experiential description of research participants’ conceptions. Marton (1976) asserted that his approach intends to describe the world from the insider’s (research subject’s) perspective, which means he sought to describe the world as a student experienced and perceived it. For phenomenographists, the focus is on experience-as-described, rather than on observable facts per se (Ashworth & Lucas, 1998; Marton, 1981). The rationale for such an approach seemed to be that the more faithful researchers are to subjects’ experience, the better we are able to understand individuals’ conceptions of learning, teaching, and other kinds of human action (Sandberg, 1997).

The outcomes of phenomenographic research are categories of subjects’ descriptions (Ashworth & Lucas, 1998; Marton, 1981; Marton & Pong, 2005). Sandberg (1997) also contended that the goal of phenomenographical research is to create and present participants’ conceptions of their experience in the form of categories of describing that experience. Sevensson (1997) pointed out that the most significant characteristics of phenomenography for supporting the creation of such categories of description are (a) the open explorative form of data collection and (b) the interpretative feature of data analysis. In this regard, Entwistle (1997) linked phenomenography to phenomenology, suggesting that the former goes beyond mere description of categories and attempts to also detect underlying meanings.

Major phenomenographical data collection methods include interviews (semi-structured or open ended, individual or group), observations (participant or non-participant), gathering artefacts produced by subjects (drawings, written responses), and searching through historical documents (Marton, 1986, 1994; Marton & Booth, 1997). Marton (1994) stressed that an interview is the preferred research technique, because it allows research participants to express how they experience and conceive of a phenomenon. He further stated that an interview should avoid
using too many questions prepared in advance. Rather, questions should begin with open-ended exploration of what subjects consider as the essence of their experience, and continue with follow-ups that probe them to further establish how they consider the phenomenon. To make the resulting categories of description rich, follow-up questions should therefore attempt to address specific aspects of the participants’ experience. This characteristic mode of interviewing is a hallmark of AoP and was followed in this study (e.g., an interview after a lesson would begin with a general prompt for the teacher to describe how the lesson was going, and follow-up questions were gleaned from the teacher’s response).

Concerning analysis of phenomenographical data, Marton and Säljö (1984) contended that categories of description should emerge from comparisons researchers perform within and across their data. Richardson (1999) pointed out that Marton’s original research seemed to have been based on creating verbatim transcripts, which were then subjected to an iterative and interactive process of identifying fundamental categories of description in the transcribed data. In this sense, Richardson (1999) noted that phenomenographic interview analysis is quite similar to the process of analysis devised by Glaser and Strauss (1967) in their grounded theory approach (discussed later in this sub-section). Such similarity is particularly found in the process of theme discovery, turned into hypotheses and theoretical constructs, and often refined against participants’ input (“member check”) about those constructs. In this study, each teacher was interviewed three times, being asked by the researcher for explanations of particular aspects of the lesson, such as her or his plan and/or ways it was implemented (or changed), students’ responses, assessment of what students knew, the importance/nature of the mathematics taught, and how students were expected to learn it. Analysis of these interviews followed the phenomenographical iterative process of transcription (and translation), comparing within cases (e.g., same teacher’s sequence of problems) and across cases (e.g., same lesson component used by all or some of them), and led to identifying fundamental categories of description (e.g., problem types that engaged students in doing mathematics they learned back in primary school).
3.2.1.3 Differences between Phenomenology and Phenomenography

Based on a review of different concepts and methods of phenomenographic research, Richardson (1999) suggested that researchers hold different views of phenomenology and phenomenography. His extensive review of the literature (e.g., the works of Gibbs, Morgan, & Taylor, 1982, McKeachie, 1984, Morgan, 1984, Prosser, 1993, and Taylor, 1983) indicated that they considered phenomenography to be essentially the same as phenomenology. Marton (1981) agreed that both approaches are relational, subjective, experience-oriented, and qualitative, but pointed out that such a view overlooks several important differences.

A major difference pointed out by Marton (1981) is that phenomenology focuses on subjects’ experience whereas phenomenography deals with both their experience and the way they seem to conceptualise it. That is, phenomenography attempts to understand how research subjects think about their experience. Moreover, he asserted that in phenomenography both experience and conceptualisation should be integrated within the same categories of description.

Most importantly, Marton (1981) contended that phenomenology’s use of the first person (singular) indicates lack of distinction between a first-order perspective—the research subject’s experience and description of the world, and a second-order perspective—the researcher’s way of describing the research subjects’ experience and description of the world. Applying this important distinction between perspectives to research in education, Marton and Svensson (1979) argued that traditional studies about learning took the researcher’s perspective as the point of departure and endeavoured to “observe the learner’s world and describe it as we see it. We frequently relate our description of the student to our description of his (sic) world and generally do this within an explanatory framework” (p. 472). Instead of taking the researcher’s perspective as the point of departure, they argued that phenomenography research should attempt to adopt the subject’s (e.g., learner) perspective on her or his experience.
In summary, phenomenology and phenomenography are two approaches that attempt to describe the phenomenon under study. They differ in the weight given to subjects’ experience and conceptualisation, and in the distinction between subject and researcher perspectives, while sharing the intention to produce categories of description that are subjective and relational. In terms of theorising about the phenomenon, phenomenology seems to steer researchers away from it and phenomenography seems to accept a rudimentary form of it in the sense of explicating the researcher’s conceptualisation of subjects’ experiences and thinking.

This study draws on these two approaches in its attempt to provide rich descriptions of the participating Chinese mathematics teachers’ classroom practices. However, this study also attempted to link this phenomenon with its plausible origins in teachers’ perspectives, which called for a methodological approach designed to generate theoretical accounts of such links. For example, this study observed and described lesson plans and the way a lesson unfolded in each of the participating teachers’ classrooms, including problems solved, students’ (or teachers’) solutions, and ways in which teachers interacted with their students (e.g., engaged them in individual work while moving about the class and checking students’ solutions). These observations and descriptions were coupled with explanation of the teachers’ rationale for their plans and execution of the lessons. The following sub-section discusses a methodology—grounded theory—that focuses on theorising from qualitative empirical data.

3.2.2 Grounded theory

In The Discovery of Grounded Theory, Glaser and Strauss (1967) introduced their seminal work on why and how to conduct rigorous qualitative research for producing credible, valuable theories about human phenomena. Simply put, a theory is grounded when it is rooted in and discovered from data. Creswell (2005) summarised grounded theory as a cohesive system of collecting data, identifying and coding themes and categories within the data, explicitly linking these themes/categories, and formulating a theory about a substantive human phenomenon in
terms of social-cultural processes that explain it. He further commented that “because a theory is ‘grounded’ in the data, it provides a better explanation than a theory borrowed “off-the-shelf” because it fits the situation, actually works in practice, is sensitive to individuals in a setting, and may represent all of the complexities actually found in the process” (p. 396). Bogdan and Biklen (1992) noted further that a grounded theory emerges from the bottom up rather than from the top down, because researchers assume that not enough is known about the phenomenon before undertaking research. Accordingly, the nature of the research process is open and flexible at the beginning, and it gradually becomes more focused. Through such a process, researchers construct an explanatory depiction of the phenomenon under study during and after they collect and examine the data.

To better depict a phenomenon, grounded theorists use a process of selecting research participants known as theoretical sampling (Creswell, 2005). This kind of sampling is guided by searching for participants who are likely to become a representative case of the phenomenon. That is, what drives the sampling process is not objectivity and randomisation, but rather a focused attempt to include subjects who manifest exemplary features of the phenomenon as conceptualised by the researchers. To increase the likelihood that such manifestation reflects participants’ voice as opposed to mainly researchers’ preconceived notions, qualitative data collection techniques used to discover a grounded theory rely heavily on interviewing, possibly as a way to best capture participants’ experiences by using their own words (Charmaz, 2000; Creswell, 2005; Creswell, 1998).

Another key feature of the grounded theory approach is the use of an emerging research design, also known as the ‘zigzag’ process. In this process, data collection and analysis are intertwined. That is, researchers do not wait until all data are collected in order to begin analysing them. Rather, from beginning to end of data collection they continually analyse the data and use this ongoing analysis to inform further data collection, as well as prepare for future (retrospective)
Figure 3.2 presents Cresswell’s (2005) depiction of the zigzag design (p. 406), using three interviews as an example of data collection events. He explained that researchers engage in initial data collection (e.g., the first interview), analyse it for preliminary categories, and then look for clues about what additional data to collect that seem needed to better address the research questions. Additional data may pertain to underdeveloped categories, or new individuals who can provide more insights. The researchers then return to the field to collect the additional data, which lead them to refine, develop, and clarify the meaning of previous categories and/or to create new ones.

This ongoing zigzag process between data collection and analysis continues until researchers determine (subjectively) that category saturation has been reached. Glaser and Strauss (1967) defined saturation as a state in which additional data provide no new insights for the categories. That is, when saturation is reached the researcher “sees similar instances over and over again” (p. 61). They further contended that studying one incident in one group is not sufficient for attaining category saturation. Furthermore, researchers are likely to realise that not all categories are equally relevant, and strive to ensure that “core theoretical categories with the most explanatory power should be saturated as completely as possible” (p. 70).
Another key feature of the grounded theory approach, which the zigzag process during data collection reflects, is the Constant Comparative Analysis method (Creswell, 2005; Glaser & Strauss, 1967). This analytic process proceeds through intentionally examining sub-sets of data and linking them to one another (e.g., comparing between individuals, between groups, and between events of the same individual at different times). Such a comparison both supports the creation/refinement of categories and is informed by those categories. Glaser and Strauss (1967) postulated that constantly comparing many data sets leads researchers to generate abstract categories and their properties, which since they emerge from the data, will clearly be important to a theory explaining the kind of behavior under observation. Lower level categories emerge rather quickly during the early phases of data collection. Higher level, overriding and integrating, conceptualizations—and the properties that elaborate them—tend to come later during the joint collection, coding and analysis of data. (p. 36)

Glaser and Strauss (1967) proposed four stages for the constant comparative method: “(1) comparing incidents applicable to each category, (2) integrating categories and their properties, (3) delimiting the theory, and (4) writing the theory” (p. 105). Cresswell (2005, p. 407) summarized this as an inductive analytic process in which generating and connecting categories grows out of comparing specific incidents (see Figure 3.3).

Cresswell (2005) explained the emergence of categories from the data. First, raw data are formed into indicators—some small segments of information that come from different participants, different sources, or the same participant at different times. These indicators are then conceptualised and grouped into several distinctive codes (A, B, and C in the Figure). These codes are then further organised into more abstract categories (e.g., category I, category II). While discovering the grounded theory, researchers constantly compare within each level (indicators to indicators, codes to codes, categories to categories), as well as across those levels (e.g., indicators that fit within another code, code within other categories, etc.). The goal is to
produce a theoretical account throughout the entire process while eliminating redundancy and providing data-grounded evidence for categories.

In summary, grounded theory is a qualitative research approach for developing theoretical accounts of a phenomenon under study. It revolves around an inductive, iterative, ongoing process of data collection and the retrospective method of constant comparative analysis. It overlaps with phenomenography in terms of the way researchers arrive at categories of description, as well as the emphasis on the use of interviews to identify and distinguish participants’ voice from the researchers’ voice and frame of reference. Grounded theory differs from phenomenology (and to less extent also from phenomenography) in that it sets out to theorise about relationship among categories of description as conceptualised by the researchers. Combined, the three approaches provided the theoretical underpinnings for depicting the participating Chinese middle school mathematics teachers’ practices (phenomenology, phenomenography) and for inferring into and theorising about perspectives that may underlie these teachers’ practices (grounded theory). In this study, the researcher focused on providing and linking rich descriptions of the middle school Chinese teachers’ lessons (plans and observed implementation), creating categories of those descriptions, and theorising about plausible ways
of thinking about mathematical knowing and learning that could explain the teachers’ work. The following sub-section presents the Account of Practice (AoP) strategy of inquiry (Simon & Tzur, 1999), which provides a coherent method for gathering and analyzing data on classroom teaching and teachers’ rationale.

3.2.3 Account of Practice – the Research Strategy of Inquiry of this Study

This study employed the Account of Practice (AoP) strategy of inquiry (Simon & Tzur, 1999), which centres on linking specific features of a teacher’s lesson—planned and implemented—with her or his pedagogical rationale. In terms of its theoretical underpinning, the AoP strategy of inquiry resides in the overlap between phenomenography, phenomenology, and grounded theory (depicted in Figure 3.1 above). Denzin and Lincoln defined this term as follows:

A strategy of inquiry comprises a bundle of skills, assumptions, and practices that researchers employ as they move from their paradigm to the empirical world. Strategies of inquiry put paradigms of interpretation into motion. At the same time, strategies of inquiry connect the researcher to specific methods of collecting and analyzing data. (p. 14)

The AoP strategy of inquiry was developed and used for studying mathematics teacher practices and perspectives in the US. It seemed suitable for describing the phenomenon of Chinese mathematics teachers’ practices (e.g. lesson structure), explaining non-observable aspects of the teaching (e.g., teacher reasoning), and theorising about possible links between the two. As explained in the first two chapters, how teachers think of learning and mathematics knowing provided the conceptual lens for accounting for their perspective in this study. The following discussion introduces the AoP strategy of inquiry and explains how it was used within the research design of this study.
3.2.3.1 Account of Practice (AoP)

Simon and Tzur (1999) proposed the *Account of Practice* (AoP) strategy of inquiry as a means to create a coherent account of teachers’ practice by explaining teachers’ perspective as it is conceptualised through the researchers’ perspectives. They defined the term *teaching practice* as consisting of not only everything the teacher does, such as planning, assessing, interacting with students, or thinking about what she or he does but also the teacher’s values, feelings, skills, and intentions. Consistent with phenomenographical emphasis on differentiating subjects’ and researchers’ perspectives, Simon and Tzur (1999) stressed that the way researchers depict and explain teachers’ practices may differ from how the teachers would depict and explain their teaching (e.g., via self-report).

Simon and Tzur (1999) emphasised that the AoP strategy of inquiry attempts to represent teachers’ practice by focusing on aspects that would be of theoretical importance for mathematics education researchers and teacher educators. To this end, and similar to constructivist researchers who stressed that how a student does mathematics makes sense to the student, the AoP assumes and sets out to account for what in a teacher’s practice seems to make sense to the teacher. They contrasted this approach with studies that focus on what teachers do not know or are unable to do. The AoP strategy of inquiry attempts to provide an understanding of the roots of teachers’ activities and to indicate important areas for teacher development.

To capture the essence of the AoP strategy of inquiry, Simon and Tzur (1999) presented a schematic diagram of the process of generating such accounts (Figure 3.4). They explained that (a) researchers’ conceptual framework and (b) aspects of a teacher’s practice researchers notice are necessarily reflexive (affecting one another). Researchers’ hypotheses and inferences about data grounded in teachers’ practices yield an account of teachers’ practice. Simultaneously, the emerging accounts influence both the researchers’ conceptual framework and their notions of what constitutes important and relevant data for the AoP.
On the basis of this schematic model, Simon and Tzur (1999) generated an extended scheme, which relates the AoP strategy of inquiry to Simon’s (1995) central notion of hypothetical learning trajectory (HLT). They argued that when a number of teachers are being studied, this approach would promote an empirically-grounded understanding of mathematics teacher development, by using the researchers’ hypothetical learning trajectory for teachers when the same teachers’ practices are being described in a number of accounts over time. This stance draws on grounded theory assumptions and analytic methods of constantly comparing data sets (e.g., different teachers, different lessons of one teacher).

In this study, this schematic explanation of the genesis of an AoP was used as follows. The researcher’s conceptual framework consisted of a constructivist view on learning and teaching as explained in Chapter 2. This perspective structured what she noticed about teachers’ practice (e.g., a repeating pattern of organising a lesson into four main components, a particular component that reactivated mathematics that students studied in the primary school, which she noticed due to the core constructivist construct of assimilation). Noticing these aspects led to the researcher’s inquiring into teachers’ rationale for those aspects, and accounting for them in terms of a plausible perspective on knowing and learning mathematics (e.g., in the teachers’
perspectives, it appeared that using primary school mathematics served the reactivation of what students supposedly knew so that a core method from that knowledge would also serve solving new problems). In turn, the teachers’ perspectives on using such aspects were explained as rooted in a view of learning conceptualised by the researcher (e.g., linking ‘new’ knowledge to ‘old’ knowledge, which is why they seemed to focus on figuring out what students knew).

Data collection in the AoP strategy of inquiry consists of video recorded classroom observations and video and audio recorded interviews. The unit of analysis of this qualitative research work is a set consisting of observing at least two consecutive, content-related mathematics lessons, and interviews with the teacher before, between, and after each lesson. A minimum set of data begins with an interview in which the researchers ask the teacher about instruction that has recently taken place and his or her plans for the first lesson that will be observed. The inquiry focuses on the teacher’s rationale for those plans, including her or his understanding of the mathematics and why particular teaching activities (e.g., problems) are likely to bring about students’ learning of that mathematics. Then, a lesson is observed and video recorded while the researcher(s) take(s) intensive field notes. After the lesson, and in preparation for the second interview, the researcher(s) review notes from the first interview and lesson, and prepare possible probing questions that can guide the second interview (consistent with the inductive, iterative, zigzag process of grounded theory). A second interview then follows in order to probe the teacher’s reflections on and explanations of his or her behaviours during the observed lesson, reasons for particular decisions (e.g., changing a planned task sequence), and the extent to which the teacher infers students’ accomplishment of her or his goals for their learning. The interviewer then asks the teacher to explain the plan for the next lesson and its rationale, using a similar approach to that in the first interview. After an observation of the second lesson takes place (video recorded, field notes), more ongoing, inductive probing questions are prepared and a third interview is then conducted to inquire into the teacher’s thinking about the second lesson in itself, as well as about the entire set, and to discuss the teacher’s plans for future instruction on the same topic.
Once a data set for a teacher is completed, video recorded classroom observation data and video/audio recorded interview data are transcribed and analysed, line-by-line. This analysis is consistent with the robust qualitative methods of constant comparative analysis (Glaser & Strauss, 1967; Strauss & Corbin, 1994) described in the previous sub-section. First, segments of data (‘indicators’) that pertain to the research questions and seem of significance are identified. Then, the researcher groups and codes these instances while creating inferential categories—conjectures about the teachers’ teaching practice and questions about possible interpretations of the data that will guide further analysis. Finally, inferences and hypotheses (categories) are synthesised into an account of the teachers’ practice, which portrays an underlying teacher perspective that seem relatively consistent over time.

Simon and Tzur (1999) developed and used the AoP strategy of inquiry to study how American teachers might change their practice along the lines of mathematics education reform in the US. Therefore, teachers who participated in their studies were recruited and selected among those who were actively engaged in reform efforts. In this present study, the AoP was adopted and adapted from the US context, via sampling and recruiting teachers who seemed to represent common approaches to teaching mathematics in Chinese middle schools. For each of the participating Year-7 mathematics teachers the researcher created a set (three interviews and observation of two consecutive mathematics lessons), transcribed and translated all data, used the zigzag, iterative process of analysis to identify categories of description and themes, and organised the themes across cases into an account of practice. The following sub-section elaborates the research aspect of participating teachers.

3.3 Participants
This study was conducted in two urban centres in Southeast China (1-million and 11-million residents, respectively). Eleven Year-7 teachers volunteered (asked by the principal and gave an explicit agreement) to participate in this study; seven of them taught in a middle school in the
first city and four in a middle school in the other city. The first group consisted of the entire Year-7 faculty in that school; the second group consisted of half the faculty. Note that only half of the teachers in the second school volunteered, which suggests teachers assumed freedom in volunteering. In terms of teaching experience, those 11 teachers spanned a range between 1 year and 30 years.

The participating teachers’ daily routines included teaching a mathematics lesson (40-50 minutes) in each of their 2 classes with about 45 students each. During the rest of the school day, they checked their students’ class work and homework individually, and co-planned the next day’s lesson as a group with veteran teachers providing guidance and critique. In addition, once a week the entire Year-7 faculty observed a lesson in each teacher’s class and discussed it later that day. The participating teachers said in the interviews that these routines are typical of Chinese mathematics teaching in big cities. The following discussion elaborates the sampling procedures.

3.3.1 Sampling
This study used a purposeful sampling strategy, which according to Welman and Kruger (1999) is an important kind of non-probabilistic sampling to identify study participants. The sample was selected based on the researcher’s judgement and the purpose of this study (Babbie, 1995; Greig & Taylor, 1999). The purpose was to include middle school mathematics teachers who teach content within the overarching topic of algebra, and who had a range of teaching experiences, while providing good access to the individuals and sites (Creswell, 2005). Among the eleven Year-7 mathematics teachers from the two participating schools, six were selected during the final phases of retrospective, constant comparison data analysis of this study. Selecting those six teachers into the final sample was based on their being a representative case of specific aspects of mathematics teaching practices and/or the rationale they used to explain that practice. Said differently, category saturation could be achieved with data from these six teachers. The
discussion below elaborates the criteria used to create the initial sample of eleven teachers (first – for schools, then – for teachers).

3.3.1.1 Selection Criteria - Schools:

(a) In China, the majority of schools are public schools; therefore, both sites in this study were public schools.

(b) Both middle schools selected were from the mainstream, academic orientation in the district (province). This means that content and teaching practices that were observed in the participating teachers’ classrooms are taken from and representative of the typical curriculum for Year-7 mathematics in China.

(c) Both schools were located in large-population metropolitan areas in a South Eastern province in China. School 1 was located in the capital city of the province; School 2 was in a county city of the same province.

(d) Both schools provided mixed-gender education, as the majority of public schools in China are co-educational.

(e) School 1 is a boarding school and is considered one of the top-achieving schools in the province. Students were admitted to this school based on their high primary school achievements. School 2 is an ordinary middle school, with levels of students in each class ranging from low to high achievements (i.e., like typical middle school classrooms in the province). The students attended this school due to vicinity.

(f) Both schools did not use level grouping or tracking in mathematics to differentiate among students. This means that a single teacher taught an average of about 47 students at different levels in each classroom. While these students are academically able, the teaching represented in this study seemed indicative of teaching seen in many other contexts in China.
Combined, these selection criteria of schools led to a sample that seemed representative of the phenomenon under study, namely, typical mathematics teachers’ practices in Chinese (urban) middle schools. Although School 1 is better resourced, the number of students per class and the essential conditions in the rooms were pretty similar (no air conditioning, one blackboard, an installed LCD projector to present PowerPoint slides, crowded rows of tables—one per student—all facing the blackboard). Like most Chinese schools, the two sampled for this study held a standard daily schedule. The first lesson began at 7:30 am, and 7-10 minutes breaks separated between two consecutive lessons. Students stayed in their classroom while teachers of different topics came to their homeroom to teach the class. A mid-day lunch break of about 1.5 hours began around 11:30, and the last lesson of the school day concluded around 5:00 pm.

### 3.3.1.2 Selection Criteria – Teachers:
School 1 had nine Year-7 mathematics teachers. Four of those nine teachers volunteered to participate in the initial, larger effort, and two of those four were selected for data analysis of this study (the final sample of six). School 2 had seven Year-7 mathematics teachers. All seven teachers volunteered to participate in the initial larger effort and four of those seven were selected for data analysis of this study (the final sample of six). Table 3.1 shows data about the final six participants selected for this study. As the table indicates, the six teachers reflect the entire spectrum of teaching experience in Chinese middle schools (one year through 30 years).

This final sample of 6 teachers includes 2 novice teachers (WK and JC), two mid-career teachers (ST and SZ), and two veteran, expert teachers (LX and ZX). It is also balanced between males (JC, SZ, and LX) and females (WK, ST, and ZX). Most importantly, the six teachers selected for this study reflect, without over repetition, the common patterns (categories of description) found

<table>
<thead>
<tr>
<th>Participants</th>
<th>Teacher (Years of Experience)</th>
<th>Class Size</th>
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Table 3.1: Participants of this study.
among all eleven teachers observed, as well as important personal deviations that are of interest to the way they taught mathematics. For example, teachers WK and LX were selected because they work together in the same school and lesson research group; WK is a female with one year of teaching experience and LX is a male with 30 years of teaching experience. Essentially, each of them taught the same content in both lessons observed, while using a similar sequence of lesson components (Reviewing, Bridging, Teaching with Variation, and Summarising—see Chapter 4) based on the group lesson planning. However, initial and retrospective analyses suggested that within those four categories of lesson components, each of these two teachers deviated in terms of specific strategies they used for each component and the reasons they provided for using specific tasks.

### 3.4 Data Collection

This study employed the AoP strategy of inquiry (Simon & Tzur, 1999), because of its emphasis on explicitly linking a teacher’s practice with her or his perspective(s) on learning and knowing mathematics. As explained in the sub-section on AoP, a data set for each of the teachers consisted of 5 video recorded session (two lessons, three 1-hour interviews). The sequence of data collection was: pre-lesson interview \(\rightarrow\) observation of Lesson 1 \(\rightarrow\) post Lesson 1 interview \(\rightarrow\) observation of Lesson 2 \(\rightarrow\) Exit interview. The following discussion elaborates this process.

#### 3.4.1 Interview Data Collection and Instruments

Consistent with the theoretical underpinnings of this study, a semi-structured interview protocol was followed. That is, a general process of proceeding from open-ended, non-specific questions
to follow-up, very focused probing questions was used. Accordingly, different questions were asked during each of the three interviews with each teacher and at different stages within each interview. Focused questions followed the particular features in the teacher’s initial, open-ended response (e.g., why would you begin with problems like “Is 3x-5 an algebraic fraction? Explain”). Such questions allowed making inferences about both the teacher’s understanding of the mathematics she or he was about to teach or already taught and about the rationale for using the planned (particular) teaching activities. This kind of questioning draws on a key assumption of the AoP strategy of inquiry, namely, that engaging a teacher in articulating very specific instances of the planned (and later implemented) lesson provides valuable data to ground an account of that teacher’s perspective. Such highly specific questions also enable constant comparisons between two critical manifestations of the teacher’s practice and perspective—the planned and the implemented (whether the teacher kept or changed them). Similarly, specific events that took place during the observed lesson would be brought up in the second and third interviews, including explicit requests from the teacher to relate planned and implemented instances.

In the first (pre-set) interview, a generic, open-ended question could be: ‘What are your plans for tomorrow’s class?’ and ‘What new mathematical concept are you going to teach?’ Once the teacher presented concepts and activities for teaching them, follow-up focused questions about each planned activity were asked, such as ‘Why would you use this [point to particular] activity (or problem, or set of problems)?’ In the second interview (between two observed classes), the generic open-ended probes (e.g., ‘Please tell me how you feel about this lesson …”) were followed by questions tied to the particular tasks, activities, and strategies the teacher actually employed in class, as well as to her or his considerations of students’ work and progress. These questions attempted to trigger the teacher’s reflections on the reasons for very specific teaching moves noticed during the observed lesson and on what led to her or his decisions during the lesson. In particular, these questions targeted moves/decisions that appeared to deviate from the
teacher’s plan. Being as specific as possible in terms of what transpired in class, teachers were asked questions such as: ‘Who among your students seem to understand (and who does not) the concept of simplifying an algebraic fraction?’; ‘What particular responses from students support your evaluation of this?’; ‘Why did you think this or that problem [named specifically], this or that strategy, could get the students to understand the taught concepts?’; ‘What are your criteria for judging whether students understood this [specified] concept?’; and ‘On what basis did you choose a specific student to answer this or that problem [specified]?’ Once probed reflection on the first lesson was completed, questions about the teacher’s plan for the next lesson and the rationale for it were asked similarly to the first interview (but specific to the contents of the next lesson). The third interview began by probing the teacher’s reflection on the second observed class similar to the previous interview. Then, further questions probed the teachers’ thinking about the two lessons as a whole (e.g., “Do you believe that students X and Y, whom you mentioned did not learn simplifying algebraic fractions in the first lesson, have learned it after the second?” and “What do you take as evidence for this?”). Finally, the questions turned to the teacher’s thinking about future instruction on the same concepts. The following six prototype interview questions (a through f) provide a generic image of the focused questions in an interview.

Before a lesson: (a) ‘What concepts are you going to teach in your lesson?’ (or, phrased differently, ‘What is your goal for students’ learning in the next lesson?’); (b) ‘What particular activities do you plan to use for teaching each of these concepts?’; and (c) ‘Why do you think that these planned activities can get your students to meet your intended goals for their learning of this-or-that [specified] concept?’

After a lesson: (d) ‘Do you think the students grasped what you wanted them to know (can you specify who and at what level)?’; (e) ‘How do you know whether they got it or not?’ (or, phrased differently, ‘What do you take as evidence they did or did not grasp it?’); (f) ‘In your opinion
and in relation to what you have done, why did some students get this and others did not?’ As explained above, the second and third interviews began with this latter set of questions and continued with the former set. The following sub-section discusses issues of video recorded data (and linked field notes).

3.4.2 Observation Data Collection and Instruments

3.4.2.1 Video Recording

Video recording of mathematics lessons in classrooms and/or of one-on-one interviews with teachers has both advantages and disadvantages (Roschelle, 2000). In particular, several issues must be considered when video recording an observed classroom in a culture and country where this is not a common practice. Unlike Western countries, in which students and teachers may be accustomed to being observed and video recorded, in the Chinese middle schools in which this study was conducted such research procedures introduced a novelty. Whereas the district and school personnel gladly allowed access to their students and teachers, for those who were observed it was the first time they ever experienced being video recorded. Thus, care had to be taken to capitalise on the advantages of video recording and minimise the impact of disadvantages.

Two key advantages of video recording classroom observations and/or interviews can be pointed out. The first pertains to capturing the complex processes that constitute classroom teaching and learning (Maher, 2008; Nemirovsky, DiMattia, Ribeiro, & Lara-Meloy, 2005). By video recording an entire lesson, the researcher can document all that is said as well as teacher and students’ actions, including bodily gestures, teaching styles, organisation of classroom activities, and student–teacher interactions. The recorded data could then be used for repeatedly reviewing any segment of the lesson retrospectively. In an interview, video recorded data provide not only what a teacher said but also bodily gestures and movement that enriched the transcripts with information that cannot be gleaned from audio recording alone. The second advantage pertains to
analysis, particularly the coding process. Once classroom practices and interviews have been video recorded, a researcher can code the data in a variety of ways, focusing on different emerging indicators and categories at different times, revisiting and reinterpreting whenever there is lack of clarity about data analysed. This is particularly important for constant comparative analysis, as the recorded data allow the needed back-and-forth shifts between data segments of significance (e.g., between real-time field notes and retrospectively noticed data).

On the other hand, a disadvantage of video recording and/or interviews should also be acknowledged, namely, the potential impact on participants. Both teachers and students, particularly in the rather conservative culture of China, may find it difficult enough to let an outsider observe the classroom, let alone record their behaviours. Consequently, both teachers and students might also adjust their behaviours and the validity of videotaped data might be questioned (Altheide, David, & Johnson, 1994).

In this study, teachers received an explanation of video recording method of data collection a few weeks prior to the start of collecting a data set in their classroom, and did not seem to be disturbed by it. However, in each of the classrooms observed it seemed that students needed a few minutes to get accustomed to the fact that a camera was at work in their classroom (e.g., when the equipment was set at the back of the room during the break between lessons, some students would turn to the camera and signal a ‘V’ with their fingers). Within a short time, however, possibly due to high level of discipline and attention to the teacher expected of them, in short time they seemed to simply ignore the camera and the researcher at the back of the room.

In relation to the above list, a few issues had to be resolved: How/when to introduce and begin video recording? Where should the video camera be located? What role would the researcher play when observing, particularly if students would try to create contact (Adler & Adler, 1994; Atkinson & Hammersley, 1994)? Another question to be dealt with was what if some students
(or their parents) were not willing to be video recorded? Most importantly, because in a 47-
student classroom multiple significant events could take place, a decision had to be made what
should the video camera focus on (The teacher? Classroom activities? Students? The board?).

These issues were addressed by following a key principle—be the least intrusive to the teaching-
learning process. Thus, the video camera was placed at the back of the classroom. This location
did not allow access to students’ individual work, but the authenticity of students’ behaviours
seemed to be compromised the least. To create a back-up audio file of the linguistic exchanges
during the lesson, the teacher was asked to carry a remote wireless microphone set, which
transmitted her or his voice, as well as students the teacher stood close to, directly into the
receiver attached to the camera. This audio-enhanced recording allowed capturing the teacher’s
communication with students as she or he moved about the class, which too often was inaudible
in the video. In addition to these two separate sets of recording, the researcher took extensive
field notes in Chinese to record such aspects as problems given to students, solutions students
wrote on the board, and major student-teacher interactions (see more about field notes below).
Field notes were taken while the researcher also moved the camera to focus mainly on the
teacher and her or his work. At times, she focused the camera on students’ work—particularly
when they were invited to share it with the entire classroom (on the blackboard or from their
seats). The decision to position the camera (and researcher) at the back of the class seemed
productive. None of the teachers or students expressed unwillingness to be included, and shortly
after the beginning of a lesson they seemed to behave quite naturally (during the interviews, a
few teachers actually confirmed this conclusion in a note about how the lesson proceeded as
usual). In this sense, the researcher played a role of non-participant observer during the
classroom observation.

In terms of validity and reliability of video recorded data, the assumption of this study is that a
teacher’s conceptions of teaching and of how students come to learn something they do not yet
know, as well as the teacher’s practice, remain fairly consistent over time (Correa, Perry, Sims, Miller, & Fang, 2008). Although teachers may wish to appear to be doing a good job, any tendency to rehearse or show off is limited by the need to effectively use their time with students. Thus, it seemed that their interactions with students and the instructional activities and strategies they planned and implemented proceeded naturally and have made sense to them. Teachers would not do what they cannot or do not believe they should do. In other words, if a teacher is observed to function in a particular way this is not a guarantee that she or he could or would do it every time, but it does mean they can and may work this way, that is, the observed practice is part of their repertoire.

Another aspect of the data collection that contributed to the validity and reliability of video recorded classroom data was that the researcher was not restricted to using data collected from interviews only. Rather, a back-and-forth check was held between what a teacher was observed to be actually doing in the classroom and what she or he said about those observations (Thompson, 1992). Using both interviews and classroom observations, along with teacher and student artefacts (e.g., PowerPoint presentations, student homework, teacher written notes), allowed triangulation of data and contributed to minimising the potential decrease in authenticity of teacher data if only video recorded classroom data were used (see more about triangulation later in this chapter, sub-section 3.8.2).

3.4.2.2 Field Notes and Artefacts
As is customary in a qualitative study, the researcher took extensive field notes during the observations (Adler & Adler, 1994; Bogdan & Biklen, 1992). The only distraction this data collection method introduced was operating the camera. Thus, at times, the teacher would be out of the camera’s focus for some time. During the interviews, to avoid distracting the teacher, the researcher minimised note taking to quickly jotting down key points the teacher made that would require further questioning later in the interview. Immediately following the interview, a more
intensive set of notes was taken, based on fresh memory of major issues that were addressed during the interview. To counteract memory decay as much as possible, a quick listing of key points was written down first. Then each of those points were elaborated to (a) provide quick access to what transpired during the interview and thus guide further data collection (without watching the video recording) and (b) indicate important segments for the retrospective, constant comparative analysis.

Artefacts for this study were collected during the interviews. These artefacts included the teachers’ plans for the observed lessons (usually prepared by the teachers’ group), the PowerPoint presentation for each lesson, a copy of the lesson plan from the National curriculum, and copies of samples of students’ solutions to in-class or homework problems (particularly those that were discussed during an interview). The field notes and artefacts informed questioning during the second and third interviews, particularly in terms of deviations the teacher made from the written plan.

3.4.3 Data Collection Considerations and Procedures
This sub-section discusses the data collection considerations that guided this study and procedures used to collect the data. It begins with a description of the data collection procedures. Then, ethics considerations for this study are discussed.

3.4.3.1 Procedures of Data Collection
Once the Ethics Committee approval for this study was received from the Australian university, the researcher contacted two middle school principals in China. According to the approved ethics guidelines, through over-the-phone and in face-to-face conversations, the purposes of the study, the methodology and procedures, and the schedules for collecting the data sets were explained to these principals. They were given time to ask any questions they might have had, and were asked whether they were satisfied with the explanation and understood how the study would proceed. At the end of the meeting, the researcher provided each principal with a formal introductory
letter in Chinese. The local protocol requires this, as well as the sharing of this letter and the information with the teachers by the principal (not by the researcher). The principal was requested to ask the teachers to schedule the time for data collection (two observations, three interviews) to lessons in which they planned to teach a new mathematical concept to their students, as opposed to, for example, reviewing materials or having a test. The assumption is that such lessons are best suited for inferring into the teacher’s perspective on learning and how it impacts her or his practice, because these lessons revolve around students’ coming from not knowing to knowing the mathematics intended by the teacher.

In line with the ethics approval, collection of all data sets strictly followed the description above, and the scope of mathematics was limited to middle school algebra lessons. These data sets were collected during the last three weeks of May 2009, which in China is about 2 months before the end of the school year. In School 1, a mathematics lesson lasted about 40-45 minutes; in School 2 it lasted about 45-50 minutes. Each teacher interview lasted about 40-70 minutes. As described above, in both schools the typical workload of a teacher is two classes per day; the rest is devoted to planning individually or in groups and to checking students’ in-class and homework. This schedule allowed the researcher to interview the teachers during the school day and to have periods of time to reflect on and plan for the next interview (see more below).

Typically, the researcher conducted four data sets during each of the three data collection weeks (i.e., one set per teacher, four teachers per week). One week was devoted to the teachers in School 1 and two weeks to the teachers in School 2. Schematically, on Monday morning the first two teachers were interviewed in the school about their first lesson to be observed. Their first lesson was observed on Tuesday morning, and the second interview was conducted about that lesson and their plans for the next lesson later in the afternoon. The second lesson of that teacher would then be observed on Wednesday morning and a separate exit interview with each teacher was conducted on that afternoon. A similar schedule was followed for the third and fourth
teachers during Wednesday, Thursday, and Friday. Of course, this description is indicative; the actual days/times for interviews and observations were selected to fit with the teachers’ schedules.

3.4.3.2 Ethics

It should be noted that ethical requirements for conducting research in China are different (less strict) from those used in Australia. In essence, all that was required in the local settings was an approval from the district’s central office, which informed the principal that the study was confirmed. Although a signed, informed consent is not required by the Chinese authorities, teacher consent forms were formulated based on the language suggested in consultation with the Ethics Committee of the Australian university, and signed by each teacher. Parent consent and student assent forms are also not customary in China and the researcher had to follow the local procedure of obtaining approval through the district’s office and principals.

In order to ensure ethical research, the researcher wrote a formal letter (in Chinese) to the principals and teachers in which the following items were explicated:

- The purpose of the research;
- The procedures of the research (e.g., videotaping, observations and interviews, etc.);
- The benefits and (minimal) risks involved in participating in the research;
- The voluntary nature of participation;
- The subjects’ right to stop their participation in the research at any given time without any ramifications;
- The procedures used to protect teachers’ and students’ confidentiality (e.g., using name initials to protect their identity);
- The information to be collected would not embarrass or harm them;
- The researcher’s obligation to report the truth when writing up the findings.
Ethics concerning the use of video recording also followed guidelines set forth in the approval received from the university in Australia, which explicated how video recording would be conducted and how it should be presented in the teacher consent form. The consent form specifically explained that video recording would be used to document the interviews and classroom observations for future analysis, but would not be shared except for research purposes. The researcher explicitly promised school principals, teachers, and students who participated in this study that the purpose of video recording and all other data collection is purely research and education, and no other uses (e.g., commercial) would be made. They were told that personal images would not be exposed and privacy would be strictly protected, and no personal information will be released to the public without their permission. In particular, the researcher guaranteed participants that their real name would not be used in any publication.

### 3.5 Data Analysis

As explained above, analysis of the data sets with teachers followed the qualitative approach of grounded theory (Glaser & Strauss, 1967; Strauss & Corbin, 1994). This included both ongoing, daily ‘zigzag’ analysis throughout the three weeks of data collection and retrospective constant comparative analysis upon completion of data collection. Ongoing analysis began with the researcher’s reviewing of field notes immediately after each observation and interview, which led to documenting intensive analytic memos (Glaser & Strauss, 1967; Glesne & Peshkin, 1992). These memos focused on teachers’ observed practices (e.g., letting students who made mistakes share their solution with the class) and comments about reasons for teaching the way they did (e.g., exposing mistakes supports learning by that student and others who might make it). These memos were organised into a short list that informed questioning of teachers during subsequent interviews—both the teacher in whose practice those memos were created and other teachers (whether or not they explicitly expressed the point). For example, soon after data collection began, a significant pattern in the way teachers seemed to organise their lessons emerged and was noted in a memo. In those lessons, after a review of previous lesson’s materials took place,
the teachers introduced a few simple problems that students were expected to already know from
the time of their primary school. When asking teachers about this pattern, they confirmed that it
was an intentional pedagogical component, translated from Chinese as ‘Bridging’ (see next
chapter), which served their purpose of reactivating a method known to students (‘old’) as a
means to learn the new concept. In turn, questioning about the teachers’ rationale for their
pedagogical activities repeatedly explored the extent to which they were aware of and
proactively used their students’ available understandings.

Once data collection was completed, consistent with the phenomenography approach that
underpins the AoP strategy of inquiry, all video and audio-recorded data—interviews and
classroom observations—were fully transcribed (verbatim, in Chinese). That is, the raw data
corpus for the analysis (eleven teachers) consisted of 22 observed lessons and 33 interviews with
their corresponding transcripts, field notes, and ongoing analytic memos. This raw data corpus
was organised into computer files that linked together all pieces of data for every teacher. Then,
the researcher translated all the data into English. These two experiences, though very labour
intensive, served a crucial role in the analysis—they provided a second and a third round of re-
experiencing the original teaching–learning processes (observations) and teacher explanations of
those processes (interviews). Combined, the three rounds (actual observation/interview, Chinese
transcription, English translation) allowed the researcher to develop close familiarity with the
data. This was particularly important for noticing and documenting behaviours of both teacher
and students, and for establishing the basis for the next analytic phase—categorising and
selecting the final pool of teachers for this study as well as segments from their work that seemed
significant for further analysis.

Because the raw data is in Chinese whereas this study was written in English, it is important to
consider how equivalence of the two languages was obtained. In this study, a process of English-
back translation was used (Brislin, 1970). In regards to prototype interview questions, the
English version was firstly established and then translated into Chinese by the researcher. University lecturers in Australia who specialise in both Chinese and English were then consulted to ensure the validity, equivalence, and consistency of the translation. In regards to translated interview and observation transcripts, the same English and Chinese language consultants’ advice was sought. They translated a few randomly selected segments of the English transcripts back to Chinese and compared that translation with the original Chinese transcript. This English-back translation confirmed the accuracy of the segments. The consultants then also directly compared the researcher’s English and Chinese transcripts and confirmed they were consistent and equivalent. In addition, two Chinese friends of the researcher, one in Australia and the other in China, were asked to check the equivalence of the translation. Both friends have a master’s degree (obtained in Australia and China, respectively) and good command of English. They both concurred with the Australian consultant’s confirmation of translation equivalence.

An important aspect of the English translation should be noted, namely, the length of words and sentences. For example, in Chinese the English phrase “Removing the denominator” is written 去分母 and expressed ‘qu fen mu’. This example highlights two additional points regarding translation and the different images one may have about interacting mathematically in different cultures. The first point concerns the dilemma between the literal translation of a word (e.g., 约分 = yue fen = cancelling the denominator) and the mathematical accepted term (e.g., removing, or eliminating the denominator). The second point concerns a particular pedagogical activity that is prevalent in China, but may not be familiar to Westerners. During a lesson, quite often a teacher may ask a question and expect students to say the response out loud and together, as if the class is a ‘speaking chorus’. Whereas the English translation of what students said as a chorus (in Chinese) may appear long and not likely (or believable) to be said together, the Chinese phrase would be much shorter and conducive to such whole-group expression. A
‘speaking chorus’ response could entail that students simply read or repeated a teacher’s definition; however, it often included generating a response that was not memorised or rehearsed.

Retrospective, constant comparative analysis proceeded mainly in English. Indicators (e.g., “calling up a student who made a mistake”) were organised into categories (e.g., “teacher pays attention to students’ available understandings” or “teacher presents mathematical problems from primary school”). As patterns among those categories emerged, more global themes were conceptualised and connected (e.g., “after reviewing a few problems from the previous lesson, teachers seemed to regularly proceed to mathematical problems that require much earlier knowledge, which the teachers termed Bridging”). All data segments pertaining to those emerging themes that seemed of particular significance were then analysed, line-by-line and in a chronological order (per teacher), through observation of the video recording while reading the transcript. This process required back-and-forth moving in the data and often resulted in refining the transcript (e.g., slightly changing the language in it and/or adding descriptions of participants’ behaviours). This process helped to focus on the way each individual teacher (case) went about teaching the intended mathematics and linking it to how she or he conceived of the learning process as a basis for choosing and changing pedagogical activities.

The case-by-case analyses led to a final round of analysis in which (a) six teachers were selected among the eleven cases and (b) the constant comparative analysis method was applied in order to relate and contrast categories/themes among all 6 teachers. The focus was on identifying what seemed to be common to all eleven Chinese teachers and represented by the six cases, as well as on unique, individual deviations from the common ways. In this sense, data collection and analysis were consistent with the inductive, iterative process advocated by grounded theory and the AoP strategy of inquiry (Glaser & Strauss, 1967; Simon & Tzur, 1999; Strauss & Corbin, 1994): coding raw data, searching for distinctive features within a case, identifying cross-case
commonalities and differences, and scrutinising the data line-by-line for confirming or disconfirming evidence.

Finally, to prepare for writing the Analysis and Discussion chapters, the researcher scrutinised the data again to identify segments that seemed (a) relevant to the research questions and (b) to provide evidence for the two central claims of this study (see below). Two main criteria were employed for deciding which of the data segments (or excerpts) should be included. First, data segments that seemed most representative of each theme (commonality) were noted (e.g., a data segment that captures well a component in the teacher’s lesson, such as Bridging). Second, data segments that highlighted teachers’ uniqueness—different organisation, approaches, and strategies of presenting the problems and activities—were added and discussed.

In each of the sub-sections (themes) of the Analysis chapter, those common themes and unique deviations by individual teachers were interwoven to support two main claims of this study. The first claim is that the participating Chinese mathematics teachers structured their lessons in a way that seems to effectively promote high level of student learning and outcomes. The second claim is that the common patterns found in those teachers’ work indicate a pedagogical perspective about learning and knowing, which is discussed in the final chapter of this study. The following sub-section discusses the trustworthiness of this analysis.

3.6 Trustworthiness of Data and Findings
This sub-section describes the trustworthiness considerations of this qualitative study. It begins with a discussion of the key constructs of reliability and validity, which combined are regarded as ‘trustworthiness’ when applied to qualitative research. It then discusses particular trustworthiness considerations of this study.
3.6.1 Reliability and Validity in Qualitative Research

Morse, Barrett, Mayan, Olson, and Spiers (2008) postulated that “without rigor, research is worthless, becomes fiction, and loses its utility” (p. 13). Therefore, as Golafshani (2003) noted, both qualitative and quantitative researchers need to ensure and demonstrate that their studies are credible. Accordingly, validity and reliability are two criteria that a qualitative researcher should be concerned about while designing a study, analysing results, and judging its merit (Patton, 2002). However, Healy and Perry (2000) asserted that the quality of a study in each paradigm (quantitative, qualitative) should be judged by its own paradigm’s terms.

Some researchers argued that reliability and validity are terms used by quantitative researchers, whereas qualitative research requires new criteria for ensuring the rigor of a study (Altheide & Johnson, 1994; Leininger, 1994; Lincoln & Guba, 1985; Rubin & Rubin, 1995). Davies and Dodd (2002) argued that the very notion of rigor in qualitative research should differ from its meaning in quantitative research. By “accepting that there is a quantitative bias in the concept of rigor, we now move on to develop our re-conception of rigor by exploring subjectivity, reflexivity, and the social interaction of interviewing” (p. 281). A particular construct that was proposed to replace reliability and validity as a parallel term for rigor in qualitative inquiry was trustworthiness (Guba & Lincoln, 1981; 1982, 1989; Lincoln & Guba, 1985). They proposed essential criteria to ensure trustworthiness in the qualitative paradigm—credibility, transferability, dependability, and conformability—and specific strategies to accomplish it, such as peer debriefing, persistent observation, and if proper also member check. Using trustworthiness as a single term is consistent with Golafshani’s (2003) assertion that in quantitative studies reliability and validity are viewed separately, but in qualitative research “terminology that encompasses both, such as credibility, transferability, and trustworthiness is used” (p. 600).
Another issue related to the quality of a study that distinguishes between a quantitative and qualitative paradigm is the research process. Morse et al. (2008) proposed that qualitative research is “iterative rather than linear, researchers should move back and forth between design and implementation to ensure congruence among question formulation, literature, recruitment, data collection strategies, and analysis” (p. 17). They suggested that researchers should systematically check the data to make sure the research focus is maintained, the conceptual framework informs the interpretations, and analyses are constantly monitored and verified (or disconfirmed). In order to ensure rigor and increase trustworthiness (or reliability and validity), they suggested verification strategies such as “ensuring methodological coherence, sampling sufficiency, developing a dynamic relationship between sampling, data collection and analysis, thinking theoretically, and theory development” (p. 17). The following discussion elaborates the issues of reliability and validity in qualitative studies.

3.6.1.1 Reliability
Examination of trustworthiness is essential to ensure reliability in qualitative research (Golafshani, 2003). Altheide and Johnson (1994) defined reliability as the stability of findings. LeCompte and Goetz (1982) proposed two types of reliability: external reliability refers to whether independent researchers would discover similar constructs in similar settings; internal reliability refers to the degree to which other researchers would agree on the match between constructs generated by the original researcher(s) and the data generated in that study.

Lincoln and Guba (1985) noted that “since there can be no validity without reliability, a demonstration of the former [validity] is sufficient to establish the latter [reliability]” (p.316). Patton (2002) also stated that reliability is a consequence of the validity in a study. The following discussion elaborates this point.
3.6.1.2 Validity

LeCompte and Goetz (1982) defined validity as concerned with the accuracy of scientific findings. Here, too, they proposed two key terms: *internal validity* refers to the extent to which research findings are authentic representations of some reality; *external validity* refers to the degree to which such representations may be compared legitimately across groups. It is important to think about validity and examine some strategies that have been developed to maximise it (Kirk & Miller, 1986; LeCompte & Goetz, 1982; Lincoln & Guba, 1985; Maxwell, 2002). Johnson (1997) postulated that in the qualitative paradigm validity refers to “research that is plausible, credible, trustworthy, and therefore, defensible” (p. 282), and Altheide and Johnson (1994) referred validity to the truthfulness of findings. Wolcott (1990b) elaborated the notions of internal and external validity for qualitative research that were first proposed by Cook and Campbell in 1979. He suggested that internal validity refers to “the degree to which a researcher is justified in concluding that an observed relationship is causal” (p. 287), which is supported by the strategy of data triangulation to maintain this (see more about triangulation below) and that external validity refers to the attempt “to generalize from a set of research findings to other people, settings, and times” (p. 289).

Johnson (1997) examined three types of validity, proposed by Maxwell in 1991, that are relevant and important to qualitative research: “descriptive validity, interpretive validity, and theoretical validity” (p.284). These three types are essential, because the two primary qualitative research activities are description of what is observed and the interpretation of participants’ thoughts and meanings. *Descriptive validity* refers to “accuracy in reporting descriptive information (e.g., description of events, objects, behaviors, people, setting, times, and places)” (p. 284). An effective strategy to obtain descriptive validity is “investigator triangulation” which allows cross checking of observations. Similarly, Wolcott (1990b) stated that “description is the foundation upon which qualitative research is built” (p. 27), and researchers should record data as accurately as possible, using participants’ words precisely. *Interpretive validity* refers to “accurately
portraying the meaning attached by participants to what is being studied by the researcher” (p. 285). Strategies for achieving interpretive validity include participant feedback, member check, and using low inference descriptors. *Theoretical validity* refers to “the degree that a theoretical explanation developed from a research study fits the data and, therefore, is credible and defensible” (p. 286). Strategies for promoting theoretical validity include extended fieldwork—spending a sufficient amount of time studying the participants and the settings.

*Triangulation* is an important strategy for ensuring trustworthiness (validity) of qualitative research. Creswell (2005) defined triangulation as a “process of corroborating evidence from different individuals, types of data, or methods of data collection in description and themes” (p. 252). Triangulation improves the validity and reliability of qualitative research, and it strengthens a study by combining data collection methods and sources (Golafshani, 2003; Patton, 2002). Golafshani (2003) further stated that using multiple methods, such as observation, interviews, recordings, and field notes leads to more valid, reliable, and diverse construction of realities. Creswell and Miller (2000) considered triangulation as “a validity procedure where researchers search for convergence among multiple and different sources of information to form themes or categories in a study” (p. 126). Wolcott (1990b) elaborated the logic of data triangulation. By using multiple methods of data collection and sources in a single study, one can reduce the weaknesses of any single method. Each method may provide better evidence for some aspects of the study, and thus the “whole is better than its parts” (p. 288).

A key issue germane to the trustworthiness of qualitative research is the impact of researchers’ own ways of thinking on how they frame a research problem, design a study, and make sense of the data. Johnson (1997) referred to this impact as the researcher bias, which “tends to result from selective observation and selective recording of information, and from allowing one’s personal views and perspectives to affect how data are interpreted and how the research is conducted” (p. 283). Such a bias presents a potential threat to trustworthiness due to the open-
ended and exploratory nature of qualitative research. He proposed two strategies to reduce such bias. *Reflectivity* implies that the researcher actively engages in self-reflection and critique about potential biases and predispositions; *Negative Case Sampling* refers to the researcher’s attempt to carefully and purposely seek examples that disconfirm their expectations and explanations about what they are studying. The latter strategy was employed in this study when, for example, the researcher compared expert and novice teachers’ use of what they might have referred to by the same term. Selection of teachers and data about them that showed deviation from the commonality partly served negative case sampling. The following discussion elaborates particular considerations of trustworthiness for this study.

3.6.2 Trustworthiness Considerations of this Study

This sub-section explicates some of the strategies used to enhance the trustworthiness of this study, including data verification and triangulation. Morse et al. (2008) suggested several such strategies, including “ensuring methodological coherence, sampling sufficiency, developing a dynamic relationship between sampling, data collection and analysis, thinking theoretically, and theory development” (p. 17).

To increase methodological coherence of this study, the research questions of this study are considered: (1) what pedagogical practices (activities, tasks, and strategies) can be identified in the participating Chinese middle school mathematics teaching? and (2) what teacher perspectives of mathematics knowing and learning may underlie these teachers’ practices? Pertaining to these questions, the AoP strategy of inquiry consists of observing two consecutive lessons of each teacher when she or he teaches a new concept. Such a pair of observations, which are video recorded and fully transcribed, provides ample evidence for addressing the first research question. It gives access to how a plan for the first lesson turns into an implemented lesson, which is then explained by the teacher as leading to the plan for what comes next (second lesson). The two lessons thus help to delineate the teacher’s conclusions about students’ learning and
understanding during the first lesson and how those, as well as the teacher’s understanding of the intended mathematics, impact the plan and implementation of the second lesson. Simply put, using a pair of lessons as a unit allowed the researcher to focus on the details of the teachers’ pedagogy from the dual, Western and Chinese perspective (having larger sets could detract the focus on details and obscure the analysis). The focus of the observations is on the concept being new to students in the teacher’s view, which means that coming from not knowing to knowing the new concept is ingrained in the teacher’s thinking about how to teach it. Furthermore, the pair of observations decreases the likelihood of collecting data that do not reflect the teacher’s typical way of teaching, along with the possibility to witness how what has been learned (or not) by students during the first lesson informs the teacher’s planning and implementation of subsequent teaching.

Likewise, interviewing the teacher three times—before, between, and after each observed lesson—increased the methodological coherence for addressing the second research question. As explained above, interview questions were tailored to very specific instances of the teacher’s pedagogical moves, during both planning and implementation. Such specific questioning provided ample evidence of the underlying meanings and reasons teachers seemed to have for specific strategies they used in their lessons. Most importantly, interweaving ongoing, zigzag analysis between interviews and lessons increased methodological congruence in addressing both research questions.

To ensure sampling sufficiency, the researcher focused on recruiting teachers who represent the phenomenon under study (middle school mathematics teachers’ practices and plausible teacher perspectives for using those practices), including a spectrum of different variables that impact teaching. As described above, school sampling considered the setting (large metropolitan areas), the type (public), academic orientation (mainstream, determined by the province), student population (both genders, elite boarding school and mixed level school), mathematics programs
(not using tracks or grouping by levels), and general conditions (class size of 42 and 50 students, 
room dimensions and organisation of tables, etc.). Teacher sampling considered gender (50-50 
between females and males), years of experience (two novices, two intermediate, and two 
veterans), and including not just a single teacher from a school but a group of teachers (half the 
faculty in one school and the entire faculty in another school) that could provide evidence for 
commonalities and individual deviations.

In this study, sampling sufficiency was also increased by first transcribing and examining the 
data sets for all eleven teachers. Once categories and themes emerged and were clearly identified, 
and category saturation was obtained, reducing the number of teachers to six was made based on 
the categories and themes. That is, the final sample of six was not decided arbitrarily, but was 
determined by the selection of data segments that represented the evolving description and 
theoretical accounts of teachers’ practice and perspectives. This ensured that saturation of 
categories was obtained first, and sufficient data to support and account for all aspects of the 
phenomenon were identified.

Dynamic relationship between sampling, data collection, and analysis was developed through the 
iterative, inductive zigzag process of data collection and ongoing analysis, and the way field 
notes and analytic memos were used to guide retrospective analysis once all data were 
transcribed and translated. The most critical aspect of this relationship was established in the 
preparation of interview questions based on observing the classroom, sifting through field notes, 
and creating initial memos. For example, to illustrate the ways that the observations and 
interviews contributed to a rich data source, the case of LX (30-year veteran) can be considered. 
The researcher noticed during the first lesson she observed in LX’s class that he engaged 
students in solving problems with numerical fractions, which they learned in the primary school. 
The researcher brought up this aspect in the second interview; LX provided the term (‘Bridging’) 
for it and explained his rationale. This aspect then guided an important focus of the researcher
when observing LX’s second lesson (and taking field notes), fed back into the questioning during
the third interview (then being considered as a potential patterns), and turned into a key
category—a pedagogical strategy that was looked for and asked about in conducting data sets
with the remaining teachers.

This example of LX’s also pertains to the increase of theoretical thinking by the researcher. The
initial puzzlement about ‘Bridging’ to primary school mathematics was rooted in the researcher’s
stance on the importance of reactivating students’ available conception (which draws on the
constructivist core construct of assimilation). Initially, LX’s use of ‘Bridging’ was interpreted as
‘focus on assimilation’. However, closer attention to how he used ‘Bridging’ and how he
explained its role in students’ learning indicated that the teacher’s stance on learning and
knowing was quite different than the one initially used by the researcher (i.e., example of
overcoming a researcher’s bias). Consequently, disconfirmation of the researcher’s initial
interpretation evolved into a new way of accounting for the teacher’s perspective, giving voice to
his rationale for the strategy as opposed to ‘seeing’ it through the researcher’s lens. Indeed, such
processes of theoretical thinking gave rise to the development of a theory that was grounded in
the data and the research process, instead of being adopted ‘off-the-shelf’ and imposed on the
data. Said differently, the theory about learning and about teachers’ perspectives that guided the
researcher in designing the study was used as a loose template for comparing and contrasting the
data and for developing theoretical explanations that were not part of the researcher’s thinking
before this study (Morse et al., 2008).

Last but certainly not least, the trustworthiness of this study was enhanced by triangulation,
which Creswell (2005) and other researchers (Creswell & Miller, 2000; Golafshani, 2003; Patton,
2002) proposed as a core strategy for making sure that qualitative research findings and
interpretations are accurate. The main triangulation methods were constant coordination among
observations, interviews, field notes, and artefacts. Conducting multiple data collection events
for each of these techniques is consistent with Johnson’s (1997) suggestion to expand data triangulation through using multiple data sources obtained via a single method. For example, the use of multiple interviews would provide multiple data sources while using a single method (i.e., the interview method). Likewise, the use of multiple observations would be another example of data triangulation; multiple data sources would be provided while using a single method (i.e., the observational method). Another important part of data triangulation involves collecting data at different times, at different places, and with different people.

A good example of this triangulation was provided above (LX’s ‘Bridging’ to primary school mathematics), as it linked data from the first interview with this teacher (plan for first observed lesson), video recording, transcript, and field notes of the first lesson, data from the second interview, video recording, transcript and field notes from the second lesson, and data from the exit interview. Another example pertains to triangulation after data collection was completed and during the transcription and translation processes. During these latter processes, some constantly occurring words (e.g., Xian Jie, the Chinese term translated to ‘Bridging’) surfaced as being commonly used by teachers from the first school. Because noticing the importance of these words occurred during data collection at the second school, teachers in the first school were not asked about them. To further triangulate this and use member check with teachers from the first school, the researcher wrote emails to those teachers with specific questions such as: Do you use Xian Jie when engaging students in solving the primary school problem of simplifying numerical fractions such as 4/8? Do you use such a strategy often? How often? And is it a typical strategy at your school or only you use it? Teachers willingly responded to those triangulating email exchanges and their responses either confirmed previous findings (e.g., the case of Bridging) or, if disconfirming, led to re-examining the data and identifying personal (or school) deviations from what initially seemed to be a commonality.
In addition, after data collection was completed the researcher revisited the first school, and talked with the principal, who is a team leader of the Grade-8 mathematics teacher group. Similar questions to those presented in the email exchanges were posed to the principal (e.g., do teachers in that school use the term and how often is this pedagogical strategy being employed?). Also, the principal was asked (member check) if he agreed with the researcher’s inference that the view of learning as a process of linking new ideas to old knowledge, which underlies the notion of Xian Jie (Bridging), is an accurate depiction of their stance (which, in this case, was confirmed). Asking the principal for confirmation although he was not a participant in this study supported the triangulation strategy, in that teachers in his school worked in a lesson-planning group under his leadership (like the second school). As a mathematics teacher and the leader of that group, he was familiar with every teacher’s work, and served as a senior teacher and mentor to the novice ones (e.g., JC) when they planned lessons based on the national curriculum and during team observation of teachers’ lessons. Triangulating data and preliminary (ongoing) analysis from the other school contributed to avoiding overgeneralisation from one school to another, and obtained what turned out to be a confirming input to the evolving categories and themes.

3.7 Summary of the Research Design and Methods
In summary, this study employed the AoP qualitative strategy of inquiry, which draws on and integrates features of phenomenology, phenomenography, and grounded theory approaches. This strategy of inquiry coordinates four major methods of data collection—observations, interviews, field notes, and artefacts—through (a) an iterative, inductive, zigzag process of ongoing analysis and (b) constant comparative process of retrospective analysis (progressing from data, to indicators, to coded categories and themes, to cross-case comparisons, to an account of practice). One data set consisting of observing two consecutive lessons and interviewing the teacher before, between and after the lessons was conducted with each of eleven teachers in two middle schools in South East China. After transcribing and translating the data, retrospective analysis focused on
segments informed by the categories and themes, led to saturation of categories, and to the selection of six teachers as representative cases for the evolving description of their teaching practice and grounded theoretical account of their perspective (rationale). Recruiting the eleven teachers and then narrowing down to the final six cases was based on school and individual criteria, and conducted according to the ethical procedures and criteria established by the Australian university. As discussed throughout this chapter, taken together these research activities helped to ensure the trustworthiness of this study (the construct used in qualitative research for integrating reliability and validity), including its descriptive validity, interpretive validity, and theoretical validity. Triangulating multiple data sources, methods, and collection events, coupled with repeated member checks and proactive search for confirming and disconfirming evidence to theoretical claims (inferences) about the data, helped to constitute methodological coherence, sampling sufficiency, theoretical thinking, and eventually saturation and credibility of description categories that comprised the findings of this study. These findings are presented in the next chapter.
Chapter 4:

Results

This chapter presents and describes data that address the two research questions of this thesis: (a) what pedagogical practices (activities, tasks, strategies) can be identified in the participating Chinese middle school mathematics teachers’ practices? and (b) what teacher perspectives of mathematics knowing and learning may underlie such practices? This chapter is organised in four sections that, combined, help to depict mathematics teaching of the Chinese teachers who participated in this study. Each of the four sections portrays one of four components that teachers used in the implementation phase of a mathematics lesson. The first section focuses on the lesson component of teaching through Reviewing problems (translated from participating teachers’ term of ‘Fu Xi 复习’), in which a teacher engages students in solving problems for activating the essence of newly learned ideas from the previous lesson. The second section focuses on the lesson component of teaching through Bridging problems, (translated from participating teachers’ terms of ‘Xian Jie 衔接’, or ‘Yin Ru 引入’). Bridging problems are designed to initiate and enable students’ learning of the day’s intended mathematical knowledge (‘new’) by posing problems or presenting examples that share common mathematical methods (‘Fang-Fa 方法’) the teacher assumes most students already know (‘old’). The third section focuses on the lesson component of teaching with Variation (translated from participating teachers’ term of ‘Bian Shi Jiao Xue 变式教学’), in which a teacher attempts to promote students’ learning of the ‘new’ mathematics via engaging them in solving problems and discussing students’ solutions. The fourth section focuses on the lesson component of teaching through Summarising (translated from participating teachers’ term of ‘Xiao Jie 小结’), in which a teacher highlights and recaps the core mathematical idea learned therein. This component is organised to accentuate the coherence of ideas learned through the first three components (henceforth, these components will be referred to as Reviewing, Bridging, Variation, and Summarising, for abbreviation).
As the participating teachers frequently emphasised, this four-component lesson organisation reflects their culturally-rooted beliefs that learning proceeds from old to new, from familiar to unfamiliar, from known to unknown, from easy to complex, and from specific to general. To promote such progressions, within each of the four components teachers utilised one or a mix of some of the following typical student activities (not necessarily in any particular order): (1) an individual student stands up and states her or his solution to a problem, (2) every student solves the problems independently, (3) three or four students solve the problems on the board while the rest solve the same problems independently, and (4) a whole class answers the question or read statements together, like a ‘talking chorus’. In the latter activity some students may be unsure of the answers and they are just whispering, but most of the time the majority of the students are talking together. In data excerpts, the term ‘chorus-read’ is used to indicate that what students said together was read from the board or a PowerPoint (PPT) slide; the term ‘chorus’ is used to indicate that what students say together is their vocal expression.

Each section provides data pertaining to two aspects of the participating Chinese middle school mathematics teachers’ practice. First, it provides data of common ways that participating teachers seemed to (a) use for promoting their students’ learning (practice) and (b) employ in thinking about their practice (perspective). Along with these commonalities, data about individual differences that reflect teachers’ personal preferences and styles are then presented and described. Such an organisation supports the two main claims of this study:

1. Chinese middle schools mathematics teachers who participated in this study structured their lessons in a way that seems to effectively promote high level of student learning;

2. Analysis of these teachers’ rationale for using such a structure reveals a pedagogical perspective about students’ learning of mathematics that is detailed in this chapter and discussed in Chapter 5.
4.1 Reviewing

To depict the role of Reviewing of previous lessons’ essence in students’ learning and how the participating Chinese teachers thought about this component, data segments were selected from four teachers: LX (30 years), WK (1 year), JC (3 years), and ZX (19 Years). Data segments were chosen from the teachers’ 2nd classroom observation. This allowed linking their teaching during that observed class to the teaching-learning processes witnessed in the previous lesson.

This section begins with LX and WK, because their work and thinking reflect the majority of teachers who participated in the study: moving from the previous day’s key concept, through particular instances of that concept, to Bridging. Then, the work of JC is introduced to highlight how a pedagogically novice teacher, with strong mathematical background (master’s degree in mathematics), proceeded from particular examples to the intended concept while centring on a single pedagogical feature—drawing on students’ errors—to benefit everyone’s learning. Finally, ZX’s specific-to-general approach is reported, to demonstrate an individual teacher’s deviation to support students’ learning—particularly her low-achieving students—via reactivating students’ existing knowledge.

Throughout this section, and the entire chapter, data presented for each teacher focus first on her or his practice. Then, an account of the teacher’s rationale is presented on the basis of interview data about what she or he (and students) did in class.

4.1.1 A Reviewing Expert (LX)

This sub-section presents data about LX, a 30-year veteran teacher, who commented in the interviews that his mathematics lessons most often started by reviewing the essence from the previous lesson. These data reveal that he did this because he appears to think that new knowledge is extended from old knowledge. From observations, analysis of lesson observation records, and from interview responses it was concluded that his typical way of teaching is to separate the 45-minute lessons into three, roughly 15-minute parts. The first part is devoted to
reviewing the topic of the previous lesson (‘old’ knowledge), which was to be extended by the
current day’s new learning. This is based on his view of learning from known to unknown. Thus,
he seemed to consider his role as a teacher as including reactivating students’ existing
knowledge into which the ‘new’ knowledge is to be linked. The following discussion elaborates
this point.

4.1.1.1 LX’s First Observed Lesson – Content of Reviewing in the Next Lesson

The first lesson observed in LX’s classroom, which was the content of Reviewing in the second
observed lesson, focused on the meaning of algebraic fractions. In it, students learned the
definition [i.e., there is a variable in the denominator], the conditions for the algebraic fraction to
be meaningful [denominator ≠ 0], and when the value of the algebraic fraction is 0 [numerator=0
and the denominator≠0]. Accordingly, in that previous lesson, students solved three different sets
of tasks (see Box 4.1.1.1).

For homework, students were assigned more problems, in which they had to determine and
explain restrictions to the value(s) of x for a given algebraic fraction and when it is equal to 0
(see Box 4.1.1.1). The homework also included two open-ended questions. In the first, students
were asked to write an algebraic fraction that, under any condition, (a) would not be equal to 0
and (b) have no restrictions. In the second, students had to create a word problem or a context for
a given algebraic fraction. It should be noted that LX, like the Chinese teachers observed in this
study, checked and returned individual students’ homework daily. This seemed to give the
teacher a good sense of students who understood and mastered the previous lessons’ materials
and those who needed more support. The participating teachers’ responses during interviews
suggested that, quite often, specific instances a teacher selected for the Reviewing component
were based on students’ checked homework.
Box 4.1.1.1: Main topics taught by LX the previous day (with examples).

1. Definition: Which of the following is an algebraic fraction?
   \[
   \frac{2m}{\pi}, \quad \frac{a+b}{ab}, \quad \frac{3x+2y}{5}
   \]

2. Restrictions: What values for the denominator are restricted?
   \[
   \frac{2x+1}{3x-5}
   \]

3. Value of fraction is 0: Which value(s) of x make the algebraic fraction 0?
   \[
   \frac{x^2-4}{x-2}
   \]  
   \(x = -2\) but not \(x = 2\)

4. Homework problems (examples):
   \[
   \frac{(x-4)(x+2)}{|x|+4}
   \]
   Restrictions? When is it 0?

Make a word problem to be solved by \(\frac{m}{a+b}\)

---

4.1.1.2 LX’s Reviewing of Algebraic Fractions: Abstract-to-Specific

LX’s stated goal for students’ learning in the second observed lesson was to understand and apply the simplification of algebraic fractions. In the first interview he said that the knowledge from the previous lesson was a foundation for learning this topic. He also said he found some common errors in students’ homework from the previous lesson. Therefore, before he moved to teaching the new knowledge, he thought it would be better to clarify student errors and make sure they knew why these mistakes were occurring. To this end, he opened the lesson by reviewing the previous lesson, proceeding from the abstract concept to concrete examples. As Excerpt 4.1.1-a indicates, the first two minutes of the lesson required students to mentally recall the three key ideas—definition, restrictions, and value = zero. In all excerpts, lines are numbered according to the time stamp on the video recording, ‘T’ stands for the teacher, ‘Students’ stands
for a group or whole class expression, and ‘$S_N$’ (e.g., S1) stands for an individual student and an index number consistently assigned to him/her throughout the lesson.

Excerpt 4.1.1-a (Teacher LX, 2009-05-20)

08:47 T: We have learned the concept of algebraic fraction yesterday. Can you recall what is an algebraic fraction?

08:50 Students (chorus): Divide two expressions, and there is a variable in the divisor; this kind of algebraic expression is called algebraic fraction.

09:19 T: The second issue is the condition for the algebraic fraction to be meaningful. How do you describe this?

09:26 Students (chorus): The denominator of the algebraic fraction cannot be 0.

09:33 T: The denominator cannot be 0, right? The third issue is the condition for when the value of the algebraic fraction is 0. How do you describe this?

09:41 Students (chorus): The numerator is 0 and the denominator cannot be 0.

09:50 T: These are the main contents we have learned yesterday. What is the key for distinguishing between an algebraic fraction and an algebraic expression? (LX looks around the class and calls a student’s name) S1?

10:04 S1 (stands up and says): An algebraic fraction has a variable in the denominator.

10:12 T: Yes, this is the key point. (He shows a PowerPoint slide with the same statement and asks students to read it together.)

10:17 Students (chorus-read): The key for distinguishing the algebraic expression from an algebraic fraction is if there is a variable in the denominator.

The data in Excerpt 4.1.1-a show that to commence a lesson on simplifying algebraic fractions LX first engaged them in reciting abstract, general mathematical ideas they learned in the previous day. His first question referred to the definition of an algebraic fraction—the ‘object’ on which they would need to operate (simplify). All teachers observed in this study used this definition of an algebraic fraction, and it seemed an important aspect of the mathematics they intended for students to learn. LX’s second question referred to a rule that governs the mathematical meaningfulness of this ‘object’. As the previous day’s lesson emphasised, a fraction is a form of division, and the divisor (hence denominator) cannot be zero. His third question referred to a specifically important value this ‘object’ can have (zero) and the division-based rule for it (dividend/numerator = 0).

Students answered the three initial questions together, as a chorus. Using this classroom activity seemed to have indicated to the students that, at this point, each and every one of them was expected to recall these basic ideas. They have not only been taught those ideas in the previous
lesson, but were supposedly applying (and likely memorising) them as homework. To further indicate this expectation, LX repeated the first, essential question in a slightly different way while directing it to a single student (S1). Such a classroom activity of obtaining a single student’s response was used frequently, and seemed to further indicate to students that each individual should be responsible for knowing the essence of the previous lesson. A third, different variation of stating the previous day’s abstract, general idea culminated those initial teacher-students exchanges—a whole class (chorus) reading of the contrast between simple (numerical) and algebraic fractions.

Close examination of the video segment revealed how lip movements of some students lagged a bit behind others’ (or were absent altogether), indicating that they could not yet produce the statements independently. Further, there is a difference between being asked to recall the principles on one’s own and being asked to simply read it from the screen. In the former case, only students who have already established the sought conception would be able to independently regenerate it. In the latter case, reading a statement allows mechanical, meaningless participation without the required knowing. But such reading could provide access to the key idea (variable in the denominator) to some students who were yet to establish it for themselves.

LX was aware that not all of his students were familiar with the abstract concepts. This awareness was indicated in his responses during the follow-up interview. As he explained, the reading technique is important for effective mathematics learning at this age, because the combination of mechanical memorisation and comprehensive understanding fits “their moderate level of logical thinking.” This is why, after the abstract level of Reviewing, he would lead the class into work on specific examples (see Excerpt 4.1.1-b).

Students answering the teacher’s abstract, prompt-less questions as a chorus could support re-
activation of what they have come to know. However, for those who were yet to conceptualise
the previous day’s knowledge, hearing their peers reciting the key principles, or reading it from
the PowerPoint slides in the abstract form themselves, could at best provide a link to the
terminology. Said differently, for the former sub-group of strong students, moving to the next
element of Reviewing, in which each student would individually have to determine if a fraction
is algebraic or not could promote application of what they already knew to particular instances of
the concept. For the latter sub-group of low-achieving students, engaging in solving the few
problems could provide an opportunity to actively and repeatedly process the previous day’s
knowledge in additional, particular instances of the abstract, general idea.

Immediately following students’ recitation of the abstract principles, LX proceeded with
Reviewing by presenting eight problems (Line 10:28, Box 4.1.1.2). It should be noted that
students did not solve these problems previously (i.e., this was not just checking homework).
Rather, LX said in the interview that these problems were chosen so that students would apply
anew the stated principles to a few examples while explicitly distinguishing among (a) the case
of algebraic fractions, (b) the case of algebraic expressions that are not fractions, and (c) the case
of fractions that are not algebraic (including the possible misconception of a fraction with a π
symbol in the denominator). As a direct follow-up to Excerpt 4.1.1-a, students’ individual work
on the problems in Excerpt 4.1.1-b and LX’s inquiry into their reasoning about problems #5 and
#8 demonstrate the case of a teacher whose Reviewing component proceeds from the general to
the specific.

Excerpt 4.1.1-b (Teacher LX, 2009-05-20)
10:28 T: think about this: Which ones are algebraic fractions?

| Box 4.1.1.2: Reviewing problems for checking definition of an algebraic fraction |
|-------------------|------------------|------------------|---------------------|
| ① 5x-7            | ② $\frac{b - 3}{2a + 1}$ | ③ 3x²-1         | ④ $\frac{4}{5b + c}$ |
| ⑤ $\frac{2}{3\pi}$ | ⑥ $\frac{x^2 - xy + y^2}{2x - 1}$ | ⑦ $\frac{m}{n}$ | ⑧ $\frac{m(n + p)}{7}$ |

10:45 T: S2 - can you share?
10:46 S2: (Stands up and says) [examples number] 2, 4, 6, 7 are algebraic fractions.
10:58 T (to class): Do you have different opinions?
10:59 Students (chorus): No.
11:06 T: Why is $\frac{2}{3\pi}$ not an algebraic fraction?
11:06 Students (many students together): because $\pi$ is a number, not a variable.
11:13 T: Because $\pi$ is a constant number, not a variable. How about $5x-7$?
11:21 T: This is an algebraic expression. How about $\frac{m(n + p)}{7}$?
11:25 Students (chorus): It is not an algebraic fraction because there is no variable in the denominator.

The first two lines in Excerpt 4.1.1-b (10:28, 10:45) reflect the level of mastery that LX seemed to expect of his students. Within about 20 seconds, they had to determine whether or not a symbolic expression is or is not an algebraic fraction (roughly 2.5 seconds per example). Moreover, as the 40-second exchanges that ensued (10:45 – 11:25) indicate, LX seemed to expect that while solving these eight problems students would also pay attention to non-examples and be able to reason about them on the basis of the general principles. For him, it appears that students’ ability to properly determine that an expression is or is not an algebraic fraction was insufficient. This is inferred from his follow-up questions to the entire class after S2 gave the correct answer alone. First, LX pointed to a common misconception of fraction (#5) and asked students why they do not consider it as algebraic. Having received the correct response (“because $\pi$ is a number, not a variable”) from a large portion of the class, he then pointed to another potentially confusing fraction (#8, variables only in the numerator). Again, he seemed to expect, and accept, a definition-based reason (“It is not an algebraic expression because there is no variable in the denominator”), which could also support understanding of the conditions for determining an algebraic fraction. These exchanges indicate that, for LX, a teacher’s responsibility to support students’ establishment of the foundation needed for the day’s learning could be accomplished via reviewing (solving) specific examples. The particular set of tasks was closely tied to the goal of his Reviewing, namely, students’ understanding of what makes a mathematical expression an
algebraic fraction. By first letting students solve problems as individuals and then sharing their answers and reasons each of them was actively involved in considering and re-considering various instances of the general idea of an algebraic fraction.

One key to making LX’s teaching potentially effective seems to be the mixed selection of examples and non-examples. As this appeared to be his typical way of Reviewing, students have long established the need to check which is which. With variables only in the numerator (#1 and #3) and denominators that included only a constant (#5 and #8), most students could engage in a mental activity of considering variable locations, which has been pointed out during the first element as the proper method (even if then it was only mechanical). Thus, LX’s selection of instances could bring forth students’ interpretation of the examples into an evolving conception—the location of a variable in an expression is a defining attribute of the ‘object’ they were about to learn to simplify. Then, by asking students to express and explain their solutions, LX could promote their way of applying the fraction classification method. Consequently, for students who might have begun conceptualising the previous day’s idea (mid-level students), the second part of Reviewing—applying to particular instances—could foster reasoning that is consistent with a socially accepted, mathematically justified method. For the low-achieving students, those who were yet to begin conceptualising the previous day’s concept, LX created one more learning opportunity. First, they listened to peers’ recalling of the abstract concept. Next, they saw and read aloud the concept formulation. Then, they attempted to apply the stated principles while identifying particular instances. Finally, they could revisit and revise their solutions by hearing peers’ and teacher solutions to the concrete examples. Thus, the low-achieving students could engage in figuring out how and why the definition is applied.

The observation excerpts above indicated four learning-supportive aspects of LX’s Reviewing. First, it could promote mid-level students’ learning by hearing some prompts from their more knowledgeable peers’ answers (Murata & Fuson, 2006; Vygotsky, 1978). Second, LX used
different classroom activities that engaged students in learning either via recitation (meaningful or meaningless) of the abstract concept or via operations on particular instances. Third, he explicitly oriented the students’ awareness onto typical errors they could make, cultivated their ability to find and correct the errors, and fostered a disposition of the need to avoid making such errors. Fourth, his teaching was well designed in that the tasks he selected served different purposes for different students (Sullivan, Mousley, & Zevenbergen, 2004). In particular, after students were reminded of the previous day’s general rule, LX introduced eight examples that could orient their reflection within each example and across situations (Tzur & Simon, 2004). These examples were more simple and accessible to students than the advanced ones used in the previous day’s lesson and homework. Thus, LX’s Reviewing instances seemed likely to bring forth conceptualisation of the intended idea at a level needed for productively participating in the present day’s lesson. The following sub-section elaborates the perspective that seemed to underlie LX’s practice.

4.1.1.3 Accounting for LX’s Perspective (in Reviewing)

To account for LX’s practice, video recorded input from him during the interviews before, between, and after the two observed lessons was examined, as explained in Chapter 3. This examination indicated that LX’s use of Reviewing focused on establishing, via student reactivation of the previous day’s concept, a foundation (conceptual ‘anchor’) he appeared to consider necessary for incorporating the current day’s new mathematics. In the follow-up interview, LX clarified that his goal in Reviewing was threefold. First, he wanted to assess if the students had a real understanding of the previous day’s concept (e.g., what is an algebraic fraction, when is it meaningful, and when is its value $= 0$). Second, he wanted to give them opportunities for applying that knowledge and thus to consolidate it. Third, he wanted to orient students’ awareness to relevant errors. This threefold approach indicates two important aspects of his pedagogy. First, LX seems to see as his main responsibility to figure out whether students are familiar with and can thus reactivate the previous day’s knowledge. Second, he seems to
think that without reestablishing the ‘old’ knowledge it is hard (or impossible) for students to meaningfully learn the new, intended knowledge. That is, it can be inferred that LX conceives of ‘new’ knowledge as an extension of the ‘old’ and of learning as gradually linking the ‘new’ to the ‘old’.

These two aspects are also evidenced in Excerpt 4.1.1-c below. In it, LX explained how he structures each of his mathematics lessons so the first part enables students to approach the learning of new, intended mathematics by working on problems they already knew. (To differentiate interview excerpts from observation excerpts, a different font is used for interviews.)

*Excerpt 4.1.1-c (Teacher LX, Intrv, 2009-05-20)*

I separate the 45 minutes lesson into three 15-minute segments. The first 15 minutes is reviewing the issues and problems from the previous lesson. Learning is from known to unknown, the new knowledge is always extended from old knowledge. From the review [we] naturally extend to a new topic. So the first 15-minute review of the lesson is not only a review of the knowledge from last lesson but also the knowledge related with today’s new lesson, and build a bridge, then propose a new concept or new knowledge. (Emphases added).

Excerpt 4.1.1-c presents a Chinese teacher’s deliberate and sustained structuring of a mathematics lesson to actively use and build on students’ existing knowledge. One may consider the use of 1/3 of a class time for working on what students already know as ineffective use of precious instructional time. For LX, however, this use of time seemed natural, sensible, and effective, because if learning entails the progression from known to unknown a teacher should proactively position both in some temporal and conceptual proximity. Thus, LX used the first 15 minutes of a lesson to position the known (‘old’) as an anchor for the ‘new’.

The key notions in Excerpt 4.1.1-c seem to be ‘bridge’ (later LX uses also ‘link’) and ‘extension’. The former indicates that for LX the two pieces of mathematical knowledge, ‘old’ and ‘new’, already exist in the world and can thus be linked to one another. Indeed, such linking requires placing both pieces in close proximity, which helps to explain why as a teacher he took responsibility for assessing and bringing forth students’ ‘old’ knowledge. The latter notion of
‘extension’ indicates that LX seems to conceive of learning as incorporation of the ‘new’ (independently existing ideas) into the ‘old’. That is, his view of learning seems to focus on mathematics (as it is known to the teacher) being extended in a piece-by-piece fashion in the students’ minds. Every new piece builds from and is linked to the ‘old’; the ‘old’ serves as the anchor for the ‘new’, whereas the ‘new’ is appended to the ‘old’.

For LX, Reviewing—the reactivation of conceptions that evolved in the previous day’s lesson—seems to serve as a means to establish and build a link between the old knowledge and the new. To this end, he must continually assess and re-assess students’ mathematics, as it is revealed both in general, abstract terms and in application to specific examples. Excerpt 4.1.1-d focuses on LX’s awareness of his need to understand what students do or do not understand. His response here followed an interview question about why he often requested answers from students who did not raise their hands.

*Excerpt 4.1.1-d (Teacher LX, Intrv, 2009-05-20)*

> Those who raised their hands are sure they can answer correctly and have real understanding. But I want to find out whether those who did not raise their hands can or cannot answer my question, or have a real understanding, and to check where they are. (Emphases added)

Excerpt 4.1.1-d indicates that LX is not simply seeking some student solutions to mathematical problems, which might be given by those who already learned the intended concepts. Rather, he asked students who did not raise their hands to share their solutions. In doing this, he seems to be interested in the understandings of those who might still need to learn the intended concepts. As data presented later in this chapter corroborate, LX and the rest of the teachers would quite often ask for solutions not just from any student who did not raise her or his hand. Rather, responses would be asked of students the teacher knew to have made a mistake. In both cases, as LX’s explanation clarified, the teacher intended to figure out what mid-level and/or low-achieving students understood. A plausible rationale for this purposeful teaching activity has already been mentioned. If a teacher conceives of learning as a process of linking ‘new’ mathematical ideas to
‘old’ knowledge, she or he has to figure out if that ‘old’ knowledge exists in students so it can be used as anchor for the ‘new’.

LX’s response in Excerpt 4.1.1-d led to a follow-up interview question regarding students’ likely embarrassment when/if they answer incorrectly. He responded that making mistakes as part of attempting to solve problems and exposing those mistakes in public are encouraged in his lessons. Most importantly, reflecting his cultural belief of the teacher’s responsibility for the learning of each and every student, he said that by asking those who did not raised their hands he provided everyone with an opportunity to learn. That is, he appears to believe that by letting a student make and expose mistakes, a good learning opportunity is created for that specific student as well as for other students. LX said he actually expects students to answer a question incorrectly. In this way, the rest of the class can correct the mistake together, which he said was likely to orient students’ attention to the underlying mathematical concept and to convince them to avoid making the same kind of mistakes. Example #5 above (π in the denominator) and data presented later about his teaching with Variation show that LX and other teachers intentionally included incorrect solutions for students’ consideration in case such errors are not exposed via their own work.

Excerpt 4.1.1-d and the ensuing information about making mistakes in public indicates LX’s focus on the role that students’ existing knowledge plays in learning new mathematical ideas. He used individual students’ responses to the particular set of examples as an intentional means to both discuss the mathematical aspects of correct and faulty solutions and to assess what students do or do not understand. He made a rough assessment of which students may already understand the previous day’s lesson (e.g., from their previous lesson’s answers and/or homework). Thus, he avoided letting strong students demonstrate the correct responses at the outset. Instead, he took the time to provide learning opportunities for the mid-level and low-achieving students via making and reflecting on their own (or others’) mistakes. In the meantime he could also
determine what they got out of the previous day’s lesson. He appeared to believe that without such assessment of the ‘old’ piece of mathematics neither he nor the students have the anchor needed for linking the ‘new’.

Further demonstration of LX’s perspective can be seen in the particular examples he used. He said in the interview that he selected those among many that were available in the national standard curriculum, on the basis of what he thought constituted students’ available knowledge and thus anticipated mistakes they might typically make. First, from their solution to the homework problem of \(\frac{x^2 - 4}{x - 2} = 0\), he knew that students’ available conceptions were likely to include a focus on finding the value (x=±2) for the denominator = 0 while overlooking the condition that the denominator x-2 can’t be 0. Because his mathematics seemed to imply that both are needed as a basis for the new (today’s) knowledge, his intention seemed to be figuring out whether and ensuring that students understand the combined condition. Moreover, this was also an example that, as he explained in the interview, could serve as the ‘link’ to this day’s concept. It could be solved without simplification, which would get rid of the extraneous value.

Yet, after the non-simplification method would be demonstrated on the board, LX could take the time to also show the simplification as a possible solution. He said that this could highlight the link from yesterday’s to this day’s lesson.

Second, by using the example of \(\frac{2}{3\pi}\), LX seemed to anticipate that some students might think π is a variable instead of a constant number. Consequently, he expected these students to incorrectly consider the expression as an algebraic fraction. By letting students who did not raise their hands to respond, and later asking the entire class if they agreed, he created an opportunity to assess who might still be thinking this way. Furthermore, he could then use the students’ error as a basis for promoting their anticipation of this kind of mistake, which was crucial if they were to link the ‘new’ to a proper understanding of ‘old’ knowledge. Simply put, gradually linking
new mathematical ideas to the ‘old’ in a way that conforms to mathematical conventions as they are understood by others in the community (e.g., teachers) necessitates that the ‘old’ is held and executed correctly. To this end, Reviewing appeared to serve LX as a first instructional step for both assessing what students know and fostering correct re-activation of the previous day’s anchor knowledge.

In summary, LX’s Reviewing seemed to target a threefold pedagogical goal. First, he sought to reactivate the foundation needed for students’ incorporation of the new knowledge that he was about to teach in the present day’s lesson. Second, to establish this foundation he assessed how students understood the previous day’s knowledge while engaging them in solving problems and doing exercises that could further such understandings. By reviewing the previous day’s lesson, he seemed to believe that more students, particularly the low-achieving students (a) were provided with an opportunity to access the knowledge, (b) could begin solving problems in the intended way, (c) might successfully complete the solution, and (d) were likely to correctly incorporate the previous day’s knowledge into what has been familiar to them. Third, his role seemed not only to teach mathematics in the lesson but also to cultivate students’ learning how to learn. To this end, he encouraged students’ development of good mathematical habits (e.g., exposing mistakes and learning from them, which could lead to a humble identity and self-improvement disposition).

4.1.2 A Reviewing Novice (WK)
This sub-section reports how WK—a young, novice teacher in her first year of teaching—used the lesson component of Reviewing. She worked with LX’s teacher group and was mentored by him. Like LX, WK also presents a case of a teacher whose Reviewing proceeds from the general, abstract mathematical ideas of the previous day’s lesson to specific instances of those ‘old’ ideas. The reason her case was selected is twofold. First, it provides another example of teachers in this study who used Reviewing to begin a lesson. They typically seemed to engage students in
reciting the general rules from the previous day’s lesson and then applying the rules to a few examples. This sequence of initial activities differs from typical constructivists’ approaches, which proceed from specific, experientially relevant examples to the general, synthesising experience. Second, WK’s case helps to show that a Chinese teacher can have the autonomy and responsibility to make individual, intentional choices. To teach a lesson on simplifying algebraic fractions (first of the two observed), which she planned together with LX and taught while being observed by him and the entire Year-7 teacher team as part of their weekly visits to one another’s classes, she chose different specific activities and tasks.

WK said in the interview between the two lessons that she usually starts her lessons by reviewing the important points from the previous lesson. Like LX, she appeared to think of new learning as an extension of the ‘old’. According to her interview responses, without establishing the old knowledge as solid foundations for the students she will not move to teaching the ‘new’. She seemed to see her role as a mathematics teacher to activate students’ existing knowledge as a basis into which the ‘new’ would be linked. The following discussion elaborates this point.

4.1.2.1 Contextualising WK’s Reviewing in the Observed Lesson
To contextualise the content and process that WK used in her Reviewing during the second lesson, the previous observed lesson is described first. That first observed lesson focused on reducing and simplifying algebraic fractions (similar to LX’s second lesson). Her students were taught the properties of algebraic fractions, application of properties, and sign conventions. They also learned to reduce two possible types of algebraic fractions, one with only monomials in both the numerator and the denominator, the other with polynomials in either or both. Students were directly instructed how to use the underlying method (Fang Fa) of simplification: first, pay attention to and confirm the sign; second, look for common factors and factor the polynomial; third, reduce the common factor(s); and fourth, make sure the answer is a fraction in its simplest form (i.e., no common factors in both numerator and denominator).
In the first observed lesson, WK’s students solved different sets of tasks pertaining to different purposes (see Box 4.1.2.1). It should be noted that the problems in Box 4.1.2.1 might appear difficult for Westerners; however, they seemed to be considered as basic expectation in the participating Chinese teachers’ lessons.

<table>
<thead>
<tr>
<th>Box 4.1.2.1: Main topics taught in WK’s first lesson with examples.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Algebraic Fractions Properties</td>
</tr>
<tr>
<td>Fill in the blanks (parentheses) so that the value of each given fraction is maintained.</td>
</tr>
<tr>
<td>(1) ( \frac{2}{xy} = \left( \right) \frac{1}{x^2y} )  (2) ( \frac{3x}{x+y} = \left( \right) \frac{15x(x+y)}{x} )  (3) ( \frac{x+y}{x^2-y^2} = \left( \right) \frac{1}{x-y} )</td>
</tr>
<tr>
<td>2. Sign Conventions:</td>
</tr>
<tr>
<td>Determine the single sign for the entire fraction so its value is maintained and place the sign outside the fraction.</td>
</tr>
<tr>
<td>(1) ( \frac{-a}{-2b} )  (2) ( \frac{-3x}{2y} )  (3) ( \frac{-x^2}{2a} )</td>
</tr>
<tr>
<td>3. Reducing Algebraic Fractions:</td>
</tr>
<tr>
<td>Reduce the following fractions to the simplest form.</td>
</tr>
<tr>
<td>(1) ( \frac{-8ab^2c}{-12a^2b} )  (2) ( \frac{a^2 + 4a + 4}{a^2 - 4} )  (3) ( \frac{4a^2b}{-6ab^2} )  (4) ( \frac{y^2 - 9}{-2y^2 + 6y} )</td>
</tr>
</tbody>
</table>

WK’s stated goal for the second observed lesson was for students to learn how and why to multiply and divide algebraic fractions. Her use of the examples that were available in her teacher group’s plan seemed to follow a script for the lesson. The following discussion turns to WK’s Reviewing during the second observed lesson.

4.1.2.2 WK’s Reviewing of Simplifying Algebraic Fractions: Abstract-to-Specific

WK opened the second observed lesson by engaging students in silently reading and reviewing points about the previous day’s learning on their own, while she stood quietly in the middle of the class (about two minutes). That is, like LX she engaged the students in reflecting upon the essence of the previous lesson, which was mostly presented in an abstract, symbolic form (before
she moved to concrete examples). After about two minutes, she began interacting with the students about what they had just read. These initial exchanges of Reviewing after students’ silent reading are presented in Excerpt 4.1.2-a. (Note about translation: In Chinese, the word for *simplification* can refer to modifying a fraction into an equivalent fraction in a simpler form, which is literally translated into ‘reduction’, and to the process of cancelling the common factors. Throughout this study, to maintain the former meaning the word reduction was used; when the reference was to operating on common factors the word ‘cancellation’ was used.)

*Excerpt 4.1.2-a (Teacher WK, 2009-05-20)*

02:12 T (after students read their notes silently): Ok, let’s have a review of yesterday’s lesson. We have studied algebraic fractions. Can you tell me what kind of algebraic expression is [also] an algebraic fraction?

02:21 Students (chorus): When two algebraic expressions are divided and the divisor contains a variable.

02:34 T: What was the important content in the previous lesson?

02:38 Students (chorus): Simplification, reduction.

02:40 T: What was the criterion for the reduction?

02:42 Students (chorus): The basic properties of the algebraic fraction.

02:50 T: Who can tell me the properties of the algebraic fraction? S1?

02:53 S1 (Stands up and answers aloud): If the numerator and denominator are multiplied or divided by an algebraic expression that is not equal to 0 simultaneously, then the value of the algebraic fraction remains the same.

03:00 T: Very good! Can anyone else tell me more about the basic properties of the algebraic fraction? S2?

03:11 S2: (Stands up and literally repeats what S1 has just said.)

03:33 T: Ok, these are the properties of the algebraic fraction, and we learned the reduction on the basis of these properties. What do we cancel from the numerator and the denominator?

03:44 Students (chorus): The common factor.

03:46 T: So reduction means cancelling the factor common to both the numerator and the denominator. How many possibilities do we have for reduction? We have learned monomial, polynomial, and the steps of the reduction. First of all, how do you reduce if the algebraic fraction is all monomial?

04:25 Students (chorus): Extract the common factor.

04:28 T: Eventually - what do you cancel?

04:29 Students (chorus): The common factor.

04:32 T: What do you do if it is a polynomial?

04:34 Students (chorus): [We] factor.

04:40 T: Then what do you cancel?

04:41 Students (chorus): The common factor.

04:58 T: What we cancel in the algebraic fraction is the common factor. One important point, the final result should be the simplest form. What is the simplest form?

05:07 Students (chorus): [There is] no common factor.
There is no common factor in the numerator and the denominator. If there is a polynomial, you should firstly factor. One important point is that the final result has to be the simplest form, or …?

Students (chorus): Algebraic expression.

05:15 T: Ok. This is the knowledge content we have learned yesterday. Now we are going to do some exercises based on the knowledge content we learned from yesterday.

WK began her lesson on multiplication and division of algebraic fractions by letting students read silently the key points from the previous day’s learning (algebraic fraction simplification). Then, as the data in Excerpt 4.1.2-a show, she engaged the students in rehearsing those key ideas. To this end, she presented a sequence of rhetorical questions that the entire class answered as a chorus or particular students she selected answered aloud.

She started by asking about the definition of the mathematical ‘object’ at issue for both lessons (line 02:12, chorus response – algebraic fraction as division with variable in the denominator). Next, she asked students to name the essential idea from the previous lesson (line 02:34, chorus - simplification) and what are the criteria for properly doing it (line 02:40, chorus – properties of algebraic fractions). At this point, turning to an individual student, WK asked the student to specify those properties (line 02:50, S1 – expresses the idea that multiplication by 1, in the form of a fraction n/n, maintains the value of the fraction). WK accepted and praised the response, then asked another student for additional properties (line, 03:00, S2 – repeated S1’s statement). Not pursuing what she might have considered additional properties, she returned to the entire class and, while specifying the two constituents (numerator, denominator), and asked what is cancelled in a fraction (line 03:33, chorus – common factors). She echoed the students’ response, pointed out types of expressions that could make up an algebraic fraction—monomial or polynomial—and asked students how a fraction made of monomials is simplified (line 03:46, chorus – extract and then reduce the common factor). Accepting their response, she asked about a fraction made of polynomials (line 04:32, chorus – factor it) and continued to what would they be reducing in such a case (line 04:40, chorus – the common factor). She paraphrased their response while emphasising the need to factorise a polynomial if given, and then repeated the
need to check that the reduced fraction is given in its simplest form. At this point, WK stated that this set of points was the mathematical essence of what they learned in the previous day and told them they would then begin solving problems that were based on that knowledge.

WK’s use of Reviewing shows a typical way in which most of the teachers who participated in this study proceeded with it, namely, from the general, abstract, symbolic form of this ‘old’ knowledge to application of it to particular instances (see Excerpt 4.1.2-b). Unlike LX, however, she provided her students with time to initially read the previous day’s properties silently. Such reading seemed to support their ability to recite the asked-for mathematics, as a chorus. That is, she enabled students to recall and (for some) possibly make a bit more sense of the ‘old’ knowledge prior to being required to state it with the entire class. Her teaching during this part of the lesson seemed rather traditional, with students responding quite mechanically to teacher’s rhetoric questions. During this initial segment, memorisation of the mathematical statements would have been all it required for students to properly participate in the lesson. Such participation might not be conductive to learning—for the high-achieving students it would not be needed and for the low-achieving students who might have not conceptualised the key ideas in the previous lesson such an activity of repeating the ideas mechanically did not seem likely to change their understanding.

It seems that to make sense of the abstract notes while reading silently, or to successfully solve the examples, students must have completed or at least began forming the previous day’s knowledge. For those strong and mid-level students, these activities could promote either more application of known concepts to novel instances (strong students) or reactivation and further processing of the given instances and possibly noticing the underlying mathematics common to all examples (mid-level students). However, it seems that for low-achieving students who were yet to form a rudimentary understanding of simplifying algebraic fractions, both activities (reading the abstract points, stating them) were not conducive to meaningful learning. In this
sense, Reviewing that begins with the abstract form of the ‘old’ might promote high-level learning by students whose available conceptions have already been instigated. As mentioned earlier, such an approach differs from a constructivist approach, in which students learn through synthesising and abstracting the general idea by first working and reflecting on specific instances.

As stated at the end of Excerpt 4.1.2-a, after WK promoted re-establishment of the previous day’s abstract, new knowledge, she sought to engage all students in further Reviewing via solving four specific problems independently. In the meantime, she invited four individual students to solve these problems on the board (one problem per student). Two of these four problems were new to students whereas the other two consisted of subtle changes to exercises she provided in the previous lesson. The latter two were not identical, because (as explained in the interview) she did not want students to simply restate or copy their previous solutions.

Excerpt 4.1.2-b below presents data about how WK continued with Reviewing—application of the abstract ‘old’ knowledge to particular instances. The first problem served as a case for cancelling common factors in monomials. The second problem was also a case of monomials while also focusing on sign conventions (multiply x-y by -1 to obtain y-x as a common factor). The latter two problems consisted of simplification that involved polynomials. Excerpt 4.1.2-b culminates with multiplication and division of simple, numerical fractions; these actually present her transition to the second component of her lesson (Bridging). (Note: When the abbreviation PPT appears in an excerpt, it refers to the PowerPoint presentation a teacher has prepared for and used during a lesson. All teachers who participated in this study combined the use of such presentations with the use of the classroom blackboard. The latter was often used to let students present their solutions to the entire class.)

Excerpt 4.1.2-b (Teacher WK, 2009-05-20)
05:35 T: (WK presents four exercises on the PPT. She assigns each of the four problems to individuals who come and solve that problem on the board. At the same time, all other students solve the problems in Box 4.1.2.2 independently.)
Simplify the following fractions:

1. \( \frac{4a^2b}{6ab^2} \)
2. \( \frac{3a^2b(x-y)}{9a^3(y-x)} \)
3. \( \frac{a-3}{a^2-6a+9} \)
4. \( \frac{a^2-4}{a^2+4a+4} \)

07:09 T: (Moves about the classroom; talks to individual students; checks the work of students who raised their hands to indicate they are finished.)

08:06 T: Please raise your hands if you have finished (looks around to see how many are still working, then continues moving about and checking students’ work).

08:57 T: Hurry up if you haven’t finished (continues checking students’ work).

09:15 T: In your solutions to the second problem I saw many mistakes. Have you finished?

09:20 Students (many reply): Yes.

09:21 T: Ok, let’s have a look [at these problems] one by one.

[Here, WK checked each of the four students’ solutions on the board. In each, she announced the methodical steps mentioned in the first part of the lesson, and indicated if those steps, as well as the entire problem, were done correctly or in error.]

11:11 T: Can you raise your hands if you solved all four problems correctly? Ok, well done! About 95% solved all problems correctly. This is the reduction of algebraic fractions that we have learned yesterday.

12:05 T: Today we are going to learn multiplication and division of the algebraic fraction. First of all, look at these two problems (see Box 4.1.2.3).

Box 4.1.2.3: First two problems used by WK as transition from Reviewing

Solve these fraction multiplication and division problems:

1. \( \left( -\frac{2}{3} \right) \times \frac{4}{5} \)
2. \( \frac{7}{6} \div \frac{14}{9} \)

The data in Excerpt 4.1.2-b show how WK followed the first element of Reviewing (recitation of the abstract, ‘old’ knowledge) with work on specific examples. As pointed out in the paragraph just above this Excerpt, each example, and WK’s specific comments about them when going over the four students’ solutions on the board (after line 09:21), reflected explicit choice of a mathematical case she appeared to see and want to accentuate. For her, this initial segment of the lesson targeted the ‘old’ knowledge, as she seemed focused on assessing individual students’ ability to apply the ‘old’ when they solved the problems silently. This claim is supported by her comment about the problem (#2) in which she saw mistakes in students’ work (line 09:15). It was further supported by her way of obtaining a ‘quick read’ about where most of her students were—asking them to compare their solutions to the one on the board (after she corrected them) and raise their hands if their solutions were correct.
WK’s way of using Reviewing appeared quite procedural. Yet, it could still provide her students with another learning opportunity of the previous day’s knowledge. In particular, such learning was supported by the pedagogical activity of letting four students solve the problems on the board while every other student solved them independently. To low-achieving and mid-level students this activity could give prompts for starting (and completing) the solution method. To strong students it gave an additional opportunity to solve new problems by applying what they already knew.

Excerpt 4.1.2-b highlights WK’s focus on student errors. This was demonstrated in her walking around the classroom to check individual students’ work. It was further demonstrated in her step-by-step review of each solution a student has created on the blackboard. She used the latter to emphasise specific issues and errors she noticed when she walked around the classroom. She oriented students’ attention to the method needed for determining signs (e.g., \(y-x\) is like \(-1^{*}(x-y)\)), to the use and cancellation of entire bracketed expressions (e.g., \(a^2-4 = (a-2)(a+2)\) and the entire \((a+2)\) has to be cancelled), and to important steps in the process (e.g., once example #1 is completed check that each factor—number, \(a\), and \(b\)—has been fully reduced).

**4.1.2.3 Comparing WK’s and LX’s Use of Reviewing**

As pointed out earlier, there were some commonalities in how WK and LX used Reviewing. First, both teachers posed questions and engaged students in answering them via two methods: whole class ‘chorus’ and individual students standing up and answering out loud. Most importantly, both teachers paraphrased students’ answers before moving to another question. Paraphrasing appeared to be an effective instructional intervention as it provided the contributing students, as well as the entire class, with an expression of their ideas by another person. Regardless if the original student responses were correct or incorrect, paraphrasing seemed to enable a comparison of two expressions and could orient students’ reflection onto underlying mathematical ideas the teacher would like to emphasise. As the observations in participating
teachers’ classes indicated, quite often hearing a teacher paraphrasing one’s incorrect solution was sufficient for that student to realise and correct the mistake.

Second, Reviewing by both LX and WK started from abstract, generalised mathematical ideas and rules and progressed to specific examples and exercises. This allowed both teachers to identify who among their students had formed the ‘old’ knowledge at a level necessary for serving as a basis for the present day’s lesson. In turn, by moving into specific examples and exercises, both teachers provided students more opportunities to regenerate and/or consolidate the anchor needed for the present day’s lesson.

Third, both LX and WK appeared to think, and intentionally use, the exposing of mistakes for repeatedly creating good learning opportunities. Exposing a mistake could orient students’ attention onto the method, outcome, and justification that make a solution mathematically correct and justified. Consequently, it had potential to trigger students’ reflection on their activities, whether they considered their own answer or their peers’ and the teacher’s answers. In this way, exposing and articulating mistakes seemed to provide one explanation for how Reviewing may contribute to students’ learning in a 50-student classroom. It seemed to help students who were yet to form the previous day’s idea with additional opportunity to actively apply and test their methods.

Fourth, both teachers’ pedagogical approaches seemed cohesive and well designed. Their tasks were selected for serving different features of the reviewed concept. By focusing on three ‘entry levels’ of sample problems, they seemed to enable most students to call up and use a method that would lead to an answer and thus active participation in the Reviewing component. This three-level selection of tasks seemed compatible with Sullivan et al.’s (2004) notions of enabling and extending prompts that a teacher can use to allow students to learn the centred-upon mathematical idea at her or his level. In this way, both LX and WK engaged students in solving
tasks that could be attempted by divergent existing understandings.

Aside from the identified commonalities with LX, the way WK’s used Reviewing introduced some unique features. Unlike LX, she engaged students in silent, independent reading (about 2 minutes). This could trigger students to review, independently of her intervention, what they have begun learning. She created a situation for the students to take the initiative and apply the knowledge to supposedly known problem types, including clarifying for themselves and WK any confusion they might still have.

WK also differed from LX in that, during Reviewing, she asked four students to solve problems on the board while the rest were working on their own. This pedagogical activity provided her with time to further infer about students’ ‘old’ knowledge at the outset of the lesson. To make such inferences, she sequenced the four Reviewing problems from simple to complex. The first exercise was quite straightforward. The second required students to pay attention to the sign of the common factor. The third and fourth were polynomials that prompted factoring of a trinomial to figure out the common factor. Through those Reviewing variations, WK intended for students to pay attention to, and grasp, the method (Fang-Fa) that she saw as underlying solutions to each of those problems: confirm the sign, isolate common factors (including handling of trinomials), cancel the common factors, check for conditions, and confirm that the answer is the simplest fraction. While four students solved the problems on the board, she moved about the class to monitor other students’ work and watched the evolution of solutions on the board. The following sub-section elaborates the perspective that seemed to underlie WK’s practice.

4.1.2.4 Accounting for WK’s Perspective (in Reviewing)

The follow-up interview with WK provided further insights into her reasoning about this ‘hybrid’ of Reviewing. To account for her perspective, the key seems to be her focus on each student’s mental engagement in problem solving as a necessary condition for linking between the ‘old’
(reviewed) and the target (present day’s) knowledge. Excerpt 4.1.2-c presents data about her rationale for requiring individual work on problems. It also indicates her awareness, similar to LX’s, of students who during whole class (chorus) responses may fake participation without real understanding.

Excerpt 4.1.2-c (Teacher WK, Intrv, 2009-05-20)

When [I let] the whole class answer the question together, there might be some students who are just making up the number, but I can’t be sure that every student is participating in the thinking and reflection. If I ask individual students to answer the question specifically, they don’t know who I am going to ask, so every student will actively participate and thus do the reflection and thinking about my question. I can reach my teaching goal by giving students sufficient time to think and reflect individually and the overall effect of teaching will be better.

[A little later in the interview, WK made another reference to the purpose of individual student work.] The middle level students in my class have a higher possibility of being called to work on the board. If the problems are easier, then I will choose low-achieving students. I don’t expect them to solve all the [Reviewing] problems correctly. The main purpose is not getting full marks, but mainly to expose mistakes or misconceptions in their daily learning or homework, so they will pay more attention to the mistakes.

(Emphases added)

Excerpt 4.1.2-c indicates that WK considered the mental processes of thinking and reflecting as key to students’ learning. In particular, she seemed to be concerned with creating, for students and herself, a situation in which a student provides her or his own genuine solution. To this end, she emphasised, students were not told who will be called; and like in LX’s class they have long learned to anticipate that not just those who know (e.g., raise their hands) are called upon.

Furthermore, in WK’s view students’ individual work seemed to enable teacher and student identification of mistakes, which was crucial for ensuring that the ‘old’ link is properly established before attempting to bring in the ‘new’. She seemed to intentionally use her knowledge of where students were (low, medium, and high) to select those (mid-level) who would solve the problems for the entire class, so other students could benefit from learning through mistakes. It appears that for her students’ understanding of the previous day’s knowledge could serve as a basis for linking the ‘new’ only if it was properly held and used in solving ‘old’ problems. Accordingly, she seemed to take responsibility for establishing this basis,
and to use Reviewing as a pedagogical means to both assess and re-teach it.

WK’s perspective also seemed evident in her selection of the four students who solved the problems on the board. Again, the key was her attention to identifying, exposing, and correcting students’ mistakes. When asked what was her criterion for this selection, her responses (Excerpt 4.1.2-c) highlighted the need to combine figuring out where students are and building on their potential errors as a means to promoting consolidation of the anchoring link for today’s lesson. She appeared to know the different levels of her students’ understanding and use it to allow students’ active participation while making different mistakes depending on that level. For her, it seemed, if students continued making mistakes it meant the anchor for linking the old and the new was not yet stable.

Excerpt 4.1.2-d provides further support for the assertion that WK paid close attention to students’ mistakes because they served to diagnose where students were and what, if anything, was missing for them to be ready for learning the new, target link.

Excerpt 4.1.2-d (Teacher WK, Intrv, 2009-05-20)

When I walked around the classroom, if I found mistakes that the majority of students have made when they solved the problems, I would raise the knowledge again for the whole class. For example, when I walked around the classroom, in the fourth Problem [(a^2-4)/(a^2+4a+4)] I found a student who got the solution of -1/4a. I assume there might be more than one student who would make this kind of mistake. So I wrote the student’s mistake on the board when I went through the student’s solution for this problem, and asked the students what was their opinion about this solution.

[A little later in the interview] For example, for simplifying (1+2)/(2+4), a student’s solution was that 2 cancels the other 2 and so it equals 1/4, while the answer should be 3/6 = 1/2. What the student misunderstood was the concept of factor, which each separate addend is not; what should be reduced is the factor instead of the addend. (Emphases added)

The data in Excerpt 4.1.2-d show the close attention that WK seemed to pay to mistakes—she noticed and recalled specific mistakes that particular students had made. She seemed to proactively look for and use those mistakes because they reflected important knowing that in her view appeared needed for the ‘new’ learning. She also said that those mistakes could be common to several students and reflect similar misapplication, which was illustrated in both of her
examples (i.e., addends were reduced separately). Thus, for her, asking a student who made such a mistake to share and expose it could help those whose same mistakes were likely committed for the same cause.

Excerpt 4.2.1-e highlights three unique and important aspects of WK’s focus on mistakes that indicate an underlying perspective on how students’ misconceptions can engender cognitive conflicts and thus serve their learning. First, WK was not merely looking for any mistake. Rather, she particularly looked for mistakes that pertained to the essence of the ‘old’ mathematics to be founded for the ‘new’ link. Second, WK appeared to consider students who made this kind of mistakes as not yet ready to learn the target knowledge (i.e., they did not have the anchoring link). Third, WK exposed these essential mathematical mistakes not from student solutions on the board, but rather from solutions given by the rest when they solved the four problems individually. She elaborated on this point in the interview: “It is better [to use student mistakes from the floor] than to directly tell them not to make this kind of mistake, because it is more impressive for them to find out and correct the mistakes. This will also motivate their interest since they can find mistakes for their peers.” That is, she seemed to take her role and responsibility as establishing the central idea as an anchor for students and get them ready for the ‘new’ link.

To better understand WK’s use of the pedagogical activity of having a few students solving problems on the board, she was asked to explain this way of Reviewing. Excerpt 4.1.2-e provide data about her response to this question, which in the interview followed her statement, “Practice is more effective than listening.”

Excerpt 4.1.2-e (Teacher WK, Intrv, 2009-05-20)

It is different to have students work on the board and then [I] demonstrate than without students work on the board. If there is no comparison, they have to simply follow my way of solving the problem, but if my thinking is different from the students’ work on the board, then there is comparison. Moreover, if you directly have the [teacher’s] demonstration on the board that specifies things for you to see, it is different from imagining and thinking on your own without any concrete thing. The latter is better, because it helps them have a more thorough understanding... And the comparison with the students’ work is important and helpful for students’ learning... After the students finished solving the problems independently, they can check if the students working on
the board have any mistakes. They may find, ‘Oh, that is a mistake’, and sometimes I would ask the students to check their peer’s answer, what was your answer since you think he was wrong, can you correct for him?... Usually when I check the work on the board, it is a whole class participation process. This can represent the overall level of understanding from the specifics, because those students were selected on purpose, so this can represent the overall level of understanding of the lesson. So this is a good way to find the places where students easily make mistakes or have misconceptions.

Excerpt 4.1.2-f further illustrates that WK used individual students’ work, and mistakes, to make sure that they all paid attention, actively reflected on the needed (‘old’) knowledge, and were ready for learning the ‘new’. Her response emphasised that just letting students follow a solution she would write on the board was not effective. Instead, she wanted each of them to experience the process, including the possibility of making mistakes. This was important because, in her view, a student needs to compare solutions and be able to identify where, and why, a solution is or is not mathematically justified. While students were solving problems on their own, WK could move about the class, assess where they are on the ‘old’ knowledge, and select students whose solutions (and mistakes) could benefit the class. As she said: “this is a good way to find the places where students easily make mistakes or have misconceptions.”

Excerpt 4.1.2-e highlights three aspects of WK’s reviewing. First, she seemed aware of the advantage for recapping the previous day’s link via two comparisons (another student, teacher) as opposed to just one (teacher). By selecting some students to solve problems on the board while others solved them on their own, they could gain additional opportunity to form the needed link while she had an opportunity to identify whether they were ready or not. Second, WK understood that in order to promote the ‘old’ for those who did not yet get it there was a chance to be prompted by others’ solutions and mistakes. This seemed to be a reason for her to pay close attention to students who were making mistakes—more of them were likely to be prompted as compared to just being shown the right way by the teacher. Third, WK made explicit her role in finding out if most students did or did not have the needed link for the present day’s lesson. As she stressed, this was a major reason for her effort to identify how prevalent essential mistakes
were in the class.

Excerpt 4.1.2-f below sheds further light on WK’s perspective. In it, she clarified what impact would her figuring out that there might still be many students struggling have on her next teaching moves.

*Excerpt 4.1.2-f (Teacher WK, Intrv, 2009-05-20)*

After I finished checking the students’ work on the board, I always asked them to raise their hands if they answered all correctly. Usually it reflects good understanding of the lesson if the majority of the students raised their hands, but sometimes *if the response wasn’t so good, then I would give them additional problems to solve.* (Emphases added)

Excerpt 4.1.2-f gives further support to the claim that in WK’s view Reviewing is geared to identifying if the previous day’s knowledge is in place for most students. This was indicated in her reasoning that otherwise she would need to provide additional tasks to consolidate that knowledge. This focus on whether the ‘old’ was in place or missing indicates that, for WK, Reviewing serves the purpose of figuring out whether students have the needed anchor into which to link the ‘new’. If she concluded the link was not there for a substantial number of students, she would choose not to move on. Rather, she would spend more time to promote the basis for that link. That is, she seemed to assume the responsibility and freedom to encourage students’ re-establishment of the link by tailoring her unique way of teaching to the specific mistakes different students made. The underlying premise is that teaching the new would be ineffective before students solidly establish the ‘old’ as a basis. The following sub-section discusses how a novice teacher (JC) used Reviewing.

4.1.3 A Reviewing Novice’s Focus on Student Mistakes (JC)

This sub-section reports how JC—a young, semi-novice teacher in his third year of teaching—used the lesson component of Reviewing. JC worked in the other middle school (not where LX and WK worked). He has a master’s degree in mathematics and is considered by his colleagues as having strong mathematical knowledge. His case is presented to further demonstrate the variety of ways in which these individual Chinese teachers used Reviewing. In particular, JC is a
case of a middle school mathematics teacher whose Reviewing proceeds from specific instances of the ‘old’ (previous day’s lesson) to the general form of this knowledge. As the data reveal, JC’s use of Reviewing and his focus on student mistakes seem to serve similar purposes to LX and WK—assessing and promoting students’ establishment of the ‘old’ as an anchor for the intended ‘new’ mathematics. The following discussion elaborates this point.

4.1.3.1 JC’s First Observed Lesson – Content of Reviewing in the Next Lesson

In the first interview with JC he said that he usually begins his lessons by reviewing the mathematical ideas from the previous lesson(s). JC’s first observed lesson focused on teaching the cross multiplication method for factoring a quadratic trinomial. He first worked on trinomials \((ax^2+px+q)\) with quadratic coefficients of \(a=1\) and then with \(a>1\). In that first observed lesson, students solved two sets of problems (see Box 4.1.3.1).

<table>
<thead>
<tr>
<th>Box 4.1.3.1: Problems solved in JC’s first observed lesson.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Coefficient (a=1)</td>
</tr>
<tr>
<td>Factor the following trinomials:</td>
</tr>
<tr>
<td>(1) (x^2-6x-7)  (2) (x^2-3xy-2y^2)  (3) (x^2(x-1)^2-4)</td>
</tr>
<tr>
<td>2. Coefficient (a&gt;1)</td>
</tr>
<tr>
<td>Factor the following trinomials</td>
</tr>
<tr>
<td>(1) (2x^2-7x+3)  (2) (12x^2-13x+3)  (3) (5x^2+6xy-8y^2)</td>
</tr>
</tbody>
</table>

In the interview before the second observed lesson in JC’s class, he said that he checked students’ homework between the two lessons and found many mistakes in their solutions to problems from the previous lesson. In that interview, he said that these mistakes caused him to conclude that the previous day’s method of cross multiplication was understood by only about half of his class. He further said that the present day’s lesson would be quite easy for the students and he thus decided to spend substantial time on students’ mistakes. This led to his design of Reviewing for the second observed lesson that proceeded from specific examples to general ideas. Such a choice is important to note from the outset, as it further highlights a teacher’s close attention to students’
existing knowledge and using it as a critical orienting criterion for planning and implementing lessons, including possible deviations from the recommended practice (e.g., teachers’ group, national curriculum). The following discussion elaborates this point.

4.1.3.2 JC’s Reviewing of Factoring Quadratic Trinomials: Specific-to-Abstract

JC’s second observed lesson was designed to teach factoring trinomials via the Sum-and-Product formulas. To begin this lesson, his Reviewing proceeded via engaging the students in solving seven problems that he wrote on the board. As JC explained in the interview, he selected those particular problems to expose the common mistakes he identified in their homework (see Excerpt 4.1.3-a, 00:37-02:30). That is, unlike LX and WK, he did not ask students to rehearse the general ideas from the previous lesson nor to solve problems on their own. Rather, he immediately turned to solving the problems himself.

It should be noted that, similar to LX and WK, students could only use factoring to solve the given polynomials if they had at least begun conceptualising the previous day’s idea. Strong and mid-level students might have grasped the method (Fang Fa) of cross-multiplication for solving quadratic equations of the form $ax^2+px+q=0$ by factoring the constant ($q=ab$) and figuring out a pair of those factors that produces the coefficient of $x$ ($p=a+b$). Further, they might have understood that such a method is directed toward the goal of obtaining two algebraic expressions that can be multiplied $[(x+a)(x+b)=0]$ and thus provide a simpler way of solving the equation $[(x+a)=0$ or $(x+b)=0]$. However, as JC himself was aware (indicated in the interview), low-achieving students and apparently many mid-level students were yet to learn this goal-directed method (Fang Fa).

Excerpt 4.1.3-a below (including Box 4.1.3.2) presents the seven problems that JC chose for Reviewing and the teacher-student exchanges that followed. As he noted during the interview before the lesson, these problems were part of what he assigned for homework and included
what he considered typical student mistakes. The problems he assigned were rather difficult and challenging, and the data will show that this claim applied to many of JC’s students. That is, unlike LX and WK, he engaged students in Reviewing not via new and simpler problems selected to accentuate particular features of the ‘old’ knowledge, but via ‘old’ problems selected because for many students they were difficult and likely to prompt misconceptions. His particular focus in this segment seemed to be on assisting students to see that each problem could be rewritten in a quadratic trinomial form, while allowing students to compare correct solutions with potential misconceptions he identified while checking their homework.

Excerpt 4.1.3-a (Teacher JC, 2009-05-15)

00:37-02:30 [Before the lesson, JC wrote on the board seven trinomial factoring problems from the previous lesson’s homework (see Box 4.1.3.2).]

| Box 4.1.3.2: Reviewing problems in JC’s second observed lesson. |
| Factor the following trinomials: |
| (1) $6x^{n+1}-7x^n-24x^{n-1}$ | (2) $\frac{1}{6}x^2y^2 + \frac{7}{6}xy + 3$ |
| (3) $(a+b)(a+b-4)-12$ | (4) $(x^2-x+1)^2-3(x^2-x)-7$ | (5) $(x^2+x+1)(x^2+x+2)-12$ |
| (6) $a^3-19a^2-216$ | (7) $(x+1)(x+2)(x+3)(x+4)+1$ |

03:30 (The bell rings for lesson start.)

03:50 T: First of all, let’s review yesterday’s homework. There are many mistakes in the homework. Let’s have a look at some of the most commonly made mistakes. The first problem, $6x^{n+1}-7x^n-24x^{n-1}$, we definitely think of using the cross multiplication method, right? But how do you deal with the exponents, $n+1$, $n$, and $n-1$?

04:17 S1: Extract the common factor.

04:18 T: We should extract the common factor if there is any. Ok, how many items in the polynomial can use the cross multiplication method that we learned yesterday.

04:28 SS (Some students say together): Three items.

04:30 T (Confirms their response): Three items. It is usually a quadratic trinomial, right? For example, if we are solving problems like $ax^2+bx+c$, we can use the cross multiplication method to do the factoring, right? We have the exponent $n+1$ here, what should we do first?

04:50 S2: Extract the common factor.

04:51 T: What is the common factor?

04:52 Students (Some students say together): $x^{n-1}$.

04:55 T: So we extract a common factor from $6x^{n+1}-7x^n-24x^{n-1}$, and we get $x^{n-1}(6x^2-7x-24)$. Can you do it [factor the trinomial] now?

04:58 Students (Many, but not all students said together): Yes.

[At this point, JC lets the students complete the trinomial factoring individually. Without checking solutions with the whole class, he then proceeds to the next problems. For each, he simply
demonstrates the initial steps needed to bring the ‘weird’ algebraic expression into its familiar, trinomial-like form, for which he seemed to assume they could then complete the factoring.]

05:26 T: [Problem] Two. \(-\frac{1}{6}x^2y^2 + \frac{7}{6}xy + 3\). We can see the coefficient of the quadratic term is \(-1/6\), we can extract the common factor and change the problem to \(-1/6 \times (x^2y^2 - 7xy - 18)\).

06:08 T: [Problem] Three. \((a+b)(a+b-4)-12\). Let’s suppose \(a+b=A\). so \((a+b)(a+b-4)=A^2-4A-12\), and we substitute \(A\) with \(a+b\) for the final result.

06:43 T: [Problem] Four. \((x^2-x+1)^2-3(x^2-x)-7\), suppose \(x^2-x+1=A\), so \((x^2-x+1)^2-3(x^2-x)-7=A^2-3A-4\).

07:08 T: [Problem] Five. \((x^2+x+1)(x^2+x+2)-12\). Let’s suppose \(x^2+x+1=A\). so \(x^2+x+2=A+1\). The problem \((x^2+x+1)(x^2+x+2)-12\) thus become \(A(A+1)-12=A^2+A-12\).

07:35 T: [Problem] Six. \(a^6 - 19a^3 - 216\). How do you deal with \(a^6\) in this problem \(a^6 - 19a^3 - 216\)?

07:49 Students (Some students say together): [Use] \((a^3)^2\).

08:00 T: [Writes on board while saying] So \(a^6 - 19a^3 - 216 = (a^3)^2 - 19a^3 - 216\).

09:01 Students (Some students say together): [We can] suppose \(a^3=A\).

09:02 T: [Writes on board while saying] So we can write \(a^6 - 19a^3 - 216 = (a^3)^2 - 19a^3 - 216 = A^2 - 19A - 216\).

09:21 T: We can get \((a^3-27)(a^3+8)\) for this problem as result. Can you continue factoring?

09:30 Students (Many students, but not all, say together): Yes.

The data in Excerpt 4.1.3-a present mathematical and pedagogical aspects of JC’s Reviewing. Mathematically, all 7 problems he chose shared a common structure—the given, sought-after quadratic trinomial was ‘masked’ within an algebraic expression. Thus, to reach the step of factoring that trinomial, students had to first ‘unmask’ the trinomial by operating on the given expression. Problem 1 masked \(6x^2 - 7x - 24\) via multiplication by \(x^n-1\), which students would ‘see’ if they were asking themselves what are all possible factors in the given expression. Problem 2 masked \(x^2y^2 - 7xy - 18\) via multiplication by \(-1/6\); it was also not just a simple instance of the \(ax^2 + px + q\) canonical form that JC stated as the underlying general expression. Problems 3-6 used substitution to mask the following trinomials (respectively): \(A^2 - 4A - 12\) was masked by substituting \(A=a+b\); \(A^2 - 3A - 4\) by \(A = x^2 - x + 1\); \(A^2 - 4A - 12\) was masked by \(A = x^2 + x + 1\) (note the identity with masked trinomial in #3); and \(A^2 - 19A - 216\) was masked by \(A=a^3\). The trinomial in problem #7 was masked by a sophisticated operation on two pairs of monomials: \((x+1)(x+4)=x^2 + 5x + 4\); \((x+2)(x+3)=x^2 + 5x + 6\). Each of the resulting trinomials could then be substituted, for example, by \(A=x^2 + 5x + 5\) leading to a new, quadratic monomial \((A+1)(A-1)((A-2)(A+3))-12=0\).
\[1 + 1 = (A^2 - 1) + 1, \] or by \[B = x^2 + 5x + 4,\] leading to a new trinomial: \[B(B+2)+1 = B^2 + 2B + 1\] (see solutions by S1 and S2 in Excerpt 4.1.3-b, respectively).

Chosen by a teacher whose own mathematics is strong, these particular instances reveal two key aspects of the Reviewing that JC had set out for his students. The first aspect concerns the ‘status’ of trinomials within a given problem. Whereas in the previous lesson students learned to factor trinomials given in the canonical form, in these seven Reviewing (also homework) problems the trinomial was not given as is. Rather, a student would have to first unmask and reformulate the trinomial in the canonical form. That is, in the previous lesson student started with trinomials as given ‘objects’ and operated on them to identify pairs of factors that produce the constant and add up to the coefficient of x. In the Reviewing (and homework) problems, to reach and revisit the factoring of a trinomial in a canonical form operation, students had to first operate on an expression (‘object’) that was not a trinomial in a canonical form. This seemed to be rooted in an attempt to emphasise and leading to the underlying structure of a quadratic equation \((ax^2 + bx + c = 0)\) they would learn later, as students had to unmask or reformulate the left side of such an equation when solving each problem.

The second aspect concerns the mathematical sophistication and abstractness of the ‘objects’ and operations students had to master in order to solve these seven problems. The three aforementioned methods of ‘masking’ a trinomial required a high level of understanding not only of the abstract ‘object’ called quadratic trinomial and its factoring (itself quite a demanding idea) but also the identification of such an ‘object’ via operations on other, abstract objects that do not bear resemblance with the canonical form. Simply put, these problems were more difficult than just Reviewing of the ‘old’. To solve these problems, a student (or a teacher) would need to have conceptualised mathematical understandings beyond the knowledge of factoring a trinomial in its canonical form—the essence of the previous lesson. Said differently, the essence of the previous lesson was not fully revisited or reactivated through the problems that JC selected for Reviewing,
and which he knew many students failed to solve on their own.

Pedagogically, two additional aspects of JC’s ‘Reviewing’ can be highlighted. The first aspect concerns his view of application of the ‘old’ to novel situations. The seven problems that JC chose indicated his attempt to challenge students by requiring application that was novel to them. Unlike Reviewing problems selected by LX or WK, which were simpler or at a similar level of the ‘old’ knowledge a student would have to apply, JC’s specific instances were more difficult, and seemed to serve as a way of building connections. Instead of revisiting the method (Fang Fa) of factoring trinomials, these were all instances of two-step problems. The first step involved revealing a ‘masked’ trinomial within a rather difficult expression and was necessary if the student would ever get to the supposedly reviewed trinomial factoring. That is, application of the ‘old’ as a second step could only be reached if a student already knew what to look for (trinomials not necessarily in the canonical form) and how to unmask them as a first step. An observer of JC’s lesson would expect only the strong, mathematically creative students to solve these problems, an expectation that was corroborated with his own findings about homework and about the work done during the present day’s lesson.

The second pedagogical aspect of JC’s Reviewing concerns his view of where students’ difficulties might lie. The previous day’s learning focused on factoring canonical-form trinomials. For him, at this point of Reviewing, that knowledge seemed to be trivial, as indicated by his stoppage short of factoring the trinomials after each example was reformulated as a quadratic trinomial. It seems that to him the remainder of the process would have been straightforward also for students. However, even to students who might have conceptualised factoring of canonical-form trinomials, knowing where to start, let alone how to complete factoring the unmasked trinomials (e.g., \( A^2-4A-12 \) with \( A = x^2+x+1 \)) could prove rather challenging. It seems that JC saw no major difficulties in the recognition of trinomials as products of two monomials and the factoring operation needed to reverse the process. Thus,
instead of engaging students in revisiting and slightly expanding the ‘old’, he used ‘Reviewing’ to substantially stretch students’ mathematical thinking. That is, JC’s ‘Reviewing’ was an attempt to teach new ideas, seemingly taking for granted that the ‘old’ is already existing as an anchor. Again, it is inferred that such a stretch could only fit within and promote the learning of the brightest students.

Excerpt 4.1.3-a highlights two key issues of JC’s Reviewing. The first issue is who among JC’s students could have gained from this initial component of the lesson. To him, the mistakes students made seemed to indicate they ‘only’ needed a prompt for transforming a non-canonical into a canonical form of a trinomial. However, in order for students to productively engage in solving the seven problems they had to conceptualise the previous day’s ‘object’ (quadratic trinomial) and method (Fang Fa) of factoring. This was evidenced in the limited number of students who actually responded to his follow-up questions. Students who have not conceptualised this ‘old’ knowledge were unable to solve or make sense of the trivial-to-teacher solutions. JC’s goal (transform the given into canonical) could not be their goal, because for them the trinomials themselves would be unfamiliar. At best, they could follow his activities and notice the outcome (canonical-form trinomials), but not yet link such an outcome to the ‘old’ knowledge of factoring a trinomial.

The second issue is JC’s sequencing of Reviewing activities. JC did not start his Reviewing via revisiting the abstract conceptions and underlying factoring method of cross-multiplication (as LX might have done). He also did not engage the students in individually completing the problem solutions, letting each of them experience the potential link between factoring and the operations on a factored (hence, canonical) trinomial (as WK might have done). Rather, students were merely shown, seven times, the repeated process of pre-factoring to reformulate an expression as a quadratic trinomial. At the beginning, this was hard for students; eventually, it turned out the students could grasp the underlying method. The following sub-section discusses
how JC’s use of these Reviewing problems fostered strong students’ learning.

4.1.3.3 JC’s Continual Support of Strong Students’ Learning via ‘Reviewing’
The analysis above highlighted how, mathematically and pedagogically, JC’s Reviewing of the first six problems was conducive to new learning by the few, high-achieving students who were ready. In this sub-section, this conclusion is examined further in data, presented in Excerpt 4.1.3-b, about how he proceeded to the solution of the seventh problem. For this last and most sophisticated instance, JC asked two individual students to stand up and share their solutions. In his calling up of different student solutions to the same problem, he created a learning opportunity for those who might have begun the much higher-level understanding of factoring via variable substitution.

Excerpt 4.1.3-b (Teacher JC, 2009-05-15)
09:35 T: [Problem Seven] (x+1)(x+2)(x+3)(x+4)+1. We are definitely going to use the holistic idea. How are you going to deal with it? [JC lets them consider the problem for a few more seconds and then selects one student among the few who raised their hands.]
09:54 T: How about S1?
09:55 S1 (Stands up and says): Put (x+1) (x+4) together.
10:00 T: Why do you want to put these two together?
10:03 S1: Because when you will multiply and expand, you can use the formula for the difference of squares.
10:11 T: What do you get by multiplying and expanding this (x+1) (x+4)?
10:12 S1: x^2+5x+4, then [I will also] multiply and expand (x+2)(x+3).
10:22 T: Ok, when you combined these two together, what did you get?
10:23 S1: Add 1 after x^2+5x+6.
10:32 T: (Writes on the board) (x+1)(x+2)(x+3)(x+4)+1= (x^2+5x+4)( x^2+5x+6)+1. Can you tell me what’s the benefit of doing this?
10:33 S1: [Now I can] look at x^2+5x+5 as A.
11:08 T: And then?
11:09 S1: It becomes [(x^2+5x+5)-1] [(x^2+5x+5)+1]+1.
11:22 T: And then?
11:40 S1: And then it becomes [(x^2+5x+5)^2-1]+1.
11:41 T: (A-1)(A+1)+1= A^2-1+1=A^2=(x^2+5x+5)^2.
12:07 T (turns to a second student): S2, you have a different opinion, right?
12:10 S2: (Stands up and says) (x^2+5x+4)(x^2+5x+6) +1= (x^2+5x+4)(x^2+5x+4+2)+1, then take 2 out [meaning multiply the first trinomial by two components of the expression in the second parenthesis—itself and +2, that is, (x^2+5x+4)(x^2+5x+4+2)], and suppose x^2+5x+4=B, and write it into B^2+2B+1, then use the cross-multiplication method.
The data in excerpt 4.1.3-b provide evidence of the high-level mathematics students in JC’s class were asked to attempt during Reviewing. As JC noted in the interview, from observing their individual work on this problem and the homework he checked, he knew who solved it correctly. Thus, he turned to one of them (S1) and, through questions about the thinking that likely guided S1’s solution, tried to provide other students a glimpse into the key aspects of the solution process. In particular, JC highlighted the goal for combining specific pairs of monomials (“Why do you want to put these two together?”), the outcomes of S1’s multiplication and expansion of the monomial pairs (x^2+5x+4, x^2+5x+6), and the mathematical ‘trick’ of substituting A= x^2+5x+5 to reformulate the original expression as (A-1)(A+1)+1= A^2-1+1= A^2=(x^2+5x+5)^2.

Not stopping there, JC turned to another student (S2) who had a different solution to the same problem, which JC appeared to know via observing students’ individual work. JC made explicit his intention to share that different way (“S2, you have a different opinion, right?”). This time, after writing S2’s solution on the board (“x^2+5x+4=B, can be written as B^2+2B+1”), JC pointed out to the creation of a trinomial that did not require factoring via cross-multiplication. Instead, as JC’s rhetoric question and immediate answer indicated, students could solve this trinomial by using the simpler formula of perfect squares.

Both students’ solutions suggested that they knew what to look for when reformulating the monomials and why such reformulation would be beneficial. In this sense, data in Excerpt 4.1.3-b indicate the sophisticated understanding that JC, and a few of his students, needed to have already conceptualised in order to see through the entire solution process. As the highlighted (italics) in that Excerpt show, the outcome of substituting an entire trinomial for a single variable (“A” or “B”) had to be known prior to solving the seventh problem. It is interesting to note that using such a problem may be pretty rare; if it would have been a part of an equation there would
be other ways to solve it instead of reformulating two pairs of monomials the way S1 and S2 have done it. That is, the focus of problem #7 as JC’s students solved it seemed to be on the same underlying canonical structure of a quadratic equation \((ax^2+bx+c=0)\), not necessarily or mainly on different ways of solving such equations.

For students who understood trinomial factoring in general and the substitution method in particular, being exposed to the two different solutions could have been beneficial. These students could identify the underlying method (Fang Fa) (substitution method) being executed in slightly different ways across the two different solutions to the same problem. Thus, this pedagogical activity of comparing a variation of solutions could be particularly beneficial for students who were in the process of conceptualising factoring-via-substitution. For them, JC’s questions (e.g., “Why do you want to put these two together?” “What do you get by multiplying and expanding this \((x+1)(x+4)\)?”; “Can you tell me what is the benefit of doing this?”) could foster new understandings. Indeed, S1’s responses to those questions suggested he already knew the substitution method ahead of solving the problem. The data available from JC observed classes and interviews do include clues to how this might have happened. It could have been taught in lessons prior to the first one observed in his class, by the student obtaining additional tutoring, or by the student exploring this method independently. The following sub-section compares JC’s use of Reviewing with LX and WK.

### 4.1.3.4 Comparing JC’s Use of ‘Reviewing’ with Other Teachers

JC’s use of Reviewing was different from that of other teachers in four important ways. First, although he focused on students’ mistakes, the Reviewing process did not share and expose them. Consequently, unlike LX and WK, JC did not engage each and every student in solving the Reviewing problems, nor did he call upon mid-level and low-achieving students to share their solutions. His Reviewing problems did not seem to provide lower-level students with realistic opportunities to further learn the previous day’s knowledge, or him with assessment of what they
understood. That is, JC did not focus his and students’ attention on learning from mistakes via recognising where the previous day’s method (Fang Fa) was applied incorrectly and plausible reasons for those unjustified solutions. Rather, he used ‘Reviewing’ to promote learning by focusing on proper ways in which the method (Fang Fa) could be applied to novel, more challenging problems. To this end, in the trivial-to-him first six problems he carried out the crucial intellectual work of properly unmasking the quadratic trinomials. Students were left with what, for him, seemed straightforward application of the ‘old’, indicating that he believed they knew it and could proceed independently. Then, in the seventh problem, he engaged two students in sharing the crucial intellectual work, while he emphasised key aspects of their proper solutions via rhetoric questions. The rest of the students were supposed to learn and correct their mistakes by following the correct solutions.

Second, unlike LX and WK, JC did not follow the sequence of first mentioning the abstract ideas and then highlighting the previous day’s underlying method (Fang Fa) via step-by-step application to solving the specific instances. Instead, after doing the unmasking himself he only pointed out what students should have recognised as ‘old’ in the resulting, canonical-form trinomials. On the basis of his own mathematics, he seemed to assume they should be able to know how to proceed from where he led them, as well as how to start the solution of future problems (i.e., recognise the need for unmasking an expression).

Third, in these lessons JC included knowledge that, he said, has been deleted from the textbook and not included in the reformed national curriculum. A remarkable example of this was the problems that required variable substitution. Thus, he spent the entire first (observed) lesson and about one third of the second to teach materials that supplement the curriculum. At issue here is the freedom, and responsibility, that JC expressed and assumed in choosing the content (highly sophisticated) and pedagogical methods (merely noting mistakes in trinomial unmasking) for student learning. This responsibility, much more then what WK had demonstrated in her
deviation from the group’s planning, is nothing like the so-called authoritarian Chinese teachers’ teaching by the textbook. It is interesting to note that JC supplemented the curriculum not by decreasing but rather by increasing the mathematical level students were expected to learn.

Fourth, by this increase of mathematical level JC gave voice to his stronger students, who presented and explained their own solutions to the entire class. As the data below indicate, he was aware of the possibility that their solutions might not be accessible to low-achieving students. However, letting the high-achieving students lead the solution for the most sophisticated problems, and hence also address other students’ mistakes, could serve to confirm the strong students’ mathematical power and set an expectation of the kind of contributions other students should aspire to—and make. The following sub-section elaborates the perspective that seemed to underlie JC’s practice.

4.1.3.5 Accounting for JC’s Perspective (in ‘Reviewing’)

An observer of JC’s lessons would quite naturally ask what led him to use Reviewing in the way he did, while other teachers observed in his school (including his mentor) were using it more like LX and WK. In the interview after the second observation, JC clarified that his Reviewing goal was to demonstrate the mistakes students made in their homework as a means to consolidate the knowledge from the previous lesson as well as to assess where students were mathematically. His perspective is further demonstrated in Excerpt 4.1.3-c.

Excerpt 4.1.3-c (Teacher JC, 2009-05-15)

This lesson is a little special, because it does not come from our textbook… It’s not included in our current textbook and curriculum. We teach [trinomial factoring] as supplementary, because we feel it is better to equip the students with this knowledge to help in their future learning. We [the teacher group] designed this homework by ourselves … No matter how different and more difficult the problems are, this kind of thinking and the invariant is the same—extract the common factor, put the same factor out. These problems are different from what they learned in the lesson, but they must know this kind of thinking… This homework is challenging and more advanced than the lesson we gave yesterday. Lots of students made many mistakes which mean they don’t have a full understanding, so I spent a fair bit of time explicating the homework, but usually I won’t spend more than 10 minutes to explicate the homework. Because I feel yesterday’s lesson can’t be finished just in one lesson, and I spent more time on the commenting of the homework. (Emphases added)
Data in Excerpt 4.1.3-c indicate that JC and his group of more experienced teachers were proactively negotiating the given, national curriculum. This curriculum took effect in the early 2000s, drawing on reform movements in the US and other Western countries (Wong, Han, & Lee, 2006). One key shift from the previous to the new curriculum was reducing the number of topics and increasing the focus on higher level thinking processes. Several teachers who participated in this study said during their interviews that trinomial factoring was a topic eliminated in the new curriculum. However, as JC stressed during the exit (third) interview, his group of teachers considered this topic important for their students, and supplemented the given materials with specific problems that they have prepared. He particularly emphasised that, to him, solving such challenging problems is important in terms of students’ thinking about the methods (Fang Fa) used for solving them—the goal he had set for their learning.

Most importantly, Excerpt 4.1.3-c highlights JC’s awareness of the difficulties and/or lack of understanding that many students seemed to have. He came to this assessment based on the quantity and nature of mistakes he found in their solutions to homework problems. As discussed above, finding those mistakes oriented him to take pedagogical action in service of ensuring they understood and could solve such problems correctly. That is, as a teacher JC seemed to take the responsibility and time for fostering students’ establishment of the needed ‘old’ knowledge. To this end, however, his way of teaching differed from that of LX and WK, who engaged their students in making similar mistakes while solving new problems during Reviewing. Instead, JC sought to engage his students in following the correct solutions to problems they already attempted. He seemed to assume that his students would compare their own, incorrect homework solutions to the correct ones (video data corroborated this assumption).

Excerpt 4.1.3-d provides data about how JC thought of his use of Reviewing. Interestingly, JC considered his reviewing of students’ mistakes on homework problems as a way of giving the lower students one more chance to get a better understanding.
Excerpt 4.1.3-d (Teacher JC, 2009-05-15)

The most important thing by doing this kind of correcting of mistakes is to help some students who have problems, or who made mistakes. We give them another chance so maybe they can understand well. You asked how we help the students with difficulties, and I think this kind of job is part of what we do to help those lower students. (Emphases added)

Excerpt 4.1.3-d indicates the overarching pedagogical goal that guides JC’s Reviewing—provide students with an opportunity to understand the mathematics at hand. To him, revisiting students’ mistakes helps low-achieving students grasp the previous day’s understanding. The goal is every student’s solid understanding of the ‘old’; mistakes entail lack of such understanding; and the means to accomplish that goal is demonstrating the correct solution.

Thus, JC’s approach to Reviewing—pointing out their pre-factoring (unmasking) mistakes—suggested that, for him, the heart of the learning needed for low-achieving students was to identify the common method (Fang Fa) of factoring a canonical trinomial. Once they did, he assumed students would be able to continue on their own instead of directly being shown the entire process—similarly to how he, as a strong mathematics knower, would do. As such a knower, JC figured out three sources of mistakes in their solution to the previous day’s problems. First, some students did not have a clear understanding of the overall mode of thinking (i.e., look at strings of $x^2$ as a whole), so they had difficulties in extracting the common factor. Second, these students might also have difficulties applying the overall mode of thinking. Third, in more sophisticated problems factoring could only be done after an initial step of reorganising an algebraic expression into a known one (e.g., canonical trinomial). This threefold analysis seemed to underlie his understanding of the link between his teaching through fixing students’ mistakes and the support for low-achieving students’ future understanding. To provide students with another chance of establishing the target, anchoring link what he appeared to consider needed was to simply point out these sources to the students.

In this sense, JC’s use of Reviewing is quite traditional. His approach, which seemed to ignore
the lack of access to the ‘old’ ideas that low-achieving students most likely experienced, seemed to grow out of his strong mathematics. He could see the relation between the previous and the new ideas and assumed that students would also see this link if mistakes were used as a ‘pointer’ to the essential, underlying method (Fang Fa). This interpretation is supported by JC’s response to the interview question about how someone comes to know a piece of mathematics they initially do not have: “If kids want to run they have to know how to walk. Once they know how to walk, there are commonalities between walking and running, and they can do it.” His metaphor implies that students who have some basic knowledge would be able to move on to the new knowledge. His role as a teacher is to assess if they do or do not have the ‘old’ knowledge and, if not, take action to fix their mistakes by comparing one’s incorrect solution to the teacher-demonstrated correct solution. The following sub-section presents the use of Bridging by another teacher (ZX).

4.1.4 A Reviewing Expert’s Focus on Particular as a Case of the General (ZX)
This sub-section reports how ZX—an experienced teacher in her 19th year of teaching—used the lesson component of Reviewing. ZX taught in the same middle school of LX and WK. Her case is presented to further demonstrate the key role that Chinese teachers who participated in the study seemed to assign to Reviewing—assessing and taking action to ensure that students understand the ‘old’, previous day’s knowledge so it can serve as an anchor for learning the ‘new’. The following discussion elaborates this point.

4.1.4.1 ZX’s Reviewing of Equations with Algebraic Fractions: Abstract-from-Specific
Like LX, WK, and JC, ZX began her observed lessons by reviewing content learned in the previous lesson. As she explained in the second interview, “the unfamiliar knowledge students learned yesterday is today’s familiar foundation for the new learning.” That is, as a teacher she appeared to assume the responsibility to activate students’ available knowledge with which the new knowledge is to be linked. The first lesson observed in her classroom focused on procedures for solving equations with algebraic fractions. The challenging problems that students solved during that
lesson are presented in Box 4.1.4.1. Although these problems look complex, students in ZX’s class did not seem to have difficulties and the class proceeded well.

**Box 4.1.4.1: Linear equations ZX’s students solved (first observed lesson)**

<table>
<thead>
<tr>
<th>During the lesson:</th>
<th>( \frac{x-1}{x+1} = \frac{1}{2} )</th>
<th>( \frac{x-1}{x+1} = \frac{5x+9}{x^2-1} )</th>
<th>( \frac{x-1}{x+1} = \frac{5x+9}{x^2-1} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Homework:</td>
<td>( \frac{2x-3}{x+6} = \frac{1}{3} )</td>
<td>( \frac{1}{x-2} + \frac{3}{2-x} = \frac{1-x}{2-x} )</td>
<td>( \frac{2}{1-x} = \frac{x}{1+x} + 1 )</td>
</tr>
</tbody>
</table>

The stated goal of the following (second observed) lesson was to promote students’ use of equations with algebraic fractions for solving word problems. ZX said in the interview before this lesson that checking students’ homework indicated that only about half of her class understood the previous day’s new method of solving such equations. Instead of starting her review by repeating the abstract rules (like LX and WK), she opened the lesson by writing two equations with algebraic fractions on the board and asking every student to solve them individually. That is, Reviewing problems she used to begin her lesson proceeded from particular instances of the previous day’s learning. However, unlike JC whose Reviewing instances were those that his students had difficulties solving for homework, her examples were new and *simpler* than those solved previously.

Excerpt 4.1.4-a (including Box 4.1.4.2) presents the two specific problems ZX used for Reviewing. The data show that her way of re-emphasising the essential steps of the method (Fang Fa) was to first let students solve instances of the mathematics at issue. Then, she would use those instances as a means to point out the general method that leads to isolating the variable.

*Excerpt 4.1.4-a (Teacher ZX, 2009-05-27)*

00:01 T: (Writes two problems on the board and asks students to solve them quietly; she walks around the class to check and assess their solutions.)
Box 4.1.4.2: Two Reviewing problems that commenced ZX’s second observed lesson

Solve the following linear equations:

\[
\begin{align*}
(1) \quad \frac{2x - 3}{x + 6} &= \frac{1}{3} \\
(2) \quad \frac{2 - x}{x - 3} &= \frac{1}{3 - x} - 2
\end{align*}
\]

00:24 T: Please raise your hands if you finished (only a few do, so she lets them continue for about 1.20 minutes).

01:44 T: Please raise your hands if you finished, I will give you another half minute. (She then asks a student from each row in the class to collect their answering sheets.)

03:00 T: Ok, we have learned yesterday …

03:03 Students (chorus-read from the board): “Solving equation with algebraic fractions.”

03:07 T: What are the basic procedures of solving an algebraic fraction equation?

03:13 Students (chorus): Removing the denominators.

03:18 T: How many unknowns are there?

03:22 Students (chorus): One.

03:23 T: If you cancel the unknown denominator, what else you can operate on? The basic procedure is …

03:25 Students (chorus): Removing the denominators and transferring the equation of the algebraic fraction into an algebraic equation.

03:30 T: Second step, solving the algebraic equation. Is that all?

03:35 Students (chorus): Not yet.

03:38 T: In the algebraic fraction equation [she reads #2: \((2-x)/(x-3)=1/(3-x)-2\)], some components are algebraic fractions. What are the criteria for confirming if an algebraic fraction is meaningful?

03:45 Students (chorus): The denominator.

03:52 T: Can you guarantee that in the solution to the algebraic fraction equation the denominator is not 0?

03:58 Students (chorus): Not necessarily.

04:04 T: It is uncertain, so you have to …

04:08 Students (chorus): Verify.

04:10 T: The third step - you must verify. After the verification, what is the last step? You have to write up the summary.

Data in Excerpt 4.1.4-a show how ZX began her Reviewing component of the lesson. She first engaged all students in solving the two examples individually. Solving the first problem seemed quite straightforward for those who knew the process of creating an equivalent equation consisting of only algebraic expressions, but not fractions. They would multiply both sides of the given equation by the denominators, remove the corresponding denominators \((x+6\) on the left and 3 on the right), and multiply the numerators accordingly—left side by 3 and right side by \(x+6\). This rather simple problem seemed to target focusing students’ attention to the need to
multiply all addends of the numerator (i.e., $3(2x-3) = 3x2x - 3x3$). Because such a process was used in the previous lesson and homework, with much more complex problems, students were likely to recall and execute it correctly. Once the first problem was solved, the second problem seemed to target students’ further attention to specific steps in the process of creating an equivalent equation. These included changing the denominator of one side to be equal to the other side (e.g., changing $1/(3-x)$ into $-1/(x-3)$) so that cross-multiplication is done by the same denominator and saves additional calculations (and possible errors); remembering to correctly multiply the numerators on each side—particularly multiplying $-2x \times x$ and $-2x \times -3$; correctly deciding the signs; and checking if the value of $x$ does not turn the denominators into 0.

While ZX’s students solved the two specific problems, she moved about the room and monitored their work. As she later explained in the interview, she was looking for their ability to recall each of the aforementioned steps and correctly execute them. Letting students solve the problems on their own seemed helpful, as she provided each student with sufficient time to complete both problems. Thus, each of them could reactivate the ‘old’ knowledge while she had an opportunity to assess where students were (which steps they did or did not employ correctly). When she concluded each student had a solution, she turned to her next pedagogical step—revisiting what to her seemed to be the key mathematical components of a proper method (Fang Fa), namely, the aforementioned steps. She particularly emphasised that finding an answer (value for $x$) is not sufficient, as such a value may turn denominators in the original equation into 0, which is improper mathematically. Thus, the first few minutes of the second observed class seemed to support reactivation of what students have learned in the previous lesson. This reactivation was initiated by the work on a pretty simple instance and then extending it to an instance that highlighted major issues one might face when solving more complex problems.

These data suggest that ZX selected the two examples purposely. Through this selection, her goal seemed to be to highlight three steps in solving equations with algebraic fractions. Her
prompts for students and their responses indicated that she saw the first step as creating an equivalent equation, consisting of two algebraic expressions that do not include a fraction (i.e., removing the denominators). The second step involved solving the equation via mathematically justified operations students had learned previously (e.g., adding the same expression to both sides of the equation). The third step, which ZX emphasised by asking students if finding the solution was the last step (and they responded “No”), involved checking for conditions, that is, none of the values found for the variable turned the denominator into 0. In her view, the two problems would require students to use the first two steps, and highlight the need for the third due to uncertainty about the possible erroneous values. In particular, $x=3$ solves the first problem with a denominator $\neq 0$, whereas $x=3$ also ‘solves’ the second problem but is an erroneous answer because then the denominator $= 0$. That is, instead of first reciting the general ideas from the previous day, ZX commenced Reviewing by engaging her students in individually solving two examples as a means to enable each student to regenerate for oneself that general method (Fang Fa).

Excerpt 4.1.4-a indicates that, for ZX, the solution method—previous day’s new idea and present day’s ‘old’—did not appear familiar to students if presented in a general way (i.e., as a list of rules). Starting with examples suggests that she was aware of the need to allow students access to the three steps via actively solving the problems. To cater for the low-achieving students, whose homework she checked, she thus began with two specific instances of the ‘old’. After students had sufficient time to solve both, she proceeded to bit-by-bit recitation of the steps that constitute the method (Fang Fa) leading to the desired effect (isolating the single variable while ensuring that only valid solutions are accepted). The following discussion elaborates this point.

4.1.4.2 ZX’s Reviewing to Support Low-Achieving Students’ Re-Learning of the ‘Old’

Excerpt 4.1.4-b below presents data about how ZX addressed the possibility that low-achieving students’ lacked the needed understanding. Once completing the general, abstract steps she
turned to a 4-minute set of exchanges about the second problem. It appears that she chose to work on this problem with the entire class as a means to instantiate each of the general steps, particularly the third, in an easy-to-follow example. She also introduced a purposeful error in her solution to orient students’ attention to a specific step in which an error was likely to be made (incorrectly ‘eliminating’ the denominators).

Excerpt 4.1.4-b (Teacher ZX, 2009-05-27)

04:21 T: Ok. [She reads problem #2] \((2-x)/(x-3)=1/(3-x)-2\). How do you transfer this algebraic fraction equation into an algebraic equation?

04:25 Students (chorus): Multiply each side of the equation by the simplest common denominator.

04:29 T: What should you pay attention to when you multiply by the simplest common denominator? [This] is one of the most challenging problems. [She calls on a student] How about S1?

04:44 S1 (stands up and says): Don’t forget any item when multiplying by the simplest common denominator.

04:45 T: What else, S2?

04:50 S2 (stands up and says): Don’t forget to change the sign.

04:55 T: what is the relationship between x-3 and 3-x?

04:57 Students (chorus): They are inverse expressions [meaning = one expression is equal to the other expression multiplied by -1].

04:59 T: …You have to pay attention to the signs. And if the denominator is polynomial, and if it can be factored, you have to firstly do the factoring. Do we need any factoring [in this problem]?

05:11 Students (chorus): No.

05:12 T: So we are removing the denominator directly. What is the simplest common denominator?

05:19 Students (chorus): x-3.

05:20 T: So the left side multiplied by (x-3) gets …?

05:28 Students (chorus): 2-x.

05:30 T: And the right side of the equation multiplied by (x-3), and you get (-1-2), right? Because just now we said (3-x) and (x-3) are inverse expressions. (She writes an incorrect equation on the board: 2-x=-1-2.)

05:49 Students (many reply together): No, this is wrong.

05:52 T: Just now S1 mentioned don’t forget to …?

05:55 Students (chorus): Multiply the common denominator for each item.

05:58 T: (She writes on the board, 2-x=-1-2(x-3), which is assumed they know why from previous learning about order of operations.)

06:00 T: This has removed the denominator. We transferred the algebraic fraction equation into an algebraic equation, and this is a …?

06:13 Students (chorus): Linear equation with one variable, 2-x=-1-2x+6.

06:21 T: You have to pay attention to the change of the sign, +6. Then you solve the linear equation and you get x=3. Are we done?

06:50 Students (chorus): Not yet.
06:53 T: we found that when x=3 the denominator equals …?
06:58 Students (chorus): Zero.
07:00 T: You have to write this step. You can simplify the process of verification, but you have to write ‘after verification, when x=3, the denominator is 0. So x=3 is an extraneous root’. And you have to summarise for the process and write ‘so there is no solution for this equation’. [Here, the teacher wrote the entire process of solving the problem on the board.]
07:40 T: This is what we have learned yesterday. When I walked around, I found that some of the students didn’t know how to begin solving the problem. You didn’t do any review at home. I keep emphasising you have to do a good job in reviewing.
08:00 T: Ok, this is what we have learned yesterday. [Today, we] are going to learn the application of the algebraic fractional equation.

Data in Excerpt 4.1.4-b show how ZX attempted to shed light on each of the steps in the ‘old’ method (Fang Fa), which she wanted students to reactivate and knew not everybody had grasped, yet. First, she explicitly asked about the first step, the one by which to initiate a solution process (“How do you transfer this algebraic fraction equation into an algebraic equation?”). She seemed to expect, and accept, a general statement of the step (as opposed to a specific suggestion about the common denominator being x-3). Before moving to that specific denominator, she continued by turning to an individual student and inquiring about an expected operation (and error), alerting students that, “Removing the denominator is one of the most challenging problems.” Again, instead of stating the specific instance (not forgetting to multiply x-3 by -2) the student provided the general rule as an expected caution they should exercise (“Don’t forget any item when multiplying by the simplest common denominator”). Unlike LX, however, she did not inquire into the student’s reasoning, nor discussed the mathematical justification herself. Perhaps this lack of inquiry reflected an assumption on her part—that students have learned order of operations long before this lesson, and that simply noting the need to maintain this order would trigger the taken-as-shared reasoning. In a similar way, ZX attempted to alert students to sign changes, while emphasising the mathematical justification for that in terms of the relationship between x-3 and 3-x (“They are inverse expressions”).

At this point, ZX turned to application of the rules (just stated) to the specific problem, while using this opportunity to act like a person who made a mistake due to not paying attention to a
rule. She pretended to have ‘forgotten’ to multiply all items, seemingly expecting students to identify, criticise, and correct her mistake. Many of them did, so she linked the mistake to the rule stated by S1 and wrote the proper solution on the board. She then turned to the last step, again asking students to state the general expectation that applying a method (Fang Fa) is not completed when a value for x has been found. Rather, she required that they not only check the value of the denominator but also explicitly write down as their final answer that “After verification, when x=3, the denominator is 0. So x=3 is an extraneous root.” At this point, almost nine minutes into the present day’s lesson and after repeatedly shifting between the general method and its application to the particular instance, she stated that this is the ‘old’ and the goal of the present day’s lesson (“learn the application of the algebraic fractional equation”).

Excerpt 4.1.4-b demonstrates how, to fulfil her responsibility to support students’ establishment of the foundation needed for the present day’s lesson, Teacher ZX attempted to prompt regeneration of the knowledge from the previous lesson. She spent almost 20% of the lesson to meticulously review each and every step of the method (Fang Fa) leading to the desired goal for both the teacher and students. Moreover, she used that time to solve a problem that, unlike problem #1, required expecting, properly carrying out, and justifying every step in the solution method. Her focus seemed to be on the general, abstract ideas that students must re-establish as anchor in order to learn the ‘new’ knowledge. She used the application of the general method to the specific instance as a means to orient students’ attention to potential mistakes. In this sense, the entire set of exchanges in ZX’s Reviewing indicated a focus on reciting a set of rules students were expected to acquire in the previous lesson. She knew that some students mastered and could apply those rules whereas others were yet struggling, as their errors indicated to her. Thus, she selected and used the specific examples, and ‘performed her mistakes’, as a means to emphasise the rules and encourage students to proactively avoid common misapplications of those rules. What seemed missing from this process was an explicit attention to reasoning about the rules and why certain steps would be considered incorrect.
Consequently, ZX’s way of Reviewing seemed to support the intended re-generation of the method (Fang Fa) for students who have at least began conceptualising the previous day’s idea. Solving both problems on their own first, then hearing their peers’ responses to imagined abstract steps (correct and incorrect), and finally following the teacher’s execution of those steps on the board oriented their attention to key aspects of the ‘old’ mathematics. For students who were yet to begin conceptualising that ‘old’ knowledge, these pedagogical moves could have served as another experience of applying the method (Fang Fa) toward a goal, established during their independent work, of isolating the variable and reflecting on intermediate (e.g., not multiplying both sides by the denominator) and final steps (e.g., identifying x=3 as an extraneous solution). However, as noted in the previous paragraph, the re-learning (or learning anew) of the key ideas from the previous lesson by students at various levels seemed to focus on a set of rules and how to properly apply them, as opposed to taking this Reviewing opportunity to emphasise deeper understandings of reasons for those rules. The following sub-section compares the use of Reviewing by all four teachers.

4.1.4.3 Comparing ZX’s Reviewing to LX, WK, and JC

Teacher ZX’s Reviewing technique had some common features with other teachers. She involved her entire class and individual students in answering the Reviewing problems. She also oriented students’ noticing of the link between the general method (Fang Fa) and specific examples, while explicitly pointing to problematic steps she did not want students to overlook and/or misapply.

On the other hand, ZX’s use of Reviewing demonstrated yet another unique approach. This approach was indicated in her atypical move from simple instances to the general, abstract method and then back to application of this method. She intentionally selected and presented two problems that fostered students’ independent bringing forth and use of the knowledge from the previous day’s lesson. First, she took the time to enable their own solution and thus recall of
steps needed for the method (Fang-Fa). Then, she made an intentional mistake that increased the potential for students’ critical examination of the process, at least those who began conceptualising the intended concept and possibly also the low-achieving students. Last but not least, she explicitly stated for the students that what they were engaged in, and what she expected them to do on their own (e.g., homework), was the Reviewing process. Thus, by using simple examples to reactivate general rules taught during the previous lesson, ZX could have fostered students’ learning through noticing commonalities in the application of the specific steps they took, correctly or incorrectly, and synthesising those commonalities into the expected rules. The following sub-section elaborates the perspective that seemed to underlie ZX’s practice.

4.1.4.4 Accounting for ZX’s Perspective (in Reviewing)
The follow-up interview with ZX after the second observed lesson provided further insights into her reasoning about her approach to Reviewing. To account for her perspective, the key seems to be her focus on each student’s application of the method (Fang Fa) as a means to explicate the general, abstract mathematics at issue.

In the interview, ZX clarified that her goal for the present day’s learning was to apply the previous day’s learning of the method for solving algebraic fraction equations to solving real word problems, including isolation of particular variables in scientific formulas. When asked why she began the lesson with those two particular problems, she responded: “To check students’ understanding of yesterday’s lesson and to review what they have learned. Some of the students may not be able to fully understand, and it is working as reviewing and consolidating.”

It should be noted that between the end of class and the follow-up interview, ZX had already checked all students’ individual solutions to those two Reviewing problems and found that 25 of them solved both problems correctly, 15 solved one problem right, and about 10 got none correct. Relating to this information, ZX said that during class she got a rough sense of students who understood the previous day’s lesson. Most importantly, she emphasised identification of
students who were still struggling. She added: “I demonstrated the second example to help the lower students to consolidate.” This response indicated two important aspects of ZX’s pedagogical approach. First, like the other three teachers, in her view students’ learning of the ‘new’ knowledge designed for the present day’s lesson cannot proceed without their ability to independently recall and make sense of the ‘old’ method (Fang Fa). Second, her role is to figure out whether the ‘old’ knowledge is, in fact, familiar to students. Yet, her way of teaching the possibly missed learning (from previous lesson) seems to fit with the ‘empty vessel’ approach. Students who could not apply the rules correctly were shown and told what to pay attention to without explicitly reasoning about why her or other students’ applications were considered correct (or incorrect). That is, reasoning about the previous day’s mathematics by students that she knew were yet to learn it was left to those students themselves.

Excerpt 4.1.4-c provides data that further support this point. These data consist of her response, during the interview, to the question about what she was looking for when moving about the class and examining students’ initial (independent) work on the two problems.

Excerpt 4.1.4-c (Teacher ZX, Intrv, 2009-05-27)

> When I walked around the class, some of the students do not even have the format of solving the problem, so I reviewed and repeated writing the steps of solving the second problem on the board… I was walking around the class and I was looking at both the good aspects and the problems exposed in the students’ work, figuring out where they are, so when I demonstrate later, I can address these issues, and know how to help them start. (Emphases added)

The data in Excerpt 4.1.4-c give further support to another key assertion about a Chinese teacher’s use of Reviewing, namely, it is used to assess and re-teach the ‘old’ as an anchor for the ‘new’ mathematics. ZX stated that she needed to figure out what students already knew. To this end, she selected the two problems and monitored students’ solutions in search of evidence that they were aware of the steps in the method. In addition, she seemed to focus on low-achieving students’ learning by assisting their ability to initiate a solution process. To this end, she emphasised her role in reviewing and enforcing repeated writing of steps they took during
Excerpt 4.1.4-c indicates the role of guiding students’ re-activation of previously taught mathematics that reviewing plays in ZX’s pedagogy. The key point is her intention to (a) infer where students are so she can figure out how to (b) promote their initiation of yesterday’s knowledge for learning today’s new knowledge. To these interrelated pedagogical ends, she distinguished three sub-groups of students in her class via (a) rough assessment during students’ silent problem solving and (b) solid checking of their work immediately after class.

The explicit link in ZX’s view between reactivating the ‘old’ and new learning was confirmed by her additional comment concerning her choice to fully demonstrate a solution to the second problem. To fulfil her responsibility for students’ learning of the present day’s new knowledge she seemed to feel responsibility for first carrying out instructional activities that accomplish a preliminary sub-goal: ensure that students, particularly the low-achieving ones, have access to the required method (Fang Fa). That is, ZX strove to promote students’ reactivation of a newly learned concept, consisting of several mathematically justified steps (operations) directed toward the goal of isolating and verifying the values of a variable that make an algebraic fraction equation true.

Additional support for ZX’s focus on students’ existing knowledge as an anchor for the new was provided via her use of purposeful mistakes. For example, while she meticulously wrote on the board every step for solving the second problem, she purposefully made a typical error that violated what S1 had stated only a few minutes earlier (“2-x=-1-2”; Line 04:58). In particular, she pointed to the key step in the method (Fang Fa): multiplying all items on both sides of an algebraic fraction equation by the common denominator. This step could orient their re-setting of the goal and trigger the steps in the method that came before and after the ‘incorrect’ one. By her taking the role of pointing to the error and the way to fix it, her teaching seemed quite unlike a
constructivist approach, in which one would attempt to let students identify the mistake as a means to reason about its source and meaningfully grasp the mathematically justified method.

Excerpt 4.1.4-d provides data about why ZX focused on students’ mistakes, as demonstrated in her use of a pretended mistake. It consists of her response to a follow-up question about why she ‘committed’ that mistake and discussed mistakes on the board.

Excerpt 4.1.4-d (Teacher ZX, Intrv, 2009-05-27)

Students have different understandings of the knowledge. I teach what I think is right for them, and they probably won’t [immediately] understand it completely and correctly. They must have their own ways of understandings. When I teach the mistakes, or ask them to expose mistakes, it doesn’t mean I look down upon them, or degrade their dignity, or humiliate them. [Rather, we] analyse students’ mistakes to find out why someone may make this kind of mistakes, and try to prevent other students from making the similar kind of mistakes. (Emphases added)

Data in Excerpt 4.1.4-d show that ZX distinguished between her and students’ understanding. Like LX, and unlike JC, she was aware that they might have their own ways of understanding. Accordingly, she appeared to see her role as teaching what is right for her students. Such a distinction is rooted in her recognition of mistakes in their solutions, and those mistakes were therefore used, continually and proactively, to support students’ learning. It appears that in her view mistakes not only provide a teacher with a glimpse into student understandings but also and most importantly with a way of orienting their attention to mathematically accepted ways of solving problems. Whereas her focus on students’ existing knowledge seemed related to a constructivist focus, the ‘empty vessel’ pedagogical approach to student mistakes and how to fix them was not.

In this sense, Excerpt 4.1.4-d further indicates the link in ZX’s perspective between reviewing, students’ mistakes, and learning. The key is her expectation of particular mistakes that students typically can make due to their way of understanding. Thus, it makes sense for her to include such plausible student mistakes in carrying out the evolving ‘old’ method (Fang Fa) so that they process and consolidate a proper way of solving a problem on their own. This use of the mistake
by ZX suggests that the reviewing component of a lesson serves the role of re-establishing the mathematical anchor required for learning the present day’s lesson. Such reestablishment could be promoted via purposely and explicitly revisiting an improper step in the method (Fang Fa) (e.g., she intended for student to expect, and avoid, failure to multiply all items by the denominator).

In summary, it seems that ZX uses Reviewing to accomplish a threefold pedagogical goal. First, she wants to re-establish the ‘old’ mathematical foundation needed for students’ learning of the new knowledge she plans to teach in the present day’s lesson. Second, to establish this foundation she needs to re-assess how students understand the previous day’s knowledge while also engaging them in solving problems that can further such understandings. Thus, she first lets them solve a few problems on their own, then guides them through re-stating of the steps in a solution process, and finally demonstrates application to a representative (in her mind) instance, including intentional mistakes, so they obtain the desired outcome (true, non-extraneous values). By such Reviewing she believes more students, particularly the low achieving ones (a) were provided with an opportunity to access the knowledge, (b) could begin solving problems in the intended way, (c) might successfully complete the solution, and (d) would reactivate yesterday’s knowledge. Third, she tries to promote in her students a metacognitive inclination toward doing reviewing on their own and hence taking initiative of their learning. This goal is compatible with LX, as they both encourage students’ sustainable development of good mathematical habits of mind. For them, the teacher’s role is not only or mainly to impart the knowledge but also to nurture students’ learning to learn.

4.1.5 Summary for Reviewing
The four teachers whose use of Reviewing was portrayed above share some common features in their practices and in their views of knowledge and learning. They also employ slightly different approaches. One key common feature seems to be their attention to what students (strong and
low-achieving) must know in order to learn the new (today’s) lesson. Their teaching starts from figuring out where students are. They conceive of themselves as having the freedom and responsibility to select the content and/or techniques of their teaching in spite of the national curriculum and a unified group lesson plan. However, the effectiveness of their instruction differs due to their experience and perspectives of teaching and learning (e.g., JC catering to the strongest students vs. ZX’s powerful support for the low-achieving while not ignoring the strong ones).

All four teachers considered the purpose of Reviewing to be consolidating the anchoring link (old knowledge) as a necessary step for teaching the new. They emphasised their attempt to make sure that no student is left behind due to the teacher failing to engender the necessary (yesterday’s) link. However, because the reviewed content consists mostly of reactivation of the newly taught content, their practice mainly promoted the work of students who have already conceptualised or at least began conceptualising this knowledge. For other students, much of the reviewing could have been inaccessible.

Among the four teachers, WK and ZX had a higher possibility of helping the low-achieving students, because their reviewing is like a miniature, condensed lesson in which prompts for students’ noticing of the intended mathematics were continually provided by the teacher and fellow students. In this sense, WK and ZX seemed to provide the lower-achieving students with some sense of what goal they need to set and which methods they should initiate to accomplish that goal.

Among the four teachers, LX and WK used Reviewing by proceeding from general, abstract concepts to concrete examples, whereas ZX and JC proceeded from concrete problems to general ideas (JC only pointed out student commencing mistakes on the board). Among these four teachers, ZX’s approach seemed to have the highest possibility for helping the low achieving
students to regenerate the target link, because she first engaged students in solving two problems while she figured out where they are by checking their work. Her meticulous introduction of the abstract concept drew on and was immediately applied to their work on the concrete example. On the other hand, the way LX used Reviewing seemed to be more conducive to students’ understanding of reasons behind the rules than the other three teachers.

Third, all four teachers’ use of Reviewing indicated a similar perspective of learning. Each of them seemed to think that without proactively re-establishing the ‘last link’ in the mathematical chain it is hard (or impossible) for students to progress to the new idea they intend to teach in the present day’s lesson. In their view, it seems, mathematical knowledge consists of closely related links; learning is a process of gradually linking a new idea into the lastly established ones. Therefore, in Reviewing, the teachers’ role is to ensure that students re-establish the newly added link (previous day’s lesson) so that the new can be securely fastened to it.
4.2 Bridging

This sub-section discusses the second component—*Bridging*—in the four-component lesson structure identified in the participating teachers’ mathematics teaching. Bridging is a translation of the teachers’ terms of *Xian Jie* 衔接 or *Yin Ru* 引入. Noticing this component was initially rooted in the researcher’s puzzlement about why, after opening a lesson with review of the previous day’s mathematics (as described in Section 4.1), most teachers in this study introduced what seemed to be intentionally chosen examples of mathematics that students have learned earlier. For example, as a bridge to teach simplifying quite complex algebraic fractions, such as \((y^2-9)/(-2y^2+6y)\), LX engaged his students in simplifying numerical fractions they studied in primary school, such as 4/8 and -16/42. This puzzlement led to ongoing inquiry throughout data collection into the various teachers’ rationale for using Bridging problems. In part, this inquiry was rooted in the researcher’s perspective on learning, particularly the role that *assimilation* of new problem situations into students’ existing conceptions plays in their learning of new mathematical ideas.

To examine in the thesis (a) the role of Bridging between the mathematics a teacher supposes students already knows (‘old’) and the mathematics learning intended for student in the current lesson (‘new’), and (b) how the participating Chinese teachers conceived of this pedagogical component, data segments from three teachers were selected: LX (30 years of teaching experience), ST (11 years), and JC (three years). In order to capture how the three teachers used Bridging, and how Bridging in the second observed lesson related to the first observed lesson, data segments from both lessons in LX’s and ST’s classrooms and a segment from one lesson in JC’s classroom were selected.

This sub-section begins with LX and ST because their work and thinking reflect the majority of teachers who participated in the study. This helps to describe a prototype use of Bridging. This prototype proceeds from knowledge students are supposed to have learned long before the
previous few lessons, focusing on solving problems that reactivate the underlying method (Fang Fa) the teacher would then draw on for solving problems involving the ‘new’ knowledge. The teachers referred to such juxtaposing of ‘old’ and ‘new’ via a similar method as the method of ‘analogy’ for teaching the intended knowledge. Between these two teachers, LX appeared to encourage and draw on students’ mathematical reasoning more than ST. Then, JC’s work is introduced to highlight how a novice teacher with strong mathematical background (masters in mathematics) proceeded from presenting a problem solvable via students’ current knowledge to a problem that requires some rearrangement of that knowledge. Later in his Bridging component, JC asked students to also explicate the geometrical meaning of an algebraic expression. This task is reported to demonstrate an individual teacher’s deviation from the ‘standard’ pedagogy that seemed to support high-achieving students’ deep mathematical understanding and reasoning.

For each of the three teachers, the use of Bridging is described first. Then, the teacher’s rationale (perspective) for using this component is examined. The following discussion presents an expert’s use of Bridging.

4.2.1 A Bridging Expert (LX)
In both observed lessons, LX used a component of Bridging after reviewing the essence of the previous lesson and as a prologue to teaching the ‘new’ ideas via Variation. In the interviews with him (see Excerpts 4.2.1-e, 4.2.1-f, and 4.2.1-g), he indicated that this component was rooted in his view that teaching must build upon students’ available knowledge, because “Learning is from known to unknown and from familiar to unfamiliar.” To this end, he explained, Bridging via the method of ‘analogy’ was used, so students can draw inferences for solving advanced problems by first solving familiar ones. As presented in the previous section (4.1) about Reviewing, LX’s first observed lesson intended to teach the concept of algebraic fraction. To lead into that concept, he used three Bridging word problems. The second observed lesson focused on simplifying algebraic fractions. To this end, in Bridging he engaged students in simplifying some simple,
numerical fractions, then drew the general principles for simplification from those simple examples, and finally oriented students’ attention onto application of those principles to algebraic fractions.

Bridging in LX’s lessons appeared to consist of two components, introduction and analogy. The following discussion elaborates this point by examining his use of Bridging in the first observed lesson. This discussion begins with Excerpt 4.2.1-a, which shows how he reviewed the ‘old’ knowledge of using an algebraic expression to express the mathematical relationships in a word problem, which he said (in the first interview) students were supposed to have learned in the primary school (this included students’ learning of the distance/time=speed formulas)

*Excerpt 4.2.1-a (Teacher LX, 2009-05-19)*

01:14 T: (Word problem #1 is presented in PPT) *To conduct research on resources of animals, a zoologist found 7 squirrels within p squares in the beauty spot. How many squirrels were in each square?*

01:42 T: Please raise your hands if you can answer the question.

01:58 Ss: (Some students raised their hands).

02:03 T: S1- can you share?

02:04 S1: Seven divided by p.

02:07 T (to the entire class): How do you express 7 divided by p? How do you express it by an algebraic expression?

02:22 Ss: (chorus) Seven over p.

02:36 T: (Word problem #2 is presented in PPT) *The River in the beauty spot is teeming with delicious fish. One of the hotels has a kg of this fish, and they sold the fish for b Yuan. What is the unit price per kg?*

02:50 T: S2 - how do you solve this problem?

02:51 S: [\( \frac{b}{a} \) Yuan.]

02:55 T: (to the entire class): It should be? 

02:56 Ss: (chorus) \( \frac{b}{a} \) divided by a, equals \( \frac{b}{a} \) Yuan.

03:03 T: The total price divided by the quantity is the unit price.

03:09 T: (Word problem #3 is presented in PPT) *A fishing boat was influenced by the stream when fishing in the lake. The fisherman sails x km/hour in still water. The speed of the stream is 2 km/hour. The fisherman is sailing upstream for 10 km. How much time does it take him to sail?*

03:43 T (to the entire class): How do you work out his speed upstream?

03:48 Ss: (chorus) Distance divided by Time equals Speed.

03:53 T: S3 - how about you solve this problem?

03:54 S3: \( \frac{x-2}{10} \) divided by 10.

04:06 Ss (about half of the class says): Wrong.

04:07 T: Please raise your hands if you have different opinions.
04:16 T: How about S4?
04:17 S4: It should be 10 divided by x-2.
04:19 T (to the entire class): Ten divided by x-2? So which one is correct?
04:23 Ss (chorus); S4 is correct.
04:26 T: Time is equal to Distance divided by Speed; the distance is 10 Km, so it should be (writes on the board) 10÷(x-2), the speed of sailing upstream should be the still speed minus the speed of the stream. Expressed by an algebraic expression, it should be \( \frac{10}{x-2} \). The line can be regarded as the sign of division as well as the line of a fraction. So, for these three problems we create these algebraic expressions: \( \frac{7}{p} \), \( \frac{b}{a} \), and \( \frac{10}{x-2} \). What are the distinctions between the algebraic expressions we have learned before and what are their common features?

Data in Excerpt 4.2.1-a show how LX used three word problems to reactivate students’ existing (‘old’) knowledge—the method (Fang Fa) of expressing a divisional relationship as a fraction. Solving these problems did not mean finding a numerical value. Rather, students were expected to respond with a proper expression. To better understand the role that solving these problems would serve in Bridging it should be noted that the subsequent task was for students to compare and contrast these three algebraic fractions with four symbolic expressions that are not (45, 1/2, 2x, a/4, and x-y; see Excerpt 4.2.1-b below).

LX’s students appeared to meet his supposition (noted in the interview) that they have long learned how to set a fraction as an expression of divisional relationship. The first problem, as solved by S1 and paraphrased by LX, initiated the translation of a simple relationship. S1 properly responded with the operation involved (“seven divided by p”), which LX immediately followed with a request to express it as an algebraic fraction (Line 02:07). The entire class responded verbally with the terms that form a fraction, if/when written (“Seven over p”). Beginning the process of Bridging with this rather straightforward problem seemed to enable students to reactivate their existing understanding of any fraction as an expression of division. Furthermore, it reactivated this understanding with division by a variable. Thus, the first problem
both added to the generality of students’ thinking about the situation (i.e., no need to specify how many squares) and created a first of three specific instances of the mathematical category (algebraic fraction) LX would then point to in contrast to other instances that do not belong in that category.

While creating another instance of the category of algebraic fractions, the second word problem seemed to focus on gradually extending the generality of the ‘old’ knowledge. In it, the ‘old’ relationship between division and its expression as a fraction had to be understood when both quantities were given as variables rather than specific numbers. By asking S2 to respond (Line 02:50), and then turning to the entire class with indication that stating the divisional relationship was also expected (Line 02:55), LX seemed to have further capitalized on their solution to the first problem. At that point, the class as a whole seemed to have understood not only the needed ‘old’ idea (fraction as an expression of division) but also the way LX expected them to fully respond to a problem (i.e., state both the division and the fractional expression of it). LX then paraphrased the acceptable (to him) expression, apparently to emphasise and situate it in the relationship as given in the word problem (“The total price divided by the quantity is the unit price”). Expressing this situated relationship was important and plausibly helpful, because the Bridging component was about to involve yet another gradual extension—to the relationship students would have to glean from and as application of a known formula (speed = distance/time).

The third problem (Line 03:09) provided a third instance of the sought category (algebraic fraction) while stretching students’ use of their ‘old’ knowledge once more. In this problem situation two relationships among quantities, one multiplicative and the other additive, had to be properly expressed. The first (multiplicative) relationship, which would constitute the intended fraction, was division of distance by speed to obtain the time of sailing. The second (additive) relationship, which would constitute the divisor and hence the denominator, was the difference between the speed in still water and the speed of the stream—a difference students would have to
conceptualise in order to set the fisherman’s actual speed upstream as a variable. As LX would later note in the interview, asking S3 to respond was intentional. As LX expected, S3 made a mistake in applying the speed formula to the quantities given in the problem. The subsequent exchange between LX and the class about S3’s answer included stating it was wrong (Line 04:06), asking students to raise their hands if they had a different solution (Line 04:07), asking S4 to share another solution (Line 04:16), hearing and then paraphrasing S4’s response (“Ten divided by x-2”), and most importantly—asking students to compare the two solutions and determine which was correct. Most students in the class responded as a chorus that S4 solved the problem correctly. This led to LX’s culmination of this initial set up of the Bridging component: He explicitly stated the formula for time (the variable asked for in the problem), and wrote on the board while stating the derived division relationship and fractional expression, and stated the categorisation of all three solutions ($\frac{7}{p}$, $\frac{b}{a}$, and $\frac{10}{x-2}$) as algebraic fractions.

Combined, the three Bridging problems that LX selected and used seemed to serve a key role in students’ learning of ‘new’ mathematics later in this lesson, namely, promoting an intended way of thinking—algebraic fractions as an expression of division that includes a variable in the divisor—by reactivating relevant ‘old’ knowledge. The problems introduced a gradual extension of the relationships students would need to glean from the word problem and then express in words and written symbols. Implicitly, this gradual extension seemed to foster comparisons across different instances as a means for capturing what was common to all three problems—algebraic fraction as a symbolic expression of division relationships needed to solve the word problem. The latter point is critical, as these expressions were not simply given symbolically, but rather created by the students, with LX’s paraphrasing, as meaningful mathematical tools for solving word problems. In turn, these problem-situated, meaningful (to students) expressions could give rise to the intended category (algebraic fraction) on which LX wanted students to think and operate.
In this sense, data in Excerpt 4.2.1-a show three important features in the way LX used Bridging. First, it seemed to focus on reactivating students’ available (‘old’) conceptions as a preparation for the new learning. Second, it seemed to promote such reactivation in many students via obtaining answers from the low-achieving students. Third, it consequently provided LX with an opportunity to get ballpark assessment of students’ preparedness for the ‘new’ learning. The following discussion elaborates each of these points.

The first feature (reactivating the ‘old’) was promoted by engaging the students in the repeated task of translating one problem situation after another into a symbolic expression. In this three-problem sequence, a student could form the goal to produce an expression that fits the problem situation, recall the known relationship between division and fractions, and carry out the activity of translating the relationship given in the problem—first into division and then into its expression as a fraction. Due to the particular situations in these three, intentionally selected problems, a likely by-product of the students’ activity might be preparedness for later learning to think also about algebraic fractions as a form of division that includes a variable in the denominator.

The second feature (reactivation in students) was shown in LX’s criterion for selecting individual students who would share their solutions. Specifically, as he noted in the interview, LX called upon low-achieving students, whom he supposed might answer incorrectly. He then oriented students to examine the incorrect solution. For students who solved the problems correctly, being exposed to an incorrect solution could bring about comparison and further justification of their correct solution. This was illustrated in Line 03:54, in which in response to a student’s incorrect solution \( \frac{10}{x-2} \) most students immediately responded that this was wrong. LX capitalised on this error to trigger further comparison and thinking in low-achieving students, by first asking students to raise their hands if they had different opinions. He intentionally waited and did not
call upon high-achieving students to share their answers, so the low-achieving students would have time to think and respond. Thus, whereas this segment of the Bridging component focused on ‘old’ knowledge that students were supposed to know, LX used the whole class discussion to help low-achieving students reactivate the understanding he knew would be needed later in the lesson. Such reactivation could be promoted further by LX’s paraphrasing and summary of the essential relationship in and meaning of each problem (e.g., “the total price divided by the quantity is the unit price”; “the speed of driving upstream should be the still speed minus the speed of the stream”; “the line can be regarded as the sign of division as well as the line of a fraction”).

The third feature (overall assessment) was shown in LX’s focus on figuring out who solved the problems correctly, who solved them incorrectly, whether students could identify the mistake, and whether they could follow justification (in his paraphrasing, Line 04:26) of the correct solution. This assessment was indicated in his distinction between students who could immediately answer the problems correctly and those who needed more time and/or were wrong. The key here was LX’s request for initial responses from students he expected might not necessarily know the answer. That is, for Bridging to produce the desired outcome of reactivating students’ ‘old’ knowledge, a teacher needs to monitor the extent to which they properly use the ‘known’ method (Fang Fa 方法). Thus, as LX stated in the interview, he usually asked low-achieving students to be the first to answer Bridging problems. Simply put, students need to know the ‘old’ in order to learn the ‘new’; thus, as a teacher, LX appeared to assume the responsibility to figure out if they did or did not.

After solving the three Bridging word problems, LX turned to orienting students’ attention to two categories of mathematical ‘objects’, those that are algebraic fractions and those that are not. To this end, he asked them to consider the three expressions they had just produced for the three word problems (, , and ). As Excerpt 4.2.1-b indicates, LX did not directly tell...
students what he intended to teach. Rather, he engaged them in comparison of features common
to several instances of a category (see line 04:26) as a means to elicit students’ thinking about
and expression of the intended differences, and hence the defining characteristics of algebraic
fractions.

Excerpt 4.2.1-b (Teacher LX, 2009-05-19)
05:10 T: (shows PPT4): We have the three algebraic expressions coming from the above
problems, $\frac{7}{p}$, $\frac{b}{a}$, and $\frac{10}{x-2}$, and the following expressions of $\frac{1}{2}$, $2x$, $\frac{a}{4}$, $x-y$. What are
the differences?

05:32 T: How about S5, can you find the differences?
05:35 S5 (stands up): The [first three, 7/p, b/a, and 10/(x-2)] are division of two algebraic
expressions.
05:36 T: Anything else? How about S6? What is your opinion?
05:48 S6 (stands up): The [first three] include a variable in the denominator.
05:49 T: Anything else? How about S7?
05:56 S7 (stands up): They [first three] are all algebraic fractions.
05:59 T: Ok, sit down… This [algebraic fraction] is what we are going to learn today. What are
the differences between these algebraic expressions and those we have learned before? Two
students found two differences, one is the feature of the algebraic fraction – the division of two
algebraic expressions; and the other is that there is a variable in the denominator. We have never
learned this before, and this is what we are interested in and what we are going to
learn today. We are familiar with these algebraic expressions $\frac{1}{2}$, $2x$, $\frac{a}{4}$, $x-y$. We are also
familiar with the difference between these algebraic expressions and the algebraic fraction
$\frac{10}{x-2}$.

Data in Excerpt 4.2.1-b show how LX engaged students in active thinking about key features of
the two sets of examples via ‘search for differences’ questions. First, without mentioning the
terms ‘fraction’ and ‘algebraic’, he asked the whole class to find differences between the two
sets of expressions. After giving everybody a few seconds to think for themselves, he asked three
individual students to share their thinking. Their responses pointed to the nature of instances in
the first set (“division of two algebraic expressions”), the defining feature of that set (“include a variable
in the denominator”), and eventually the term used to categorise them (they “are all algebraic fractions”).
As a result of (a) juxtaposing two sets of expressions, one of which *students had generated* to solve the initial three word problems and (b) engaging students in identifying differences between the *sets*, LX seemed to foster students’ contribution of the key features and term of the ‘new’ concept intended for their learning in that lesson. He then used paraphrasing to emphasise these key features and term, and stated that the latter was the focus of their upcoming learning.

In this sense, starting with the three word problems and proceeding to the task of comparing two sets of instances, LX seemed to serve as a facilitator of students’ thinking. Initially, he reactivated the ‘old’ understanding of *any* fraction as a mathematical expression of divisional relationship. While gradually complicating the particular divisional relationship, he fostered *students’* creation of a meaningful set of fractions that were all algebraic. Then, he fostered a mental activity of comparison between these three examples as instances of a single kind and four other instances that were not of this kind. One aspect of how he seemed to promote students’ noticing of the intended differences concerns the instances of the second set—an integer, a fraction that is not algebraic, an algebraic expression that is not a fraction, and another algebraic expression that consists of operation other than division. Combined, those four instances could help to orient students’ attention to the special combination of conditions that underlie the meaning of an algebraic fraction (division; expressed as a fraction; *and* a variable in the divisor/denominator).

The response of S7 and LX’s follow-up statement (“We have never learned this before”) suggest that some of the students were already familiar with this concept before he would then teach it during that lesson. They could have learned it by themselves (e.g., reading ahead in the book) or through outside classroom opportunities (e.g., tutoring). From his familiarity with students’ knowledge, it seems that LX considered such a possibility. Thus, the Bridging process was structured in two phases to allow those for whom this concept was truly new to (a) inductively develop a category out of a few instances and (b) further conceptualise this category via
comparing its members to a set of non-member instances. That is, LX’s goal was to teach the concept of algebraic fraction; his practice was not to just introduce the definition and foster its memorisation, but rather to engage students in first meaningfully generating and then thinking about a set of algebraic fractions. Consequently, in about 7.5 minutes (1/6 of the entire lesson), LX seemed to enable many of his students to reactivate the ‘old’ knowledge as a means to prepare their thinking for the new concept.

Because this section (4.2) of the analysis focuses on the lesson component of Bridging, the way LX capitalised on the reactivation of students’ knowledge in the first observed lesson is returned to in Section 4.3 (about teaching with Variation). The discussion below further elaborates the use of Bridging by examining LX’s expert use of it in the second observed lesson.

LX’s second observed lesson followed a similar pattern. To set the stage for teaching the concept of that day—simplification of algebraic fractions, he began by explicitly juxtaposing the complex algebraic fraction \( \frac{x^2 - 9}{x - 3} \) and \( x + 3 \) and asked, “Which of the two is simpler?” That is, LX used this juxtaposition to orient the students’ noticing onto the desired effect of simplifying a complex algebraic fraction. Excerpt 4.2.1-c shows how LX capitalized on this comparison further.

Excerpt 4.2.1-c (Teacher LX, 2009-05-20)

13:53 T: There is a close link between division and an algebraic fraction.

Box 4.2.1.1: LX’s first example of simplifying an algebraic fraction

PPT-3: Do you think that \( \frac{x^2 - 9}{x - 3} = \frac{(x - 3)(x + 3)}{x - 3} = x + 3? \)

14:26 Ss: (chorus) Yes.

14:29 T: So [considering] \( \frac{x^2 - 9}{x - 3} \) and \( x + 3 \), which one is simpler? Of course, \( x + 3 \) is simpler. So this involves simplification of the algebraic fraction, right? Today, we are going to learn simplification of algebraic fractions.
14:50 T: Before we start learning the simplification of the algebraic fraction, let’s review some fractional problems that we learned in the primary school. Can you simplify these fractions? (Shows the following fractions on PPT-4)

<table>
<thead>
<tr>
<th>PPT-4: Simplify the following fractions</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{4}{8} )</td>
</tr>
</tbody>
</table>

15:00 Ss: (chorus) Yes, \( \frac{1}{2} \).

15:07 T: How about the others?

15:07 Ss: (chorus) \( \frac{8}{21} \), \( \frac{-2}{3} \), \( \frac{1}{4} \).

15:21 T: So we can simplify the fractions, what do we call the simplification of the fractions?

15:27 Ss: (chorus) Reduction of fraction.

15:29 T: Reduction of the fraction to the lowest terms. [Let’s look at] \( \frac{-16}{42} \).

Negative divided by positive gets?

15:37 Ss: (chorus) Negative.

15:38 T: This \( \frac{6}{9} \) is negative itself; how about, \( \frac{3}{-12} \), positive divided by negative gets?

15:41 Ss: (chorus) Negative.

15:51 T: What are the essential properties of fractions that we learned in the primary school?

15:54 Ss: (chorus) The value of the fraction won’t change if the numerator and the denominator of the fraction is simultaneously multiplied or divided by a number not equal to 0.

[A little later] T: Do you think the algebraic fraction has similar properties?

Data in Excerpt 4.2.1-c show how LX promoted reactivation of hopefully known concepts in students. First, he conveyed the goal of the day’s learning to students via a rather straightforward, short activity following the process of simplifying an algebraic fraction (lines 13:53 – 14:29). He restated a conventional meaning for the notion of ‘simplification’ via an example of the process by which the result (a simpler expression) is obtained and the obvious conclusion (“of course, \( x+3 \) is simpler”) being the goal of that process—and their lesson. However, to teach toward this goal LX did not proceed to more examples of simplifying algebraic fractions. Rather, he first engaged students in simplifying numerical fractions, including attention to the signs (i.e., directed numbers), which they learned in primary school (see Line 14:50 and Box 4.2.1.2).
In response to his question about whether they could simplify those numerical fractions, students said yes and gave the answer to the first instance (1/2). LX requested the answers to the three other instances, and the entire class responded as a chorus (-8/21, -2/3, and -1/4). These data indicated that most students knew how to solve the four fraction problems he gave. Once being prompted for the goal, they accomplished it via reactivating a method (Fang Fa) they could independently call upon and use to produce an equivalent fraction in its simplest form. The data do not indicate what specific processes students used, as LX did not inquire into these processes. At the time, due to the researcher’s focus on the fact that LX began with numerical fractions, she asked him why he used those simple instances but not about the lack of his inquiry into how they simplified them. Such a question by the teacher would have been beneficial, because it could allow him, and students, to figure out how they operate on fractions and justify their solutions. In retrospect, it can only be speculated that LX’s supposition that students would know this, and their immediate correct answers, indicated to him that the students were using the intended, correct method (finding and cancelling all common factors in the numerator and denominator). That is, in the Bridging problems LX had used the ‘old’ seemed (to him) ‘painfully obvious’ and thus no further inquiry would be necessary.

After the students solved all four problems, LX further oriented their attention to the issue of determining the sign for a fraction as a single entity, as opposed to as a part of the numerator or the denominator alone. Similarly to the way he introduced the term ‘simpler’ via a solution process, he emphasised the sign conventions via discussing the three instances that contained a negative sign. For each of those, he stated the fraction, made explicit the signs and the operation (e.g., “negative divided by positive gets?”), and seemingly expected, and accepted, students’ chorus (correct) responses.

At that point (Line 15:51) LX engaged students in stating key aspects of the method of simplification and the resulting equivalent fraction. This request gives some support to the
speculation that he considered their reasoning when solving the four problems to be rather obvious. If they could properly state the general process—it would indicate they have likely applied it to each of the four instances. As the solid ‘chorus’ of responses (“The value of the fraction won’t change if the numerator and the denominator of the fraction is simultaneously multiplied or divided by a number not equal to 0”) indicated to the researcher (and apparently also to the teacher), the five examples seemed to reactivate existing (‘old’) knowledge in students, including low-achieving ones.

Once students stated the essential property of fraction equivalence, which underlies expanding or simplifying any fraction, LX turned to promoting further linkages between ‘old’ (numerical) and ‘new’ (algebraic) fractions. The following excerpt, 4.1.2-d, shows he attempted to accomplish such linkage by pointing out that algebraic fractions extend the category of numerical fractions by including variables while maintaining the essential properties of any fraction (see line 16:27). In particular, he pointed out that a fraction, which they reconceptualised in the previous lesson as an expression of division, could not include a divisor (hence, denominator) that is equal to zero.

Excerpt 4.2.1-d (Teacher LX, 2009-05-20)

16:27 T: The fraction and the algebraic fraction are closely related; [the latter] only expands the number into the field of variables, so the question is does the algebraic fraction has the same feature as fractions?

16:36 Ss (chorus): Yes.

16:37 T: The answer is, for sure, yes. We can use the method of analogy, and we mentioned yesterday when we solved the problem of algebraic fraction, that we can think of the fraction, and the essential properties of the algebraic fraction, as closely related. So we can change the properties of the fraction into the properties of the algebraic fraction by using the analogy.

PPT5: Does \( \frac{a}{2a} \) equal to \( \frac{1}{2} \)?

17:04 Ss (chorus): Yes.

PPT5: Does \( \frac{n^2}{mn} \) equal \( \frac{n}{m} \)?

17:10 Ss (chorus): They are equal.

17:12 T: What condition you should pay attention to?

17:16 Ss (chorus) That m and n are not equal to 0.
17:18 T: Under the condition that \( m, n \neq 0 \). So the problem of fraction is changed into the problem of algebraic fraction, and the properties of the fraction can be changed into the properties of the algebraic fraction. (LX shows PP-6 and asks students to read it.)

PPT-6: The properties of the algebraic fractions: If the numerator and the denominator are simultaneously multiplied or divided by an algebraic expression not equal to 0, the value of the algebraic fraction remains the same.

18:09 T: Expressed algebraically, if the numerator and denominator are multiplied by \( m \) simultaneously, the value of the fraction remains the same.

18:16 T: What is the condition that \( m \) must meet?
18:17 Ss (chorus): \( m \neq 0 \).
18:20 T: Why can’t it be 0?
18:29 Ss (chorus): It is undefined when it is 0.
18:33 T: So \( \frac{0}{0} \) is undefined, right? \( \frac{a}{b} = \frac{a \pm m}{b \pm m} \) if the divisor is not 0.

Data in Excerpt 4.2.1-d show how LX attempted to capitalise on the initial Bridging step of actually simplifying numerical fractions to gradually extend students’ learning. At the core of his attempt seemed to be orienting students’ attention to an essential property of the category of fractions, which comprises both numerical and algebraic fractions.

To this end, LX first introduced two more instances of simplifying algebraic fractions (\( \frac{a}{2a} \) and \( \frac{n^2}{mn} \)). The first instance was rather basic and straightforward, as it required cancelling a visibly common factor (‘a’). The second one slightly extended the application of the simplification process by masking the common factor within an algebraic expression (\( n^2=n \times n \), and so one of these would be the common factor, ‘n’). This extension seemed to target, and likely to reactivate, two other aspects of the method (Fang Fa) on which his teaching would focus later. The first aspect concerned searching for every common factor—masked or given directly; the second aspect concerned properly isolating and cancelling the common factors (e.g., it would be incorrect to simplify \( n^2/mn \) into \( 2/m \) by ‘just cancelling’ \( n \) in the numerator and denominator).
To understand the significance of how LX used Bridging via the two instances of simplifying algebraic fractions it is important to emphasise two interrelated points. First, he selected rudimentary examples \((a/2a, n^2/mn)\) that students were supposedly able to solve. However, this point alone does not seem too insightful, as their work during the previous lesson suggested that most students could solve these problems without first being asked to simplify the four numerical fractions. Thus, the second point stresses his sequencing of all Bridging instances—similar to the analysis of the three Bridging word problems during his first observed lesson. This sequence proceeded from a rather sophisticated example by which he introduced the target ("\(x+3\) is simpler"), through simplifying numerical fractions with sign determination, to simplifying rudimentary algebraic fractions (direct and then masked). In this way, he seemed to create an opportunity for students to compare across all seven instances and identify what he appeared to consider as common aspects of the process when operating on different ‘members’ of the two fractional sub-categories (numerical and algebraic). In turn, students’ solutions of these representative cases seemed to lead quite naturally into (a) their statement of the condition (Line 17:16); (b) LX’s paraphrasing of their statement and his follow-up statement that the properties apply to both sub-categories (Line 17:18); (c) engaging them in reading the summary rule about equivalence of any fraction (Line 17:18, PPT-6); and (d) the symbolic expressions of this rule which, when read from right to left, provide the basis for simplifying fractions (Box 4.2.1.3 and Line 18:33). That is, this sequence of instances seemed to reactivate students’ knowledge of simplifying fractions and situating it in the generalised forms of extending or reducing a fraction without changing its value through multiplying or dividing it by an algebraic fraction that is equal to ‘1’ (same numerator and denominator).

To summarise LX’s use of Bridging, he attempted to initiate learning of the ‘new’ knowledge by reactivating students’ ‘old’ knowledge through engaging them in solving carefully chosen examples. Besides choosing and sequencing these examples, the main role he seemed to play was orienting students’ reactivation of their existing processes (e.g., fraction is an expression of
division, simplifying any fraction occurs via cancelling common factors). Then, he used the reactivated ‘old’ conceptions as a basis for the link to the ‘new’. Consequently, LX seemed to help his students, especially the low-achieving ones, to transform a piece of mathematics they already knew into a related, new piece he intended for them to learn.

4.2.1.1 Accounting for the Role of Bridging in LX’s Perspective

In the interviews after the observations, LX was asked to elaborate on the goal that Bridging served in his teaching. In response, he emphasised the support it provided for low-achieving students to link the new to the old. He said that students have different conceptual levels of understanding. Therefore, his teaching proceeded from what he supposed students would understand. In turn, he used those understandings as a basis to stress the underlying method (Fang Fa 方法) as it is applied to algebraic fractions. To further clarify his perspective, the researcher asked him whether Bridging works differently for low- and high-achieving students.

Excerpt 4.2.1-e (Teacher LX, Intrv, 2009-05-19)

I always introduce very simple and commonly used problems in the Bridging process to make sure the low achieving students can also understand. For this lesson, the low-achieving students also know the real word problems and they can understand, and from what they know [I can] lead to the introduction of the new concept of algebraic fraction. I assume the low-achieving students can understand, and the upper students have no problem understanding. The only difference is that the upper students don’t know what is the link between the old and the new knowledge, [but] once I introduced or pointed it out, they immediately know the link between the old and the new. (Emphases added).

Data in Excerpt 4.2.1-e show that LX considered the role of Bridging as a pedagogical method for supporting low-achieving students’ learning. In this excerpt, he referred to the three word problems from the first observed lesson. He appeared to conclude that low-achieving students understood how those problems were solved (i.e., creating a fraction to express divisional relationships, with variables, given in the word problems). That is, the Bridging problems appeared to enable, in his view, reactivating of students’ available (‘old’) knowledge. In turn, he seemed to reason that such reactivation allowed him to lead students’ thinking to the new concept of algebraic fractions.
In this sense, data in Excerpt 4.2.1-e provide evidence for three aspects of LX’s perspective. First, he seemed to assume responsibility for helping students learn the intended (new) mathematics by first reactivating what they already knew. Thus, he seemed to select Bridging problems that would be simple enough. This allows low-achieving students to understand the problem or question, make them feel engaged, and nurture their sense of being part of the lesson from the outset. Second, he seemed to use Bridging as a means to create an anchor for the link he would then introduce to the new concept. This suggests a view of mathematics as a related body of knowledge and of learning as a process in which what is known should be used to open the way (link) to a new piece of that knowledge. Third, he seemed to consider Bridging, and accordingly selected specific instances, as a pedagogical move that supports low-achieving students’ learning, whereas high-achieving students could effortlessly make the link. That is, Bridging seems to entail a perspective on knowing in which what has been learned previously must be reactivated in order to support successful linking to new concepts. But the teacher seemed to conceive of ‘things’ that are being linked and the link itself as existing outside the knower.

Another point shown in Excerpt 4.2.1-e is that LX was aware of students being at different levels of understanding, and the impact such differences have on how they may interpret the ‘same’ problem. Accordingly, he might also have different requirements and/or expectations for students at different levels. By juxtaposing algebraic expressions and algebraic fractions, for example, and by asking students to find out the characteristics and differences between the ‘old’ and the ‘new’, in Bridging he seemed to focus on promoting the essential link for the students. Later in the lesson (component of Variation, see Section 4.3), he would add problems that could serve to challenge high-achieving students to learn more difficult aspects of the ‘new’ concept.

In the observed lessons and interviews, LX stated that he used the method of analogy for Bridging. The researcher asked him to expound on this notion and he responded: ‘I use the analogy, for example, that an algebraic fraction is very similar to a fraction… firstly we introduce the fraction, because they
can accept fractions, when we give the examples, the sign conventions were also included… it is a smooth transfer from the fraction to the algebraic fraction’. He further explained that in this way students feel they can solve the problem via intuitive thinking, and was asked to elaborate this.

*Excerpt 4.2.1-f (Teacher LX, Intrv, 2009-05-20)*

The so-called intuitive thinking is that *teaching* starts from what the students already know. Give them some simple examples that they can consider directly, without requiring profound or abstract thinking. The students can directly perceive from sensing it and understand it immediately as soon as they see it. (Emphasis added)

Excerpt 4.2.1-f and the quote from teacher LX just above it further demonstrate his focus on teaching through reactivation of students’ available knowledge. Aside for his view being contestable (as the process may include other steps), for him intuitive thinking seemed to mean being able to use the available knowledge in interpreting and solving problems. However, it is not used to rehearse ‘old’ knowledge for promoting students’ competence and confidence in applying it (e.g., to solve word problems). Rather, a teacher uses it to focus students’ thinking on and use of mathematical processes that are relevant to interpreting and solving new problems as what to LX seems intuitive extension of the ‘old’. Thus, as a teacher his role is to select Bridging problems that are simple enough, requiring no profound or abstract thinking, so students can solve them as soon as she or he sees it. For example, students who have supposedly mastered and are familiar with the simplification of numerical fractions can ‘intuitively’ solve a problem as soon as they see it precisely because they can independently interpret it via their existing knowledge. In turn, this reactivated ‘intuition’ can lead them to successfully simplify rudimentary algebraic fractions.

LX’s teaching seems to capitalise on students’ independent, immediate grasp of ‘old’ problems by intentionally juxtaposing successful solutions to those with the active, mental work needed to solve problems that slightly extend that knowledge. The key here seems to be his apparent attempt to foster students’ use of the mathematical method (Fang Fa) that he considers to underlie both the ‘old’ and the ‘new’ (e.g., asking students to simplify $a/2a$ and $n^2/mn$
immediately after they simplified 4/8, -16/42, -6/9, and 3/-12). This apparent focus on the crucial role that reactivating students’ existing knowledge as a starting point for teaching new concepts is further shown Excerpt 4.2.1-g.

Excerpt 4.2.1-g (Teacher LX, Intrv, 2009-05-20)
My teaching philosophy is, when you teach something new, try to build/develop the new knowledge on where the students are and what they already know, then introduce the new knowledge, and later grasp the new knowledge, … because the human cognition is developed from the old to the new. In order to get new knowledge, we definitely should link it with the old... From my perspective, learning should start from what students already know [and proceed] to learning of the new knowledge, which is a direct transfer and guidance from what they know to unknown and assimilate the old and the new. It is more natural. For example, when we teach the algebraic fraction, we use the method of the analogy. Like these algebraic fractions, we haven’t taught the simplification of algebraic fractions, but once we relate the algebraic fraction with [numerical] fractions students are naturally cancelling ‘a’ in a/2a. Initially, cancelling ‘a’ from the algebraic fraction is today’s new knowledge, because this is an algebraic fraction with a variable in the denominator, but for me I think it is intuitive thinking. (Emphases added).

Data in Excerpt 4.2.1-g show the rationale about mathematics learning that seems to underlie LX’s pedagogical practice of Bridging. In this rationale, the process of coming to know something ‘new’ builds on what students already know. Because learning should start from what students already know, the teacher’s role is to reactivate this ‘old’ knowledge in such a way that would ‘intuitively’ proceed to the ‘new’. Furthermore, he appears to think that, once and if the ‘old’ has been properly reactivated by the teacher, students’ progress to the ‘new’ occurs via linking the ‘new’ to it (“In order to get new knowledge, we definitely should link it with the old”). Such progress, according to LX, can be fostered via orienting students’ attention to analogous features of the ‘old’ and the ‘new’. For example, in his view such a method of analogy appears to allow students’ “naturally cancelling” of a common factor given by a variable similar to how they cancelled a common numerical factor in the Bridging instances.

4.2.2 Intermediate Level of Bridging (ST)
Like LX, both lessons observed in ST’s class showed distinct features of using Bridging as a pedagogical means to reactivate the ‘old’ and build on it as an anchor for introducing and linking to the ‘new’. Similarly to LX, ST stressed that learning starts from what students already know.
Likewise, ST said she regularly use the method of analogy near the beginning of a lesson to promote Bridging.

ST’s first observed lesson was about the multiplication and division of algebraic fractions. She firstly presented two typical problems involving numerical fractions for students to solve, which then led onto using the same operations on algebraic fractions. Her second lesson was about adding and subtracting algebraic fractions, where she used the same strategy—presenting two typical problems of numerical fractions followed by adding and subtracting algebraic fractions. That is, Bridging in ST’s lessons consisted of two components, reactivating the old and then juxtaposing it with the new by the method of analogy. However, her approach differed from LX’s in terms of the extent to which each teacher encouraged students’ mathematical reasoning via posing reflective questions. Excerpt 4.2.2-a shows how ST opened her first lesson by seemingly attempting to reactivate students’ presumed ‘old’ knowledge (taught in primary school) for multiplying and dividing numerical fractions.

Excerpt 4.2.2-a (Teacher ST, 2009-05-21)

00:12 T (shows PPT-1 and asks): Ok, let’s do some simple calculations; can you do it?

Box 4.2.2.1: ST’s two Bridging problems (multiply/divide numerical fractions)

PPT-1: Solve the problems: (1) \( \frac{2}{3} \times \frac{4}{5} \) (2) \( \frac{7}{6} \div \frac{14}{9} \)

00:14 Ss (chorus): Yes.

00:15 T: Raise your hands if you can work out these problems.

00:18 Ss: (Many students raise their hands.)

00:23 T: They are very simple, right. The first [problem] – how about S1?

00:25 S1: Eight-fifteenths (\( \frac{8}{15} \)).

00:35 T: Eight-fifteenths, ok. Please sit down. (Once S1 sits, she shows the solution in PPT-1.)

Box 4.2.2.2: Solution to problem #1 (above)

PPT-1 (continues): \( \frac{2}{3} \times \frac{4}{5} = \frac{2 \times 4}{3 \times 5} = \frac{8}{15} \)

00:36 T: The second problem? (Many students raise their hands.) Ok, S2.
S2: Three-fourths.
T: (to the whole class) Three-fourths; is that right?
Ss (chorus): Yes.
T: In this operation, you should first change the division into?
Ss (chorus): Change the division into multiplication.
T: So firstly change the division into multiplication (shows PPT-1).

<table>
<thead>
<tr>
<th>Box 4.2.2.3: Solution to problem #2 (above)</th>
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| PPT-1 (continues): \[
\frac{7}{6} \div \frac{14}{9} = \frac{7}{6} \times \frac{9}{14} = \frac{3}{4}
\] |

T: In these two problems, what are we operating on?
Ss (chorus): A fraction.
T: Yes, right, these are operations on fractions. What principles are we using from primary school by doing operations on fractions?
Ss (chorus): The essential properties of fractions.
T: Yes, right.... Let’s recall the principles. I would invite one student to read the principles loudly. Any volunteers? Ok, S3.
S3: (Stands up and reads the principles of multiplication and division of fractions from the PPT).

Data in Excerpt 4.2.2-a show that ST started her Bridging component by presenting two instances of operating on numerical fractions, one for multiplication and another for division, as a means to reactivate students’ use and later statement of the abstract principles. Like she seemed to expect, these two problems were simple enough for students to solve by using their available conceptions (and in their heads). Regardless of her intention behind the management request for students to raise their hands, this pedagogical move seemed to provide more time for low-achieving students to reactivate their ‘old’ knowledge. However, unlike LX she did not ask the students to explain why and how they got their answers. For example, after S1 solved the first problem she simply showed (PPT) the full process of solving it (Box 4.2.2.2). Later, she did not prompt reflection on and explaining the recalled principles of fraction multiplication and division. Instead, she directly showed the PPT and asked volunteers to read these principles out loud (Line 01:13).

ST’s approach to Bridging was similar to LX’s in that she began students’ path to operating on algebraic fractions (‘new’) via first fostering reactivation of the supposedly known, analogous
operations on numerical fractions (‘old’). However, her way of using Bridging deviated from LX, who repeatedly (a) created a situation for problem solving before he put forth the answers to his own questions and (b) gave enough time for low-achieving students to think and solve the problems on their own. In comparison to LX, ST’s pace of moving through the Bridging instances and the mathematical justification for their solutions seemed to miss out on a good opportunity to promote reactivation of old knowledge by students at different conceptual levels. In particular, LX asked his students to regenerate the principle of simplifying fractions before presenting it on the PPT; ST directly presented the principles and merely asked students to read them. Thus, ST’s Bridging did not combine reactivation and use of available conceptions with reflection on and justification of the relationship between their solution processes. LX’s students could reactivate a higher level of understanding by firstly using and reorganising the principle in thought and then verbally expressing it. As von Glasersfeld (1995) asserted, spontaneously expressing an idea is cognitively more demanding than interpreting it from given prompts (let alone literally reading it). ST asked the students only to read from the PPT, which at best seemed to promote mechanical linking between the reactivated process and its abstract, symbolised mathematics justification.

After ST used concrete examples and read-out-loud summarising of the principles, she asked students to synthesise the principles for algebraic expressions. Excerpt 4.2.2-b shows how she used Bridging from numerical to algebraic fractions.

*Excerpt 4.2.2-b (Teacher ST, 2009-05-21)*

01:44 T: Can you use an algebraic expression with variables to express the principles described above?

PPT-1 (continues): Think: can you use variables to express the principles described above?

02:12 T: Ok, S4.

02:13 S4: (says while T writes on the board) $\frac{b}{a} \times \frac{d}{c}$ equals $\frac{bd}{ac}$

02:23 T: Ok, the second one.
Data in Excerpt 4.2.2-b show how ST attempted to build on students’ solutions of the two Bridging instances of numerical fractions (Excerpt 4.2.2-a) to foster application of the rules while expressing them algebraically. She asked one student to state the rule for multiplication of fractions (Line 02:13) and wrote on the board what the student said. She then asked another student to state the rule for division. This time, she waited until the student completed the statement, and then paraphrased it while she wrote the rule on the board. At that point, she stated what seemed to be a key in her thinking about the shift from the numerical instances, namely, the use of variables to stand for numbers. To culminate this shift, she restated the fact that they have learned to operate on numerical fractions and would then move on to operating on fractions with a variable in the denominator. In response to her request to name such a fraction, students responded in chorus, “The operation of the algebraic fraction.” This led to ST’s stating of the goal for that day’s lesson—learning to multiply and divide algebraic fractions.

Combining the exchanges in Excerpt 4.2.2-a and 4.2.2-b, it appears that ST’s use of Bridging consisted of two components, reactivating the ‘old’ (Excerpt 4.2.2-a) and leading into the ‘new’ (Excerpt 4.2.2-b). In the former, she could orient low-achieving students’ reactivation of available knowledge without much reflection. In the latter, she oriented more reflection by avoiding just telling and showing. She called on a student to express the principle of operating on fractions algebraically, and wrote the answers on the board instead of showing them on the PPT (lines 02:23, 02:47). By proceeding from numerical instances to stating and expressing the
principles and rules she could promote re-establishment of the principles on the basis of first actively applying them to supposedly known examples. This pedagogical process, however, seemed to focus on quickly moving students from a consideration of simple examples to gleaning the rules. For high-achieving students, her rather quick shift to general, symbolic rules could be trivial. However, unlike LX, for ST’s low-achieving students the shift from numerical instances to general rules seemed to lack the additional emphasis on the essence of the ‘old’ and the use of reflection on why the ‘old’ method (Fang Fa) works as a basis for students’ regeneration of the rules.

ST’s second observed lesson, in which she taught teaching addition and subtraction of algebraic fractions, followed a similar pattern. To bridge between students’ ‘old’ knowledge and the intended learning she first presented two problems of adding and subtracting numerical fractions, and proceeded to stating the principles for analogous operations on algebraic fractions. Because Bridging in both lessons was conducted similarly (from supposedly known instances to abstract principles) and consisted of two parallel components (reactivating the ‘old’ leading to stating the ‘new’ with variables), the second observed lesson is briefly summarised below.

To reactivate ‘old’ knowledge from the primary school, ST asked students to individually solve the problems: (1) \( \frac{1}{5} + \frac{2}{5} \) and (2) \( \frac{7}{10} - \frac{3}{10} \). Then, a solution to each problem was shared by one student, which ST followed by asking the rest of the class if they agreed. After each solution, she commented that this was “Very simple, right?” Then, she engaged the students in stating the principles for adding like-denominator fractions and the need to simplify the answer to the second solution (“\( \frac{4}{10} \), it should be \( \frac{2}{5} \)).

For the second component of leading into the new (additive operations on algebraic fractions), teacher ST first provided a problem analogous to the first numerical instance of adding like-
denominator fractions \( \frac{1}{a} + \frac{3}{a} \) and asked an individual student to solve it. As soon as he stated the answer \( \frac{4}{a} \), she asked students to instantiate the general number with two different values, because “… we said the variables represent numbers.” She solicited two different values from the students (a=6 and later a=2) and in solving the numerical problems on the board emphasised that equivalence of both sides of equation had to be maintained. Finally, she called on another student to state the principles for adding fractions while she wrote them on the board and later also asked students to read them out loud: \( \frac{b}{a} + \frac{c}{a} = \frac{b+c}{a} \). Finally, she asked students about subtraction of like-denominator fractions and wrote the shorthand version of the double rule (addition and subtraction): \( \frac{b}{a} \pm \frac{c}{a} = \frac{b\pm c}{a} \). In all, similar to her work during the first observed lesson, ST appeared to use the two supposedly known instances as a direct vehicle to stating the general principle, without engaging students in much reflection and reasoning about the analogy. In this sense, she might have assumed that the analogy was trivially ingrained in the shift from numerical to algebraic operations, and there was no need to spend time on elaborating the underlying method (Fang Fa) of this shift.

4.2.2.1 Accounting for the Role of Bridging in ST’s Perspective

In the interviews after observing ST’s lessons, she clarified that her purpose for using Bridging was to teach from what students already knew to the unknown. That is, quite similar to LX, her rationale for using Bridging seemed to focus on making the transition to the ‘new’ (e.g., operations on algebraic fractions) smooth and gradual. She said that students at this age (12-13) have relatively greater difficulty in receiving new knowledge without such anchoring. Therefore, she said, new knowledge is not introduced in her lessons without first reactivating the ‘old’ knowledge. Not activating the ‘old’, in her view, appeared to hamper students’ learning because, as she said, learning of the ‘new’ builds upon the ‘old’ and should thus be explicitly related to
the ‘old’. Accordingly, she took her role as a teacher to include reactivation of the ‘old’ via use of particular problems students were supposed to successfully solve. She asserted that Bridging helps the students see and form the *link* between the ‘new’ and the ‘old’; it makes accepting, digesting, grasping, and understanding the ‘new’ easier for students. That is, ST’s teaching built on student available knowledge, which the teacher needs to reactivate in order to serve as the foundation to which the ‘new’ knowledge would then be linked.

A good example of ST’s notion of smooth transfer she attempted to promote between what students already knew and the new knowledge was her asking students to select some numbers to substitute for the variable (they suggested $a=6$ and $a=2$). In the interview, she was asked why she did this. ST responded she wanted to let the students experience the use of variable in substitution for the denominator while the fraction remained equivalent. She further explicated that the purpose of doing this was to, “let the students know the denominator can be substituted, then the next step, [substitute for] the numerator, it is a transition from fraction to algebraic fraction.” As the data set with ST was conducted after the data set with LX was completed, she was asked if the purpose of using Bridging this way was to help students see that the same method (Fang Fa) was used for solving the ‘old’ and ‘new’ problems. She concurred, “Yes”. Excerpt 4.2.2-c expounds on this view further.

*Excerpt 4.2.2-c (Teacher ST, Intrv 2009-05-22)*

What I want the students to see is the Fang Fa 方法. If the bridging is close enough, or simple enough, the students will recognize it on their own, and if the bridging is too far, I will go through it and give the students some prompt or demonstrate it to help the students start seeing or solving the problem.

Data in Excerpt 4.2.2-c show two key points in ST’s rationale for Bridging, and hence her perspective on mathematics knowing and learning. The first point is that, for ST, Bridging appears to serve as a means to reactivate students’ available knowledge as a starting point for the transition to the ‘new’. The key is her emphasis on making sure that students “recognize it on their own,” where ‘it’ refers to the underlying method (Fang Fa). Accordingly, she seems to think that
if Bridging from the ‘old’ to the new is close enough students would have no difficulty solving problems by reactivating the ‘old’ method (Fang Fa), which in turn enables them to form links to the ‘new’ for themselves. She also appeared to think that if the distance between the ‘old’ and the ‘new’ is too far, a prompt on her part could lead the students to form the link.

The second point was ST’s focus on the role that Bridging should serve in student learning, namely, highlighting for them that the mathematical core is the common solution method (Fang Fa). This focus indicates the integration between (a) the teacher’s reliance on profound understanding (Ma, 1999) of the particular mathematics and (b) her expectation that juxtaposing an operation on familiar objects (e.g., numerical fractions) with the same operation on ‘new’ objects (e.g., algebraic fractions with one variable) would promote linking the ‘new’ to the ‘old’. The latter seems to be a crucial aspect of this perspective. She did not seem to distinguish between her and the students’ mathematics; the link she saw was also the link students would come to see if Bridging was properly designed (“close enough”). Said differently, the mathematical link does not change from knower to knower—it ‘exists’ independent of the teacher or the learner; Bridging simply affords students with seeing that same link. Therefore, the first priority in her pedagogical approach seemed to be figuring out where students are (e.g., make sure they knew how to add like-denominator fractions). This allowed her to select problem situations that would reveal the link (method) to the new knowledge (e.g., adding an algebraic fraction with a single variable like-denominator). She used the word ‘transition’ to imply students’ application of the familiar method to the new situation and appeared to promote such transition not through directly telling and showing the intended link. Rather, she selected simple examples that students could solve with the intended method as a means to promote students’ learning in a natural, intuitive way (which she called ‘smooth transfer’).

ST’s practice and perspective differed from LX’s in an important way, which supports coding her use of Bridging as intermediate. Unlike her, LX appeared to more deeply appreciate the
problematic nature of the intended transition for the low-achieving students. Thus, he focused his attention not only on low-achieving students’ use of the supposedly known instances but also on engaging those students in explicitly reasoning and reflecting on the common features of the method (Fang Fa) that underlie its application to the ‘old’ and then to the ‘new’. Such reasoning processes in low-achieving students were further supported by his intentional requests for sharing solutions by these students and capitalising on their mistakes to foster students’ learning. That is, both teachers used quite similar words to explain why Bridging was helpful, but LX seemed to better differentiate and intentionally use this lesson component to support the learning of low-achieving students.

In summary, the approach to teaching by both LX and ST was based on reactivating students’ available knowledge as a means to foster linking the ‘new’ concept to the ‘old’. To this end, they used well-designed problem situations of supposedly familiar instances as a bridge. They engaged students in actively applying thinking processes first to the ‘old’ and then to the ‘new’ mathematical objects. The following sub-section discusses a novice teacher’s use of Bridging in ways that indicate how a similar rationale gave rise to a dissimilar practice.

4.2.3 A Bridging Novice (JC)
Like LX and ST, the lessons observed in JC’s classroom indicated the use of Bridging. He, too, used this term during the interviews and clarified that Bridging allows a smooth transition from the ‘old’ to the ‘new’ as extension of the ‘old’. He saw this link and expected his students to apply the method (Fang Fa) from ‘old’ to ‘new’ problems. The data and discussions in this sub-section show commonalities that JC’s Bridging shared with LX and ST. In particular, he used the techniques of calling on individual students to solve problems and introducing problems that he supposed students could solve. On the other hand, the data reveal some unique features in JC’s use of Bridging. First, whereas JC’s initial component of Bridging was also reactivating the ‘old’ knowledge, his second component was not the method of analogy. Instead, he posed new, quite
challenging problems, which differed from the simple problems students could solve using their available knowledge. Second, whereas LX’s and ST’s Bridging oriented students’ noticing of the method (Fang Fa) across situations, JC appeared to use Bridging as a means to advance the students to a higher level of mathematical reasoning and thus catering more to high-achieving students’ learning than to low-achieving ones. The discussion below elaborates both points.

Based on JC’s first interview, in the first lesson observed in his classroom he intended to teach the factoring of generic quadratic trinomials via the method of cross-multiplication. Before JC introduced this method, he oriented students’ focus onto the special case of quadratic trinomials that can be factored by using the formulas that the students have already learned (for \(a^2-b^2\) and for \((a\pm b)^2\)). Excerpt 4.2.3-a begins with the classroom work on reviewing these formulas.

Excerpt 4.2.3-a (Teacher JC, 2009-05-14)

02:43 T: Can we factor the quadratic trinomial \(x^2+4x+4\) by the method of a formula?
02:58 Ss (chorus): Yes.
03:01 T: How do you factor it? S3 - can you solve it?
03:02 S3: It is \((x+2)^2\).
03:07 T: Why?
03:09 Ss (chorus): Because it is the formula for a perfect square.
03:10 T: Because it is a quadratic trinomial that can be factored by the formula of a perfect square, right? Let’s look at this problem: \(x^2+4x+3\).
03:23 S: (Some students raised their hands.)
03:24 T: Can you use your available knowledge to factor this quadratic trinomial? What is the available knowledge? How about S4 answer this question?
03:37 S4: Use the formula of a perfect square and formula for the difference of squares respectively.
03:38 T: How do you adjust it?
03:39 S4: \(x^2+4x+3 = x^2+4x+4-1\)
03:44 T (Writes on the board what S4 said): So you changed this 3 into +4-1. Why did you change it?
03:46 S4: Because one [1] can be looked as one-squared \([1^2]\), and the front part \(x^2+4x+4\) can use the formula for a perfect square to factor.
03:52 T: So the front part can use the formula for a perfect square, and 1 can be looked at as \(1^2\).
03:58 S4: \((x+2)^2 - 1\).
03:58 T: (Writes on the board what S4 said)
04:03 S4: Then use the formula for the difference of squares.
04:04 T: Then use of the formula for the difference of squares.
Data in Excerpt 4.2.3-a show that the first problem JC chose for Bridging (“factor the quadratic trinomial \(x^2+4x+4\) by the method of a formula”) was supposedly familiar to students. He asked S3 to share the answer, \((x+2)^2\), and followed by asking the entire class for justification. The students responded as a chorus by stating the formula that allows solving this single instance (“Because it is the formula for a perfect square”). In JC’s class, it appears that such justification was acceptable, as he immediately proceeded to a problem for which application of the formulas was not obvious (“Let’s look at this problem: \(x^2+4x+3\)”).

That this second problem was not another simple instance of what students already knew was indicated by the fact that not even half of the class raised hands. Apparently sensing that this might be an issue, JC repeated the task while emphasising the focus on applying one’s available knowledge (“Can you use your available knowledge to factor this quadratic trinomial? What is the available knowledge?”), and followed by asking a student (S4) to provide the solution out loud. S4’s response of how a combination of two formulas could be used indicated high-level application of the formulas. JC asked S4 to explicate that combination, and wrote on the board what S4 had stated (“\(x^2+4x+3 = x^2+4x+4-1\)”). JC apparently understood the focus of S4’s adjustment, as indicated by his request to explain the change from 3 to 4-1. What remained implicit, and possibly not understood by other students who were unable to independently produce a solution, was the potential purpose S4 had in making this change, that is, what made such a change useful. Making this purpose explicit could be accomplished, for example, had JC expressed the alteration as \((x^2+4x+4) - 1\) and highlighted the trinomial, so that S4’s explanation of why such a step was useful could be ‘seen’. Instead, JC verbally paraphrased S4’s reason for the change and proceeded to writing S4’s verbal answer of the resulting expression on the board (“\((x+2)^2- 1\)”).

On one hand, the first problem JC used for Bridging tapped into knowledge students did seem to
know and were able to use. This first instance, including JC’s explicit request of students to use their available knowledge, indicated what he intended their work on this instance to accomplish, namely, reactivating ‘old’ knowledge in students. In this sense, JC’s rationale for using Bridging seemed similar with LX’s and ST’s (at least face-value). On the other hand, already at the point of introducing and working on the first problem, a deviation from the way Bridging was used by LX and ST can be noticed. In JC’s case, the ‘old’ referred to recently taught knowledge (the three general, symbolic formulas); in the other teachers’ cases the ‘old’ referred to rudimentary knowledge of operation on numbers that students have learned much earlier (at primary school). Compared to LX’s first observed lesson, JC’s use of Bridging also did not situate the meaning of these formulas in contextualised word problems. And JC did not build on the ‘old’ to gradually progress to the goal of the day’s lesson—the general method (Fang Fa) of factoring any trinomial—via analogous operations on rudimentary instances of such a method. That is, in spite of face-value similarity in the purpose for using Bridging, JC’s way of using it indicated a deviation in the meaning of ‘old’ knowledge the teacher appeared to have.

A second, substantial deviation presented by JC’s case is gleaned from the way he attempted to foster transition from ‘old’ to ‘new’. As the cases of LX and ST indicated, their way of using Bridging introduced slight, gradual changes in terms of the mathematical ‘object’ students were operating on while purposely maintaining the essential features of the method (Fang Fa) used to solve the problems. The known (‘old’) method, and its features, seemed to constitute the core of student learning by way of analogous operations on different objects. In contrast, the second Bridging problem that JC used was not just a simple extension of the method used in the first. Rather, it required a non-trivial coordination of two formulas with some anticipation of how reorganising the given trinomial can make use of such coordination. That is, a student in JC’s class would have to not only consider the possibility of using more than one formula but also envision the usefulness of a particular coordination and organise the given trinomial accordingly. This coordination seemed quite challenging, and only some students in JC’s class raised their
hands to indicate they might have a solution. At best, the rest of the class could follow and appreciate the sophistication of S4’s solution. However, the second instance that JC used for Bridging did not seem to fit with a pedagogical component geared toward reactivation of knowledge that is *supposedly ‘old’ for all students.*

Once the second problem was solved, JC’s Bridging proceeded not to similar, simple examples, but to the general form of factoring (or generating) a quadratic trinomial with coefficient of 1 of the quadratic term: \(x^2+(a+b)x+ab=(x+a)(x+b)\) (see Excerpt 4.2.3-b). This showed another deviation in JC’s approach to Bridging.

**Excerpt 4.2.3-b (Teacher JC, 2009-05-14)**

04:19 T: Let’s see if you can expand this algebraic expression \((x+a)(x+b)\) by the multiplication of polynomials?

04:48 Ss: (Working in their notebooks for about 30 seconds.)

05:20 T (to S5): What is your answer?

05:23 S5 (stands up): \((x+a)(x+b) = x^2+bx+ax+ab\)

05:49 T: (Writes S5’s answer on the board while slightly organising the right side) \((x+a)(x+b)=x^2+(a+b)x+ab\), because we know the left and right side of a equal sign can be swapped, so it can be written like \(x^2+(a+b)x+ab=(x+a)(x+b)\). We call the first expression ‘the multiplication of the binomials’, and the second ‘factoring of the trinomial’. Which means this kind of quadratic trinomial, we can factor it like this: \(x^2+(a+b)x+ab=(x+a)(x+b)\). Can we work out the geometrical meaning for this algebraic expression? Can you draw a geometrical graph to express the geometrical meaning for this expression?

06:27 Ss: (Drawing diagrams in their notebooks for about 20 seconds.)

06:47 T: Have you finished? S6, can you tell me how to draw the diagram and I will draw what you said on the board?

07:38 T: (Drawing the representation of what S6 described on the board)

<table>
<thead>
<tr>
<th></th>
<th>x</th>
<th>b</th>
</tr>
</thead>
<tbody>
<tr>
<td>x</td>
<td>x^2</td>
<td>bx</td>
</tr>
<tr>
<td>a</td>
<td>ax</td>
<td>ab</td>
</tr>
</tbody>
</table>

07:39 T: This is the geometrical meaning of the algebraic formula. Can you understand?

07:47 Ss (chorus): Yes

07:48 T: The area of the rectangle equals \((x+a)(x+b) = x^2+(a+b)x+ab\). Does it make sense?

08:09 Ss (chorus): Yes.

08:13 T: We have extended the formula of the perfect square and the formula for the difference of squares into this formula. Can you apply this formula to solve the problem \(x^2+4x+3\)? How about \(S7\)?

08:32 S7: In \(x^2+4x+3 = x^2+(1+3)x+(1\times3)\). I took 1 as a, 3 as b, so it becomes \((x+1)(x+3)\).
Data in Excerpt 4.2.3-b further demonstrate how JC’s use of Bridging was shifting rather quickly to advanced understandings of the intended mathematics and thus being supportive mainly of linking ‘old’ and ‘new’ knowledge for the high-achieving students. Conceptually, the transition from the first two instances (factoring \( x^2+4x+4 \) and then \( x^2+4x+3 \)) to the general, symbolic form of any trinomial (“\( x^2+(a+b)x+ab \)” seemed rather abrupt. There was not even a mention of the purpose in moving to this general form as a means to overcome the limitations of factoring via the three formulas. For example, JC could have (but did not) allude to the shift from the first to the second instance as an example of having to deal with trinomials that do not directly fit any of the formulas. This, he said in the interview, was his underlying mathematical rationale for using the second instance, but it remained implicit in his teaching. Thus, when JC asked if they could “expand this algebraic expression \((x+a)(x+b)\) by the multiplication of polynomials?” he appeared to consider the new task as part of Bridging but for students it would have likely been interpreted as new, unrelated topic.

To JC’s students’ credit, he allowed them about 30 seconds to work on the task individually in their notebooks. For students whose ‘old’ knowledge included mastery of the distributive property of multiplication over addition, solving this general, symbolic task could have served as an experience of Bridging, albeit not necessarily linked in their minds to the previous two instances. What also seemed helpful for students in later understanding of the general method for factoring a trinomial (the day’s new knowledge) was (a) JC’s asking a student to share the creation of a trinomial out of the multiplication of two (binomial) factors (Lines 05:20-05:23) and then (b) emphasising the equivalence of both sides of the equation—hence the possibility to swap the sides to show the move from a trinomial to its factored form (Line 05:49). This exposition, however, led to another transition that seemed quite abrupt for students, especially
the low-achieving ones.

Instead of capitalising on students’ development of a general, symbolic expression for factoring any trinomials, which JC paraphrased, he turned to presenting an entirely new way of thinking about the issue. This new way included a geometrical form of finding the area of a rectangle that comprised of $x+b$ as its length and $x+a$ as its width. Interestingly, for JC the example appeared to represent the concept of factoring, whereas for a student who worked it out this could be understood as an example of multiplication of factors. That is, to understand the geometrical and algebraic forms as two different presentations of a single mathematical structure, which seemed to figure prominently in how JC saw the mathematics, students needed to already have grasped this structure—a grasp that he stated as the goal of that lesson. Said differently, in order to productively participate in the Bridging component of JC’s lesson (two instances of trinomials $\rightarrow$ general algebraic form $\rightarrow$ geometric form) the ‘old’ knowledge he attempted to reactivate in students was essentially more abstract and likely more difficult for low-achieving students than their later work during Variation (see Section 4.3). In this sense, JC seemed to be unaware of the learning paradox as explained by Tzur (2008).

All in all, JC appeared to use Bridging as a means to encourage students’ deep understanding of known formulas by orienting their attention to its justification (e.g., using geometrical representation of area), its twofold meanings (multiplication and factoring as inverses), and under what circumstances factoring could be applied. Compared to LX and ST, JC’s Bridging sought to promote novel (to students) mathematical reasoning, not just applying the ‘old’ method to the ‘new’. However, in order to solve the problems he posed for the students, especially the one for drawing the geometrical representation of the algebraic expression, students needed to already understand much of what JC intended to teach them later. For Bridging to take place from the first two examples into the general form, a student must have been able to operate on the algebraic representation of trinomial factoring and then apply it to a specific example. For
such a student, factoring via the general formula and via the completed square, as it was applied
to \(x^2+4x+3\) would be linked as two ways to solve the same problem, one more general than the
other. But for low-achieving students, such an understanding and ability to solve the problems
seemed above and beyond their ‘old’ knowledge. In this sense, JC’s strong mathematical
knowledge and little pedagogical experience seemed to underlie the use of Bridging in a way that
supported learning by high-achieving students at the expense of low-achieving students.

4.2.3.1 Accounting for JC’s Perspective

In the interview after the observation, JC clarified that his goal for bridging was twofold. First,
he wanted to orient students’ attention to the link between the ‘new’ and the ‘old’, as he expected
them to independently adjust the new problem \((x^2+4x+3)\) into a reformulated version of the ‘old’
instance \((x^2+4x+4)\). Second, he wanted to encourage students’ deep meaning of the algebraic
expressions they used and reasoning behind the formulas. JC’s approach has two important
implications. First, like LX and ST he seemed to believe the ‘new’ knowledge is an extension of
and can be linked to the ‘old’, but this extension is not simple or straightforward. Rather,
intellectual effort and application beyond what has been taught and practiced was required of
students. Second, JC’s purpose in using Bridging appeared to be not only reactivating the ‘old’,
but also to deepen students’ understanding and reasoning of/with the formulas. These two points
are shown in his interview response below (Excerpt 4.2.3-c).

Excerpt 4.2.3-c (Teacher JC, Intrv 2009-05-14)

Usually the first problem serves as Bridging for introducing the new method, but the
problem is a kind of problem that the students can use their old knowledge or old method
to solve it, like here using the method of completing the square, that the students can solve
the new problem by the old method but with some variations. I won’t present a
problem to them that there is no way to solve it, they can still use their old method to
solve it with some variations, and certainly I support them using the old method. Based
on that, I will follow up introducing today’s new knowledge in this lesson; this is the
issue of bridging. (Emphases added).

Data in Excerpt 4.2.3-c show how, from his perspective, the Bridging problems JC had selected
were solvable via the ‘old’ knowledge students supposedly possessed (“I won’t present a problem to
them that there is no way to solve it, they can still use their old method to solve it with some variations”).
Accordingly, these data also show that, to him, the purpose of Bridging was to enable students’ reactivation and application of the ‘old’ as passage to the ‘new’. In terms of JC’s rationale for Bridging, his perspective on learning and knowing seems quite comparable with that of LX and ST. Like them, therefore, he appeared to assume the responsibility for selecting specific problems and engaging students in solving them so that the ‘old’ is available. To learn, the ‘new’ students must first establish the foundation link (the ‘old’) and he used Bridging to foster reactivation of such a foundation. Unlike LX and ST, however, JC seemed to have a different meaning for the ‘old’ (e.g., he taught the formulas for factoring trinomials in the previous 2 lessons) and a different expectation about how students would use the ‘old’ knowledge (adjust it to the unfamiliar problem and thus make it a familiar one). The latter point, and how it was related to JC’s apparent focus on promoting deep understandings, was further elaborated in his response to the interview question about his purpose for introducing the geometrical meaning of the trinomial (see Excerpt 4.2.3-d).

Excerpt 4.2.3-d (Teacher JC, Intrv 2009-05-14)

For example, the formula for a perfect square and the formula for the difference of squares have their geometrical meaning, and naturally for the other trinomials algebraic expression, they also have their geometrical expression. It helps deepen their impression of the understanding of the formula. (Emphases added).

Data in Excerpt 4.2.3-d show two interrelated points. The first is obvious—JC’s pedagogy seemed to focus on developing students’ understanding (in spite of the fact that, for an observer, ‘deepening the impression’ he talked about is hard to believe). The second has to do with his particular way of accomplishing such understandings, that is, by letting students coordinate between geometrical and algebraic representations. It seems that, for JC, mechanical understanding or memorization of the formula was insufficient. Rather, mathematical reasoning was expected as outcome of students’ learning and thus required in his lesson. To this end, JC engaged his students in solving and justifying mathematically sophisticated problems, first on their own and then as he wrote what the students stated on the board. Furthermore, his Bridging seemed to foster learning of the ‘new’, intended knowledge on the part of high-achieving
students—those who could already interpret and solve the non-routine problems. In this sense, JC served like a guide by posing problems for students to reason and reflect about, instead of telling and showing the students step by step how to solve the problem.

4.2.4 Summary of Bridging

Bridging in the practices of LX, ST, and JC seemed to be rooted in a view of learning as a process of linking what students already know (‘old’) with the ‘new’, intended knowledge. Consequently, Bridging as a pedagogical lesson component used prior to moving to teaching the ‘new’ has three distinct features of drawing on the students’ supposedly available knowledge. First, in order to learn the ‘new’ concept students have to reactivate the ‘old’, which occurs via solving a few problems that serve as instances of that knowledge. Thus, the teacher has a responsibility to organise learning situations that deliberately reactivate the needed ‘old’ knowledge. Second, in order to properly reactivate the ‘old’ a teacher needs to continually assess what students’ already know and explicitly build on this knowledge when moving on to the ‘new’. As the problems (instances) selected by each of the three teachers indicated, understanding what students understood played a prominent role in determining the specific Bridging instances the teachers used. Third, to the best understanding of the teacher, Bridging instances must be close enough to both what students know and what the teacher intended for their ‘new’ learning. Such a desired conceptual proximity would be an asset particularly if the teacher attempts to foster transition to the ‘new’ by way of analogy, using the ‘old’ method (Fang Fa) with which students are familiar as a vehicle for operating on and making sense of the ‘new’.

In summary, this section articulated Bridging as a distinct lesson component used by the Chinese teachers who participated in this study, and it consists of the following features:

Bridging starts from what the students already know, as it enables students to accept and
progress from the ‘old’ to the ‘new’. ST called it ‘smooth transfer’ and LX called it ‘intuitive thinking’. Therefore, Bridging problems should be simple enough for the majority of the students in the class, particularly the low-achieving students. The teacher’s role is to guide reactivation of the ‘old’ and help students see the links between the ‘old’ and the ‘new’.

In Bridging, the method (Fang Fa 方法) used for solving a class of problems underlies the mathematics teachers are trying to promote in students’ thinking. The method can be used as a means to link the ‘new’ with the ‘old’.

If the Bridging gap (old-new) is close enough, the teacher is likely to let students recognise the link on their own; otherwise the teacher is likely to go through the entire problem and provide some prompts to help the students see the links.

The teachers’ rationale for using Bridging is to reactivate the ‘old’, because without it being reactivated and available to the students—learning (linking) of the ‘new’ is not likely to happen. The ‘old’ knowledge is the anchor onto which the ‘new’ knowledge will be linked, and the teacher assumes responsibility for properly reactivating it during every lesson prior to proceeding to teaching the ‘new’.

The next section elaborates on how, once the ‘old’ was reactivated via Bridging, the participating teachers attempted to and thought about teaching the ‘new’ via Variation of problems and/or solutions.
4.3 Teaching with Variation

This section examines participating teachers’ use of the pedagogical lesson component of teaching with Variation. In itself, teaching with Variation has been studied previously (Gu, Huang, & Marton, 2006). As these researchers showed, it is a prevalent practice used by Chinese teachers to promote their students’ learning of the intended mathematics. In this dissertation study, however, teaching with Variation is examined further by situating it within the context of the four-component lesson structure. In particular, this section attempts to highlight the effectiveness of teaching with Variation after teachers used Bridging to reactivate ‘old’ knowledge. That is, in this study the significance of teaching with Variation is considered through both the nature and sequence of mathematical questions and activities in which a Chinese teacher may engage her or his students.

This section discusses two main types of teaching with Variation: (a) solving the same problem in different ways and (b) solving different problems with the same underlying methods. As the data in this section show, in both types teachers use purposely-selected and sequenced problems, which proceed from easy and simple to gradually more difficult and complicated ones. Such sequencing is designed by the teacher to accomplish in each lesson a specified goal for student learning (the day’s mathematics) at three distinct levels: Basic Point (‘Zhi Shi Dian’, 知识点), Focal Point (‘Zhong Dian’, 重点), and Demanding/Difficult Point (‘Nan Dian’, 难点). Through teaching with Variation at all three levels, participating teachers seemed to attempt to promote students’ understanding and use of an invariant method (Fang Fa, 方法) for solving a category of structurally-similar problems. Using teaching with Variation after Bridging seemed to support students’ learning in terms of knowing how to independently start the solution process for novel (Variation) problems by using the method reactivated during Bridging. Moving along the sequence of gradually more difficult problems then seemed to further students’ understanding of when and why to use the method.
To depict the role that Variation of problems and/or solutions may play in students’ learning the
day’s novel mathematics, as well as the participating Chinese teachers’ perspective of this
component, data segments were selected from the following teachers: SZ (nine years of teaching
experience), LX (30), WK (one year), and ST (12). The ways in which these teachers used
Variation help to convey both the common aspects of this central lesson component (for teaching
the ‘new’) and individual deviations from it. The section begins with SZ, because his work
demonstrates key features of the type of Variation in which several solutions to a single problem
are considered. Then, data from SZ are used to depict how the other type of Variation—solving
several problems by the same method—is used.

To further highlight the significance of using Variation after Bridging (and Reviewing) while
also revealing individual teacher deviations, data are then presented from LX, WK, and ST.
Their cases further substantiate the common, general pattern: (a) provide a few, related problems
that can accentuate the new idea (method), (b) engage students in solving some simple problems,
and (c) progress to gradually more demanding examples. When students solve the various
problems, the teacher moves about the class and focuses on difficulties or mistakes they may
have, so these problems are later solved on the board with the whole class. The following
discussion uses the case of SZ to elaborate the variation of solutions to a single problem.

4.3.1 One Problem - Multiple Solutions
This sub-section presents and analyses data from the work of a teacher (SZ) who, when
considering teaching with Variation, seemed to fit the virtuoso model (Paine, 1990). His case
was chosen particularly because of what appeared as an expert use of Variation of solutions to a
single problem. In fact, one of the two lessons observed in his class was entirely devoted to
working on a single problem with variations. The following sub-section discusses his case.

4.3.1.1 A Variation Expert (SZ)
SZ used teaching with Variation in both lessons observed in his class. Apparently to promote
students’ learning of the novel, mathematical idea (lesson’s goal), he encouraged them to come up with multiple solutions to a single problem. This was done due to his belief that for most mathematics problems there is more than one solution (see data in Excerpt 4.3.1.1-a from an interview with SZ) and his intention to inspire students’ interests and divergent thinking. He appeared to assume that there will always be unknown aspects of students’ future mathematics learning, and they have to learn how to dig deep and explore these unknown aspects on their own. To figure out the unknown, he seemed to expect students to find a link between the new problem and their available knowledge—and transform the new problem into an old problem (make the unfamiliar familiar).

During the second interview with SZ, he stated that the goal for the second observed lesson was for students to learn methods of factoring any trinomial (ax²+bx+c) when the coefficient of the quadratic term is larger than 1. At the end of the previous, first observed lesson, in which students factored trinomials with a=1, he assigned a homework problem of factoring 2x²-7x+3, which was novel to the students. That is, prior to any introduction of the new mathematics (a>1), SZ provided an opportunity for students to think and reflect on activities of solving a problem that, for them, was novel and not yet routine. Upon assigning this homework problem, he explicitly told students his expectation: “Try applying your old knowledge to the new situation as a way to solve the novel problem.” Interestingly, and contrary to his plan, the entire lesson turned into the sharing and exploration of seven different solutions to that single problem that students presented on the board, including articulation of their reasoning. Due to the lengthy exchanges, these seven solutions are separated into two groups, based on the nature of teaching intervention, and are presented as summary of each student’s solution. Box 4.3.1.1-a presents the first four students’ homework solutions; Box 4.3.1.1-b summarises the remaining three solutions after SZ engaged students in figuring out a way of factoring that builds on their ‘old’ knowledge of particular formulas (e.g., perfect squares).
Box 4.3.1.1-a: Four students’ solutions to SZ’s homework problem (2009-05-15)

T (01:19): Yesterday, we introduced factoring of the trinomial where the coefficient of the quadratic term is 1. I gave the problem $2x^2-7x+3$ for you to think as homework, where the coefficient of the quadratic term is 2, not 1 anymore. How do you solve this problem? Can you share your solutions?

S1 (01:58 – 05:47): Because the first item is $2x^2$, the second item is $7x$, so these two can’t be added together, so $x^2$ can be separated into two [factoring] ‘x’s. (He goes to the board and writes the trinomial, and factors below it, as he speaks): The $x^2$ is becoming $x\times x$; and [the coefficient] 2 has two factoring options, it can be regarded as $1\times 2$, or $( -1)\times(-2)$, and then $+3$ can be factored as $1\times 3$, or $( -1)\times(-3)$. (He writes the trinomial, $2x^2-7x+3$, and two rows of factors below it, 1 2 1 3 and -1 -2 -1 -3, and explains): Then I disregard the x, because it is 7x. (He points to the x in $2x^2$ and says) and let’s do some matching. These two groups (points to the row of negative factors) can make -7. (Here, he first factored them into $( -x+1)(-2x-3)$, then realized his own mistake, and corrected it to $(x-3)(2x-1)$).

T (05:47 – 06:57): (Paraphrases how S1 factored 2 in ‘2$x^2$’ and the constant ‘+3’ simultaneously, asked and received students’ (chorus) response that there are four ways to organise these factors, said that S1 checked all four of them and finally found the correct grouping that gives ‘-7x’. At this point, he asks if students have different solutions and invites S2 to share.

S2 (06:57 – 09:24): (As S2 speaks he writes the equations on the board, below the trinomial, $2x^2-7x+3$): I can look at 2 as $a\times b=2$, so we can separate this item into $ax\times bx=2x^2$, and take [the constant] 3 as $c\times d=3$, take -7x as $acx+bhx=7x$, here we can divide both sides by x and get $a+c=7$. Because they are all integers, then we can get $(x-3)(2x-1)$ or $(x+3)(-2x+1)$. Because these two are equivalent, we can select any one of them as the solution.

T (09:24 – 13:28): Paraphrases another student who said that S3 used the method of undetermined coefficients, then invites the fourth student to share his solution on the board.

S3 (09:24 – 13:28): (Goes to the board and writes, (x-3)(2x-1), then says): I got the idea from the solution [the trinomial]. I think of this solution in a reverse order. I was thinking whether there are some generic formulas to solve this kind of problems. Because there is a coefficient in front of both ‘x’s, I substitute the four constants a, b, c, d in it and it becomes $(ax+b)(cx+d)=acx^2+(ad+bc)x+bd$. It is easy to solve this problem if we now substitute the problem $2x^2-7x+3$ into $acx^2+(ad+bc)x+bd$. (Continues writing on the board): a=1, b=-3, c=2, d=-1, and underneath it also $(x-3)(2x-1)$. He then draws arrows from the general formula $acx^2+(ad+bc)x+bd$ to $1\times -1$ and $2\times -3$, so that a points to 1, d points to -1, c points to 2, and b points to -3, and underneath it writes $-7=(-1)+(-6)$ [seemingly the ‘solution’ he referred to earlier].

S4 (14:03 – 16:45): (Goes to the board and writes while saying): Suppose $(ax+b)(cx+d)=2x^2-7x+3$; when we expand [the left side], we get $acx^2+(ad+bc)x+bd$. Because we suppose these two are equivalent, ‘a’ and ‘c’ can be considered as coefficients of $x^2$, it means we separate 2 into ‘a’ times ‘c’, $bd=3$, and $-7=ad+bc$. We get $ad+bc$, which is the coefficient of x. So this is my solution. (SZ asks the student how did he come up with this method and S4 continues): It is the same as the previous one with the quadratic coefficient a=1. [There, $x^2+px+q$] we separated q into ab, then in fact the coefficient [of $x^2$] is 1, which means we separated the coefficient 1 into two 1s [$1\times 1$], and p=a+b as in $(1\times a)+(1\times b)$, so I get the solution.

T (15:59 – 17:00)
Let’s have a look together. S4’s solution has some similarities with S3 in their way of thinking, and has some creative thinking in terms of the written format. His written method is in a vertical way to write the two constants factored from the coefficient of the quadratic term at the first line, and he wrote the second line as the factors of the constant in the quadratic trinomial, and then put the sign of cross. Let’s give this method a name; we call it the cross multiplication method. (The entire class echoes SZ in chorus: “The cross-multiplication method.”) This is a simple format and method to solve this problem. This is a very good method. The previous four students gave us four different methods. Do you have any other ways of solving this problem?
portion of the lesson, three important aspects should be taken into consideration. The first two were stated by SZ in the follow-up interview: (a) he did not know how these four students solved the problem prior to the lesson and (b) he planned for the sharing of solutions to take only a few minutes. The third aspect concerns SZ’s location in the room and interactions with students during the first 17:00 minutes—he stood at the side of the class and let the students lead the work, while only interjecting the few comments and final words. In return, he and the entire class seemed to gain insight into a variety of ways by which the same trinomial could be factored.

For sharing the first way, SZ called on a student (S1) who seemed unsure of the way to do the factoring. In particular, S1 did not seem to understand the need to determine not only the correct-but-separate factoring of the coefficient of $x^2$ and the constant, but also what combination of the pairs of factors results in the given coefficient of $x$ (e.g., -7). Considering the sophisticated levels of the other three solutions, having S1’s solution first seemed like a good choice for helping other students’ follow the use of the underlying method. S1’s apparent hesitation after writing an incorrect answer $((-x+1)(-2x-3))$ and while trying out the other three combinations enabled students to follow the target of and process for determining the correct pairing. In turn (Line 05:47 – 06:57), SZ could capitalise on S1’s mistake and exhaustive trials to orient students’ attention to the multiple combinations (here, four) and the need to determine which of them meets all three numbers in a given trinomial.

The second (S2’s) way for factoring the homework trinomial extended the focus on specific numbers to an algebraic representation of each coefficient. S2 further explained the coordination between coefficients and constants in the binomials and the resulting coefficients (or constant) in the given trinomial, including the cancellation of $x$ via division. Furthermore, S2 noticed and explicated the fact that in the binomials the ‘-’ sign could be appended to the constants or to the coefficients, hence accentuating the multiplicity of solutions. Without any further intervention from SZ, the third way was then presented by S3.
S3 began his sharing by emphasising his way of thinking about the solution, that is, starting from the trinomial as a result of multiplying two binomials and trying to figure out what were those originating binomials. This inference builds on S3’s statements (“I got the idea from the solution”) and the following explanation (“Because there is a coefficient in front of both ‘x’s’”) that referred to the coefficients (‘a’ and ‘c’) in the left side of the equation written on the board (“(ax+b)(cx+d)= acx^2 + (ad+bc)x + bd”). This way of thinking led S3 to present a substitution process that was a generalisation of S2’s method. To clarify his thinking and assist the class, S3 drew arrows that explicated this link—each number in the pairs of factors was pointed to in terms of the binomial coefficient/constant from which it came, as well as the factoring pairs (“1 × -1 and 2 × -3”) and the resulting sum (“-7 = (-1) + (-6)”). At this point, SZ intervened again to emphasise the name for S3’s method—undetermined coefficients (proposed by another student). Because the previous two solutions were not called this, SZ’s short interjection could support students’ interpretation of S3’s method as being different than the previous two solutions (which themselves were markedly different in generality). It also led quite smoothly to the fourth solution.

The way S4 solved the same problem (factoring 2x^2-7x+3) included two important contributions that differed from the already general, abstract solutions shared by S2 and S3. First, it seemed to indicate a consideration of all possible ways by which two binomials could be expressed (“(ax+b)(cx+d)”), which were then expanded into a general form for any trinomial of which the given one would be an instance and hence equivalent if coefficients match (indicated by S4’s explicit expression of three equations for those matches). Second, S4 also explained how this solution met the teacher’s (SZ) challenge to use their ‘old’ knowledge. To this end, the student made explicit how the canonical form of a trinomial with coefficient of ‘1’ for the quadratic term (i.e., x^2+px+q), which the class was taught in the previous days, was generalised. SZ then concluded (Line 15:59-17:00) this first portion of teaching with Variation of solutions to the same problem by (a) explicitly comparing the two latter solutions (S3 and S4), (b) differentiating the two by giving a different label to S4’s (“the cross-multiplication method”) than he gave to S3’s
(“undetermined coefficients”), (c) explicitly emphasising S4’s method, and (d) explicitly pointing out the multiplicity of solutions given thus far and asking for more ways to solve the same problem.

It seems that the four students who solved the novel homework problem, particularly S2, S3, and S4, were among the high-achieving students in SZ’s class, as they have already grasped and could apply the general method he intended to teach. That is, they have accomplished the teacher’s goal for students’ learning prior to the lesson. In particular, S3 could think about and explain his solution as starting from the end; S4 could provide a generalisation of the special case of trinomials with quadratic coefficient of ‘1’ and apply it to the given instance from homework. They could independently understand the need to find two binomials that, when multiplied, would produce the given trinomial. To this end, they also understood the need to coordinate finding all possible factors of both the quadratic coefficient and the constant number with additive combinations of those factors that yield the coefficient of x (here, -7). Said differently, all four students (S1 included) seemed to understand that in factoring $x^2+px+q$, the key was $q=a\times b$ and $a+b=p$, and they were able to apply this idea to the quadratic coefficient ($a=2$).

Asking those four students to share and explain their solutions seemed to serve three purposes in SZ’s teaching with Variation of solutions to the same problem. First, he could check students’ understanding of yesterday’s lesson (factoring trinomials with quadratic coefficient $a=1$). Second, he could anticipate his students’ understanding of the novel generalisation to any trinomial ($a=1 \text{ and } a>1$). Indeed, at least 4 students already knew this (perhaps via reading ahead in their book and/or tutoring) and could thus lead their peers’ learning. Third and most importantly, by letting high-achieving students articulate multiple ways in which they solved the unfamiliar problem, he appeared to create opportunities for cross-situation comparisons that might help to consolidate the intended generalised method.
SZ’s teaching with Variation could also enable reactivating the day’s novel generalisation (a>1) in students who have grasped the previous day’s lesson (for a=1) even if they could not solve the homework problem on their own. Those middle-level students in his class could follow and interpret S1’s solution, because it specified both (a) the numbers and signs for each coefficient’s factoring (1×2 and 1×3, or -1×-2 and -1×-3) and (b) the additive combination required to obtain the specific coefficient of ‘x’ (-7). That is, S1’s solution process explicated for students each and every step in the intended method for a>1. Moreover, through trial-and-error while at the board S1 spent about 3 minutes organising his ideas and explicitly articulating his thought processes, including the resultant, simultaneous grouping of the coefficient of the quadratic term (+2) and the constant (+3). He shared how he tried each of the four possibilities of grouping and explained why each of them did or did not work. Thus, SZ’s choice of S1 as the student who would initiate the solution process for the entire class could reactivate low- and middle-achieving students’ thinking about a trinomial (only a constant ‘q’, or both the constant and the quadratic coefficient) as the outcome of multiplying two binomials that also produce the given coefficient of ‘x’. In turn, these students might have been able to follow the more general solutions of S2, S3, and S4. Consequently, SZ’s use of teaching with Variation—multiple student solutions to a non-routine problem—seemed to support first-rate mathematics understandings at least in his high- and middle-achieving students.

In addition, asking S1 to provide the first of several solutions could support SZ’s low-achieving students’ learning. S1 experienced a difficulty in his attempt to express how he solved the problem (see Pirie & Kieren, 1994 about the difference between understanding as acting and as expressing). When S1 began sharing, he appeared to have only a vague (implicit) view of the link between the multi-step mental process required to identify all possible factors of two different numbers (+2 and +3), consider their grouping, and determine the correct one so it also matches the coefficient of ‘x’. Working at the board, S1 was oriented by SZ to reflect on his process of factoring those numbers, focus on finding the binomials, and consolidate the method.
through generating an explanation to the rest of the class. Thus, S1’s sharing, combined with
SZ’s summary of the solution, could trigger similar processes for any student in the class,
particularly those who were yet to understand how the process of finding factors (to the quadratic
coefficient and constant) and then properly combining them can produce two suitable binomials.

SZ’s few, short restatements of students’ contributions indicated his role as a facilitator of
students’ learning. He appeared to capitalise on their different solutions as a good opportunity for
teaching the intended mathematics to the entire class. The goal would be for each of them to
understand and be able to factor any trinomial. To this end, sharing and comparing different
solutions to a simple case ($2x^2-7x+3$), which consisted of a quadratic coefficient and a constant
that are small primes and thus easily factored, could foster student understandings not only of
this particular instance but also of the intended generalisation. In this sense, SZ used teaching
with Variation of solutions to a single problem as a means to let students lead the learning
process instead of telling and showing students what he intended for them to learn. At the same
time, he could assess how the high-achieving students in his class understood the intended
mathematics.

After the class had followed the four students’ solutions, SZ continued to challenge other
students for more, different solutions (“The previous four students gave us four different methods. Do you
have any other ways of solving this problem”). As this was a homework problem, the expectation would
be that each student at least tried to solve it outside class. By asking for more solutions SZ could
therefore figure out what other students understood, and (if needed) enable each student to solve
and understand the extension to the previous day’s learning. The following discussion (including
data in Box 4.3.1.1-b) elaborates this point.
Box 4.1.1.3-b: SZ reactivates a known method to elicit three additional solutions (2009-05-15)

T: (17:34 – 20:34, after about 35 seconds of wait-time since SZ challenged students to find additional, different ways, and received no response, he gives a prompt for a trinomial instance from the previous day and asks for the method used to factor it): Do you have any other methods? Let’s recall what we have learned yesterday. We solved the problem of $x^2+4x+3$; what method have we adopted for solving this problem? (Students responded in chorus, “The method of completing the squares”, and SZ continued): Can we use that method to solve the problem $2x^2-7x+3$? How about you try to solve this problem by using this method? (For about a minute, he moved about the students as they worked on the homework problem in their notebooks; about 30 seconds into their work he prompted again): Pay attention to the fact that when you consider using the method of completing the square, you have to make sure the change you make satisfies the formula for the perfect square. There is no perfect square in this problem, so [the question you have to ask yourself is] how can you operate on it and change it into a problem for which you can adopt the method of completing the square? (About a minute after students had begun working, 19:08, SZ prompts further): We have to continuously explore the links in mathematics. The essence of thinking is in searching the links between pieces of knowledge. There are lots of regularities and rules in these links. (SZ gives students about 2.5 more minutes to work on the problem while moving about and glancing at their solutions, then 22:37 he calls their attention): Ok, let’s share. [Walking around] I found this is still a challenging problem for many students. This is the first time we use our old knowledge or available knowledge to explore this problem, to analyse this problem. We are not so experienced. Now let’s share your methods. S5 adapted this method; let’s have a look at his method.

S5 (23:04 – 26:18): (The student hands his notebook to SZ; the teacher writes S5’s solution on the board, $2x^2-7x+3=2x^2-4x+3x+4-1= x^2-x^2-4x+3x+4-1$, and then says to the class): This kind of thinking is a method in mathematics we commonly call transfer. (Another student yells this doesn’t work, which SZ paraphrases and then suggest looking at the expression in parts): Let’s have a look at $x^2-4x+4$; what is this format? (The class responds as a chorus, “Formula for perfect squares,” and SZ continues): So this is a formula for a perfect square, and this is done (SZ writes on the board, the equation $2x^2-7x+3 = x^2-4x+4+ x^2-3x-1$, and underneath it the equivalent expression with the first trinomial factored via the formula, $(x-2)^2-12+x(x-3)$, then continues): S5 solved it in this way (SZ writes on the board below the last expression: $(x-2)^2-12+x(x-3)$, to which the entire class responds with “Wow!” Without commenting on the final expression not yet being a multiplication of two binomials, SZ says that another student, S6, has a different solution and invites to share).

S6 (26:20 – 29:11): There is another, simpler solution. (S6 goes to the board and writes each of the following expressions underneath one another (see left side of Picture 4.1, while saying out loud): $2x^2-7x+3; =x^2-7x+12+(x-9); =(x-4)(x-3)+(x+3)(x-3); =(2x-1)(x-3)$. (The class responds as a chorus, “Good,” and SZ follows with “Shall we applaud him?” to which all students respond with excitement.)

S7 (29:11 – 31:13): (The student stands up and says): If we follow S6’s solution, he makes the number larger; I am solving by making the number smaller and simpler. (SZ asks, “How do you make the number smaller?” and S7 continues to explain while also writing on the board the trinomial and each of the following expressions, one below the other): $=x^2-4x+3+(x-3); =(x-1)(x-3)+x(x-3); =(2x-1)(x-3)$. (NOTE: S7 essentially completed the solution of S5.)

T (31:13 – 32:00, summarises the entire portion of seven different solutions): All of these methods are very good. When we meet a new problem, we should always transfer the new problem into an old problem, and find the link between the new and the old, find the linkage point. If you cannot find it immediately, try again and again. Like [when] we are digging for water, if you dig a little bit deeper, you will find the water is just there. Now, let me provide another method. (Here, SZ demonstrated on the board how to solve the trinomial by turning it into one with quadratic coefficient of 1: $2x^2-7x+3 =2(x^2-7/2x+3/2)=2(x-3)(x-1/2)$.)
Data in Box 4.1.1.3-b show how SZ continued using teaching with Variation of solutions to foster students’ productive participation and learning. Initially, when he challenged students to share more solutions and gave them about 30 seconds to think about different ways, no one raised their hands to indicate having such a solution. Getting a rough sense that most students did not yet consider the method used in the previous lesson, he prompted them further. The prompt included presenting a simple instance of a trinomial with quadratic coefficient of 1 ($x^2+4x+3$), asking the class what method they learned in the previous lesson, and engaging them in individual work on factoring the homework trinomial via this method. During the following 3.5 minutes, SZ moved about the class to figure out what students were doing, while providing one more prompt—orienting their attention to the need to change the given trinomial so it does have a perfect square formula and encouraging their exploration of mathematical links that can help solving a novel problem. Having seen students’ work during that time, he then requested S5’s notebook and began sharing more solutions.

As SZ said in the interview, choosing to commence the whole class discussion with S5’s unfinished solution was intentional. On one hand, this solution provided specific steps of reorganising the homework trinomial into parts that included a perfect square and could thus be factored via the formula. Furthermore, these steps included creating a “difference of squares” with ‘$1^2$', which seemed to match SZ’s quest for students’ creative, thoughtful use of what they knew. On the other hand, S5’s solution, which seemed to impress students (“Wow”), stopped...
short of completing the factoring of $2x^2-7x+3$ into two binomials and enabled SZ to foster more exploration via sharing different solutions. To this end, he invited S6 to share a different way.

S6’s solution did not draw on the ‘old’ knowledge of a perfect square formula. Rather, the given trinomial was reorganised into two expressions, each consisting of quadratic term with coefficient of 1: a simple trinomial $(x^2-7x+12)$ and a difference of squares $(x^2-9)$ that contained a common binomial factor $(x-3)$. This reorganisation led to the next step of factoring each expression ($(x-4)(x-3)+(x+3)(x-3)$), and finally to obtaining the target factoring ($(2x-1)(x-3)$). From SZ’s reaction—facial expression and inviting the entire class to applaud, it appeared that S6’s solution truly impressed the teacher and matched the quest for students’ independent exploration of links to what they already knew. To continue the teaching with Variation, SZ then let S7 present yet another different solution.

S7 began by stating the essence of the difference between his and S6’s solution (“If we follow S6’s solution, he makes the number larger; I am solving by making the number smaller and simpler”). This statement indicated the norm that SZ appeared to encourage in his class, namely solving the same problem in different ways while explicating the difference. The latter claim was inferred from SZ’s request that S7 would explain the meaning of the noted difference (“How do you make the number smaller”), to which S7 responded by saying and writing on the board a solution in which the constant was not increased. In essence, S7 reorganised the given trinomial $(2x^2-7x+3)$ in the way S5 did it, but followed the factoring process to its conclusion. Interestingly, SZ did not point out this similarity, which could have led to further articulation of what makes a solution different from or similar to another solution. Instead, after about 32 minutes of unplanned exploration of 7 different solutions to the same problem, and making sure students had time to attempt this case-problem on her or his own, SZ summarised the key aspects of their learning: transfer and link what they know to a novel situation and persist until they find such a link—persistence he encouraged via the Chinese metaphor about digging deep enough to find the water.
SZ’s teaching with variation of solutions, via the first four methods found by students prior to the lesson and additional three solutions found during the lesson (with the teacher’s prompts), indicated a fourfold pedagogical approach. First, SZ let his students actively and independently explore before any demonstration or discussion. This could promote their attempts to solve problems and appreciation for the challenges and rewarding insights involved in their intellectual work. Second, he capitalised on students’ solutions as a pedagogical strategy for teaching their peers. This could promote learning of the intended mathematics by the students who shared their solution (e.g., S1’s learning as a result of having to express his thinking) as well as the other students in the class (e.g., by following the trial-and-error process experienced by those who shared). Third, SZ appeared to encourage students’ reasoning and divergent thinking. This could convey a message not only about their individual capacity to attempt novel problems, but also a view of mathematics as a complex, open-to-exploration body of knowledge. Fourth, SZ’s approach seemed rooted in a similar perspective to those reported in the previous sections on Reviewing (4.1) and Bridging (4.2), namely, seeing learning as a process of exploring and linking between what is novel to students and what they already know (the ‘old’). Combined, these four aspects seemed to serve a critical role in students’ work and possible progress—creating repeated changes in ways of solving a single problem as context for orienting students’ attention to the common core of the mathematics at issue (here, the structure of a generalised trinomial as an outcome of multiplying two binomials). The data in Box 4.3.1.1-a indicated how SZ had done this via students’ diverging solutions of factoring the quadratic coefficient and the constant, and then by additively combining the factors to fit with the coefficient of ‘x’.

In regards to the first and third aspects of SZ’s fourfold approach, he first encouraged students to independently apply the known method (completing squares) for solving the novel problem (quadratic coefficient a>1). He used this occasion to walk around the classroom and assess individual students’ work. This pedagogical strategy created an opportunity for students’ active, divergent/creative thinking, which was demonstrated in the three solutions that students then
presented and explained. As highlighted in the Reviewing (4.1) and Bridging (4.2) sections, letting students solve on their own a problem that has multiple possible solution methods could simultaneously serve their potential learning and the teacher’s assessment of their evolving understandings.

In regards to the second and third aspects of SZ’s fourfold approach, giving every student an opportunity to solve independently could assist creating the intellectual basis that middle- and low-achieving students would need to make sense of other students’ solutions. Based on SZ’s ongoing assessment during the independent work, he took S5’s notebook and copied the solution to the board, because it seemed to match the teacher’s goal—to present a solution that was justified but incomplete. This seemed like an important pedagogical move, because SZ could orient students’ attention to the ‘unfinished’ process. As a student (or observer) of SZ’s class would expect, when S5’s solution \((2x^2-7x+3=2x^2-4x-3x+4-1=x^2+x^2-4x-3x+4-1)\) appeared on the board, a more knowledgeable peer, of which SZ was aware based on his ongoing assessment, noted: “No, it doesn’t work”. This comment prompted further work, which eventually yielded the completion of this solution by S7. Regardless of S5’s solution being partial, SZ appeared to provide support for the student’s thinking and contribution by linking it to what is commonly called transfer in mathematics learning.

During the entire lesson (essentially—Variation of solutions to a single problem), SZ did not directly tell or show students the intended mathematics. Rather, he let students lead the teaching of the lesson, while providing prompts and summaries. Through the practice of juxtaposing and comparing their solutions he seemed to focus on fostering students’ divergent, critical thinking. That is, he created a learning environment where students had the freedom to express different opinions, and a sense of striving for mathematical sophistication. As pointed out earlier, students who shared their solutions could gain from articulating their solutions on the board. For other students, following peers’ solutions on the board after attempting a solution on their own, as well
as SZ’s summaries, seemed to provide situations conducive to their learning.

In regards to the fourth aspect of SZ’s fourfold approach, his explicit summary of the whole lesson after the students finished their exploration provided a window into SZ’s perspective: “When we meet a new problem, we should always transfer the new problem into an old problem, and find the link between the new and the old, find the linkage point. If you can’t find out immediately, try again and again. Like [when] we are digging for water, if you dig a little bit deeper, you will find the water is just there.” Using a vivid, culturally embedded metaphor, he seemed to consider learning as a process of linking new ideas to what students already know. His words further suggested that the ‘new’ is an extension of the ‘old’, and by transferring (reorganising) the ‘new’ to fit within the ‘old’ students could at least know how to start solving the novel problem. The following discussion elaborates this point.

**4.3.1.2 Accounting for SZs’ Perspective**

Teaching with Variation seemed to be the lesson component in which teachers who participated in this study attempted to teach new mathematics. Therefore, the interview with SZ after his lesson in which students shared multiple solutions to a single problem focused on his perspective about learning and knowing. At the start of the interview, SZ was asked to discuss the difference between his original plan for this lesson and the way it actually unfolded. He said that the plan was to focus on the method of cross-multiplication. However, he realised that students had solved the homework problem in interesting, different ways, and thus decided to let them lead the flow of the lesson. He particularly mentioned the seven different solutions they shared, which included the method of cross-multiplication he intended for them. SZ said he thought that letting students lead the lesson was more effective than what he had prepared, “because all these [solutions] came up by students taking the initiative.” He further explained: “I did not fully accomplish my teaching goal, but I feel I gained a lot because the students’ divergent thinking exceeded my expectations, and were wider than those methods I could think of.” This indicated SZ’s flexibility with his lesson plan, and his focus on letting students take the initiative for independent, divergent thinking. He appeared to prefer
changing the lesson plan and let students have the opportunity to explore and share than sticking to the plan at the expense of their ideas.

To further inquire into the perspective that underlies SZ’s use of students’ multiple solutions as a means to promote their learning, he was asked to explain this teaching approach. The three data segments from the interview (see Excerpt 4.3.1.1-a below) give further support to an evolving, common theme in the participating teachers’ view—learning is a process by which students link between the ‘old’ and the ‘new’ and the corresponding responsibility of the teacher to figure out what their ‘old’ knowledge is. In addition, these data segments help to substantiate SZ’s view of the active role that students must take in order for the intended learning to take place, and the corresponding responsibility of the teacher to set the ‘new’ at a proper, challenging distance from the ‘old’.

Excerpt 4.3.1.1-a (Teacher SZ, Intrv 2009-05-15)

For this problem [factoring 2x^2-7x+3], many people might have just stopped here, but there are still many unknown things with regularities for us. Sometimes what we should do is to dig deep into and discover those unknowns fully. During the process of [intellectual] digging, some regularity might appear … there are lots of variations, or new ways of dealing with the problem. It will help students when they will go on to learning higher mathematics, because the more mathematics they learn the higher requirement will be placed on their logical thinking. The creativity in logical thinking, as we mentioned creative thinking, means that when you face unknown problems or situations you will know how to find the ‘old’ tools [related, familiar knowledge] to solve the problem... to utilise what you have already come to know. I feel that this [future requirement] places a big responsibility on teachers’ shoulders. (Emphases added)

I will select some problems or exercises based on the actual level of the students in the class, based on how much interest I can arouse in the students, and I am very clear of this now. I have to put forward some problems that can spark their interests, not just present them the problem as it is, and there will be some flexible variations of the solutions … in the field of education in China, there is a phrase that ‘you have to set up a peach that they can see on the tree, which can [only] be reached by some jumping, but not too difficult to reach … we should set this kind of [mathematical] ‘peach’ all the time.

Mathematics is not just out there, it depends on our exploration, and there is a variety of ways to solve a problem, or to look at a problem. We cannot just solve the problem within a certain small domain. We should widely open the students’ mind. In fact, in mathematics teaching, if teachers can do this and lead their class to open up, they can widen their students’ thinking … if I [directly] teach them all the time, they might feel they are passively accepting knowledge, instead of digging knowledge by themselves, and they may think it is meaningless. The new problem is challenging for the students, and sometimes kids like challenges. If I teach them [directly], probably I will only be
teaching one or two ways, but students solve the problem from different perspectives and it widens their minds … the students’ potentials/capacities are unlimited and they have different prior knowledge and understandings. Only if teachers are good at inspiring them or giving them proper guidance, they can learn better. (Emphases added)

The data segments in Excerpt 4.3.1.1-a indicate four key aspects of SZ’s perspective. First, he appears to believe that mathematics is not out there to be directly captured by students. Rather, coming to know mathematics requires students’ exploration via solving problems that challenge their current thinking. Second, he seems to think that there are many ways to solve or approach a problem. In particular, he appears to be aware of the fact that students may come up with solutions he did not consider, which indicate a twofold recognition—students’ thinking may differ from and can thus extend the teacher’s knowledge. Accordingly, it seems that SZ assumes the responsibility to widen students’ minds, inspire their divergent thinking, and give them proper (flexible) guidance. Third, far from a traditional, authoritative pedagogy, SZ appears to view the teacher’s key responsibility as creating learning opportunities for students by letting them take the initiative for ‘digging out’ the intended knowledge. The data in the excerpt above indicate that he rejects a view of students passively accepting the teacher’s transmission of knowledge. Fourth, he emphasised that the teacher should design challenging problems for students as a principal method for promoting their learning.

The data in Excerpt 4.3.1.1-a also show that SZ’s view of learning focuses on the need to reactivate students’ ‘old’ knowledge, which can consist of different understandings. This is inferred from the linkage he made between what students know and the guidance a teacher needs to provide (e.g., the peach metaphor). For him, it appears that the Chinese way of teaching includes attention to selection of problems that suit students’ current knowledge (and levels). It also requires proper sequencing of student solutions so that ‘reaching a peach’ (i.e., learning) is supported through intermediate, gradual extensions. Accordingly, he appears to assume the responsibility to select problems and solutions so they are reasonably distanced from his students’ ‘old’ knowledge, which requires of him to continually infer what students know (e.g., closely
monitoring their independent work).

A related issue in SZ’s statement about the selection of a problem is that it should spark the students’ interests (this is similar to findings about elementary teachers in China; see Correa et al., 2008). Here, the ‘proper distance’ (neither too difficult nor too easy) seems to serve the role of a motivator in his view of learning. The metaphor about setting the problem like a peach that the students can see and reach by some ‘jumping’ is telling. First, to set a problem in such a way the teacher needs to know where students are mathematically. Second, the teacher also needs to select problems that are carefully tailored to different students’ conceptual level in his class. It is this combination between learning via problem solving and sparking students’ motivation that seemed to underlie his teaching with Variation. In this way, he could provide instruction so that low-achieving students would not feel discouraged, while high-achieving students would not feel bored or doing meaningless work.

Another aspect of the role that SZ appeared to assign to teaching with Variation is preparing the students’ minds for mathematical unknowns they may face in the future. Thus, he seemed to emphasise the teacher’s responsibility for fostering in students a mindset of creative thinking, persistence, and intentional use of and alteration to their ‘old’ knowledge. As he stated, the tools for coping with future unknowns are found in the person’s available (‘old’) knowledge. Students can and should learn via creatively linking ‘new’, unknown mathematics to their available knowledge. Therefore, his responsibility is to ensure such learning via Variation of solutions and problems, so when his students face future unknowns they will know (a) to search for and (b) how to find the proper available knowledge for solving a problem. A good illustration of this responsibility was indicated in his prompts (Box 4.3.1.1-b) for students to consider the example of $x^2+4x+3$ and to use the method (completing the square) that they have encountered in the previous lesson. As he explained in the interview, this specific prompt was presented to give students a chance to start from their current knowledge and to think of ways of solving problems
when they face an unfamiliar situation.

In summary, SZ’s teaching with Variation of various solutions to a single problem seemed like that of a non-authoritative facilitator of students’ learning. His use of this lesson component seemed rooted in a perspective about knowing in which ‘new’ knowledge is an extension of the ‘old’, and about learning as the process of linking the ‘new’ to the ‘old’. Accordingly, he seemed to both assess and capitalise on students’ available knowledge as a means to promote their creative and divergent thinking. Furthermore, he seemed to think that students would benefit from exploring multiple ways of solving open-ended, novel problems as this process fosters linking the ‘new’ to their current knowledge. The following sub-section inquires further into teaching with Variation of problems.

4.3.2 Teaching with Variation of Problems in a Single Lesson

This sub-section discusses how teaching with Variation of problems was used to foster students’ learning of an underlying mathematical method (Fang Fa). First, SZ’s and then LX’s practices and perspectives are reported to convey a sense of how this form of Variation was used by expert teachers. In Appendix A, two additional cases (WK and ST) are presented briefly to depict the typical sequencing of problems in an entire mathematics lesson and highlight the juxtaposing of Bridging and Variation. The following discussion elaborates teaching with Variation of problems by SZ.

4.3.2.1 A Variation Expert (SZ)

In the interview before and after the first observed lesson in SZ’s classroom, he explained that his teaching with Variation typically consists not only of multiple solutions to a single problem (sub-section above), but also of variation of problems for teaching the goal concept. To contextualise this way of teaching with Variation, his first observed lesson is reported. SZ’s stated goal for students’ new learning in that lesson (the one he taught just a day before the lesson described above) was to teach factoring of trinomials with a quadratic coefficient of 1. He
wanted students to understand and flexibly apply the generic formula $x^2+px+q = x^2+(a+b)x+ab = (x+a)(x+b)$, where $q=a\times b$ and $a+b=p$ for factoring specific problems under different situations.

SZ started this first observed lesson by two examples followed by establishing the formula for the students, then two more examples, four exercises, and four problems with some variations. Box 4.3.2.1-a below presents the sequence of problems used for the entire lesson and the timing of their presentation to the students; later excerpts present specific classroom interactions in this sequence.

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The data in Box 4.3.2.1-a show how SZ lesson plan unfolded during the first observed lesson, while teaching with Variation of problems. These data indicate a progression from simple and direct application of the method, through an exposition of the symbolic, abstract, and general relationship between a trinomial and its binomial factors, to gradually more challenging instances of the concept—factoring a trinomial as an inverse operation of multiplying two binomials. These data also indicate how problems were selected to include the three levels of understanding this concept—the basic point (items 1-3), focal point (items 4-5), and demanding point (items 6-9).

SZ opened the lesson by engaging students in solving two simple instances of a trinomial. The first \((x^2+4x+4)\) reactivated students’ ‘old’ knowledge of the perfect square formula; the second slightly extended this ‘old’ knowledge to a trinomial that requires factoring of a small, constant (3), which is also a prime number and thus has only two factors. Importantly, besides starting students off by solving the perfect square problem, SZ did not tell them how to solve the slightly novel (second) problem. Some students, apparently the high-achieving group who could flexibly apply the perfect square formula as well as the formula for differences of squares (e.g., \((a+b)(a-b)=a^2-b^2\)), immediately reorganised the second problem: \(x^2+4x+3 =x^2+4x+4-1 = (x+2)^2-1^2 = (x+1)(x+3)\). They changed the constant (3) into 4-1 to obtain the expression from the first problem (‘old’) with a number (1) that is also a perfect square, both of which they apparently knew how to solve. Most importantly, this simple adjustment enabled them to notice the two factors of 3 in the binomial multiplication answer.

Engaging students in factoring the first two, rudimentary instances enabled SZ to introduce the generic formula through another instance that began with the factors \(((x+2)(x+3)\) and the Reversed: \(x^2+5x+6)\) and was juxtaposed with the general form \((p, q, a, \text{and } b)\). This introduction is presented
Box 4.3.2.1-b: SZ’s introduction of a general method of factoring (2009-05-14)

01:03 T: The title of today’s lesson is, Factoring the Quadratic Trinomial of the general form: \(x^2+px+q\). [In this form] \(p\) is a coefficient of \(x\); \(q\) is a constant [number]. Let’s see how we can get some method of factoring for this generic format.

03:09 – 04:08 (particular)
SZ asked what do they call the operation of \((x+2)(x+3)\). Students responded, “multiplication of binomial expressions.” He continued, “What can we modify this into?” and they responded, “\(x^2+5x+6\),” which he wrote on the board. SZ then stated that they can reverse this multiplication process, asked what would \(x^2+5x+6\) be equal to, and the class responded as a chorus “\((x+2)(x+3)\).” He continued, “The result, or the process, is called …?” and students responded, “Factoring the trinomial.”

04:08 – 10:15 (general)
SZ continued, “Ok, this is a specific example. Can we make it more generic?” Some students responded, “Yes,” he asked how, and then invited S1 to share. S1 stood up and said, “This is factoring of the trinomial, and both of the coefficients for the constant \(x\) are 1.” SZ paraphrased it, and then capitalised on S1’s further sharing (factoring 6 into 2×3 and hence \(x^2+5x+6= (x+2)(x+3)\)) to ask, “What do we get for \((x+a)(x+b)\)?” A majority of the students responded in chorus while SZ wrote their statement on the board: “\((x+a)(x+b) = x^2+(a+b)x+ab\).” He followed with a statement that the factor of the coefficient in the quadratic trinomial is one, and asked, “Is this \(x^2+(a+b)x+ab= (x+a)(x+b)\) factoring of the trinomial?” Students concurred and SZ asked, “Can we transfer this more generalised quadratic trinomial \(x^2+px+q\) into this form as well?” S2 responded, “\(q\) should satisfy \(a\times b\), and then we have to make sure \((a+b)=p\).” SZ paraphrased the answer while writing the following on the board:

\[
(x+a)(x+b) = x^2+(a+b)x+ab \\
\text{Reversed: } x^2+(a+b)x+ab = (x+a)(x+b)
\]

If \(q=ab\), \(a+b=p\)  
Then \(x^2+px+q = x^2+(a+b)x+ab = (x+a)(x+b)\)

The summary of data in Box 4.3.2.1-b shows how SZ capitalised on the reactivation of students’ ‘old’ knowledge via the Bridging problems, and on another specific instance of a multiplication of two binomials, to produce a trinomial and lead into meaningful presentation of the generic form of any trinomial. First, he let students express the result of multiplying \((x+2)(x+3)\) and the reversed operation of factoring \(x^2+5x+6\) back into the binomials. Then, he asked S1 to suggest a way of generalising this example, but received another expression of the particular example, though with emphasis on the key step of factoring the constant \((6=2\times3)\). Thus, SZ took upon himself to prompt for the generalised form by asking what would be the result of multiplying \((x+a)(x+b)\). He followed students’ correct response by a short exchange with S2 about the fact that the coefficient of each ‘\(x\)’ in the binomials was 1, and then led to presentation of the link between the two general forms (for coefficients \(p\), \(q\), \(a\), and \(b\)).

SZ then asked students to factor two more simple instances \((x^2+3x+2\) and \(x^2+2x-15\), see item #3 in Box
The difference in the process between item #1 and item #3 was that variation of instances in the former was used as a means to focus students’ attention on the underlying method so it could be generalised (in item #2), whereas in the latter students were expected to apply the ‘new’, general method to the given instances. For example, they needed to consider the constants ‘2’ and ‘15’ as ‘q’ and figure out a pair of factors for each that could be considered as a×b, and then test if this pair also satisfied the other coefficient (a+b=p). Again, the first instance selected by SZ was straightforward, because the constant (2) was a small prime and could only be factored into either 1×2 or -1×-2. This instance could therefore also indirectly refocus students’ attention to the sign, a consideration that SZ would explicate later in the lesson. The second instance provided another example of slight variation of the general form—it extended the thinking needed to determine the correct pair of factors by introducing a composite number (15) that is a multiple of two primes. Thus, only four pairs of factors would have to be considered (-1×15, 1×-15, -3×5, and 3×-5) and finding the coefficient p would be quite simple. Consequently, teaching with Variation via only two instances seemed to enable many students’ understanding of how the method of factoring begins and proceeds.

In all, teaching with Variation during the opening 14 minutes of SZ’s first observed lesson encompassed back-and-forth shifting among Bridging problems, general (‘new’) formulas, and application of the ‘new’ method to simple examples. This process seemed to provide a basis for students’ grasp of why and how the general method of factoring a trinomial works (it’s a reverse process of binomial multiplication). Xie and Carspecken (2008) asserted that such back-and-forth shifting between particulars and general forms is rooted in the dialectic materialism approach and is a hallmark of Chinese curricula and lesson design. The following discussion focuses on how SZ capitalised on this lesson opening via what the researcher came to think of as a good example of teaching with Variation.

Item #4 in SZ’s lesson plan consisted of four additional examples with simple variations of q and
p, mainly focusing on signs ($x^2+7x+6$, $x^2-7x+6$, $x^2-6x-7$, and $x^2+6x-7$; see Picture 4.2). These instances were part of what he considered the basic point (factor the constant numbers) and focal point (determine signs and sums) of his lesson.

Apparently, SZ’s monitoring of students’ work indicated to him that, at least within the lesson, most students grasped the basic point. Thus, about 19 minutes into the lesson (see Excerpt 4.3.2.1-a below), he turned to fostering students’ learning of the importance of determining the sign of p, while distinguishing between two principal categories ($q>0$ and $q<0$). Again, this underlying method was taught via capitalising on the variation of four instances of trinomials that students have just completed factoring.

**Excerpt 4.3.2.1-a (Teacher SZ 2009-05-14)**

19:01 T: How many possibilities for the sign do you have for factoring the constant q?

19:10 Ss (chorus): Two.

19:13 T: There are only two possibilities, right? If $q>0$ [sign is ‘+] , what are the relationships between the two numbers you take apart?

19:20 Ss (chorus): [Both numbers should have] the same sign.

19:22 T: The same sign (he writes on the board, if $q>0$, a and b have the same sign, then asks): When a and b have the same sign, what is the relationship between these two signs and the coefficient p? Is there any relationship? (He underlines the solved example $x^2+7x+6 = (x+1)(x+6)$, and writes next to it $+1=a$, $+6=b$). The sign for p is also positive. (He points to $+7$ in $+7x$ and says), it is positive, $+1$, what is the relationship between $+6$ and $+7$?

19:57 Ss (chorus): They have the same sign.

19:58 T: The same sign, right. What about when [p] is negative? Let’s have a look. (He points to the solved instance of $x^2-7x+6 = (x-1)(x-6)$, and underlines $-1$ and $-6$). The coefficient for p is negative as well. Can we find some rules or patterns for the sign? How do we summarise the pattern for the sign? If $q>0$, then the two constants, a and b, should have the same sign like p, right?
20:23 Ss (chorus): Yes.
20:26 T: Ok, we have gained some common features for the sign when q>0. Let’s think about the situation when q<0 (sign is ‘-’), what about the signs for a and b?
20:43 Ss (chorus): Opposite signs.
20:44 T: It is easy to figure out a and b have different signs. Let’s have a look at this [solved example], \(x^2-6x-7=(x-7)(x+1)\), (he underlines -7 and +1). -7=a, and +1=b, is there any relationship between these two signs and p? If they have, what is the relationship?
21:02 T: Ok, S5 – can you share?
21:06 S5: I think the sign of p should be the same sign as the number with the larger absolute value.
21:28 T: How should we summarise? S6- Can you refine S5’s opinion?
21:38 S6: if \(|a|>|b|\) [then] p has the same sign as a.
21:52 S6: if \(|a|<|b|\), then p has the same sign as b.
21:55 T: (writes on the board) [And] if \(|a|<|b|\), then p has the same sign as b.

The data in Excerpt 4.3.2.1-a show how SZ used teaching with Variation of problems to support an understanding and application of the underlying method of trinomial factoring. In particular, SZ appeared to focus on students’ grasp of a structural relationship among factors and their signs. It should be recalled that the data in this excerpt followed the five minutes of students’ work on factoring those four instances, first individually in their notebooks (about 3 minutes) and then sharing and explaining their answers on the board (each of four students presented a solution to one trinomial). At this point, having worked through 9 examples (two in Bridging, one in developing the generalised form, two simple applications, and these four), SZ turned the focus of their thinking to the issue of signs and how it could be generalised.

Apparently to foster students to move beyond mere trial-and-error of all possible options and use a systematic consideration of sign (+/-) and additive combination needed to fit both q and p in any given trinomial, SZ began by asking students a simple (rhetoric) question (“How many possibilities of the sign do you have for factoring the constant q?”). Students responded (“Two”) and he continued to explore the simpler possibility of a positive number, as it entails that the signs of both factors, and of the resulting p (coefficient of x) are the same. To foster generalised thinking he would later expound on, SZ presented the question in terms of the generic constant (e.g., q>0).
He then pointed to the particular trinomial \((x^2+7x+6 = (x+1)(x+6),\) factored by a student) that in his selection of four instances served as an exemplary case of this category, and pointed out the compatibility between the general and the specific \((+1=a, +6=,\) and the sign of \(p=+7\) is also positive). He then continued to the second general category, in which \(q>0\) but \(p<0\), asked and answered his own question ("Can we find some rules or patterns for the sign? How do we summarise the pattern for the sign? If \(q>0,\) then the two constants, \(a\) and \(b,\) should have the same sign like \(p,\) right?") and moved on to the other, more complex category of \(q<0\). For this category, he returned to the corresponding Variation instance \((x^2-6x-7=(x-7)(x+1)),\) used it to ask students about a general relationship, and invited S5 (and then S6) to share. Based on their contributions, SZ finally summarised the two sub-categories of a constant \(q<0\), while using the absolute value symbols to express the relationship that determine the sign of \(p\) (e.g., if \(q=-7\) and \(p=-6,\) \(|a|=7\) and \(|b|=1,\) then the negative number should be assigned to the larger factor so that the sum \(a+b=p<0\)).

In and of itself, data in Excerpt 4.3.2.1-a could be interpreted as manifestation of authoritarian, teacher-led, transmission kind of practice. However, considered within the 9-problem Variation sequence in which it was employed—it can be seen differently. In particular, the last four instances, which consisted of simple numbers for \(p\) and \(q\) and variation of their symbols, seemed to give students an opportunity to experience the decision process (e.g., trial-and-error) about factors—numbers and signs to obtain \(p\). Then, the exchanges in Excerpt 4.3.2.1-a could help them distinguish and consolidate the sign component of a generalised factoring process. To this end, SZ also elevated the discussion to a higher level of abstraction, considering signs not in their particular appearance (‘+’ and ‘-’) but as mathematical categories consisting of all possible values the factors of \(q\) could assume (larger or smaller than ‘0’). The way by which SZ introduced this categorisation (absolute value) indicated that students were familiar with it prior to the lesson. Based on the analysis above, it seems that SZ created a potential for students’ learning, particularly through inviting four individual students to share their solutions on the board.
The last part of the first observed lesson in SZ’s classroom focused on extending students’ learning to more complex understanding of the day’s concept (‘Demanding Point’, 难点). This extension was attempted by Variation of problems that (a) changed the ‘object’ on which students were operating via the taught method and/or (b) required coordination of ‘old’ (e.g., difference of squares) and ‘new’ (trinomial factoring) methods, which he appeared to assume were then known to the students. These Variation problems (see items 6-9 in Box 4.3.2.1-a) included: \(x^2+3xy+2y^2\); \(x^2(x-1)^2-4\); \(x^2\_\_\_ \_ \_ x+60\) (to which students had to insert a sign, ‘+’ or ‘-’, above the blank space after \(x^2\) and the coefficient for \(x\) in the parentheses); and finding a value for \(k\) in the expression \(x^2+2kx-3k^2\) given that it contains the factor of \((x-1)\). The following discussion summarises the classroom exchanges when students solved the first two problems.

As SZ introduced the next problem, he explicitly stated the nature of the ‘object’ in the variation: “Let’s do some exercises with variations of the first example, \(x^2+3x+2\); we change it into \(x^2+3xy+2y^2\); how do you solve this problem?” He asked a student (S7) to stand up and present the solution (out loud), while writing what S7 said on the board: \(x^2+3xy+2y^2 = (x+2y)(x+y)\). SZ then noted: “There is a variable in this problem; S7 took this variable as a constant in order to solve it” and proceeded to the open-ended problem (see Excerpt 4.3.2.1-e below), which was followed by the problem \(x^2(x-1)^2-4\).

To solve the problem of factoring \(x^2(x-1)^2-4\), SZ invited another student (S8) while writing on the board what S8 had said: \(x^2(x-1)^2-4 = [x(x-1)]^2-4 = (x^2-x+2)(x^2-x+2)\). SZ then asked the class: “Are we done?” In chorus, the students replied, “Not yet,” which SZ echoed by: “Anything else?” and two other students yelled, “The first parenthesis can still be factored.” SZ asked S8 to sit down and continue thinking of what can be further factored in his answer, then turned to another student (S9), who said (while SZ wrote on the board): \(= [x(x-1)]^2-4 = (x^2-x-2)(x^2-x+2) = (x+1)(x-2)(x^2-x+2)\). SZ did not simply accept this complete solution and move on. Rather, he continued orienting students’ attention to the expectation of being explicitly exhaustive in their factoring activities: “Can we continue factoring?” Most of the students in the class replied in chorus, “No more,” which he
followed by: “Why can’t it be factored anymore? This means the quadratic trinomial $x^2-x+2$ cannot be factored any more within the domain of integers.” At this point SZ himself took upon the explicit factoring of ‘+2’, stating that both pairs (1 & 2, or -1 & -2) cannot produce the coefficient of ‘x’ in the trinomial of S9’s answer ($p=-1$). Finally, SZ asked another student (S10) to explain, in general terms, why no further factoring could take place, and paraphrased S10’s response that both conditions for factoring a trinomial, $q=a\times b$ and $a+b=p$, must be met. As planned for this example, SZ capitalised on the counter example—a trinomial that cannot be factored into integers and/or variables—to summarise the lesson’s intended learning: “These two conditions necessarily come together. Firstly factor, and then verify if they satisfy the condition [for $p$] … Pay attention! Not all the quadratic trinomials can be factored within the domain of integers.” The following discussion elaborates on the open-ended problem that SZ used.

The Variation problem that SZ’s selected and presented ($x^2-(\_\_)x+60$), in which students had to supply the signs and numbers for the coefficient of x, was of special significance as it was an instance of problem Variation that, itself, allowed multiple solutions to the same problem. That is, this problem was an example of how participating Chinese teachers might integrate Variation of problems and Variation of Solutions. Moreover, to solve this non-routine problem SZ asked students to work in small groups—a teaching technique considered as highly effective in Western countries (AAMT, 2006; National Council of Teachers of Mathematics, 2000). This task seemed to consist of two key features: (a) the constant he chose ($q=60$) had a large number of factor pairs and (b) the sign of the coefficient $p$ was unknown. Such a problem was likely to prompt multiple examples of the intended method (including through checking with peers). Excerpt 4.3.2.1-b elaborates the unfolding of their work on this multi-solution Variation problem.

Excerpt 4.3.2.1-b (Teacher SZ 2009-05-14)

29:43 T: Now, let’s think about this problem: $x^2-(\_\_)x+60=$. Firstly, fill in the blank; then [find the] factors. Discuss this problem with your neighbours at the front and back desk.

30:13 Ss: (Form small groups with neighbour peers and discuss solutions to the problem, while SZ walks around the class and monitors their work. Some students are heard saying, via the microphone that SZ carries, that there are 10 possible solutions, while others say there are more than 10, maybe 12 ways.)
In how many ways can we factor 60?

12 ways.

I heard some have said 10, and others who said 12. So in how many ways can we factor 60? How about S7 – can you share with us?

1×60, 2×30, 3×20, 4×15, 5×12, 6×10.

(SZ writes on the board what S7 says, see Picture 4.3, and then asks with a prompt to the specific sign (+) that S11 had used): What else for positive?

No more.

These are 6 ways with two positives. Can we have the same 6 groups with negatives?

Yes.

How many ways in total?

12 ways.

So we can choose whichever way to fill in the blank. For example, what should the sign be if we select 60=3×20? We re-write the original problem $x^2+cx+60$ into $x^2+(23)x+60$; Then, how do we factor?

$(x+3)(x+20)$.

We can also write eleven other ways like this. Now, if I let you begin by factoring the trinomial $x^2+23x+60$, how do you think of factoring this problem? We still need to think how to factor 60, and there are 12 ways for factoring 60, but how many ways are satisfying this problem?

One (1).

So you have to continue trying until you find the right one. This is the basic method for factoring the quadratic trinomial with larger constant.

Data in Excerpt 4.3.2.1-b show two key ways in which SZ seemed to provide students at various conceptual levels with ample opportunities for learning the target concept via Variation of solutions to the same problem that itself was an instance of multiple problem Variation. First, by asking students to discuss the problem in groups, each of them could be mentally engaged in applying the newly learned method of factoring a trinomial ($q=a\times b$, $a+b=p$, select values of $a$ and $b$ that simultaneously satisfy $p$ and $q$). They were also gaining multiple examinations of each
possible factoring, as well as how to methodically exhaust all of them, via comparisons to peers’ solutions, so they would be able to respond to the question, “In how many ways can you factor 60? This was further prompted by SZ (Line 31:20), when he oriented students’ attention to the fact that in factoring 60 one should also consider the sign in order to determine and justify the total number of ways.

Second, SZ invited an individual student to stand up and share not only his multiple answers (factor pairs), but also his group’s justification for the list being exhaustive. This pedagogical move provided an additional opportunity for students at different conceptual levels to compare solutions, particularly due to SZ’s request that S7 regenerates the entire list of factor pairs. Like any person who would monitor the list that S7 regenerated (1×60, 2×30, 3×20, 4×15, 5×12, 6×10) and be further prompted by SZ (“What else for positive?”)—a mindful reply of “No more” required a systematic checking of the component q=ab. Similarly, SZ’s next prompt (“Can we have the same 6 groups with negatives?”) and the students’ chorus response (“Yes”) would have to rely on a general method as opposed to non-methodical trial-and-error. It seems that even low-achieving students could follow this specific listing of factor pairs for 60. Most importantly, SZ did not just leave the general solution (12 ways) as is. Rather, he demonstrated a specific example of factoring 60=3×20 and generated the corresponding trinomial and its factored form, 
\[ x^2 + 23x + 60 = (x+3)(x+20). \]
This last move increased the likelihood that most of SZ’s students might grasp the intended concept—factoring a trinomial, including an understanding of the reverse relationship between factoring and binomial multiplication.

In all, the way SZ used teaching with Variation to foster students’ learning in a single lesson included two key features: (a) the task/problem sequence he designed—from simple to complex, from direct application to gradual changes while sharing the same method (“Fang Fa” 方法) of solving the problem and (b) the orientation of students’ application of and thinking about this method across particular instances. In his task selection and sequencing, SZ seemed to pay
attention to motivation of students at different conceptual levels and corresponding levels of mathematical difficulty. Via small-group work, following peers’ solutions, and then teacher summaries, the middle- and low-achieving students were possibly less likely to feel threatened than if the ‘new’, demanding problems and symbolic generalisations would be presented abruptly. By gradually increasing the difficulty level of the tasks SZ seemed to help these students to know how to start and proceed with the method of factoring the trinomial. This teaching-learning process seemed consistent with Tzur’s (2008) 7-step teaching cycle—a point that is elaborated in the Discussion chapter. The following discussion elaborates the perspective that seemed to underlie SZ’s teaching with Variation of problems.

4.3.2.2 Accounting for SZ’s use of Variation of Problems

The interview with SZ after this first observed lesson provided further insights into his perspective of knowing and learning through Variation. In essence, this perspective has already been addressed earlier (Sub-section 4.3.1.2). Thus, three short segments from the second interview with SZ (see Excerpt 4.3.2.1-b below) are presented to provide more data about his teaching with Variation of problems.

Excerpt 4.3.2.1-b (Teacher SZ Intrv 2009-05-14)

At the very beginning, when they have not yet found the solution, I might only give some very simple prompts. The perfect square is what they have learned in the last two days. I gave them a new problem that could not be solved by directly using [it, so they had to think] how to link the new knowledge with the [known] perfect square formula? The students immediately found they could use this method ... I feel that in this problem they were given an opportunity for flexible application of their available knowledge... we have to follow the developmental order, which is from simple to complex, from specific to general/abstract, and let students experience the process ... We need to refer to some basic methods [that] they have met before, or have learned. We can solve some very complex problems using those basic methods, but we have to find the point of link. In this example, what was important is their real understanding of the structure of the perfect square formula ... If they [do] and can flexibly apply it, they will try using it when facing some similar problems with different forms—compare the differences between the new and old problem, and find the link.

If you look at these specific numbers, 7, 6, -7, and -6, the variation of these is relatively not complicated, but these four numbers included all the four cases of the quadratic term with the coefficient 1 for the quadratic trinomial … As the students are just starting to learn the new method ... it is better to keep the numbers simple. I thought of these small numbers, so students will use their mind and concentrate on experiencing the process of [first] factoring the \( q = a \times b \), then think about how to make these two numbers meet the other requirement, which is \( p = a + b \).
[Concerning the problem with \( q=60 \)], it required a comprehensive analysis because it involves both steps of factoring a number and considering the signs. To factor a positive number ... there are two types of same-symbol options (two positive or two negative), so we need some proper prompt here because some students might have difficulties in dealing with the symbols [on their own]. For example, I think there are 12 ways to factor 60 ... for some students this might be a bit harder to think of, and they may need the teacher’s guidance ... Those who thought of 10 ways probably haven’t comprehensively expressed all the ways that we expected. In essence, [they] only missed out one way that only required the changing of the symbol... Actually when I designed this problem I also considered they might miss out some sets [factor-pairs]. (Emphases added)

Data in the first segment in Excerpt 4.3.2.1-b provide further evidence for the two key features inferred earlier as underlying SZ’s perspective on learning and knowing. These features include capitalising on students’ available (‘old’) knowledge (e.g., “learned in the last two days” and “flexible application of their available knowledge”) and linking ‘new’ understandings to the ‘old’ via thinking about ways to solve the new with what they already know (e.g., “compare the differences between the new and old problem, and find the link”). This perspective was further shown in the second segment, in which SZ explained that his selection of Variation problems, particularly numbers, played the role of making the ‘new’ accessible to the students and guide their thinking to the intended link (e.g., “the variation of these is relatively not complicated, but these four numbers included all the four features” and “concentrate on experiencing the process [of \( q=a\times b \quad a+b=p \)]”). The third segment further shows that SZ used Variation of problems to intentionally expose, assess, and help students to rectify plausible/typical difficulties (e.g., “when I designed this problem I also considered they might miss out some sets like these” and “so they need the teacher’s guidance”).

Combined, the three segments thus help to shed light on SZ’s apparent perspective about the central role that his teaching with Variation of problems (and/or solutions) could play in students’ learning. His perspective seemed to focus on both reactivating what they have learned in the past and their understanding that new knowledge in mathematics can and should be linked to the ‘old’. In particular, he appeared to intend for them to realise that, across the variation of problem situations, the underlying method (Fang Fa, 方法) remains the same although at times it must be
applied flexibly. And it is this ‘old’ method that they extend when a new problem cannot be directly solved by the ‘old’ one. For example, he said in the interview: “No matter how the form changes, they can see that the essential structure remains the same.” To foster their grasp of the method, he appeared to take as his responsibility the selection and sequencing of a variety of tasks that gradually progressed from the ‘old’ method (e.g., difference of squares), through particular instances of factoring the constant q into a×b (e.g., x^2+4x+3, x^2+3x+2, x^2+2x-15) followed by adding the factor pair (with signs) to match the given p, to making sense of the structure that he saw as underlying those different instances.

The key to inferring that SZ’s perspective capitalises on students’ available knowledge and attempts to foster linking of the ‘old’ to the ‘new’ is the notion of gradual variation. As demonstrated in the entire task sequence (Box 4.3.2.1-a), in the classroom exchanges that transpired, and in SZ’s explanations (Excerpt 4.3.2.1-b), he appeared to expect that slight changes from simple to complex and from specific to general, accompanied by his and peers’ contributions, would allow students to start and complete a solution process when a slightly advanced problem was presented as ‘new knowledge’. In turn, each slight advance seemed, in SZ’s view, to provide an intermediate link from the lesson’s starting point to its ultimate goal—students forming a structural understanding of the intended method (here, factoring a trinomial with the quadratic coefficient of 1). The example of the four problems he discussed in the second segment of Excerpt 4.3.2.1-b is telling. As he explained, they first had to factor the constant (q=a×b), then check the possible combinations (a+b=p), and finally verify which factor pair fit the condition, while the variation of symbols for p and q was gradually revealed. In his words (from the interview): “[these four problems] reflected the learning of the focal and difficult points. The difficult point is the dealing with the symbols. [My] purpose was for them to summarise the generic situation for factoring the model, like x^2+px+q, so they would be able to use this model to both factor q (the constant) and p, and determine the symbol.”
In support of the inference above, it is important to consider SZ’s reasons for engaging students in solving problems with their peers. He said that, “Every student has different conceptual understanding levels and they have their own views and understandings.” Accordingly, he seemed to focus on creating an opportunity for them to have some open-ended exploration of and communication about a problem so they could “get some inspiration from their peers and learn from each other.” The point is that his awareness of students’ different conceptual levels at the outset of the process seemed to underlie his thinking of how group work could foster their learning. Accordingly, as a teacher he seemed to assume responsibility to structure and sequence the mathematical problems so they fit with those different levels and provide students with opportunities to express and share their ideas as a means for learning from each other. This approach seemed to reflect a learning-centred, mentally active, task-based pedagogy consistent with the aforementioned Western reform recommendations. The following discussion examines how another teacher used and thought about teaching with Variation.

4.3.2.3 A Variation Expert (LX)

In Sections 4.1 (Reviewing) and 4.2 (Bridging), the practice and perspective of LX have been discussed. In particular, the discussion focused on how—and why—an expert, veteran teacher used Bridging from ‘old’ to ‘new’. In this sub-section about teaching with Variation of problems, two aspects of his practice are highlighted further: (a) the juxtaposing of what he supposed students already knew with gradual introduction of the mathematics he intended for them to learn and (b) his use of the three-point levels (basic point-知识点, focal point -重点, and demanding point-难点) to guide selection and sequencing of tasks for the lesson. It should be recalled that LX said in an interview that he typically devoted about 1/3 of the 45-minute class time to each lesson component—Reviewing, Bridging, and Variation (with a brief lesson summary).

This sub-section further addresses how LX’s teaching with Variation unfolds after the Bridging
lesson component through data segments from his second observed lesson. To recall, the stated goal (basic point) for students’ new learning in that lesson was the simplification of algebraic fractions. As he stressed in the interviews, he wanted students to understand why (and how) the value of an algebraic fraction remains the same when the numerator and the denominator are simultaneously multiplied or divided by a common factor. To situate LX’s use of Variation within the overall flow of his lesson, the entire sequence of tasks he used during the lesson, organised into 10 phases and the start time of each, is presented first (see Box 4.3.2.3-a). In the interview prior to this lesson, LX said that Variation would take place in phases 4, 5, 7, 8, 9, and 10.

The data in Box 4.3.2.3-a show the 10-phase sequence of problems that LX selected for this second observed lesson. The problems in the first two phases have already been analysed in the previous sections of this chapter (4.1 and 4.2). Once the Bridging problems in Phase 2 were used to reactivate students’ ‘old’ method of simplifying numerical fractions, the two problems in Phase 3 seemed to provide a first extension of this method into the realm of algebraic fractions. In the first problem, students had to identify that the fraction on the left was multiplied by y/y; in the second, they had to identify that a common factor (x) was cancelled. Through this set of Variation problems LX appeared to intend two key extensions of the method—from numerical to algebraic (variable) common factors and thus from no consideration of particular values to ensuring that the value of the denominator is not zero.

These two problems seemed to also emphasise for students the underlying operation, and reason, for creating equivalent fractions, namely, multiplying any given fraction by 1, presented in the form of a fraction with the same numerator and denominator (e.g., y/y or x/x). Indirectly, those two problems could also foster students’ attention to the two-way, equivalent process by which a fraction could be expanded or simplified.
Box 4.3.2.3-a: Ten phases in LX’s second observed lesson (2009-05-20)

1. **Reviewing (the concept of the algebraic fraction)**

10:28 Think about it: Which ones are algebraic fractions?

1. \[ \frac{5x-7}{2a+1} \]
2. \[ \frac{b-3}{5b+c} \]
3. \[ \frac{3x^2-1}{2} \]
4. \[ \frac{4}{3\pi} \]
5. \[ \frac{2}{2x-1} \]
6. \[ \frac{x^2-xy+y^2}{m} \]
7. \[ \frac{m}{n} \]
8. \[ \frac{m(n+p)}{7} \]

11:35 Competing, who is faster?

- When \( \frac{1}{x-1} \) is meaningful
- When \( \frac{1-x}{4x-8} \) is undefined
- When \( \frac{3x-9}{x-2} \) is 0
- When \( \frac{|x|-3}{x-3} \) is 0
- When \( \frac{x^2-9}{x-3} \) is 0

2. **Bridging (simplification of common fractions)**

14:56 Can you simplify the following fractions?

\[ \frac{4}{8} \]
\[ \frac{-16}{42} \]
\[ \frac{6}{9} \]
\[ \frac{3}{-12} \]

3. **Problem-based introduction of the essential properties of the algebraic fraction—the basis for simplification**

18:39 How do you get the value on the right from what’s given on the left?

1. \( \frac{b}{2x} = \frac{by}{2xy} \) (y)
2. \( \frac{ax}{bx} = \frac{a}{b} \) (x≠0)

4. **Variation #1: Simple application of the essential properties**

19:58 (Fill in the blanks)

\[ \frac{2}{xy} = \frac{\_}{x^2y^2} \]
\[ \frac{3x}{x+y} = \frac{15x(x+y)}{\_} \]
\[ \frac{x+y}{x^2-y^2} = \frac{\_}{x-y} \]

5. **Variation #2: Multi-step simplification**

22:37(1) \[ \frac{-8ab^2c}{-12a^2b} \]
2. \[ \frac{a^2 + 4a + 4}{a^2 - 4} \]
6. Direct Instruction: The Sign Convention

33:00 What is your opinion of these two solutions?

1) Xiao Ying: \( \frac{5xy}{20x^3y} = \frac{5x}{20x^2} \)  
   Xiao Ming: \( \frac{5xy}{20x^2y} = \frac{5y}{4x \times 5xy} = \frac{1}{4x} \)  

2) Xiao Ying: \( \frac{-5xy}{20x^3y} = \frac{-5xy}{4x \times 5xy} = \frac{-1}{4x} \)  
   Xiao Ming: \( \frac{-5xy}{20x^2y} = \frac{-5xy}{4x \times 5xy} = \frac{-1}{4x} \)

38:08 The sign convention \( \frac{-b}{a} = \frac{b}{a} \)  
\( \frac{-b}{a} = \frac{b}{a} \)  
\( \frac{-b}{a} = \frac{b}{a} \)  
\( \frac{-b}{a} = \frac{b}{a} \)

7. Variation #3: Simple problems involving sign convention

38:43 Application: Don’t change the value of the algebraic fraction, just change the numerator and the denominator so they do not contain the ‘-’ sign

(1) \( \frac{-a}{2b} \)  
(2) \( \frac{-3x}{2y} \)  
(3) \( \frac{-x^2}{2a} \)

8. Variation #5: Coordinating sign conventions with algebraic expressions

39:36 Application: Don’t change the value of the algebraic fraction; make the highest exponential item positive in both the numerator and the denominator in the following algebraic fractions.

(1) \( \frac{3x}{1-x^2} \)  
(2) \( \frac{-2x+1}{x^2-3x+2} \)  
(3) \( \frac{1-x}{2x-x^2+3} \)

9. Variation #6: Comprehensive application of simplification with signs

47:46 Simplify the following algebraic fractions.

(1) \( \frac{4a^2b}{-6ab^2} \)  
(2) \( \frac{-4m^3n^2}{2m^2n} \)  
(3) \( \frac{3x^2+x}{-x+x^2} \)  
(4) \( \frac{y^2-9}{-2y^2+6y} \)

10. Variation #7: Complex extension

66:03 Application: Don’t change the value of the algebraic fraction; change the fraction and the decimal fraction coefficients of both the numerator and the denominator into integers.

(1) \( \frac{0.2a+0.5b}{0.7a-b} \)  
(2) \( \frac{x+\frac{1}{3}}{\frac{1}{2}x-y} \)

The three problems in Phase 4 essentially capitalised on the same extensions as Phase 3, but
without disclosing the common factor. Thus, problem variation then engaged students in a process of thinking that would proceed from an assumption about the multiplication by the identify fraction \( \frac{x}{x}=1 \), through figuring out what factors produced the change in the given denominators (problems #1 and #3) or numerators (problem #2), to properly including those factors to maintain the equivalence. Problem #3 also extended their thinking in that the common factor was masked by a supposedly known formula (difference of squares). Similar masking was used also in problem #2 of Phase 5, after students extended the underlying method to simplify an algebraic fraction in which common factors were masked by squared variables (along with sign and numerical considerations).

In Phase 6, LX seemed to focus his teaching with Variation of problems on two aspects of the simplification method that, he said in the interview, were commonly done incorrectly by students. One aspect was the proper determination of the sign for the fraction as a single entity; the other was the understanding that a fraction in its simplest form would include no common factors in the numerator and the denominator. Their solutions to those problems were shared by students and paraphrased by LX, who then progressed to the general, symbolic conventions for signs. This was another example of how LX used juxtaposition of an instance (problem #2) and conventions (Line 38:08) to seemingly foster students’ substantiation of the intended method in its application to specific examples. This inference is supported by LX’s shift from the introduction of conventions back to three different problems in which the only requested operation on the given algebraic fractions was the determination of the sign. It was also supported by the wording LX used to set the three problems in Phase 8: “Coordinating sign conventions with algebraic expressions.” In these problems, students were engaged in identifying the highest exponential of a variable in the entire given fraction, and using simultaneous multiplication by \(-1/-1\) to turn that expression into a positive one (if it was negative). Eventually, students’ use of the different aspects of the methods had to be combined and used completely—for solving three problems with variable (or binomial) masking in Phase 9 and two problems
with operations on fractions so they include only integer coefficients in Phase 10. As LX explained in the interview, he considered these last two phases as the demanding point of the lesson. To an observer, they seemed like a set of loosely related problems (mainly connected by the need to multiply the numerator and denominator by a common factor as a means to change the fractions).

Taking all ten phases together, and the way LX engaged students in solving these problems, suggested a pattern in his use of Variation. In each phase, he first let students solve each of the Variation problems individually. Then, he asked a student to share her or his answer and probed into the student’s reasoning. Third, he confirmed or disconfirmed the student’s answer and explicitly restated the reasoning in his own (teacher) language. Finally, he checked with the entire class to figure out if they understood. This pattern, and the way Variation problems were sequenced, seemed designed to optimise the chance for students’ learning. Whether or not a student knew how to solve a problem individually, even the low-achieving students could follow LX’s restatement of the entire process and link it with the answer. Box 4.3.2.3-b below provides data from the solution to problem 2 in Phase 5 to illustrate this pattern in LX’s teaching.

**Box 4.3.2.3-b: An illustration of the pattern in LX’s teaching with Variation (2009-05-20)**

26:34 T (after students worked individually on Phase 5 problems): Let’s look at Example 2 (he reads the expression fully): \((a^2+4a+4)/(a^2-4)\). How do we simplify this fraction? Let’s recall what we have learned in the previous chapter; we used the method of factoring the trinomial to do the division of the integral expression. According to the concept of the algebraic fraction, the fraction bar can be considered as division. So can we write this problem into \((a^2+4a+4)/(a^2-4)\)?

27:18 Ss (chorus): Yes.
27:20 T: How do you deal with it? Do you have any ways to do it?
27:21 Ss (chorus): Yes.
27:26 T: S15 – can you share?
27:28 S15: We change \(a^2+4a+4\) into \((a+2)^2\)
27:32 T: Why?
27:34 S15 (states the rule): Because we can use the formula of perfect square.
27:35 T (repeats the rule): Yes, we use the formula of perfect square to …
27:39 Ss (chorus): Factor the trinomial.
27:42 T: So it is (teacher writes on the board what S15 said) \((a+2)^2\) …?
27:51 Ss (chorus): \((a+2)\) times \((a-2)\).
27:54 T: (Writes on the board what students said) \((a^2+4a+4)/(a^2-4)\) = \((a+2)^2/(a+2)(a-2)\).
Besides the aforementioned pattern in LX’s teaching with Variation of problems, data in Box 4.3.2.3-b show how he attempted to explicate what in students’ solutions was based on their ‘old’ knowledge. For example, in Lines 26:36, 27:54, and 29:01 he required of students to be flexible in the application of the knowledge they have learned in a new situation. He also asked them to explain their reasoning for each step (Lines 27:32). In particular, in Line 28:43 he purposely wrote an error contributed by a student and checked if they recognised it. In this sense, LX supported the same expectations identified in SZ’s teaching with Variation.

4.3.2.4 Accounting for LX’s Perspective

In the previous sections of this chapter (4.1 and 4.2), segments from the interviews with LX were presented and analysed to infer into the perspective that might underlie his practice. Excerpt 4.3.2.4-a below presents a short additional segment from the interview about the second
observed lesson that further examines his rationale for using students’ mistakes.

*Excerpt 4.3.2.4-a (Teacher LX, Intrv 2009-05-20)*

Firstly, to find out the students’ level of understanding and their courage, some of them dare not come to the board. Secondly, if a student demonstrates well on the board with correct answer, it is a good model for the students working on their own; if it is a mistake, probably this is a mistake that many other students would also make … So we pointed out the mistakes on the board, and the other students can learn from the mistakes. Sometimes the students who … didn’t make this kind of mistake, probably they were correct when [they solved the problem] on their own, but may not know the other student’s mistake on the board; after the explanation they will think, “ok, if I didn’t pay attention to this I would also have made the same mistake”… Thirdly, to standardise the procedures of writing by doing the demonstration on the board … So it is also modelling or educating the students how to do it when they solve on their own. (Emphases added)

Date in Excerpt 4.3.2.4-a indicate two aspects of LX’s perspective. First, he seemed routinely attuned to students’ ways of interpreting the problems as a means to figure out the students’ levels of understanding. He took student incorrect processes of writing a solution as indicators of their thinking, because in his view such student thinking seemed to drive what, how, and why they would do independently.

Second, his teaching seemed to build on such mistakes not just for those who made them but also for other students. As shown in the Reviewing section (4.1), if a student made a mistake, it indicated to LX that the student was not ready for the new learning, because he appeared to believe that without a solid anchor of the ‘old’ knowledge the student would get stuck. If the other students solved the problem correctly, they may not be aware of that possible mistake, which indicates the nature of knowing of a ‘link’ he expected students to grasp. Just knowing how to solve problems correctly did not seem sufficient for him; rather, students needed to include in their understanding what is considered wrong and why.

Excerpt 4.3.2.4-a also indicates LX’s practice of letting students teach their peers. By first engaging students in solving the problems independently and then in sharing incorrect solutions on the board (which he deliberately invited), LX seemed to encourage students to take the initiative of thinking and doing mathematics. He appeared to assume a role of guiding them
through their initiatives. In this role, he did not tell and show students what and how they should do correctly. Rather, he let them solve and figure out what is the mistake, why it was a mistake, and how to correct it.

In summary, LX’s lesson component of teaching mathematics with Variation of problems consisted of three key features. First, he seemed strategic in preparing and sequencing the varieties of problems. Moving from simple to highly challenging problems could enable him to include the basic, focal, and demanding points of the day’s ‘new’ knowledge. This helped catering for students at different levels, and seemed consistent with the use of enabling and challenging prompts advocated by Sullivan, Mousley, and Zevenbergen (2004). The basic and focal points were those LX expected his students to grasp. The demanding points, promoted via Variation of advanced problems (Phases 9 and 10), catered to the high-achieving students’ learning.

Second, LX appeared to use Variation of problems as a means to guide student grasp of what remains the same (the intended method). No matter how easy or challenging a problem is, it would require application and/or transformation of the method of simplifying an algebraic fraction: find all common factors (extract monomials, factor polynomials), determine the sign, cancel the common factor(s), and make sure the resulting fraction is in its simplest form (includes no common factors). For LX, the intended mathematics revolves around understanding and applying the method (Fang Fa, 方法)—know where it comes from, why it works (including the explanation of mistakes), under what conditions it may not work, and why it is useful.

Third, he invited students who, he seemed to anticipate, would make mistakes to solve the various problems on the board. In this way, he figured out their conceptions while promoting further thinking for his students. To further substantiate the claim that teaching with Variation plays a central role in promoting students’ learning of the ‘new’ through gradual extension of the
‘old’, teaching with Variation of problems from two other teachers is presented in Appendix B.

4.3.3 Summary of Teaching with Variation
All four teachers whose lessons were analysed in this section (including Appendix B) used Variation of problems in a single lesson. Five features seemed common to their practices. First, they attempted to select and sequence tasks from simple to complex and from direct application of the day’s new method to more complex operations with it. Consequently, the problems seemed to be conducive for the learning of students at different conceptual levels, as all teachers believed that problems should be reachable (but not too easy) to students via their available knowledge. By including easy and difficult tasks, these teachers appeared to attempt to motivate the students’ interests and thus to fulfil their teaching goal. Second, teaching with Variation seemed to typically proceed from concrete examples that reactivated intended thinking processes in students, through summaries of the formal, abstract rules, to application of the underlying method across situations. Third, no matter how many tasks they presented in a lesson, they appeared highly focused on the core content and the three levels of learning it (basic, focal, and demanding points). The core content was the method (Fang Fa, 方法) they wanted students to use and understand why it is used across the variety of problem situations. Fourth, these teachers’ pedagogical approaches for teaching the new seemed quite similar—moving from students’ independent work on the problems while the teacher monitors their work, through individual students’ sharing of their solutions (answers + reasons), to the teacher’s review of students’ solutions. Fifth, a similar perspective about knowing and learning seemed to underlie the practice of all five teachers. This perspective was indicated by: (a) their focus on reactivating students’ available knowledge, (b) their view of the mathematical links between the ‘old’ and the ‘new’ as perceivable to students via gradual extension of the method to slightly more complex problems, (c) a view of the learning process as linking the ‘new’ to the ‘old’, which appeared to drive their constant attempts to assess learning and to ensure that students had a solid anchor before moving on. Accordingly, all four teachers seemed to consider their role as helping students to make the
link between the ‘new’ and the ‘old’ via variation of problems that accentuate the underlying method.

Along with these commonalities, the four teachers also displayed individual deviations in teaching with Variation. SZ stood out in his use of group discussions of an open-ended problem (constant=60) and in initiating the learning of the ‘new’ via engaging students in solving a non-routine, novel (homework) problem. LX seemed particularly keen on organising the lesson into three comparable periods, where Variation of problems characterised also the Reviewing and Bridging components. WK seemed to centre on anticipating and rectifying students’ mistakes when they applied the steps of the underlying method. And ST introduced the work on realistic word problems as part of the learning-through-Variation process. These individual deviations do not seem to fit with depictions of Chinese mathematics teaching as a standardised, textbook-driven, one-method-fits-all practice.

The teachers who participated in this study used Variation of problems and/or solutions in their observed lessons to teach the intended mathematics to their students. To culminate this lesson component and the entire lesson, they then typically took a short time in which summary of the lesson’s central mathematical goal was recapitulated. The following section discusses that fourth lesson component, namely, Summarising.
4.4 Summarising

This fourth section of the Analysis chapter focuses on the final lesson component identified in the participating teachers’ practice, namely, Summarising the essence of the mathematics that students were supposedly learning during a lesson (Summarising is translated from the Chinese term ‘Xiao Jie 小结’). The data and analysis presented in the previous sections of this chapter showed that, in fact, summarising key mathematical points was used quite often throughout a lesson. Teachers summarised as a culminating activity when reviewing the previous day’s materials, as transition between Bridging and Variation, during Variation when linking particular instances to the general, symbolic methods and ideas, and at the end of the lesson. In the previous sections, Summarising during a lesson was discussed within each of the three lesson components. This section elaborates the ways teachers used Summarising at the end of their lessons, by presenting and discussing data from two teachers: LX (30 years, expert) and ST (11 years, expert). Appendix B presents additional data and analysis of a novice teacher (JC, 3 years).

4.4.1 A Summarising Expert (LX)

In each of the three interviews with LX, he said that in most lessons he would include a component of Summarising the mathematical essence at the end of the lesson. He clarified that his purpose in Summarising was to help students to recap what they have learned during the lesson. Via this pedagogical strategy, he appeared to hope to foster in every student a deep, lasting grasp of key points of the lesson—at least the basic and focal points. Data in Excerpt 4.4.1-a show how LX used Summarising at the end of the first observed lesson in his classroom.

Excerpt 4.4.1-a (Teacher LX 2009-05-19)

45:05 T: This is the end of the introduction of today’s lesson. What did you gain from this lesson? S39 – can you share what did you gain?

45:21 (The school’s bell rings for end-of-class. No one’s behaviour, teacher or students, indicates that the lesson is over.)

45:21 S39: If there is a variable in the denominator then it is called algebraic fraction.

45:24 T: Ok, [we learned] the concept of algebraic fraction. [Let’s look at] the first problem. Will the entire class please read together?

45:31 Ss (chorus read): When two algebraic expressions are divided, with a variable in the divisor, they are called an algebraic fraction.
45:45 T: What else have you learned? S40 – can you share?
45:46 S40: The condition for the algebraic fraction to be meaningful.
45:46 T: Ok, the condition for the algebraic fraction to be meaningful, how do you describe it?
45:48 S40: The denominator cannot be 0.
45:52 T: Correct; the denominator cannot be 0.
45:56 T: Is there anything else? S41 – can you share?
45:57 S41: The conditions for the value of the algebraic fraction to be 0.
46:01 T: Oh, the conditions for the value of the algebraic fraction to be 0.
46:04 S41: When the numerator is 0, and the denominator is not 0, the value of the algebraic fraction is 0.
46:15 T: Yes, you are correct. Pay attention to the word ‘and’, the numerator is 0 and the denominator is not 0. We learned one more point, which is applying the algebraic fraction to solve the real word problem. Try to remember what you have learned in the lesson. (Emphases added)

The data in Excerpt 4.4.1-a show how, at the end of an observed lesson, LX tried to foster students’ recapitulation of the mathematical essence of the lesson. One key aspect was the way he engaged students in providing the key aspects of their learning. First, he asked one student (S39) to share what was the essence of their learning. The student captured it quite succinctly (“If there is a variable in the denominator then it is called algebraic fraction”). This led to LX’s request that they all read (chorus) a statement of a rule from the PPT (“When two algebraic expressions are divided, with a variable in the divisor, they are called an algebraic fraction”). LX turned to another student (S40) and asked what else was important to capture. That student contributed one of the two main properties of algebraic fractions they have learned via the idea that a fraction is an expression of division and thus the divisor (denominator) cannot be 0 (“The conditions for the value of the algebraic fraction to be 0”). It should be noted that the student’s response depicted the general idea (conditions) as opposed to (possibly) giving an example, or talking about how she solved particular problems. Similarly, LX continued to inquire if there were other important ideas and another student (S41) responded with a general statement about the second property (“The conditions for the value of the algebraic fraction to be 0,” which he further specified, “When the numerator is 0, and the denominator is not 0, the value of the algebraic fraction is 0”).

Three aspects of how LX used Summarising seemed significant. First, by inviting individual students to offer summaries of particular (basic) points, LX plausibly triggered more thinking
about the meaning and importance of what they learned. Most students seemed capable of doing this recap, because their work during the lesson provided specific experiences (Reviewing, Bridging, and ample Variation problems) alongside generalisations that drew on those particular instances. For these students, and particularly for the three who stood up and shared their thinking, verbally summarising could promote consolidation of their learning due to comparing their and the teacher’s paraphrasing of key properties of algebraic fractions.

Second, by inviting the entire class to read the concept of algebraic fraction after S39 shared it, LX seemed to provide an opportunity for yet another reflective thinking experience in students. This last inference rests on what transpired during the entire lesson prior to the chorus reading of the Summarising statement. At the time students read the statement from the PPT, they have just completed numerous operations on algebraic (and non-algebraic) fractions. Such reading could help to elucidate the essence of their activities and hence assist students in better reorganising, retaining, and applying the key concept. That is, LX did not simply ask students to read rules and procedures. Rather, he asked them to do so after engaging them in activities and reflections that enabled their grasp of the intended conceptions. At this point, it seemed that their reading of the Summarising statement was an opportunity to give a brief title that captured what their learning experiences were all about.

Third, LX did not tell and show the essence of the lesson. Instead, he engaged students in considering what was essential via his repeated requests for more summaries. In this regard, the open-ended nature of his prompts should be noted. By asking questions such as “What did you gain from this lesson?” or “What else have you learned?” or “Is there anything else?” he seemed to avoid funnelling students to (or ‘fishing for’) a specific answer. Rather, students contributed to the flow and the sequence of the summary, while LX appeared to facilitate their learning through paraphrasing.
Another important aspect shown in Excerpt 4.4.1-a concerns a cultural norm of mathematics teaching and learning in China. This norm was indicated by student and teacher behaviours when and after the school’s bell rang (Line 45:21), which occurred in at least half of the observations in this study. For two more minutes, during which all three students actually shared their ideas and LX concluded the lesson, they all continued as if no bell ringing was heard. It appeared that students’ high regard for the teacher and for learning governed the lesson continuation until LX finally dismissed the class. In the interview with LX, he was asked to discuss why the lesson continued after the bell rang. His response (Excerpt 4.4.1-b) shows his perspective about norms that should govern students’ learning and a teacher’s practice.

*Excerpt 4.4.1-b (Teacher LX Intrv 2009-05-19)*

Normally, when the bell rings I should finish the lesson and dismiss the class so students can get ready for the next lesson. Students should respect teachers, and teachers should also respect students, both are important. This [bell ring] creates a *dilemma* between the official and the individual/personal needs. I sometimes think I only need 2-3 more minutes to finish my lesson plan, only one or two sentences, so I don’t need to repeat the content in the next lesson. [Stopping] may not be so conducive to the students’ learning, it might be the same if I stopped teaching when the bell rings, however, *I feel much more relieved if I have finished the plan and [accomplished] the goal of today’s teaching*. In fact this is a *sense of responsibility*. I think we [teachers] are *doing goodness to the students*, and I am *taking the responsibility for the students’ well being*. If I run overtime, I intrude the students’ right for a break, but *students are here to learn*. In order to teach knowledge to the students, the *students might gain more than what they lose* [by staying after the bell]. From my perspective, the students should *respect the knowledge they could gain*, and it is reasonable to sacrifice some break time. *The students probably don’t like it, but they can understand*. After all, what they gain is more than what they lose. The students *respect teachers* and show their respect by sitting and listening quietly until the teacher dismisses the class. (Emphases added)

The data in Excerpt 4.4.1-b show the norms and nature of relationships between students, teachers, and knowing/learning that LX appeared to foster. On one hand, within the school’s formal culture, the bell serves the purpose of reminding the teacher that the class time is over, and it is the responsibility of the teacher to dismiss the students. A very subtle distinction must be made here between ‘the lesson (and learning) is over’ and the bell indicating to a teacher that ‘time is over’. In the Chinese culture, the lesson/learning is never really over; it pauses and resumes based on the teacher indication. This subtle distinction was indicated in LX’s own words, when he naturally expressed his responsibility: ‘When the bell rings I should finish the lesson and
dismiss the students’.

On the other hand, as LX pointed out, in the Confucius Heritage Culture knowledge and learning primarily benefit students. Thus, a teacher’s chief responsibility is to ensure they learn. For this reason, teachers are highly respected in China. Accordingly, LX’s students appeared to accept the situation in which the bell rang and the learning continued. They also seemed to abide by the norm that, first and foremost, they are in school to learn with/from the teacher. Not only was the lesson not concluded (because the teacher did not say so) but also learning was not concluded. Their behaviours demonstrated culturally driven discipline and respect—they should keep learning regardless of the bell. This norm, seemingly held by both the teacher and the students, is captured in a Chinese metaphor—knowledge and knowing is a key to life and it is the teacher who ‘holds and uses’ the key to open the doors of knowledge for her or his students. Consequently, Chinese people, young or old, are expected to respect the person who holds the key of knowledge, namely, the teacher. In this culturally embedded way, the relationships between students and teachers seem to be regulated by both parties’ yielding to a higher-order authority—knowing and learning. Their mutual agreement is that students are in schools and lessons to learn, and whatever teachers do is helping the students learn better. Picture 4.4, taken at the centre of LX’s school, symbolises this cultural metaphor (note the icon of a ‘book’ within the ‘key’). The following discussion elaborates the use of Summarising by another expert teacher.

Picture 4.4: The central statue of key to knowing in LX’s school.
4.4.2 A Summarising Expert (ST)

In the exit interview with ST, she said that she most often concluded her mathematics lessons by Summarising the essence of students’ learning. Indeed, she used this lesson component in both lessons observed in her class. She said that she uses Summarising to motivate students during the lesson, because they expect to possibly be the person called upon toward the end of the lesson to summarise what they have learned and gained in it. This, she said, helps her to keep students more focused during the lesson and to cultivate their competence to capture the gist of the mathematics she taught.

The data in Excerpt 4.4.2-a below show that ST used Summarising in a similar way to how LX did it. Both of them invited individual students to summarise before the teacher restated and elaborated student responses. As explained above, this approach promoted reflective processes in students. Like LX, she also guided students’ thinking and reflection by asking open-ended questions such as “Can you summarise?” or “What else?” or “Is there anything else?” However, data in Excerpt 4.4.3-a also show unique features in how ST used Summarising.

Excerpt 4.4.2-a (Teacher ST 2009-05-21)

42:30 T: Ok, today we have learned the multiplication and division of algebraic fractions. What did you gain from this lesson?
42:45 Ss (chorus): The multiplication and division of algebraic fractions.
42:46 T: The multiplication and division of the algebraic fraction, and the principles. What should you pay attention to when you solve problems? Can you summarise? S26 – can you summarise?
43:15 S26: Firstly, determine the sign of the result from all operations.
43:25 T: Firstly determine the sign of the result from all operations. Very good! What else?
43:28 S26: One algebraic fraction divided by another algebraic fraction is equivalent to multiplying the first [fraction] by the inverse of the second [fraction].
43:35 T: In division, we should firstly change the division into multiplication, specifically pay attention to the divisor where the numerator and the denominator should be inversed, or change their position. Don’t just change the sign of the division into multiplication without changing the position for the numerator and the denominator. Is there anything else? S27 – can you share?
43:55 S27: The result has to be [expressed] in the simplest form.
44:08 S28: [In] multiplication and division of polynomials we should firstly factor the polynomial and then cancel the common factors.
44:18 T: Very good! The multiplication and the division of polynomials may not reduce directly, like monomials. Like in the example, the a² in the denominator and a² in the numerator can’t be
cancelled directly. They should first be factored. Very good! So there are mainly two other points we mentioned earlier, the first is to confirm the sign of the result, and the second is to express final result in the simplest algebraic fraction or algebraic expression.

Besides the aforementioned similarities to LX’s use of Summarising, the data in Excerpt 4.4.2-a show how ST’s use of this lesson component slightly differed from his practice. When ST paraphrased individual students’ responses, she often times elaborated the point the student made. For example, in Line 43:35 she added extra prompts that focused on points she appeared to think (a) were important and (b) could be overlooked by students. Moreover, unlike LX’s general statement of rules, she pointed out specific examples from the lesson to call students’ attention to an issue she seemed to be aware of in their work. For example, to illustrate the general rule stated by S28 (“[In] multiplication and division of the polynomials we should firstly factor the polynomials and then cancel the common factors”) she linked it to the example: \((a^2-4)/(a^2+4a+4)\). During the lesson, a few students simply and incorrectly cancelled \(a^2\) in the denominator and the numerator. She prompted students’ reflection onto that operation (firstly factor both, then cancel as proper) with the mathematically justified outcome (simplest form) that should underlie S28’s generic statement of the rule for the students.

In the exit (third) interview, ST was asked why she invited students to summarise instead of doing it herself. Her response (see Excerpt 4.4.2-b) indicates that, as a teacher, she seemed to focus on cultivating students’ mathematical thinking via letting them take the initiative for their own learning.

*Excerpt 4.4.2-b (Teacher ST Intrv 2009-05-21)*

If I summarise by myself, the students feel bored of listening to me the entire class. If I ask the students to summarise, they will be more engaged, and they like listening to their peers’. Another aspect is that I ask them to think about the problem together, so they will be more focused in the lesson … the main purpose is to cultivate their summarising skills; if they form the habit of summarising during a lesson, they will pay more attention to the lesson because they know the teacher may ask them to summarise. They can do the summary only if they listen carefully … [this] promotes their motivation in learning, and their summarisation always exceeds what I prepared I could think of, which is good. (Emphases added)

The data in Excerpt 4.4.2-b highlight four issues about ST’s use of Summarising at the end of a
lesson. First, she seemed aware of the fact that students would be more motivated and engaged if she creates an opportunity for them to take the initiative and do the reflection and summary. This approach is consistent with the findings of Correa et al. (2008) about Chinese elementary teachers’ ongoing focus on students’ interest and motivation. For her, it appears that learning and knowing were intimately linked with internal drive to know and learn, and Summarising should serve this purpose. Second, ST also seemed aware of the fact that students may prefer listening to their peers (over listening to her). Thus, during Summarising she seemed to work as a non-authoritative facilitator who tried to create opportunities for students to learn from one another the essential content that constituted her goal for their learning. Third, her pedagogical approach seemed to establish a norm that contributed to the productive nature of mathematics learning and teaching via instances that served as cases of general ideas. Students in her class could anticipate being invited to summarise at the end of the lesson, and she appeared to assume they knew they could only do so if they listen attentively and reflect throughout the lesson. This indicates that, in Summarising, a teacher’s goal is to foster students’ consolidation and expression of the mathematical essence; keeping students ready for the task of summarising is a means to accomplish this goal. Fourth, ST seemed to consider that letting students do the summarising is likely to yield understandings and ideas that exceed what she would have done. This indicates a humble stance of a person who feels responsible (but not authoritative) for fostering students’ learning. Appendix B elaborates the use of Summarising by a novice teacher.

4.4.3 Summary of Summarising
The use of Summarising by Chinese middle school mathematics teachers who participated in this study seemed to focus on promoting students’ reflection on and consolidation of the essence of the day’s ‘new’ knowledge. Most teachers seemed to provide this opportunity by (a) inviting students to share their own reflection and (b) elaborating and capitalising on those contributions. Like in the other lesson components, the effectiveness of using Summarising differed between novice and expert teachers. These differences seemed related to varied levels of confidence the
teachers had in their students, which brought about different degrees of student freedom expressed by the teacher’s utterances. The expert teachers seemed to give more freedom to the students by asking open-ended questions. Accordingly, they served more like facilitators who keep the flow of the lesson while letting students take the initiative of summarising and learning. They intervened to elaborate students’ contributions, to further clarify possible confusions, and to orient students’ attention to features of the intended mathematics that students might overlook. In contrast, novice teachers seemed to keep the flow of Summarising under one’s control by asking relatively more fixed questions.

In all, the participating teachers used Summarising at the end of a lesson as a strategy to conclude the four-component lesson structure: Reviewing, Bridging, Variation, and Summarising. It allowed teachers to engage students in actively reflecting on and recapping the essential mathematics that they had set as a goal for student learning in that lesson. Every student was supposedly aware of the possibility that she or he might be called upon to provide the summary, and teachers appeared to believe that this awareness increased both motivation and competence to grasp the gist of a lesson. Due to the timing of Summarising, it often continued after the school’s bell rang to signify the end of a lesson. Teachers explained that taking a few minutes beyond the official end of the lesson reflected the Chinese cultural belief, shared by students and teachers, that knowing and learning is the key to students’ futures. The next chapter discusses and elaborates the four-component lesson structure and the perspectives inferred to underlie teachers’ use of this pedagogical practice.
Chapter 5:
Discussion

This dissertation addressed two research questions: (1) What pedagogical practices (activities, tasks, strategies) can be identified in the participating Chinese middle school mathematics teachers? and (2) What teacher perspectives of mathematics knowing and learning may underlie these teachers’ practices? The data were collected in two urban schools in a South Eastern province in China. In the previous section, four components of a mathematics lesson were analysed on the basis of case studies with 6 teachers in those middle schools. In this final chapter of the study, these four components are discussed along two central claims:

1. The comparison across case studies reveals a distinctive pedagogical practice that revolves around those four components;

2. This practice seems to be rooted in a particular pedagogical perspective that differs from the three perspectives (traditional, perception-and conception-based) identified by Simon et al. (2000).

Accordingly, this chapter consists of two sections. First, a five-component teaching cycle that seems common to middle school teachers that were studied is depicted and linked to possible students’ learning. This cycle highlights how teachers created learning opportunities for the students through tasks/problems they selected and posed and ways in which they implemented their teaching. Then, the particular perspective hypothesised to underlie those Chinese teachers’ practices is articulated. Two key aspects of this newly distinguished perspective are teachers’ views of the role that students’ existing conceptions play in their learning (and teaching) and of the process of learning as linking ‘new’, intended mathematics to ‘old’, prior knowledge. These views are discussed in terms of potential impact on mathematics teaching and teacher development. Finally, further implications of the study for theory, research, and practice are discussed.
5.1 A Chinese Practice: The Five-Component Teaching Cycle

To promote Chinese students’ learning of mathematics, participating teachers’ work seemed to be organised in a cycle consisting of five key components (see Figure 5.1): Reviewing, Bridging, Variation, Summary, and teacher Reflection/Planning. The first four components are implemented in this order during a lesson; the last component takes place between lessons and is typically done in teacher groups; it was not part of the data collected but should be considered as part of the cycle. As data and analysis in the previous chapter showed, each of the first four components may include the following student activities: (1) every student solves the problem(s) independently, (2) a few students (2-4) solve the problem(s) on the board while the rest solve the same problems independently, (3) an individual student stands up and solves a problem/task (sharing with the entire class), and (4) the whole class answers the question together as if in a ‘talking chorus’. The key to making the teaching cycle conducive to students’ learning and successful outcomes was not those components per se. Rather, it seemed to be the way in which teachers thought about and carried out the cyclic process to reactivate students’ available (‘old’)
knowledge and capitalising on it to make sense and actively solve gradually more challenging problems.

The participating Chinese teachers’ cyclic practice of teaching mathematics was common to all teachers in both schools and evidenced in the work of six cases reported in this dissertation. The five components of this cycle seem congruent with the 7-step cycle proposed by Tzur (2008) as a pedagogical means to provide students with solid opportunities to conceptualise new mathematical concepts (see Figure 5.3 later in the chapter). Learning opportunities that this teaching cycle seemed to provide were linked to the way ‘old’ and ‘new’ knowledge were juxtaposed by teachers.

Figure 5.2 below illustrates how the teaching cycle is used to interweave the previous day’s new learning, with knowledge available to students from past studies, and with the current day’s intended, ‘new’ learning (numbers for days are schematic). On Day Two, Reviewing the essence of Day One’s new, intended concepts (indicated in red) seems to target consolidation of that
emerging understanding. Bridging on Day Two then reactivates available conceptions that can support today’s learning for all students, particularly low-achieving students who may have not yet formed Day One’s new understandings even at a rudimentary level. Variation and then Summary prompt students to use the solution method (Fang Fa, 方法) that was re-activated during both Reviewing and Bridging and transform it, via reflection on their use of this method (particularly errors), into Day Two’s intended learning. In turn, Day Two’s new learning sets the stage for Day Three’s ‘new’ learning, as teachers reflect (in planning groups) on Day Two’s lesson, make explicit the students’ available understandings, and plan Day Three’s lesson to fit with those understandings as well as the curricular objectives. If a teacher feels that Day Two’s learning raised too many issues and difficulties, she or he may spend more time to elaborate students’ mistakes during Day Three’s Reviewing. The following discussion elaborates the four-component lesson structure as used within this five-component teaching cycle.

5.1.1 Reviewing
Reviewing was used by the Chinese teachers in this study to bring the students back to the learning of the recent concepts that took place in the previous lesson(s), so the essence of the knowledge from the previous day’s lesson is both re-activated and further consolidated. Most often, this component did not consist of reviewing homework problems or going over the previous day’s concepts, but rather of solving more, different problems that required application of content taught during the previous day. Teachers appeared to consider the purpose of Reviewing to be consolidating the anchoring link (‘old’ knowledge) as a necessary step for teaching the ‘new’. Most of them repeatedly emphasised their attempt to make sure no students are left behind due to the teacher’s failing to engender the necessary previous day’s link. The reviewed content consisted mostly of guided work on the previous day’s ‘new’ knowledge. Yet, for the low-achieving students, quite often the review could have been inaccessible, unless it included rudimentary problems (usually observed in the more experienced teachers’ classrooms).
In the latter case, it could also help lower-achieving students re-process yesterday’s new understandings, so they could at least know how/where to start solving the Reviewing problems.

Reviewing seemed to suit best the learning of students who have already began grasping yesterday’s knowledge, because once they have been prompted by the teacher’s examples for the understandings, via solving a few specific instances, they were likely to make sense of and apply the method on their own. For example, teacher WK first asked students to solve problems individually—a first iteration of using the intended method. Then, she invited four students to solve the same problems on the board—a second iteration that enabled comparing across solutions, including noticing errors either in one’s own or in peers’ solutions. Finally, she went through each and every step of those four students’ solutions while repeatedly pointing to potential/actual errors—a third and critical iteration of reflection where students could compare across previous ‘runs’ of the method.

The four teachers whose Reviewing practices were analysed in the previous chapter (LX, WK, JC, and ZX) used Reviewing to begin their lessons also as a means to figure out where the students are and, if needed, adjusting instruction to better fit within the students’ available knowledge. Each of those teachers emphasised that where students were conceptually drove the curricular objectives they have set for students’ learning (as opposed to the given textbook’s objectives alone). Therefore, in spite of the national curriculum and a unified group lesson plan, they seemed to feel the responsibility and freedom to select the content and proper techniques for their lesson to cater for the students in their specific class. In this sense, Reviewing, coupled with Bridging, indicated that participating Chinese mathematics teachers seemed to promote reactivation of and capitalises on students’ available concepts. The following discussion elaborates this point.
5.1.2 Bridging

The second lesson component in the teachers’ cyclic practice, Bridging, indicated the teachers’ close attention to promoting learning of new ideas based on what students already know and can reactivate in service of the intended concepts. Each teacher (LX, JC, and ST) whose Bridging practice was analysed seemed to use it in a conscious attempt to initiate new learning by bringing forth knowledge with which students were assumed to be familiar. This was evident in the lessons observed in the expert teachers’ classes (LX and ST). In the novice teacher’s class, the problems selected for Bridging appeared challenging even for high-achieving students and not conducive to low-achieving students’ reactivation of known concepts. However, he appeared to consider the role of Bridging similarly to the way other teachers did. The teachers seemed to presume that reactivating students’ available knowledge was needed to support engagement in situations for guiding their use of the underlying, ‘old’ solution method also in the new situations. For example, to teach algebraic fractions a few teachers used problems that could reactivate their students’ knowledge of numerical fractions (learned in primary school). In this sense, the teachers capitalised on students’ available knowledge in designing tasks/problems that prompted long-known methods for students - particularly the lower-achieving ones. Consequently, the Chinese teachers seemed to enable students’ active participation in the current lesson’s ‘new’ learning at an entry point that fit the student’s knowing. In other words, Chinese teachers’ task/problem design, selection, and/or adjustment seemed compatible with the recommendations for layered difficulty levels of enabling and challenging prompts made by Sullivan and his colleagues (Sullivan et al., 2003). LX referred to Bridging as ‘intuitive thinking’; ST called it ‘smooth transfer’ from ‘old’ to ‘new’. Both terms indicate an explicit awareness by the teacher of the responsibility for providing students with ample, solid opportunities to reactivate available knowledge (‘old’) as a basis for learning the ‘new’ by solving structurally similar mathematical problems.
It cannot be overemphasised that both Reviewing and Bridging are practices that seem geared toward reactivating students’ prior knowledge. Reviewing attempts to bring back the novel concept established in the previous lesson(s); Bridging attempts to bring students back far enough, even to primary school, to make sure that students reactivate a relevant, ‘old’ method and use it for making the intended link to the day’s lesson. In essence, Bridging seems to belong in the domain of Reviewing, but uses much older understandings and the technique of analogy to promote students’ smooth transition into learning the ‘new’. Whereas the analysis of Reviewing suggested that it fosters learning by students who were already grasping the previous lesson’s new ideas, Bridging seemed to assure reactivation, goal setting, and method initiating by low-achieving students. This distinction helps to explain why Bridging was more frequently conducted as a whole class technique whereas Reviewing engaged students in both individual and whole class activities: Bridging was set to enable students getting ready for learning the ‘new’. This finding is consistent with researchers’ (An et al., 2004; Correa et al., 2008; Ma, 1999) suggestion that the teachers’ view of teaching focuses on enabling students to make connections between new and prior knowledge. It also seems consistent with Correa et al.’s (2008) conclusion that, in terms of building on student’s prior knowledge, Chinese upper-elementary teachers’ beliefs are aligned with constructivist views on learning. Likewise, in this study of teachers’ at the middle school, students’ prior knowledge seemed to play a prominent role in their rationale for pedagogical activities they planned and used. In particular, Bridging problems helped to depict how this rationale was translated into deliberate reactivation of prior knowledge that was also relevant to the intended learning. The following discussion elaborates teaching of ‘new’ ideas via Variation of problems and/or solutions based on students’ prior knowledge.

5.1.3 Teaching with Variation
If Bridging is about reactivation of students’ prior knowledge, Variation is about orienting students’ gradual transition to using this ‘old’ knowledge (method, Fang Fa) for solving new problems. The Chinese teachers who participated in this study created various situations that
seemed to orient students’ comparison between the outcomes they expected to obtain via the use of an ‘old’ method and the actually obtained outcome (answer and/or explanation). Most importantly for learning the day’s new mathematics at a robust, long-lasting level, in Variation the teachers were found to orient students’ comparisons across different problems and/or solutions. The teachers appeared to purposely design problem sequences according to their belief of how learning takes place: from simple to complex and from direct application to variation of problems and/or solutions. Via the use of basic points (知识点), focal points (重点), and demanding points (难点), the Variation tasks seemed to cater for students at different conceptual entry levels and thus to promote emergence and consolidation of the ‘new’. This task design and implementation demonstrate, in practice, Sullivan et al.’s (2003, 2004) proposition that teachers should learn to teach the whole class via engaging students in solving non-routine problems/tasks at different entry levels. Similarly, a Chinese mathematics lesson contains a large variety of problems, which require various adjustments to operations on the problems, while clearly focusing on a core goal for students’ learning.

Indeed, the findings of this study corroborated what previous research on teaching with Variation and the three difficulty levels in Chinese mathematics lessons have demonstrated (Gu, 1994; Gu et al., 2006; Paine, 2002). This study’s findings are also consistent with An et al.’s (An, 2008; An et al., 2004) conclusions that Chinese teachers focus on students’ different cognitive levels and thus they usually design a layered practice format to help students achieve proficiency in mathematics. However, this study further articulated the significance of juxtaposing teaching with Variation and Bridging. This juxtaposition helped to explain how the participating Chinese mathematics teachers systematically fostered their middle school students’ grasp of new ideas due to the reactivation and application of methods available to the students via their prior knowledge. That is, this study indicated that, in and of itself, teaching with Variation might not fully explain Chinese students’ high levels of mathematical outcomes. Rather, it seemed to be
the explicit focus on reactivating students’ prior knowledge, through Bridging and to a lesser extent Reviewing, coupled with the orientation of students’ comparison across problems and solutions, that set forth the significant impact of Variation. In essence, this study suggests that Bridging (coupled with Reviewing) endows Variation with the pedagogical power for producing Chinese students’ first-rate mathematical outcomes. This synergistic Reviewing-Bridging-Variation (R-B-V) approach seems consistent with constructivists’ pedagogical recommendations (Steffe, 1990a; 1990b; von Glasersfeld, 1995a) and particularly with Tzur’s (2008) 7-step cycle (this point is elaborated later in this section). Key to these recommendations is the need for a teacher to analyse students’ available conceptions (e.g., simplifying common fractions) and bring them forth as a starting point for students’ learning of new, intended mathematics (e.g., simplifying algebraic fractions). The participating teachers’ selection and sequencing of Reviewing, Bridging, and Variation problems seemed to focus on this transition from known to new ideas. The following discussion elaborates the Summarising lesson component.

5.1.4 Summarising
The role of Summarising in the Chinese teachers’ cyclic practice seems to be providing students at different conceptual levels with additional opportunities for *timely reflection* on the day’s novel mathematics. The Chinese teachers who participated in this study used Summarising during a lesson as well as at the end. Summarising during a lesson usually followed introduction of exemplary instances so that the underlying, general idea is captured. End-of-lesson Summarising was intended to re-orient reflection on new understandings gained through an entire lesson.

As the data and analysis in the previous chapter showed, just like with the Reviewing-Bridging-Variation triplet, the effectiveness of summarising differed greatly between novice and expert teachers. Such differences appeared to be rooted in (1) teachers’ different pedagogical
approaches, (2) the level of confidence they seemed to have in their students, and consequently (3) the degree of freedom students were seen to experience. For example, the expert teachers LX and ST appeared to attempt more open-ended questions and less rigid direction of student reflection, because they seemed to have more confidence in their students and in themselves. Accordingly, these teachers functioned more like facilitators whose role is to keep the lesson’s flow while letting students summarise key points of the lesson (which could support reflective thinking). Expert teachers often intervened during Summarising only when students’ work and confusion required it, particularly orienting students’ attention onto features of the novel concept that students might have overlooked. On the other hand, the novice teacher JC appeared (see Appendix B) to exercise more control over the flow of Summarising, as indicated by his asking of questions with a fixed nature that seemed to limit students’ thinking (although he invited individual students to share their different opinions). JC’s seeming need for control indicated how a novice teacher’s lack of pedagogical confidence in himself, in spite of being coupled with profound understanding of the mathematics, led to lower levels of confidence in his students and the ensuing lower level of students’ freedom to explore and inquire. This finding is consistent with Cooney’s (1999) observation that beginning teachers commonly confound their authority in terms of classroom social norms with authority for mathematical knowledge. The following discussion links the four-component lesson structure with the teachers’ work between lessons (which ‘closes’ the five-component teaching cycle).

5.1.5 Reflection/Planning
Compatible with previous research (Li, Chen, & Kulm, 2009; Li & Li, 2009; Ma, 1999; Paine & Ma, 1993), this study found that Chinese teachers typically use group lesson planning as the foundation for their practice. Yet, this study highlighted that the teachers made individual adjustments to fit with their classroom students’ evolving understandings (and misunderstandings) and with their own teaching preferences/styles. Time and again, the agreed upon group plan seemed to be altered by individual teachers based on their constant awareness
and assessment of where the students are conceptually. By attending to students’ responses to questions the teacher posed, she or he appeared to judge what students have gained, what they were still confused about, and what they were lacking. To substantiate such judgments, during the reflection and planning phases the teachers meticulously checked students’ homework to figure out whether or not students grasped the day’s new ideas. Quite often during the interviews, the teachers explained their selection of specific tasks for the next lesson’s Reviewing (and Bridging) based on this assessment. In this sense, it seems that the Chinese teachers’ awareness of and attention to students’ thinking at any given time were consistent with Mason’s (1998, 2008) assertions about what it takes to make ‘real’ (effective) teachers.

During the Reflection-Planning phase, the Chinese teachers appeared to pay particular attention to identifying and recording typical mistakes in students’ work (in class and at home). These mistakes were used as a main vehicle not only for planning but more so for implementing the next lesson, specifically as a guide for selection of contents/problems of the next lesson’s Reviewing and/or the students to call upon. For example, JC’s first observed lesson was based on his expectation of students’ mistaken responses to trinomial-factoring problems. He adjusted the lesson plan based on his judgment that, in class, most students have grasped the new, cross-multiplication method. Accordingly, he moved on to teaching the new idea via more complex problems. Initially, senior teachers suggested to him during the group lesson plan to separate the coefficient for the quadratic terms (=1 and >1) into two lessons. However, JC found that his students fluently operated on quadratic expressions with coefficient=1. Instead of following the group lesson plan he adjusted his lesson and finished both coefficient types in one lesson. Contrary to his expectation, however, after the lesson he realised that students’ homework included many mistakes, which seemed to mean for him that the students did not quite understand the lesson. So in his next lesson, instead of teaching new content according to the group lesson plan, JC ended up spending about one third of the lesson revisiting students’ mistakes.
Another example of how a teacher adjusted the group lesson plan was observed in SZ’s class. In his case, the entire teaching plan for the second observed lesson was altered. The students were exploring and sharing their different solutions for the new problem of factoring $2x^2-3x+7$, which SZ had not taught before. Instead of following the planned lesson (mere 10 minutes for this novel problem), he kept its flow to fit with students’ evolving conceptions and engagement, while letting them lead the class and thus take the initiative for their own learning.

Due to limitations of time and research focus, data collection did not include the Chinese teachers’ group lesson planning. Collecting and analysing such data would be important because, as Cai and Wang (2006) pointed out, Chinese mathematics teachers share many similarities in their lesson plans. These similarities (with deviations) were shown in the data about an entire lesson plan used by LX, WK, and ZX. Because group lesson planning plays an important role in China (Ma, 1999; Paine & Ma, 1993; Stigler & Stevenson, 1991), a future study would greatly benefit from such a focus, particularly linking it to the weekly observations that teachers make in one another’s classrooms. This suggestion is consistent with Li, Chen, and Kulm’s (2009) argument about the need to explore teachers’ practices in lesson planning and their thinking behind their practices.

Regardless of this limitation of this study, however, the examples of teachers who repeatedly deviated from the group’s plan (and the national curriculum) indicated a key claim. The participants’ teaching of mathematics did not strictly follow the textbook plan in a mode of transmitting the knowledge to passive students. Rather, they all appeared to adjust the agreed upon plans to fit with their students’ available understandings and needs. Both their planning and implementation seemed to focus on coordinating between attention to students and to the intended mathematics.
This study is consistent with An’s (2006) findings that “most Chinese teachers would use one or two hours for daily planning” (p. 121). The teachers in this study reported they spent this amount of time for lesson planning. If teachers can design well thought-out, high-quality lesson plans, then it can serve as a solid basis for high quality classroom implementation. This is consistent with the NCTM (2000) PSSM assertion that “Effective teachers must know how to ask questions and plan lessons that reveal student prior knowledge; they can then design experiences and lessons that respond to, and build on, this knowledge” (p. 12). Furthermore, the interviews with the teachers in this study corroborated An’s (2006) findings that Chinese teachers’ lesson plans consist of detailed teaching notes of the objectives, materials, teaching methods, types of questions to be asked, examples to be given, alternative ways to solve problems, and summary activities. Planning done by teachers in this study also seemed to corroborate Li et al.’s (2009) findings that Chinese mathematics teachers plan their lessons through (a) intensive study of the content to be taught, (b) anticipating students’ possible responses and difficulties, and (c) discussing/sharing one’s lesson plans with colleagues. In addition, this dissertation study shed further light on how Chinese teachers repeatedly made adjustments to the lessons that each of them planned with their group, which supported the following conclusions about the importance of the Reflection/Planning component in the teaching cycle:

1. As is consistent with recommendations for pedagogy arising from a constructivist perspective (Simon, 1995; Steffe, 1990a, 2008; von Glasersfeld, 1995a), any plan and its implementation should proceed from and be progressively adjusted to fit within students’ available conceptions.

2. Chinese teachers seemed to use student errors proactively—they look for them as an opportunity to assess where students are and as a vehicle for fostering new learning opportunities. Chinese students seemed to accept this approach—they never appeared embarrassed by being asked to expose their mistakes in public (Wang & Murphy, 2006).
3. The Chinese national (standard) curriculum and group lesson plans provide general guidance but they do not determine what an individual teacher eventually does in her or his classroom. Both the intended and implemented curriculum of Chinese teachers seemed to be continually tailored to students’ needs, progress, and difficulties (An et al., 2004; Blömeké et al., 2008; Li et al., 2009).

4. The actual lesson a Chinese teacher executes does not seem to follow the plan like a script, which was a key misinterpretation in depicting Chinese teaching as traditional, authoritative, and textbook driven (An et al., 2004; Li et al., 2009). Rather, the observed Chinese teachers proceeded from group and personal analyses of the intended mathematics, through continual consideration of students’ available/evolving understandings, to meticulous selection (and adjustment) of problems for Reviewing, Bridging, and Variation.

The following discussion elaborates the classroom activities that participating teachers used throughout Reviewing, Bridging, Variation, and Summarising to promote their student learning.

5.1.6 How Might Chinese Teachers’ Classroom Activities Promote Student Learning?
As shown in the previous chapter, each of the four implementation components of the teachers’ cyclic practice may include the following student activities: (1) every student solves the problem(s) independently, (2) a few students (2-4) solve the problem(s) on the board while the rest solve the same problems independently, (3) an individual student stands up and solves a problem/task (sharing with the entire class), and (4) the whole class answers the question together as if in a ‘talking chorus’. All four activities seemed to foster students’ learning of the intended mathematics. The discussion below elaborates these points.

The first activity (independent problem solving) seemed to promote students’ reactivation of their available conceptions. Through their attempts to solve the problems independently, students had to set for themselves sub-goals and a process of solving a given problem. Thus, by asking
students to solve problems individually the Chinese teachers provided each student with an opportunity to bring forth and apply their prior knowledge. That is, students were engaged in actively processing the intended method, first for ‘old’ and then for ‘new’ problems.

In addition to the advantages of solving problems individually, the second activity (a few students solve on the board) had the advantage of giving numerous prompts for students who were just beginning to grasp the ‘new’ knowledge, so they could at least know how/where to start. For low-achieving students, the solutions on the board could promote further comparison between one’s own solutions and a peer’s (or the teacher’s) solutions. Thus, the low-achieving students were repeatedly given prompts from peers, and later from teachers, so they could at least initiate a solution method. Students’ solutions on the board also seemed supportive for mid- and high-achieving students because they could spontaneously and independently reactivate the intended method in service of solving the ‘new’ problems. In turn, their solutions could bring forth reflections across problem situations, which could promote grasping of the underlying mathematical structure common to those situations. The key to the teacher’s support via this activity seemed twofold: (1) walking around the classroom to assess individual students’ work, and (2) explicitly checking each of the solutions a student has created on the board. In these two ways the teacher not only saw what students were doing but could also interject further prompts based on their work—first to individuals and later to the entire class.

The third activity—a student stands up and solves a problem/task after the entire class solved those individually—had an advantage for low-achieving students. They were given an opportunity to first be actively (mentally) engaged on their own, and later listening to their peers’ and teacher’s confirmation (or not) of one’s own solution. Because a student would often be called upon due to her or his mistakes, this activity seemed to promote high level of attention on the part of low-achieving students, as well as opportunities to re-work their solutions and further their grasp of the intended method.
The fourth activity—the entire class answering as a ‘chorus’—could also promote reactivation and comparison of one’s own answer to the class stated one. This seemed particularly powerful for mid-level students, as their talking out loud with the chorus provided another opportunity to express their answers. In case their answers differed from their peers’, they would then listen more attentively to the teacher’s explanation and could thus reflect more.

Besides the aforementioned four common activities, the Analysis chapter reported on some activities that were unique to particular teachers. ST and WK let the students silently read mathematical conventions or key aspects of previous day’s lesson; and LX, ST, WK, and ZX frequently paraphrased students’ responses. The following discussion elaborates these two additional activities.

By letting individual students read independently and silently, ST and WK potentially triggered students’ reactivation of and reflection on what they have learned in the previous day. This created an opportunity for students to clarify for themselves and for the teacher possible confusions they might have. Such independent review could also promote students’ attention to questions/prompts that ST and WK would provide later, and make them more ready for interpreting and effectively using those prompts. Students who have only began grasping the previous day’s ‘new’ might have gotten stuck on one or more steps of using it as an ‘old’ method. Setting their goal via reading could have increased the probability they would recognise the teachers’ (and peers’) prompts and thus make better sense of how the solution was obtained via the intended method.

By paraphrasing students’ responses, LX, ST, WK, and ZX gave credit to these contributions while using these responses to potentially redirect students’ attention to the particular aspects of the intended method. Those restatements could thus promote (a) reactivation of the method among low-achieving students, and (b) comparison across situations that could lead to
consolidation of the ‘new’ method among mid- and high-achieving students. Mentally following a peer’s statement and then a teacher’s restatement provided potential triggers for comparisons across problems and thus the potential to grasp the underlying method. It should be noted that teachers’ paraphrasing differed between a convention and a conceptualisation of a new mathematical idea. For conventions, the teachers seemed to simply restate; to promote conceptualisation, they would paraphrase by also discussing students’ errors and emphasise the correct solution so that the intended method could be correctly justified. This latter approach seemed consistent with von Glasersfeld’s (1995a) recommendation for how teachers should deal with student errors, namely, using them as a basis for orienting students’ reflective processes via accentuating the cognitive conflict the errors can trigger in students as a source for resolution of such a conflict and hence learning.

5.1.7 Summary of the Five-Component Cyclic Practice

This dissertation study confirmed and expanded findings of previous studies about mathematics teaching in China (Biggs & Watkins, 1996, 2001; Huang & Leung, 2006; Leung, 1995, 2001; Ma, 1999; Paine, 1990; Watkins & Biggs, 2001), which separately depicted components such as Reviewing, Variation or Reflection-Planning. This dissertation study, however, organised these components, while distinguishing the particular component of Bridging, into a cyclic process. This cycle seems to emphasise three key features.

First, the five components seemed to be agreed upon and used by all participating teachers in this study as the overarching approach for successfully teaching mathematics to their students. Key to this feature is how teachers abide by and deviate from the standard and/or group planned curricula. They all seemed to agree with the underlying premises and implications of the structure of a good lesson (hence, the regular cyclic pattern) and the core mathematical ideas to be taught. They all also seemed to agree that it is the teacher’s utmost responsibility to make sure each component is tailored to learning opportunities suitable for her or his particular students.
(hence, the deviations). Thus, for example, when ST decided to include a real-world problem in her lesson (Appendix A) she skipped the Reviewing part of the group-planned lesson. Taken together, a dialectical approach to structure and flexibility seemed mutually accepted by the teachers.

Second, this teaching cycle seemed to be rooted in Chinese values on educational processes that the mathematics teachers employed to co-produce a learning environment with their students. In China, students and parents hold high regard for learning (Gao, 1998; Ho, 1993; Holloway, 1988; Salili, 1996; Watkins, 2000). Consequently, they have high respect for the teacher (Biggs, 1996; Biggs & Watkins, 1996). A powerful example of this high regard for learning and teachers was observed at the end of the mathematics lessons. In spite of the bell ringing for class end, students did not behave as if, for them, the lesson was over. Rather, they continued working on the assigned problems or following the teacher’s directions for up to 10 more minutes—until the teacher explicitly stated that class was dismissed. As this example indicates, as well as the non-embarrassing spirit of working on student errors, the students and the teacher seemed to have constituted a mutual agreement: students are here to learn; teachers are here to ensure students’ success (Gao, 1998; Gao & Watkins, 2001; Watkins, 2000). Such a spirit ascribes to the teacher a significant responsibility for students’ learning (or failure). For example, LX said in one of the interviews ‘if the students have hard time understand the Bridging problem, this lesson is considered as a failure lesson, and the teaching is a failure’. This disposition seems consistent with Cooney et al.’s (1998) observation of teachers who put themselves in the core of students’ learning.

Moreover, Chinese teachers appeared to deeply believe in their responsibility not only for students’ learning of the subject matter, but also students’ sustainable development and well being (Gao, 1998; Gao & Watkins, 2001; Watkins, 2000). The students seemed to understand, trust, and humbly accept the teacher’s help for their learning. For example, LX encouraged students’ motivation and interests of pursuing higher academic degrees by inspiring their
curiosity of how to solve problems they could not presently solve (e.g., “Why is $0/0$ undefined?”). Similarly, SZ told students they should continuously explore the problem and dig deeper into it. In the teachers’ own words, they appeared to see their role not simply as ‘imparting math knowledge’ but as nurturing students’ learning how to learn. This co-production of learning opportunities by the students and the teacher seemed consistent with the co-construction approach advocated by Bauersfeld (1988) and Tzur (2004).

Third, the five components in the teachers’ cyclic practice seemed to fit well with the 7-step teaching cycle proposed by Tzur (2008): (1) specifying students’ current (assimilatory) conceptions; (2) specifying the intended mathematics students need to learn next; (3) identifying a goal students may set, a mental process they may carry out, effects of these processes students may notice, and comparisons they may carry out to transform #1 into #2; (4) selecting tasks/problems that are likely to promote the intended conceptions; (5) engaging students in those tasks; (6) monitoring students’ progress and altering/renegotiating planned tasks as needed; (7) introducing reflection-orienting follow-up questions and prompts (planned or adjusted to students’ progress). The participating teachers’ cyclic practice seems consistent with the cognitive processes postulated by Tzur to underlie students’ construction of new (to them) mathematical conceptions. The five-component teaching cycle and Tzur’s (2008) 7-step cycle do not, however, fully overlap. Step 3 in Tzur’s cycle refers to a view of learning as mental transformation in existing conceptions that was not found in Chinese teachers’ thinking. With this lack of fit in mind, the following discussion elaborates the similarity between the two cycles.

Chinese middle school mathematics teachers seemed to use the first step in the 7-step cycle by constantly raising questions for individual students and the whole class, solidly checking students’ work both in class and homework, and considering it in all five components. In their teaching cycle, Reviewing seemed to consist of six out of Tzur’s 7-step cycle. Reviewing could be considered like a miniature cycle in which the teacher firstly re-identifies students’ current
conceptions, specifies the intended mathematics, and constantly figures out follow-up questions and prompts to use for promoting learning. Bridging seems to consist of steps (1), (2), (4), (5), and (6) in Tzur’s 7-step cycle. Variation, which draws on the selection of tasks that pertain to the triple point level (i.e., step #4), seems to consist of steps (1), (5), (6), and (7) in Tzur’s 7-step cycle. Summarising at the end of the lesson addresses the first step of the 7-step cycle, and it indirectly addresses step #2, but it does not seem to be part of the 7-step cycle as is. It is proposed that Summarising could augment the 7-step cycle by making explicit the teacher’s role in communicating with students what has been learned during the day’s lesson, thus possibly fostering further reflection in students. Finally, the Reflection/Planning component seems to consist of steps (1), (2), and (4) in the 7-step cycle. Figure 5.3 below illustrates the relationship between the five-component teaching cycle and Tzur’s (2008) constructivist 7-step cycle.

![Diagram](image)

**Figure 5.3: Participating teachers’ cyclic practice fit with Tzur’s 7-step cycle.**

The critical realisation about Chinese teachers’ practices that this study offers arose out of the analysis of the five components in all six teachers’ practice. This practice ensures and repeatedly fosters students’ reflection on the basis of reactivating and capitalising on their prior knowledge. The teachers’ selection of tasks seemed to provide entry points for most students and promoted
reflection across various instances of using the underlying method by students who began grasping the intended, ‘new’ mathematics during Reviewing and Bridging.

It may be that the construction of robust mathematical understandings can be explained by the common emphases in the 5-component teaching cycle and Tzur’s 7-step model. Chinese students’ first-rate outcomes do not seem to stand in contrast to, but rather in accord with the teaching they receive. Within the five-component teaching cycle, Chinese teachers selected not too easy and not too difficult tasks that seemed to engender learning of students at any conceptual entry level, so most students had an opportunity to be highly engaged in mathematical thinking and mentally involved in the lesson (Xie & Carspecken, 2008). That is, the Chinese teachers did not appear to attempt to simply transmit knowledge, but rather to promote conceptual growth via engaging students in solving various problems, orienting students’ reflective processes, and encouraging students to express/notice different opinions and divergent thinking. For example, SZ provided his entire class with opportunities to explore a novel problem he had never taught (e.g., $2x^2-7x+3$). Acting like a facilitator, he appeared to encourage students’ divergent thinking for solving that problem, as well as for letting them cope in small groups and then as a whole class with the highly challenging, open-ended problem of factoring $x^2(x+60$. Moreover, both SZ (expert) and JC (novice) seemed to foster students’ critical thinking by providing them with a problem that could not be solved within their current knowledge (e.g., factoring $x^2-x+2$). Similarly, LX (expert) and JC (novice) continuously probed students with ‘why’ questions and with different approaches (e.g., use a geometrical approach to justify the formula $x^2+(a+b)x+ab=(x+a)(x+b)$) to foster high level mathematical reasoning. In short, although Chinese teachers do encourage rote memorisation of formulas, their focus seems to be on students’ deep understanding of where these formula ‘come from’, what they mean, and in which situations they work and why.

Likewise, the findings of this study seem to eschew a view of Chinese teachers’ practices as authoritarian and textbook-driven. The participating teachers did follow the textbooks and
teachers’ instruction reference book. However, they frequently extended and/or adjusted the curriculum, which differed from Li et al.’s (2009) assertion that, under the national unified curriculum standards, teachers in China rely too heavily on textbooks. For example, JC and SZ taught the cross-multiplication method, and WK and ST (see Appendix A) taught exponentiation—topics that are not included in the standard curriculum. Each and every teacher in this study indicated she or he has the responsibility, freedom, and encouragement to teach based on what suits best for their students, particularly students’ prior knowledge. Although Chinese teachers work in the context of a unified national curriculum and unified group lesson plans, every teacher seemed to end up teaching differently because the premise is that he or she is responsible for the learning of a different group of students. This finding seems to differ with Cai and Wang’s (2006) proposition that Chinese mathematics teachers’ lesson plans are used intact in real classrooms. The Chinese teachers who participated in this study seemed to adjust their lesson plans to better suit students’ prior knowledge. This finding is more consistent with Blömeke et al.’s (2008) finding that the East Asian high-achieving education systems focus on student cognition in classroom instruction. This finding is also consistent with An’s (2006) finding that Chinese teachers design a layered practice format at different cognitive levels to help students, based on their prior knowledge, to achieve proficiency in mathematics. It is also consistent with Li et al.’s (2009) finding that in group lesson planning Chinese teachers consider students’ needs and knowledge as seriously as they consider the content. The following discussion turns to the articulation of the perspective that seemed to underlie the teachers’ five-component teaching cycle.

5.2 Identifying A Particular pedagogical Approach: Progressive Incorporation Perspective

This study helped to distinguish a particular pedagogical approach, which the researcher termed Progressive Incorporation Perspective (PIP), hypothesised to underlie Chinese teachers’ practice. Identifying the PIP addresses the second research question of this study: What teacher
perspectives of mathematics knowing and learning may underlie these teachers’ practices? The importance of PIP is explained via elaborating its two key characteristics: (1) the central role that students’ prior knowledge plays in teachers’ view of knowing and learning, and (2) teachers’ apparent thinking about mathematics learning as a process of linking ‘new’ knowledge to ‘old’.

The analyses of the participating teachers’ practices (previous chapter) indicated they hold a view of *learning* that should proceed from where students are, which is why their teaching should be based on students’ ability to interpret mathematical problems provided by the teacher via their available knowledge. However, one should be careful with this statement, because the teachers (except for SZ) seemed to consider mathematics as independent of the learner. This specific interpretation was evident in the teachers’ belief, similar to a perception-based perspective, that their role is to help students *see* the link between the ‘new’ and the ‘old’. In their view, it seemed that if problems were selected properly throughout Reviewing, Bridging, and Variation, the students would finally grasp the intended link similarly to how the teachers saw it. For the teachers, a new, intended mathematical idea seemed to have an existence of its own, ‘out there’ and independent of the teacher (or the students). Accordingly, the teachers appeared to consider *learning* as a process in which students *incorporate* the ‘new’ (independently existing ideas) into the ‘old’ (knowledge students ‘came to see’ previously). This view of learning differs from the constructivist’s assertion that new knowledge evolves as transformation in old (available) assimilatory schemes (Heinz et al., 2000; Simon et al., 2000; Tzur et al., 2001). For the Chinese teacher, it seems that every new piece of knowledge can and should be linked to the ‘old’; the ‘old’ serves as an anchor to which the ‘new’ is added (incorporated).

A teacher who adheres to the PIP seems to seriously consider students’ available knowledge as a starting point (hence, the daily need for Bridging), which differs from a perception-based perspective. On the other hand, the PIP seems to depict learning as a process of linking existing
pieces of mathematics to one another. In this sense, the Chinese teachers’ inferred perspective seems different from perspectives identified by Simon et al. (2000). Adhering to PIP, a Chinese teacher seems to pay explicit and close attention to how her or his students’ available conceptions may afford and constrain what they may learn anew. To promote students’ seeing the link of the ‘new’ to the ‘old’, a Chinese teacher appears to conceive of her or his main roles as being to: (a) ensure students’ reactivation and consolidation of the anchoring (‘old’) piece of mathematics and (b) pointing out the underlying method by which these pieces can be linked.

Distinguishing and describing PIP seems an important contribution for teaching and teacher development. Simon et al. (2000) and Tzur (2008) pointed out to the difficult paradigm shift involved in the progress to a conception-based perspective. In contrast, developing a PIP, which when used effectively (e.g., via the five-component teaching cycle) can promote high level mathematics in students, seems to be more feasible. The reason is quite straightforward. PIP, with its focus on assessing and capitalising on students’ prior knowledge (e.g., via Bridging), could be developed as transformation in teachers’ assessment practices. The five-component teaching cycle uses assessment to inform task selection based not only on curricular objectives (what students do not yet know) but also and mainly on what students do know and can use to initiate and productively participate in the new learning. A PIP, even though a teacher may still consider mathematics as existing ‘out there’ and learning as linking new to old (hence, no paradigm shift), seems to hold a higher potential to (a) minimise the number of students who are left behind and (b) promote more students’ progress from the rudimentary understanding to consolidating the ‘new’ knowledge. In this sense, PIP seems a more powerful perspective than a perception-based perspective, because PIP does not overlook the key role that student prior knowledge plays in their learning. As mentioned above, it is hard and challenging to foster teachers’ progress toward a conception-based perspective; however, it seems much more reasonable to expect teachers to adopt teaching practices, such as Bridging and Reviewing,
which support students’ learning via reactivation of their prior knowledge. The following discussion suggests implications of this study for theory, practice, and future research.

5.3 Implications for Theory, Practice, and Future Research
In this section, implications of this study for the field of mathematics education are discussed. It begins by articulating contributions to theory building in the field, continues to ways in which this study can inform mathematics teaching and teacher education, and culminates with potential impact on future research.

5.3.1 Contribution to theory
This study demonstrated the potential offered through using a Western, constructivist theory to examine Chinese mathematics teachers’ practice beneath the surface of observable teaching behaviours. In particular, it articulated why the five-component teaching cycle may support students’ construction of first-rate mathematical understandings. One critical contribution that this study seems to make is the possibility for adopting an additional perspective (PIP) within the constructivist conceptual framework to closely examine the learning processes on the part of students that may be engendered by the five-component teaching cycle.

Moreover, the findings of this study seem to give support to Wang and Murphy’s (2006) assertion that Chinese teachers consider students’ errors as a valuable learning opportunity to benefit the entire class. The mathematics teachers in this study appeared to pay close attention to students’ mistakes in their lessons, and grabbed and exposed the students’ errors in public in a way that seemed to promote students’ reflection via comparing one’s solutions to those of others (peers and/or teacher). From a constructivist perspective, exposing a mistake implies re-orienting students’ reflection onto their own activity (Simon & Tzur, 2004) and on the mathematical justification it requires in order to be accepted as ‘correct’ by others in one’s community. Consequently, exposing students’ mistakes seems to trigger socially-guided student reflections on mathematically viable relationships between one’s and others’ solutions. In other words,
exposing and thoroughly reasoning about errors seems to be a powerful (Chinese) practice because it promotes students’ learning of problem solving methods that are compatible with those accepted in the community of mathematics knowers (Cobb, Boufi, McClain, & Whitenack, 1997).

A second theoretical contribution of this study is the identification of the perspective—PIP—that seems to offer a way of describing Chinese teachers’ practice. As explained above, this PIP-rooted practice consists of five components executed in a teaching cycle. The theoretical significance of PIP lies in the coordination between the Chinese (Eastern) way of thinking about and carrying out mathematics teaching and the constructivist, 7-step teaching cycle proposed by Tzur (2008). This coordination (see Figure 5.3) helps to explain what cognitive role each of the five components may play in fostering learning by students. Prior to this study, the existing literature explicated Chinese mathematics teaching from a social-cultural perspective, like the variation and scaffolding introduced by Gu et al. (2006). However, to date none of the studies on Chinese mathematics teachers’ work adopted a cognitive lens. Depicting the five-component practice in terms of cognitive processes it may empower opens the way for further scrutiny and integration of mathematics teaching with the learning it engenders in students. This leads to the third theoretical contribution of this study.

The attempt to link Chinese teachers’ behaviours and their explanations of these behaviours led to distinguishing a new pedagogical approach, namely, the Progressive Incorporation Perspective (PIP). This perspective seems to provide an important contrast to a perception- and a conception-based perspective. Theoretically, the PIP highlights an aspect of teachers’ thinking that is first-of-its-kind. On one hand, a Chinese teacher seems to maintain an epistemological stance of mathematics that is consistent with traditional and perception-based perspectives (i.e., mathematics is ‘out there’). On the other hand, she or he appears to seriously consider, in each component of the five-component teaching cycle, not only what students already know but also
how that knowing affords and constrains what and how students may learn next. Thus, a PIP seems to afford teaching practices that explicitly and systematically promote coordination of students’ extant conceptions to the intended curricular objectives. Such coordination seems (a) missing in practices rooted in perception-based perspectives and (b) to make the Chinese practice conducive to proactively fostering students’ learning through reactivating their prior knowledge. The following discussion elaborates an implication of this study for practice (i.e., clarifying plausible goals for mathematics teacher development).

5.3.2 Contribution to Practice

This study explored how the participating Chinese teachers’ work might promote learning of new mathematical conceptions among a sizeable number of students. This highly desired pedagogical accomplishment is rooted in a practice—the ‘five-component teaching cycle’—that can quite readily be adopted by teachers elsewhere. Indeed, this practice revolves around (and requires) profound mathematical knowledge for teaching (Ma, 1999). It also requires teachers’ focus on assessing students’ available knowledge as a necessary first step for deliberately reactivating it. Otherwise, Reviewing, Bridging, and Variation simply cannot be designed and/or adjusted to fit within students’ available mathematics. With these two considerations in mind, the five-component Chinese practice seems to provide a feasible goal for teacher development because it does not demand the difficult-to-promote, radical paradigm shift in a teacher’s epistemological stance required by a conception-based perspective. That is, the five-component teaching cycle demonstrates the possibility of having ‘a good enough pedagogy’, where students’ conceptions are explicitly linked to curricular objectives, while teachers may not have a profound awareness of the learning paradox (PALP) (Tzur, 2008). In this sense, a PALP, while certainly desired, does not seem necessary to instigate the powerful learning processes it was considered to support. By adopting the five-component teaching cycle, a mathematics teacher has a higher possibility to reduce the number of her or his students who are left behind, because this cycle promotes students’ reactivation of and capitalising on their available concepts.
Accordingly, the newly identified PIP, which seems to underlie the participating Chinese mathematics teachers’ practices, highlights two practical directions for mathematics teacher education. First and foremost, PIP can be set as a feasible goal for the development of teachers who may adhere to perception-based perspectives (PBP). For the reasons noted above (e.g., no need for epistemological paradigm shift), one would expect PBP teachers to construct such a pedagogical rationale (PIP) in support of and via their adoption of the five-component teaching cycle. A view of mathematics learning that explicates the role of students’ available conceptions via using Bridging seems simple enough for teachers to act upon and grasp. Second, determining that a teacher’s practice seems to be rooted in PIP opens the way to designing professional development tasks for him or her that are likely to foster development of a conception-based perspective. Several researchers proposed useful tasks that can foster such a transition (Tzur, Zaslavsky, & Sullivan, 2008). The PIP can inform teacher educators’ decisions concerning for whom, for what developmental purpose, and how might these tasks be effective. The following discussion elaborates this study’s contribution to future research.

5.3.3 Contribution to Future Research

This study may contribute to future research in four ways. First, it highlights the advantage of employing a cognitive lens to study the five-component teaching practices and teacher professional development. Future research can focus on the dialectic relationship between (a) advances in the teachers’ practices and changes in their rationale (e.g., from perception-based perspective to PIP), and (b) the impact of such changes on students’ learning.

Second, this study indicates ways in which future research can address teaching improvements. For example, studies can address the problem of how and the extent to which employing Bridging intensifies students’ use of their available conceptions and thus decreases the number of students who are left behind. Another example will be to study how a teacher-group’s process of Reflecting/Planning may foster coordination of the mathematics to be taught and students’
current understandings, so that one’s plans support larger number of students’ consolidation of the intended, ‘new’ knowledge.

Third, this study demonstrates the need to further explore how comprehensive, conceptual understanding and mechanical/rote memorisation may be integrated to benefit student learning (Sfard, 2000). For example, LX emphasised that combining the mechanical/rote memorisation with the comprehensive understanding is needed for Year-7 students. He explained that these students are at an age when higher order and logical thinking may not be so profound. Thus, he appeared to consider it necessary to use combined techniques to promote students’ understanding of abstract concepts (e.g., ‘chorus’ reading, visual assistance, silent reading). This view seems consistent with Xie and Carspecken’s (2008) recent analysis of the dialectical roots of Chinese pedagogy. Furthermore, all six teachers in this study mentioned the variable of students’ age and its impact on their choice of combined methods (rote, comprehension). It seems important to study what proportion of this combination may fit with students at different conceptual and/or age levels (e.g., primary, secondary).

Fourth, this study seemed to support the viability of the Account of Practice (AoP) strategy of inquiry, which was developed in the West, for studying links between teacher thinking and practice in an Eastern culture. As the researcher learned through conducting this dissertation study, the AoP strategy can capture the flow of teaching-learning processes (e.g., the 3-day schematic progression of five-component lessons) because it focuses on two or more consecutive lessons (Simon & Tzur, 1999). This strategy of inquiry seems conducive to building a strong case of how and why teachers do in their lessons. By combining interviews before and after each lesson and asking questions specifically tied to what the teacher is going to teach, how she or he is going teach it, why a teacher thinks this way can accomplish students’ learning, and how does the teacher judge if students have grasped the intended mathematics—a researcher can better capture what drives the teacher’s work and how it might be improved. More importantly, by
triangulating data from observations and interviews one could capture how mathematics teachers create learning opportunities for students (in China or elsewhere).

This triangulated strategy of inquiry to data collection and analysis, recommended by Simon and Tzur (1999), seems to offer three main advantages. First, the proximity between interviews and observations and the tailoring of interview questions to what a teacher actually does seem to ease the teacher’s disposition toward the researcher. In the AoP strategy of inquiry, both the researcher and the teacher work together to enquire into the teacher’s practice, instead of the teacher being (or feeling) judged about his or her teaching. At least in the Chinese context, the questions asked during this study data collection matched the teachers’ eagerness to improve students’ learning and hence to improve their own practice. These questions were familiar to the teachers because they were anchored in the problems/tasks the teacher used in class, and then moved on to the teacher’s pedagogical approach in relation to students’ responses. If questions were general—about rationales, terms, or teaching philosophies—many teachers might not have had a strong theoretical background to answer them, might have felt nervous and threatened, might have made up ‘stories’, or provided data they might have thought could make them look good. Without observing their lessons and tightly relating the interview questions to what happened in the lesson, there would have been no ground to link what a teacher said with what she or he did. If the researcher asks specific enough questions rooted in the teacher’s lesson, there is a higher possibility of getting a genuine teacher’s response. Indeed, this increases the trustworthiness of the data as compared to only doing the interview and asking generic questions or to only observing the lessons.

Second, by asking specific enough questions that are tied to what happened in the lesson the researcher helped teachers to recall and reflect on their own teaching. For example, in JC’s second interview, he was asked why he selected the four examples and ordered them in this sequence during the first lesson. The novice and honest teacher JC seemed to be, he responded
that he had never thought about this question. He simply followed what the senior teacher told him to do, or the way it was sequenced in the book. Clearly, the interview not only provided data for the study, but also fostered teacher learning, which JC indicated was a highly desirable event for him.

Third, audio taping and videotaping the lessons and the interviews enabled the researcher to go back to the data, and hence grounding the findings and claims in the data. In this sense, the AoP strategy of inquiry helps to capture a rich data set for future analysis of links between teacher actions and thinking (Simon & Tzur, 1999). The video captured not only the student-teacher interactions during class, but also their mathematical responses (e.g., what they wrote on the board and said about their answers). Such data are not available if a researcher only takes field notes or only audio records a lesson/interview.

5.4 Concluding Remarks
This study took the researcher through a novel and worthy journey—using a Western, constructivist lens about teaching and learning to examine how Chinese middle school mathematics teachers attempt to promote their students’ mathematical understandings. In particular, this study was novel in its application of a Western framework for identifying mathematics teachers’ development (Heinz et al., 2000; Simon et al., 2000; Tzur et al., 2001) and the corresponding AoP strategy of inquiry (Simon & Tzur, 1999) to study the perspectives that may underlie Chinese teachers’ practices. The latter entailed conducting a set of videotaped observations of at least two consecutive lessons along with interviews before and after each lesson. The conclusion of these particular approaches amounts to two central contributions: (1) an empirically grounded depiction of a Chinese five-component teaching cycle and a four-component lesson structure, and (2) the postulation of a perspective that seem to underlie these practices—the newly distinguished Progressive Incorporation Perspective (PIP).
The depiction of the five-component teaching cycle seemed to provide a window onto potential ways in which Chinese students’ learning of mathematics is promoted. The Chinese teaching cycle depicted in this study seemed to systematically create opportunities for students to carry out (a) mathematical methods with which they were partly familiar and (b) focused, repeated comparisons among solutions, including student mistakes (Tzur & Simon, 2004). Thus, the five-component teaching cycle seemed beneficial for both minimising the number of students who are left behind and maximising the number of students who progress to the consolidation of new concepts. Substantial overlaps were found between the five-component teaching cycle and Tzur’s (2008) 7-step teaching cycle. This study indicated a way to augment the latter via explicit addition of a Summarising component and the deliberate use of errors.

The second contribution (distinction of PIP) drew on the analyses of ways in which the six teachers thought about their classroom practices, which were captured in the teaching cycle. Amidst personal deviations, all six teachers appeared to commonly employ the five components of Reviewing, Bridging, Variation, Summarising, and Reflection/Planning. In each of those components, they flexibly used four typical pedagogical techniques: (1) student independent problem solving, (2) a few students solve the problems at the board, (3) an individual student solves a problem out loud, and (4) a whole class answers the question like a ‘talking chorus’. Accounting for the perspective that seemed to underlie these teachers’ explanation and implementation of those practices yielded the distinction of a pedagogical approach—Progressive Incorporation Perspective (PIP). In terms of teacher development, the two key characteristics of PIP (close attention to and proactively capitalizing on students’ prior knowledge; view of learning as linking ‘new’ to ‘old’) seem to make it a feasible goal for mathematics teachers’ development.

Identifying the PIP makes an important contribution to the field in that such a perspective does not require the difficult-to-promote epistemological shift entailed by a conception-based
perspective. Rather, teachers’ adoption of a focus on student available conceptions via using and reflecting on the practice of Bridging (and to a less extent also Reviewing) seems to be a more feasible goal for mathematics teacher education (Simon, 2006a, 2006b). It also seems consistent with Sullivan’s (2008a) recommendations of tasks that teacher educators can use to this end. All in all, a PIP that employs the Chinese five-component teaching cycle can inform, in the East and the West, mathematics teacher development of pedagogical practices that are likely to support students’ accomplishment of profound mathematics.
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Paper presented at the American Education Research Association Annual Meeting, New Orleans, USA.


Appendix A:

Two Additional Cases of Problem Variation

This appendix briefly presents a lesson plan from each of two additional teachers, ST (female, 12 years) and WK (female, one year). The purpose of this presentation is to provide further evidence to the central role that teaching with Variation plays in the four-component lesson structure—Reviewing, Bridging, Variation, and Summary—that participating teachers seemed to use commonly. Thus, the essence of the two lessons is presented in Box 4.3.2.5-a, followed by a short analysis of key features those lessons manifest. Both lessons seem pretty similar to LX’s lesson, because they all planned together as a group. However, slight differences could be found that reflect individual preferences and teaching styles.

<table>
<thead>
<tr>
<th>Box 4.3.2.5-a: Two teachers’ use of Variation</th>
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</thead>
<tbody>
<tr>
<td><strong>ST (female, 12 yrs)</strong></td>
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<tr>
<td><strong>Goal</strong></td>
</tr>
<tr>
<td><strong>Reviewing</strong></td>
</tr>
<tr>
<td>1)</td>
</tr>
<tr>
<td>2)</td>
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<tr>
<td>3)</td>
</tr>
<tr>
<td><strong>Bridging</strong></td>
</tr>
<tr>
<td>1)</td>
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<td>2)</td>
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<tr>
<td><strong>Variation</strong></td>
</tr>
<tr>
<td>$\frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}$</td>
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<tr>
<td>$\frac{b}{a} \div \frac{d}{c} = \frac{b \times c}{a \times d}$</td>
</tr>
<tr>
<td>(3) Examples:</td>
</tr>
<tr>
<td>1)</td>
</tr>
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<td>2)</td>
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<tr>
<td>(4) Exercises:</td>
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<td></td>
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</tbody>
</table>
1) \[ \frac{3a}{4b^2} \times \frac{16b^3}{-9a^2} \]
2) \[ \frac{b^2}{a^3c} \div \frac{-b}{a} \times \frac{a^2}{b^3} \]

(5) Examples:
1) \[ \frac{a^2 + 2a}{9 - 6a + a^2} \div \frac{a^2 - 4}{a^2 - 6a + 9} \]
2) \[ \frac{m^2 - 16}{12 - 3m} \div (m^2 + 4m) \]

(6) Exercises:
1) \[ (xy - x^2) \div \frac{x - y}{xy} \]
2) \[ -2 - a \times \frac{1}{a - 2} \div \frac{a^2 + 2a}{a^2 + 2a} \]
3) \[ \frac{x^2 - 10x + 25}{x - 1} \div \frac{5x - 1}{x^2 - 1} \times \frac{1}{x + 1} \]

(7) Application to real-life:
There is a rectangle box with length, width, and height of a, b, and h, respectively. The box is filled with cylindrical aluminum cans with the height of h and radius r (see figure). Work out the utilization rate of the box (work out the accuracy rate of the ratio of the volume of the aluminum can and the dimensions of the box to 1%).

(8) Extension (power operations)
Use your head: fill in the blanks (same as #8 in T4)

\[
\left( \frac{a}{b} \right)^2 = \left( \frac{a}{b} \right) \times \left( \frac{a}{b} \right) = \frac{a}{b} \times \frac{a}{b} = \frac{a^2}{b^2}
\]
\[
\left( \frac{a}{b} \right)^3 = \left( \frac{a}{b} \right) \times \left( \frac{a}{b} \right) \times \left( \frac{a}{b} \right) = \frac{a}{b} \times \frac{a}{b} \times \frac{a}{b} = \frac{a^3}{b^3}
\]
\[
\left( \frac{a}{b} \right)^4 = \left( \frac{a}{b} \right) \times \left( \frac{a}{b} \right) \times \left( \frac{a}{b} \right) \times \left( \frac{a}{b} \right) = \frac{a}{b} \times \frac{a}{b} \times \frac{a}{b} \times \frac{a}{b} = \frac{a^4}{b^4}
\]
\[
\left( \frac{a}{b} \right)^n = \left( \frac{a}{b} \right) \times \left( \frac{a}{b} \right) \times \left( \frac{a}{b} \right) \cdots \left( \frac{a}{b} \right) = \frac{a^n}{b^n}
\]

(9) Exercises - power operations:
1) \[ \left( \frac{3m^2n^2}{2mn} \right)^2 \times \left( \frac{4mn}{9m^2n^2} \right)^3 \]
2) \[ \frac{y - x}{x + y} \times \left( \frac{x - y}{y + x} \right)^2 \]

As discussed in the Reviewing component (Section 4.1), WK commonly let her students solve
the problems as she walked around to find what mistakes they were making. She said in the interview that mistakes indicated to her the students were not ready for learning the ‘new’, because the ‘old’ knowledge was not yet solidly anchored. Thus, after Bridging via Variation of two problems of common fractions (Phase 2) and introduction of the rules for multiplication and division of fractions (Phase 3) took place, she proceeded to the ‘new’ learning.

Unlike LX’s gradual sequence, however, phases 4, 5, 6, and 7 in WK’s sequence engaged students in solving problems that seemed to ‘jump’ from straightforward operations on variables to factoring algebraic expressions, and back to factoring via operations on variables (albeit the change to division, which at that point did not seem to constitute a ‘demanding point’ for them). Her lack of experience in sequencing was particularly evident in the placement of problems where students were asked to find mistakes (Phase 4). These two examples required a comprehensive application of (a) the principle of multiplication and division of algebraic fractions, (b) simplification, and (c) sign conventions. For students to find the mistakes in the two given problems they must have known how to solve correctly and how to explain what/why was wrong in the given solutions. This paradoxical situation could be partly rectified when those students, and WK, worked on the solutions and made explicit the correct process (solve on their own, compare to peers’ solutions, consolidate via comparison to WK’s solution, and follow the correction of mistakes on the board). Like LX, when WK was asked why she presented the mistakes in advance she explained that this would help students to have a deeper impression of the solution process needed. However, in WK’s class such learning opportunities seemed to be curbed, because she used only one basic and one focal point, whereas the other problems were instances of the demanding point.

Concerning ST’s lesson, an interesting feature of her teaching with Variation of problems was the introduction of a realistic problem for teaching the demanding point. As she said in the interview, this was as an advanced problem and was thus used toward the end of her lesson. Her
teaching was quite consistent with LX’s work in terms of sequencing tasks, engaging students in individual work before seeing peers’ solutions and then teacher’s work, and the use of students’ mistakes to promote learning. However, to compensate for the extra time needed to solve the realistic problem, ST had skipped the Reviewing component in both of the observed lessons and moved directly to Bridging.

The task sequence in ST’s lesson and her explanations of why she used it (presented in the Bridging sub-section, 4.2) provide a telling story. Just like LX and SZ, she was an expert teacher who said she used Variation of problems consistently. Like them, she also appeared to focus on figuring out what her students knew so she could build on their understanding, including their mistakes, to establish the ‘old’ as anchor for linking the ‘new’. This linking, she emphasised in the interview, was taking place via students’ solutions to problems that she sequenced to be gradually more challenging, hence spanning the three difficulty levels.
Appendix B:

A Case of Summarising Novice (JC)

This appendix presents an additional case of how Summarising was used by a novice teacher (JC). To recall, JC was the teacher with the strongest mathematical background of all participating teachers in this study (master’s degree in mathematics). However, his teaching indicated less confidence in students’ ability to come up with what he intended for their learning. Unlike LX and ST, in Summarising he showed more inclination toward leading the students’ thinking, for example by asking more direct questions such as, “What are the features of these types of problems that we can use these [cubic sum/difference] formulas?” He also appeared to focus on specific goals and try to control the flow of classroom interactions so he could lead the students’ thinking in the direction he intended. In spite of the relatively more controlled approach, by letting students do the Summarisation JC seemed to trigger their reflection and open up the possibility for them to express different ideas at the end of the lesson. Excerpt 4.4.3-a below shows how JC did Summarisation at the end of one of the observed lessons.

Excerpt 4.4.3-a (Teacher JC 2009-05-15)

42:21 T: Let’s have a brief review of today’s lesson. What have we learned? We have learned the formulas for cubic sum and cubic difference, and how to apply these formulas to do factoring. Generally, what are the properties of these types of problems that allow us to use these formulas? S18?

42:58 S18: They include cubic (exponents) and only two items.

43:02 (The bell rings for lesson end.)

43:05 T: So it has to do with cubic exponents and generally has two items. It can be several items put together with a bracket, and it can also be counted as two items, and each item is cubic. S19, do you have different ideas?

43:23 S19: I think it should be more than cubic.

43:36 T: So it does not necessarily have to be cubic, right? It can be transformed into cubic in many different cases, such as take those items, and look at them as a whole, or extract the common factor.

The data in Excerpt 4.4.3-a show two aspects of JC’s use of Summarising. First, while being pretty direct in his first question (Line 42:21), he emphasised the rules used for solving the problems. Thus, like in other lesson components in JC’s teaching, the level of understanding
required to provide the summary seemed to fit with high-achieving students. For low-achieving students, however, it would be not only difficult to generate such a summary but also to make sense of the explanation provided by peers. Second, like LX (and all teachers in whose class the bell rang before the lesson concluded), JC continued the learning after the bell. As explained above, this seemed rooted in the cultural high respect for learning, knowledge, and thus teachers.

In the interview after this observed lesson, JC was asked why summarising is important. Excerpt 4.4.3-b shows his response.

Excerpt 4.4.3-b (Teacher JC Intrv 2009-05-15)
I think summarising is very important. Many experienced teachers told me you should summarise at the end of every lesson. Initially, I just do it at random or casually, but now I feel the importance of summarising. Personally, I always like to add in more challenging problems in the lesson, and dig deeply into the topic, so I frequently forget summarising. Sometimes after I did lots of exercises in the lesson, and I would look back, and give some feedback, and reflect on the lesson, what kind of methods we have learned today to solve what types of problems, and I feel this is actually important. In fact, I always reflect on my teaching and often conclude I didn’t do well in summarising at the end of the lesson. Sometimes, if I realised this, I probably will do the summarising; Other times, if I don’t have enough time, I will simply ignore the summary part. For example, after I finished introducing the generic quadratic trinomial algebraic like $x^2+px+q$ with the coefficient of the quadratic term being 1, and the other category like $ax^2+bx+c$ where the coefficient for the quadratic terms is not 1, it is better if I had a brief summary of the features and rules after the introduction of these two categories, and the teaching would be more effective…The summary doesn’t have to be at the end of the lesson, the summary can be in the middle of the lesson when I finish introducing one category and start introducing another. I can have another summary at the end of the lesson. (Emphases added)

The data in Excerpt 4.4.3-b show three important aspects of how JC seemed to think about Summarising. First, he appeared to think that the content of Summarising is the features, rules, and the underlying method (Fang Fa, 方法) at the core of a lesson. Furthermore, he seemed flexible for using Summarising so it flows with the content and student learning (hence, also during a lesson). His flexibility seemed linked to the importance he saw in Summarising: at different phases of the lesson it might serve different content purposes, while always giving students an opportunity to reflect on and consolidate their learning of the day’s mathematics. Second, the importance of Summarising did not yet seem to be his own idea, integrated within a system of pedagogical understandings. Rather, he learned it from and followed the advices of
senior teachers who had the responsibility to mentor him as a novice teacher. Thus, as JC said, he might forget, or consider other aspects of teaching to be more important, and realise only in retrospect that he did not summarise the essence of ‘new’ learning. Third, as a mentored, novice teacher JC humbly and honestly shared his struggle to become an effective teacher and improve his pedagogical understanding and practice. The data suggest that he learns via both reflecting on his own teaching and listening to senior teachers’ suggestions. His reference to time constraints seemed typical of a novice. What for the expert teachers came as natural conclusion of their well-organised lessons, to him seemed in conflict with the ‘work (problems) he had to cover’. Although he was clearly aware of the importance of summarising, other goals seemed to compete with this component of a lesson instead of being mutually supportive.

When JC was asked why he continued teaching after the bell rang, his response echoed that of LX—focusing on the sense of responsibility he felt for students’ learning. He said he would feel guilty (to students) if he did not finish demonstrating the content as planned. In his view, it appeared that students show their respect to the teacher by sitting quietly until the teacher dismisses the class. Partly, the teacher’s and students’ behaviour could reflect the researcher’s presence, but the essence of the teacher’s response seems to indicate his rationale.