Closed-form approximations for option prices in stochastic volatility models via the mixing solution

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Abstract

We consider the classical European option pricing problem in a general stochastic volatility framework with time-dependent parameters. It is possible to express the price of a European option as the expectation of a functional of the integrated variance process. In particular, this functional itself is similar to that of a Black-Scholes formula, which possesses many well studied properties. This is referred to as the *mixing solution* methodology by Hull and White [The Journal of Finance, 42, 1987]. From there, it is possible to utilise expansion techniques to approximate the option price in a closed-form manner. We achieve this using two different types of approaches, one contingent on change of measure techniques, the other on Malliavin calculus machinery. As an error term is generated through the expansion, we explore the possibility of achieving sharp bounds on it. Furthermore, we investigate the numerical implementation of our approximation formulas in application. We devise a fast calibration scheme for both approximation procedures. In addition, for the change of measure methodology, we carry out a numerical error and sensitivity analysis for the Heston and GARCH models. For the Malliavin calculus methodology, we perform a numerical error and sensitivity analysis in the stochastic Verhulst model, which possesses a quadratic drift term. In all cases, we find that the numerical errors are well within the range for application purposes. In addition, with respect to certain parameter ranges, we find that the approximations exhibit behaviour that we expect.

Declaration

This thesis contains no material which has been accepted for the award of any other degree or diploma at any university or equivalent institution and that, to the best of my knowledge and belief, this thesis contains no material previously published or written by another person, except where due reference is made in the text of the thesis.

Publications during enrolment

The bulk of this thesis is comprised of two academic manuscripts that have been written during my candidature.

- Chapter 3 and Chapter 4 are based off joint work with Nicolas Langrené. The manuscript, titled 'Closed-form expansions with respect to the mixing solution for option pricing under stochastic volatility', has been submitted to the journal of Finance and Stochastics. A version is currently on the arXiv under the same name.

- Chapter 5 and Chapter 6 are based off joint work with Nicolas Langrené. Currently, a draft of this manuscript is in progress, titled 'Closed-form expansions with respect to the mixing solution for option pricing via Malliavin calculus'. We intend to have this manuscript on the arXiv within the next month.

For my family

Acknowledgements

Contents

Chapter 1

Introduction

The purpose of this thesis is to investigate methodology pertaining to the closed-form approximation of the price of a European option (here on in referred to as simply an option) in a general framework of stochastic volatility models with time-dependent parameters. To achieve this, we express the price of an option as the expectation of a functional of the underlying integrated volatility or variance process. Conveniently, this functional is similar to a Black-Scholes formula, which is well studied in the literature. This methodology is referred to as the *mixing solution* by Hull and White [38]. In view of this, we are able to appeal to expansion techniques in order to obtain an explicit expression for the approximation of the option price. In addition, our general framework consists of a volatility or variance process driven by arbitrary drift and diffusion coefficients with time-dependent parameters, which satisfy some regularity conditions. Due to the approximation procedure, an error term is induced. We give mathematical results on bounding this error, as well as numerical analyses. This thesis not only details the derivation of explicit formulas for approximating option prices, but also their numerical implementation in application. For both procedures, we devise a fast calibration scheme under the assumption of piecewise-constant parameters.

This thesis is comprised of seven chapters:

1. Chapter 1 and Chapter 2 are the introduction and background chapters respectively. We advise that all readers refer to the notation section (Section 1.1) so as to familiarise themselves with the notation used in this thesis.

2. Chapter 3 and Chapter 4 are based on our paper 'Closed-form expansions with respect to the mixing solution for option pricing under stochastic volatility'. This work details the approximation of a put option price in a variety of stochastic volatility models with time-dependent parameters via the use of change of measure techniques. Chapter 3 involves deriving the explicit representation of the approximation formula and error term, as well as bounding the error term. Chapter 4 is dedicated to numerical implementation and analysis of the approximation formula in the Heston and GARCH models. This paper has been submitted to the journal of Finance and Stochastics and is currently on the arXiv.

3. Chapter 5 and Chapter 6 are based on our draft academic manuscript 'Closed form expansions with respect to the mixing solution for option pricing via Malliavin calculus'. This work details the approximation of a put option price in a general framework of stochastic volatility models with time-dependent parameters through the use

of Malliavin calculus machinery. Chapter 5 involves obtaining the explicit expression for the approximation formula and error term, as well as obtaining a bound on the latter. Chapter 6 is dedicated to numerical implementation and analysis of our approximation formula in the stochastic Verhulst model.

4. Chapter 7 gives concluding remarks for the two expansion procedures explored in thesis. We also comment on further work.

1.1 Notation

The purpose of this section is to fix notation that will be used throughout the rest of the thesis. The majority of notation used in this thesis is common throughout the fields of stochastic analysis and derivative pricing.

1.1.1 Spaces

1. $\mathbb{R} = (-\infty, \infty)$.

2. $\mathbb{R}_+ = (0, \infty)$.

3. $\mathbb{R}_- = (-\infty, 0)$.

4. $\overline{\mathbb{R}} = [-\infty, \infty]$.

5. $\overline{\mathbb{R}}_+ = (0, \infty]$.

6. $\overline{\mathbb{R}}_- = [-\infty, 0)$.

7. $\mathbb{N} = \{1, 2, \dots\}$.

8. $\mathbb{Z} = \{\dots, -1, 0, 1, \dots\}$.

9. $C^n(A; B)$ denotes the space of functions with domain A and codomain B which are n times differentiable over A with a continuous n-th derivative.

10. $C^\infty(A; B)$ denotes the space of functions with domain A and codomain B which are infinitely differentiable (or *smooth*) over A.

1.1.2 Analysis

Let $f : [0, \infty) \to \overline{\mathbb{R}}$ be an extended real valued function.

1. We denote by $\underline{f}(t) := \inf_{u \leq t}\{f(u)\}$ and $\overline{f}(t) := \sup_{u \leq t}\{f(u)\}$ its running infimum and supremum respectively.

2. $f_-(x) := \max(-f(x), 0)$ and $f_+(x) := \max(f(x), 0)$ denote the negative and positive part of f respectively.

3. $x \wedge y := \min(x, y)$ and $x \vee y := \max(x, y)$.

4. We say that $f = o(g)$ (read: 'f is little-oh of g') if $\lim_{x \to \infty} \frac{f(x)}{g(x)} = 0$.

5. We say that $f = o_0(g)$ (read: 'f is little-oo of g') if $\lim_{x \downarrow 0} \frac{f(x)}{g(x)} = 0$.

6. Let $(E_1, \mathcal{E}_1)$ and $(E_2, \mathcal{E}_2)$ be measurable spaces. We write $g : E_1/\mathcal{E}_1 \to E_2/\mathcal{E}_2$ to denote a function g with domain E_1 and codomain E_2, which is measurable in the sense that any preimage of a measurable set in $\mathcal{E}_2$ under g is a measurable set in $\mathcal{E}_1$.

7. Let $g : X \to Y$. We say that g obeys property P on $(A; B)$ if it obeys property P on $A \subseteq X$, and the image of A under g is B.

1.1.3 Probability and stochastic analysis

1. A stochastic process $\{X_t : t \in [0, T]\}$ will be denoted by (X_t) or simply X, when the context is clear.

2. $\mathbb{Q}$ will refer to a risk-neutral measure rather than the set of rationals.

3. B and W will refer to standard one-dimensional Brownian motions.

4. For a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $L^p := L^p(\Omega, \mathcal{F}, \mathbb{P})$ denotes the space of equivalence classes of random variables with finite L^p-norm, where the L^p norm is given by $\|\cdot\|_p := (\mathbb{E}|\cdot|^p)^{1/p}$ for $p \geq 1$. In particular, $\|\cdot\|_1 \equiv \|\cdot\|$.

5. For two stochastic processes X and Y, their quadratic covariation from time 0 to t is denoted by $\langle X, Y \rangle_t$. None of the stochastic processes in this thesis occur with jumps, and so the need to distinguish between sharp bracket and square bracket processes is superfluous.

6. Suppose M is a martingale on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$. To distinguish the filtration and probability measure M exists with respect to, we will simply refer to it as a $((\mathcal{F}_t), \mathbb{P})$ martingale. For a sub/super/semi/local martingale or Brownian motion, we will do the same.

7. We write $X \sim \mathcal{N}(\mu, \sigma^2)$ to signify that X is normally distributed with mean μ and variance σ^2.

8. We write $Y \sim \mathcal{LN}(\mu, \sigma^2)$ to signify that Y is log-normally distributed, such that $\ln(Y)$ has mean μ and variance σ^2.

9. Confusingly but common to the field, $\mathcal{N}$ will also denote the standard normal distribution function. That is, $\mathcal{N}(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} \mathrm{d}u$.

10. ϕ denotes the standard normal density function. That is, $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$.

1.2 Elements of the general theory

In this section, we present some well known results from the general theory of stochastic calculus and derivative pricing. The purpose of this section is to list results that are pertinent to the rest of this thesis. We make no claims that this is an exhaustive nor comprehensive treatment of either of these subjects. As these are standard theorems from the literature, we do not present the proofs. However, we do provide references to the sources of the original proofs, alongside others. In addition, we refer the reader to the works of Klebaner [41], Shreve [59], Rogers and Williams [55] and Cherny and Engelbert [17] for a comprehensive treatment on these topics.

1.2.1 Stochastic calculus

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$ be a filtered probability space. In the following, consider the one-dimensional SDE

$$\mathrm{d}X_t = \mu(t, X_t)\mathrm{d}t + \sigma(t, X_t)\mathrm{d}B_t, \quad X_0 = x_0, \tag{1.1}$$

where $\mu, \sigma : [0, T] \times \mathbb{R} \to \mathbb{R}$ are both measurable and B is a $((\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$ Brownian motion. Many of the following definitions and results hold for multidimensional SDEs. However, we will only require results concerning one-dimensional SDEs in this thesis.

Definition 1.2.1 (Weak solution). A weak solution to eq. (1.1) is a pair of processes $(\tilde{X}, \tilde{B})$ on a filtered probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{0 \leq t \leq T}, \tilde{\mathbb{P}})$ where $\tilde{B}$ is a $((\tilde{\mathcal{F}}_t)_{0 \leq t \leq T}, \tilde{\mathbb{P}})$ Brownian motion and $\tilde{X}$ is adapted to $(\tilde{\mathcal{F}}_t)_{0 \leq t \leq T}$ such that

$$\tilde{X}_t = x_0 + \int_0^t \mu(u, \tilde{X}_u)\mathrm{d}u + \int_0^t \sigma(u, \tilde{X}_u)\mathrm{d}\tilde{B}_u.$$

Definition 1.2.2 (Strong solution). A strong solution to eq. (1.1) is a pair of processes (X, B) on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$ where X is adapted to $(\mathcal{F}_t)_{0 \leq t \leq T}$ such that

$$X_t = x_0 + \int_0^t \mu(u, X_u)\mathrm{d}u + \int_0^t \sigma(u, X_u)\mathrm{d}B_u.$$

Remark 1.2.1. A solution to eq. (1.1) can refer to either a weak or strong solution.

Definition 1.2.3 (Solution unique in law). Suppose $(X^{(1)}, B^{(1)})$ and $(X^{(2)}, B^{(2)})$ are solutions to eq. (1.1). Then the two solutions are unique in law if the finite dimensional distributions of $X^{(1)}$ and $X^{(2)}$ agree.

Definition 1.2.4 (Pathwise unique solution). Suppose $(X^{(1)}, B^{(1)})$ and $(X^{(2)}, B^{(2)})$ are solutions to eq. (1.1) on the same filtered probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{0 \leq t \leq T}, \tilde{\mathbb{P}})$. Then the two solutions are pathwise unique if

$$\tilde{\mathbb{P}}(X_t^{(1)} = X_t^{(2)}, \forall t \in [0, T]) = 1.$$

Theorem 1.2.1 (Sufficient conditions for pathwise uniqueness). Suppose there exists a solution to eq. (1.1). Then the solution is pathwise unique if the following both hold:

1. $|\mu(t, x) - \mu(t, y)| \leq \kappa(|x - y|)$ uniformly in t, where $\kappa : \mathbb{R}_+ \to \mathbb{R}_+$ is convex, strictly increasing and satisfies $\int_0^\varepsilon (\kappa(u))^{-1} \, \mathrm{d}u = \infty$ for some $\varepsilon > 0$.

2. $|\sigma(t, x) - \sigma(t, y)| \leq \rho(|x - y|)$ uniformly in t, where $\rho : \mathbb{R}_+ \to \mathbb{R}_+$ is strictly increasing and satisfies $\int_0^\varepsilon (\rho(u))^{-2} \, \mathrm{d}u = \infty$ for some $\varepsilon > 0$.

Proof. See Yamada et al. [65]. $\square$

Corollary 1.2.1. Suppose there exists a solution to eq. (1.1). Then the solution is pathwise unique if the following both hold:

1. μ is Lipschitz continuous in x, uniformly in t.

2. σ is Hölder continuous in x of order $\geq 1/2$, uniformly in t.

Proof. This is just a direct consequence of Theorem 1.2.1 with $\kappa(u) = u$ and $\rho(u) = u^\alpha$ for $\alpha \geq 1/2$. $\square$

Theorem 1.2.2 (Yamada-Watanabe)**.** Suppose there exists a weak solution to eq. (1.1). In addition, assume the solution is pathwise unique. Then the solution is strong.

Proof. See Yamada et al. [65]. □

Theorem 1.2.3 (Girsanov's theorem for Brownian motion)**.** Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$ be a filtered probability space and B a $((\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$ Brownian motion. Let H be an arbitrary $(\mathcal{F}_t)_{0 \leq t \leq T}$ adapted process and define $M_t := \int_0^t H_u \mathrm{d}B_u$. Denote by $\mathfrak{z}^{(M)}$ the Doléans-Dade exponential of M. That is

$$\mathfrak{z}_t^{(M)} := e^{\int_0^t H_s \mathrm{d}B_s - \frac{1}{2} \int_0^t H_s^2 \mathrm{d}s}.$$

Suppose $\mathfrak{z}^{(M)}$ is a true martingale. Then it defines a Radon-Nikodym derivative, meaning

$$\tilde{\mathbb{P}}(A) := \int_A \mathfrak{z}_T^{(M)}(\omega) \mathbb{P}(\mathrm{d}\omega), \quad A \in \mathcal{F}$$

is a probability measure on $(\Omega, \mathcal{F})$. Then the following holds:

1. $\tilde{\mathbb{P}}$ is equivalent to $\mathbb{P}$.

2. The process $\tilde{B}$ defined as $\tilde{B}_t := B_t - \int_0^t H_u \mathrm{d}u$ is an $((\mathcal{F}_t)_{0 \leq t \leq T}, \tilde{\mathbb{P}})$ Brownian motion.

Proof. See Girsanov [32]. □

Remark 1.2.2. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$ be a filtered probability space and M a $((\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$ continuous local martingale with $M_0 = 0$. Denote by $\mathfrak{z}^{(M)}$ its Doléans-Dade exponential. That is

$$\mathfrak{z}_t^{(M)} := e^{M_t - \frac{1}{2} \langle M, M \rangle_t}.$$

As $\mathfrak{z}^{(M)}$ is a positive local martingale, then it is a supermartingale. Sufficient conditions for $\mathfrak{z}^{(M)}$ to be a true martingale are:

1. *Novikov's condition:*

   $\mathbb{E}e^{\frac{1}{2} \langle M, M \rangle_T} < \infty.$

2. *Kazamaki's condition:*

   $\sup_{t < T} \mathbb{E}e^{\frac{1}{2} M_t} < \infty.$

Notice that Novikov's condition implies Kazamaki's condition, and so the latter is a less stringent condition.

Theorem 1.2.4 (Beneš' conditions)**.** Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$ be a filtered probability space and B an $((\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$ Brownian motion. Define the continuous local martingales

$$M_t := \int_0^t \sigma(B_u) \mathrm{d}B_u,$$

$$N_t := \int_0^t \beta_u(B_{[0,u]}) \mathrm{d}B_u,$$

where $\sigma : \mathbb{R} \to \mathbb{R}$ and β_u is a functional of B up to time u. Specifically, $\beta : [0, T] \times C([0, T]; \mathbb{R}) \to \mathbb{R}$. Denote by $\mathfrak{z}^{(M)}$ and $\mathfrak{z}^{(N)}$ the Doléans-Dade exponentials of M and N respectively. That is

$$\mathfrak{z}_t^{(M)} := e^{\int_0^t \sigma(B_u) \mathrm{d}B_u - \frac{1}{2} \int_0^t \sigma^2(B_u) \mathrm{d}u},$$

$$\mathfrak{z}_t^{(N)} := e^{\int_0^t \beta_u(B_{[0,u]}) \mathrm{d}B_u - \frac{1}{2} \int_0^t \beta_u^2(B_{[0,u]}) \mathrm{d}u}.$$

1. Suppose $\sigma^2(y) \leq C(1+y^2)$. Then $\mathfrak{z}^{(M)}$ is a true martingale, or equivalently $\mathbb{E}(\mathfrak{z}_T^{(M)}) = 1$.

2. Suppose $\beta_u^2(y_{[0,u]}) \leq C(1+\sup_{s\leq u} y_{[0,s]}^2)$. Then $\mathfrak{z}^{(N)}$ is a true martingale, or equivalently $\mathbb{E}(\mathfrak{z}_T^{(N)}) = 1$.

Proof. See Klebaner and Liptser [42]. $\qquad\square$

1.2.2 Black-Scholes formulas and results

In the following, let $K, k, (r_t^d)_{0\leq t\leq T}$ and $(r_t^f)_{0\leq t\leq T}$ correspond to the strike, log-strike, time-dependent domestic interest rate and foreign interest rate respectively. Furthermore, assume $K > 0, k \in \mathbb{R}$ and $r_t^d, r_t^f \in [0,1]$ for all $t \in [0,T]$. Proofs of Proposition 1.2.1, Proposition 1.2.2 and Proposition 1.2.3 are standard results, and are thus not included.

Definition 1.2.5 (Black-Scholes put formula for spot, strike and integrated variance)**.** The Black-Scholes put formula parametrised for spot, strike and integrated variance is defined as

$$\text{Put}_{\text{BS}}(x,y) := Ke^{-\int_0^T r_t^d \mathrm{d}t}\mathcal{N}(-d_-) - xe^{-\int_0^T r_t^f \mathrm{d}t}\mathcal{N}(-d_+),$$

where

$$d_{\pm}(x,y) := d_{\pm} := \frac{\ln(x/K) + \int_0^T (r_t^d - r_t^f)\mathrm{d}t}{\sqrt{y}} \pm \frac{1}{2}\sqrt{y}.$$

Proposition 1.2.1. Suppose that S solves the SDE

$$\mathrm{d}S_t = (r_t^d - r_t^f)S_t\mathrm{d}t + \sqrt{y/T}S_t\mathrm{d}W_t, \quad S_0 = x.$$

Its explicit pathwise unique strong solution is given by

$$S_T = x\exp\left(\int_0^T (r_u^d - r_u^f)\mathrm{d}u - \frac{1}{2}y + \sqrt{y/T}W_T\right).$$

Then

$$e^{-\int_0^T r_u^d \mathrm{d}u}\mathbb{E}(K - S_T)_+ = \text{Put}_{\text{BS}}(x,y).$$

Definition 1.2.6 (Black-Scholes put formula for log-spot, log-strike and integrated variance)**.** The Black-Scholes put formula parametrised for log-spot, log-strike and integrated variance is defined as

$$P_{\text{BS}}(x,y) := e^k e^{-\int_0^T r_t^d \mathrm{d}t}\mathcal{N}(-d_-^{\ln}) - e^x e^{-\int_0^T r_t^f \mathrm{d}t}\mathcal{N}(-d_+^{\ln}),$$

where

$$d_{\pm}^{\ln}(x,y) := d_{\pm}^{\ln} := \frac{x - k + \int_0^T (r_t^d - r_t^f)\mathrm{d}t}{\sqrt{y}} \pm \frac{1}{2}\sqrt{y}.$$

Proposition 1.2.2. Suppose that X solves the SDE

$$\mathrm{d}X_t = \left((r_t^d - r_t^f) - \frac{1}{2}\frac{y}{T}\right)\mathrm{d}t + \sqrt{y/T}\mathrm{d}W_t, \quad X_0 = x.$$

Then

$$e^{-\int_0^T r_u^d \mathrm{d}u}\mathbb{E}(e^k - e^{X_T})_+ = P_{\text{BS}}(x,y).$$

Lemma 1.2.1. $\mathrm{Put}_{\mathrm{BS}}$ is smooth on $(\mathbb{R}_+^2; \mathbb{R})$ and P_{BS} is smooth on $(\mathbb{R} \times \mathbb{R}_+; \mathbb{R})$.

Proof. This is a clear consequence of the fact that the composition of smooth functions is again smooth. $\qquad\square$

Proposition 1.2.3 (Put-Call parities)**.** The Black-Scholes call option formulas analogous to $\mathrm{Put}_{\mathrm{BS}}$ and P_{BS} are given by

$$\mathrm{Call}_{\mathrm{BS}}(x, y) := xe^{-\int_0^T r_t^f \mathrm{d}t}\mathcal{N}(d_+) - Ke^{-\int_0^T r_t^d \mathrm{d}t}\mathcal{N}(d_-),$$
$$C_{\mathrm{BS}}(x, y) := e^x e^{-\int_0^T r_t^f \mathrm{d}t}\mathcal{N}(d_+^{\ln}) - e^k e^{-\int_0^T r_t^d \mathrm{d}t}\mathcal{N}(d_-^{\ln}),$$

respectively. Then the Put-Call parities are the following equations:

$$\mathrm{Call}_{\mathrm{BS}}(x, y) - \mathrm{Put}_{\mathrm{BS}}(x, y) = xe^{-\int_0^T r_t^f \mathrm{d}t} - Ke^{-\int_0^T r_t^d \mathrm{d}t},$$
$$C_{\mathrm{BS}}(x, y) - P_{\mathrm{BS}}(x, y) = e^x e^{-\int_0^T r_t^f \mathrm{d}t} - e^k e^{-\int_0^T r_t^d \mathrm{d}t}.$$

For our numerical experiments we will require the notion of strikes corresponding to the value of a put option Delta. This is done by solving the Black-Scholes put option Delta (parametrised for spot, volatility and strike) for the strike value.

Remark 1.2.3. Denote by Δ the Black-Scholes put option Delta, where $\Delta \in (0, -1)$. The strike value corresponding to the value of Δ is given by

$$K_\Delta = S_0 e^{\sigma\sqrt{T} + \frac{1}{2}\sigma^2 T} e^{\int_0^T (r_t^d - r_t^f)\mathrm{d}t} \exp\left(\mathcal{N}^{-1}\left(-\Delta e^{\int_0^T r_t^f \mathrm{d}t}\right)\right),$$

where $\mathcal{N}^{-1}$ denotes the normal quantile function. Notice all quantities are known except the volatility σ. This will be estimated by the initial value of the volatility process in the stochastic volatility model.

1.2.3 Well known inequalities

We will make use of the following inequalities common in analysis and stochastic analysis. Let X and Y be random variables, M a continuous local martingale null at $t = 0$ and $f : \mathbb{R} \to \overline{\mathbb{R}}$ an extended real valued function.

1. *Cauchy-Schwarz:*

   $\|XY\| \le \|X\|_2 \|Y\|_2.$

2. *Hölder:*

   $\|XY\| \le \|X\|_p \|Y\|_q$ for $1/p + 1/q = 1$, where $p, q \ge 1$.

3. *Minkowski:*

   $\|X + Y\|_p \le \|X\|_p + \|Y\|_p$ for $p \ge 1$.

4. *Jensen:*

   For convex g and concave h, $g(\mathbb{E}(X)) \le \mathbb{E}(g(X))$ and $\mathbb{E}(h(X)) \le h(\mathbb{E}(X))$.

5. *Burkholder-Davis-Gundy:*

   $c_p \mathbb{E}\left(\langle M, M\rangle_t^{p/2}\right) \le \mathbb{E}\left(\left(\overline{M}_t\right)^p\right) \le C_p \mathbb{E}\left(\langle M, M\rangle_t^{p/2}\right)$ for $p \in (0, \infty)$ and some constants c_p and C_p solely depending on p.

6. $\left(\int_0^t |f(s)|\mathrm{d}s\right)^p \le t^{p-1} \int_0^t |f^p(s)|\mathrm{d}s$ for $p \ge 1$.

1.2.4 Taylor's theorem

The expansion procedures used in this thesis will rely on Taylor expansions, and thus Taylor's theorem will be paramount. We list it here predominantly to fix the notation.

Theorem 1.2.5. Let $A \subseteq \mathbb{R}, B \subseteq \mathbb{R}$ and $f : A \to B$ be a C^3 function in a closed interval about the point $a \in A$. Then the Taylor series of f around the point a is given by

$$f(x) = f(a) + f'(a)(x - a) + \frac{1}{2}f''(a)(x - a)^2 + R(x),$$

where

$$R(x) = \frac{1}{2}\int_a^x (x - u)^2 f'''(u)\mathrm{d}u = \frac{1}{2}(x - a)^3 \int_0^1 (1 - u)^2 f'''(a + u(x - a))\mathrm{d}u.$$

Theorem 1.2.6. Let $A \subseteq \mathbb{R}^2, B \subseteq \mathbb{R}$ and $g : A \to B$ be a C^3 function in a closed ball about the point $(a, b) \in A$. Then the Taylor series of g around the point (a, b) is given by

$$g(x, y) = g(a, b) + g_x(a, b)(x - a) + g_y(a, b)(y - b)$$
$$+ \frac{1}{2}g_{xx}(a, b)(x - a)^2 + \frac{1}{2}g_{yy}(a, b)(y - b)^2 + g_{xy}(a, b)(x - a)(y - b) + R(x, y),$$

where

$$R(x, y) = \sum_{|\alpha|=3} \frac{|\alpha|}{\alpha_1!\alpha_2!}E_\alpha(x, y)(x - a)^{\alpha_1}(y - b)^{\alpha_2}$$

$$E_\alpha(x, y) = \int_0^1 (1 - u)^2 \frac{\partial^3}{\partial x^{\alpha_1}\partial y^{\alpha_2}}g(a + u(x - a), b + u(y - b))\mathrm{d}u,$$

with $\alpha := (\alpha_1, \alpha_2)$ and $|\alpha| := \alpha_1 + \alpha_2$.

Chapter 2

Background

Financial derivatives are contracts whose values are contingent on an underlying asset. Amongst the myriad of such contracts, the European put option is a classic example, which gives the owner the right but not the obligation to sell the underlying asset at a predetermined price and date. A natural question that arises is, at its conception, what is the price of this contract? This is the question that the area of derivative pricing is devoted to answering.

2.1 Derivative pricing

Before one can consider establishing the price of a derivative, a model for the underlying asset needs to be built. Through his thesis dissertation in 1900, Bachelier [6] was the first in the literature to propose a model for an asset price process driven by Brownian motion. Specifically, this process corresponded to the arithmetic Brownian motion process. As the marginals of a such a process are normally distributed, this implies that the process takes negative values with positive probability. Clearly, such a feature is not desirable in a model for an asset price. Nonetheless, the idea to model the dynamics of an asset price process driven by Brownian motion was novel and paved the path for the future of financial modelling. Samuelson [56] in 1965 rediscovered the work of Bachelier, extending his ideas by proposing that log-returns of the asset price process should be normally distributed, or equivalently, that returns are log-normally distributed. This results in the asset price being governed by a geometric Brownian motion process, guaranteeing its positivity.

The influential paper by Black and Scholes [10] in 1973 expanded upon Samuelson's idea. Here Black and Scholes assume a geometric Brownian motion for the asset price process, and in addition invoke a dynamic hedging strategy in order to price a European option. Specifically, this involves constructing a replicating portfolio whose sensitivities[1] with respect to certain parameters perfectly offset the same sensitivities of the option. Such a replicating portfolio is called a dynamic hedge, and the corresponding procedure is called dynamic replication. This in turns leads to the option price being expressed as the solution to a PDE. Black and Scholes then solve this PDE explicitly, yielding their famous Black-Scholes formula. Indeed, the concept of dynamic replication was such a significant contribution to the field of derivative pricing that it resulted in Black and Scholes being awarded the Nobel prize in economics. Although intuitively straightforward, dynamic replication leads to derivative prices being expressed as solutions to PDEs. It is well known

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[1]Sensitivity of a portfolio with respect to some parameter is the change in the value of the portfolio over an infinitesimal time period with respect to this arbitrary parameter. Clearly, this is analogous to the partial derivative of the portfolio with respect to the parameter.

that existence of an explicit solution to a PDE is rare. If one moves away from the Black-Scholes framework, there is no guarantee that the induced PDE will be easily solvable.

A turning point for the theory of derivative pricing was the seminal paper by Harrison and Pliska [36] in 1981. Their work lead to the culmination of the fundamental theorems of asset pricing, which made the notions of arbitrage mathematically rigorous and unified the mathematical theory of derivative pricing. The fundamental theorems of asset pricing give sufficient and necessary conditions for when a financial model does not contain an arbitrage opportunity, and also an alternative and often more convenient representation for the price of a derivative. Specifically, the price of derivative can be represented as the expectation of the payoff of the derivative under a risk-neutral measure[2]. Thus, in the field of derivative pricing, one can represent the price of a derivative as either the solution to a PDE or as a risk-neutral expectation of a functional of the underlying asset price process. Evidently, the fields of PDEs and stochastic analysis are both paramount when one considers the area of derivative pricing.

2.2 Volatility smiles and skews: extensions to the classical framework

From the last few decades, empirical evidence has been used to establish that volatility is highly dependent on the strike and maturity of option contracts. In theory, the Black-Scholes model assumes that volatility is constant in all parameters. However, what is observed is that implied volatility[3] is a function of both strike and maturity. By fixing maturity and varying strike, one can observe that implied volatility usually exhibits a shape akin to that of either a smile or a skewed trajectory. This phenomenon is called the volatility smile or skew, clearly an attribute the Black-Scholes model fails to address. In response, there have been a number of frameworks proposed to model the volatility smiles and skews observed in the market. Although chronologically not the first type of framework proposed to explain the smile phenomenon, local volatility models are the simplest extension to the classic modelling framework. They were pioneered in 1994 by the works of Dupire et al. [28], Dupire [27] and are specifically of the form

$$dS_t = r_t dt + \sigma(t, S_t) S_t dW_t, \quad S_0.$$

The novel feature of this model is the volatility function σ, which itself depends on the spot. Dupire showed that under a local volatility model, one can derive a PDE and solve for the volatility function as

$$\sigma^2(T, K) = \frac{\partial_T C + r_t K \partial_K C}{\frac{1}{2} K^2 \partial_{KK} C}, \tag{2.1}$$

where $C(T, K)$ is the price of a European call option with strike K and maturity T.[4] Evidently, the RHS of eq. (2.1) can be rewritten in terms of Black-Scholes implied volatilities. Thus, calibration of a local volatility model from the market observed data is rather straightforward from Dupire's equation.

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[2]A risk-neutral measure is a probability measure equivalent to the real world probability measure, such that the discounted asset price process is a martingale.

[3]Implied volatility is defined as the volatility required to be plugged into the Black-Scholes formula to reproduce an option price.

[4]Although the derivation of eq. (2.1) is relatively straightforward, the focus of this thesis is on stochastic volatility models. We refer the reader to Gatheral [31] chapter 1 for a concise derivation of Dupire's equation.

Stochastic volatility models[5] were introduced a few years before local volatility models
by Hull and White [38] in 1987. Here the volatility itself is a stochastic process possibly
correlated with the spot. For most purposes, the general framework follows the diffusion

$$\mathrm{d}S_t = S_t((r_t^d - r_t^f)\mathrm{d}t + \sqrt{\sigma_t}\mathrm{d}W_t), \quad S_0,$$
$$\mathrm{d}\sigma_t = \alpha(t, \sigma_t)\mathrm{d}t + \beta(t, \sigma_t)\mathrm{d}B_t, \quad \sigma_0,$$
$$\mathrm{d}\langle W, B \rangle_t = \rho_t \mathrm{d}t,$$

or

$$\mathrm{d}S_t = S_t((r_t^d - r_t^f)\mathrm{d}t + V_t \mathrm{d}W_t), \quad S_0,$$
$$\mathrm{d}V_t = \alpha(t, V_t)\mathrm{d}t + \beta(t, V_t)\mathrm{d}B_t, \quad V_0 = v_0,$$
$$\mathrm{d}\langle W, B \rangle_t = \rho_t \mathrm{d}t,$$

where σ and V are called the variance and volatility process respectively. Some classic
models include the Heston and SABR models [37, 35], and there has been extensive research
done on these models in the literature. As opposed to local volatility models, one could
postulate that stochastic volatility models are more realistic, as they capture the random
behaviour of volatility. Hagan et al. [35] argue that local volatility models model the
smile behaviour of volatility ineffectively. This is because as the underlying assets price
increases or decreases, it can be observed empirically that the smile effect moves in the
same direction. Instead, local volatility models tend to display the opposite behaviour.
However, due to the added source of randomness in stochastic volatility models, the market
becomes incomplete[6]. This can be seen via Girsanov's theorem; there exists infinitely many
risk-neutral measures in such a framework. One often has to make some assumption about
the market price of volatility risk in order to choose a unique risk-neutral measure. In some
applications, for example calibration, this is not necessary. The intuition behind this is
the following: as practitioners and investors assume that there exists a unique risk-neutral
measure under which the real market operates, then parameters calibrated from real market
data will uniquely determine the risk-neutral measure.

Evidently, financial practitioners will prefer models which are realistic, such as local or
stochastic volatility models. However, when one describes the movement of asset prices
through these more sophisticated models, this in turn complicates the mathematical prop-
erties of the asset price process. What is paramount to practitioners is not only the realism
of the model, but also the potential the model possesses for use in application. For ex-
ample, obtaining a closed-form price for a derivative involves solving a PDE or calculating
a risk-neutral expectation explicitly. When one uses these more sophisticated models, the
associated PDE or risk-neutral expectation representing a derivative price becomes increas-
ingly complicated to solve or compute. This is because, as is well known, the majority of
PDEs and expectations of functionals of processes cannot be evaluated explicitly. Clearly
this is an issue; although an expression for the price of a derivative is relatively simple to
obtain, representing this expression in terms of elementary functions is usually non-trivial.
For this reason, numerical procedures to evaluate PDEs (numerical PDE solvers) and ex-
pectations (Monte Carlo simulation) have been substantially developed in the literature,
see for example Van der Stoep et al. [62], Andersen [3]. However, these numerical proce-
dures have their own issues. Although under most circumstances they will 'work', meaning
the numerical value obtained through the numerical procedure does indeed represent the

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[5]By which we mean one-factor stochastic volatility models.
[6]Precisely, an incomplete market is one where, for each derivative, there may not exist a dynamic hedge
which replicates it.

arbitrage free price of a derivative[7], the time complexity with implementing the numerical procedure can be many orders of magnitude higher than that of evaluating an explicit expression. Depending on the application, this can be detrimental. For example, if the derivative price needs to be evaluated many times, then having a slow numerical procedure is undesirable. One specific application where this does occur is calibration of a stochastic volatility model. Calibration involves choosing parameters in such a way that the model agrees with observed market data as best as possible. In order to achieve this, usually a least squares algorithm is implemented, in which the derivative price must be computed several times. This is unlike a local volatility model, where Dupire's equation can instead be utilised for calibration.

In a stochastic volatility model, if we assume that the characteristic function of the log-spot is known explicitly, then the option price can be computed quasi-explicitly[8], albeit under the restrictive assumption of constant or piecewise-constant parameters [37, 15, 50]. One class of models where this occurs are the affine models. Affine stochastic volatility models are those such that $\ln \mathbb{E}\left(e^{iu \ln(S_t)}\right)$ is affine in $\ln S_0$, that is, the log of the characteristic function of the log-spot is an affine function in $\ln S_0$ [24, 1]. For example, consider the classical Heston model [37]

$$\mathrm{d}S_t = S_t(r_t\mathrm{d}t + \sqrt{\sigma_t}\mathrm{d}W_t), \quad S_0,$$
$$\mathrm{d}\sigma_t = \kappa(\theta - \sigma_t)\mathrm{d}t + \lambda\sqrt{\sigma_t}\mathrm{d}B_t, \quad \sigma_0,$$
$$\mathrm{d}\langle W, B\rangle_t = \rho\mathrm{d}t.$$

Heston shows that in his model, the price of a put option P with strike K and maturity T can be written as

$$P = Ke^{-rT}(1 - P_2) - S_0(1 - P_1),$$

where

$$P_1 = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathfrak{R}\left(\frac{e^{-iu\ln(K)}\varphi_T(u)}{iu}\right)\mathrm{d}u,$$
$$P_2 = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathfrak{R}\left(\frac{e^{-iu\ln(K)}\varphi_T(u-i)}{iu\varphi_T(-i)}\right)\mathrm{d}u.$$

Here $\varphi_T(u) := \mathbb{E}(e^{iu\ln(S_T)})$ and $\mathfrak{R}$ denotes the real part. Thus, P_1 and P_2 are one-dimensional complex integrals of expressions involving the characteristic function of the log-spot. As long as this characteristic function is known explicitly, then these integrals can be numerically computed. It is tempting to want to perform a fast Fourier transform to calculate P_1 and P_2. However, the integrands are unfortunately singular at $u = 0$. For this reason, Carr and Madan [15] instead establish that the Fourier transform of the call option price (with perturbation term) in the log-strike k is a quasi closed-form expression involving the characteristic function of the log-spot.

Indeed, this representation of the option price is not unique to the affine models; under any stochastic volatility model it is possible to write the option price as an expression involving integrals of the characteristic function of the log-spot. However, obtaining an explicit expression for the characteristic function of the log-spot is usually only possible in the affine models. For non-affine models, the characteristic function of the log-spot is

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[7]To be precise, the approximation of the arbitrage free price, as numerical methods are inherently approximations of the true quantity.

[8]Quasi-explicit meaning in terms of at most one-dimensional complex integrals, where the integrands are explicit functions.

rarely known explicitly, and such a procedure will not be effective. Non-affine models, although usually intractable compared to their affine counterparts, are often far more realistic. This has been shown in a number of studies, see for example Gander and Stephens [30], Christoffersen et al. [18], Kaeck and Alexander [40].

2.3 Closed-form approximations

Closed-form approximations are an alternative methodology for derivative pricing, where the derivative price is approximated by an explicit expression. Since the expression is explicit, the derivative price can be computed rapidly. Additionally, as transform methods are not employed, time-dependent parameters can usually be handled well. In a sense, they combine the best of both worlds. One can utilise a realistic model for their asset price process, and additionally the pricing of a derivative can be executed rapidly. Unfortunately, there are disadvantages to closed-form approximation methodologies. Firstly, obtaining a closed-form approximation is almost always very difficult and tedious, requiring a plethora of sophisticated mathematical machinery. Furthermore, it is usually only possible to obtain closed-form approximations for contracts with simple payoffs, for example European options. Secondly, the approximation procedure often makes some sort of assumption on parameter ranges or magnitudes. For example, one might assume vol-vol is small and perform expansion techniques to yield a closed-form approximation. Intuitively, one would then expect that the approximation formula will result in large errors for when the vol-vol size is large, and in practice this is often observed.

There have been a plethora of results on closed-form approximations in the literature. The most famous closed-form approximation result is due to Hagan et al. [35] in their SABR model. Although closed-form approximation methods did exist before, their approach was arguably the first to garner widespread attention from both academics and practitioners. Specifically, the SABR model is given by

$$\mathrm{d}F_t = \alpha_t F_t^\beta \mathrm{d}W_t, \quad F_0 = f,$$
$$\mathrm{d}\alpha_t = \nu \alpha_t \mathrm{d}B_t, \quad \alpha_0 = \alpha,$$
$$\mathrm{d}\langle W, B \rangle_t = \rho \mathrm{d}t,$$

where $\nu \geq 0$, $\beta \in [0,1]$ and $F_t = e^{\int_t^T r_u \mathrm{d}u} S_t$ is the forward asset price process. By use of singular perturbation expansion techniques, they derive the well known closed-form approximation for implied volatility, denoted by σ_{Im}, as

$$\sigma_{Im}(f,K) = \frac{\alpha}{(fK)^{(1-\beta)/2}\left\{1 + \frac{(1-\beta)^2}{24}\ln^2(f/K) + \frac{(1-\beta)^4}{1920}\ln^4(f/K) + \dots\right\}} \cdot \left(\frac{z}{x(z)}\right)$$

$$\cdot \left\{1 + \left[\frac{(1-\beta)^2}{24}\frac{\alpha^2}{(fK)^{1-\beta}} + \frac{1}{4}\frac{\rho\beta\nu\alpha}{(fK)^{(1-\beta)/2}} + \frac{2-3\rho^2}{24}\nu^2\right]T + \dots,$$

$$z := \frac{\upsilon}{\alpha}(fK)^{(1-\beta)/2}\ln(f/K),$$

$$x(z) := \ln\left\{\frac{\sqrt{1-2\rho z + z^2} + z - \rho}{1-\rho}\right\}.$$

Although this expression for implied volatility seems formidable, it is indeed entirely explicit. The success of the methodology behind the SABR model spurred interest in the

area of closed-form approximations. For example, Lorig et al. [47] derive a general closed-form approximation for the price of an option via a PDE approach, as well as its corresponding implied volatility. In Pagliarani and Pascucci [53], the authors derive the exact implied volatility in a region close to both expiry and money. Alòs [2] show that from the mixing solution[9], one can approximate the option price by decomposing it into a sum of two terms, one being completely correlation independent and the other dependent on correlation. However, neither terms are explicit. Under the Heston model, Sartorelli [57] implement the so-called Edgeworth expansion in order to obtain a closed-form approximation for the option price. However, their method relies on the affine nature of the Heston model, as the characteristic function of the log-spot is required in closed-form. Utilisation of the Edgeworth expansion has also been studied by Fukasawa et al. [29]. Furthermore, Antonelli and Scarlatti [5], Antonelli et al. [4] show that under the assumption of small correlation, an expansion can be performed with respect to the mixing solution, where the resulting expectations can be computed using Malliavin calculus techniques. Benhamou et al. [7, 8] and Gobet and Suleiman [33] develop the so-called proxy model methodology in local volatility models. This involves assuming a proxy model in order to approximate the true model's dynamics. In addition, this proxy model is chosen to be either normal or log-normal so that the price of the derivative in the proxy model is explicit. Thus, the price of a derivative in the true model can then be expressed as the price in the proxy model, with the addition of some correction terms which can be made explicit. In addition, Bompis and Gobet [12] establish higher order approximations for option prices via the proxy model methodology. Moreover, in his thesis dissertation, Bompis [11] extends the proxy model methodology by considering pricing in a stochastic local volatility framework, where the volatility is driven by a square root process. This is further elaborated upon in Bompis and Gobet [14].

2.4 Model framework for this thesis

This thesis focuses on obtaining a closed-form approximation for an option price in a stochastic volatility framework, where parameters are assumed to be time-dependent. Stochastic volatility models usually model either the volatility directly, or indirectly via the variance process. A critical assumption is that volatility or variance has some sort of mean reversion behaviour, and this is supported by empirical evidence, see for example Gatheral [31], Clark [19], Kaeck and Alexander [40]. Specifically, for modelling the variance, this class of stochastic volatility models is given by

$$dS_t = S_t((r_t^d - r_t^f)dt + \sqrt{\sigma_t}dW_t), \quad S_0,$$
$$d\sigma_t = \kappa_t(\theta_t \sigma_t^{\hat{\mu}} - \sigma_t^{\tilde{\mu}})dt + \lambda_t \sigma_t^{\mu}dB_t, \quad \sigma_0,$$
$$d\langle W, B\rangle_t = \rho_t dt,$$

whereas for modelling the volatility, this class is of the form

$$dS_t = S_t((r_t^d - r_t^f)dt + V_t dW_t), \quad S_0,$$
$$dV_t = \kappa_t(\theta_t V_t^{\hat{\mu}} - V_t^{\tilde{\mu}})dt + \lambda_t V_t^{\mu}dB_t, \quad V_0 = v_0,$$
$$d\langle W, B\rangle_t = \rho_t dt,$$

---

[9]The mixing solution will be introduced in later sections of this thesis. Briefly, this result states that the price of a European option in a specific class of stochastic volatility models can be expressed as the expectation of a Black-Scholes like formula, whose arguments are functionals of the underlying volatility/variance process.

for some $\tilde{\mu}, \hat{\mu}$ and $\mu \in \mathbb{R}$.[10][11] In this thesis, we will focus on these classes of stochastic volatility models. Some popular models from the literature that belong in this class include:

Model	Variance/Volatility	Dynamics	$\hat{\mu}$	$\tilde{\mu}$	μ
Heston [37]	Variance	$d\sigma_t = \kappa_t(\theta_t - \sigma_t)dt + \lambda_t\sqrt{\sigma_t}dB_t$	0	1	1/2
Schöbel and Zhu [58]	Volatility	$dV_t = \kappa_t(\theta_t - V_t)dt + \lambda_t dB_t$	0	1	0
GARCH [51, 64]	Variance	$d\sigma_t = \kappa_t(\theta_t - \sigma_t)dt + \lambda_t\sigma_t dB_t$	0	1	1
Inverse Gamma [44]	Volatility	$dV_t = \kappa_t(\theta_t - V_t)dt + \lambda_t V_t dB_t$	0	1	1
3/2 Model [45]	Variance	$d\sigma_t = \kappa_t(\theta_t\sigma_t - \sigma_t^2)dt + \lambda_t\sigma_t^{3/2}dB_t$	1	2	3/2
Verhulst[12][46, 16]	Volatility	$dV_t = \kappa_t(\theta_t V_t - V_t^2)dt + \lambda_t V_t dB_t$	1	2	1

---

[10]There exist other classes of stochastic volatility models. For example, the exponential Ornstein-Uhlenbeck model (Wiggins [63]) is not a part of either of these classes.

[11]Our model formulation here is for FX market purposes, but can be easily adjusted for equity and fixed income markets purposes.

[12]Also known as the Logistic or XGBM model.

Chapter 3

Change of measure methodology

3.1 Introduction

In this chapter, we explore how a second-order expansion of the mixing solution, alongside change of measure techniques, can result in a closed-form approximation for the price of a European put option. The tractability of our methodology relies largely on the dynamics of the underlying variance process. Our method extends that of Drimus [23], in which the Heston model is considered with constant parameters. Our contribution is as follows. We apply the expansion methodology to obtain the explicit approximation of a European put option price in a variety of of stochastic volatility models including the Heston, GARCH and Inverse-Gamma with time-dependent parameters. We also include a robust error analysis section. The sections are organised as follows:

- Section 3.2 details preliminary calculations, where we express the put option price as the mixing solution. Once done, a second-order Taylor expansion is performed, giving the approximation formula in terms of a number of expectations of functionals of the underlying variance process.

- Section 3.3 details how to derive more convenient expressions for these expectations obtained in Section 3.2 through change of measure techniques. Specifically, we rewrite the spot S_T as a convenient expression so as to construct a term which is a Doléans-Dade exponential, thereby defining a Radon-Nikodym derivative. This term allows us to change measure, allowing for more convenient calculations.

- Section 3.4 introduces specific models. As precise dynamics are assumed, the objective is to derive explicit expressions for the expectations from Section 3.3. In particular, we consider the Heston, GARCH and Inverse-Gamma models.

- In Section 3.5 we perform an error analysis, bounding the error in the expansion in terms of higher order moments of the underlying variance process.

3.2 Preliminary calculations

Suppose the spot S with variance σ follows the dynamics

$$\mathrm{d}S_t = S_t((r_t^d - r_t^f)\mathrm{d}t + \sqrt{\sigma_t}\mathrm{d}W_t), \quad S_0,$$
$$\mathrm{d}\sigma_t = \alpha(t, \sigma_t)\mathrm{d}t + \beta(t, \sigma_t)\mathrm{d}B_t, \quad \sigma_0,$$
$$\mathrm{d}\langle W, B \rangle_t = \rho_t \mathrm{d}t,$$

16

where W and B are Brownian motions with instantaneous correlation $(\rho_t)_{0 \leq t \leq T}$, defined on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{Q})$. Here T is a finite time horizon, where $(r_t^d)_{0 \leq t \leq T}$ and $(r_t^f)_{0 \leq t \leq T}$ are the deterministic, time-dependent domestic and foreign interest rates respectively. Furthermore, $(\mathcal{F}_t)_{0 \leq t \leq T}$ is the filtration generated by (W, B) which satisfies the usual assumptions.[1] In the following, $\mathbb{E}(\cdot)$ denotes the expectation under $\mathbb{Q}$, where $\mathbb{Q}$ is a risk-neutral measure which we assume to be chosen. We assume that the drift and and diffusion coefficients of σ are such that σ has a pathwise unique strong solution. However, this is not particularly important, as our approximation methodology eventually requires assuming specific models, which will always have a pathwise unique strong solution. This is in contrast with the methodology which will be seen in Chapter 5, where the approximation procedure is not model dependent. In that case, it becomes critical that we state specific conditions on the drift and diffusion coefficients.

Definition 3.2.1 (Geometric Brownian motion process). A process Y is called a Geometric Brownian motion (GBM) process if it solves the SDE

$$\mathrm{d}Y_t = \mu_t Y_t \mathrm{d}t + \nu_t Y_t \mathrm{d}\tilde{B}_t, \quad Y_0 = y_0.$$

Assuming μ and ν are adapted to the Brownian filtration and satisfy some regularity conditions[2], Y has the well known explicit pathwise unique strong solution

$$Y_t = y_0 \exp \left\{ \int_0^t \left(\mu_u - \frac{1}{2}\nu_u^2 \right) \mathrm{d}u + \int_0^t \nu_u \mathrm{d}\tilde{B}_u \right\}.$$

We call the process Y a $\mathrm{GBM}(y_0; \mu_t, \nu_t)$.

We decompose the Brownian motion W as $W_t = \int_0^t \rho_u \mathrm{d}B_u + \int_0^t \sqrt{1 - \rho_u^2} \mathrm{d}Z_u$, where Z is a Brownian motion under $\mathbb{Q}$ independent of B. Then, noticing S is a $\mathrm{GBM}(S_0; r_t^d - r_t^f, \sqrt{\sigma_t})$, we obtain the expression

$$S_T = S_0 \xi_T \exp \left\{ \int_0^T (r_t^d - r_t^f) \mathrm{d}t - \frac{1}{2} \int_0^T \sigma_t (1 - \rho_t^2) \mathrm{d}t + \int_0^T \sqrt{\sigma_t (1 - \rho_t^2)} \mathrm{d}Z_t \right\},$$

$$\xi_t := \exp \left\{ \int_0^t \rho_u \sqrt{\sigma_u} \mathrm{d}B_u - \frac{1}{2} \int_0^t \rho_u^2 \sigma_u \mathrm{d}u \right\}.$$

3.2.1 Pricing a put option

Denote by $(\mathcal{F}_t^B)_{0 \leq t \leq T}$ the filtration generated by the Brownian motion B.

Proposition 3.2.1. The price of a put option on S, denoted by Put, can be expressed as

$$\mathrm{Put} = e^{-\int_0^T r_t^d \mathrm{d}t} \mathbb{E}(K - S_T)_+ = \mathbb{E}\left\{ e^{-\int_0^T r_t^d \mathrm{d}t} \mathbb{E}\left[(K - S_T)_+ | \mathcal{F}_T^B \right] \right\}$$

$$= \mathbb{E}\left(\mathrm{Put}_{\mathrm{BS}}\left(S_0 \xi_T, \int_0^T \sigma_t (1 - \rho_t^2) \mathrm{d}t \right) \right), \tag{3.1}$$

where

$$\mathrm{Put}_{\mathrm{BS}}(x, y) := K e^{-\int_0^T r_t^d \mathrm{d}t} \mathcal{N}(-d_-) - x e^{-\int_0^T r_t^f \mathrm{d}t} \mathcal{N}(-d_+),$$

$$d_\pm(x, y) := d_\pm := \frac{\ln(x/K) + \int_0^T (r_t^d - r_t^f) \mathrm{d}t}{\sqrt{y}} \pm \frac{1}{2}\sqrt{y}.$$

Proof. This is a consequence of the mixing solution methodology, which is detailed in Appendix C.1. $\square$

---

[1]Meaning that $(\mathcal{F}_t)_{0 \leq t \leq T}$ is right continuous and augmented by $\mathbb{Q}$ null-sets.

[2]For example, μ and ν bounded on $[0, T]$ is sufficient.

3.2.2 Expansion

To obtain an explicit approximation to the put option price, we utilise a Taylor expansion of the function Put_{BS}.

Assumption 3.2.1. ξ is a martingale, or equivalently, that $\mathbb{E}(\xi_T) = 1$.

Since ξ is a Doléans-Dade exponential of an Itô integral process, sufficient conditions for when Assumption 3.2.1 is true are given by Theorem 1.2.4 (Beneš' conditions), or Remark 1.2.2.

Proposition 3.2.2 (Second-order put option price approximation)**.** The second-order put option price approximation, denoted by $\text{Put}^{(2)}$, is given by

$$
\begin{aligned}
\text{Put}^{(2)} = {}& \text{Put}_{\text{BS}}(\hat{x}, \hat{y}) \\
& + \frac{1}{2}\partial_{xx}\text{Put}_{\text{BS}}(\hat{x}, \hat{y})S_0^2\mathbb{E}(\xi_T - 1)^2 + \frac{1}{2}\partial_{yy}\text{Put}_{\text{BS}}(\hat{x}, \hat{y})\mathbb{E}\left(\int_0^T (1 - \rho_t^2)(\sigma_t - \mathbb{E}(\sigma_t))\mathrm{d}t\right)^2 \\
& + \partial_{xy}\text{Put}_{\text{BS}}(\hat{x}, \hat{y})S_0\mathbb{E}\left\{(\xi_T - 1)\left(\int_0^T (1 - \rho_t^2)(\sigma_t - \mathbb{E}(\sigma_t))\mathrm{d}t\right)\right\},
\end{aligned}
$$

$$(3.2)$$

where $(\hat{x}, \hat{y}) := (S_0, \int_0^T (1 - \rho_t^2)\mathbb{E}(\sigma_t)\mathrm{d}t)$.

Proof. Recall from Lemma 1.2.1 that Put_{BS} is smooth on $(\mathbb{R}_+^2; \mathbb{R}_+)$. We expand around the mean of $\left(S_0\xi_T, \int_0^T \sigma_t(1 - \rho_t^2)\mathrm{d}t\right)$. Under Assumption 3.2.1, the expansion point is $(\hat{x}, \hat{y}) = (S_0, \int_0^T (1 - \rho_t^2)\mathbb{E}(\sigma_t)\mathrm{d}t)$. Thus

$$
\begin{aligned}
\text{Put}_{\text{BS}}\left(S_0\xi_T, \int_0^T \sigma_t(1 - \rho_t^2)\mathrm{d}t\right) \approx {}& \text{Put}_{\text{BS}}(\hat{x}, \hat{y}) \\
& + \partial_x\text{Put}_{\text{BS}}(\hat{x}, \hat{y})S_0(\xi_T - 1) + \partial_y\text{Put}_{\text{BS}}(\hat{x}, \hat{y})\left(\int_0^T (1 - \rho_t^2)(\sigma_t - \mathbb{E}(\sigma_t))\mathrm{d}t\right) \\
& + \frac{1}{2}\partial_{xx}\text{Put}_{\text{BS}}(\hat{x}, \hat{y})S_0^2(\xi_T - 1)^2 + \frac{1}{2}\partial_{yy}\text{Put}_{\text{BS}}(\hat{x}, \hat{y})\left(\int_0^T (1 - \rho_t^2)(\sigma_t - \mathbb{E}(\sigma_t))\mathrm{d}t\right)^2 \\
& + \partial_{xy}\text{Put}_{\text{BS}}(\hat{x}, \hat{y})S_0(\xi_T - 1)\left(\int_0^T (1 - \rho_t^2)(\sigma_t - \mathbb{E}(\sigma_t))\mathrm{d}t\right).
\end{aligned}
$$

Taking expectation gives a second-order approximation to the put option price, that is, $\text{Put}^{(2)}$. Notice that $\text{Put}_{\text{BS}}(\hat{x}, \hat{y})$ is a deterministic quantity, thus the first order terms will vanish. $\qquad\square$

Remark 3.2.1 (Second-order Greeks approximation)**.** The Put Delta is obtained via partial differentiation of the Put price with respect to the underlying S_0.

$$
\begin{aligned}
\partial_{S_0}\text{Put}^{(2)} = {}& \partial_x\text{Put}_{\text{BS}}(\hat{x}, \hat{y}) \\
& + \frac{1}{2}\left[2S_0\partial_{xx}\text{Put}_{\text{BS}}(\hat{x}, \hat{y}) + S_0^2\partial_{xxx}\text{Put}_{\text{BS}}(\hat{x}, \hat{y})\right]\mathbb{E}(\xi_T - 1)^2 \\
& + \frac{1}{2}\partial_{xyy}\text{Put}_{\text{BS}}(\hat{x}, \hat{y})\mathbb{E}\left(\int_0^T (1 - \rho_t^2)(\sigma_t - \mathbb{E}(\sigma_t))\mathrm{d}t\right)^2 \\
& + \left[\partial_{xy}\text{Put}_{\text{BS}}(\hat{x}, \hat{y}) + S_0\partial_{xxy}\text{Put}_{\text{BS}}(\hat{x}, \hat{y})\right]\mathbb{E}\left\{(\xi_T - 1)\left(\int_0^T (1 - \rho_t^2)(\sigma_t - \mathbb{E}(\sigma_t))\mathrm{d}t\right)\right\}.
\end{aligned}
$$

$$(3.3)$$

The Put Gamma is obtained via partial differentiation of the Delta with respect to the underlying S_0.

$$
\begin{aligned}
\partial_{S_0 S_0} \mathrm{Put}^{(2)} = {}& \partial_{xx} \mathrm{Put}_{\mathrm{BS}}(\hat{x}, \hat{y}) \\
& + \frac{1}{2} \left[2\partial_{xx} \mathrm{Put}_{\mathrm{BS}}(\hat{x}, \hat{y}) + 2S_0 \partial_{xxx} \mathrm{Put}_{\mathrm{BS}}(\hat{x}, \hat{y}) + S_0^2 \partial_{xxxx} \mathrm{Put}_{\mathrm{BS}}(\hat{x}, \hat{y}) \right] \mathbb{E}(\xi_T - 1)^2 \\
& + \frac{1}{2} \partial_{xxyy} \mathrm{Put}_{\mathrm{BS}}(\hat{x}, \hat{y}) \mathbb{E} \left(\int_0^T (1 - \rho_t^2)(\sigma_t - \mathbb{E}(\sigma_t)) \mathrm{d}t \right)^2 \\
& + \left[2\partial_{xxy} \mathrm{Put}_{\mathrm{BS}}(\hat{x}, \hat{y}) + S_0 \partial_{xxxy} \mathrm{Put}_{\mathrm{BS}}(\hat{x}, \hat{y}) \right] \mathbb{E} \left\{ (\xi_T - 1) \left(\int_0^T (1 - \rho_t^2)(\sigma_t - \mathbb{E}(\sigma_t)) \mathrm{d}t \right) \right\}.
\end{aligned}
$$
(3.4)

The above partial derivatives of $\mathrm{Put}_{\mathrm{BS}}$ are given in Appendix A.1. What remains to be done is the calculation of each of the expectations, which are

$$
\mathbb{E}(\xi_T - 1)^2, \tag{3.5}
$$

$$
\mathbb{E} \left(\int_0^T (1 - \rho_t^2)(\sigma_t - \mathbb{E}(\sigma_t)) \mathrm{d}t \right)^2, \tag{3.6}
$$

$$
\mathbb{E} \left\{ (\xi_T - 1) \left(\int_0^T (1 - \rho_t^2)(\sigma_t - \mathbb{E}(\sigma_t)) \mathrm{d}t \right) \right\}. \tag{3.7}
$$

3.3 Calculation of expectations

The following lemmas will be useful in order to calculate eqs. (3.5) to (3.7). These lemmas are clear consequences of Girsanov's theorem, and so we omit the proofs.

Lemma 3.3.1. There exists a probability measure $\mathbb{Q}_1 \sim \mathbb{Q}$ defined by the Radon-Nikodym derivative

$$
\frac{\mathrm{d}\mathbb{Q}_1}{\mathrm{d}\mathbb{Q}} := \xi_T = \exp \left\{ \int_0^T \rho_u \sqrt{\sigma_u} \mathrm{d}B_u - \frac{1}{2} \int_0^T \rho_u^2 \sigma_u \mathrm{d}u \right\},
$$

such that $B_t^1 := B_t - \int_0^t \rho_u \sqrt{\sigma_u} \mathrm{d}u$ is a $\mathbb{Q}_1$ Brownian motion. Furthermore, expectations can be calculated under the new measure by the equation $\mathbb{E}(X\xi_T) = \mathbb{E}_{\mathbb{Q}_1}(X)$ or $\mathbb{E}(X) = \mathbb{E}_{\mathbb{Q}_1}(X \frac{1}{\xi_T})$.

We can extend the above idea to a sequence of equivalent measures.

Lemma 3.3.2. Let $(\mathbb{Q}_n)_{n \geq 0}$ be a sequence of probability measures equivalent to $\mathbb{Q}$, defined by the Radon-Nikodym derivatives

$$
\frac{\mathrm{d}\mathbb{Q}_{n+1}}{\mathrm{d}\mathbb{Q}_n} := \xi_T^{(n)} := \exp \left\{ \int_0^T \rho_u \sqrt{\sigma_u} \mathrm{d}B_u^n - \frac{1}{2} \int_0^T \rho_u^2 \sigma_u \mathrm{d}u \right\}, \quad \xi_T^{(0)} := \xi_T, \quad n \geq 0,
$$

where $\mathbb{Q}_0 := \mathbb{Q}$ and $B^0 := B$. Under $\mathbb{Q}_n$, $B_t^n := B_t^{n-1} - \int_0^t \rho_u \sqrt{\sigma_u} \mathrm{d}u$ is a Brownian motion. Furthermore, we have the relationship between densities as

$$
\xi_T^{(n)} = \xi_T^{(n-1)} e^{-\int_0^T \rho_u^2 \sigma_u \mathrm{d}u}, \quad n \geq 1. \tag{3.8}
$$

Expectations can also be calculated as

$$\mathbb{E}_{\mathbb{Q}_n}(X) = \mathbb{E}_{\mathbb{Q}_{n-1}}(X\xi_T^{(n-1)}),$$

$$\mathbb{E}_{\mathbb{Q}_{n-1}}(X) = \mathbb{E}_{\mathbb{Q}_n}\left(X\frac{1}{\xi_T^{(n-1)}}\right). \tag{3.9}$$

The two above relationships, eq. (3.8) and eq. (3.9), allow for alternative and often more convenient calculations of expectations under $\mathbb{Q}$.

Using these tools, we can now give alternative expressions for the expectations seen in eq. (3.5), eq. (3.6) and eq. (3.7).

3.3.1 $\mathbb{E}(\xi_T - 1)^2$

First, expanding eq. (3.5) gives

$$\mathbb{E}(\xi_T - 1)^2 = \mathbb{E}(\xi_T^2) - 1.$$

This second moment can be dealt with a number of changes of measures

$$\mathbb{E}(\xi_T^2) = \mathbb{E}_{\mathbb{Q}_1}(\xi_T) = \mathbb{E}_{\mathbb{Q}_1}(\xi_T^{(1)} e^{\int_0^T \rho_t^2 \sigma_t \mathrm{d}t})$$

$$= \mathbb{E}_{\mathbb{Q}_2}(e^{\int_0^T \rho_t^2 \sigma_t \mathrm{d}t}). \tag{3.10}$$

Under the assumption of constant parameters[3] we may calculate eq. (3.10) explicitly via the Laplace transform for certain processes σ. However to our knowledge, there exists no explicit solution when parameters are time-dependent, see Hurd and Kuznetsov [39]. Instead, we approximate eq. (3.10) by expanding the exponential around the mean of $\int_0^T \rho_t^2 \sigma_t \mathrm{d}t$ to second-order.

$$\mathbb{E}_{\mathbb{Q}_2}(e^{\int_0^T \rho_t^2 \sigma_t \mathrm{d}t})$$

$$\approx \mathbb{E}_{\mathbb{Q}_2}\left\{e^{\int_0^T \rho_t^2 \mathbb{E}_{\mathbb{Q}_2}(\sigma_t)\mathrm{d}t}\left[1 + \int_0^T \rho_t^2\left(\sigma_t - \mathbb{E}_{\mathbb{Q}_2}(\sigma_t)\right)\mathrm{d}t + \frac{1}{2}\left(\int_0^T \rho_t^2\left(\sigma_t - \mathbb{E}_{\mathbb{Q}_2}(\sigma_t)\mathrm{d}t\right)\right)^2\right]\right\}$$

$$= e^{\int_0^T \rho_t^2 \mathbb{E}_{\mathbb{Q}_2}(\sigma_t)\mathrm{d}t}\left\{1 + \frac{1}{2}\mathbb{E}_{\mathbb{Q}_2}\left[\left(\int_0^T \rho_t^2\left(\sigma_t - \mathbb{E}_{\mathbb{Q}_2}(\sigma_t)\mathrm{d}t\right)\right)^2\right]\right\}$$

$$= e^{\int_0^T \rho_t^2 \mathbb{E}_{\mathbb{Q}_2}(\sigma_t)\mathrm{d}t}\left\{1 + \int_0^T \rho_t^2\int_0^t \rho_s^2 \mathrm{Cov}_{\mathbb{Q}_2}(\sigma_s, \sigma_t)\mathrm{d}s\mathrm{d}t\right\},$$

where we have used the fact that $\left(\int_0^T f(t)\mathrm{d}t\right)^2 = 2\int_0^T f(t)\left(\int_0^t f(s)\mathrm{d}s\right)\mathrm{d}t$.

3.3.2 $\mathbb{E}\left(\int_0^T (1 - \rho_t^2)(\sigma_t - \mathbb{E}(\sigma_t))\mathrm{d}t\right)^2$

To calculate eq. (3.6), we use the same trick from Section 3.3.1.

$$\mathbb{E}\left(\int_0^T (1 - \rho_t^2)(\sigma_t - \mathbb{E}(\sigma_t))\mathrm{d}t\right)^2 = 2\int_0^T (1 - \rho_t^2)\left(\int_0^t (1 - \rho_s^2)\mathrm{Cov}(\sigma_s, \sigma_t)\mathrm{d}s\right)\mathrm{d}t.$$

---

[3]That is, $\alpha(t, \sigma_t) = \alpha(\sigma_t)$, $\beta(t, \sigma_t) = \beta(\sigma_t)$ and $\rho_t = \rho$.

3.3.3 $\quad \mathbb{E}\left\{ (\xi_T - 1)\left(\int_0^T (1 - \rho_t^2)(\sigma_t - \mathbb{E}(\sigma_t))\mathrm{d}t \right) \right\}$

Calculation of the mixed expectation eq. (3.7) gives

$$
\mathbb{E}\left\{ (\xi_T - 1)\left(\int_0^T (1 - \rho_t^2)(\sigma_t - \mathbb{E}(\sigma_t))\mathrm{d}t \right) \right\} = \int_0^T (1 - \rho_t^2)\big(\mathbb{E}(\xi_T \sigma_t) - \mathbb{E}(\sigma_t)\big)\mathrm{d}t
$$

$$
= \int_0^T (1 - \rho_t^2)\big(\mathbb{E}_{\mathbb{Q}_1}(\sigma_t) - \mathbb{E}(\sigma_t)\big)\mathrm{d}t.
$$

3.4 Pricing equations for specific models

We now introduce specific dynamics for both the spot and its underlying variance process. From Section 3.3, it is apparent that a closed-form expression for Put$^{(2)}$ will largely depend on the tractability of the variance process σ under the original measure $\mathbb{Q}$, as well as the artificial measures $\mathbb{Q}_1$ and $\mathbb{Q}_2$.

3.4.1 Heston model

Suppose the spot S with variance V follows the Heston dynamics

$$
\begin{aligned}
\mathrm{d}S_t &= S_t((r_t^d - r_t^f)\mathrm{d}t + \sqrt{V_t}\mathrm{d}W_t), \quad S_0, \\
\mathrm{d}V_t &= \kappa_t(\theta_t - V_t)\mathrm{d}t + \lambda_t \sqrt{V_t}\mathrm{d}B_t, \quad V_0 = v_0, \\
\mathrm{d}\langle W, B\rangle_t &= \rho_t \mathrm{d}t,
\end{aligned} \tag{3.11}
$$

where $(\kappa_t)_{0 \le t \le T}$, $(\theta_t)_{0 \le t \le T}$ and $(\lambda_t)_{0 \le t \le T}$ are time-dependent, deterministic, strictly positive and bounded on $[0, T]$. Here we model the variance directly, that is, in the language of the initial sections, $\sigma_t = V_t$. This is convenient for the calculations.

Definition 3.4.1 (CIR process). A process $\tilde{V}$ is called a CIR process if it solves the SDE

$$
\mathrm{d}\tilde{V}_t = \kappa_t(\theta_t - \tilde{V}_t)\mathrm{d}t + \lambda_t \sqrt{\tilde{V}_t}\mathrm{d}\tilde{B}_t, \quad \tilde{V}_0 = \tilde{v}_0,
$$

where we assume $(\kappa_t)_{0 \le t \le T}$, $(\theta_t)_{0 \le t \le T}$ and $(\lambda_t)_{0 \le t \le T}$ are time-dependent, deterministic, strictly positive and bounded on $[0, T]$. It can be integrated to obtain

$$
\tilde{V}_t = \tilde{v}_0 e^{-\int_0^t \kappa_z \mathrm{d}z} + \int_0^t e^{-\int_u^t \kappa_z \mathrm{d}z} \kappa_u \theta_u \mathrm{d}u + \int_0^t e^{-\int_u^t \kappa_z \mathrm{d}z} \lambda_u \sqrt{\tilde{V}_u}\mathrm{d}B_u.
$$

We call the process $\tilde{V}$ a CIR$(\tilde{v}_0; \kappa_t, \theta_t, \lambda_t)$.

It is clear that the variance process V in eq. (3.11) is a CIR$(v_0; \kappa_t, \theta_t, \lambda_t)$.

Lemma 3.4.1. Let $(\mathbb{Q}_n)_{n \ge 0}$ be a sequence of probability measures equivalent to $\mathbb{Q}$, defined by the Radon-Nikodym derivatives

$$
\frac{\mathrm{d}\mathbb{Q}_{n+1}}{\mathrm{d}\mathbb{Q}_n} := \xi_T^{(n)} := \exp\left\{ \int_0^T \rho_u \sqrt{V_u}\mathrm{d}B_u^n - \frac{1}{2}\int_0^T \rho_u^2 V_u \mathrm{d}u \right\}, \quad \xi_T^{(0)} := \xi_T, \quad n \ge 0,
$$

where $\mathbb{Q}_0 := \mathbb{Q}$ and $B^0 := B$. Under $\mathbb{Q}_n$, $B_t^n := B_t^{n-1} - \int_0^t \rho_u \sqrt{V_u}\mathrm{d}u$ is a Brownian motion. For $n \ge 0$, the dynamics of V under the measure $\mathbb{Q}_n$ are

$$
\mathrm{d}V_t = (\kappa_t - n\lambda_t \rho_t)\left(\frac{\theta_t \kappa_t}{\kappa_t - n\lambda_t \rho_t} - V_t \right)\mathrm{d}t + \lambda_t \sqrt{V_t}\mathrm{d}B_t^n,
$$

which is a CIR$(v_0; \kappa_t - n\lambda_t \rho_t, \frac{\theta_t \kappa_t}{\kappa_t - n\lambda_t \rho_t}, \lambda_t)$.

Proof. This lemma is simply obtained through Lemma 3.3.2, then expressing V under the new measures $\mathbb{Q}_n$. $\square$

Thus, the variance process V is a CIR process under all measures considered, and we have explicit expressions for its moments and covariance. All the terms needed can be calculated explicitly.

Pricing under the Heston framework

The second-order approximation of the put option price in the Heston framework is given by the following theorem.

Theorem 3.4.1 (Second-order Heston put option price). *The second-order approximation to the put option price in the Heston model, denoted by $\mathrm{Put}_H^{(2)}$, is*

$$
\begin{aligned}
\mathrm{Put}_H^{(2)} = {} & \mathrm{Put}_{BS}(\hat{x}, \hat{y}) \\
& + \frac{1}{2} \partial_{xx} \mathrm{Put}_{BS}(\hat{x}, \hat{y}) S_0^2 \mathbb{E}(\xi_T - 1)^2 + \frac{1}{2} \partial_{yy} \mathrm{Put}_{BS}(\hat{x}, \hat{y}) \mathbb{E}\left(\int_0^T (1 - \rho_t^2)(V_t - \mathbb{E}(V_t)) \mathrm{d}t \right)^2 \\
& + \partial_{xy} \mathrm{Put}_{BS}(\hat{x}, \hat{y}) S_0 \mathbb{E}\left\{ (\xi_T - 1) \left(\int_0^T (1 - \rho_t^2)(V_t - \mathbb{E}(V_t)) \mathrm{d}t \right) \right\},
\end{aligned}
$$

$$(3.12)$$

where

$$
\mathbb{E}(\xi_T - 1)^2 \approx e^{\int_0^T \rho_t^2 \mathbb{E}_{\mathbb{Q}_2}(V_t) \mathrm{d}t} \left\{ 1 + \int_0^T \rho_t^2 \int_0^t \rho_s^2 \mathrm{Cov}_{\mathbb{Q}_2}(V_s, V_t) \mathrm{d}s \mathrm{d}t \right\} - 1,
$$

$$
\mathbb{E}_{\mathbb{Q}_2}(V_t) = v_0 e^{-\int_0^t \kappa_z - 2\lambda_z \rho_z \mathrm{d}z} + \int_0^t e^{-\int_u^t \kappa_z - 2\lambda_z \rho_z \mathrm{d}z} \kappa_u \theta_u \mathrm{d}u,
$$

$$
\mathrm{Cov}_{\mathbb{Q}_2}(V_s, V_t) = e^{-\int_s^t \kappa_z - 2\lambda_z \rho_z \mathrm{d}z} \int_0^s \lambda_u^2 e^{-2\int_u^s \kappa_z - 2\lambda_z \rho_z \mathrm{d}z} \left[v_0 e^{-\int_0^u \kappa_z - 2\lambda_z \rho_z \mathrm{d}z} + \int_0^u e^{-\int_p^u \kappa_z - 2\lambda_z \rho_z \mathrm{d}z} \kappa_p \theta_p \mathrm{d}p \right] \mathrm{d}u,
$$

$$
\begin{aligned}
& \mathbb{E}\left(\int_0^T (1 - \rho_t^2)(V_t - \mathbb{E}(V_t)) \mathrm{d}t \right)^2 \\
& = 2 \int_0^T (1 - \rho_t^2) \left(\int_0^t (1 - \rho_s^2) \left[e^{-\int_s^t \kappa_z \mathrm{d}z} \int_0^s \lambda_u^2 e^{-2\int_u^s \kappa_z \mathrm{d}z} \left\{ v_0 e^{-\int_0^u \kappa_z \mathrm{d}z} + \int_0^u e^{-\int_p^u \kappa_z \mathrm{d}z} \kappa_p \theta_p \mathrm{d}p \right\} \mathrm{d}u \right] \mathrm{d}s \right) \mathrm{d}t,
\end{aligned}
$$

and

$$
\begin{aligned}
& \mathbb{E}\left\{ (\xi_T - 1) \left(\int_0^T (1 - \rho_t^2)(V_t - \mathbb{E}(V_t)) \mathrm{d}t \right) \right\} \\
& = \int_0^T (1 - \rho_t^2) \left\{ v_0 \left(e^{-\int_0^t \kappa_z - \lambda_z \rho_z \mathrm{d}z} - e^{-\int_0^t \kappa_z \mathrm{d}z} \right) + \int_0^t \left(e^{-\int_u^t \kappa_z - \lambda_z \rho_z \mathrm{d}z} - e^{-\int_u^t \kappa_z \mathrm{d}z} \right) \kappa_u \theta_u \mathrm{d}u \right\} \mathrm{d}t.
\end{aligned}
$$

Furthermore, $\hat{x} = S_0$ and $\hat{y} = \int_0^T (1 - \rho_t^2) \mathbb{E}(V_t) \mathrm{d}t = \int_0^T (1 - \rho_t^2) \left\{ v_0 e^{-\int_0^t \kappa_z \mathrm{d}z} + \int_0^t e^{-\int_u^t \kappa_z \mathrm{d}z} \kappa_u \theta_u \mathrm{d}u \right\} \mathrm{d}t.$

Proof. Use Proposition 3.2.2 and adapt Section 3.3 to the Heston framework. Furthermore, the CIR moments are obtained from Appendix B.1. $\square$

3.4.2 GARCH diffusion model

Suppose the spot S with variance V follows the GARCH diffusion dynamics

$$
\begin{aligned}
\mathrm{d}S_t &= S_t((r_t^d - r_t^f)\mathrm{d}t + \sqrt{V_t}\mathrm{d}W_t), \quad S_0, \\
\mathrm{d}V_t &= \kappa_t(\theta_t - V_t)\mathrm{d}t + \lambda_t V_t \mathrm{d}B_t, \quad V_0 = v_0, \\
\mathrm{d}\langle W, B\rangle_t &= \rho_t \mathrm{d}t,
\end{aligned}
\tag{3.13}
$$

where $(\kappa_t)_{0 \leq t \leq T}$, $(\theta_t)_{0 \leq t \leq T}$ and $(\lambda_t)_{0 \leq t \leq T}$ are time-dependent, deterministic, strictly positive and bounded on $[0, T]$. Like the Heston model, we model the variance directly.

Definition 3.4.2 (Inverse-Gamma process). A process $\tilde{V}$ is called an Inverse-Gamma (IGa) process if it solves the SDE

$$
\mathrm{d}\tilde{V}_t = \kappa_t(\theta_t - \tilde{V}_t)\mathrm{d}t + \lambda_t \tilde{V}_t \mathrm{d}\tilde{B}_t, \quad \tilde{V}_0 = \tilde{v}_0,
$$

where we assume $(\kappa_t)_{0 \leq t \leq T}$, $(\theta_t)_{0 \leq t \leq T}$ and $(\lambda_t)_{0 \leq t \leq T}$ are time-dependent, deterministic, strictly positive and bounded on $[0, T]$. Let Y be a $\mathrm{GBM}(1; -\kappa_t, \lambda_t)$. Then the explicit pathwise unique strong solution of $\tilde{V}$ is

$$
\tilde{V}_t = Y_t\left(\tilde{v}_0 + \int_0^t \frac{\kappa_u \theta_u}{Y_u}\mathrm{d}u\right).
$$

We call the process $\tilde{V}$ an $\mathrm{IGa}(\tilde{v}_0; \kappa_t, \theta_t, \lambda_t)$.

It is evident that the variance process V in eq. (3.13) is an $\mathrm{IGa}(v_0; \kappa_t, \theta_t, \lambda_t)$.

Lemma 3.4.2. Let $(\mathbb{Q}_n)_{n \geq 0}$ be a sequence of probability measures equivalent to $\mathbb{Q}$, defined by the Radon-Nikodym derivatives

$$
\frac{\mathrm{d}\mathbb{Q}_{n+1}}{\mathrm{d}\mathbb{Q}_n} := \xi_T^{(n)} := \exp\left\{\int_0^T \rho_u \sqrt{V_u}\mathrm{d}B_u^n - \frac{1}{2}\int_0^T \rho_u^2 V_u \mathrm{d}u\right\}, \quad \xi_T^{(0)} := \xi_T, \quad n \geq 0,
$$

where $\mathbb{Q}_0 := \mathbb{Q}$ and $B^0 := B$. Under $\mathbb{Q}_n$, $B_t^n := B_t^{n-1} - \int_0^t \rho_u \sqrt{V_u}\mathrm{d}u$ is a Brownian motion. For $n \geq 0$, the dynamics of V under the measure $\mathbb{Q}_n$ are

$$
\mathrm{d}V_t = \kappa_t\left(\theta_t - V_t + \frac{n\lambda_t \rho_t}{\kappa_t}V_t^{3/2}\right)\mathrm{d}t + \lambda_t V_t \mathrm{d}B_t^n.
$$

Proof. This lemma is simply obtained through Lemma 3.3.2, then expressing V under the new measures $\mathbb{Q}_n$. $\qquad\square$

Lemma 3.4.3. Suppose the arbitrary diffusion U solves the SDE

$$
\mathrm{d}U_t = f(t, U_t)\mathrm{d}t + \nu_t U_t \mathrm{d}\tilde{B}_t, \quad U_0 = u_0,
\tag{3.14}
$$

where $(\nu_t)_{0 \leq t \leq T}$ is adapted to the Brownian filtration and f and ν satisfy some regularity conditions[4] so that a pathwise unique strong solution for U exists. Then if an explicit solution exists, it is given by

$$
U_t = Y_t/F_t,
$$

---

[4]f being Lipschitz in x, uniformly in t and ν bounded on $[0, T]$ (and thus $\beta(t, x) = \nu_t x$ is Lipschitz in x uniformly in t) is enough for this to be true.

where F is a $\mathrm{GBM}(1; \nu_t^2, -\nu_t)$, That is,

$$\mathrm{d}F_t = \nu_t^2 F_t \mathrm{d}t - \nu_t F_t \mathrm{d}B_t, \quad F_0 = 1,$$

$$\Rightarrow F_t = \exp\left\{\int_0^t \frac{1}{2}\nu_u^2 \mathrm{d}u - \int_0^t \nu_u \mathrm{d}\tilde{B}_u\right\},$$

and Y solves the integral equation (written in differential form)

$$\mathrm{d}Y_t = F_t f\left(t, \frac{Y_t}{F_t}\right)\mathrm{d}t, \quad Y_0 = u_0. \tag{3.15}$$

Proof. We essentially verify that this form of U satisfies the SDE eq. (3.14).

$$
\begin{aligned}
\mathrm{d}\left(\frac{Y_t}{F_t}\right) &= \mathrm{d}\left(1/F_t\right)Y_t + \frac{1}{F_t}\mathrm{d}Y_t + \mathrm{d}\left(1/F_t\right)\mathrm{d}Y_t \\
&= \left(\left\{\frac{\nu_t}{F_t}\mathrm{d}B_t - \frac{\nu_t^2}{F_t}\mathrm{d}t\right\} + \frac{\nu_t^2}{F_t}\mathrm{d}t\right)Y_t + f\left(t, Y_t/F_t\right)\mathrm{d}t + 0 \\
&= \frac{Y_t}{F_t}\nu_t \mathrm{d}B_t + f\left(t, Y_t/F_t\right)\mathrm{d}t \\
&= \nu_t U_t \mathrm{d}B_t + f(t, U_t)\mathrm{d}t.
\end{aligned}
$$

$\square$

Remark 3.4.1. Let $\mathbb{Q}_n$ be defined as in Lemma 3.4.2. Under the measures $\mathbb{Q}_n$, $n \geq 1$, V has no known explicit solution, nor known explicit moments.

Validity of Remark 3.4.1. The SDE in Lemma 3.4.2 is a linear diffusion type SDE. From Lemma 3.4.3, it is known that if an explicit solution exists, it is given by

$$V_t = Y_t/F_t,$$

where F is a $\mathrm{GBM}(1; \lambda_t^2, -\lambda_t)$ and Y is the solution to the integral equation (written in differential form)

$$\mathrm{d}Y_t = \left(\kappa_t \theta_t F_t - \kappa_t Y_t + \frac{n\lambda_t \rho_t}{\kappa_t}Y_t^{3/2}F_t^{-1/2}\right)\mathrm{d}t.$$

Define $A_t := \kappa_t \theta_t F_t$ and $C_t := \frac{n\lambda_t \rho_t}{\kappa_t}F_t^{-1/2}$. Then first note that A_t and C_t are both non-differentiable in t. Thus

$$\mathrm{d}Y_t = \left(A_t - \kappa_t Y_t + C_t Y_t^{3/2}\right)\mathrm{d}t.$$

As far as we know, there is no explicit solution to these types of integral equations in the literature, even when A and C are differentiable. As for explicit moments, it is unclear how to approach this problem. There seems to be no approach to this problem in the literature, especially in the case of time-dependent parameters, see for example Kloeden and Platen [43] chapter 4.4 for a comprehensive list of explicitly solvable SDEs. Furthermore, as an explicit solution does not exist, we cannot use the method of approximating moments via the SDE's solution.

Pricing under the GARCH diffusion framework: $\rho = 0$

The change of measure technique gives an intractable dynamic for V; we cannot appeal to it for calculating expectations. However, in the case of $\rho = 0$ a.e., this implies $\xi_T = 1$ $\mathbb{Q}$ a.s., and one will notice that the terms in the expansion requiring a change of measure will disappear. Of course, the cost is the unrealistic assumption that spot and volatility movements are uncorrelated. We hope to mitigate this issue in future work by combining this approach with small correlation expansion methods, see Antonelli and Scarlatti [5], Antonelli et al. [4].

Theorem 3.4.2 (Second-order GARCH put option price). Assume $\rho = 0$ a.e.. Then the second-order put option price in the GARCH diffusion model, denoted by $\text{Put}_{\text{GARCH}}^{(2)}$, is

$$\text{Put}_{\text{GARCH}}^{(2)} = \text{Put}_{\text{BS}}(\hat{x}, \hat{y}) + \frac{1}{2} \partial_{yy} \text{Put}_{\text{BS}}(\hat{x}, \hat{y}) \mathbb{E} \left(\int_0^T (V_t - \mathbb{E}(V_t)) \mathrm{d}t \right)^2. \tag{3.16}$$

Here the expectation is

$$\mathbb{E} \left(\int_0^T (V_t - \mathbb{E}(V_t)) \mathrm{d}t \right)^2 = 2 \int_0^T \left(\int_0^t \text{Cov}(V_s, V_t) \mathrm{d}s \right) \mathrm{d}t.$$

Furthermore, $\hat{x} = S_0$ and $\hat{y} = \int_0^T \mathbb{E}(V_t) \mathrm{d}t$.

Proof. Use Proposition 3.2.2 under the assumption of $\rho = 0$ a.e.. Both $\text{Cov}(V_s, V_t)$ and $\mathbb{E}(V_t)$ are given in Appendix B.2. $\qquad \square$

3.4.3 Inverse-Gamma model

Suppose the spot S with volatility V follows the Inverse-Gamma dynamics

$$\mathrm{d}S_t = S_t((r_t^d - r_t^f)\mathrm{d}t + V_t \mathrm{d}W_t), \quad S_0,$$
$$\mathrm{d}V_t = \kappa_t(\theta_t - V_t)\mathrm{d}t + \lambda_t V_t \mathrm{d}B_t, \quad V_0 = v_0,$$
$$\mathrm{d}\langle W, B \rangle_t = \rho_t \mathrm{d}t,$$

where $(\kappa_t)_{0 \le t \le T}$, $(\theta_t)_{0 \le t \le T}$ and $(\lambda_t)_{0 \le t \le T}$ are time-dependent, deterministic, strictly positive and bounded on $[0, T]$. Unlike the Heston model, we are no longer modelling the variance directly. Instead, we model its square root, the volatility. To arrive at the desired framework, one replaces σ_t with V_t^2 from the initial sections. Immediately, it is clear that the calculations are less straightforward, as the process V^2 is not nearly as convenient as V.

Lemma 3.4.4. Let $(\mathbb{Q}_n)_{n \ge 0}$ be a sequence of probability measures equivalent to $\mathbb{Q}$, defined by the Radon-Nikodym derivatives

$$\frac{\mathrm{d}\mathbb{Q}_{n+1}}{\mathrm{d}\mathbb{Q}_n} := \xi_T^{(n)} := \exp \left\{ \int_0^T \rho_u V_u \mathrm{d}B_u^n - \frac{1}{2} \int_0^T \rho_u^2 V_u^2 \mathrm{d}u \right\}, \quad \xi_T^{(0)} := \xi_T, \quad n \ge 0,$$

where $\mathbb{Q}_0 := \mathbb{Q}$ and $B^0 := B$. Under $\mathbb{Q}_n$, $B_t^n := B_t^{n-1} - \int_0^t \rho_u V_u \mathrm{d}u$ is a Brownian motion. For $n \ge 0$, the dynamics of V under the measure $\mathbb{Q}_n$ are

$$\mathrm{d}V_t = \kappa_t \left(\theta_t - V_t + \frac{n\lambda_t \rho_t}{\kappa_t} V_t^2 \right) \mathrm{d}t + \lambda_t V_t \mathrm{d}B_t^n.$$

Proof. This lemma is simply obtained through Lemma 3.3.2, then expressing V under the new measures $\mathbb{Q}_n$. $\qquad \square$

Under the measures $\mathbb{Q}_n$, $n \geq 1$, V has no explicit solution, nor explicit moments. This can seen in a similar way of Remark 3.4.1; the resulting integral equation needed to be solved has no known explicit solution. Thus, we cannot explicitly calculate some of the terms in the expansion for the IGa model.

Pricing under the Inverse-Gamma framework: $\rho = 0$

Again, the dynamics of V are intractable under $\mathbb{Q}_n$. Assuming $\rho = 0$ a.e. will eliminate the terms we cannot calculate.

Theorem 3.4.3 (Second-order IGa put option price)**.** Assume $\rho = 0$ a.e.. The second-order put option price in the IGa model, denoted by $\mathrm{Put}_{\mathrm{IGa}}^{(2)}$, is

$$\mathrm{Put}_{\mathrm{IGa}}^{(2)} = \mathrm{Put}_{\mathrm{BS}}(\hat{x}, \hat{y}) + \frac{1}{2}\partial_{yy}\mathrm{Put}_{\mathrm{BS}}(\hat{x}, \hat{y})\mathbb{E}\left(\int_0^T (V_t^2 - \mathbb{E}(V_t^2))\mathrm{d}t\right)^2. \qquad (3.17)$$

Here the expectation is

$$\mathbb{E}\left(\int_0^T (V_t^2 - \mathbb{E}(V_t^2))\mathrm{d}t\right)^2 = 2\int_0^T \left(\int_0^t \mathrm{Cov}(V_s^2, V_t^2)\mathrm{d}s\right)\mathrm{d}t.$$

Furthermore, $\hat{x} = S_0$ and $\hat{y} = \int_0^T \mathbb{E}(V_t^2)\mathrm{d}t$.

Proof. Use Proposition 3.2.2 under the assumption of $\rho = 0$ a.e.. Both $\mathrm{Cov}(V_s^2, V_t^2)$ and $\mathbb{E}(V_t^2)$ are given in Appendix B.2. $\qquad\square$

3.5 Error analysis

We present an explicit bound on the error term in our expansion in terms of higher order moments of the corresponding variance process. Specifically, this means bounding the remainder term in the second-order expansion of the function $\mathrm{Put}_{\mathrm{BS}}$, and for the case when $\rho \neq 0$, the error term associated with the expansion of $e^{\int_0^T \rho_u^2 \sigma_u \mathrm{d}u}$.

We will need explicit expressions for the error terms. These are given by Taylor's theorem, which is presented in Section 1.2.4. As the expansion is second-order, we only consider the results up to second-order.

3.5.1 Explicit expression for error term

The representation for the total error due to the expansion can be summarised by the following theorem.

Theorem 3.5.1 (Total expansion error)**.** As a functional of the underlying variance process σ, let $\mathcal{E}_{\mathrm{BS}}(\sigma)$ and $\tilde{\mathcal{E}}(\sigma)$ correspond to the error induced by the expansion of $\mathrm{Put}_{\mathrm{BS}}$ and $e^{\int_0^T \rho_u^2 \sigma_u \mathrm{d}u}$ respectively. The error due to Taylor expansions for a general variance process σ is given by

$$\mathcal{E}(\sigma) = \mathcal{E}_{\mathrm{BS}}(\sigma) + \tilde{\mathcal{E}}(\sigma),$$

where

$$\mathcal{E}_{\mathrm{BS}}(\sigma) = \sum_{|\alpha|=3} \frac{|\alpha|}{\alpha_1! \alpha_2!} E_\alpha \left(S_0 \xi_T, \int_0^T \sigma_t (1 - \rho_t^2) \mathrm{d}t \right) S_0^{\alpha_1} (\xi_T - 1)^{\alpha_1} \left(\int_0^T (1 - \rho_u^2)(\sigma_u - \mathbb{E}(\sigma_u)) \mathrm{d}u \right)^{\alpha_2},$$

$$E_\alpha \left(S_0 \xi_T, \int_0^T \sigma_t (1 - \rho_t^2) \mathrm{d}t \right) = \int_0^1 (1 - u)^2 \frac{\partial^3}{\partial x^{\alpha_1} \partial y^{\alpha_2}} \mathrm{Put}_{\mathrm{BS}} \left(F(u), G(u) \right) \mathrm{d}u,$$

$$F(u) := S_0 + u S_0 (\xi_T - 1),$$

$$G(u) := \int_0^T (1 - \rho_t^2) \mathbb{E}(\sigma_t) \mathrm{d}t + u \left(\int_0^T (1 - \rho_t^2)(\sigma_t - \mathbb{E}(\sigma_t)) \mathrm{d}t \right),$$

and

$$\tilde{\mathcal{E}}(\sigma) = \frac{1}{4} \partial_{xx} \mathrm{Put}_{\mathrm{BS}}(\hat{x}, \hat{y}) S_0^2 \xi_T^2 e^{-\int_0^T \rho_u^2 \sigma_u \mathrm{d}u} \left(\int_0^T \rho_u^2 (\sigma_u - \mathbb{E}_{\mathbb{Q}_2}(\sigma_u)) \mathrm{d}u \right)^3$$

$$\cdot \int_0^1 (1 - u)^2 e^{\int_0^T \rho_m^2 \mathbb{E}_{\mathbb{Q}_2}(\sigma_m) \mathrm{d}m} e^{u \int_0^T \rho_m^2 (\sigma_m - \mathbb{E}_{\mathbb{Q}_2}(\sigma_m)) \mathrm{d}m} \mathrm{d}u.$$

Proof. First, we deal with the error term associated with the function $\mathrm{Put}_{\mathrm{BS}}$, that is, $\mathcal{E}_{\mathrm{BS}}(\sigma)$. Recall the expansion of $\mathrm{Put}_{\mathrm{BS}}$ around the point $(\hat{x}, \hat{y}) := (S_0, \int_0^T \mathbb{E}(\sigma_t)(1 - \rho_t^2) \mathrm{d}t)$ evaluated at $(S_0 \xi_T, \int_0^T \sigma_t (1 - \rho_t^2) \mathrm{d}t)$ for a general variance process σ:

$$\mathrm{Put}_{\mathrm{BS}} \left(S_0 \xi_T, \int_0^T \sigma_t (1 - \rho_t^2) \mathrm{d}t \right) = \mathrm{Put}_{\mathrm{BS}}(\hat{x}, \hat{y})$$

$$+ \partial_x \mathrm{Put}_{\mathrm{BS}}(\hat{x}, \hat{y}) S_0 (\xi_T - 1) + \partial_y \mathrm{Put}_{\mathrm{BS}}(\hat{x}, \hat{y}) \left(\int_0^T (1 - \rho_t^2)(\sigma_t - \mathbb{E}(\sigma_t)) \mathrm{d}t \right)$$

$$+ \frac{1}{2} \partial_{xx} \mathrm{Put}_{\mathrm{BS}}(\hat{x}, \hat{y}) S_0^2 (\xi_T - 1)^2 + \frac{1}{2} \partial_{yy} \mathrm{Put}_{\mathrm{BS}}(\hat{x}, \hat{y}) \left(\int_0^T (1 - \rho_t^2)(\sigma_t - \mathbb{E}(\sigma_t)) \mathrm{d}t \right)^2$$

$$+ \partial_{xy} \mathrm{Put}_{\mathrm{BS}}(\hat{x}, \hat{y}) S_0 (\xi_T - 1) \left(\int_0^T (1 - \rho_t^2)(\sigma_t - \mathbb{E}(\sigma_t)) \mathrm{d}t \right) + \mathcal{E}_{\mathrm{BS}}(\sigma).$$

Using Theorem 1.2.6 for the function $\mathrm{Put}_{\mathrm{BS}}$, this gives the error term as

$$\mathcal{E}_{\mathrm{BS}}(\sigma) = \sum_{|\alpha|=3} \frac{|\alpha|}{\alpha_1! \alpha_2!} E_\alpha \left(S_0 \xi_T, \int_0^T \sigma_t (1 - \rho_t^2) \mathrm{d}t \right) S_0^{\alpha_1} (\xi_T - 1)^{\alpha_1} \left(\int_0^T (1 - \rho_u^2)(\sigma_u - \mathbb{E}(\sigma_u)) \mathrm{d}u \right)^{\alpha_2},$$

$$E_\alpha \left(S_0 \xi_T, \int_0^T \sigma_t (1 - \rho_t^2) \mathrm{d}t \right) = \int_0^1 (1 - u)^2 \frac{\partial^3}{\partial x^{\alpha_1} \partial y^{\alpha_2}} \mathrm{Put}_{\mathrm{BS}} \left(F(u), G(u) \right) \mathrm{d}u,$$

$$F(u) := S_0 + u S_0 (\xi_T - 1),$$

$$G(u) := \int_0^T (1 - \rho_t^2) \mathbb{E}(\sigma_t) \mathrm{d}t + u \left(\int_0^T (1 - \rho_t^2)(\sigma_t - \mathbb{E}(\sigma_t)) \mathrm{d}t \right).$$

We now investigate the error term associated with the calculation of $\mathbb{E}\xi_T^2$, that is, $\tilde{\mathcal{E}}(\sigma)$. Let us look at this term without the expectation.

$$\xi_T^2 = \left(\xi_T^2 e^{-\int_0^T \rho_u^2 \sigma_u \mathrm{d}u} \right) e^{\int_0^T \rho_u^2 \sigma_u \mathrm{d}u}.$$

We expand $e^{\int_0^T \rho_u^2 \sigma_u \mathrm{d}u}$ around the expectation of the exponential's argument under $\mathbb{Q}_2$. Note that $\xi_T^2 e^{-\int_0^T \rho_u^2 \sigma_u \mathrm{d}u}$ is the Radon-Nikodym derivative which changes measure from $\mathbb{Q}$ to $\mathbb{Q}_2$.

Expanding to second-order gives

$$e^{\int_0^T \rho_u^2 \sigma_u du} = e^{\int_0^T \rho_u^2 \mathbb{E}_{\mathbb{Q}_2}(\sigma_u)du} \left(1 + \int_0^T \rho_u^2(\sigma_u - \mathbb{E}_{\mathbb{Q}_2}(\sigma_u))du + \frac{1}{2}\left(\int_0^T \rho_u^2(\sigma_u - \mathbb{E}_{\mathbb{Q}_2}(\sigma_u))du\right)^2 \right)$$

$$+ \frac{1}{2}\left(\int_0^T \rho_u^2(\sigma_u - \mathbb{E}_{\mathbb{Q}_2}(\sigma_u)du\right)^3 \int_0^1 (1-u)^2 e^{\int_0^T \rho_m^2 \mathbb{E}_{\mathbb{Q}_2}(\sigma_m)dm} e^{u\int_0^T \rho_m^2(\sigma_m - \mathbb{E}_{\mathbb{Q}_2}(\sigma_m))dm} du.$$

Finally, the coefficient in front of ξ_T^2 in the pricing formula is $\frac{1}{2}\partial_{xx}\mathrm{Put}_{\mathrm{BS}}(\hat{x},\hat{y})S_0^2$. Thus, the error term $\tilde{\mathcal{E}}(\sigma)$ can be written as

$$\tilde{\mathcal{E}}(\sigma) = \frac{1}{4}\partial_{xx}\mathrm{Put}_{\mathrm{BS}}(\hat{x},\hat{y})S_0^2 \xi_T^2 e^{-\int_0^T \rho_u^2 \sigma_u du} \left(\int_0^T \rho_u^2(\sigma_u - \mathbb{E}_{\mathbb{Q}_2}(\sigma_u))du \right)^3$$

$$\cdot \int_0^1 (1-u)^2 e^{\int_0^T \rho_m^2 \mathbb{E}_{\mathbb{Q}_2}(\sigma_m)dm} e^{u\int_0^T \rho_m^2(\sigma_m - \mathbb{E}_{\mathbb{Q}_2}(\sigma_m))dm} du.$$

$\square$

Corollary 3.5.1 (Total expansion error: $\rho = 0$)**.** The error due to Taylor expansions for a general variance process σ with $\rho = 0$ a.e., denoted by $\mathcal{E}_0(\sigma)$, is given by

$$\mathcal{E}_0(\sigma) = \frac{1}{2}\left(\int_0^T (\sigma_t - \mathbb{E}(\sigma_t))dt\right)^3 \int_0^1 (1-u)^2 \partial_{yyy}\mathrm{Put}_{\mathrm{BS}}\left(S_0, \tilde{G}(u)\right)du,$$

$$\tilde{G}(u) := \int_0^T \mathbb{E}(\sigma_t)dt + u\left(\int_0^T (\sigma_t - \mathbb{E}(\sigma_t)dt\right),$$

which is just $\mathcal{E}_{\mathrm{BS}}(\sigma)$ when $\rho = 0$ a.e..

3.5.2 Bounding error term

The hope now is to be able to bound $\mathbb{E}(\mathcal{E}(\sigma))$ in terms of the higher order moments of the variance process σ. To do this, we will need to show that the partial derivatives $\frac{\partial^3 \mathrm{Put}_{\mathrm{BS}}}{\partial x^{\alpha_1} \partial y^{\alpha_2}}(F(u), G(u))$ for $u \in (0,1)$ where $\alpha_1 + \alpha_2 = 3$, are functions of T and K which are bounded. First, we notice the following is true.

Lemma 3.5.1. Consider the third-order partial derivatives of $\mathrm{Put}_{\mathrm{BS}}$, $\frac{\partial^3 \mathrm{Put}_{\mathrm{BS}}}{\partial x^{\alpha_1} \partial y^{\alpha_2}}$, where $\alpha_1 + \alpha_2 = 3$. Let $\hat{f}(x) := \ln(x/K) + \int_0^T \left(r_t^d - r_t^f\right) dt$. Then

$$\lim_{y \downarrow 0} \left| \frac{\partial^3 \mathrm{Put}_{\mathrm{BS}}}{\partial x^{\alpha_1} \partial y^{\alpha_2}} \right|_{\hat{f}(x)=0} = \infty.$$

Furthermore, this is the only case where the partial derivatives explode.

Proof. In the following, we will repeatedly denote as F to be an arbitrary polynomial of some degree, as well as A to be an arbitrary constant. That is, they may be different on each use. From Appendix A.1, it can seen that as a function of x and y, the third-order partial derivatives are of the form

$$A\frac{\phi(d_+)}{x^n y^{m/2}}F(d_+, d_-, \sqrt{y}), \quad n \in \mathbb{Z}, m \in \mathbb{N}. \tag{3.18}$$

Recall

$$d_\pm = d_\pm(x, y) = \frac{\ln(x/K) + \int_0^T \left(r_t^d - r_t^f \right) \mathrm{d}t}{\sqrt{y}} \pm \frac{1}{2}\sqrt{y},$$

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$

Written in this form eq. (3.18), it is evident that the partial derivatives could only blow up if either x or y tend to 0 or infinity. We need only look at these limits independently of the other variable.

1. For fixed x: From eq. (3.18) the partial derivatives are of the form $A\frac{\phi(d_+)F(d_+,d_-,\sqrt{y})}{y^{m/2}}$. It can be shown that

$$\phi(d_+) = Ae^{-D_2 \frac{1}{y} - D_1 y},$$

where $D_2 = \frac{1}{2}(\hat{f}(x))^2$ and $D_1 = 1/8$. Hence both D_2 and D_1 are non-negative. However, there will be two cases to consider, when $D_2 > 0$ or $D_2 = 0$.

(a) Suppose $D_2 > 0$, then $\hat{f}(x) \neq 0$. As F is a polynomial in d_+, d_-, and $\sqrt{y}$, we can say that $F(d_+, d_-, \sqrt{y}) = o_0(1/y^{M_0/2})$ and $F(d_+, d_-, \sqrt{y}) = o(y^{M/2})$ for some $M, M_0 \in \mathbb{N}$. Thus

$$\left| \frac{\phi(d_+)F(d_+, d_-, \sqrt{y})}{y^{m/2}} \right| = |A| \frac{e^{-D_2 \frac{1}{y} - D_1 y} o_0(1/y^{M_0/2})}{y^{m/2}}$$

and also

$$\left| \frac{\phi(d_+)F(d_+, d_-, \sqrt{y})}{y^{m/2}} \right| = |A| \frac{e^{-D_2 \frac{1}{y} - D_1 y} o(y^{M/2})}{y^{m/2}}.$$

Then as $y \downarrow 0$ or $y \to \infty$, the partial derivatives tend to 0.

(b) Suppose $D_2 = 0$, then $\hat{f}(x) = 0$. Thus $d_+ = \frac{1}{2}\sqrt{y}$ and $\phi(d_+) = Ae^{-D_1 y}$. Evidently, $F(d_+, d_-, \sqrt{y}) = \sum_{i=0}^N C_i y^{i/2}$ for some $N \in \mathbb{N}$ and constants $C_0, \ldots, C_N$. Thus

$$\left| \frac{\phi(d_+)F(d_+, d_-, \sqrt{y})}{y^{m/2}} \right| = |A| \frac{e^{-D_1 y} |\sum_{i=0}^N C_i y^{i/2}|}{y^{m/2}}.$$

This quantity tends to 0 as $y \to \infty$, as the exponential decay makes the polynomial growth/decay irrelevant. However, when $y \downarrow 0$, then this limit depends on the polynomial F. If $N > m$ and if one of the $C_0, C_1, \ldots, C_m$ are non-zero then this quantity tends to ∞ as $y \downarrow 0$. If $N < m$ then this quantity tends to ∞ as $y \downarrow 0$. For each of the partial derivatives, it can be shown that either of these cases are satisfied. Thus the partial derivatives tend to ∞ when $y \downarrow 0$.

To conclude, for fixed x, if $\hat{f}(x) = 0$, then the partial derivatives tend to 0 if $y \to \infty$, and to ∞ if $y \downarrow 0$. When $\hat{f}(x) \neq 0$, the partial derivatives tend to 0 if $y \downarrow 0$ or $y \to \infty$.

2. For fixed y: From eq. (3.18), the partial derivatives are of the form $\frac{\phi(d_+)F(d_+,d_-)}{x^n}$. It can be shown that

$$\phi(d_+) = Ax^{-E_2 \ln(x) - E_1},$$

29

where $E_2 > 0$ and $E_1 \in \mathbb{R}$. Furthermore, as F is a polynomial in d_+ and d_-, then

$$|F(d_+, d_-)| \leq \sum_{i=0}^{N} |C_i| \left| \ln^i(x) \right|,$$

for some $N \in \mathbb{N}$ and constants $C_0, \ldots C_N$. It is clear $F = o(x)$ and $F = o_0(\ln^{N+1}(x))$. Thus

$$\left| \frac{\phi(d_+) F(d_+, d_-)}{x^n} \right| = |A| x^{-E_2 \ln(x) - E_1 - n} o(x).$$

Then as $x \to \infty$, this quantity tends to 0. In addition

$$\left| \frac{\phi(d_+) F(d_+, d_-)}{x^n} \right| = |A| x^{-E_2 \ln(x) - E_1 - n} o_0(\ln^{N+1}(x)).$$

Then as $x \downarrow 0$ this quantity tends to 0. So for fixed y, the partial derivatives tend to 0 as $x \downarrow 0$ or $x \to \infty$.

$\square$

We will however, be concerned with the behaviour of $u \mapsto \frac{\partial^3}{\partial x^{\alpha_1} \partial y^{\alpha_2}} \mathrm{Put}_{\mathrm{BS}}(F(u), G(u))$, meaning we will have to consider both arguments simultaneously, as they are both linear functions of u.

Lemma 3.5.2. Consider the third-order partial derivatives of $\mathrm{Put}_{\mathrm{BS}}$, $\frac{\partial^3 \mathrm{Put}_{\mathrm{BS}}}{\partial x^{\alpha_1} \partial y^{\alpha_2}}$, where $\alpha_1 + \alpha_2 = 3$ as well as the linear functions $h_1, h_2 : [0, 1] \to \mathbb{R}_+$ such that $h_1(u) = u(d_1 - c_1) + c_1$ and $h_2(u) = u(d_2 - c_2) + c_2$. Assume there exists no point $a \in (0, 1)$ such that

$$\lim_{u \to a} \frac{\ln(h_1(u)/K) + \int_0^T (r_t^d - r_t^f) dt}{\sqrt{h_2(u)}} = 0 \quad \text{and} \quad \lim_{u \to a} h_2(u) = 0.$$

Then there exists functions M_α bounded on $\mathbb{R}_+^2$ such that

$$\sup_{u \in (0,1)} \left| \frac{\partial^3 \mathrm{Put}_{\mathrm{BS}}}{\partial x^{\alpha_1} \partial y^{\alpha_2}} (h_1(u), h_2(u)) \right| = M_\alpha(T, K).$$

Furthermore, the behaviour of M_α for fixed K and T is characterised by the functions ζ and η respectively, where

$$\zeta(T) = \hat{A} e^{-\int_0^T r_t^f dt} e^{-E_2 \tilde{r}^2(T)} e^{-E_1 \tilde{r}(T)} \sum_{i=0}^{n} c_i \tilde{r}^i(T),$$

with $\tilde{r}(T) := \int_0^T (r_t^d - r_t^f) dt$ and $E_2 > 0$, $E_1 \in \mathbb{R}$, $\hat{A} \in \mathbb{R}$, $n \in \mathbb{N}$ and $c_0, \ldots, c_n$ are constants, and

$$\eta(K) = \tilde{A} K^{-D_2 \ln(K) + D_1} \sum_{i=0}^{N} C_i (-1)^i \ln^i(K),$$

with $D_2 > 0$, $D_1 \in \mathbb{R}$, $\tilde{A} \in \mathbb{R}$, $N \in \mathbb{N}$ and $C_0, \ldots, C_N$ are constants.

Proof. Under the assumptions of h_1 and h_2, by a direct application of Lemma 3.5.1, this supremum will be bounded. Next, we need to show that M_α are bounded on $\mathbb{R}_+^2$ and behave as ζ and η for fixed K and T respectively.

In the following, A denotes an arbitrary constant and F an arbitrary polynomial of some degree. They may be different on each use.

1. Behaviour in T: Fix all variables as constant except T. Then we can write the partial derivatives as

$$Ae^{-\int_0^T r_t^f \mathrm{d}t}\phi(d_+)F(d_+, d_-).$$

Expanding and collecting terms with T, we can write the partial derivatives in the form

$$Ae^{-\int_0^T r_t^f \mathrm{d}t}e^{-E_2\tilde{r}^2(T)}e^{-E_1\tilde{r}(T)}\sum_{i=0}^n c_i\tilde{r}^i(T) = \zeta(T),$$

where $E_2 > 0$, $E_1 \in \mathbb{R}$, $n \in \mathbb{N}$ and $c_0, \ldots, c_n$ are constants. As ζ is a composition of polynomials and exponentials of $\tilde{r}(T)$, then it is bounded for any closed interval not containing 0. Now since $\sup_{t\in[0,T]}(|r_t^d - r_t^f|) =: R < 1$ then $|\tilde{r}(T)| < RT$. Thus $\tilde{r}^i(T) = o(T^i)$ and $\tilde{r}^i(T) = o_0(T^i)$. Hence ζ tends to 0 as $T \downarrow 0$ or $T \to \infty$. Thus ζ is bounded on $\mathbb{R}_+$.

2. Behaviour in K: Now fix all variables as constant except K. Then the partial derivatives can be written as

$$A\phi(d_+)F(d_+, d_-).$$

Expanding and collecting terms with K, the partial derivatives can be written in the form

$$AK^{-D_2\ln(K)+D_1}F(\ln(1/K)) = \eta(K),$$

where $D_2 > 0$ and $D_1 \in \mathbb{R}$. Then writing out the polynomial explicitly

$$\eta(K) = AK^{-D_2\ln(K)+D_1}\sum_{i=0}^N C_i(-1)^i\ln^i(K),$$

where $N \in \mathbb{N}$ and $C_0, \ldots, C_N$ are constants. Thus

$$|\eta(K)| \leq |A|K^{-D_2\ln(K)+D_1}\sum_{i=0}^N |\ln^i(K)|.$$

η is bounded for any closed interval not containing 0 since it is a composition of exponentials and logarithms. Then as $\ln^i(K) = o(K)$ and $\ln^i(K) = o_0(\ln^{N+1}(K))$, η tends to 0 as $K \downarrow 0$ or $K \to \infty$. Thus η is bounded on $\mathbb{R}_+$.

$\square$

Proposition 3.5.1. There exists functions M_α as in Lemma 3.5.2 such that

$$\sup_{u\in(0,1)}\left|\frac{\partial^3}{\partial x^{\alpha_1}\partial y^{\alpha_2}}\mathrm{Put_{BS}}\left(F(u), G(u)\right)\right| \leq M_\alpha(T, K) \quad \mathbb{Q} \text{ a.s..}$$

Proof. Since F and G are linear functions, then from Lemma 3.5.2, this claim is immediately true if we can show that G is bounded away from 0. Recall

$$G(u) = (1 - u)\left(\int_0^T (1 - \rho_t^2)\mathbb{E}(\sigma_t)\mathrm{d}t\right) + u\int_0^T (1 - \rho_t^2)\sigma_t\mathrm{d}t.$$

G corresponds to the linear interpolation of $\int_0^T (1 - \rho_t^2)\mathbb{E}(\sigma_t)\mathrm{d}t$ and $\int_0^T (1 - \rho_t^2)\sigma_t\mathrm{d}t$. It is clear $\sup_{t\in[0,T]}(1 - \rho_t^2) > 0$. As σ corresponds to the variance process, in application this is always chosen to be a non-negative process such that the set $\{t \in [0, T] : \sigma_t > 0\}$ has non-zero Lebesgue measure. Thus, these integrals are strictly positive and hence G is bounded away from 0 $\mathbb{Q}$ a.s.. $\qquad\square$

We obtain the following corollary.

Corollary 3.5.2. There exists a function M as in Lemma 3.5.2 such that

$$\sup_{u\in(0,1)} \left|\partial_{yyy}\mathrm{Put}_{\mathrm{BS}}\left(S_0, \tilde{G}(u)\right)\right| \leq M(T, K) \quad \mathbb{Q} \text{ a.s..}$$

Proof. Recall

$$\tilde{G}(u) = (1 - u)\left(\int_0^T \mathbb{E}(\sigma_t)\mathrm{d}t\right) + u\int_0^T \sigma_t\mathrm{d}t.$$

Then by the same argument in the proof of Proposition 3.5.1, $\tilde{G}$ is bounded away from 0 $\mathbb{Q}$ a.s.. Hence by Lemma 3.5.2, the claim is true. $\qquad\square$

Theorem 3.5.2 (Error bounds for general σ)**.** The error term in the pricing formula is bounded as

$$|\mathbb{E}(\mathcal{E}(\sigma))| \leq \sum_{|\alpha|=3} C_\alpha M_\alpha(T, K)T^{\alpha_2-\frac{1}{2}}S_0^{\alpha_1}\left(\mathbb{E}(\xi_T - 1)^{2\alpha_1}\right)^{1/2}\left\{\int_0^T (1 - \rho_u^2)^{2\alpha_2}\mathbb{E}|\sigma_u - \mathbb{E}(\sigma_u)|^{2\alpha_2}\mathrm{d}u\right\}^{1/2}$$

$$+ C\tilde{M}(T, K)S_0^2 T^{5/2}e^{\int_0^T \rho_m^2\mathbb{E}_{\mathbb{Q}_2}(\sigma_m)\mathrm{d}m}\left(\mathbb{E}_{\mathbb{Q}_2}e^{\int_0^T 2\rho_m^2|\sigma_m - \mathbb{E}_{\mathbb{Q}_2}(\sigma_m)|\mathrm{d}m}\right)^{1/2}$$

$$\cdot\left(\int_0^T \rho_u^{12}\mathbb{E}_{\mathbb{Q}_2}|\sigma_u - \mathbb{E}_{\mathbb{Q}_2}(\sigma_u)|^6\mathrm{d}u\right)^{1/2},$$

where $\tilde{M}(T, K) = \partial_{xx}\mathrm{Put}_{\mathrm{BS}}(\hat{x}, \hat{y})$ is bounded on $\mathbb{R}_+^2$ and $C = 1/12, C_\alpha = \frac{1}{3}\frac{|\alpha|}{\alpha_1!\alpha_2!}$ are constants, the latter depending on α.

Proof. First, by Proposition 3.5.1, we have that $E_\alpha(S_0\xi_T, \int_0^T \sigma_t(1 - \rho_t^2)) \leq \frac{1}{3}M_\alpha(T, K)$. By Theorem 3.5.1, the error is decomposed as $\mathcal{E}(\sigma) = \mathcal{E}_{\mathrm{BS}}(\sigma) + \tilde{\mathcal{E}}(\sigma)$. We will make use of the integral inequality

$$\left(\int_0^T |f(u)|\mathrm{d}u\right)^p \leq T^{p-1}\int_0^T |f(u)|^p\mathrm{d}u, \quad p \geq 1. \tag{3.19}$$

For the term $\mathcal{E}_{\mathrm{BS}}(\sigma)$, we have

$$|\mathcal{E}_{\mathrm{BS}}(\sigma)| \leq \sum_{|\alpha|=3} C_\alpha M_\alpha(T, K)S_0^{\alpha_1}|\xi_T - 1|^{\alpha_1}T^{\alpha_2-1}\left(\int_0^T (1 - \rho_u^2)^{\alpha_2}|\sigma_u - \mathbb{E}(\sigma_u)|^{\alpha_2}\right),$$

where we have used the integral inequality eq. (5.31). Applying the Cauchy-Schwarz inequality, we obtain

$$\mathbb{E}|\mathcal{E}_{\mathrm{BS}}(\sigma)| \leq \sum_{|\alpha|=3} C_\alpha M_\alpha(T,K) S_0^{\alpha_1} T^{\alpha_2-1} \left(\mathbb{E}|\xi_T - 1|^{2\alpha_1}\right)^{1/2} \left\{\mathbb{E}\left(\int_0^T (1-\rho_u^2)^{\alpha_2}|\sigma_u - \mathbb{E}(\sigma_u)|^{\alpha_2}\right)^2\right\}^{1/2}$$

$$\leq \sum_{|\alpha|=3} C_\alpha M_\alpha(T,K) S_0^{\alpha_1} T^{\alpha_2-1} \left(\mathbb{E}|\xi_T - 1|^{2\alpha_1}\right)^{1/2} T^{1/2} \left\{\mathbb{E}\left(\int_0^T (1-\rho_u^2)^{2\alpha_2}|\sigma_u - \mathbb{E}(\sigma_u)|^{2\alpha_2}\right)\right\}^{1/2},$$

where we have used the integral inequality eq. (5.31) for the second inequality.

For the term $\tilde{\mathcal{E}}(\sigma)$, notice that

$$\mathbb{E}(\tilde{\mathcal{E}}(\sigma)) = \mathbb{E}_{\mathbb{Q}_2}\left(\tilde{\mathcal{E}}(\sigma)\xi_T^{-2} e^{\int_0^T \rho_u^2 \sigma_u \mathrm{d}u}\right) = \frac{1}{4}\partial_{xx}\mathrm{Put}_{\mathrm{BS}}(\hat{x},\hat{y}) S_0^2 \mathbb{E}_{\mathbb{Q}_2}\left\{\left(\int_0^T \rho_u^2(\sigma_u - \mathbb{E}_{\mathbb{Q}_2}(\sigma_u))\mathrm{d}u\right)^3\right.$$

$$\left.\cdot \int_0^1 (1-u)^2 e^{\int_0^T \rho_m^2 \mathbb{E}_{\mathbb{Q}_2}(\sigma_m)\mathrm{d}m} e^{u\int_0^T \rho_m^2(\sigma_m - \mathbb{E}_{\mathbb{Q}_2}(\sigma_m))\mathrm{d}m}\mathrm{d}u\right\}.$$

Now for $u \in (0,1)$, $e^{u\int_0^T \rho_m^2(\sigma_m - \mathbb{E}_{\mathbb{Q}_2}(\sigma_m))\mathrm{d}m} \leq e^{u\int_0^T \rho_m^2|\sigma_m - \mathbb{E}_{\mathbb{Q}_2}(\sigma_m)|\mathrm{d}m}$. Thus

$$\sup_{u\in(0,1)} e^{u\int_0^T \rho_m^2(\sigma_m - \mathbb{E}_{\mathbb{Q}_2}(\sigma_m))\mathrm{d}m} \leq e^{\int_0^T \rho_m^2|\sigma_m - \mathbb{E}_{\mathbb{Q}_2}(\sigma_m)|\mathrm{d}m}.$$

Hence

$$\int_0^1 (1-u)^2 e^{\int_0^T \rho_m^2 \mathbb{E}_{\mathbb{Q}_2}(\sigma_m)\mathrm{d}m} e^{u\int_0^T \rho_m^2(\sigma_m - \mathbb{E}_{\mathbb{Q}_2}(\sigma_m))\mathrm{d}m}\mathrm{d}u \leq \frac{1}{3}e^{\int_0^T \rho_m^2\left(\mathbb{E}_{\mathbb{Q}_2}(\sigma_m) + |\sigma_m - \mathbb{E}_{\mathbb{Q}_2}(\sigma_m)|\right)\mathrm{d}m}.$$

Thus

$$\mathbb{E}|\tilde{\mathcal{E}}(\sigma)| \leq C\partial_{xx}\mathrm{Put}_{\mathrm{BS}}(\hat{x},\hat{y}) S_0^2 e^{\int_0^T \rho_m^2 \mathbb{E}_{\mathbb{Q}_2}(\sigma_m)\mathrm{d}m}\mathbb{E}_{\mathbb{Q}_2}\left\{\left(\int_0^T \rho_u^2(\sigma_u - \mathbb{E}_{\mathbb{Q}_2}(\sigma_u))\mathrm{d}u\right)^3 e^{\int_0^T \rho_m^2|\sigma_m - \mathbb{E}_{\mathbb{Q}_2}(\sigma_m)|\mathrm{d}m}\right\}.$$

Finally, using the Cauchy-Schwarz inequality and the integral inequality eq. (5.31), we obtain

$$\mathbb{E}|\tilde{\mathcal{E}}(\sigma)| \leq C\partial_{xx}\mathrm{Put}_{\mathrm{BS}}(\hat{x},\hat{y}) S_0^2 e^{\int_0^T \rho_m^2 \mathbb{E}_{\mathbb{Q}_2}(\sigma_m)\mathrm{d}m} T^{5/2}\left(\int_0^T \rho_u^{12}\mathbb{E}_{\mathbb{Q}_2}|\sigma_u - \mathbb{E}_{\mathbb{Q}_2}(\sigma_u)|^6 \mathrm{d}u\right)^{1/2}$$

$$\cdot \left(\mathbb{E}_{\mathbb{Q}_2}\left(e^{\int_0^T 2\rho_m^2|\sigma_m - \mathbb{E}_{\mathbb{Q}_2}(\sigma_m)|\mathrm{d}m}\right)\right)^{1/2}.$$

Furthermore, notice that $\partial_{xx}\mathrm{Put}_{\mathrm{BS}}(\hat{x},\hat{y}) = \tilde{M}(T,K)$, where $\tilde{M}(T,K)$ is a function which behaves like $M(T,K)$. $\qquad \square$

Corollary 3.5.3 (Error bounds for general σ: $\rho = 0$)**.** For $\rho = 0$ a.e., the error term in the pricing formula is bounded as

$$|\mathbb{E}(\mathcal{E}_0(\sigma))| \leq CM(T,K)T^2 \int_0^T \mathbb{E}|\sigma_t - \mathbb{E}(\sigma_t)|^3 \mathrm{d}t,$$

where $C = 1/6$ is a constant.

Proof. From Corollary 3.5.1, we have

$$\mathcal{E}_0(\sigma) = \frac{1}{2} \left(\int_0^T (\sigma_t - \mathbb{E}(\sigma_t)) \mathrm{d}t \right)^3 \int_0^1 (1-u)^2 \partial_{yyy} \mathrm{Put}_{\mathrm{BS}} \left(S_0, \tilde{G}(u) \right) \mathrm{d}u.$$

Notice that $(\int_0^T |f(t)| \mathrm{d}t)^3 \leq T^2 \int_0^T |f(t)|^3 \mathrm{d}t$ and use Corollary 3.5.2. Then the result is immediate. $\qquad\square$

In this section, we have derived a bound on the error term that depends directly on the higher moments of the underlying variance process, and not through the partial derivatives of $\mathrm{Put}_{\mathrm{BS}}$. To do this, we notice that the partial derivatives appear in integrals with arguments that are linear functions in the dummy variable u. We show the supremum of the partial derivatives in the dummy variable u are functions in T and K whose behaviour for fixed K and T is characterised by the functions ζ and η defined in Lemma 3.5.2. Then, standard inequalities from stochastic analysis are used to obtain the final form of the bound.

Chapter 4

Change of measure methodology: numerical implementation

In this chapter, we devise a fast calibration scheme using the second-order approximation formulas from Chapter 3 under the assumption of piecewise-constant parameters. We also perform a numerical error and sensitivity analysis for the Heston and GARCH models in order to assess our approximation formula in application.

- Section 4.1 details our fast calibration scheme. In particular, we rewrite the pricing functions found for the Heston and GARCH models in Chapter 3 in terms of specific integral operators, which can be shown to satisfy some convenient recursive properties when parameters are assumed to be piecewise-constant.

- Section 4.2 is dedicated to a numerical error and sensitivity analysis for the Heston and GARCH models.

4.1 Fast calibration

To this end, define the integral operator

$$\omega_T^{(k,l)} := \int_0^T l_u e^{\int_0^u k_z \mathrm{d}z} \mathrm{d}u. \tag{4.1}$$

In addition, we define the n-fold integral operator using the following recurrence:

$$\omega_T^{(k^{(n)},l^{(n)}),(k^{(n-1)},l^{(n-1)}),\dots,(k^{(1)},l^{(1)})} := \omega_T^{\left(k^{(n)},l^{(n)}w_\cdot^{(k^{(n-1)},l^{(n-1)}),\cdots,(k^{(1)},l^{(1)})}\right)}, \quad n \in \mathbb{N}. \tag{4.2}$$

[1]Let $\mathcal{T} = \{0 = T_0, T_1, \dots, T_{N-1}, T_N = T\}$ be a collection of maturity dates on $[0,T]$, with $\Delta T_i := T_{i+1} - T_i$ and $\Delta T_0 \equiv 1$. When the dummy functions are piecewise-constant, that is, $l_t^{(n)} = l_i^{(n)}$ on $t \in [T_i, T_{i+1})$ and similarly for $k^{(n)}$, then we can recursively calculate the integral operators eq. (4.1) and eq. (4.2). Define

$$e_t^{(k^{(n)},\dots,k^{(1)})} := e^{\int_0^t \sum_{j=1}^n k_z^{(j)} \mathrm{d}z},$$

$$\varphi_{T_i,t}^{(k,p)} := \int_{T_i}^t \gamma_i^p(u) e^{\int_{T_i}^u k_z \mathrm{d}z} \mathrm{d}u,$$

---

[1]For example

$$\omega_T^{(k^{(3)},l^{(3)}),(k^{(2)},l^{(2)}),(k^{(1)},l^{(1)})} = \int_0^T l_{u_3}^{(3)} e^{\int_0^{u_3} k_z^{(3)} \mathrm{d}z} \left(\int_0^{u_3} l_{u_2}^{(2)} e^{\int_0^{u_2} k_z^{(2)} \mathrm{d}z} \left(\int_0^{u_2} l_{u_1}^{(1)} e^{\int_0^{u_1} k_z^{(1)} \mathrm{d}z} \mathrm{d}u_1 \right) \mathrm{d}u_2 \right) \mathrm{d}u_3.$$

where $\gamma_i(u) := (u - T_i)/\Delta T_i$ and $p \in \mathbb{N} \cup \{0\}$. In addition, define recursively

$$\varphi_{T_i,t}^{(k^{(n)},p_n),\ldots,(k^{(2)},p_2),(k^{(1)},p_1)} := \int_{T_i}^t \gamma_i^{p_n}(u) e^{\int_{T_i}^u k_z^{(n)} \mathrm{d}z} \varphi_{T_i,u}^{(k^{(n-1)},p_{n-1}),\ldots,(k^{(2)},p_2),(k^{(1)},p_1)} \mathrm{d}u,$$

where $p_n \in \mathbb{N} \cup \{0\}$.[2] With the assumption that the dummy functions are piecewise-constant, we can obtain the integral operator at time T_{i+1} expressed by terms at time T_i.

$$\omega_{T_{i+1}}^{(k^{(1)},l^{(1)})} = \omega_{T_i}^{(k^{(1)},l^{(1)})} + l_i^{(1)} e_{T_i}^{(k^{(1)})} \varphi_{T_i,T_{i+1}}^{(k^{(1)},0)},$$

$$\begin{aligned}
\omega_{T_{i+1}}^{(k^{(2)},l^{(2)}),(k^{(1)},l^{(1)})} = {}& \omega_{T_i}^{(k^{(2)},l^{(2)}),(k^{(1)},l^{(1)})} \\
&+ l_i^{(2)} e_{T_i}^{(k^{(2)})} \varphi_{T_i,T_{i+1}}^{(k^{(2)},0)} \omega_{T_i}^{(k^{(1)},l^{(1)})} \\
&+ l_i^{(2)} l_i^{(1)} e_{T_i}^{(k^{(2)},k^{(1)})} \varphi_{T_i,T_{i+1}}^{(k^{(2)},0),(k^{(1)},0)},
\end{aligned}$$

$$\begin{aligned}
\omega_{T_{i+1}}^{(k^{(3)},l^{(3)}),(k^{(2)},l^{(2)}),(k^{(1)},l^{(1)})} = {}& \omega_{T_i}^{(k^{(3)},l^{(3)}),(k^{(2)},l^{(2)}),(k^{(1)},l^{(1)})} \\
&+ l_i^{(3)} e_{T_i}^{(k^{(3)})} \varphi_{T_i,T_{i+1}}^{(k^{(3)},0)} \omega_{T_i}^{(k^{(2)},l^{(2)}),(k^{(1)},l^{(1)})} \\
&+ l_i^{(3)} l_i^{(2)} e_{T_i}^{(k^{(3)},k^{(2)})} \varphi_{T_i,T_{i+1}}^{(k^{(3)},0),(k^{(2)},0)} \omega_{T_i}^{(k^{(1)},l^{(1)})} \\
&+ l_i^{(3)} l_i^{(2)} l_i^{(1)} e_{T_i}^{(k^{(3)},k^{(2)},k^{(1)})} \varphi_{T_i,T_{i+1}}^{(k^{(3)},0),(k^{(2)},0),(k^{(1)},0)},
\end{aligned}$$

$$\begin{aligned}
\omega_{T_{i+1}}^{(k^{(4)},l^{(4)}),\ldots,(k^{(1)},l^{(1)})} = {}& \omega_{T_i}^{(k^{(4)},l^{(4)}),(k^{(3)},l^{(3)}),(k^{(2)},l^{(2)}),(k^{(1)},l^{(1)})} \\
&+ l_i^{(4)} e_{T_i}^{(k^{(4)})} \varphi_{T_i,T_{i+1}}^{(k^{(4)},0)} \omega_{T_i}^{(k^{(3)},l^{(3)}),(k^{(2)},l^{(2)}),(k^{(1)},l^{(1)})} \\
&+ l_i^{(4)} l_i^{(3)} e_{T_i}^{(k^{(4)},k^{(3)})} \varphi_{T_i,T_{i+1}}^{(k^{(4)},0),(k^{(3)},0)} \omega_{T_i}^{(k^{(2)},l^{(2)}),(k^{(1)},l^{(1)})} \\
&+ l_i^{(4)} l_i^{(3)} l_i^{(2)} e_{T_i}^{(k^{(4)},k^{(3)},k^{(2)})} \varphi_{T_i,T_{i+1}}^{(k^{(4)},0),(k^{(3)},0),(k^{(2)},0)} \omega_{T_i}^{(k^{(1)},l^{(1)})} \\
&+ l_i^{(4)} l_i^{(3)} l_i^{(2)} l_i^{(1)} e_{T_i}^{(k^{(4)},k^{(3)},k^{(2)},k^{(1)})} \varphi_{T_i,T_{i+1}}^{(k^{(4)},0),(k^{(3)},0),(k^{(2)},0),(k^{(1)},0)},
\end{aligned}$$

$$\begin{aligned}
\omega_{T_{i+1}}^{(k^{(5)},l^{(5)}),\ldots,(k^{(1)},l^{(1)})} = {}& \omega_{T_i}^{(k^{(5)},l^{(5)}),(k^{(4)},l^{(4)}),(k^{(3)},l^{(3)}),(k^{(2)},l^{(2)}),(k^{(1)},l^{(1)})} \\
&+ l_i^{(5)} e_{T_i}^{(k^{(5)})} \varphi_{T_i,T_{i+1}}^{(k^{(5)},0)} \omega_{T_i}^{(k^{(4)},l^{(4)}),(k^{(3)},l^{(3)}),(k^{(2)},l^{(2)}),(k^{(1)},l^{(1)})} \\
&+ l_i^{(5)} l_i^{(4)} e_{T_i}^{(k^{(5)},k^{(4)})} \varphi_{T_i,T_{i+1}}^{(k^{(5)},0),(k^{(4)},0)} \omega_{T_i}^{(k^{(3)},l^{(3)}),(k^{(2)},l^{(2)}),(k^{(1)},l^{(1)})} \\
&+ l_i^{(5)} l_i^{(4)} l_i^{(3)} e_{T_i}^{(k^{(5)},k^{(4)},k^{(3)})} \varphi_{T_i,T_{i+1}}^{(k^{(5)},0),(k^{(4)},0),(k^{(3)},0)} \omega_{T_i}^{(k^{(2)},l^{(2)}),(k^{(1)},l^{(1)})} \\
&+ l_i^{(5)} l_i^{(4)} l_i^{(3)} l_i^{(2)} e_{T_i}^{(k^{(5)},k^{(4)},k^{(3)},k^{(2)})} \varphi_{T_i,T_{i+1}}^{(k^{(5)},0),(k^{(4)},0),(k^{(3)},0),(k^{(2)},0)} \omega_{T_i}^{(k^{(1)},l^{(1)})} \\
&+ l_i^{(5)} l_i^{(4)} l_i^{(3)} l_i^{(2)} l_i^{(1)} e_{T_i}^{(k^{(5)},k^{(4)},k^{(3)},k^{(2)},k^{(1)})} \varphi_{T_i,T_{i+1}}^{(k^{(5)},0),(k^{(4)},0),(k^{(3)},0),(k^{(2)},0),(k^{(1)},0)}.
\end{aligned}$$

---

[2]For example

$$\varphi_{T_i,t}^{(k^{(3)},p_3),(k^{(2)},p_2),(k^{(1)},p_1)} = \int_{T_i}^t \gamma_i^{p_3}(u_3) e^{\int_{T_i}^{u_3} k_z^{(3)} \mathrm{d}z} \left(\int_{T_i}^{u_3} \gamma_i^{p_2}(u_2) e^{\int_{T_i}^{u_2} k_z^{(2)} \mathrm{d}z} \left(\int_{T_i}^{u_2} \gamma_i^{p_1}(u_1) e^{\int_{T_i}^{u_1} k_z^{(1)} \mathrm{d}z} \mathrm{d}u_1 \right) \mathrm{d}u_2 \right) \mathrm{d}u_3.$$

[3]The only terms here that are not explicit are the functions $e_{\cdot}^{(\cdot,\dots,\cdot)}$ and $\varphi_{T_i,\cdot}^{(\cdot,\cdot),\dots,(\cdot,\cdot)}$. For $t \in (T_i, T_{i+1}]$, we can derive the following:

$$e_t^{(k^{(n)},\dots,k^{(1)})} = e_{T_i}^{(k^{(n)},\dots,k^{(1)})} e^{\Delta T_i \gamma_i(t) \sum_{j=1}^n k_i^{(j)}} = e^{\sum_{m=0}^{i-1} \Delta T_m \sum_{j=1}^n k_m^{(j)}} e^{\Delta T_i \gamma_i(t) \sum_{j=1}^n k_i^{(j)}},$$

where $e_0^{(k^{(n)},\dots,k^{(1)})} = 1$. By using integration by parts and basic integration properties, we find that

$$\varphi_{T_i,t}^{(k,p)} = \begin{cases} \frac{1}{k_i}\left(\gamma_i^p(t) e^{k_i \Delta T_i \gamma_i(t)} - \frac{p}{\Delta T_i}\varphi_{T_i,t}^{(k,p-1)}\right), & k_i \neq 0, p \geq 1, \\[2mm] \frac{1}{k_i}\left(e^{k_i \Delta T_i \gamma_i(t)} - 1\right), & k_i \neq 0, p = 0, \\[2mm] \frac{1}{p+1}\Delta T_i \gamma_i^{p+1}(t), & k_i = 0, p \geq 0. \end{cases}$$

In addition, for $n \geq 2$,

$$\varphi_{T_i,t}^{(k^{(n)},p_n),\dots,(k^{(1)},p_1)} = \begin{cases} \frac{1}{k_i^{(n)}}\left(\gamma_i^{p_n}(t) e^{k_i^{(n)}\Delta T_i \gamma_i(t)}\varphi_{T_i,t}^{(k^{(n-1)},p_{n-1}),\dots,(k^{(1)},p_1)}\right. \\ \quad -\frac{p_n}{\Delta T_i}\varphi_{T_i,t}^{(k^{(n)},p_n-1),(k^{(n-1)},p_{n-1}),\dots,(k^{(1)},p_1)} \\ \quad \left.-\varphi_{T_i,t}^{(k^{(n)}+k^{(n-1)},p_n+p_{n-1}),(k^{(n-2)},p_{n-2}),\dots,(k^{(1)},p_1)}\right), & k_i^{(n)} \neq 0, p_n \geq 1, \\[4mm] \frac{1}{k_i^{(n)}}\left(e^{k_i^{(n)}\Delta T_i \gamma_i(t)}\varphi_{T_i,t}^{(k^{(n-1)},p_{n-1}),\dots,(k^{(1)},p_1)}\right. \\ \quad \left.-\varphi_{T_i,t}^{(k^{(n)}+k^{(n-1)},p_{n-1}),(k^{(n-2)},p_{n-2}),\dots,(k^{(1)},p_1)}\right), & k_i^{(n)} \neq 0, p_n = 0, \\[4mm] \frac{\Delta T_i}{p_n+1}\left(\gamma_i^{p_n+1}(t)\varphi_{T_i,t}^{(k^{(n-1)},p_{n-1}),\dots,(k^{(1)},p_1)}\right. \\ \quad \left.-\varphi_{T_i,t}^{(k^{(n-1)},p_n+p_{n-1}+1),(k^{(n-2)},p_{n-2}),\dots,(k^{(1)},p_1)}\right), & k_i^{(n)} = 0, p_n \geq 0. \end{cases}$$

To implement our fast calibration scheme, one executes the following algorithm. Let $\mu_t \equiv \mu = (\mu^{(1)}, \mu^{(2)}, \dots, \mu^{(n)})$ be an arbitrary set of parameters and denote by ω_t an arbitrary integral operator.

- Calibrate μ over $[0, T_1)$ to obtain μ_0. This involves computing ω_{T_1}.

- Calibrate μ over $[T_1, T_2)$ to obtain μ_1. This involves computing ω_{T_2} which is in terms of ω_{T_1}, the latter already being computed in the previous step.

- Repeat until time T_N.

4.1.1 Heston calibration scheme

From Theorem 3.4.1, recall the second-order put option price in the Heston model,

$$\mathrm{Put}_{\mathrm{H}}^{(2)} = \mathrm{Put}_{\mathrm{BS}}(\hat{x}, \hat{y})$$

$$+ \frac{1}{2}\partial_{xx}\mathrm{Put}_{\mathrm{BS}}(\hat{x}, \hat{y})S_0^2 \mathbb{E}(\xi_T - 1)^2 + \frac{1}{2}\partial_{yy}\mathrm{Put}_{\mathrm{BS}}(\hat{x}, \hat{y})\mathbb{E}\left(\int_0^T (1 - \rho_t^2)(V_t - \mathbb{E}(V_t))\mathrm{d}t\right)^2$$

$$+ \partial_{xy}\mathrm{Put}_{\mathrm{BS}}(\hat{x}, \hat{y})S_0 \mathbb{E}\left\{(\xi_T - 1)\left(\int_0^T (1 - \rho_t^2)(V_t - \mathbb{E}(V_t))\mathrm{d}t\right)\right\}.$$

---

[3]In general

$$\omega_{T_{i+1}}^{(k^{(n)},l^{(n)}),\dots,(k^{(2)},l^{(2)}),(k^{(1)},l^{(1)})} = \sum_{m=1}^{n+1} \omega_{T_i}^{(k^{(n-m+1)},l^{(n-m+1)}),\dots,(k^{(1)},l^{(1)})} \left(\prod_{j=0}^{m-2} l_i^{(n-j)}\right) e_{T_i}^{(k^{(n-m+2)},\dots,k^{(1)})} \varphi_{T_i,T_{i+1}}^{(k^{(n-m+2)},0),\dots,(k^{(1)},0)},$$

where whenever the index goes outside of $\{1, \dots, n\}$, then that term is equal to 1.

The three expectations were calculated in Section 3.4.1. We can write them in terms of the integral operators eq. (6.1) and eq. (6.2).

$$\mathbb{E}(\xi_T - 1)^2 \approx \exp\left\{ v_0\omega_T^{(-(\kappa-2\lambda\rho),\rho^2)} + \omega_T^{(-(\kappa-2\lambda\rho),\rho^2),(\kappa-2\lambda\rho,\kappa\theta)} \right\}\left\{ 1 + v_0\omega_T^{(-(\kappa-2\lambda\rho),\rho^2),(-(\kappa-2\lambda\rho),\rho^2),(\kappa-2\lambda\rho,\lambda^2)} \right. $$
$$\left. + \omega_T^{(-(\kappa-2\lambda\rho),\rho^2),(-(\kappa-2\lambda\rho),\rho^2),(\kappa-2\lambda\rho,\lambda^2),(\kappa-2\lambda\rho,\kappa\theta)} \right\} - 1.$$

$$\mathbb{E}\left(\int_0^T (1-\rho_t^2)(V_t - \mathbb{E}(V_t))\mathrm{d}t \right)^2 = 2v_0\omega_T^{(-\kappa,1-\rho^2),(-\kappa,1-\rho^2),(\kappa,\lambda^2)} + 2\omega_T^{(-\kappa,1-\rho^2),(-\kappa,1-\rho^2),(\kappa,\lambda^2),(\kappa,\kappa\theta)}.$$

$$\mathbb{E}\left\{ (\xi_T - 1)\left(\int_0^T (1-\rho_t^2)(V_t - \mathbb{E}(V_t))\mathrm{d}t \right) \right\} = v_0\left(\omega_T^{(-(\kappa-\lambda\rho),1-\rho^2)} - \omega_T^{(-\kappa,1-\rho^2)} \right) + \omega_T^{(-(\kappa-\lambda\rho),1-\rho^2),(\kappa-\lambda\rho,\kappa\theta)} $$
$$- \omega_T^{(-\kappa,1-\rho^2),(\kappa,\kappa\theta)}.$$

Furthermore $\hat{x} = S_0$ and $\hat{y} = v_0\omega_T^{(-\kappa,1-\rho^2)} + \omega_T^{(-\kappa,1-\rho^2),(\kappa,\kappa\theta)}$.

Assuming the parameters are all piecewise-constant on $\{0 = T_0, T_1, \ldots, T_{N-1}, T_N = T\}$, that is,

$$(\kappa_t, \theta_t, \lambda_t, \rho_t) = (\kappa_i, \theta_i, \lambda_i, \rho_i), \quad t \in [T_i, T_{i+1}), \quad i = 0, \ldots N - 1,$$

then we can use the scheme presented in Section 4.1 to calibrate the Heston parameters forwards in time.

4.1.2 GARCH calibration scheme: $\rho = 0$

From Theorem 3.4.2, recall the second-order put option price in the GARCH model,

$$\mathrm{Put}^{(2)}_{\mathrm{GARCH}} = \mathrm{Put}_{\mathrm{BS}}(\hat{x}, \hat{y}) + \frac{1}{2}\partial_{yy}\mathrm{Put}_{\mathrm{BS}}(\hat{x}, \hat{y})\mathbb{E}\left(\int_0^T (V_t - \mathbb{E}(V_t))\mathrm{d}t \right)^2.$$

We can write the expectation in terms of the integral operators eq. (6.1) and eq. (6.2).

$$\mathbb{E}\left(\int_0^T (V_t - \mathbb{E}(V_t))\mathrm{d}t \right)^2 = 2\left(v_0^2\omega_T^{(-\kappa,1),(-\kappa,1),(\lambda^2,\lambda^2)} + 2v_0\omega_T^{(-\kappa,1),(-\kappa,1),(\lambda^2,\lambda^2),(-(\lambda^2-\kappa),\kappa\theta)} \right.$$
$$\left. + 2\omega_T^{(-\kappa,1),(-\kappa,1),(\lambda^2,\lambda^2),(-(\lambda^2-\kappa),\kappa\theta),(\kappa,\kappa\theta)} \right).$$

Furthermore $\hat{x} = S_0$ and

$$\hat{y} = \int_0^T \mathbb{E}(V_t)\mathrm{d}t = v_0\omega_T^{(-\kappa,1)} + \omega_T^{(-\kappa,1),(\kappa,\kappa\theta)}.$$

Assuming the parameters are all piecewise-constant on $\{0 = T_0, T_1, \ldots, T_{N-1}, T_N = T\}$, that is,

$$(\kappa_t, \theta_t, \lambda_t) = (\kappa_i, \theta_i, \lambda_i) \quad t \in [T_i, T_{i+1}), \quad i = 0, \ldots N - 1,$$

then we can use the scheme presented in Section 4.1 to calibrate the GARCH diffusion parameters forwards in time.

4.2 Numerical tests and sensitivity analysis

We test our approximation method by considering the sensitivity of our approximation with respect to one parameter at a time. Specifically, for an arbitrary parameter set $(\mu^{(1)}, \mu^{(2)}, \dots, \mu^{(n)})$, we vary only one of the $\mu^{(i)}$ at a time and keep the rest fixed. Then, we compute implied volatilities via our approximation method as well as the Monte Carlo for strikes corresponding to Put 10, 25 and ATM deltas. Specifically,

$$\text{Error}(\mu) = \sigma_{IM-Approx}(\mu, K) - \sigma_{IM-Monte}(\mu, K)$$

for K corresponding to Put 10, Put 25 and ATM strikes.

For all our simulations, we use 2,000,000 Monte Carlo paths, and 24 time steps per day. This is to reduce the Monte Carlo and discretisation errors sufficiently well.

4.2.1 Heston sensitivity analysis

We consider maturity times $T \in \{1/12, 3/12, 6/12, 1\}$. We start from a 'safe' parameter, which are parameters calibrated by Bloomberg USD/JPY FX option price data on 9/07/18. The safe parameter set is $(S_0, v_0, r_d, r_f) = (100.00, 0.0036, 0.02, 0)$ with

$$(\kappa, \theta, \lambda, \rho) = \begin{cases} (5.000, 0.019, 0.414, -0.391), & T = 1/12, \\ (5.000, 0.011, 0.414, -0.391), & T = 3/12, \\ (5.000, 0.009, 0.414, -0.391), & T = 6/12, \\ (5.000, 0.009, 0.414, -0.391), & T = 1. \end{cases}$$

In our analysis, we vary one of the $(\kappa, \theta, \lambda, \rho)$ and keep the rest fixed.[4]

Varying κ

We vary κ over the values $\{1, 2, 3, 4, 5, 6, 7, 8\}$.

Table 4.1: κ: Error for ATM implied volatilities in basis points

κ	1	2	3	4	5	6	7	8
1M	-1.31	0.10	1.12	1.85	2.40	2.82	3.14	3.39
3M	-51.25	-28.87	-16.79	-9.69	-5.25	-2.35	-0.39	0.97
6M	-125.36	-60.43	-31.45	-16.74	-8.64	-3.92	-1.07	0.70
1Y	-198.02	-72.32	-29.82	-12.27	-4.22	-0.32	1.66	2.67

---

[4]The Feller condition is

$$2\kappa\theta > \lambda^2.$$

We will use red text to indicate when the Feller condition is not satisfied. Note that in application, this condition is almost always violated. That is, parameters calibrated from market data almost always violate the Feller condition, see for example Clark [19], Da Fonseca and Grasselli [21], Ribeiro and Poulsen [54].

Table 4.2: κ: Error for Put 25 implied volatilities in basis points

κ	1	2	3	4	5	6	7	8
1M	-0.86	0.60	1.61	2.33	2.86	3.25	3.55	3.78
3M	-49.82	-27.87	-16.03	-9.08	-4.74	-1.92	-0.02	1.29
6M	-121.83	-57.64	-29.37	-15.03	-7.19	-2.66	0.05	1.70
1Y	-194.14	-69.80	-27.98	-10.85	-3.11	0.60	2.41	3.31

Table 4.3: κ: Error for Put 10 implied volatilities in basis points

κ	1	2	3	4	5	6	7	8
1M	-0.79	0.77	1.84	2.59	3.14	3.55	3.86	4.09
3M	-48.42	-26.74	-15.11	-8.30	-4.09	-1.35	0.47	1.72
6M	-118.47	-55.45	-27.71	-13.80	-6.19	-1.81	0.80	2.37
1Y	-191.35	-68.26	-27.22	-10.51	-2.94	0.65	2.41	3.27

Varying θ

We vary θ over the values
{7e-03, 10e-03, 13e-03, 16e-03, 19e-03, 22e-03, 25e-03, 28e-03}.

Table 4.4: θ: Error for ATM implied volatilities in basis points

θ	7e-03	10e-03	13e-03	16e-03	19e-03	22e-03	25e-03	28e-03
1M	-0.26	0.70	1.42	1.97	2.40	2.75	3.04	3.28
3M	-12.55	-6.34	-2.78	-0.55	0.95	2.02	2.80	3.39
6M	-14.60	-7.29	-3.45	-1.14	0.34	1.35	2.07	2.60
1Y	-8.21	-3.30	-0.81	0.62	1.51	2.10	2.50	2.79

Table 4.5: θ: Error for Put 25 implied volatilities in basis points

θ	7e-03	10e-03	13e-03	16e-03	19e-03	22e-03	25e-03	28e-03
1M	-0.20	0.77	1.51	2.07	2.50	2.86	3.15	3.40
3M	-11.62	-5.76	-2.38	-0.25	1.17	2.17	2.91	3.47
6M	-13.50	-6.47	-2.76	-0.56	0.86	1.82	2.50	3.00
1Y	-6.48	-1.69	0.75	2.17	3.06	3.67	4.10	4.41

Table 4.6: θ: Error for Put 10 implied volatilities in basis points

θ	7e-03	10e-03	13e-03	16e-03	19e-03	22e-03	25e-03	28e-03
1M	-0.09	0.87	1.58	2.12	2.54	2.88	3.16	3.39
3M	-11.88	-6.22	-3.01	-1.01	0.31	1.23	1.90	2.41
6M	-11.89	-4.97	-1.31	0.87	2.29	3.26	3.95	4.48
1Y	-6.33	-1.89	0.29	1.50	2.24	2.71	3.03	3.25

Varying λ

We vary λ over the values $\{0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9\}$.

Table 4.7: λ: Error for ATM implied volatilities in basis points

λ	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
1M	2.80	2.97	2.24	-0.56	-6.59	-16.77	-31.73	-51.86
3M	2.95	1.50	-4.39	-17.01	-37.79	-67.67	-107.02	-155.90
6M	2.77	0.53	-6.99	-22.03	-45.92	-79.29	-122.64	-176.50
1Y	3.08	1.78	-3.14	-13.33	-29.85	-53.44	-84.54	-123.02

Table 4.8: λ: Error for Put 25 implied volatilities in basis points

λ	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
1M	3.80	4.11	3.51	0.86	-4.98	-14.88	-29.54	-49.41
3M	3.59	2.42	-3.08	-15.25	-35.61	-64.91	-103.80	-152.11
6M	3.60	1.70	-5.50	-20.04	-43.33	-76.10	-118.78	-171.52
1Y	3.32	2.10	-2.73	-12.74	-29.21	-52.35	-82.97	-121.09

Table 4.9: λ: Error for Put 10 implied volatilities in basis points

λ	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
1M	3.56	3.91	3.40	0.90	-4.73	-14.36	-28.73	-48.09
3M	4.04	3.06	-2.19	-13.96	-33.74	-62.50	-100.87	-148.80
6M	3.84	2.06	-4.91	-19.30	-42.33	-74.85	-117.27	-169.57
1Y	3.71	2.78	-1.70	-11.29	-26.99	-49.50	-79.77	-117.51

Varying ρ

We vary ρ over the values $\{-0.7, -0.6, -0.5, -0.4, -0.3, -0.2, -0.1, 0\}$.

Table 4.10: ρ: Error for ATM implied volatilities in basis points

ρ	-0.7	-0.6	-0.5	-0.4	-0.3	-0.2	-0.1	0
1M	52.52	24.09	9.93	2.59	-1.25	-3.20	-4.08	-4.28
3M	53.60	21.85	4.84	-4.89	-10.63	-13.97	-15.71	-16.17
6M	52.02	20.13	2.84	-7.24	-13.33	-17.01	-18.99	-19.55
1Y	48.20	19.17	4.09	-4.33	-9.24	-12.11	-13.60	-13.91

Table 4.11: ρ: Error for Put 25 implied volatilities in basis points

ρ	-0.7	-0.6	-0.5	-0.4	-0.3	-0.2	-0.1	0
1M	55.43	26.08	11.36	3.66	-0.42	-2.54	-3.55	-3.85
3M	57.58	24.55	6.73	-3.55	-9.66	-13.29	-15.24	-15.89
6M	54.73	21.62	3.61	-6.90	-13.27	-17.11	-19.19	-19.83
1Y	49.97	19.92	4.28	-4.47	-9.55	-12.50	-14.02	-14.34

Table 4.12: ρ: Error for Put 10 implied volatilities in basis points

ρ	-0.7	-0.6	-0.5	-0.4	-0.3	-0.2	-0.1	0
1M	56.28	26.32	11.26	3.37	-0.79	-2.92	-3.91	-4.17
3M	60.03	26.03	7.61	-3.05	-9.42	-13.20	-15.28	-16.01
6M	58.74	24.43	5.63	-5.45	-12.23	-16.40	-18.75	-19.61
1Y	54.35	23.14	6.70	-2.62	-8.14	-11.45	-13.27	-13.87

The above sensitivity analysis is consistent with what we expect. For example, for large maturity T, large vol-vol λ or large correlation $|\rho|$, the component-wise variance of the difference in the expansion and evaluation point increases. Thus, when these parameters are large, we expect the approximation to break down. As we can see, this indeed occurs. For realistic parameter values we see that the magnitude of error is around 10-50bps, which is reasonable for application purposes.

4.2.2 GARCH sensitivity analysis

We start from the same 'safe' parameter set from Section 4.2.1, albeit with $\rho = 0$ always.

Varying κ

We vary κ over the values $\{1, 2, 3, 4, 5, 6, 7, 8\}$.

Table 4.13: κ: Error for ATM implied volatilities in basis points

κ	1	2	3	4	5	6	7	8
1M	0.044	0.090	0.122	0.146	0.163	0.174	0.182	0.186
3M	0.003	0.019	0.026	0.028	0.027	0.025	0.022	0.019
6M	0.020	0.027	0.027	0.024	0.021	0.019	0.016	0.015
1Y	-0.047	-0.015	-0.004	0.000	0.002	0.002	0.003	0.003

Table 4.14: κ: Error for Put 25 implied volatilities in basis points

κ	1	2	3	4	5	6	7	8
1M	0.044	0.092	0.127	0.152	0.171	0.183	0.192	0.197
3M	0.045	0.056	0.060	0.059	0.056	0.052	0.048	0.043
6M	-0.031	-0.013	-0.006	-0.003	-0.003	-0.002	-0.002	-0.002
1Y	-0.064	-0.029	-0.016	-0.010	-0.006	-0.004	-0.002	-0.002

Table 4.15: κ: Error for Put 10 implied volatilities in basis points

κ	1	2	3	4	5	6	7	8
1M	0.072	0.118	0.151	0.175	0.192	0.204	0.212	0.216
3M	-0.074	-0.052	-0.040	-0.034	-0.032	-0.031	-0.031	-0.031
6M	-0.004	0.004	0.005	0.003	0.001	0.000	-0.001	-0.002
1Y	-0.044	-0.013	-0.003	0.001	0.003	0.003	0.003	0.003

Varying θ

We vary θ over the values
$\{7e\text{-}03, 10e\text{-}03, 13e\text{-}03, 16e\text{-}03, 19e\text{-}03, 22e\text{-}03, 25e\text{-}03, 28e\text{-}03\}$.

Table 4.16: θ: Error for ATM implied volatilities in basis points

θ	7e-03	10e-03	13e-03	16e-03	19e-03	22e-03	25e-03	28e-03
1M	0.039	0.077	0.112	0.143	0.171	0.198	0.223	0.246
3M	0.046	0.061	0.073	0.085	0.095	0.104	0.113	0.121
6M	0.012	0.019	0.024	0.029	0.033	0.036	0.040	0.043
1Y	-0.040	-0.045	-0.050	-0.054	-0.058	-0.062	-0.066	-0.069

Table 4.17: θ: Error for Put 25 implied volatilities in basis points

θ	7e-03	10e-03	13e-03	16e-03	19e-03	22e-03	25e-03	28e-03
1M	0.051	0.089	0.124	0.156	0.185	0.212	0.237	0.261
3M	0.019	0.031	0.042	0.051	0.059	0.066	0.072	0.079
6M	0.013	0.017	0.020	0.023	0.026	0.029	0.031	0.033
1Y	-0.047	-0.055	-0.063	-0.069	-0.075	-0.081	-0.086	-0.091

Table 4.18: θ: Error for Put 10 implied volatilities in basis points

θ	7e-03	10e-03	13e-03	16e-03	19e-03	22e-03	25e-03	28e-03
1M	0.043	0.081	0.116	0.147	0.176	0.203	0.228	0.252
3M	0.024	0.038	0.049	0.059	0.067	0.075	0.082	0.089
6M	-0.010	-0.009	-0.008	-0.008	-0.007	-0.007	-0.007	-0.007
1Y	-0.034	-0.038	-0.042	-0.045	-0.049	-0.052	-0.055	-0.058

Varying λ

We vary λ over the values $\{0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9\}$.

Table 4.19: λ: Error for ATM implied volatilities in basis points

λ	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
1M	-0.184	-0.182	-0.179	-0.176	-0.173	-0.168	-0.161	-0.153
3M	-0.030	-0.026	-0.020	-0.012	0.000	0.016	0.040	0.073
6M	-0.015	-0.017	-0.018	-0.015	-0.007	0.010	0.039	0.083
1Y	0.004	0.008	0.013	0.021	0.034	0.055	0.088	0.135

Table 4.20: λ: Error for Put 25 implied volatilities in basis points

λ	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
1M	-0.194	-0.197	-0.200	-0.203	-0.204	-0.205	-0.204	-0.201
3M	-0.032	-0.028	-0.023	-0.015	-0.004	0.013	0.036	0.070
6M	-0.019	-0.024	-0.026	-0.025	-0.017	-0.002	0.026	0.069
1Y	0.008	0.014	0.021	0.032	0.048	0.073	0.109	0.160

Table 4.21: λ: Error for Put 10 implied volatilities in basis points

λ	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
1M	-0.185	-0.184	-0.183	-0.180	-0.177	-0.173	-0.167	-0.159
3M	-0.036	-0.033	-0.029	-0.023	-0.012	0.003	0.026	0.059
6M	-0.010	-0.010	-0.007	-0.001	0.011	0.031	0.063	0.112
1Y	-0.008	-0.011	-0.012	-0.009	-0.002	0.014	0.040	0.082

The GARCH error behaves well, with most errors being less than 1bp in magnitude. In contrast to the Heston analysis, this is most likely due to the case that the correlation ρ is assumed to be 0 always. Otherwise, the approximation behaves as we expect, with errors being larger for large maturity T and vol-vol λ, as the variance of the difference in the expansion and evaluation point will grow with these parameters.

Chapter 5

Malliavin calculus methodology

5.1 Introduction

In this chapter, we explore how a second-order expansion of the mixing solution, coupled with Malliavin calculus machinery can give a closed-form approximation for the price of a European put option. The reader who is not familiar with Malliavin calculus should not worry; we give a brief overview of the tools we will need in Section 5.4.1. Our method extends that of both Benhamou et al. [9] and Langrené et al. [44], where the Heston and Inverse-Gamma models with time-dependent parameters are considered respectively.[1] Specifically:

1. For the Heston model

$$
\begin{aligned}
\mathrm{d}S_t &= S_t((r_t^d - r_t^f)\mathrm{d}t + \sqrt{\sigma_t}\mathrm{d}W_t), \\
\mathrm{d}\sigma_t &= \kappa(\theta_t - \sigma_t)\mathrm{d}t + \lambda_t\sqrt{\sigma_t}\mathrm{d}B_t, \\
\mathrm{d}\langle W, B\rangle_t &= \rho_t\mathrm{d}t,
\end{aligned}
$$

   this has been studied by Benhamou et al. [9].

2. For the Inverse-Gamma model

$$
\begin{aligned}
\mathrm{d}S_t &= S_t((r_t^d - r_t^f)\mathrm{d}t + V_t\mathrm{d}W_t), \\
\mathrm{d}V_t &= \kappa_t(\theta_t - V_t)\mathrm{d}t + \lambda_t V_t\mathrm{d}B_t, \\
\mathrm{d}\langle W, B\rangle_t &= \rho_t\mathrm{d}t,
\end{aligned}
$$

   this has been tackled by Langrené et al. [44].

The purpose of this chapter is to extend the methodology used in these aforementioned papers to a framework where the volatility process is driven by an arbitrary drift and diffusion which satisfy the regularity conditions given in Assumption 5.2.1. The sections are structured as follows:

- Section 5.2 details some preliminary calculations. First, we reparametrise the volatility process in terms of a small perturbation parameter, obtaining the process $\left(V_t^{(\varepsilon)}\right)$. Additionally, we rewrite the expression for the price of a put option via the mixing solution.

---

[1]For the time-dependent Heston model in [9], the mean reversion speed κ is assumed to be constant.

- Section 5.3 investigates our expansion procedure, where we combine a Taylor expansion of a Black-Scholes formula with a small vol-vol expansion of the function $\varepsilon \mapsto V_t^{(\varepsilon)}$ and its variants. This gives a second-order approximation to the price of a put option.

- Section 5.4 is dedicated to the explicit calculation of terms induced by our expansion procedure from Section 5.3. In particular, we use Malliavin calculus techniques in order to reduce the corresponding terms down into expressions which are explicit.

- In Section 5.5 we give an explicit form for the error in our expansion methodology. In particular, we comment on the feasibility of bounding this error term in the stochastic Verhulst model.

5.2 Preliminary calculations

Consider the following general stochastic volatility model with volatility process V

$$
\begin{aligned}
\mathrm{d}S_t &= (r_t^d - r_t^f)S_t\mathrm{d}t + V_tS_t\mathrm{d}W_t, \quad S_0, \\
\mathrm{d}V_t &= \alpha(t, V_t)\mathrm{d}t + \beta(t, V_t)\mathrm{d}B_t, \quad V_0 = v_0, \\
\mathrm{d}\langle W, B\rangle_t &= \rho_t\mathrm{d}t,
\end{aligned}
\tag{5.1}
$$

where W and B are Brownian motions with instantaneous correlation $(\rho_t)_{0 \leq t \leq T}$, defined on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{Q})$. Here T is a finite time horizon, where $(r_t^d)_{0 \leq t \leq T}$ and $(r_t^f)_{0 \leq t \leq T}$ are the deterministic time-dependent domestic and foreign interest rates respectively. Furthermore, $(\mathcal{F}_t)_{0 \leq t \leq T}$ is the filtration generated by (W, B) which satisfies the usual assumptions.[2] In the following, $\mathbb{E}(\cdot)$ denotes the expectation under $\mathbb{Q}$, where $\mathbb{Q}$ is a risk-neutral measure which we assume to be chosen.

Notice that the drift and diffusion coefficients for V in eq. (5.1) are currently arbitrary. We will only need to restrict the regularity of the drift and diffusion coefficients if the expansion procedure demands it. In anticipation of this, we will make the following assumptions on the regularity of the drift and diffusion coefficients of V in eq. (5.1).

Assumption 5.2.1. For $t \in [0, T]$

1. α is Lipschitz continuous in x, uniformly in t.

2. β is Hölder continuous of order $\geq 1/2$ in x, uniformly in t.

3. There exists a weak solution of V.

4. α is twice differentiable a.e. in x.

5. β is differentiable a.e. in x.

Notice that by items 1 and 2 in Assumption 5.2.1, if a solution for V in eq. (5.1) exists, it will be pathwise unique, see Theorem 1.2.1. Coupled with item 3, this guarantees a pathwise unique strong solution for V, see Theorem 1.2.2. We will comment on the reasoning behind items 4 and 5 in full detail in Remark 5.3.2.

---

[2]Meaning that $(\mathcal{F}_t)_{0 \leq t \leq T}$ is right continuous and augmented by $\mathbb{Q}$ null-sets.

5.2.1 Pricing a put option

Denote the price of a put option on S in the general model eq. (5.1) by $\mathrm{Put_G}$, so that

$$\mathrm{Put_G} = e^{-\int_0^T r_t^d \mathrm{d}t} \mathbb{E}(e^k - S_T)_+.$$

Let the process X denote the log-spot. That is, $X_t := \ln S_t$. Now perturb X in the following way: for $\varepsilon \in [0, 1]$,

$$\begin{aligned}
\mathrm{d}X_t^{(\varepsilon)} &= \left(r_t^d - r_t^f - \frac{1}{2}(V_t^{(\varepsilon)})^2\right)\mathrm{d}t + V_t^{(\varepsilon)}\mathrm{d}W_t, \quad X_0^{(\varepsilon)} = \ln S_0 =: x_0, \\
\mathrm{d}V_t^{(\varepsilon)} &= \alpha(t, V_t^{(\varepsilon)})\mathrm{d}t + \varepsilon\beta(t, V_t^{(\varepsilon)})\mathrm{d}B_t, \quad V_0^{(\varepsilon)} = v_0, \\
\mathrm{d}\langle W, B\rangle_t &= \rho_t\mathrm{d}t.
\end{aligned} \tag{5.2}$$

We can recover the original diffusion from eq. (5.2) by noticing $(S, V) = (\exp(X^{(1)}), V^{(1)})$.

Denote the filtration generated by B as $(\mathcal{F}_t^B)_{0 \leq t \leq T}$ and let $\tilde{X}_t^{(\varepsilon)} := X_t^{(\varepsilon)} - \int_0^t (r_u^d - r_u^f)\mathrm{d}u$. By writing $W_t = \int_0^t \rho_u \mathrm{d}B_u + \int_0^t \sqrt{1 - \rho_u^2}\mathrm{d}Z_u$, where Z is a Brownian motion independent of B, it can be seen that

$$\tilde{X}_T^{(\varepsilon)}|\mathcal{F}_T^B \overset{d}{=} \mathcal{N}(\hat{\mu}_\varepsilon(T), \hat{\sigma}_\varepsilon^2(T)),$$

with

$$\begin{aligned}
\hat{\mu}_\varepsilon(T) &:= x_0 - \int_0^T \frac{1}{2}(V_t^{(\varepsilon)})^2\mathrm{d}t + \int_0^T \rho_t V_t^{(\varepsilon)}\mathrm{d}B_t, \\
\hat{\sigma}_\varepsilon^2(T) &:= \int_0^T (1 - \rho_t^2)(V_t^{(\varepsilon)})^2\mathrm{d}t.
\end{aligned}$$

Let

$$g(\varepsilon) := e^{-\int_0^T r_u^d \mathrm{d}u} \mathbb{E}(e^k - e^{X_T^{(\varepsilon)}})_+.$$

Then $g(1)$ is the price of a put option in the general model eq. (5.1). That is, $g(1) = \mathrm{Put_G}$.

Proposition 5.2.1. The function g can be written as

$$g(\varepsilon) = \mathbb{E}\left\{e^{-\int_0^T r_u^d \mathrm{d}u}\mathbb{E}[(e^k - e^{X_T^{(\varepsilon)}})_+|\mathcal{F}_T^B]\right\} = \mathbb{E}\left[P_{\mathrm{BS}}\left(\hat{\mu}_\varepsilon(T) + \frac{1}{2}\hat{\sigma}_\varepsilon^2(T), \hat{\sigma}_\varepsilon^2(T)\right)\right],$$

where explicitly

$$\begin{aligned}
\hat{\mu}_\varepsilon(T) + \frac{1}{2}\hat{\sigma}_\varepsilon^2(T) &= x_0 - \int_0^T \frac{1}{2}\rho_t^2(V_t^{(\varepsilon)})^2\mathrm{d}t + \int_0^T \rho_t V_t^{(\varepsilon)}\mathrm{d}B_t, \\
\hat{\sigma}_\varepsilon^2(T) &= \int_0^T (1 - \rho_t^2)(V_t^{(\varepsilon)})^2\mathrm{d}t,
\end{aligned}$$

and

$$\begin{aligned}
P_{\mathrm{BS}}(x, y) &= e^k e^{-\int_0^T r_t^d \mathrm{d}t}\mathcal{N}(-d_-^{\mathrm{ln}}) - e^x e^{-\int_0^T r_t^f \mathrm{d}t}\mathcal{N}(-d_+^{\mathrm{ln}}), \\
d_\pm^{\mathrm{ln}} = d_\pm^{\mathrm{ln}}(x, y) &= \frac{x - k + \int_0^T (r_t^d - r_t^f)\mathrm{d}t}{\sqrt{y}} \pm \frac{1}{2}\sqrt{y}.
\end{aligned} \tag{5.3}$$

Proof. This is a consequence of the mixing solution methodology. A derivation can be found in Appendix C.2. $\qquad\square$

5.3 Expansion procedure

In this section, we detail our expansion procedure. The notation is similar to that in Benhamou et al. [9], however there are some differences. The expansion procedure can be briefly summarised by two main steps:

1. First, we expand the function P_{BS} up to second-order. This step is given in Section 5.3.2.

2. Then, we expand the functions $\varepsilon \mapsto V_t^{(\varepsilon)}$ and $\varepsilon \mapsto \left(V_t^{(\varepsilon)}\right)^2$ up to second-order. This step is given in Section 5.3.1 and Section 5.3.3.

We then combine both these expansions in order to obtain a second-order approximation for the put option price, which is given in Theorem 5.3.1. However, this approximation is not explicit. We will obtain the explicit second-order approximation in Theorem 5.4.1.

Remark 5.3.1. Suppose $(t, \varepsilon) \mapsto \xi_t^{(\varepsilon)}$ is a $C([0,T] \times [0,1]; \mathbb{R})$ function smooth in ε. Denote by $\xi_{i,t}^{(\varepsilon)} := \frac{\partial^i \xi}{\partial \varepsilon^i}$ its i-th derivative in ε, and let $\xi_{i,t} := \xi_{i,t}^{(\varepsilon)}|_{\varepsilon=0}$. Then by a second-order Taylor expansion around $\varepsilon = 0$, we have the representation

$$\xi_t^{(\varepsilon)} = \xi_{0,t} + \xi_{1,t} + \frac{1}{2}\xi_{2,t} + \Theta_{2,t}^{(\varepsilon)}(\xi),$$

where Θ is the error term given by Taylor's theorem. Specifically, for $i \geq 0$

$$\Theta_{i,t}^{(\varepsilon)}(\xi) := \int_0^\varepsilon \frac{1}{i!}(\varepsilon - u)^i \xi_{i+1,t}^{(u)} \mathrm{d}u.$$

5.3.1 Expanding processes $\varepsilon \mapsto V_t^{(\varepsilon)}$ and $\varepsilon \mapsto \left(V_t^{(\varepsilon)}\right)^2$

Using the notation from Remark 5.3.1, we can now represent the functions $\varepsilon \mapsto V_t^{(\varepsilon)}$ and $\varepsilon \mapsto \left(V_t^{(\varepsilon)}\right)^2$ via a Taylor expansion around $\varepsilon = 0$ to second-order.

$$\begin{aligned}
V_t^{(\varepsilon)} &= v_{0,t} + \varepsilon V_{1,t} + \frac{1}{2}\varepsilon^2 V_{2,t} + \Theta_{2,t}^{(\varepsilon)}(V), \\
(V_t^{(\varepsilon)})^2 &= v_{0,t}^2 + 2\varepsilon v_{0,t} V_{1,t} + \varepsilon^2 \left(V_{1,t}^2 + v_{0,t} V_{2,t}\right) + \Theta_{2,t}^{(\varepsilon)}(V^2),
\end{aligned} \tag{5.4}$$

where $v_{0,t} := V_{0,t}$.

Lemma 5.3.1. The processes $(V_{1,t})$ and $(V_{2,t})$ satisfy the SDEs

$$\mathrm{d}V_{1,t} = \alpha_x(t, v_{0,t}) V_{1,t} \mathrm{d}t + \beta(t, v_{0,t}) \mathrm{d}B_t, \quad V_{1,0} = 0, \tag{5.5}$$

$$\mathrm{d}V_{2,t} = \left(\alpha_{xx}(t, v_{0,t})(V_{1,t})^2 + \alpha_x(t, v_{0,t}) V_{2,t}\right)\mathrm{d}t + 2\beta_x(t, v_{0,t}) V_{1,t} \mathrm{d}B_t, \quad V_{2,0} = 0, \tag{5.6}$$

with explicit solutions

$$V_{1,t} = e^{\int_0^t \alpha_x(z, v_{0,z})\mathrm{d}z} \int_0^t \beta(s, v_{0,s}) e^{-\int_0^s \alpha_x(z, v_{0,z})\mathrm{d}z} \mathrm{d}B_s, \tag{5.7}$$

$$V_{2,t} = e^{\int_0^t \alpha_x(z, v_{0,z})\mathrm{d}z} \left\{ \int_0^t \alpha_{xx}(s, v_{0,s})(V_{1,s})^2 e^{-\int_0^s \alpha_x(z, v_{0,z})\mathrm{d}z} \mathrm{d}s + \int_0^t 2\beta_x(s, v_{0,s}) V_{1,s} e^{-\int_0^s \alpha_x(z, v_{0,z})\mathrm{d}z} \mathrm{d}B_s \right\}. \tag{5.8}$$

Proof. We give a sketch of the proof for $(V_{1,t})$. First, we write

$$\mathrm{d}V_{1,t}^{(\varepsilon)} = \mathrm{d}\left(\partial_\varepsilon V_t^{(\varepsilon)}\right) = \partial_\varepsilon(\mathrm{d}V_t^{(\varepsilon)}).$$

The SDE for $V_t^{(\varepsilon)}$ is given in eq. (5.2). By differentiating, we obtain

$$\mathrm{d}V_{1,t}^{(\varepsilon)} = \alpha_x(t, V_t^{(\varepsilon)})V_{1,t}^{(\varepsilon)}\mathrm{d}t + \left[\varepsilon\beta_x(t, V_t^{(\varepsilon)})V_{1,t}^{(\varepsilon)} + \beta(t, V_t^{(\varepsilon)})\right]\mathrm{d}B_t, \quad V_{1,0}^{(\varepsilon)} = 0.$$

Letting $\varepsilon = 0$ gives the SDE eq. (5.5). Since the SDE is linear, it can be solved explicitly (see for example Klebaner [41]). This gives the result eq. (5.7). The calculations for $(V_{2,t})$ are similar. $\qquad\square$

Remark 5.3.2. We now comment on the extra clauses given in Assumption 5.2.1.

1. As explained before, items 1 to 3 guarantee a pathwise unique strong solution for V.

2. For Lemma 5.3.1 to be valid, it is clear that we will require the existence of α_{xx} and β_x. This is assumed via items 4 and 5.

5.3.2 Expanding P_{BS}

Let

$$\tilde{P}_T^{(\varepsilon)} := x_0 - \int_0^T \frac{1}{2}\rho_t^2\left(V_t^{(\varepsilon)}\right)^2 \mathrm{d}t + \int_0^T \rho_t V_t^{(\varepsilon)}\mathrm{d}B_t,$$

$$\tilde{Q}_T^{(\varepsilon)} := \int_0^T (1 - \rho_t^2)\left(V_t^{(\varepsilon)}\right)^2 \mathrm{d}t.$$

Immediately we have $\tilde{P}_T^{(\varepsilon)} = \hat{\mu}_\varepsilon(T) + \frac{1}{2}\hat{\sigma}_\varepsilon^2(T)$ and $\tilde{Q}_T^{(\varepsilon)} = \hat{\sigma}_\varepsilon^2(T)$. Hence from Proposition 5.2.1,

$$g(\varepsilon) = \mathbb{E}\left(P_{\mathrm{BS}}\left(\tilde{P}_T^{(\varepsilon)}, \tilde{Q}_T^{(\varepsilon)}\right)\right). \tag{5.9}$$

As $g(1)$ corresponds to the price of a put option, we are interested in approximating the expression P_{BS} at $\left(\tilde{P}_T^{(1)}, \tilde{Q}_T^{(1)}\right)$. To do this, we will expand P_{BS} around the point

$$\left(\tilde{P}_T^{(0)}, \tilde{Q}_T^{(0)}\right) = \left(x_0 - \int_0^T \frac{1}{2}\rho_t^2 v_{0,t}^2\mathrm{d}t + \int_0^T \rho_t v_{0,t}\mathrm{d}B_t, \int_0^T (1 - \rho_t^2)v_{0,t}^2\mathrm{d}t\right)$$

and evaluate at $\left(\tilde{P}_T^{(1)}, \tilde{Q}_T^{(1)}\right)$. Additionally, introduce the functions

$$\begin{aligned}
P_T^{(\varepsilon)} &:= \tilde{P}_T^{(\varepsilon)} - \tilde{P}_T^{(0)} \\
&= \int_0^T \rho_t(V_t^{(\varepsilon)} - v_{0,t})\mathrm{d}B_t - \frac{1}{2}\int_0^T \rho_t^2\left(\left(V_t^{(\varepsilon)}\right)^2 - v_{0,t}^2\right)\mathrm{d}t, \\
Q_T^{(\varepsilon)} &:= \tilde{Q}_T^{(\varepsilon)} - \tilde{Q}_T^{(0)} \\
&= \int_0^T (1 - \rho_t^2)\left(\left(V_t^{(\varepsilon)}\right)^2 - v_{0,t}^2\right)\mathrm{d}t,
\end{aligned}$$

and the shorthand

$$\tilde{P}_{\mathrm{BS}} := P_{\mathrm{BS}}\left(\tilde{P}_T^{(0)}, \tilde{Q}_T^{(0)}\right),$$

$$\frac{\partial^{i+j}\tilde{P}_{\mathrm{BS}}}{\partial x^i \partial y^j} := \frac{\partial^{i+j}P_{\mathrm{BS}}\left(\tilde{P}_T^{(0)}, \tilde{Q}_T^{(0)}\right)}{\partial x^i \partial y^j}.$$

Proposition 5.3.1. By a second-order Taylor expansion, the expression $P_{\mathrm{BS}}\left(\tilde{P}_T^{(1)}, \tilde{Q}_T^{(1)}\right)$ can be approximated to second-order as

$$
\begin{aligned}
P_{\mathrm{BS}}\left(\tilde{P}_T^{(1)}, \tilde{Q}_T^{(1)}\right) \approx{} & \tilde{P}_{\mathrm{BS}} + \left(\partial_x \tilde{P}_{\mathrm{BS}}\right) P_T^{(1)} + \left(\partial_y \tilde{P}_{\mathrm{BS}}\right) Q_T^{(1)} \\
& + \frac{1}{2}\left(\partial_{xx} \tilde{P}_{\mathrm{BS}}\right)\left(P_T^{(1)}\right)^2 + \frac{1}{2}\left(\partial_{yy} \tilde{P}_{\mathrm{BS}}\right)\left(Q_T^{(1)}\right)^2 + \left(\partial_{xy} \tilde{P}_{\mathrm{BS}}\right) P_T^{(1)} Q_T^{(1)}.
\end{aligned}
$$

5.3.3 Expanding functions $\varepsilon \mapsto P_T^{(\varepsilon)}$, $\varepsilon \mapsto Q_T^{(\varepsilon)}$ and its variants

The next step in our expansion procedure is to approximate the functions $\varepsilon \mapsto P_T^{(\varepsilon)}, \varepsilon \mapsto \left(P_T^{(\varepsilon)}\right)^2, \varepsilon \mapsto Q_T^{(\varepsilon)}, \varepsilon \mapsto \left(Q_T^{(\varepsilon)}\right)^2$ and $\varepsilon \mapsto P_T^{(\varepsilon)} Q_T^{(\varepsilon)}$. By Remark 5.3.1, we can write

$$
\begin{aligned}
P_T^{(\varepsilon)} &= P_{0,T} + \varepsilon P_{1,T} + \frac{1}{2}\varepsilon^2 P_{2,T} + \Theta_{2,T}^{(\varepsilon)}(P), \\
(P_T^{(\varepsilon)})^2 &= P_{0,T}^2 + 2\varepsilon P_{0,T} P_{1,T} + \varepsilon^2\left(P_{1,T}^2 + P_{0,T} P_{2,T}\right) + \Theta_{2,T}^{(\varepsilon)}(P^2),
\end{aligned}
\tag{5.10}
$$

$$
\begin{aligned}
Q_T^{(\varepsilon)} &= Q_{0,T} + \varepsilon Q_{1,T} + \frac{1}{2}\varepsilon^2 Q_{2,T} + \Theta_{2,T}^{(\varepsilon)}(Q), \\
(Q_T^{(\varepsilon)})^2 &= Q_{0,T}^2 + 2\varepsilon Q_{0,T} Q_{1,T} + \varepsilon^2\left(Q_{1,T}^2 + Q_{0,T} Q_{2,T}\right) + \Theta_{2,T}^{(\varepsilon)}(Q^2),
\end{aligned}
\tag{5.11}
$$

and

$$
\begin{aligned}
P_T^{(\varepsilon)} Q_T^{(\varepsilon)} ={} & P_{0,T} Q_{0,T} + \varepsilon\left(Q_{0,T} P_{1,T} + P_{0,T} Q_{1,T}\right), \\
& + \frac{1}{2}\varepsilon^2\left(Q_{0,T} P_{2,T} + P_{0,T} Q_{2,T} + 3\left(Q_{1,T} P_{2,T} + P_{1,T} Q_{2,T}\right)\right) + \Theta_{2,T}^{(\varepsilon)}(PQ).
\end{aligned}
\tag{5.12}
$$

This gives the following lemma.

Lemma 5.3.2. Equations (5.10) to (5.12) can be rewritten as

$$
\begin{aligned}
P_T^{(\varepsilon)} &= \varepsilon P_{1,T} + \frac{1}{2}\varepsilon^2 P_{2,T} + \Theta_{2,T}^{(\varepsilon)}(P), \\
(P_T^{(\varepsilon)})^2 &= \varepsilon^2 P_{1,T}^2 + \Theta_{2,T}^{(\varepsilon)}(P^2),
\end{aligned}
\tag{5.13}
$$

$$
\begin{aligned}
Q_T^{(\varepsilon)} &= \varepsilon Q_{1,T} + \frac{1}{2}\varepsilon^2 Q_{2,T} + \Theta_{2,T}^{(\varepsilon)}(Q), \\
(Q_T^{(\varepsilon)})^2 &= \varepsilon^2 Q_{1,T}^2 \Theta_{2,T}^{(\varepsilon)}(Q^2),
\end{aligned}
\tag{5.14}
$$

and

$$
P_T^{(\varepsilon)} Q_T^{(\varepsilon)} = \varepsilon^2 P_{1,T} Q_{1,T} + \Theta_{2,T}^{(\varepsilon)}(PQ),
\tag{5.15}
$$

respectively, where

$$
\begin{aligned}
P_{1,T} &= \int_0^T \rho_t V_{1,t}\,\mathrm{d}B_t - \int_0^T \rho_t^2 v_{0,t} V_{1,t}\,\mathrm{d}t, \\
P_{2,T} &= \int_0^T \rho_t V_{2,t}\,\mathrm{d}B_t - \int_0^T \rho_t^2\left(V_{1,t}^2 + v_{0,t} V_{2,t}\right)\mathrm{d}t, \\
Q_{1,T} &= 2\int_0^T (1 - \rho_t^2) v_{0,t} V_{1,t}\,\mathrm{d}t, \\
Q_{2,T} &= 2\int_0^T (1 - \rho_t^2)\left(V_{1,t}^2 + v_{0,t} V_{2,t}\right)\mathrm{d}t.
\end{aligned}
$$

Proof. First, notice that by their definitions, $P_{0,T} = P_T^{(0)} = \tilde{P}_T^{(0)} - \tilde{P}_T^{(0)} = 0$, and similarly $Q_{0,T} = 0$. We will show how to obtain the form of $P_{1,T}$, the rest being similar. By definition

$$P_{1,T}^{(\varepsilon)} = \partial_\varepsilon \left(P_T^{(\varepsilon)} \right) = \partial_\varepsilon \left(x_0 - \int_0^T \frac{1}{2} \rho_t^2 \left(V_t^{(\varepsilon)} \right)^2 \mathrm{d}t + \int_0^T \rho_t V_t^{(\varepsilon)} \mathrm{d}B_t \right)$$

$$= \int_0^T \rho_t V_{1,t}^{(\varepsilon)} \mathrm{d}B_t - \int_0^T \rho_t^2 V_t^{(\varepsilon)} V_{1,t}^{(\varepsilon)} \mathrm{d}t.$$

By putting $\varepsilon = 0$ we obtain $P_{1,T}$, that is

$$P_{1,T} = \int_0^T \rho_t V_{1,t} \mathrm{d}B_t - \int_0^T \rho_t^2 v_{0,t} V_{1,t} \mathrm{d}t.$$

$\square$

Theorem 5.3.1 (Second-order put option price approximation). Denote by $\mathrm{Put}_{\mathrm{G}}^{(2)}$ the second-order approximation to the price of a put option in eq. (5.1). Then

$$\mathrm{Put}_{\mathrm{G}}^{(2)} = \mathbb{E}\tilde{P}_{\mathrm{BS}}$$

$$(C_x :=) \qquad + \mathbb{E}\partial_x \tilde{P}_{\mathrm{BS}} \left(\int_0^T \rho_t \left(V_{1,t} + \frac{1}{2} V_{2,t} \right) \mathrm{d}B_t \right.$$

$$\left. - \frac{1}{2} \int_0^T \rho_t^2 \left(2v_{0,t} V_{1,t} + \left(V_{1,t}^2 + v_{0,t} V_{2,t} \right) \right) \mathrm{d}t \right)$$

$$(C_y :=) \qquad + \mathbb{E}\partial_y \tilde{P}_{\mathrm{BS}} \left(\int_0^T (1 - \rho_t^2)\left(2v_{0,t} V_{1,t} + \left(V_{1,t}^2 + v_{0,t} V_{2,t} \right) \right) \mathrm{d}t \right)$$

$$(C_{xx} :=) \qquad + \frac{1}{2}\mathbb{E}\partial_{xx} \tilde{P}_{\mathrm{BS}} \left(\int_0^T \rho_t V_{1,t} \mathrm{d}B_t - \int_0^T \rho_t^2 v_{0,t} V_{1,t} \mathrm{d}t \right)^2$$

$$(C_{yy} :=) \qquad + \frac{1}{2}\mathbb{E}\partial_{yy} \tilde{P}_{\mathrm{BS}} \left(\int_0^T (1 - \rho_t^2)(2v_{0,t} V_{1,t}) \mathrm{d}t \right)^2$$

$$(C_{xy} :=) \qquad + \mathbb{E}\partial_{xy} \tilde{P}_{\mathrm{BS}} \left(\int_0^T \rho_t V_{1,t} \mathrm{d}B_t - \int_0^T \rho_t^2 v_{0,t} V_{1,t} \mathrm{d}t \right)$$

$$\left(\int_0^T (1 - \rho_t^2)(2v_{0,t} V_{1,t}) \mathrm{d}t \right).$$

Additionally, $\mathrm{Put}_{\mathrm{G}} = \mathrm{Put}_{\mathrm{G}}^{(2)} + \mathbb{E}(\mathcal{E})$, where $\mathcal{E}$ denotes the error in the expansion.

Proof. From Proposition 5.3.1, consider the two-dimensional Taylor expansion of P_{BS} around

$$\left(\tilde{P}_T^{(0)}, \tilde{Q}_T^{(0)} \right)$$

evaluated at

$$\left(\tilde{P}_T^{(1)}, \tilde{Q}_T^{(1)} \right).$$

Then, substitute in the second-order expansions of $P_T^{(1)}, \left(P_T^{(1)} \right)^2, Q_T^{(1)}, \left(Q_T^{(1)} \right)^2$ and $P_T^{(1)} Q_T^{(1)}$ from Lemma 5.3.2. As this is a second-order expression, the remainder terms Θ are neglected. Taking expectation yields $\mathrm{Put}_{\mathrm{G}}^{(2)}$. $\square$

The explicit expression for $\mathcal{E}$ and the analysis of it is left for Section 5.5.

The goal now is to make the terms $C_x, C_y, C_{xx}, C_{yy}, C_{xy}$ explicit, which is the purpose of Section 5.4. This will then yield Theorem 5.4.1, the explicit second-order put option price. Thus, one can think of Section 5.4 as the proof for Theorem 5.4.1.

5.4   Explicit price

In order to make the second-order approximation $\text{Put}_{\text{G}}^{(2)}$ explicit, we will make use of some machinery from Malliavin calculus. In the following subsection we give a short excerpt on Malliavin calculus. We point the reader towards the lecture notes by Nualart [52] for a complete and accessible source on Malliavin calculus.

5.4.1   Malliavin calculus machinery

The underlying framework of Malliavin calculus involves a zero-mean Gaussian process $\tilde{W}$ induced by an underlying separable Hilbert space H. Specifically, we have that $\tilde{W} = \{\tilde{W}(h) : h \in H\}$ is a zero-mean Gaussian process such that $\mathbb{E}(W(h)W(g)) = \langle h, g \rangle_H$.

We need only make use of Malliavin calculus when the underlying Hilbert space is

$$H = L^2([0, T]) := L^2([0, T], \mathcal{B}([0, T]), \lambda^*),$$

where λ^* is the one-dimensional Lebesgue measure. Thus, the inner product on H is

$$\langle h, g \rangle_H = \int_0^T h_t g_t \lambda^*(\mathrm{d}t) = \int_0^T h_t g_t \mathrm{d}t.$$

Our Gaussian process $\tilde{W}$ will be explicitly given as $\tilde{W}(h) := \int_0^T h_t \mathrm{d}B_t$, for any $h \in L^2([0, T])$. By use of the zero-mean and Itô isometry properties of the Itô integral, it can be seen that such a Hilbert space H and Gaussian process $\tilde{W}$ satisfy the framework for Malliavin calculus.

Definition 5.4.1 (Malliavin derivative)**.** Let

$$\mathcal{S}_n := \left\{ F = f\left(\int_0^T h_{1,t} \mathrm{d}B_t, \ldots, \int_0^T h_{n,t} \mathrm{d}B_t \right) : f \in C_p^\infty(\mathbb{R}^n; \mathbb{R}), h_{i,\cdot} \in H \right\}$$

and $\mathcal{S} := \bigcup_{n \geq 1} \mathcal{S}_n$. Here $C_p^\infty(\mathbb{R}^n; \mathbb{R})$ is the space of smooth Borel measurable functions $f : \mathbb{R}^n/\mathcal{B}(\mathbb{R}^n) \to \mathbb{R}/\mathcal{B}(\mathbb{R})$ which have at most polynomial growth. Thus, the elements of $\mathcal{S}_n$ are random variables. For $F \in \mathcal{S}_n$, the Malliavin derivative D is an operator from $\mathcal{S} \to L^0([0, T] \times \Omega)$ and is given by

$$(DF)_t := \sum_{i=1}^n \partial_i f\left(\int_0^T h_{1,t} \mathrm{d}B_t, \ldots, \int_0^T h_{n,t} \mathrm{d}B_t \right) h_{i,t}.$$

Proposition 5.4.1 (Extending domain of D)**.** Define the space $\mathbb{D}^{1,p}$ as the completion of $\mathcal{S}$ with respect to the norm

$$\|F\|_{1,p} := \left(\mathbb{E}|F|^p + \mathbb{E}\left[\int_0^T (D_t F)^p \mathrm{d}t \right]^p \right)^{1/p},$$

where $F \in \mathcal{S}$ and $p \geq 1$. Then the operator D is closable to $\mathbb{D}^{1,p}$, and $D : \mathbb{D}^{1,p} \to L^p([0, T] \times \Omega)$.

Proof. See Nualart [52]. □

The Malliavin derivative satisfies a duality relationship.

Proposition 5.4.2 (Malliavin duality relationship)**.** Let $G \in \mathbb{D}^{1,2}$ and $\alpha \in L^2([0,T] \times \Omega)$ such that α is adapted to the filtration $(\mathcal{F}_t^B)_{0 \leq t \leq T}$. Then

$$\mathbb{E}\left(\int_0^t \alpha_s(DG)_s \mathrm{d}s\right) = \mathbb{E}\left(G \int_0^t \alpha_s \mathrm{d}B_s\right)$$

for any $t < T$.

Another way of phrasing the above proposition is as follows. Let $\langle J, K \rangle_{L^2([0,T] \times \Omega)} :=$ $\mathbb{E}\left(\int_0^T J_t K_t \mathrm{d}t\right)$ and $\langle U, V \rangle_{L^2(\Omega)} := \mathbb{E}(UV)$ denote the inner products on $L^2([0,T] \times \Omega)$ and $L^2(\Omega)$ respectively. Let G and α satisfy the assumptions in Proposition 5.4.2. Then Proposition 5.4.2 can be restated as

$$\langle DG, \alpha \rangle_{L^2([0,T] \times \Omega)} = \langle G, I(\alpha) \rangle_{L^2(\Omega)},$$

where $I(\alpha) := \int_0^T \alpha_t \mathrm{d}B_t$. Essentially, when $H = L^2([0,T])$, $\tilde{W}(h) = \int_0^T h_t \mathrm{d}B_t$ and u is adapted to $(\mathcal{F}_t^B)_{0 \leq t \leq T}$, then the adjoint of the Malliavin derivative corresponds to the Itô integral.

Proof. See Nualart [52]. □

This results in the following lemma.

Lemma 5.4.1 (Malliavin integration by parts)**.** Let $\tilde{T} \leq T$ and $\hat{T} \leq T$. If $G = l\left(\int_0^{\tilde{T}} h_u \mathrm{d}B_u\right)$, then $G \in \mathcal{S}_1 \subseteq \mathbb{D}^{1,2}$ and $DG = l^{(1)}\left(\int_0^{\tilde{T}} h_u \mathrm{d}B_u\right) h.\mathbf{1}_{\{\cdot \leq \tilde{T}\}}$. In addition, let $\alpha \in L^2([0,T] \times \Omega)$ such that α is adapted to $(\mathcal{F}_t^B)_{0 \leq t \leq T}$. Consequently, by Proposition 5.4.2

$$\mathbb{E}\left[l\left(\int_0^{\tilde{T}} h_u \mathrm{d}B_u\right)\left(\int_0^{\hat{T}} \alpha_u \mathrm{d}B_u\right)\right] = \mathbb{E}\left[l^{(1)}\left(\int_0^{\tilde{T}} h_u \mathrm{d}B_u\right)\left(\int_0^{\tilde{T} \wedge \hat{T}} h_u \alpha_u \mathrm{d}u\right)\right].$$

In particular, for $\tilde{T} = T$ and $\hat{T} = t < T$, we obtain

$$\mathbb{E}\left[l\left(\int_0^{T} h_u \mathrm{d}B_u\right)\left(\int_0^{t} \alpha_u \mathrm{d}B_u\right)\right] = \mathbb{E}\left[l^{(1)}\left(\int_0^{T} h_u \mathrm{d}B_u\right)\left(\int_0^{t} h_u \alpha_u \mathrm{d}u\right)\right].$$

Furthermore, we will make use of the following relationship between partial derivatives.

Proposition 5.4.3 (P_{BS} partial derivative relationship)**.**

$$\partial_y P_{\mathrm{BS}}(x,y) = \frac{1}{2}\left(\partial_{xx} P_{\mathrm{BS}}(x,y) - \partial_x P_{\mathrm{BS}}(x,y)\right).$$

Proof. Simply rearranging the relevant partial derivatives from Appendix A.2 gives the result. □

In addition, we will make extensive use of the stochastic integration by parts formula, which we will list here for convenience.

Remark 5.4.1 (Stochastic integration by parts). Let X and Y be semimartingales with respect to a filtration $(\tilde{\mathcal{F}}_t)$. Then we have

$$X_T Y_T = \int_0^T X_t \mathrm{d}Y_t + \int_0^T Y_t \mathrm{d}X_t + \int_0^T \mathrm{d}\langle X, Y \rangle_t,$$

given that the above Itô integrals exist. In particular, if $X_t = \int_0^t x_u \mathrm{d}\tilde{X}_u$ and $Y_t = \int_0^t y_u \mathrm{d}\tilde{Y}_u$, where $\tilde{X}$ and $\tilde{Y}$ are semimartingales and x and y are stochastic processes adapted to the underlying filtration $(\tilde{\mathcal{F}}_t)$ such that X and Y exist, then the stochastic integration by parts formula reads as

$$\int_0^T x_t \mathrm{d}\tilde{X}_t \int_0^T y_t \mathrm{d}\tilde{Y}_t = \int_0^T \left(\int_0^t x_u \mathrm{d}\tilde{X}_u \right) y_t \mathrm{d}\tilde{Y}_t + \int_0^T \left(\int_0^t y_u \mathrm{d}\tilde{Y}_u \right) x_t \mathrm{d}\tilde{X}_t + \int_0^T x_t y_t \mathrm{d}\langle \tilde{X}, \tilde{Y} \rangle_t.$$

We now have all the machinery necessary in order to calculate the terms in Theorem 5.3.1 explicitly.

5.4.2 $\mathbb{E}\tilde{P}_{\mathbf{BS}}$

Notice that $\mathbb{E}\tilde{P}_{\mathrm{BS}} = g(0) = \mathbb{E}(e^k - e^{X_T^{(0)}})_+$. Since the perturbed volatility process $V_t^{(\varepsilon)}$ is deterministic when $\varepsilon = 0$, then $g(0)$ will just be a Black-Scholes formula. Thus we have

$$\mathbb{E}\tilde{P}_{\mathrm{BS}} = P_{\mathrm{BS}} \left(x_0, \int_0^T v_{0,t}^2 \mathrm{d}t \right).$$

5.4.3 C_x

Using Lemma 5.4.1 (Malliavin integration by parts)

$$\mathbb{E}\partial_x \tilde{P}_{\mathrm{BS}} \int_0^T \rho_t \left(V_{1,t} + \frac{1}{2}V_{2,t} \right) \mathrm{d}B_t = \mathbb{E}\partial_{xx} \tilde{P}_{\mathrm{BS}} \int_0^T \rho_t^2 v_{0,t} \left(V_{1,t} + \frac{1}{2}V_{2,t} \right) \mathrm{d}t.$$

Furthermore, using Proposition 5.4.3 (P_{BS} partial derivative relationship)

$$\mathbb{E}\partial_{xx} \tilde{P}_{\mathrm{BS}} \int_0^T \rho_t^2 v_{0,t} \left(V_{1,t} + \frac{1}{2}V_{2,t} \right) \mathrm{d}t = \mathbb{E}(2\partial_y + \partial_x) \tilde{P}_{\mathrm{BS}} \int_0^T \rho_t^2 v_{0,t} \left(V_{1,t} + \frac{1}{2}V_{2,t} \right) \mathrm{d}t.$$

Thus

$$C_x = 2\mathbb{E}\partial_y \tilde{P}_{\mathrm{BS}} \int_0^T \rho_t^2 v_{0,t} \left(V_{1,t} + \frac{1}{2}V_{2,t} \right) \mathrm{d}t - \frac{1}{2}\mathbb{E}\partial_x \tilde{P}_{\mathrm{BS}} \int_0^T \rho_t^2 V_{1,t}^2 \mathrm{d}t.$$

5.4.4 C_{xx}

For C_{xx} we first use Remark 5.4.1 (stochastic integration by parts) to reduce this expression.

$$
\begin{aligned}
C_{xx} &= \frac{1}{2}\mathbb{E}\partial_{xx}\tilde{P}_{\mathrm{BS}}\left(\int_0^T \rho_t V_{1,t}\mathrm{d}B_t - \int_0^T \rho_t^2 v_{0,t}V_{1,t}\mathrm{d}t\right)^2 \\
&= \frac{1}{2}\mathbb{E}\partial_{xx}\tilde{P}_{\mathrm{BS}}\left(\int_0^T \rho_t V_{1,t}\mathrm{d}B_t\right)^2 - \mathbb{E}\partial_{xx}\tilde{P}_{\mathrm{BS}}\left(\int_0^T \rho_t^2 v_{0,t}V_{1,t}\mathrm{d}t\right)\left(\int_0^T \rho_t V_{1,t}\mathrm{d}B_t\right) \\
&\quad + \frac{1}{2}\mathbb{E}\partial_{xx}\tilde{P}_{\mathrm{BS}}\left(\int_0^T \rho_t^2 v_{0,t}V_{1,t}\mathrm{d}t\right)^2 \\
&= \mathbb{E}\partial_{xx}\tilde{P}_{\mathrm{BS}}\left(\int_0^T \left\{\int_0^t \rho_s^2 v_{0,s}V_{1,s}\mathrm{d}s\right\}\rho_t^2 v_{0,t}V_{1,t}\mathrm{d}t\right) \\
&\quad - \mathbb{E}\partial_{xx}\tilde{P}_{\mathrm{BS}}\left(\int_0^T \left\{\int_0^t \rho_s^2 v_{0,s}V_{1,s}\mathrm{d}s\right\}\rho_t V_{1,t}\mathrm{d}B_t + \int_0^T \left\{\int_0^t \rho_s V_{1,s}\mathrm{d}B_s\right\}\rho_t^2 v_{0,t}V_{1,t}\mathrm{d}t\right) \\
&\quad + \mathbb{E}\partial_{xx}\tilde{P}_{\mathrm{BS}}\left(\int_0^T \left\{\int_0^t \rho_s V_{1,s}\mathrm{d}B_s\right\}\rho_t V_{1,t}\mathrm{d}B_t\right) + \frac{1}{2}\mathbb{E}\partial_{xx}\tilde{P}_{\mathrm{BS}}\left(\int_0^T \rho_t^2 V_{1,t}^2\mathrm{d}t\right) \\
&= \mathbb{E}\partial_{xx}\tilde{P}_{\mathrm{BS}}\left(\int_0^T \left\{\left(\int_0^t \rho_s V_{1,s}\mathrm{d}B_s - \int_0^t \rho_s^2 v_{0,s}V_{1,s}\mathrm{d}s\right)\left(\rho_t V_{1,t}\mathrm{d}B_t - \rho_t^2 v_{0,t}V_{1,t}\mathrm{d}t\right)\right\}\right) \\
&\quad + \frac{1}{2}\mathbb{E}\partial_{xx}\tilde{P}_{\mathrm{BS}}\left(\int_0^T \rho_t^2 V_{1,t}^2\mathrm{d}t\right).
\end{aligned}
$$

Using Lemma 5.4.1 (Malliavin integration by parts),

$$
\begin{aligned}
C_{xx} &= -\mathbb{E}\partial_{xx}\tilde{P}_{\mathrm{BS}}\left(\int_0^T \left\{\left(\int_0^t \rho_s V_{1,s}\mathrm{d}B_s - \int_0^t \rho_s^2 v_{0,s}V_{1,s}\mathrm{d}s\right)\rho_t^2 v_{0,t}V_{1,t}\mathrm{d}t\right\}\right) \\
&\quad + \mathbb{E}\partial_{xxx}\tilde{P}_{\mathrm{BS}}\left(\int_0^T \left(\int_0^t \rho_s V_{1,s}\mathrm{d}B_s - \int_0^t \rho_s^2 v_{0,s}V_{1,s}\mathrm{d}s\right)\rho_t^2 v_{0,t}V_{1,t}\mathrm{d}t\right) + \frac{1}{2}\mathbb{E}\partial_{xx}\tilde{P}_{\mathrm{BS}}\left(\int_0^T \rho_t^2 V_{1,t}^2\mathrm{d}t\right) \\
&= \mathbb{E}(\partial_{xxx}\tilde{P}_{\mathrm{BS}} - \partial_{xx}\tilde{P}_{\mathrm{BS}})\left(\int_0^T \left(\int_0^t \rho_s V_{1,s}\mathrm{d}B_s - \int_0^t \rho_s^2 v_{0,s}V_{1,s}\mathrm{d}s\right)\rho_t^2 v_{0,t}V_{1,t}\mathrm{d}t\right) \\
&\quad + \frac{1}{2}\mathbb{E}\partial_{xx}\tilde{P}_{\mathrm{BS}}\left(\int_0^T \rho_t^2 V_{1,t}^2\mathrm{d}t\right).
\end{aligned}
$$

Then using Proposition 5.4.3 (P_{BS} partial derivative relationship)

$$
\begin{aligned}
C_{xx} &= 2\mathbb{E}\partial_{xy}\tilde{P}_{\mathrm{BS}}\left(\int_0^T \left(\int_0^t \rho_s V_{1,s}\mathrm{d}B_s - \int_0^t \rho_s^2 v_{0,s}V_{1,s}\mathrm{d}s\right)\rho_t^2 v_{0,t}V_{1,t}\mathrm{d}t\right) \\
&\quad + \frac{1}{2}\mathbb{E}\partial_{xx}\tilde{P}_{\mathrm{BS}}\left(\int_0^T \rho_t^2 V_{1,t}^2\mathrm{d}t\right).
\end{aligned}
$$

Adding the terms C_x, C_{xx} and C_y, one gets

$$
\begin{aligned}
C_x + C_{xx} + C_y &= \mathbb{E}\partial_y\tilde{P}_{\mathrm{BS}}\left(\int_0^T 2v_{0,t}V_{1,t} + V_{1,t}^2 + v_{0,t}V_{2,t}\mathrm{d}t\right) \\
&\quad + 2\mathbb{E}\partial_{xy}\tilde{P}_{\mathrm{BS}}\left(\int_0^T \left(\int_0^t \rho_s V_{1,s}\mathrm{d}B_s - \int_0^t \rho_s^2 v_{0,s}V_{1,s}\mathrm{d}s\right)\rho_t^2 v_{0,t}V_{1,t}\mathrm{d}t\right).
\end{aligned}
$$

5.4.5 C_{xy}

For C_{xy} we use Remark 5.4.1 (stochastic integration by parts) to obtain

$$\mathbb{E}\partial_{xy}\tilde{P}_{\mathrm{BS}}\left(\int_0^T \rho_t V_{1,t}\mathrm{d}B_t - \int_0^T \rho_t^2 v_{0,t}V_{1,t}\mathrm{d}t\right)\left(\int_0^T (1-\rho_t^2)(2v_{0,t}V_{1,t})\mathrm{d}t\right)$$

$$= 2\mathbb{E}\partial_{xy}\tilde{P}_{\mathrm{BS}}\left(\int_0^T\left(\int_0^t (1-\rho_s^2)v_{0,s}V_{1,s}\mathrm{d}s\right)\left(\rho_t V_{1,t}\mathrm{d}B_t - \rho_t^2 v_{0,t}V_{1,t}\mathrm{d}t\right)\right)$$

$$+ 2\mathbb{E}\partial_{xy}\tilde{P}_{\mathrm{BS}}\int_0^T\left(\int_0^t \rho_s V_{1,s}\mathrm{d}B_s - \int_0^t \rho_s^2 v_{0,s}V_{1,s}\mathrm{d}s\right)(1-\rho_t^2)v_{0,t}V_{1,t}\mathrm{d}t$$

$$= 2\mathbb{E}\partial_{xy}\tilde{P}_{\mathrm{BS}}\left(\int_0^T\left(\int_0^t (1-\rho_s^2)v_{0,s}V_{1,s}\mathrm{d}s\right)\left(\rho_t V_{1,t}\mathrm{d}B_t - \rho_t^2 v_{0,t}V_{1,t}\mathrm{d}t\right)\right)$$

$$- 2\mathbb{E}\partial_{xy}\tilde{P}_{\mathrm{BS}}\int_0^T\left(\int_0^t \rho_s^2 v_{0,s}V_{1,s}\mathrm{d}s\right)v_{0,t}V_{1,t}\mathrm{d}t$$

$$+ 2\mathbb{E}\partial_{xy}\tilde{P}_{\mathrm{BS}}\left(\int_0^T\left(\int_0^t \rho_s V_{1,s}\mathrm{d}B_s\right)v_{0,t}V_{1,t}\mathrm{d}t\right)$$

$$- 2\mathbb{E}\partial_{xy}\tilde{P}_{\mathrm{BS}}\left(\int_0^T\left(\int_0^t \rho_s V_{1,s}\mathrm{d}B_s - \int_0^t \rho_s^2 v_{0,s}V_{1,s}\mathrm{d}s\right)\rho_t^2 v_{0,t}V_{1,t}\mathrm{d}t\right).$$

Furthermore, using Proposition 5.4.2 (Malliavin duality relationship)

$$\hat{C}_{xy} := 2\mathbb{E}\partial_{xy}\tilde{P}_{\mathrm{BS}}\left(\int_0^T v_{0,t}V_{1,t}\left(\int_0^t \rho_s V_{1,s}\mathrm{d}B_s\right)\mathrm{d}t\right) = 2\int_0^T \mathbb{E}\partial_{xy}\tilde{P}_{\mathrm{BS}}v_{0,t}V_{1,t}\left(\int_0^t \rho_s V_{1,s}\mathrm{d}B_s\right)\mathrm{d}t$$

$$= 2\int_0^T \mathbb{E}\left(\int_0^t \rho_s V_{1,s}D_s^B(\partial_{xy}\tilde{P}_{\mathrm{BS}}v_{0,\cdot}V_{1,\cdot})\mathrm{d}s\right)\mathrm{d}t.$$

Using the definition of the Malliavin derivative, we obtain

$$D_s^B(\partial_{xy}\tilde{P}_{\mathrm{BS}}v_{0,\cdot}V_{1,\cdot}) = \partial_{xxy}\tilde{P}_{\mathrm{BS}}v_{0,t}V_{1,t}\rho_s v_{0,s}\mathbf{1}_{\{s\leq T\}} + \partial_{xy}\tilde{P}_{\mathrm{BS}}D_s^B(v_{0,\cdot}V_{1,\cdot})$$

$$= \partial_{xxy}\tilde{P}_{\mathrm{BS}}v_{0,t}V_{1,t}\rho_s v_{0,s}\mathbf{1}_{\{s\leq T\}}$$

$$+ \partial_{xy}\tilde{P}_{\mathrm{BS}}v_{0,t}\left(e^{\int_0^t \alpha_x(u,v_{0,u})\mathrm{d}u}\beta(s,v_{0,s})e^{-\int_0^s \alpha_x(z,v_{0,z})\mathrm{d}z}\mathbf{1}_{\{s\leq t\}}\right),$$

where we have used the explicit form for $V_{1,t}$ from eq. (5.7). Thus using Proposition 5.4.2 (Malliavin duality relationship)

$$2\int_0^T \mathbb{E}\left(\int_0^t \rho_s V_{1,s}D_s^B(\partial_{xy}\tilde{P}_{\mathrm{BS}}v_{0,\cdot}V_{1,\cdot})\mathrm{d}s\right)\mathrm{d}t$$

$$= 2\int_0^T \mathbb{E}\partial_{xxy}\tilde{P}_{\mathrm{BS}}\left(\int_0^t \rho_s^2 v_{0,s}V_{1,s}\mathrm{d}s\right)v_{0,t}V_{1,t}\mathrm{d}t$$

$$+ 2\int_0^T \mathbb{E}\partial_{xy}\tilde{P}_{\mathrm{BS}}\left(\int_0^t \rho_s V_{1,s}\beta(s,v_{0,s})e^{-\int_0^s \alpha_x(z,v_{0,z})\mathrm{d}z}\mathrm{d}s\right)v_{0,t}e^{\int_0^t \alpha_x(z,v_{0,z})\mathrm{d}z}\mathrm{d}t$$

$$= 2\mathbb{E}\partial_{xxy}\tilde{P}_{\mathrm{BS}}\int_0^T\left(\int_0^t \rho_s^2 v_{0,s}V_{1,s}\mathrm{d}s\right)v_{0,t}V_{1,t}\mathrm{d}t$$

$$+ 2\mathbb{E}\partial_y\tilde{P}_{\mathrm{BS}}\int_0^T e^{\int_0^t \alpha_x(z,v_{0,z})\mathrm{d}z}v_{0,t}\left(\int_0^t v_{0,s}^{-1}\beta(s,v_{0,s})V_{1,s}e^{-\int_0^s \alpha_x(z,v_{0,z})\mathrm{d}z}\mathrm{d}B_s\right)\mathrm{d}t.$$

Assumption 5.4.1. Let $\beta(t,x) = \lambda_t x^\mu$ for $\mu \in [1/2, 1]$, where λ is bounded on $[0,T]$.

There are a couple of reasons behind Assumption 5.4.1:

1. β is Hölder continuous of order $\geq 1/2$ in x, uniformly in $t \in [0, T]$. In addition, it is differentiable a.e. in x, so Assumption 5.2.1 is satisfied.

2. Such a diffusion coefficient is common in application, see for example the SABR model [35] and CEV model [20].

Truthfully, we could leave β as an arbitrary diffusion coefficient that obeys the clauses in Assumption 5.2.1. However, in terms of application purposes and also for our fast calibration scheme in Section 6.1, it will be more insightful to have this form for β. For the interested reader, all the following calculations still remain valid solely under Assumption 5.2.1.

In view of Assumption 5.4.1, we can rewrite $V_{1,t}$ and $V_{2,t}$ from Lemma 5.3.1 as

$$V_{1,t} = e^{\int_0^t \alpha_x(z, v_{0,z}) \mathrm{d}z} \int_0^t \lambda_s v_{0,s}^\mu e^{-\int_0^s \alpha_x(z, v_{0,z}) \mathrm{d}z} \mathrm{d}B_s, \tag{5.16}$$

$$V_{2,t} = e^{\int_0^t \alpha_x(z, v_{0,z}) \mathrm{d}z} \left\{ \int_0^t \alpha_{xx}(s, v_{0,s})(V_{1,s})^2 e^{-\int_0^s \alpha_x(z, v_{0,z}) \mathrm{d}z} \mathrm{d}s + \int_0^t 2\mu \lambda_s v_{0,s}^{\mu-1} V_{1,s} e^{-\int_0^s \alpha_x(z, v_{0,z}) \mathrm{d}z} \mathrm{d}B_s \right\}. \tag{5.17}$$

Then we obtain

$$\hat{C}_{xy} = 2\mathbb{E}\partial_{xxy}\tilde{P}_{\mathrm{BS}} \int_0^T \left(\int_0^t \rho_s^2 v_{0,s} V_{1,s} \mathrm{d}s \right) v_{0,t} V_{1,t} \mathrm{d}t$$
$$+ 2\mathbb{E}\partial_y \tilde{P}_{\mathrm{BS}} \int_0^T v_{0,t} e^{\int_0^t \alpha_x(z, v_{0,z}) \mathrm{d}z} \left(\int_0^t \lambda_s v_{0,s}^{\mu-1} V_{1,s} e^{-\int_0^s \alpha_x(z, v_{0,z}) \mathrm{d}z} \mathrm{d}B_s \right) \mathrm{d}t.$$

Hence

$$C_{xy} = 2\mathbb{E}\partial_{xy}\tilde{P}_{\mathrm{BS}} \left(\int_0^T \left(\int_0^t (1 - \rho_s^2) v_{0,s} V_{1,s} \mathrm{d}s \right) \left(\rho_t V_{1,t} \mathrm{d}B_t - \rho_t^2 v_{0,t} V_{1,t} \mathrm{d}t \right) \right)$$
$$- 2\mathbb{E}\partial_{xy}\tilde{P}_{\mathrm{BS}} \int_0^T \left(\int_0^t \rho_s^2 v_{0,s} V_{1,s} \mathrm{d}s \right) v_{0,t} V_{1,t} \mathrm{d}t$$
$$+ 2\mathbb{E}\partial_{xxy}\tilde{P}_{\mathrm{BS}} \int_0^T \left(\int_0^t \rho_s^2 v_{0,s} V_{1,s} \mathrm{d}s \right) v_{0,t} V_{1,t} \mathrm{d}t$$
$$+ 2\mathbb{E}\partial_y \tilde{P}_{\mathrm{BS}} \int_0^T v_{0,t} e^{\int_0^t \alpha_x(z, v_{0,z}) \mathrm{d}z} \left(\int_0^t \lambda_s v_{0,s}^{\mu-1} V_{1,s} e^{-\int_0^s \alpha_x(z, v_{0,z}) \mathrm{d}z} \mathrm{d}B_s \right) \mathrm{d}t$$
$$- 2\mathbb{E}\partial_{xy}\tilde{P}_{\mathrm{BS}} \left(\int_0^T \left(\int_0^t \rho_s V_{1,s} \mathrm{d}B_s - \int_0^t \rho_s^2 v_{0,s} V_{1,s} \mathrm{d}s \right) \rho_t^2 v_{0,t} V_{1,t} \mathrm{d}t \right).$$

5.4.6 C_{yy}

C_{yy} is given by Remark 5.4.1 (stochastic integration by parts) as

$$4\mathbb{E}\partial_{yy}\tilde{P}_{\mathrm{BS}} \left(\int_0^T \left\{ \int_0^t (1 - \rho_s^2) v_{0,s} V_{1,s} \mathrm{d}s \right\} (1 - \rho_t^2) v_{0,t} V_{1,t} \mathrm{d}t \right).$$

5.4.7 Adding C_x, C_y, C_{xx}, C_{xy} and C_{yy}

Now we add up all the terms after manipulation from Sections 5.4.2 to 5.4.6.

$$
(C_x + C_y + C_{xx}) + C_{xy} + C_{yy}
$$

$$
= \mathbb{E}\partial_y \tilde{P}_{\mathrm{BS}} \left(\int_0^T 2v_{0,t}V_{1,t} + V_{1,t}^2 + v_{0,t}V_{2,t}\mathrm{d}t \right)
$$

$$
+ 2\mathbb{E}\partial_y \tilde{P}_{\mathrm{BS}} \int_0^T v_{0,t} e^{\int_0^t \alpha_x(z,v_{0,z})\mathrm{d}z} \left(\int_0^t \lambda_s v_{0,s}^{\mu-1} V_{1,s} e^{-\int_0^s \alpha_x(z,v_{0,z})\mathrm{d}z}\mathrm{d}B_s \right) \mathrm{d}t
$$

$$
+ 2\mathbb{E}\partial_{xy} \tilde{P}_{\mathrm{BS}} \left(\int_0^T \left(\int_0^t (1-\rho_s^2)v_{0,s}V_{1,s}\mathrm{d}s \right) \left(\rho_t V_{1,t}\mathrm{d}B_t - \rho_t^2 v_{0,t}V_{1,t}\mathrm{d}t \right) \right)
$$

$$
- 2\mathbb{E}\partial_{xy} \tilde{P}_{\mathrm{BS}} \int_0^T \left(\int_0^t \rho_s^2 v_{0,s}V_{1,s}\mathrm{d}s \right) v_{0,t}V_{1,t}\mathrm{d}t
$$

$$
+ 2\mathbb{E}\partial_{xxy} \tilde{P}_{\mathrm{BS}} \int_0^T \left(\int_0^t \rho_s^2 v_{0,s}V_{1,s}\mathrm{d}s \right) v_{0,t}V_{1,t}\mathrm{d}t
$$

$$
+ 4\mathbb{E}\partial_{yy} \tilde{P}_{\mathrm{BS}} \left(\int_0^T \left\{ \int_0^t (1-\rho_s^2)v_{0,s}V_{1,s}\mathrm{d}s \right\} (1-\rho_t^2)v_{0,t}V_{1,t}\mathrm{d}t \right).
$$

Then

$$
\begin{aligned}
&C_x + C_y + C_{xx} + C_{xy} + C_{yy} \\
&= \mathbb{E}\partial_y \tilde{P}_{\mathrm{BS}} \left(\int_0^T 2v_{0,t}V_{1,t} + V_{1,t}^2 + v_{0,t}V_{2,t}\mathrm{d}t \right) \\
&\quad + 2\mathbb{E}\partial_y \tilde{P}_{\mathrm{BS}} \int_0^T v_{0,t}e^{\int_0^t \alpha_x(z,v_{0,z})\mathrm{d}z} \left(\int_0^t \lambda_s v_{0,s}^{\mu-1}V_{1,s}e^{-\int_0^s \alpha_x(z,v_{0,z})\mathrm{d}z}\mathrm{d}B_s \right) \mathrm{d}t \\
&\quad + 2\mathbb{E}\partial_{xy}\tilde{P}_{\mathrm{BS}} \left(\int_0^T \left(\int_0^t (1-\rho_s^2)v_{0,s}V_{1,s}\mathrm{d}s \right) \left(\rho_t V_{1,t}\mathrm{d}B_t - \rho_t^2 v_{0,t}V_{1,t}\mathrm{d}t \right) \right) \\
&\quad + 4\mathbb{E}\partial_{yy}\tilde{P}_{\mathrm{BS}} \int_0^T \left(\int_0^t \rho_s^2 v_{0,s}V_{1,s}\mathrm{d}s \right) v_{0,t}V_{1,t}\mathrm{d}t \\
&\quad + 4\mathbb{E}\partial_{yy}\tilde{P}_{\mathrm{BS}} \left(\int_0^T \left\{ \int_0^t (1-\rho_s^2)v_{0,s}V_{1,s}\mathrm{d}s \right\} (1-\rho_t^2)v_{0,t}V_{1,t}\mathrm{d}t \right) \\
&= \mathbb{E}\partial_y \tilde{P}_{\mathrm{BS}} \left(\int_0^T 2v_{0,t}V_{1,t} + V_{1,t}^2 + v_{0,t}V_{2,t}\mathrm{d}t \right) \\
&\quad + 2\mathbb{E}\partial_y \tilde{P}_{\mathrm{BS}} \int_0^T v_{0,t}e^{\int_0^t \alpha_x(z,v_{0,z})\mathrm{d}z} \left(\int_0^t \lambda_s v_{0,s}^{\mu-1}V_{1,s}e^{-\int_0^s \alpha_x(z,v_{0,z})\mathrm{d}z}\mathrm{d}B_s \right) \mathrm{d}t \\
&\quad + 4\mathbb{E}\partial_{yy}\tilde{P}_{\mathrm{BS}} \left(\int_0^T \left(\int_0^t (1-\rho_s^2)v_{0,s}V_{1,s}\mathrm{d}s \right) \rho_t^2 v_{0,t}V_{1,t}\mathrm{d}t \right) \\
&\quad + 4\mathbb{E}\partial_{yy}\tilde{P}_{\mathrm{BS}} \int_0^T \left(\int_0^t \rho_s^2 v_{0,s}V_{1,s}\mathrm{d}s \right) v_{0,t}V_{1,t}\mathrm{d}t \\
&\quad + 4\mathbb{E}\partial_{yy}\tilde{P}_{\mathrm{BS}} \left(\int_0^T \left\{ \int_0^t (1-\rho_s^2)v_{0,s}V_{1,s}\mathrm{d}s \right\} (1-\rho_t^2)v_{0,t}V_{1,t}\mathrm{d}t \right) \\
&= \mathbb{E}\partial_y \tilde{P}_{\mathrm{BS}} \left(\int_0^T 2v_{0,t}V_{1,t} + V_{1,t}^2 + v_{0,t}V_{2,t}\mathrm{d}t \right) \\
&\quad + 2\mathbb{E}\partial_y \tilde{P}_{\mathrm{BS}} \int_0^T v_{0,t}e^{\int_0^t \alpha_x(z,v_{0,z})\mathrm{d}z} \left(\int_0^t \lambda_s v_{0,s}^{\mu-1}V_{1,s}e^{-\int_0^s \alpha_x(z,v_{0,z})\mathrm{d}z}\mathrm{d}B_s \right) \mathrm{d}t \\
&\quad + 4\mathbb{E}\partial_{yy}\tilde{P}_{\mathrm{BS}} \left(\int_0^T \left(\int_0^t v_{0,s}V_{1,s}\mathrm{d}s \right) v_{0,t}V_{1,t}\mathrm{d}t \right),
\end{aligned}
$$

where we have used the partial derivative relationship Proposition 5.4.3 (P_{BS} partial derivative relationship), Proposition 5.4.2 (Malliavin duality relationship) and partial derivative relationship Proposition 5.4.3 (P_{BS} partial derivative relationship), then simplification for the first, second and third equalities respectively. Lastly, notice by Remark 5.4.1 (stochastic integration by parts)

$$
2\left(\int_0^T \left(\int_0^t v_{0,s}V_{1,s}\mathrm{d}s \right) v_{0,t}V_{1,t}\mathrm{d}t \right) = \left(\int_0^T v_{0,t}V_{1,t}\mathrm{d}t \right)^2 .
$$

Proposition 5.4.4. In view of the calculations from Sections 5.4.2 to 5.4.7, and under Assumption 5.4.1, we obtain the simpler form of the second-order approximation $\mathrm{Put}_{\mathrm{G}}^{(2)}$

from Theorem 5.3.1 as

$$
\begin{aligned}
\text{Put}_{\text{G}}^{(2)} &= P_{\text{BS}}\left(x_0, \int_0^T v_{0,t}^2 \text{d}t\right) \\
&\quad + \mathbb{E}\partial_y \tilde{P}_{\text{BS}}\left(\int_0^T 2v_{0,t}V_{1,t} + V_{1,t}^2 + v_{0,t}V_{2,t} \text{d}t\right) \\
&\quad + 2\mathbb{E}\partial_y \tilde{P}_{\text{BS}}\left(\int_0^T v_{0,t}e^{\int_0^t \alpha_x(z,v_{0,z})\text{d}z}\left(\int_0^t \lambda_s v_{0,s}^{\mu-1}V_{1,s}e^{-\int_0^s \alpha_x(z,v_{0,z})\text{d}z}\text{d}B_s\right)\text{d}t\right) \\
&\quad + 2\mathbb{E}\partial_{yy}\tilde{P}_{\text{BS}}\left(\int_0^T v_{0,t}V_{1,t}\text{d}t\right)^2.
\end{aligned}
\tag{5.18}
$$

5.4.8 Eliminating the processes $(V_{1,t})$ and $(V_{2,t})$

The next step is to reduce these remaining expectations in Proposition 5.4.4 down by eliminating the processes $(V_{1,t})$ and $(V_{2,t})$. To this end, define the following integral operator

$$
\omega_{t,T}^{(k,l)} := \int_t^T l_u e^{\int_0^u k_z \text{d}z} \text{d}u.
$$

In addition, we define the n-fold integral operator using the following recurrence.

$$
\omega_{t,T}^{(k^{(n)},l^{(n)}),(k^{(n-1)},l^{(n-1)}),\ldots,(k^{(1)},l^{(1)})} := \omega_{t,T}^{\left(k^{(n)},l^{(n)}w_{\cdot,T}^{(k^{(n-1)},l^{(n-1)}),\cdots,(k^{(1)},l^{(1)})}\right)}, \quad n \in \mathbb{N}.
$$

[3]

Lemma 5.4.2. Let Z be a semimartingale such that $Z_0 = 0$ and let f be a Lebesgue integrable deterministic function. Then

$$
\int_0^T f_t Z_t \text{d}t = \int_0^T \omega_{t,T}^{(0,f)} \text{d}Z_t.
$$

Proof. A simple application of Remark 5.4.1 (stochastic integration by parts) gives the desired result. $\qquad\square$

Using Lemma 5.4.2 we can obtain the following lemma.

Lemma 5.4.3. The following equalities hold:

$$
\mathbb{E}\left(l\left(\int_0^T \rho_t v_{0,t}\text{d}B_t\right)\int_0^T \xi_t V_{1,t}\text{d}t\right) = \omega_{0,T}^{(-\alpha_x,\rho\lambda v_{0,\cdot}^{\mu+1}),(\alpha_x,\xi)}\mathbb{E}\left(l^{(1)}\left(\int_0^T \rho_t v_{0,t}\text{d}B_t\right)\right), \tag{5.19}
$$

$$
\begin{aligned}
\mathbb{E}\left(l\left(\int_0^T \rho_t v_{0,t}\text{d}B_t\right)\int_0^T \xi_t V_{1,t}^2\text{d}t\right) &= 2\omega_{0,T}^{(-\alpha_x,\rho\lambda v_{0,\cdot}^{\mu+1}),(-\alpha_x,\rho\lambda v_{0,\cdot}^{\mu+1}),(2\alpha_x,\xi)}\mathbb{E}\left(l^{(2)}\left(\int_0^T \rho_t v_{0,t}\text{d}B_t\right)\right) \\
&\quad + \omega_{0,T}^{(-2\alpha_x,\lambda^2 v_{0,\cdot}^{2\mu}),(2\alpha_x,\xi)}\mathbb{E}\left(l\left(\int_0^T \rho_t v_{0,t}\text{d}B_t\right)\right),
\end{aligned}
\tag{5.20}
$$

---

[3]For example

$$
\omega_{t,T}^{(k^{(3)},l^{(3)}),(k^{(2)},l^{(2)}),(k^{(1)},l^{(1)})} = \int_t^T l_{u_3}^{(3)}e^{\int_0^{u_3} k_z^{(3)}\text{d}z}\left(\int_{u_3}^T l_{u_2}^{(2)}e^{\int_0^{u_2} k_z^{(2)}\text{d}z}\left(\int_{u_2}^T l_{u_1}^{(1)}e^{\int_0^{u_1} k_z^{(1)}\text{d}z}\text{d}u_1\right)\text{d}u_2\right)\text{d}u_3.
$$

$$\mathbb{E}\left(l\left(\int_0^T \rho_t v_{0,t}\mathrm{d}B_t\right)\int_0^T \xi_t V_{2,t}\mathrm{d}t\right) = \omega_{0,T}^{(-2\alpha_x,\lambda^2 v_{0,\cdot}^{2\mu}),(\alpha_x,\alpha_{xx}),(\alpha_x,\xi)}\mathbb{E}\left(l\left(\int_0^T \rho_t v_{0,t}\mathrm{d}t\right)\right)$$

$$+\left\{2\omega_{0,T}^{(-\alpha_x,\rho\lambda v_{0,\cdot}^{\mu+1}),(-\alpha_x,\rho\lambda v_{0,\cdot}^{\mu+1}),(\alpha_x,\alpha_{xx}),(\alpha_x,\xi)} + 2\mu\omega_{0,T}^{(-\alpha_x,\rho\lambda v_{0,\cdot}^{\mu+1}),(0,\rho\lambda v_{0,\cdot}^{2\mu-1}),(\alpha_x,\xi)}\right\}\mathbb{E}\left(l^{(2)}\left(\int_0^T \rho_t v_{0,t}\mathrm{d}B_t\right)\right),$$

$$(5.21)$$

$$\mathbb{E}\left(l\left(\int_0^T \rho_t v_{0,t}\mathrm{d}B_t\right)\left\{\int_0^T \xi_t V_{1,t}\mathrm{d}t\right\}^2\right) = 2\omega_{0,T}^{(-2\alpha_x,\lambda^2 v_{0,\cdot}^{2\mu}),(\alpha_x,\xi),(\alpha_x,\xi)}\mathbb{E}\left(l\left(\int_0^T \rho_t v_{0,t}\mathrm{d}B_t\right)\right)$$

$$+\left(\omega_{0,T}^{(-\alpha_x,\rho\lambda v_{0,\cdot}^{\mu+1}),(\alpha_x,\xi)}\right)^2\mathbb{E}\left(l^{(2)}\left(\int_0^T \rho_t v_{0,t}\mathrm{d}B_t\right)\right).$$

$$(5.22)$$

Here we write $\alpha_x := \alpha_x(\cdot, v_{0,\cdot})$ and $\alpha_{xx} := \alpha_{xx}(\cdot, v_{0,\cdot})$ for readability purposes.

Proof. We will only show how to obtain eq. (5.19). Equations (5.20) to (5.22) can be obtained in a similar way. First, we replace $V_{1,t}$ with its explicit form from eq. (5.16). Thus, we can write the left hand side of eq. (5.19) as

$$\mathbb{E}\left(l\left(\int_0^T \rho_t v_{0,t}\mathrm{d}B_t\right)\int_0^T \xi_t e^{\int_0^t \alpha_x(z,v_{0,z})\mathrm{d}z}\left(\int_0^t \lambda_s v_{0,s}^\mu e^{-\int_0^s \alpha_x(z,v_{0,z})\mathrm{d}z}\mathrm{d}B_s\right)\mathrm{d}t\right).$$

Using Lemma 5.4.2 with $f_t = \xi_t e^{\int_0^t \alpha_x(z,v_{0,z})\mathrm{d}z}$ and $Z_t = \int_0^t \lambda_s v_{0,s}^\mu e^{-\int_0^s \alpha_x(z,v_{0,z})\mathrm{d}z}\mathrm{d}B_s$, we get

$$\mathbb{E}\left(l\left(\int_0^T \rho_t v_{0,t}\mathrm{d}B_t\right)\int_0^T \xi_t e^{\int_0^t \alpha_x(z,v_{0,z})\mathrm{d}z}\left(\int_0^t \lambda_s v_{0,s}^\mu e^{-\int_0^s \alpha_x(z,v_{0,z})\mathrm{d}z}\mathrm{d}B_s\right)\mathrm{d}t\right)$$

$$= \mathbb{E}\left(l\left(\int_0^T \rho_t v_{0,t}\mathrm{d}B_t\right)\int_0^T \omega_{t,T}^{(\alpha_x,\xi)}\lambda_t v_{0,t}^\mu e^{-\int_0^t \alpha_x(z,v_{0,z})\mathrm{d}z}\mathrm{d}B_t\right).$$

Lastly, appealing to the Malliavin integration by parts Lemma 5.4.1 we obtain

$$\mathbb{E}\left(l^{(1)}\left(\int_0^T \rho_t v_{0,t}\mathrm{d}B_t\right)\int_0^T \omega_{t,T}^{(\alpha_x,\xi)}\rho_t \lambda_t v_{0,t}^{\mu+1} e^{-\int_0^t \alpha_x(z,v_{0,z})\mathrm{d}z}\mathrm{d}t\right)$$

$$= \left(\int_0^T \omega_{t,T}^{(\alpha_x,\xi)}\rho_t \lambda_t v_{0,t}^{\mu+1} e^{-\int_0^t \alpha_x(z,v_{0,z})\mathrm{d}z}\mathrm{d}t\right)\mathbb{E}\left(l^{(1)}\left(\int_0^T \rho_t v_{0,t}\mathrm{d}B_t\right)\right)$$

$$= \omega_{0,T}^{(-\alpha_x,\rho\lambda v_{0,\cdot}^{\mu+1}),(\alpha_x,\xi)}\mathbb{E}\left(l^{(1)}\left(\int_0^T \rho_t v_{0,t}\mathrm{d}B_t\right)\right).$$

In addition, to obtain eq. (5.22), notice the following integral property holds:

$$\left(\omega_{0,T}^{(k^{(2)},l^{(2)}),(k^{(1)},l^{(1)})}\right)^2 = 2\omega_{0,T}^{(k^{(2)},l^{(2)}),(k^{(1)},l^{(1)}),(k^{(2)},l^{(2)}),(k^{(1)},l^{(1)})}$$

$$+ 4\omega_{0,T}^{(k^{(2)},l^{(2)}),(k^{(2)},l^{(2)}),(k^{(1)},l^{(1)}),(k^{(1)},l^{(1)})}.$$

$$\square$$

Then, using Lemma 5.4.1 (Mallavin integration by parts) and eq. (5.19) we get

$$2\mathbb{E}\partial_y \tilde{P}_{\mathrm{BS}}\left(\int_0^T v_{0,t} e^{\int_0^t \alpha_x(z,v_{0,z})\mathrm{d}z}\left(\int_0^t \lambda_s v_{0,s}^{\mu-1}V_{1,s} e^{-\int_0^s \alpha_x(z,v_{0,z})\mathrm{d}z}\mathrm{d}B_s\right)\mathrm{d}t\right)$$

$$= 2\omega_{0,T}^{(-\alpha_x,\rho\lambda v_{0,\cdot}^{\mu+1}),(0,\rho\lambda v_{0,\cdot}^\mu),(\alpha_x,v_{0,\cdot})}\mathbb{E}\partial_{xxy}\tilde{P}_{\mathrm{BS}}.$$

Finally, using Lemma 5.4.3 we obtain the explicit second-order put option price.

Theorem 5.4.1 (Explicit second-order put option price). Under Assumption 5.4.1, the explicit second-order price of a put option in the general model eq. (5.1) is given by

$$
\begin{aligned}
\mathrm{Put}_{\mathrm{G}}^{(2)} = {} & P_{\mathrm{BS}}\left(x_0, \int_0^T v_{0,t}^2 \mathrm{d}t\right) \\
& + 2\omega_{0,T}^{(-\alpha_x, \rho\lambda v_{0,\cdot}^{\mu+1}), (\alpha_x, v_{0,\cdot})} \partial_{xy} P_{\mathrm{BS}}\left(x_0, \int_0^T v_{0,t}^2 \mathrm{d}t\right) \\
& + \omega_{0,T}^{(-2\alpha_x, \lambda^2 v_{0,\cdot}^{2\mu}), (2\alpha_x, 1)} \partial_y P_{\mathrm{BS}}\left(x_0, \int_0^T v_{0,t}^2 \mathrm{d}t\right) \\
& + 2\omega_{0,T}^{(-\alpha_x, \rho\lambda v_{0,\cdot}^{\mu+1}), (-\alpha_x, \rho\lambda v_{0,\cdot}^{\mu+1}), (2\alpha_x, 1)} \partial_{xxy} P_{\mathrm{BS}}\left(x_0, \int_0^T v_{0,t}^2 \mathrm{d}t\right) \\
& + \omega_{0,T}^{(-2\alpha_x, \lambda^2 v_{0,\cdot}^{2\mu}), (\alpha_x, \alpha_{xx}), (\alpha_x, v_{0,\cdot})} \partial_y P_{\mathrm{BS}}\left(x_0, \int_0^T v_{0,t}^2 \mathrm{d}t\right) \\
& + \left\{ 2\omega_{0,T}^{(-\alpha_x, \rho\lambda v_{0,\cdot}^{\mu+1}), (-\alpha_x, \rho\lambda v_{0,\cdot}^{\mu+1}), (\alpha_x, \alpha_{xx}), (\alpha_x, v_{0,\cdot})} \right. \\
& \left. \quad + 2\mu\omega_{0,T}^{(-\alpha_x, \rho\lambda v_{0,\cdot}^{\mu+1}), (0, \rho\lambda v_{0,\cdot}^{2\mu-1}), (\alpha_x, v_{0,\cdot})} \right\} \partial_{xxy} P_{\mathrm{BS}}\left(x_0, \int_0^T v_{0,t}^2 \mathrm{d}t\right) \\
& + 2\omega_{0,T}^{(-\alpha_x, \rho\lambda v_{0,\cdot}^{\mu+1}), (0, \rho\lambda v_{0,\cdot}^{\mu}), (\alpha_x, v_{0,\cdot})} \partial_{xxy} P_{\mathrm{BS}}\left(x_0, \int_0^T v_{0,t}^2 \mathrm{d}t\right) \\
& + 4\omega_{0,T}^{(-2\alpha_x, \lambda^2 v_{0,\cdot}^{2\mu}), (\alpha_x, v_{0,\cdot}), (\alpha_x, v_{0,\cdot})} \partial_{yy} P_{\mathrm{BS}}\left(x_0, \int_0^T v_{0,t}^2 \mathrm{d}t\right) \\
& + 2\left(\omega_{0,T}^{(-\alpha_x, \rho\lambda v_{0,\cdot}^{\mu+1}), (\alpha_x, v_{0,\cdot})}\right)^2 \partial_{xxyy} P_{\mathrm{BS}}\left(x_0, \int_0^T v_{0,t}^2 \mathrm{d}t\right),
\end{aligned}
$$

where the partial derivatives of P_{BS} are given in Appendix A.2.

Proof. As stated before, Section 5.4 is devoted to the proof of this theorem. $\square$

Remark 5.4.2 (Greeks). The explicit second-order approximation of the put option Delta and Gamma can be obtained via partial differentiation of the second-order pricing formula in Theorem 5.4.1 with respect to $S_0 = e^{x_0}$. One will notice that

$$
\begin{aligned}
\partial_{S_0}\left(\frac{\partial^{i+j}}{\partial x^i \partial y^j} P_{\mathrm{BS}}\left(x_0, \int_0^T v_{0,t}^2 \mathrm{d}t\right)\right) &= \partial_{S_0}\left(\frac{\partial^{i+j}}{\partial x^i \partial y^j} P_{\mathrm{BS}}\left(\ln S_0, \int_0^T v_{0,t}^2 \mathrm{d}t\right)\right) \\
&= e^{-x_0}\left(\frac{\partial^{i+1+j}}{\partial x^{i+1} \partial y^j} P_{\mathrm{BS}}\left(x_0, \int_0^T v_{0,t}^2 \mathrm{d}t\right)\right)
\end{aligned}
$$

for $i, j \in \mathbb{N} \cup \{0\}$. The partial derivatives of P_{BS} are given in Appendix A.2.

In this derivation, the only assumption made was on the diffusion coefficient from Assumption 5.4.1. Specifically, $\beta(t, x) = \lambda_t x^\mu$ for $\mu \in [1/2, 1]$. This means we can obtain the second-order pricing formula for different models by choosing a specific $\alpha(t, x)$ that adheres to Assumption 5.2.1 and a $\mu \in [1/2, 1]$, then appealing to Theorem 5.4.1. For instance, if we choose $\alpha(t, x) = \kappa_t(\theta_t - x)$, then this drift satisfies Assumption 5.2.1. By choosing some $\mu \in [1/2, 1]$, we will obtain the explicit second-order price of a put option where the volatility obeys the dynamics

$$
\mathrm{d}V_t = \kappa_t(\theta_t - V_t)\mathrm{d}t + \lambda_t V_t^\mu \mathrm{d}B_t, \quad V_0 = v_0.
$$

In particular, to obtain the explicit second-order put option price in the Inverse-Gamma model, choose $\alpha(t, x) = \kappa_t(\theta_t - x)$ and $\mu = 1$, so that $\alpha_x(t, x) = -\kappa_t$ and $\alpha_{xx}(t, x) = 0$. Indeed, this gives the desired result for the second-order price of a put option in the Inverse-Gamma model as seen in Langrené et al. [44].

5.4.9 Stochastic Verhulst model explicit price

We now assume the so-called stochastic Verhulst model, here on in known as the Verhulst model. Specifically, the framework is given by

$$
\begin{aligned}
\mathrm{d}S_t &= (r_t^d - r_t^f)S_t\mathrm{d}t + V_tS_t\mathrm{d}W_t, \quad S_0, \\
\mathrm{d}V_t &= \kappa_t(\theta_t - V_t)V_t\mathrm{d}t + \lambda_tV_t\mathrm{d}B_t, \quad V_0 = v_0, \\
\mathrm{d}\langle W, B\rangle_t &= \rho_t\mathrm{d}t.
\end{aligned}
\tag{5.23}
$$

The time-dependent, deterministic parameters $(\kappa_t)_{0\le t\le T}, (\theta_t)_{0\le t\le T}$ and $(\lambda_t)_{0\le t\le T}$ are all assumed to be strictly positive and bounded for all $t \in [0, T]$.

Remark 5.4.3 (Verhulst model heuristics)**.** The process V occurring in the SDE for the volatility we call the stochastic Verhulst process, here on in known as the Verhulst process. This process is reminiscent of the deterministic Verhulst/Logistic model which most famously arises in population growth models, but also in many other areas of sciences, see Tuckwell and Koziol [61]. The process behaves intuitively in the following way. Focusing on the drift term of the volatility V in eq. (5.23), specifically $\kappa(\theta - V)V$, we notice that there is a quadratic term. The interpretation here is that V mean reverts to level θ at a speed of κV. That is, the mean reversion speed of V depends on V itself, and is thus stochastic. Let us compare this with the regular linear type mean reversion drifts seen in stochastic volatility models, $\kappa(\theta - V)$. Here, the mean reversion level is still θ, however the mean reversion speed is κ, and is not directly influenced by V. For an in depth discussion of the Verhulst model for option pricing, we refer the reader to Lewis [46] and Carr and Willems [16].

Notice that the drift term for V is quadratic and thus does not adhere to Assumption 5.2.1; the drift is non-Lipschitz. However, the next proposition shows that this is not a problem.

Proposition 5.4.5. Let Y be a Verhulst process. That is, Y solves the SDE

$$
\mathrm{d}Y_t = a_t(b_t - Y_t)Y_t\mathrm{d}t + c_tY_t\mathrm{d}B_t, \quad Y_0 = y_0,
$$

where $(a_t)_{0\le t\le T}, (b_t)_{0\le t\le T}$ and $(c_t)_{0\le t\le T}$ are deterministic, strictly positive and bounded for all $t \in [0, T]$. The explicit pathwise unique strong solution of Y is given by

$$
Y_t = \left(F_t\left(y_0^{-1} + \int_0^t \frac{a_u}{F_u}\mathrm{d}u \right) \right)^{-1},
$$

where

$$
F_t = \exp\left(-\int_0^t \left(a_ub_u - \frac{1}{2}c_u^2 \right)\mathrm{d}u - \int_0^t c_u\mathrm{d}B_u \right).
$$

Proof. Notice the SDE for the process (Y_t^{-1}) solves a linear SDE. Specifically, by use of Itô's formula we obtain

$$
\mathrm{d}Y_t^{-1} = \left[a_t - (a_tb_t - c_t^2)Y_t^{-1} \right]\mathrm{d}t - c_tY_t^{-1}\mathrm{d}B_t, \quad Y_0^{-1} = y_0^{-1}.
$$

Thus the SDE for (Y_t^{-1}) is linear and can be solved explicitly, see for example Klebaner [41]. $\square$

Lemma 5.4.4 (Verhulst model explicit second-order put option price). Under the Verhulst model eq. (5.23), the explicit second-order price of a put option is given by

$$
\begin{aligned}
\mathrm{Put}^{(2)}_{\mathrm{Verhulst}} = {} & P_{\mathrm{BS}}\left(x_0, \int_0^T v_{0,t}^2 \mathrm{d}t\right) \\
& + 2\omega_{0,T}^{(-(\kappa\theta-2\kappa v_{0,.}),\rho\lambda v_{0,.}^2),(\kappa\theta-2\kappa v_{0,.},v_{0,.})} \partial_{xy} P_{\mathrm{BS}}\left(x_0, \int_0^T v_{0,t}^2 \mathrm{d}t\right) \\
& + \omega_{0,T}^{(-2(\kappa\theta-2\kappa v_{0,.}),\lambda^2 v_{0,.}^2),(2(\kappa\theta-2\kappa v_{0,.}),1)} \partial_y P_{\mathrm{BS}}\left(x_0, \int_0^T v_{0,t}^2 \mathrm{d}t\right) \\
& + 2\omega_{0,T}^{(-(\kappa\theta-2\kappa v_{0,.}),\rho\lambda v_{0,.}^2),(-(\kappa\theta-2\kappa v_{0,.}),\rho\lambda v_{0,.}^2),(2(\kappa\theta-2\kappa v_{0,.}),1)} \partial_{xxy} P_{\mathrm{BS}}\left(x_0, \int_0^T v_{0,t}^2 \mathrm{d}t\right) \\
& + \omega_{0,T}^{(-2(\kappa\theta-2\kappa v_{0,.}),\lambda^2 v_{0,.}^2),(\kappa\theta-2\kappa v_{0,.},-2\kappa),(\kappa\theta-2\kappa v_{0,.},v_{0,.})} \partial_y P_{\mathrm{BS}}\left(x_0, \int_0^T v_{0,t}^2 \mathrm{d}t\right) \\
& + \left\{ 2\omega_{0,T}^{(-(\kappa\theta-2\kappa v_{0,.}),\rho\lambda v_{0,.}^2),(-(\kappa\theta-2\kappa v_{0,.}),\rho\lambda v_{0,.}^2),(\kappa\theta-2\kappa v_{0,.},-2\kappa),(\kappa\theta-2\kappa v_{0,.},v_{0,.})} \right. \\
& \left. + 2\omega_{0,T}^{(-(\kappa\theta-2\kappa v_{0,.}),\rho\lambda v_{0,.}^2),(0,\rho\lambda v_{0,.}),(\kappa\theta-2\kappa v_{0,.},v_{0,.})} \right\} \partial_{xxy} P_{\mathrm{BS}}\left(x_0, \int_0^T v_{0,t}^2 \mathrm{d}t\right) \\
& + 2\omega_{0,T}^{(-(\kappa\theta-2\kappa v_{0,.}),\rho\lambda v_{0,.}^2),(0,\rho\lambda v_{0,.}),(\kappa\theta-2\kappa v_{0,.},v_{0,.})} \partial_{xxy} P_{\mathrm{BS}}\left(x_0, \int_0^T v_{0,t}^2 \mathrm{d}t\right) \\
& + 4\omega_{0,T}^{(-2(\kappa\theta-2\kappa v_{0,.}),\lambda^2 v_{0,.}^2),(\kappa\theta-2\kappa v_{0,.},v_{0,.}),(\kappa\theta-2\kappa v_{0,.},v_{0,.})} \partial_{yy} P_{\mathrm{BS}}\left(x_0, \int_0^T v_{0,t}^2 \mathrm{d}t\right) \\
& + 2\left(\omega_{0,T}^{(-(\kappa\theta-2\kappa v_{0,.}),\rho\lambda v_{0,.}^2),(\kappa\theta-2\kappa v_{0,.},v_{0,.})}\right)^2 \partial_{xxyy} P_{\mathrm{BS}}\left(x_0, \int_0^T v_{0,t}^2 \mathrm{d}t\right).
\end{aligned}
$$

Proof. Under Assumption 5.4.1, the approximation formula in the general model eq. (5.1) is given in Theorem 5.4.1. Notice that

$$\alpha(t, x) = \kappa_t(\theta_t - x)x.$$

In addition,

$$
\begin{aligned}
\alpha_x(t, x) &= \kappa_t\theta_t - 2\kappa_t x, \\
\alpha_{xx}(t, x) &= -2\kappa_t.
\end{aligned}
$$

Substituting these expressions into the formula from Theorem 5.4.1 gives the result. □

5.5 Error analysis

This section is dedicated to the explicit representation and analysis of the error induced by our expansion procedure in Section 5.3. The section is divided into three parts.

1. Section 5.5.1 is devoted to the explicit representation of the error term induced by the expansion procedure.

2. Section 5.5.2 details how one would approach bounding the error term induced by the expansion procedure in terms of the remainder terms generated by the approximation of the underlying volatility/variance process.

3. Section 5.5.3 explores obtaining an explicit bound on the error term in the Verhulst model.

5.5.1 Explicit expression for error term

Recall from Theorem 5.3.1 that the price of a put option in the general model eq. (5.1) is $\text{Put}_{\text{G}} = \text{Put}_{\text{G}}^{(2)} + \mathbb{E}(\mathcal{E})$, where $\text{Put}_{\text{G}}^{(2)}$ is the second-order closed-form price. As our expansion methodology was contingent on the use of Taylor polynomials, the term $\mathcal{E}$ evidently appears due to the truncation of Taylor series. To represent $\mathcal{E}$, we will need explicit expressions for the error terms. These are given by Taylor's theorem, which is presented in Section 1.2.4. As the expansion is second-order, we will only consider results up to second-order. Recall from Section 5.3 the functions

$$\tilde{P}_T^{(\varepsilon)} = x_0 - \int_0^T \frac{1}{2}\rho_t^2 \left(V_t^{(\varepsilon)}\right)^2 \mathrm{d}t + \int_0^T \rho_t V_t^{(\varepsilon)} \mathrm{d}B_t,$$

$$\tilde{Q}_T^{(\varepsilon)} = \int_0^T (1 - \rho_t^2) \left(V_t^{(\varepsilon)}\right)^2 \mathrm{d}t,$$

and

$$\begin{aligned}
P_T^{(\varepsilon)} &= \tilde{P}_T^{(\varepsilon)} - \tilde{P}_T^{(0)} \\
&= \int_0^T \rho_t (V_t^{(\varepsilon)} - v_{0,t}) \mathrm{d}B_t - \frac{1}{2}\int_0^T \rho_t^2 \left(\left(V_t^{(\varepsilon)}\right)^2 - v_{0,t}^2\right) \mathrm{d}t, \\
Q_T^{(\varepsilon)} &= \tilde{Q}_T^{(\varepsilon)} - \tilde{Q}_T^{(0)} \\
&= \int_0^T (1 - \rho_t^2)\left(\left(V_t^{(\varepsilon)}\right)^2 - v_{0,t}^2\right) \mathrm{d}t.
\end{aligned}$$

Furthermore, recall the shorthand

$$\tilde{P}_{\text{BS}} = P_{\text{BS}}\left(\tilde{P}_T^{(0)}, \tilde{Q}_T^{(0)}\right),$$

$$\frac{\partial^{i+j}\tilde{P}_{\text{BS}}}{\partial x^i \partial y^j} = \frac{\partial^{i+j}P_{\text{BS}}\left(\tilde{P}_T^{(0)}, \tilde{Q}_T^{(0)}\right)}{\partial x^i \partial y^j}.$$

Theorem 5.5.1 (Explicit error term). The error term $\mathcal{E}$ induced from the expansion procedure in Theorem 5.3.1 can be decomposed as

$$\mathcal{E} = \mathcal{E}_P + \mathcal{E}_V$$

where $\mathcal{E}_P$ corresponds to error in the approximation of the function P_{BS}, and $\mathcal{E}_V$ corresponds to the error in the approximation of the functions $\varepsilon \mapsto V_t^{(\varepsilon)}$ and $\varepsilon \mapsto \left(V_t^{(\varepsilon)}\right)^2$. Additionally, $\mathcal{E}_P$ and $\mathcal{E}_V$ can be written explicitly as

$$\mathcal{E}_P = \sum_{|\alpha|=3} \frac{|\alpha|}{\alpha_1!\alpha_2!} E_\alpha\left(\tilde{P}_T^{(1)}, \tilde{Q}_T^{(1)}\right) \left(P_T^{(1)}\right)^{\alpha_1} \left(Q_T^{(1)}\right)^{\alpha_2},$$

$$E_\alpha\left(\tilde{P}_T^{(1)}, \tilde{Q}_T^{(1)}\right) = \int_0^1 (1-u)^2 \frac{\partial^3 P_{\text{BS}}}{\partial x^{\alpha_1} \partial y^{\alpha_2}}\left((1-u)\tilde{P}_T(0) + u\tilde{P}_T(1), (1-u)\tilde{Q}_T(0) + u\tilde{Q}_T(1)\right) \mathrm{d}u,$$

and

$$\mathcal{E}_V = \sum_{|\alpha|=1} \frac{\partial \tilde{P}_{\text{BS}}}{\partial x^{\alpha_1} \partial y^{\alpha_2}} \Theta_{2,T}^{(1)}(P^{\alpha_1}Q^{\alpha_2}) + \frac{1}{2}\sum_{|\alpha|=2} \frac{|\alpha|}{\alpha_1!\alpha_2!} \frac{\partial^2 \tilde{P}_{\text{BS}}}{\partial x^{\alpha_1} \partial y^{\alpha_2}} \Theta_{2,T}^{(1)}(P^{\alpha_1}Q^{\alpha_2}),$$

where $\Theta_{2,T}^{(1)}(P^0 Q^x) := \Theta_{2,T}^{(1)}(Q^x)$ and $\Theta_{2,T}^{(1)}(P^y Q^0) := \Theta_{2,T}^{(1)}(P^y)$, for $x, y \in \{1, 2\}$. Here $\alpha := (\alpha_1, \alpha_2)$ and $|\alpha| = \alpha_1 + \alpha_2$.

Proof. The decomposition $\mathcal{E} = \mathcal{E}_P + \mathcal{E}_V$ is a clear consequence of Taylor's theorem. The next two subsections are devoted to representing $\mathcal{E}_P$ and $\mathcal{E}_V$ explicitly.

Explicit $\mathcal{E}_P$

First we will derive $\mathcal{E}_P$ explicitly, the error term corresponding to the second-order approximation of P_{BS}. In our expansion procedure, we expand P_{BS} up to second-order around the point

$$\left(\tilde{P}_T^{(0)}, \tilde{Q}_T^{(0)}\right) = \left(x_0 - \int_0^T \frac{1}{2}\rho_t^2 v_{0,t}^2 \mathrm{d}t + \int_0^T \rho_t v_{0,t} \mathrm{d}B_t, \int_0^T (1 - \rho_t^2) v_{0,t}^2 \mathrm{d}t\right)$$

and evaluate at $\left(\tilde{P}_T^{(1)}, \tilde{Q}_T^{(1)}\right)$. Hence, in the Taylor expansion of P_{BS}, the terms will be of the form

$$\frac{\partial^{|\alpha|} \tilde{P}_{\mathrm{BS}}}{\partial x^{\alpha_1} \partial y^{\alpha_2}} \left(P_T^{(1)}\right)^{\alpha_1} \left(Q_T^{(1)}\right)^{\alpha_2}$$

for $|\alpha| = 0, 1, 2$. By Theorem 1.2.6 (Taylor's theorem) we can write the second-order Taylor polynomial of $P_{\mathrm{BS}}\left(\tilde{P}_T^{(1)}, \tilde{Q}_T^{(1)}\right)$ with error term as

$$\begin{aligned}
P_{\mathrm{BS}}\left(\tilde{P}_T^{(1)}, \tilde{Q}_T^{(1)}\right) &= \tilde{P}_{\mathrm{BS}} + \left(\partial_x \tilde{P}_{\mathrm{BS}}\right) P_T^{(1)} + \left(\partial_y \tilde{P}_{\mathrm{BS}}\right) Q_T^{(1)} \\
&\quad + \frac{1}{2}\left(\partial_{xx} \tilde{P}_{\mathrm{BS}}\right)\left(P_T^{(1)}\right)^2 + \frac{1}{2}\left(\partial_{yy} \tilde{P}_{\mathrm{BS}}\right)\left(Q_T^{(1)}\right)^2 + \left(\partial_{xy} \tilde{P}_{\mathrm{BS}}\right) P_T^{(1)} Q_T^{(1)} \\
&\quad + \underbrace{\sum_{|\alpha|=3} \frac{|\alpha|}{\alpha_1! \alpha_2!} E_\alpha\left(\tilde{P}_T^{(1)}, \tilde{Q}_T^{(1)}\right)\left(P_T^{(1)}\right)^{\alpha_1}\left(Q_T^{(1)}\right)^{\alpha_2}}_{\text{Error term}}
\end{aligned} \tag{5.24}$$

with

$$\begin{aligned}
E_\alpha\left(\tilde{P}_T^{(1)}, \tilde{Q}_T^{(1)}\right) &= \int_0^1 (1-u)^2 \frac{\partial^3 P_{\mathrm{BS}}}{\partial x^{\alpha_1} \partial y^{\alpha_2}}\left(\tilde{P}_T(0) + u P_T(1), \tilde{Q}_T(0) + u Q_T(1)\right) \mathrm{d}u \\
&= \int_0^1 (1-u)^2 \frac{\partial^3 P_{\mathrm{BS}}}{\partial x^{\alpha_1} \partial y^{\alpha_2}}\left((1-u)\tilde{P}_T(0) + u\tilde{P}_T(1), (1-u)\tilde{Q}_T(0) + u\tilde{Q}_T(1)\right) \mathrm{d}u.
\end{aligned}$$

Taking expectation gives $\mathrm{Put}_{\mathrm{G}}$. Thus the explicit form for the error term $\mathcal{E}_P$ is

$$\mathcal{E}_P = \sum_{|\alpha|=3} \frac{|\alpha|}{\alpha_1! \alpha_2!} E_\alpha\left(\tilde{P}_T^{(1)}, \tilde{Q}_T^{(1)}\right)\left(P_T^{(1)}\right)^{\alpha_1}\left(Q_T^{(1)}\right)^{\alpha_2}.$$

Explicit $\mathcal{E}_V$

Now we derive $\mathcal{E}_V$ explicitly, the error corresponding to the second-order approximation of the functions $\varepsilon \mapsto V_t^{(\varepsilon)}$ and $\varepsilon \mapsto \left(V_t^{(\varepsilon)}\right)^2$. Recall from Lemma 5.3.2, since $P_{0,T} = \tilde{P}_T^{(0)} - \tilde{P}_T^{(0)} = 0$ and similarly $Q_{0,T} = 0$, we could write

$$P_T^{(\varepsilon)} = P_{1,T} + \frac{1}{2}\varepsilon^2 P_{2,T} + \Theta_{2,T}^{(\varepsilon)}(P),$$

$$(P_T^{(\varepsilon)})^2 = \varepsilon^2 P_{1,T}^2 + \Theta_{2,T}^{(\varepsilon)}(P^2),$$

$$Q_T^{(\varepsilon)} = \varepsilon Q_{1,T} + \frac{1}{2}\varepsilon^2 Q_{2,T} + \Theta_{2,T}^{(\varepsilon)}(Q),$$

$$(Q_T^{(\varepsilon)})^2 = \varepsilon^2 Q_{1,T}^2 + \Theta_{2,T}^{(\varepsilon)}(Q^2),$$

and

$$P_T^{(\varepsilon)}Q_T^{(\varepsilon)} = \varepsilon^2 P_{1,T}Q_{1,T} + \Theta_{2,T}^{(\varepsilon)}(PQ),$$

where

$$P_{1,T} = \int_0^T \rho_t V_{1,t}\mathrm{d}B_t - \int_0^T \rho_t^2 v_{0,t}V_{1,t}\mathrm{d}t,$$

$$P_{2,T} = \int_0^T \rho_t V_{2,t}\mathrm{d}B_t - \int_0^T \rho_t^2 \left(V_{1,t}^2 + v_{0,t}V_{2,t}\right)\mathrm{d}t,$$

$$Q_{1,T} = 2\int_0^T (1 - \rho_t^2)v_{0,t}V_{1,t}\mathrm{d}t,$$

$$Q_{2,T} = 2\int_0^T (1 - \rho_t^2)\left(V_{1,t}^2 + v_{0,t}V_{2,t}\right)\mathrm{d}t.$$

The idea then was to approximate the functions $\varepsilon \mapsto P_T^{(\varepsilon)}$, $\varepsilon \mapsto Q_T^{(\varepsilon)}$ and their variants by their second-order expansions. For example, in the expansion of P_{BS} in eq. (5.24), if we focus on the term corresponding to the first derivative of P_{BS} in its second argument, we have

$$(\partial_y \tilde{P}_{\mathrm{BS}})Q_T^{(1)} = (\partial_y \tilde{P}_{\mathrm{BS}})(Q_{1,T} + \frac{1}{2}Q_{2,T} + \Theta_{2,T}^{(1)}(Q))$$

$$= (\partial_y \tilde{P}_{\mathrm{BS}})(Q_{1,T} + \frac{1}{2}Q_{2,T}) + \underbrace{(\partial_y \tilde{P}_{\mathrm{BS}})(\Theta_{2,T}^{(1)}(Q))}_{\text{Error term}}.$$

For the term corresponding to the second derivative of P_{BS} in its second argument, we have

$$\frac{1}{2}(\partial_{yy}\tilde{P}_{\mathrm{BS}})\left(Q_T^{(1)}\right)^2 = \frac{1}{2}(\partial_{yy}\tilde{P}_{\mathrm{BS}})\left(Q_{1,T}^2 + \Theta_{2,T}^{(1)}(Q^2)\right)$$

$$= \frac{1}{2}(\partial_{yy}\tilde{P}_{\mathrm{BS}})\left(Q_{1,T}^2\right) + \underbrace{\frac{1}{2}(\partial_{yy}\tilde{P}_{\mathrm{BS}})\left(\Theta_{2,T}^{(1)}(Q^2)\right)}_{\text{Error term}}.$$

Following this pattern, we can see that the error term $\mathcal{E}_V$ can be written explicitly as

$$\mathcal{E}_V = \sum_{|\alpha|=1} \frac{\partial \tilde{P}_{\mathrm{BS}}}{\partial x^{\alpha_1}y^{\alpha_2}}\Theta_{2,T}^{(1)}(P^{\alpha_1}Q^{\alpha_2}) + \frac{1}{2}\sum_{|\alpha|=2} \frac{|\alpha|}{\alpha_1!\alpha_2!}\frac{\partial^2 \tilde{P}_{\mathrm{BS}}}{\partial x^{\alpha_1}y^{\alpha_2}}\Theta_{2,T}^{(1)}(P^{\alpha_1}Q^{\alpha_2}).$$

$\square$

As the second-order price of a put option is the expectation of our expansion, the goal is to bound $\mathcal{E}$ in L^1 for a specific volatility process V.

5.5.2 Bounding error term

Our objective is to appeal to the explicit representation of the error term $\mathcal{E}$ as seen in Theorem 5.5.1 and bound it in L^1 under the general model eq. (5.1), where the volatility process V adheres to Assumption 5.2.1. Afterwards, in Section 5.5.3, we will comment on obtaining an explicit error bound under the Verhulst model. In order to obtain an L^1 bound on the error term $\mathcal{E}$, it is sufficient to obtain ingredients given in the following proposition.

Proposition 5.5.1. In order to obtain an L^1 bound on the error term $\mathcal{E}$, it is sufficient to obtain:

1. Bounds on $\|\Theta_{2,T}^{(1)}(P^{\alpha_1}Q^{\alpha_2})\|$, where $|\alpha| = 1, 2$.

2. Bounds on $\|P_T^{(1)}\|_p$ and $\|Q_T^{(1)}\|_p$ for $p \geq 2$.

The purpose of the next part of this section is to validate Proposition 5.5.1.

Lemma 5.5.1. Consider the third-order partial derivatives of P_{BS}, $\frac{\partial^3 P_{\mathrm{BS}}}{\partial x^{\alpha_1} \partial y^{\alpha_2}}$, where $\alpha_1 + \alpha_2 = 3$ as well as the linear functions $h_1, h_2 : [0,1] \to \mathbb{R}_+$ such that $h_1(u) = u(d_1 - c_1) + c_1$ and $h_2(u) = u(d_2 - c_2) + c_2$. Assume there exists no point $a \in (0,1)$ such that

$$\lim_{u \to a} \frac{h_1(u) - k + \int_0^T (r_t^d - r_t^f)\mathrm{d}t}{\sqrt{h_2(u)}} = 0 \quad \text{and} \quad \lim_{u \to a} h_2(u) = 0. \tag{5.25}$$

Then there exists functions B_α bounded on $\mathbb{R}_+ \times \mathbb{R}$ such that

$$\sup_{u \in (0,1)} \left| \frac{\partial^3 P_{\mathrm{BS}}}{\partial x^{\alpha_1} \partial y^{\alpha_2}}(h_1(u), h_2(u)) \right| = B_\alpha(T, k).$$

Furthermore, the behaviour of B_α for fixed k and T is characterised by the functions ζ and ν respectively, where

$$\zeta(T) = \hat{A}e^{-\int_0^T r_t^f \mathrm{d}t} e^{-E_2 \tilde{r}^2(T)} e^{-E_1 \tilde{r}(T)} \sum_{i=0}^n c_i \tilde{r}^i(T),$$

with $\tilde{r}(T) := \int_0^T (r_t^d - r_t^f)\mathrm{d}t$ and $E_2 > 0$, $E_1 \in \mathbb{R}$, $\hat{A} \in \mathbb{R}$, $n \in \mathbb{N}$ and $c_0, \ldots, c_n$ are constants, and

$$\nu(k) = \tilde{A}e^{-D_2 k^2 + D_1 k} \sum_{i=0}^N C_i (-1)^i k^i,$$

with $D_2 > 0$, $D_1 \in \mathbb{R}$, $\tilde{A} \in \mathbb{R}$, $N \in \mathbb{N}$ and $C_0, \ldots, C_N$ are constants.

Proof. Lemma 5.5.1 is very similar to Lemma 3.5.2, where the latter is the equivalent lemma for the function $\mathrm{Put}_{\mathrm{BS}}$. In fact, we will show that Lemma 3.5.2 implies Lemma 5.5.1. In the following, we will repeatedly denote by F or G to be an arbitrary polynomial of some degree, as well as A to be an arbitrary constant. That is, they may be different on each use.

First, as a function of x and y, notice from Appendix A.2 that the third-order partial derivatives $\frac{\partial^3 P_{\mathrm{BS}}}{\partial x^{\alpha_1} \partial y^{\alpha_2}}$, where $\alpha_1 + \alpha_2 = 3$ can be written as

$$A \frac{e^x \phi(d_+^{\mathrm{ln}})}{y^{m/2}} G(d_+^{\mathrm{ln}}, d_-^{\mathrm{ln}}, \sqrt{y}), \quad m \in \mathbb{N} \tag{5.26}$$

except for when $\alpha = (3,0)$, in which case the partial derivative can be written as

$$A \frac{e^x \phi(d_+^{\mathrm{ln}})}{y^{m/2}} G(d_+^{\mathrm{ln}}, d_-^{\mathrm{ln}}, \sqrt{y}) + \overline{A}e^x \phi(d_+^{\mathrm{ln}})(\mathcal{N}(d_+^{\mathrm{ln}}) - 1), \quad m \in \mathbb{N}. \tag{5.27}$$

Furthermore, from Appendix A.1, it can seen that as a function of x and y, the third-order partial derivatives of $\mathrm{Put}_{\mathrm{BS}}$ can be written as

$$A \frac{\phi(d_+)}{x^n y^{m/2}} F(d_+, d_-, \sqrt{y}), \quad n \in \mathbb{Z}, m \in \mathbb{N}. \tag{5.28}$$

Recall

$$d_\pm = d_\pm(x, y) = \frac{\ln(x/K) + \int_0^T \left(r_t^d - r_t^f \right) \mathrm{d}t}{\sqrt{y}} \pm \frac{1}{2}\sqrt{y},$$

$$d_\pm^{\ln} = d_\pm^{\ln}(x, y) = \frac{x - k + \int_0^T \left(r_t^d - r_t^f \right) \mathrm{d}t}{\sqrt{y}} \pm \frac{1}{2}\sqrt{y},$$

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$

Let us consider the cases for which $\alpha \neq (3,0)$. Without loss of generality, set $k = \ln(K)$. Notice that $d_\pm^{\ln}(x, y) = d_\pm(e^x, y)$. Take $n = -1$ in eq. (5.28). Roughly speaking, we will say that two functions f and g are 'of the same form' if they are equal up to constant values.[4] Furthermore, we will denote this relation by $f \stackrel{C}{\sim} g$. Then comparing eq. (5.26) and eq. (5.28), the form of the partial derivatives of P_{BS} are the same as the partial derivatives of $\mathrm{Put}_{\mathrm{BS}}$ composed with the function e^x in its first argument. Specifically, we can write

$$\frac{\partial^3 P_{\mathrm{BS}}}{\partial x^{\alpha_1} \partial y^{\alpha_2}}(x, y) \stackrel{C}{\sim} \frac{\partial^3 \mathrm{Put}_{\mathrm{BS}}}{\partial x^{\alpha_1} \partial y^{\alpha_2}}(e^x, y).$$

Now, consider arbitrary functions $f, b : \mathbb{R} \to \mathbb{R}$ such that

$$\sup_{x \in \mathbb{R}} |f(x)| = L < \infty.$$

Then it is true that

$$\sup_{x \in \mathbb{R}} |f(b(x))| = \tilde{L} \leq L < \infty.$$

Thus

$$\sup_{u \in (0,1)} \left| \frac{\partial^3 P_{\mathrm{BS}}}{\partial x^{\alpha_1} \partial y^{\alpha_2}}(h_1(u), h_2(u)) \right| \stackrel{C}{\sim} \sup_{u \in (0,1)} \left| \frac{\partial^3 \mathrm{Put}_{\mathrm{BS}}}{\partial x^{\alpha_1} \partial y^{\alpha_2}}(e^{h_1(u)}, h_2(u)) \right|.$$

Under the assumption in eq. (5.25), and then using Lemma 3.5.2, this supremum will not blow up. Clearly, $\sup_{u \in (0,1)} \frac{\partial^3 \mathrm{Put}_{\mathrm{BS}}}{\partial x^{\alpha_1} \partial y^{\alpha_2}}(e^{h_1(u)}, h_2(u))$ is a function of T and K. By substituting $k = \ln(K)$ in the result of Lemma 3.5.2, we obtain the form of ζ and ν.

Now for the case of $\alpha = (3,0)$, we have that

$$\frac{\partial^3 P_{\mathrm{BS}}}{\partial x^3}(x, y) \stackrel{C}{\sim} \frac{\partial^3 \mathrm{Put}_{\mathrm{BS}}}{\partial x^3}(e^x, y) + A \underbrace{e^x \phi(d_+^{\ln})(\mathcal{N}(d_+^{\ln}) - 1)}_{=:H(x,y)}.$$

Now

$$|H(x, y)| = |e^x \phi(d_+^{\ln})(\mathcal{N}(d_+^{\ln}) - 1)| \leq e^x \phi(d_+^{\ln}).$$

Thus

$$\sup_{x \in \mathbb{R}} |H(x, y)| = \sup_{x \in \mathbb{R}} |e^x \phi(d_+^{\ln})(\mathcal{N}(d_+^{\ln}) - 1)| \leq \sup_{x \in \mathbb{R}} e^x \phi(d_+^{\ln}) < \infty$$

---

[4]It is possible to give a formal mathematical definition of 'is of the form', however this is actually much more involved that one would initially think. For this reason, we will not define this notion rigorously, and it should be clear to the reader what is meant.

and also

$$\sup_{y\in\mathbb{R}_+}|H(x,y)| = \sup_{y\in\mathbb{R}_+}|\phi(d_+^{\ln})(\mathcal{N}(d_+^{\ln})-1)| \leq \sup_{y\in\mathbb{R}_+}\phi(d_+^{\ln}) < \infty.$$

Hence

$$\sup_{u\in(0,1)} H(h_1(u),h_2(u)) \leq \sup_{u\in(0,1)} e^{h_1(u)}\phi(d_+^{\ln}(h_1(u),h_2(u))) = \hat{m}(T,k),$$

where $\hat{m}$ is a bounded function on $\mathbb{R}_+\times\mathbb{R}$. By direct computation, it is clear that for fixed T the form of $\hat{m}$ is given by

$$Ae^{-\hat{D}_2 k^2}e^{\hat{D}_1 k},$$

where $\hat{D}_2 > 0$ and $\hat{D}_1 \in \mathbb{R}$. For fixed k, it is given by

$$Ae^{-\hat{E}_2\tilde{r}^2(T)}e^{\hat{E}_1\tilde{r}(T)},$$

where $\hat{E}_2 > 0$ and $\hat{E}_1 \in \mathbb{R}$. Thus

$$\sup_{u\in(0,1)}\left|\frac{\partial^3 P_{\text{BS}}}{\partial x^3}(h_1(u),h_2(u))\right| \overset{C}{\sim} \sup_{u\in(0,1)}\left|\frac{\partial^3\text{Put}_{\text{BS}}}{\partial x^3}(e^{h_1(u)},h_2(u)) + AH(h_1(u),h_2(u))\right|$$

$$\leq \sup_{u\in(0,1)}\left|\frac{\partial^3\text{Put}_{\text{BS}}}{\partial x^3}(e^{h_1(u)},h_2(u))\right| + A\sup_{u\in(0,1)}|H(h_1(u),h_2(u))|$$

$$\overset{C}{\sim} B_{(3,0)}(T,k) + A\hat{m}(T,k).$$

But the form of $\hat{m}$ is exactly that of B_α without the polynomial expression. Thus, the sum of them is again of the form of B_α. $\qquad\square$

Bounding $\mathcal{E}_V$

We first consider bounding the term $\mathcal{E}_V$ from Theorem 5.5.1 in L^1. The terms of interest to bound are

$$\frac{\partial^{|\alpha|}\tilde{P}_{\text{BS}}}{\partial x^{\alpha_1}\partial y^{\alpha_2}}\Theta_{2,T}^{(1)}(P^{\alpha_1}Q^{\alpha_2}), \quad |\alpha| = 1,2.$$

Now the second argument of $\tilde{P}_{\text{BS}}$ is $\tilde{Q}_T^{(0)}$, which is strictly positive. By considering the trivial linear function $u \mapsto (1-u)Q_T^{(0)} + uQ_T^{(0)}$, then by Lemma 5.5.1 this implies $\frac{\partial^{|\alpha|}\tilde{P}_{\text{BS}}}{\partial x^{\alpha_1}\partial y^{\alpha_2}} \leq B_\alpha(T,k)$. Thus

$$\left\|\frac{\partial^{|\alpha|}\tilde{P}_{\text{BS}}}{\partial x^{\alpha_1}\partial y^{\alpha_2}}\Theta_{2,T}^{(1)}(P^{\alpha_1}Q^{\alpha_2})\right\| \leq B_\alpha(T,k)\left\|\Theta_{2,T}^{(1)}(P^{\alpha_1}Q^{\alpha_2})\right\|. \tag{5.29}$$

Equation (5.29) suggests that obtaining an L^1 bound on the remainder term $\Theta_{2,T}^{(1)}(P^{\alpha_1}Q^{\alpha_2})$ for $|\alpha| = 1,2$ is sufficient. This validates item 1 in Proposition 5.5.1.

Bounding $\mathcal{E}_P$

The terms of interest are

$$E_\alpha \left(\tilde{P}_T^{(1)}, \tilde{Q}_T^{(1)} \right) \left(P_T^{(1)} \right)^{\alpha_1} \left(Q_T^{(1)} \right)^{\alpha_2}, \quad |\alpha| = 3.$$

We now write

$$J(u) := (1 - u)\tilde{P}_T(0) + u\tilde{P}_T(1),$$
$$K(u) := (1 - u)\tilde{Q}_T(0) + u\tilde{Q}_T(1),$$

so that

$$\frac{\partial^3 P_{\mathrm{BS}}}{\partial x^{\alpha_1} \partial y^{\alpha_2}} (J(u), K(u)) = \frac{\partial^3 P_{\mathrm{BS}}}{\partial x^{\alpha_1} \partial y^{\alpha_2}} \left((1 - u)\tilde{P}_T(0) + u\tilde{P}_T(1), (1 - u)\tilde{Q}_T(0) + u\tilde{Q}_T(1) \right).$$

Proposition 5.5.2. There exists functions B_α with $\alpha_1 + \alpha_2 = 3$ as in Lemma 5.5.1 such that

$$\sup_{u \in (0,1)} \left| \frac{\partial^3 P_{\mathrm{BS}}}{\partial x^{\alpha_1} \partial y^{\alpha_2}} (J(u), K(u)) \right| \le B_\alpha(T, k) \quad \mathbb{Q} \text{ a.s..}$$

Proof. Since J and K are linear functions, then from Lemma 5.5.1, this claim is immediately true if we can show that K is bounded away from 0. Recall

$$K(u) = (1 - u)\left(\int_0^T (1 - \rho_t^2)v_{0,t}^2 \mathrm{d}t \right) + u \int_0^T (1 - \rho_t^2)V_t^2 \mathrm{d}t.$$

K corresponds to the linear interpolation of $\int_0^T (1 - \rho_t^2)v_{0,t}^2 \mathrm{d}t$ and $\int_0^T (1 - \rho_t^2)V_t^2 \mathrm{d}t$. It is clear $\sup_{t \in [0,T]}(1 - \rho_t^2) > 0$. As V^2 corresponds to the variance process, in application this is always chosen to be a non-negative process such that the set $\{t \in [0, T] : V_t^2 > 0\}$ has non-zero Lebesgue measure. Thus these integrals are strictly positive and hence K is bounded away from 0 $\mathbb{Q}$ a.s.. $\qquad\square$

By Proposition 5.5.2

$$\left| E_\alpha \left(\tilde{P}_T^{(1)}, \tilde{Q}_T^{(1)} \right) \right| = \left| \int_0^1 (1 - u)^2 \frac{\partial^{|\alpha|} P_{\mathrm{BS}}}{\partial x^{\alpha_1} \partial y^{\alpha_2}} (J(u), K(u)) \, \mathrm{d}u \right|$$
$$\le \frac{1}{3} B_\alpha(T, k).$$

Thus

$$\left\| E_\alpha \left(\tilde{P}_T^{(1)}, \tilde{Q}_T^{(1)} \right) \left(P_T^{(1)} \right)^{\alpha_1} \left(Q_T^{(1)} \right)^{\alpha_2} \right\| \le \frac{1}{3} B_\alpha(T, k) \left\| \left(P_T^{(1)} \right)^{\alpha_1} \right\|_2 \left\| \left(Q_T^{(1)} \right)^{\alpha_2} \right\|_2. \tag{5.30}$$

Looking at the second and third term on the RHS of eq. (5.30), it is clear one of our objectives is to bound $P_T^{(1)}$ and $Q_T^{(1)}$ in L^p for $p \ge 2$. This validates item 2 in Proposition 5.5.1.

Lemma 5.5.2. The terms from Proposition 5.5.1 can be bounded if the following quantities can be bounded:

1. $\|\Theta_{0,t}^{(1)}(V)\|_p$ and $\|\Theta_{0,t}^{(1)}(V^2)\|_p$ for $p \ge 2$.

2. $\|\Theta_{1,t}^{(1)}(V)\|_p$ and $\|\Theta_{1,t}^{(1)}(V^2)\|_p$ for $p \ge 2$.

3. $\|\Theta_{2,t}^{(1)}(V)\|_p$ and $\|\Theta_{2,t}^{(1)}(V^2)\|_p$ for $p \geq 2$.

Proof. We will make extensive use of the following integral inequality:

$$\left(\int_0^T |f(u)|\mathrm{d}u\right)^p \leq T^{p-1}\int_0^T |f(u)|^p \mathrm{d}u, \quad p \geq 1. \tag{5.31}$$

For the rest of this proof, assume that $p \geq 2$. We will denote by C_p and D_p generic constants that solely depend on p. They may be different on each use. Notice

$$P_T^{(1)} = \int_0^T \rho_t \Theta_{0,t}^{(1)}(V)\mathrm{d}B_t - \frac{1}{2}\int_0^T \rho_t^2 \Theta_{0,t}^{(1)}(V^2)\mathrm{d}t,$$

$$Q_T^{(1)} = \int_0^T (1 - \rho_t^2)\Theta_{0,t}^{(1)}(V^2)\mathrm{d}t.$$

Applying the Minkowski and Burkholder-Davis-Gundy inequalities, as well as the integral inequality eq. (5.31), we obtain

$$\|P_T^{(1)}\|_p \leq C_p T^{\frac{1}{2}-\frac{1}{p}}\left(\int_0^T \rho_t^p \|\Theta_{0,t}^{(1)}(V)\|_p^p \mathrm{d}t\right)^{1/p} + \frac{1}{2}D_p T^{1-\frac{1}{p}}\left(\int_0^T \rho_t^{2p}\|\Theta_{0,t}^{(1)}(V^2)\|_p^p \mathrm{d}t\right)^{1/p}$$

and

$$\|Q_T^{(1)}\|_p \leq C_p T^{1-\frac{1}{p}}\left(\int_0^T (1-\rho_t^2)^p\|\Theta_{0,t}^{(1)}(V^2)\|_p^p \mathrm{d}t\right)^{1/p}.$$

Now also $\left(V_t^{(1)}\right)^2 = \left(v_{0,t} + \Theta_{0,t}^{(1)}(V)\right)^2 = v_{0,t}^2 + 2v_{0,t}\Theta_{0,t}^{(1)}(V) + \left(\Theta_{0,t}^{(1)}(V)\right)^2$, so that

$$\Theta_{0,t}^{(1)}(V^2) = 2v_{0,t}\Theta_{0,t}^{(1)}(V) + \left(\Theta_{0,t}^{(1)}(V)\right)^2.$$

This suggests that finding an L^p bound on the remainder term $\Theta_{0,t}^{(1)}(V)$ is sufficient in order to bound $P_T^{(1)}$ and $Q_T^{(1)}$ in L^p. This validates item 1.

We can write the following remainder terms of P and Q as

$$\Theta_{2,T}^{(1)}(P) = \int_0^T \rho_t \Theta_{2,t}^{(1)}(V)\mathrm{d}B_t - \frac{1}{2}\int_0^T \rho_t^2 \Theta_{2,t}^{(1)}(V^2)\mathrm{d}t,$$

$$\Theta_{2,T}^{(1)}(Q) = \int_0^T (1-\rho_t^2)\Theta_{2,t}^{(1)}(V^2)\mathrm{d}t,$$

$$\begin{aligned}
\Theta_{2,T}^{(1)}(P^2) &= \left(P_T^{(1)}\right)^2 - P_{1,T}^2 \\
&= (P_T^{(1)} - P_{1,T})(P_T^{(1)} + P_{1,T}) \\
&= \left(\int_0^T \rho_t \Theta_{1,t}^{(1)}(V)\mathrm{d}B_t - \frac{1}{2}\int_0^T \rho_t^2\Theta_{1,t}^{(1)}(V^2)\mathrm{d}t\right) \\
&\quad \cdot \left(\int_0^T \rho_t(2\Theta_{0,t}^{(1)}(V) - \Theta_{1,t}^{(1)}(V))\mathrm{d}B_t - \frac{1}{2}\int_0^T \rho_t^2(2\Theta_{0,t}^{(1)}(V^2) - \Theta_{1,t}^{(1)}(V^2))\mathrm{d}t\right),
\end{aligned} \tag{5.32}$$

$$\begin{aligned}
\Theta_{2,T}^{(1)}(Q^2) &= \left(Q_T^{(1)}\right)^2 - Q_{1,T}^2 \\
&= (Q_T^{(1)} - Q_{1,T})(Q_T^{(1)} + Q_{1,T}) \\
&= \left(\int_0^T (1-\rho_t^2)\Theta_{1,t}^{(1)}(V^2)\mathrm{d}t\right)\left(\int_0^T (1-\rho_t^2)\left[2\Theta_{0,t}^{(1)}(V^2) - \Theta_{1,t}^{(1)}(V^2)\right]\mathrm{d}t\right).
\end{aligned}$$

Furthermore, notice

$$\Theta_{1,t}^{(1)}(V^2) = \Theta_{0,t}^{(1)}(V^2) + 2v_{0,t}\left(\Theta_{1,t}^{(1)}(V) - \Theta_{0,t}^{(1)}(V)\right),$$

$$\Theta_{2,t}^{(1)}(V^2) = \Theta_{1,t}^{(1)}(V^2) - 2v_{0,t}\left(\Theta_{1,t}^{(1)}(V) - \Theta_{2,t}^{(1)}(V)\right) - \left(\Theta_{0,t}^{(1)}(V) - \Theta_{1,t}^{(1)}(V)\right)^2.$$

Then, by application of the Minkowski, Burkholder-Davis-Gundy and Cauchy-Schwarz inequalities, it is sufficient to obtain L^p bounds on $\Theta_{1,t}^{(1)}(V)$ and $\Theta_{2,t}^{(1)}(V)$ in order to obtain L^p bounds on the remainders of P and Q from eq. (5.32). For the cross remainder term, we have

$$\|\Theta_{2,T}^{(1)}(PQ)\|_p \le \|P_T^{(1)}\|_{2p}\|Q_T^{(1)}\|_{2p} + \|P_{1,T}^{(1)}\|_{2p}\|Q_{1,T}^{(1)}\|_{2p}.$$

We just need to check how to obtain L^p bounds on $P_{1,T}^{(1)}$ and $Q_{1,T}^{(1)}$. Notice

$$\|P_{1,T}\|_p \le C_p T^{\frac{1}{2}-\frac{1}{p}}\left(\int_0^T \rho_t^p \|\Theta_{0,t}^{(1)}(V) - \Theta_{1,t}^{(1)}(V)\|_p^p dt\right)^{1/p}$$
$$+ \frac{1}{2}D_p T^{1-\frac{1}{p}}\left(\int_0^T \rho_t^{2p}\|\Theta_{0,t}^{(1)}(V^2) - \Theta_{1,t}^{(1)}(V^2)\|_p^p dt\right)^{1/p}$$

and

$$\|Q_{1,T}\|_p \le D_p T^{1-\frac{1}{p}}\left(\int_0^T (1-\rho_t^2)^p\|\Theta_{0,t}^{(1)}(V^2) - \Theta_{1,t}^{(1)}(V^2)\|_p^p dt\right)^{1/p}.$$

Again, all we need to obtain L^p bounds on the cross remainder term are L^p bounds on $\Theta_{1,t}^{(1)}(V)$ and $\Theta_{2,t}^{(1)}(V)$. This validates item 2 and item 3. □

5.5.3 Verhulst model error analysis remarks

In this section, we consider obtaining an explicit bound on the error induced by our expansion procedure under the Verhulst model eq. (5.23). Recall that the volatility process for the Verhulst model is the Verhulst process, which is given as the solution to the SDE

$$dV_t = \kappa_t(\theta_t - V_t)V_t dt + \lambda_t V_t dB_t, \quad V_0 = v_0. \tag{5.33}$$

Furthermore, recall that the perturbed volatility process is given as the solution to the SDE

$$dV_t^{(\varepsilon)} = \kappa_t(\theta_t - V_t^{(\varepsilon)})V_t^{(\varepsilon)} dt + \varepsilon\lambda_t V_t^{(\varepsilon)} dB_t, \quad V_0^{(\varepsilon)} = v_0,$$

where $\varepsilon \in [0,1]$. In view of Lemma 5.5.2, to obtain an L^1 bound on the error term resulting from the approximation procedure, it is sufficient to obtain L^p bounds on the remainder terms of the approximated volatility process. For example, analysing the SDE for the remainder term $\Theta_{0,t}^{(1)}(V)$, we get

$$d\Theta_{0,t}^{(1)}(V) = dV_t^{(1)} - dv_{0,t}$$
$$= A_t\Theta_{0,t}^{(1)}(V)dt + \lambda_t V_t dB_t, \quad \Theta_{0,0}^{(1)}(V) = 0,$$

where $A_t := \kappa_t\theta_t - \kappa_t(V_t + v_{0,t})$. Thus, an expression for $\Theta_{0,t}^{(1)}(V)$ can be obtained as

$$\Theta_{0,t}^{(1)}(V) = e^{\int_0^t A_z dz}\int_0^t \lambda_u V_u e^{-\int_0^u A_z dz} dB_u. \tag{5.34}$$

Although eq. (5.34) is not really an explicit expression for $\Theta_{0,t}^{(1)}(V)$, it is easily obtained by use of the linear relationship of the SDE. By inspecting eq. (5.34), it is evident that obtaining an L^p bound on $\Theta_{0,t}^{(1)}(V)$ will at least require bounds on the moments of the Verhulst process V. A similar situation occurs when exploring the higher order remainder terms. Unfortunately, to our knowledge, there has been little research done on the Verhulst process. In fact, from scouring the literature, the only results that can currently be obtained are the explicit solution and stationary distribution of the process. The explicit solution was obtained in Proposition 5.4.5. The stationary distribution can be shown to be Gamma. That is, as $t \to \infty, V_t \sim$ Gamma, see Mackevičius [48]. However, there does not exist explicit expressions for its moments nor an explicit distribution for any fixed time t. These however, can be approximated, see again Mackevičius [48]. The lack of results on this process is mostly due to the quadratic nature of the drift term.

A crude bound on the p-th moment can be obtained in the following way. From Proposition 5.4.5, the explicit solution to the Verhulst process is

$$V_t = \left(F_t \left(v_0^{-1} + \int_0^t \frac{\kappa_u}{F_u} \mathrm{d}u \right) \right)^{-1},$$

where

$$F_t = \exp\left(-\int_0^t \left(\kappa_u \theta_u - \frac{1}{2}\lambda_u^2 \right) \mathrm{d}u - \int_0^t \lambda_u \mathrm{d}B_u \right).$$

Let $Z_t := V_t^{-1} = F_t \left(v_0^{-1} + \int_0^t \frac{\kappa_u}{F_u} \mathrm{d}u \right)$. Then for $p > 0$,

$$\mathbb{E}(V_t^p) = \mathbb{E}\left(\frac{1}{Z_t^p} \right) \leq \frac{1}{(\mathbb{E}(Z_t))^p},$$

where we have used Jensen's inequality with the concave function $x \mapsto x^{-p}$. Notice for $u \leq t$,

$$F_t/F_u \sim \mathcal{LN}\left(-\int_u^t \left(\kappa_z \theta_z - \frac{1}{2}\lambda_z^2 \right) \mathrm{d}z, \int_u^t \lambda_z^2 \mathrm{d}z \right).$$

Hence we obtain

$$\mathbb{E}(Z_t) = e^{-\int_0^t \left(\kappa_z \theta_z - \lambda_z^2 \right)\mathrm{d}z} \left(v_0^{-1} + \int_0^t \kappa_u e^{-\int_0^u \left(\kappa_z \theta_z - \lambda_z^2 \right)\mathrm{d}z} \mathrm{d}u \right).$$

Thus

$$\mathbb{E}(V_t^p) \leq \frac{e^{p \int_0^t \left(\kappa_z \theta_z - \lambda_z^2 \right)\mathrm{d}z}}{\left(v_0^{-1} + \int_0^t \kappa_u e^{-\int_0^u (\kappa_z \theta_z - \lambda_z^2)\mathrm{d}z} \mathrm{d}u \right)^p}.$$

Notice

$$\frac{e^{\int_0^t \left(\kappa_z \theta_z - \lambda_z^2 \right)\mathrm{d}z}}{v_0^{-1} + \int_0^t \kappa_u e^{-\int_0^u (\kappa_z \theta_z - \lambda_z^2)\mathrm{d}z} \mathrm{d}u} \leq \frac{e^{\int_0^t \left(\kappa_z \theta_z - \lambda_z^2 \right)\mathrm{d}z}}{v_0^{-1} + \underline{\kappa}_t e^{-\int_0^t |\kappa_z \theta_z - \lambda_z^2|\mathrm{d}z} t} \sim v_0 e^{\int_0^t \left(\kappa_z \theta_z - \lambda_z^2 \right)\mathrm{d}z}.$$

Thus, we obtain the bound

$$\mathbb{E}(V_t^p) \leq \frac{e^{p \int_0^t \left(\kappa_z \theta_z - \lambda_z^2 \right)\mathrm{d}z}}{\left(v_0^{-1} + \underline{\kappa}_t e^{-\int_0^t |\kappa_z \theta_z - \lambda_z^2|\mathrm{d}z} t \right)^p} \sim v_0^p e^{p \int_0^t \left(\kappa_z \theta_z - \lambda_z^2 \right)\mathrm{d}z}.$$

This bound on the p-th moment of the Verhulst process is asymptotically exponentially increasing in t. Clearly, this is not an ideal bound.

To summarise the properties of the Verhulst process:

1. An explicit strong solution exists, which is given in Proposition 5.4.5.

2. The stationary distribution is known explicitly and is Gamma.

3. The explicit distribution is not known for any fixed time t.

4. An explicit expression for moments of any order do not exist. However, if we define $M_n(t) := \mathbb{E}(X_t^n)$, then an ODE in t for M_n can be derived which depends on M_{n+1}. Thus a backwards recurrence exists.

In contrast, if one considers the Heston model, we notice that the variance process is modelled by a CIR process. The CIR process has been researched extensively and there exists a plethora of results in the literature, see for example Dufresne [26], Göing-Jaeschke et al. [34], Maghsoodi [49] or essentially any book on stochastic calculus/financial modelling. This is mainly due to the fact that the CIR process is an affine process. These results were largely exploited in Benhamou et al. [9] for the expansion methodology on the time-dependent Heston model, and thus a theoretical error analysis was achievable. For example, the use of the explicit form of the Laplace transform of the integrated CIR process is exploited in their error analysis, which fortunately is known explicitly in the literature, see again Dufresne [26]. In addition, there exists a well known correspondence between squared Bessel processes and the CIR process, which can be further exploited. But in the case of the Verhulst model (or any stochastic volatility model with an unexplored process for the volatility/variance), such a mathematical analysis on the error term is difficult, if not infeasible. We leave this analysis for future work. We refer the reader to Chapter 6, where we include a robust numerical error and sensitivity analysis for our approximation procedure.

Chapter 6

Malliavin calculus methodology: numerical implementation

In this chapter, we develop a fast calibration scheme for our second-order approximation formula from Chapter 5 in the stochastic Verhulst model. Furthermore, we present a numerical error and sensitivity analysis for the approximation formula in the stochastic Verhulst model.

1. Section 6.1 details our fast calibration scheme, where we exploit a convenient recursive property of our integral operators when parameters are assumed to be piecewise-constant.

2. Section 6.2 is dedicated to a numerical sensitivity analysis of the second order approximation of the put option price in the stochastic Verhulst model given by Lemma 5.4.4.

6.1 Fast calibration

Recall from Section 5.4.8 the integral operator

$$\omega_{t,T}^{(k,l)} := \int_t^T l_u e^{\int_0^u k_z \mathrm{d}z} \mathrm{d}u, \tag{6.1}$$

and its n-fold extension

$$\omega_{t,T}^{(k^{(n)},l^{(n)}),(k^{(n-1)},l^{(n-1)}),\dots,(k^{(1)},l^{(1)})} := \omega_{t,T}^{\left(k^{(n)},l^{(n)}w_{\cdot,T}^{(k^{(n-1)},l^{(n-1)}),\cdots,(k^{(1)},l^{(1)})}\right)}, \quad n \in \mathbb{N}. \tag{6.2}$$

Let $\mathfrak{T} = \{0 = T_0, T_1, \dots, T_{N-1}, T_N = T\}$ be a collection of maturity dates on $[0, T]$, with $\Delta T_i := T_{i+1} - T_i$ and $\Delta T_0 \equiv 1$. When the dummy functions are piecewise-constant, that is, $l_t^{(n)} = l_i^{(n)}$ on $t \in [T_i, T_{i+1})$ and similarly for $k^{(n)}$, we can recursively calculate the integral operators eq. (6.1) and eq. (6.2). Furthermore, let $\tilde{\mathfrak{T}} = \{0, \tilde{T}_1, \dots, \tilde{T}_{\tilde{N}-1}, T\}$ such that $\tilde{\mathfrak{T}} \supseteq \mathfrak{T}$. Let $\Delta \tilde{T}_i := \tilde{T}_{i+1} - \tilde{T}_i$ with $\Delta \tilde{T}_0 \equiv 1$. Then, consider the ODE for $(v_{0,t})$ in the Verhulst model eq. (5.23).

$$\mathrm{d}v_{0,t} = \kappa_t(\theta_t - v_{0,t})v_{0,t}\mathrm{d}t. \tag{6.3}$$

We have the following Euler approximation to the ODE eq. (6.3):

$$v_{0,t} \approx v_{0,\tilde{T}_i} + \kappa_i(\theta_i - v_{0,\tilde{T}_i})v_{0,\tilde{T}_i}\Delta\tilde{T}_i\tilde{\gamma}_i(t), \quad t \in [\tilde{T}_i, \tilde{T}_{i+1}),$$

where $\tilde{\gamma}_i(t) := (t - \tilde{T}_i)/\Delta\tilde{T}_i$. It is true that an explicit solution exists for the ODE eq. (6.3). However, the solution is non-linear, which is problematic as we will need to utilise the linearity of the Euler approximation for our calibration scheme. Define

$$
\begin{aligned}
e_t^{(k^{(n)},\dots,k^{(1)})} &:= e^{\int_0^t \sum_{j=1}^n k_z^{(j)} \mathrm{d}z}, \\
e_{v,t}^{(h^{(n)},\dots,h^{(1)})} &:= e^{\int_0^t v_{0,z} \sum_{j=1}^n h_z^{(j)} \mathrm{d}z}, \\
\varphi_{t,T_{i+1}}^{(k,h,p)} &:= \int_t^{T_{i+1}} \gamma_i^p(u) e^{\int_{T_i}^u k_z + h_z v_{0,z} \mathrm{d}z} \mathrm{d}u,
\end{aligned}
$$

where $\gamma_i(u) := (u - T_i)/\Delta T_i$ and $p \in \mathbb{N} \cup \{0\}$. In addition, define recursively

$$
\varphi_{t,T_{i+1}}^{(k^{(n)},h^{(n)},p_n),\dots,(k^{(1)},h^{(1)},p_1)} := \int_t^{T_{i+1}} \gamma_i^{p_n}(u) e^{\int_{T_i}^u k_z^{(n)} + h_z^{(n)} v_{0,z} \mathrm{d}z} \varphi_{u,T_{i+1}}^{(k^{(n-1)},h^{(n-1)},p_{n-1}),\dots,(k^{(2)},h^{(2)},p_2),(k^{(1)},h^{(1)},p_1)} \mathrm{d}u,
$$

where $p_n \in \mathbb{N} \cup \{0\}$.[1] We now assume that the dummy functions are piecewise-constant on $\mathcal{T}$. However, to make this recursion simpler, we will assume that we are working on the finer grid $\tilde{\mathcal{T}}$ rather than $\mathcal{T}$, since if the dummy functions are piecewise-constant on $\mathcal{T}$, then there exists an equivalent parameterisation on $\tilde{\mathcal{T}}$. For example, let k_i be the constant value of k on $[T_i, T_{i+1})$. Then there exists $\tilde{T}_{\tilde{i}}, \tilde{T}_{\tilde{i}+1}, \dots, \tilde{T}_{\tilde{j}}$ such that $\tilde{T}_{\tilde{i}} = T_i$ and $\tilde{T}_{\tilde{j}} = T_{i+1}$. Then let $\tilde{k}_m := k_i$ for $m = \tilde{i}, \dots, \tilde{j}$. Thus, without loss of generality, we can assume that we are working on $\tilde{\mathcal{T}}$ and we will suppress the tilde from now on.

With the assumption that the dummy functions are piecewise-constant on $\mathcal{T}$, we can obtain

---

[1] For example

$$
\varphi_{t,T_{i+1}}^{(k^{(3)},p_3,q_3),(k^{(2)},p_2,q_2),(k^{(1)},p_1,q_1)}
$$

$$
= \int_t^{T_{i+1}} \gamma_i^{p_3}(u_3) v_{0,u_3}^{q_3} e^{\int_{T_i}^{u_3} k_z^{(3)} \mathrm{d}z} \left(\int_{u_3}^{T_{i+1}} \gamma_i^{p_2}(u_2) v_{0,u_2}^{q_2} e^{\int_{T_i}^{u_2} k_z^{(2)} \mathrm{d}z} \left(\int_{u_2}^{T_{i+1}} \gamma_i^{p_1}(u_1) v_{0,u_1}^{q_1} e^{\int_{T_i}^{u_1} k_z^{(1)} \mathrm{d}z} \mathrm{d}u_1 \right) \mathrm{d}u_2 \right) \mathrm{d}u_3.
$$

the integral operator at time T_{i+1} expressed by terms at time T_i.

$$\omega_{0,T_{i+1}}^{(k^{(1)}+h^{(1)}v_{0,\cdot},l^{(1)}v_{0,\cdot}^{q_1})}$$
$$= \omega_{0,T_i}^{(k^{(1)}+h^{(1)}v_{0,\cdot},l^{(1)}v_{0,\cdot}^{q_1})} + l_i^{(1)}v_{0,T_i}^{q_1}e_{T_i}^{(k^{(1)})}e_{v,T_i}^{(h^{(1)})}\varphi_{T_i,T_{i+1}}^{(k^{(1)},h^{(1)},0)},$$

$$\omega_{0,T_{i+1}}^{(k^{(2)}+h^{(2)}v_{0,\cdot},l^{(2)}v_{0,\cdot}^{q_2}),(k^{(1)}+h^{(1)}v_{0,\cdot},l^{(1)}v_{0,\cdot}^{q_1})}$$
$$= \omega_{0,T_i}^{(k^{(2)}+h^{(2)}v_{0,\cdot},l^{(2)}v_{0,\cdot}^{q_2}),(k^{(1)}+h^{(1)}v_{0,\cdot},l^{(1)}v_{0,\cdot}^{q_1})}$$
$$+ l_i^{(1)}v_{0,T_i}^{q_1}e_{T_i}^{(k^{(1)})}e_{v,T_i}^{(h^{(1)})}\varphi_{T_i,T_{i+1}}^{(k^{(1)},h^{(1)},0)}\omega_{0,T_i}^{(k^{(2)}+h^{(2)}v_{0,\cdot},l^{(2)}v_{0,\cdot}^{q_2})}$$
$$+ l_i^{(2)}l_i^{(1)}v_{0,T_i}^{q_2+q_1}e_{T_i}^{(k^{(2)},k^{(1)})}e_{v,T_i}^{(h^{(2)},h^{(1)})}\varphi_{T_i,T_{i+1}}^{(k^{(2)},h^{(2)},0),(k^{(1)},h^{(1)},0)},$$

$$\omega_{0,T_{i+1}}^{(k^{(3)}+h^{(3)}v_{0,\cdot},l^{(3)}v_{0,\cdot}^{q_3}),\ldots,(k^{(1)}+h^{(1)}v_{0,\cdot},l^{(1)}v_{0,\cdot}^{q_1})}$$
$$= \omega_{0,T_i}^{(k^{(3)}+h^{(3)}v_{0,\cdot},l^{(3)}v_{0,\cdot}^{q_3}),\ldots,(k^{(1)}+h^{(1)}v_{0,\cdot},l^{(1)}v_{0,\cdot}^{q_1})}$$
$$+ l_i^{(1)}v_{0,T_i}^{q_1}e_{T_i}^{(k^{(1)})}e_{v,T_i}^{(h^{(1)})}\varphi_{T_i,T_{i+1}}^{(k^{(1)},h^{(1)},0)}\omega_{0,T_i}^{(k^{(3)}+h^{(3)}v_{0,\cdot},l^{(3)}v_{0,\cdot}^{q_3}),(k^{(2)}+h^{(2)}v_{0,\cdot},l^{(2)}v_{0,\cdot}^{q_2})}$$
$$+ l_i^{(2)}l_i^{(1)}v_{0,T_i}^{q_2+q_1}e_{T_i}^{(k^{(2)},k^{(1)})}e_{v,T_i}^{(h^{(2)},h^{(1)})}\varphi_{T_i,T_{i+1}}^{(k^{(2)},h^{(2)},0),(k^{(1)},h^{(1)},0)}\omega_{0,T_i}^{(k^{(3)}+h^{(3)}v_{0,\cdot},l^{(3)}v_{0,\cdot}^{q_3})}$$
$$+ l_i^{(3)}l_i^{(2)}l_i^{(1)}v_{0,T_i}^{q_3+q_2+q_1}e_{T_i}^{(k^{(3)},k^{(2)},k^{(1)})}e_{v,T_i}^{(h^{(3)},h^{(2)},h^{(1)})}\varphi_{T_i,T_{i+1}}^{(k^{(3)},h^{(3)},0),(k^{(2)},h^{(2)},0),(k^{(1)},h^{(1)},0)},$$

$$\omega_{0,T_{i+1}}^{(k^{(4)}+h^{(4)}v_{0,\cdot},l^{(4)}v_{0,\cdot}^{q_4}),\ldots,(k^{(1)}+h^{(1)}v_{0,\cdot},l^{(1)}v_{0,\cdot}^{q_1})}$$
$$= \omega_{0,T_i}^{(k^{(4)}+h^{(4)}v_{0,\cdot},l^{(4)}v_{0,\cdot}^{q_4}),\ldots,(k^{(1)}+h^{(1)}v_{0,\cdot},l^{(1)}v_{0,\cdot}^{q_1})}$$
$$+ l_i^{(1)}v_{0,T_i}^{q_1}e_{T_i}^{(k^{(1)})}e_{v,T_i}^{(h^{(1)})}\varphi_{T_i,T_{i+1}}^{(k^{(1)},h^{(1)},0)}\omega_{0,T_i}^{(k^{(4)}+h^{(4)}v_{0,\cdot},l^{(4)}v_{0,\cdot}^{q_4}),(k^{(3)}+h^{(3)}v_{0,\cdot},l^{(3)}v_{0,\cdot}^{q_3}),(k^{(2)}+h^{(2)}v_{0,\cdot},l^{(2)}v_{0,\cdot}^{q_2})}$$
$$+ l_i^{(2)}l_i^{(1)}v_{0,T_i}^{q_2+q_1}e_{T_i}^{(k^{(2)},k^{(1)})}e_{v,T_i}^{(h^{(2)},h^{(1)})}\varphi_{T_i,T_{i+1}}^{(k^{(2)},h^{(2)},0),(k^{(1)},h^{(1)},0)}\omega_{0,T_i}^{(k^{(4)}+h^{(4)}v_{0,\cdot},l^{(4)}v_{0,\cdot}^{q_4}),(k^{(3)}+h^{(3)}v_{0,\cdot},l^{(3)}v_{0,\cdot}^{q_3})}$$
$$+ l_i^{(3)}l_i^{(2)}l_i^{(1)}v_{0,T_i}^{q_3+q_2+q_1}e_{T_i}^{(k^{(3)},k^{(2)},k^{(1)})}e_{v,T_i}^{(h^{(3)},h^{(2)},h^{(1)})}\varphi_{T_i,T_{i+1}}^{(k^{(3)},h^{(3)},0),(k^{(2)},h^{(2)},0),(k^{(1)},h^{(1)},0)}\omega_{0,T_i}^{(k^{(4)}+h^{(4)}v_{0,\cdot},l^{(4)}v_{0,\cdot}^{q_4})}$$
$$+ l_i^{(4)}l_i^{(3)}l_i^{(2)}l_i^{(1)}v_{0,T_i}^{q_4+q_3+q_2+q_1}e_{T_i}^{(k^{(4)},k^{(3)},k^{(2)},k^{(1)})}e_{v,T_i}^{(h^{(4)},h^{(3)},h^{(2)},h^{(1)})}\varphi_{T_i,T_{i+1}}^{(k^{(4)},h^{(4)},0),(k^{(3)},h^{(3)},0),(k^{(2)},h^{(2)},0),(k^{(1)},h^{(1)},0)}.$$

[2]The only terms here that are not explicit are the functions $e_\cdot^{(\cdot,\ldots,\cdot)}$, $e_{v,\cdot}^{(\cdot,\ldots,\cdot)}$ and $\varphi_{t,T_{i+1}}^{(\cdot,\cdot,\cdot),\ldots,(\cdot,\cdot,\cdot)}$. For $t \in (T_i, T_{i+1}]$, we can derive the following:

$$e_t^{(k^{(n)},\ldots,k^{(1)})} = e_{T_i}^{(k^{(n)},\ldots,k^{(1)})}e^{\Delta T_i\gamma_i(t)\sum_{j=1}^n k_i^{(j)}} = e^{\sum_{m=0}^{i-1}\Delta T_m\sum_{j=1}^n k_m^{(j)}}e^{\Delta T_i\gamma_i(t)\sum_{j=1}^n k_i^{(j)}},$$

$$e_{v,t}^{(h^{(n)},\ldots,h^{(1)})} = e_{v,T_i}^{(h^{(n)},\ldots,h^{(1)})}e^{\Delta T_i\gamma_i(t)v_{0,T_i}\sum_{j=1}^n h_i^{(j)}} = e^{\sum_{m=0}^{i-1}\Delta T_m v_{0,T_m}\sum_{j=1}^n h_m^{(j)}}e^{\Delta T_i\gamma_i(t)v_{0,T_i}\sum_{j=1}^n h_i^{(j)}},$$

where $e_0^{(k^{(n)},\ldots,k^{(1)})} = 1$ and $e_{v,0}^{(h^{(n)},\ldots,h^{(1)})} = 1$. Let $\tilde{k}_i := k_i + h_iv_{0,T_i}$ and $\tilde{k}_i^{(n)} := k_i^{(n)} + h_i^{(n)}v_{0,T_i}$. Then

$$\varphi_{t,T_{i+1}}^{(k,h,p)} = \begin{cases} \frac{1}{\tilde{k}_i}\left(e^{\tilde{k}_i\Delta T_i} - \gamma_i^p(t)e^{\tilde{k}_i\Delta T_i\gamma_i(t)} - \frac{p}{\Delta T_i}\varphi_{t,T_{i+1}}^{(k,h,p-1)}\right), & \tilde{k}_i \neq 0, p \geq 1, \\ \frac{1}{\tilde{k}_i}\left(e^{\tilde{k}_i\Delta T_i} - e^{\tilde{k}_i\Delta T_i\gamma_i(t)}\right), & \tilde{k}_i \neq 0, p = 0, \\ \frac{1}{p+1}\Delta T_i\left(1 - \gamma_i^{p+1}(t)\right), & \tilde{k}_i = 0, p \geq 0. \end{cases}$$

---

[2]In general

$$\omega_{0,T_{i+1}}^{(k^{(n)}+h^{(n)}v_{0,\cdot},l^{(n)}v_{0,\cdot}^{q_n}),\ldots,(k^{(1)}+h^{(1)}v_{0,\cdot},l^{(1)}v_{0,\cdot}^{q_1})} = \sum_{m=1}^{n+1}\omega_{0,T_i}^{(k^{(n)}+h^{(n)}v_{0,\cdot},l^{(n)}v_{0,\cdot}^{q_n}),\ldots,(k^{(m)}+h^{(m)}v_{0,\cdot},l^{(m)}v_{0,\cdot}^{q_m})}$$

$$\cdot\left(\prod_{j=0}^{m-2}l_i^{(j+1)}\right)v_{0,T_i}^{\sum_{j=1}^{m-1}q_j}e_{T_i}^{(k^{(n-m+2)},\ldots,k^{(1)})}e_{v,T_i}^{(h^{(n-m+2)},\ldots,h^{(1)})}\varphi_{T_i,T_{i+1}}^{(k^{(n-m+2)},h^{(n-m+2)},0),\ldots,(k^{(1)},h^{(1)},0)},$$

where whenever the index goes outside of $\{1,\ldots,n\}$, then that term is equal to 1.

In addition, for $n \geq 2$,

$$
\varphi_{t,T_{i+1}}^{(k^{(n)},h^{(n)},p_n),\ldots,(k^{(1)},h^{(1)},p_1)} =
$$

$$
\begin{cases}
\begin{aligned}
\frac{1}{\tilde{k}_i^{(n)}} \Big(& \varphi_{t,T_{i+1}}^{(k^{(n)}+k^{(n-1)},h^{(n)}+h^{(n-1)},p_n+p_{n-1}),(k^{(n-2)},h^{(n-2)},p_{n-2}),\ldots,(k^{(1)},h^{(1)},p_1)} \\
& - \frac{p_n}{\Delta T_i} \varphi_{t,T_{i+1}}^{(k^{(n)},h^{(n)},p_n-1),(k^{(n-1)},h^{(n-1)},p_{n-1}),\ldots,(k^{(1)},h^{(1)},p_1)} \\
& - \gamma_i^{p_n}(t) e^{\tilde{k}_i^{(n)}\Delta T_i \gamma_i(t)} \varphi_{t,T_{i+1}}^{(k^{(n-1)},h^{(n-1)},p_{n-1}),\ldots,(k^{(1)},h^{(1)},p_1)} \Big),
\end{aligned} & \tilde{k}_i^{(n)} \neq 0, p_n \geq 1, \\[2em]
\begin{aligned}
\frac{1}{\tilde{k}_i^{(n)}} \Big(& \varphi_{t,T_{i+1}}^{(k^{(n)}+k^{(n-1)},h^{(n)}+h^{(n-1)},p_{n-1}),(k^{(n-2)},h^{(n-2)},p_{n-2}),\ldots,(k^{(1)},h^{(1)},p_1)} \\
& - e^{\tilde{k}_i^{(n)}\Delta T_i \gamma_i(t)} \varphi_{t,T_{i+1}}^{(k^{(n-1)},h^{(n-1)},p_{n-1}),\ldots,(k^{(1)},h^{(1)},p_1)} \Big),
\end{aligned} & \tilde{k}_i^{(n)}, \neq 0, p_n = 0, \\[2em]
\begin{aligned}
\frac{\Delta T_i}{p_n+1} \Big(& \varphi_{t,T_{i+1}}^{(k^{(n-1)},h^{(n-1)},p_n+p_{n-1}+1),(k^{(n-2)},h^{(n-2)},p_{n-2}),\ldots,(k^{(1)},h^{(1)},p_1)} \\
& - \gamma_i^{p_n+1}(t) \varphi_{t,T_{i+1}}^{(k^{(n-1)},h^{(n-1)},p_{n-1}),\ldots,(k^{(1)},h^{(1)},p_1)} \Big),
\end{aligned} & \tilde{k}_i^{(n)} = 0, p_n \geq 0.
\end{cases}
$$

To implement our fast calibration scheme, one executes the following algorithm. Let $\mu_t \equiv \mu = (\mu^{(1)}, \mu^{(2)}, \ldots, \mu^{(n)})$ be an arbitrary set of parameters and denote by ω_t an arbitrary integral operator.

- Calibrate μ over $[0, T_1)$ to obtain μ_0. This involves computing ω_{T_1}.

- Calibrate μ over $[T_1, T_2)$ to obtain μ_1. This involves computing ω_{T_2} which is in terms of ω_{T_1}, the latter already being computed in the previous step.

- Repeat until time T_N.

Remark 6.1.1. In the general model eq. (5.1), it is clear that as long as a unique solution for $(v_{0,t})$ exists, then the same procedure outlined above will result in a fast calibration scheme. This is due to the fact that the Euler approximation to the ODE for $(v_{0,t})$ gives a linear equation, which is all that is necessary to obtain the fast calibration scheme. Thus, this fast calibration scheme can be adapted to the general model eq. (5.1) immediately. This extends the fast calibration scheme presented in Langrené et al. [44], where they require the ODE for $(v_{0,t})$ to be linear.

6.2 Numerical tests and sensitivity analysis

We test our approximation method by considering the sensitivity of the approximation formula with respect to one parameter at a time. Specifically, for an arbitrary parameter set $(\mu^{(1)}, \mu^{(2)}, \ldots, \mu^{(n)})$, we vary only one of the $\mu^{(i)}$ at a time and keep the rest fixed. Then, we compute implied volatilities via our approximation method as well as the Monte Carlo for strikes corresponding to Put 10, 25 and ATM deltas. Specifically,

$$
\text{Error}(\mu) = \sigma_{IM-Approx}(\mu, K) - \sigma_{IM-Monte}(\mu, K)
$$

for K corresponding to Put 10, Put 25 and ATM.

For all our simulations, we use 2,000,000 Monte Carlo paths, and 24 time steps per day. This is to reduce the Monte Carlo and discretisation errors sufficiently well.

The parameter set we start from is $(S_0, v_0, r_d, r_f) = (100.00, 0.18, 0.02, 0)$ with

$$(\kappa, \theta, \lambda, \rho) = \begin{cases} (8.00, 0.15, 0.92, -0.63), & T = 1\text{M}, \\ (8.00, 0.15, 0.92, -0.63), & T = 3\text{M}, \\ (8.00, 0.15, 0.92, -0.63), & T = 6\text{M}, \\ (8.00, 0.15, 0.92, -0.63), & T = 1\text{Y}. \end{cases}$$

In our analysis, we vary one of the $(\kappa, \theta, \lambda, \rho)$ and keep the rest fixed.

Varying κ

We vary κ over $\{6, 7, 8, 9, 10, 11, 12, 13\}$.

Table 6.1: κ: Error for ATM implied volatilities in basis points

κ	6	7	8	9	10	11	12	13
1M	7.99	7.83	7.66	7.50	7.34	7.18	7.03	6.87
3M	16.48	15.65	14.85	14.06	13.30	12.56	11.85	11.16
6M	30.10	27.35	24.84	22.55	20.47	18.57	16.83	15.24
1Y	37.23	31.34	26.52	22.57	19.29	16.56	14.25	12.30

Table 6.2: κ: Error for Put 25 implied volatilities in basis points

κ	6	7	8	9	10	11	12	13
1M	7.03	6.89	6.75	6.60	6.46	6.33	6.19	6.05
3M	20.33	19.37	18.45	17.57	16.72	15.90	15.12	14.37
6M	33.47	30.58	27.90	25.44	23.16	21.07	19.15	17.38
1Y	41.45	34.15	28.06	22.97	18.69	15.08	12.01	9.40

Table 6.3: κ: Error for Put 10 implied volatilities in basis points

κ	6	7	8	9	10	11	12	13
1M	-0.03	-0.07	-0.12	-0.16	-0.21	-0.26	-0.31	-0.37
3M	9.16	8.87	8.55	8.21	7.86	7.50	7.13	6.75
6M	27.16	25.28	23.44	21.66	19.96	18.35	16.81	15.37
1Y	53.69	46.54	40.34	34.96	30.28	26.21	22.64	19.52

Varying θ

We vary θ over the set $\{0.10, 0.13, 0.16, 0.19, 0.22, 0.25, 0.28, 0.31\}$.

Table 6.4: θ: Error for ATM implied volatilities in basis points

θ	0.10	0.13	0.16	0.19	0.22	0.25	0.28	0.31
1M	5.98	6.62	7.26	7.90	8.54	9.19	9.84	10.49
3M	14.23	16.32	18.44	20.58	22.73	24.89	27.06	29.23
6M	18.36	22.26	26.18	30.07	33.91	37.69	41.38	44.96
1Y	14.99	20.16	25.10	29.81	34.33	38.71	43.00	47.26

Table 6.5: θ: Error for Put 25 implied volatilities in basis points

θ	0.10	0.13	0.16	0.19	0.22	0.25	0.28	0.31
1M	4.85	5.52	6.18	6.86	7.53	8.21	8.89	9.57
3M	12.55	14.84	17.15	19.48	21.82	24.17	26.52	28.87
6M	20.41	24.69	28.91	33.04	37.07	40.96	44.69	48.26
1Y	22.03	26.91	31.16	34.86	38.14	41.14	44.04	46.96

Table 6.6: θ: Error for Put 10 implied volatilities in basis points

θ	0.10	0.13	0.16	0.19	0.22	0.25	0.28	0.31
1M	-0.62	0.12	0.85	1.59	2.34	3.09	3.84	4.59
3M	2.65	5.30	7.97	10.64	13.32	16.00	18.68	21.36
6M	17.42	22.63	27.63	32.43	37.00	41.33	45.41	49.24
1Y	31.98	37.57	41.45	44.00	45.63	46.72	47.60	48.57

Varying λ

We vary λ over the set $\{0.5, 0.6, 0.7, 0.8, 0.9, 1.0, 1.1, 1.2\}$.

Table 6.7: λ: Error for ATM implied volatilities in basis points

λ	0.5	0.6	0.7	0.8	0.9	1.0	1.1	1.2
1M	-1.60	-0.68	0.43	1.72	3.19	4.84	6.68	8.71
3M	3.65	6.07	8.94	12.26	16.05	20.30	25.03	30.24
6M	5.59	9.23	13.51	18.41	23.93	30.05	36.76	44.04
1Y	5.31	9.17	13.46	18.06	22.87	27.79	32.71	37.53

Table 6.8: λ: Error for Put 25 implied volatilities in basis points

λ	0.5	0.6	0.7	0.8	0.9	1.0	1.1	1.2
1M	0.96	1.69	2.59	3.65	4.90	6.33	7.96	9.78
3M	4.06	6.30	9.05	12.37	16.27	20.80	25.97	31.81
6M	5.35	9.10	13.74	19.31	25.84	33.36	41.89	51.46
1Y	6.53	11.20	16.75	23.15	30.36	38.33	47.00	56.31

Table 6.9: λ: Error for Put 10 implied volatilities in basis points

λ	0.5	0.6	0.7	0.8	0.9	1.0	1.1	1.2
1M	-1.22	-1.25	-1.17	-0.99	-0.65	-0.16	0.52	1.41
3M	-1.44	-0.37	1.28	3.61	6.69	10.60	15.39	21.12
6M	-0.36	2.79	7.24	13.12	20.53	29.56	40.25	52.65
1Y	1.83	7.40	14.85	24.26	35.66	49.06	64.45	81.77

Varying ρ

We vary ρ over the set $\{-0.7, -0.6, -0.5, -0.4, -0.3, -0.2, -0.1, 0\}$.

Table 6.10: ρ: Error for ATM implied volatilities in basis points

ρ	-0.7	-0.6	-0.5	-0.4	-0.3	-0.2	-0.1	0
1M	7.93	5.67	3.75	2.18	0.97	0.10	-0.41	-0.56
3M	19.34	13.81	9.17	5.41	2.54	0.54	-0.59	-0.83
6M	30.76	22.13	14.87	8.97	4.40	1.17	-0.73	-1.30
1Y	29.67	19.23	10.51	3.49	-1.84	-5.49	-7.47	-7.77

Table 6.11: ρ: Error for Put 25 implied volatilities in basis points

ρ	-0.7	-0.6	-0.5	-0.4	-0.3	-0.2	-0.1	0
1M	6.79	5.27	3.92	2.73	1.71	0.87	0.21	-0.26
3M	17.74	14.28	11.09	8.18	5.59	3.34	1.48	0.04
6M	31.23	25.86	20.76	15.97	11.55	7.54	4.02	1.07
1Y	35.19	28.44	21.91	15.68	9.81	4.39	-0.49	-4.72

Table 6.12: ρ: Error for Put 10 implied volatilities in basis points

ρ	-0.7	-0.6	-0.5	-0.4	-0.3	-0.2	-0.1	0
1M	-0.36	0.38	0.90	1.23	1.34	1.24	0.92	0.38
3M	5.57	7.09	8.05	8.43	8.22	7.38	5.88	3.65
6M	21.95	23.90	24.78	24.58	23.27	20.77	17.01	11.85
1Y	37.53	40.42	41.55	40.86	38.30	33.72	26.97	17.75

The numerical analysis displays errors that are on average around 10-50bps out, with small errors being exhibited for reasonable parameter values, and large errors for more unreasonable parameter values. The errors also behave as we expect. For example, large vol-vol intuitively should result in an error which is high, since the expansion procedure was contingent on vol-vol being small. This behaviour is exhibited in the above numerical results. A high mean reversion speed should intuitively result in an error which is lower, which is also seen in the κ numerical sensitivity analysis.

Chapter 7

Conclusion

In this thesis, we explored the potential of obtaining closed-form approximations for option prices in a variety of stochastic volatility models with time-dependent parameters. This was executed through two different methodologies. Common to both methodologies is the mixing solution. This involves writing the put option price as an expectation of a functional of the integrated variance, where this functional is similar to a Black-Scholes formula.

In Chapter 3, we started from the mixing solution, and then performed a second-order Taylor expansion around the mean of the argument of the put Black-Scholes formula. Using change of measure techniques, we were able to express the closed-form approximation as a sum of terms involving moments of the variance process under some artificial probability measures. We found that it was only possible to compute these moments in a closed-form fashion under the Heston framework. This is due to the change of measure technique resulting in variance processes driven by SDEs that are not well understood in the literature. By assuming a correlation of 0 a.e., it is then possible to obtain the closed-form expression. We also bound the error induced by the expansion by higher order moments of the underlying variance process.

In Chapter 4, we devise a fast calibration scheme for the approximation formula derived in Chapter 3. To do so, we exploit the recursive properties of the integral operators in which the closed-form approximation is written in terms of. We also perform a numerical error analysis under the Heston and GARCH models (the latter with correlation 0 everywhere), and find that the numerical error is well within the range for application purposes. Furthermore, we numerically investigate the sensitivity of the approximation formula by varying parameters one at a time. We establish that the approximation formula behaves as we expect it to. For example, we observe that the approximation breaks down as maturity increases. This aligns with our intuition, as the variance of the expansion point increases with respect to this parameter.

Our framework in Chapter 5 involves a volatility process being driven by arbitrary drift and diffusion coefficients which are restricted by some regularity conditions. Starting from the mixing solution, we use a small vol-vol expansion of the underlying volatility process combined with a Taylor expansion of the put Black-Scholes formula in order to obtain an expression for the second-order approximation. Using Malliavin calculus machinery, we are able to write the put option price in terms of our integral operators. We attempt to bound the error term in the stochastic Verhulst model, which exhibits a quadratic drift. We find that such a task is currently unachievable, as the mathematical properties of the Verhulst process are not well understood in the literature.

Similar to Chapter 4, we devise a fast calibration scheme in Chapter 6 for the approximation formula derived in Chapter 5. This involves exploiting recursive properties of the integral operators that the approximation formula is written in terms of. This fast calibra-

tion scheme is also quite general as opposed to that in Langrené et al. [44], which requires linearity of the ODE governing the volatility process without its diffusion coefficient. To do so, we implement an Euler scheme to approximate this ODE. We also perform a numerical error and sensitivity analysis in the stochastic Verhulst model. We find that the approximation error is well within in the range for application purposes. Furthermore, the approximation formula behaves as we expect. For example, when vol-vol becomes large, the error in the approximation rises. This aligns with our expectations; clearly a small vol-vol expansion procedure should result in an approximation formula that performs poorer as vol-vol becomes large.

7.1   Further work

The use of the mixing solution to obtain closed-form approximations for option prices is clearly dependent on Taylor expansion techniques: as the mixing solution involves writing the price as an expectation of a Black-Scholes like formula, the most reasonable way forward is to perform Taylor expansions on this function. There are most likely a plethora of procedures one could concoct in order to make the approximation explicit. These procedures would either be model dependent, or depend on approximating the underlying volatility or variance process. In the author's opinion, we feel that the capabilities of the mixing solution methodology (in one-factor stochastic volatility models) have been pushed to its limit. Indeed, the approximation formula derived in Chapter 5 is very general, and could be used in almost any popular stochastic volatility model. Further work that would need to be done is a concise error analysis for individual models. An obvious extension to the expansion methodologies implemented in this thesis would be performing an expansion of an order higher than two, with the hopes that the closed-form attribute of the approximation is preserved. Although this is not necessarily infeasible, the calculations would most likely become increasingly complicated. Furthermore, there is no guarantee that a higher order expansion would result in a significantly better approximation. In addition, even if a higher order expansion could be made explicit, it is not clear how these extra terms would influence the implementation of the approximation.

It is plausible that the mixing solution methodology could be adapted to multifactor stochastic volatility models of dimension $d > 2$, see for example Duffie and Kan [25], Da Fonseca et al. [22]. For instance, a stochastic volatility model with vol-vol itself stochastic would be such an example. It would be quite interesting to explore the possibility of extending the Malliavin calculus methodology from Chapter 5 to a general multifactor stochastic volatility model setting.

If we consider pricing derivatives which are not European options, one can no longer rely on the mixing solution. In this case, the payoff is a functional of the underlying asset price process, and one essentially needs to consider the approximation of its expectation. This usually involves approximating the SDE (possibly multidimensional) governing the asset price process and any auxiliary processes (volatility, etc) in some sense. For example, approximating the SDE directly via perturbation techniques or the process' finite-dimensional distributions, see for example Takahashi et al. [60] and Bompis and Gobet [13]. In addition, one must be careful with the regularity of the payoff function. In such a general setup, it is reasonable to believe that one could devise a myriad of approximation techniques to obtain closed-form expressions.

Bibliography

[1] Alfonsi, A. et al. (2015). *Affine diffusions and related processes: simulation, theory and applications*, volume 6. Springer.

[2] Alòs, E. (2006). A generalization of the Hull and White formula with applications to option pricing approximation. *Finance and Stochastics*, 10(3):353–365.

[3] Andersen, L. B. (2007). Efficient simulation of the Heston stochastic volatility model. *Available at SSRN 946405*.

[4] Antonelli, F., Ramponi, A., and Scarlatti, S. (2010). Exchange option pricing under stochastic volatility: a correlation expansion. *Review of Derivatives Research*, 13(1):45–73.

[5] Antonelli, F. and Scarlatti, S. (2009). Pricing options under stochastic volatility: a power series approach. *Finance and Stochastics*, 13(2):269–303.

[6] Bachelier, L. (1900). *Théorie de la spéculation*. Gauthier-Villars.

[7] Benhamou, E., Gobet, E., and Miri, M. (2009). Smart expansion and fast calibration for jump diffusions. *Finance and stochastics*, 13(4):563–589.

[8] Benhamou, E., Gobet, E., and Miri, M. (2010a). Expansion formulas for European options in a local volatility model. *International Journal of Theoretical and Applied Finance*, 13(04):603–634.

[9] Benhamou, E., Gobet, E., and Miri, M. (2010b). Time Dependent Heston model. *SIAM Journal on Financial Mathematics*, 1(1):289–325.

[10] Black, F. and Scholes, M. (1973). The pricing of options and corporate liabilities. *The Journal of Political Economy*, pages 637–654.

[11] Bompis, R. (2013). *Stochastic expansion for the diffusion processes and applications to option pricing*. PhD thesis, Ecole Polytechnique X.

[12] Bompis, R. and Gobet, E. (2013). Asymptotic and non asymptotic approximations for option valuation. In *Recent Developments in Computational Finance: Foundations, Algorithms and Applications*, pages 159–241. World Scientific.

[13] Bompis, R. and Gobet, E. (2014). Stochastic approximation finite element method: Analytical formulas for multidimensional diffusion process. *SIAM Journal on Numerical Analysis*, 52(6):3140–3164.

[14] Bompis, R. and Gobet, E. (2018). Analytical approximations of local-Heston volatility model and error analysis. *Mathematical Finance*, 28(3):920–961.

[15] Carr, P. and Madan, D. (1999). Option valuation using the fast Fourier transform. *Journal of computational finance*, 2(4):61–73.

[16] Carr, P. and Willems, S. (2019). A lognormal type stochastic volatility model with quadratic drift. *arXiv preprint arXiv:1908.07417*.

[17] Cherny, A. S. and Engelbert, H.-J. (2005). *Singular stochastic differential equations*. Number 1858. Springer Science & Business Media.

[18] Christoffersen, P., Jacobs, K., and Mimouni, K. (2010). Volatility dynamics for the S&P500: evidence from realized volatility, daily returns, and option prices. *The Review of Financial Studies*, 23(8):3141–3189.

[19] Clark, I. J. (2011). *Foreign exchange option pricing: a practitioner's guide*. John Wiley & Sons.

[20] Cox, J. C. (1996). The constant elasticity of variance option pricing model. *Journal of Portfolio Management*, page 15.

[21] Da Fonseca, J. and Grasselli, M. (2011). Riding on the smiles. *Quantitative Finance*, 11(11):1609–1632.

[22] Da Fonseca, J., Grasselli, M., and Tebaldi, C. (2008). A multifactor volatility Heston model. *Quantitative Finance*, 8(6):591–604.

[23] Drimus, G. G. (2011). Closed-form convexity and cross-convexity adjustments for Heston prices. *Quantitative Finance*, 11(8):1137–1149.

[24] Duffie, D., Filipović, D., Schachermayer, W., et al. (2003). Affine processes and applications in finance. *The Annals of Applied Probability*, 13(3):984–1053.

[25] Duffie, D. and Kan, R. (1996). A yield-factor model of interest rates. *Mathematical finance*, 6(4):379–406.

[26] Dufresne, D. (2001). *The integrated square-root process*.

[27] Dupire, B. (1997). Pricing and hedging with smiles. *Mathematics of derivative securities*, 1(1):103–111.

[28] Dupire, B. et al. (1994). Pricing with a smile. *Risk*, 7(1):18–20.

[29] Fukasawa, M. et al. (2011). Asymptotic analysis for stochastic volatility: Edgeworth expansion. *Electronic Journal of Probability*, 16:764–791.

[30] Gander, M. P. and Stephens, D. A. (2007). Stochastic volatility modelling in continuous time with general marginal distributions: inference, prediction and model selection. *Journal of Statistical Planning and Inference*, 137(10):3068–3081.

[31] Gatheral, J. (2011). *The volatility surface: a practitioner's guide*, volume 357. John Wiley & Sons.

[32] Girsanov, I. V. (1960). On transforming a certain class of stochastic processes by absolutely continuous substitution of measures. *Theory of Probability & Its Applications*, 5(3):285–301.

[33] Gobet, E. and Suleiman, A. (2014). New approximations in local volatility models. In *Inspired by Finance*, pages 305–330. Springer.

[34] Göing-Jaeschke, A., Yor, M., et al. (2003). A survey and some generalizations of Bessel processes. *Bernoulli*, 9(2):313–349.

[35] Hagan, P. S., Kumar, D., Lesniewski, A. S., and Woodward, D. E. (2002). Managing smile risk. *The Best of Wilmott*, page 249.

[36] Harrison, J. and Pliska, S. (1981). Martingales and stochastic integrals in the theory of continuous trading. *Stochastic Processes and their Applications*, 11(3):215–260.

[37] Heston, S. L. (1993). A closed-form solution for options with stochastic volatility with applications to bond and currency options. *Review of Financial Studies*, 6(2):327–343.

[38] Hull, J. and White, A. (1987). The pricing of options on assets with stochastic volatilities. *The Journal of Finance*, 42(2):281–300.

[39] Hurd, T. R. and Kuznetsov, A. (2008). Explicit formulas for Laplace transforms of stochastic integrals. *Markov Processes and Related Fields*, 14(2):277–290.

[40] Kaeck, A. and Alexander, C. (2012). Volatility dynamics for the s&p 500: further evidence from non-affine, multi-factor jump diffusions. *Journal of Banking & Finance*, 36(11):3110–3121.

[41] Klebaner, F. (2012). *Introduction to stochastic calculus with applications*. Imperial College Press.

[42] Klebaner, F. and Liptser, R. (2014). When a stochastic exponential is a true martingale. Extension of the Beneš method. *Theory of Probability & Its Applications*, 58(1):38–62.

[43] Kloeden, P. E. and Platen, E. (2013). *Numerical solution of stochastic differential equations*, volume 23. Springer Science & Business Media.

[44] Langrené, N., Lee, G., and Zhu, Z. (2016). Switching to non-affine stochastic volatility: a closed-form expansion for the inverse gamma model. *International Journal of Theoretical and Applied Finance*, 19(05):1650031.

[45] Lewis, A. (2002). Option valuation under stochastic volatility with mathematica code. *International Review of Economics & Finance*, 11(3):331–333.

[46] Lewis, A. L. (2019). Exact solutions for a GBM-type stochastic volatility model having a stationary distribution. *Wilmott magazine*, 2019(101):20–41.

[47] Lorig, M., Pagliarani, S., and Pascucci, A. (2015). Explicit implied volatilities for multi-factor local-stochastic volatility models. *Mathematical Finance*.

[48] Mackevičius, V. (2015). Verhulst versus CIR. *Lithuanian Mathematical Journal*, 55(1):119–133.

[49] Maghsoodi, Y. (1996). Solution of the extended CIR term structure and bond option valuation. *Mathematical finance*, 6(1):89–109.

[50] Mikhailov, S. and Nögel, U. (2004). *Heston's stochastic volatility model: implementation, calibration and some extensions*. John Wiley and Sons.

[51] Nelson, D. B. (1990). ARCH models as diffusion approximations. *Journal of Econometrics*, 45(1-2):7–38.

[52] Nualart, D. (2006). *The Malliavin calculus and related topics*, volume 1995. Springer.

[53] Pagliarani, S. and Pascucci, A. (2017). The exact Taylor formula of the implied volatility. *Finance and Stochastics*, 21(3):661–718.

[54] Ribeiro, A. and Poulsen, R. (2013). Approximation behoves calibration. *Quantitative Finance Letters*, 1(1):36–40.

[55] Rogers, L. C. G. and Williams, D. (2000). *Diffusions, Markov processes and martingales: Volume 2, Itô calculus*, volume 2. Cambridge university press.

[56] Samuelson, P. (1965). Rational theory of warrant pricing. *Industrial Management Review*, 6:13–31.

[57] Sartorelli, G. (2010). Edgeworth expansion for the computation of plain vanilla prices in the Heston model. *Available at SSRN 1614703*.

[58] Schöbel, R. and Zhu, J. (1999). Stochastic volatility with an Ornstein–Uhlenbeck process: an extension. *Review of Finance*, 3(1):23–46.

[59] Shreve, S. E. (2004). *Stochastic calculus for finance II: Continuous-time models*, volume 11. Springer Science & Business Media.

[60] Takahashi, A., Yamada, T., et al. (2016). A weak approximation with asymptotic expansion and multi-dimensional Malliavin weights. *The Annals of Applied Probability*, 26(2):818–856.

[61] Tuckwell, H. C. and Koziol, J. A. (1987). Logistic population growth under random dispersal. *Bulletin of mathematical biology*, 49(4):495–506.

[62] Van der Stoep, A. W., Grzelak, L. A., and Oosterlee, C. W. (2014). The Heston stochastic-local volatility model: efficient Monte Carlo simulation. *International Journal of Theoretical and Applied Finance*, 17(07):1450045.

[63] Wiggins, J. B. (1987). Option values under stochastic volatility: theory and empirical estimates. *Journal of financial economics*, 19(2):351–372.

[64] Willard, G. (1997). Calculating prices and sensitivities for path-independent derivative securities in multifactor models. *Journal of Derivatives*, 5(1):45–61.

[65] Yamada, T., Watanabe, S., et al. (1971). On the uniqueness of solutions of stochastic differential equations. *Journal of Mathematics of Kyoto University*, 11(1):155–167.

Appendix A

Black-Scholes formula partial derivatives

This appendix contains some partial derivatives for the Black-Scholes put option formulas Put_{BS} and P_{BS}. One can think of these partial derivatives as being analogous to Black-Scholes Greeks. However, they are slightly different since our Black-Scholes formulas are parametrised with respect to integrated variance, rather than a constant volatility.

A.1 Put$_{\text{BS}}$ partial derivatives

Some of the partial derivatives of Put_{BS} are:

A.1.1 First-order Put$_{\text{BS}}$

$$\partial_x \text{Put}_{\text{BS}} = e^{-\int_0^T r_u^f \mathrm{d}u} \left(\mathcal{N}(d_+) - 1 \right),$$

$$\partial_y \text{Put}_{\text{BS}} = \frac{x e^{-\int_0^T r_u^f \mathrm{d}u} \phi(d_+)}{2\sqrt{y}}.$$

A.1.2 Second-order Put$_{\text{BS}}$

$$\partial_{xx} \text{Put}_{\text{BS}} = \frac{e^{-\int_0^T r_u^f \mathrm{d}u} \phi(d_+)}{x\sqrt{y}},$$

$$\partial_{yy} \text{Put}_{\text{BS}} = \frac{x e^{-\int_0^T r_u^f \mathrm{d}u} \phi(d_+)}{4 y^{3/2}} (d_- d_+ - 1),$$

$$\partial_{xy} \text{Put}_{\text{BS}} = (-1) \frac{e^{-\int_0^T r_u^f \mathrm{d}u} \phi(d_+) d_-}{2y}.$$

A.1.3   Third-order Put$_{\text{BS}}$

$$\partial_{xxx}\text{Put}_{\text{BS}} = (-1)\frac{e^{-\int_0^T r_u^f \mathrm{d}u}\phi(d_+)}{x^2 y}(d_+ + \sqrt{y}),$$

$$\partial_{xxy}\text{Put}_{\text{BS}} = \frac{e^{-\int_0^T r_u^f \mathrm{d}u}\phi(d_+)}{2y}(d_- d_+ - 1),$$

$$\partial_{xyy}\text{Put}_{\text{BS}} = (-1)\frac{e^{-\int_0^T r_u^f \mathrm{d}u}\phi(d_+)}{2y^2}\left(\frac{d_-^2 d_+}{2} - \frac{d_+}{2} - d_-\right),$$

$$\partial_{yyy}\text{Put}_{\text{BS}} = \frac{x e^{-\int_0^T r_u^f \mathrm{d}u}\phi(d_+)}{8y^{5/2}}\left((d_- d_+ - 1)^2 - (d_- + d_+)^2 + 2\right).$$

A.1.4   Fourth-order Put$_{\text{BS}}$

$$\partial_{xxxx}\text{Put}_{\text{BS}} = \frac{e^{-\int_0^T r_u^f \mathrm{d}u}\phi(d_+)}{x^3 y^{3/2}}(d_+^2 + 3d_+\sqrt{y} + 2y + 1),$$

$$\partial_{xxxy}\text{Put}_{\text{BS}} = \frac{e^{-\int_0^T r_u^f \mathrm{d}u}\phi(d_+)}{2x y^{3/2}}(d_-(1 - d_+^2)),$$

$$\partial_{xxyy}\text{Put}_{\text{BS}} = (-1)\frac{e^{-\int_0^T r_u^f \mathrm{d}u}\phi(d_+)}{2x y^{5/2}}\left(\frac{1}{2}(d_- + d_+)^2 + d_- d_+\left(1 - \frac{d_- d_+}{2}\right) - \frac{3}{2}\right),$$

$$\partial_{xyyy}\text{Put}_{\text{BS}} = \frac{e^{-\int_0^T r_u^f \mathrm{d}u}\phi(d_+)}{8y^3}\Big((\sqrt{y} - d_+)\left[(d_- d_+ - 1)^2 - (d_- + d_+)^2 + 2\right]$$
$$+ 4\left[d_+(d_- - d_+) - d_- - d_+\right]\Big),$$

$$\partial_{yyyy}\text{Put}_{\text{BS}} = \frac{x e^{-\int_0^T r_u^f \mathrm{d}u}\phi(d_+)}{8y^{7/2}}\bigg(\frac{1}{2}(d_- d_+ - 1)^2(d_- d_+ - 5) - (d_- d_+ - 1)(d_- + d_+)$$
$$- \frac{1}{2}(d_- + d_+)^2(d_- d_+ - 7) + (d_- d_+ - 1)\bigg).$$

A.2 P_{BS} partial derivatives

A.2.1   First-order P_{BS}

$$\partial_x P_{\text{BS}} = e^x e^{-\int_0^T r_u^f \mathrm{d}u}\left(\mathcal{N}(d_+^{\ln}) - 1\right),$$

$$\partial_y P_{\text{BS}} = \frac{e^x e^{-\int_0^T r_u^f \mathrm{d}u}\phi(d_+^{\ln})}{2\sqrt{y}}.$$

A.2.2   Second-order P_{BS}

$$\partial_{xx}P_{\text{BS}} = \frac{e^x e^{-\int_0^T r_u^f \mathrm{d}u}\phi(d_+^{\ln})}{\sqrt{y}} + \partial_x P_{\text{BS}}$$

$$= \frac{e^x e^{-\int_0^T r_u^f \mathrm{d}u}\phi(d_+^{\ln})}{\sqrt{y}} + e^x e^{-\int_0^T r_u^f \mathrm{d}u}\left(\mathcal{N}(d_+^{\ln}) - 1\right),$$

$$\partial_{xy}P_{\text{BS}} = (-1)\frac{e^x e^{-\int_0^T r_u^f \mathrm{d}u}\phi(d_+^{\ln})d_-^{\ln}}{2y},$$

$$\partial_{yy}P_{\text{BS}} = \frac{e^x e^{-\int_0^T r_u^f \mathrm{d}u}\phi(d_+^{\ln})}{4y^{3/2}}(d_-^{\ln}d_+^{\ln} - 1).$$

A.2.3   Third-order P_{BS}

$$\partial_{xxx}P_{\text{BS}} = \frac{e^x e^{-\int_0^T r_u^f \mathrm{d}u}\phi(d_+^{\ln})}{y}(\sqrt{y} - d_+^{\ln}) + \partial_{xx}P_{\text{BS}}$$

$$= \frac{e^x e^{-\int_0^T r_u^f \mathrm{d}u}\phi(d_+^{\ln})}{y}(2\sqrt{y} - d_+^{\ln}) + e^x e^{-\int_0^T r_u^f \mathrm{d}u}(\mathcal{N}(d_+^{\ln}) - 1),$$

$$\partial_{xxy}P_{\text{BS}} = (-1)\frac{e^x e^{-\int_0^T r_u^f \mathrm{d}u}\phi(d_+^{\ln})}{2y^{3/2}}\left(d_-^{\ln}\sqrt{y} + (1 - d_-^{\ln}d_+^{\ln})\right),$$

$$\partial_{xyy}P_{\text{BS}} = \frac{e^x e^{-\int_0^T r_u^f \mathrm{d}u}\phi(d_+^{\ln})}{4y^2}\left((2d_+^{\ln} - \sqrt{y}) + (1 - d_-^{\ln}d_+^{\ln})(d_+^{\ln} - \sqrt{y})\right),$$

$$\partial_{yyy}P_{\text{BS}} = \frac{e^x e^{-\int_0^T r_u^f \mathrm{d}u}\phi(d_+^{\ln})}{8y^{5/2}}\left((d_-^{\ln}d_+^{\ln} - 1)^2 - (d_-^{\ln} + d_+^{\ln})^2 + 2\right).$$

A.2.4 Fourth-order $P_{\mathbf{BS}}$

$$\partial_{xxxx}P_{\mathrm{BS}} = (-1)\frac{e^x e^{-\int_0^T r_u^f \mathrm{d}u}\phi(d_+^{\ln})}{y^{3/2}}(1 - (d_+^{\ln} - \sqrt{y})^2) + \partial_{xxx}P_{\mathrm{BS}}$$

$$= \frac{e^x e^{-\int_0^T r_u^f \mathrm{d}u}\phi(d_+^{\ln})}{y^{3/2}}\left[(d_+^{\ln} - \sqrt{y})^2 + 2y - d_+^{\ln}\sqrt{y} - 1\right] + e^x e^{-\int_0^T r_u^f \mathrm{d}u}(\mathcal{N}(d_+^{\ln}) - 1),$$

$$\partial_{xxxy}P_{\mathrm{BS}} = \frac{e^x e^{-\int_0^T r_u^f \mathrm{d}u}\phi(d_+^{\ln})}{2y^2}\left[(\sqrt{y} - d_+^{\ln})(d_-^{\ln}d_+^{\ln} - 2) + (\sqrt{y} + d_-^{\ln}) - d_-^{\ln}y - \sqrt{y}(1 - d_-^{\ln}d_+^{\ln})\right],$$

$$\partial_{xxyy}P_{\mathrm{BS}} = (-1)\frac{e^x e^{-\int_0^T r_u^f \mathrm{d}u}\phi(d_+^{\ln})}{2y^{5/2}}\left[3d_-^{\ln}d_+^{\ln} + \frac{1}{2}(d_-^{\ln})^2 d_+^{\ln}\sqrt{y} - \frac{1}{2}(d_-^{\ln})^2(d_+^{\ln})^2\right.$$

$$\left. + \frac{1}{2}y - \frac{1}{2}\sqrt{y}\left(2d_-^{\ln} + d_+^{\ln}\right) - \frac{3}{2}\right],$$

$$\partial_{xyyy}P_{\mathrm{BS}} = \frac{e^x e^{-\int_0^T r_u^f \mathrm{d}u}\phi(d_+^{\ln})}{8y^{7/2}}\left[2y^{3/2}(d_-^{\ln}d_+^{\ln} - 1)(2d_+^{\ln} - \sqrt{y}) - 4\sqrt{y}(d_-^{\ln} + d_+^{\ln})\right.$$

$$\left. + \sqrt{y}(\sqrt{y} - d_+^{\ln})\left((d_-^{\ln}d_+^{\ln} - 1)^2 - (d_-^{\ln} + d_+^{\ln})^2 + 2\right)\right],$$

$$\partial_{yyyy}P_{\mathrm{BS}} = \frac{e^x e^{-\int_0^T r_u^f \mathrm{d}u}\phi(d_+^{\ln})}{8y^{7/2}}\left(\frac{1}{2}(d_-^{\ln}d_+^{\ln} - 1)^2(d_-^{\ln}d_+^{\ln} - 5) - (d_-^{\ln}d_+^{\ln} - 1)(d_-^{\ln} + d_+^{\ln})\right.$$

$$\left. - \frac{1}{2}(d_-^{\ln} + d_+^{\ln})^2(d_-^{\ln}d_+^{\ln} - 7) + (d_-^{\ln}d_+^{\ln} - 1)\right).$$

Appendix B

Explicit moments for some processes

In this appendix, we derive expressions for some of the moments, mixed moments and covariances of the CIR and IGa processes used in this thesis. Although the results for the CIR process are well known, one could deem the calculation of such terms for the IGa process as non-trivial. In fact, according to our knowledge, we have not seen a derivation of IGa moments with time-dependent parameters in the literature. In the following, we will use notation introduced in Chapter 3.

B.1 Moments of the CIR process

Let V be a $\mathrm{CIR}(v_0; \kappa_t, \theta_t, \lambda_t)$. It satisfies the SDE

$$\mathrm{d}V_t = \kappa_t(\theta_t - V_t)\mathrm{d}t + \lambda_t\sqrt{V_t}\mathrm{d}B_t, \quad V_0 = v_0,$$

where we assume κ, θ and λ are time-dependent and deterministic and satisfy some regularity conditions. For $s < t$, it can be integrated to obtain

$$V_t = V_s e^{-\int_s^t \kappa_z \mathrm{d}z} + \int_s^t e^{-\int_u^t \kappa_z \mathrm{d}z}\kappa_u\theta_u\mathrm{d}u + \int_s^t e^{-\int_u^t \kappa_z \mathrm{d}z}\lambda_u\sqrt{V_u}\mathrm{d}B_u. \tag{B.1}$$

In particular, for $s = 0$,

$$V_t = v_0 e^{-\int_0^t \kappa_z \mathrm{d}z} + \int_0^t e^{-\int_u^t \kappa_z \mathrm{d}z}\kappa_u\theta_u\mathrm{d}u + \int_0^t e^{-\int_u^t \kappa_z \mathrm{d}z}\lambda_u\sqrt{V_u}\mathrm{d}B_u. \tag{B.2}$$

V has the following moments:

$$\mathbb{E}(V_t^n) = e^{-\int_0^t n\kappa_z \mathrm{d}z}\left(v_0^n + \int_0^t e^{\int_0^u n\kappa_z \mathrm{d}z}\left(n\kappa_u\theta_u + \frac{1}{2}n(n-1)\lambda_u^2\right)\mathbb{E}(V_u^{n-1})\mathrm{d}u\right)$$

$$\mathrm{Var}(V_t) = \int_0^t \lambda_u^2 e^{-2\int_u^t \kappa_z \mathrm{d}z}\left\{v_0 e^{-\int_0^u \kappa_z \mathrm{d}z} + \int_0^u e^{-\int_p^u \kappa_z \mathrm{d}z}\kappa_p\theta_p\mathrm{d}p\right\}\mathrm{d}u.$$

$$\mathrm{Cov}(V_s, V_t) = e^{-\int_s^t \kappa_z \mathrm{d}z}\int_0^s \lambda_u^2 e^{-2\int_u^s \kappa_z \mathrm{d}z}\left\{v_0 e^{-\int_0^u \kappa_z \mathrm{d}z} + \int_0^u e^{-\int_p^u \kappa_z \mathrm{d}z}\kappa_p\theta_p\mathrm{d}p\right\}\mathrm{d}u$$

$$\mathbb{E}(V_s^m V_t^n) = e^{-\int_0^t n\kappa_z \mathrm{d}z}\left(\mathbb{E}(V_s^{m+n}) + \int_s^t e^{\int_0^u n\kappa_z \mathrm{d}z}\left(n\kappa_u\theta_u + \frac{1}{2}n(n-1)\lambda_u^2\right)\mathbb{E}(V_s^m V_u^{n-1})\mathrm{d}u\right)$$

$$\mathrm{Cov}(V_s^m, V_t^n) = \mathbb{E}(V_s^m V_t^n) - \mathbb{E}(V_s^m)\mathbb{E}(V_t^n),$$

all for $m, n \geq 1$ and $s < t$.

We give an outline for obtaining $\mathrm{Var}(V_t)$ and $\mathrm{Cov}(V_s, V_t)$. The other terms follow a similar methodology.

Proof. Notice that $\text{Var}(V_t) = \mathbb{E}(V_t - \mathbb{E}(V_t))^2$. Then using eq. (B.2) and $\mathbb{E}(V_t)$,

$$\text{Var}(V_t) = \mathbb{E}\left(\int_0^t e^{-\int_u^t \kappa_z \mathrm{d}z} \lambda_u \sqrt{V_u} \mathrm{d}B_u\right)^2 = \int_0^t e^{-2\int_u^t \kappa_z \mathrm{d}z} \lambda_u^2 \mathbb{E}(V_u) \mathrm{d}u.$$

Assume $s < t$. Using the representation of V_t in terms of V_s eq. (B.1), we have

$$\text{Cov}(V_s, V_t) = \text{Cov}\left(V_s, V_s e^{-\int_s^t \kappa_z \mathrm{d}z} + \int_s^t e^{-\int_u^t \kappa_z \mathrm{d}z} \kappa_u \theta_u \mathrm{d}u + \int_s^t e^{-\int_u^t \kappa_z \mathrm{d}z} \lambda_u \sqrt{V_u} \mathrm{d}B_u\right)$$

$$= e^{-\int_s^t \kappa_u \mathrm{d}u} \text{Var}(V_s),$$

where we have used that V_s is independent of the Itô integral $\int_s^t e^{-\int_u^t \kappa_z \mathrm{d}z} \lambda_u \sqrt{V_u} \mathrm{d}B_u$. $\qquad\square$

B.2 Moments of the IGa process

Let V be an $\text{IGa}(v_0; \kappa_t, \theta_t, \lambda_t)$. It satisfies the SDE

$$\mathrm{d}V_t = \kappa_t(\theta_t - V_t)\mathrm{d}t + \lambda_t V_t \mathrm{d}B_t, \quad V_0 = v_0,$$

where we assume κ, θ and λ are time-dependent and deterministic and satisfy some regularity conditions. Let Y be a $\text{GBM}(1; -\kappa_t, \lambda_t)$. Then for $s < t$, V has the explicit pathwise unique strong solution

$$V_t = V_s \frac{Y_t}{Y_s}\left(\frac{v_0 + \int_0^t \kappa_u \theta_u / Y_u \mathrm{d}u}{v_0 + \int_0^s \kappa_u \theta_u / Y_u \mathrm{d}u}\right).$$

In particular, for $s = 0$,

$$V_t = Y_t\left(v_0 + \int_0^t \frac{\kappa_u \theta_u}{Y_u}\mathrm{d}u\right).$$

V has the following moments:

$$\mathbb{E}(V_t^n) = e^{\int_0^t \frac{n(n-1)}{2}\lambda_z^2 - n\kappa_z \mathrm{d}z}\left(v_0^n + n\int_0^t \kappa_u \theta_u e^{-\int_0^u \frac{n(n-1)}{2}\lambda_z^2 - n\kappa_z \mathrm{d}z}\mathbb{E}(V_u^{n-1})\mathrm{d}u\right)$$

$$\text{Var}(V_t) = e^{-2\int_0^t \kappa_z \mathrm{d}z}\int_0^t \lambda_u^2 \mathbb{E}(V_u^2)e^{2\int_0^u \kappa_z \mathrm{d}z}\mathrm{d}u$$

$$\text{Cov}(V_s, V_t) = \text{Var}(V_s)e^{-\int_s^t \kappa_z \mathrm{d}z}$$

$$\mathbb{E}(V_s^m V_t^n) = e^{\int_0^t \frac{n(n-1)}{2}\lambda_z^2 - n\kappa_z \mathrm{d}z}\left(\mathbb{E}(V_s^{m+n})e^{-\int_0^s \frac{n(n-1)}{2}\lambda_z^2 - n\kappa_z \mathrm{d}z}\right.$$

$$\left. + n\int_s^t \kappa_u \theta_u e^{-\int_0^u \frac{n(n-1)}{2}\lambda_z^2 - n\kappa_z \mathrm{d}z}\mathbb{E}(V_s^m V_u^{n-1})\mathrm{d}u\right)$$

$$\text{Cov}(V_s^m, V_t^n) = \mathbb{E}(V_s^m V_t^n) - \mathbb{E}(V_s^m)\mathbb{E}(V_t^n),$$

all for $m, n \geq 1$ and $s < t$. We show how to obtain $\mathbb{E}(V_s^n V_t^m)$. The other terms follow a similar methodology.

Proof. We consider the differential of V^n.

$$\mathrm{d}(V_t^n) = \left(n\kappa_t \theta_t V_t^{n-1} + \left(\frac{1}{2}n(n-1)\lambda_t^2 - n\kappa_t\right)V_t^n\right)\mathrm{d}t + n\lambda_t V_t^n \mathrm{d}B_t$$

$$\Rightarrow V_t^n = V_s^n + \int_s^t n\kappa_u \theta_u V_u^{n-1} + \left(\frac{1}{2}n(n-1)\lambda_u^2 - n\kappa_u\right)V_u^n \mathrm{d}u + \int_s^t n\lambda_u V_u^n \mathrm{d}B_u.$$

Multiplying both sides by V_s^m and taking expectation yields

$$\mathbb{E}(V_s^m V_t^n) = \mathbb{E}(V_s^{n+m}) + \int_s^t n\kappa_u \theta_u \mathbb{E}(V_s^m V_u^{n-1}) + \left(\frac{1}{2}n(n-1)\lambda_u^2 - n\kappa_u\right)\mathbb{E}(V_s^m V_u^n)\mathrm{d}u.$$

Differentiating both sides in t and letting $M_s^{m,n}(t) := \mathbb{E}(V_s^m V_t^n)$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}M_s^{m,n}(t) = n\kappa_t\theta_t M_s^{m,n-1}(t) + \left(\frac{1}{2}n(n-1)\lambda_t^2 - n\kappa_t\right)M_s^{m,n}(t).$$

This is a first order ODE, which can be solved with the integrating factor method by integrating from s to t. $\qquad\square$

Appendix C

Mixing solution

In this appendix, we give a derivation of the result referred to as the mixing solution by Hull and White [38]. This result is crucial for the two expansion methodologies seen in this thesis. Hull and White first established the expression for the case of independent Brownian motions W and B. Later on, this was extended by Willard for the correlated Brownian motions case [64]. We give the derivations for when the model is parameterised by either spot or log-spot. However, the derivations for either are essentially the same.

C.1 Mixing solution for the spot

Under the risk-neutral measure $\mathbb{Q}$, suppose that the spot S and variance process σ follow the dynamics

$$\mathrm{d}S_t = S_t((r_t^d - r_t^f)\mathrm{d}t + \sqrt{\sigma_t}\mathrm{d}W_t), \quad S_0,$$
$$\mathrm{d}\sigma_t = \alpha(t, \sigma_t)\mathrm{d}t + \beta(t, \sigma_t)\mathrm{d}B_t, \quad \sigma_0,$$
$$\mathrm{d}\langle W, B\rangle_t = \rho_t\mathrm{d}t.$$

We give an outline of the result

$$\mathrm{Put} = e^{-\int_0^T r_t^d \mathrm{d}t}\mathbb{E}(K - S_T)_+ = \mathbb{E}\left\{e^{-\int_0^T r_t^d \mathrm{d}t}\mathbb{E}\left[(K - S_T)_+|\mathcal{F}_T^B\right]\right\}$$
$$= \mathbb{E}\left(\mathrm{Put}_{\mathrm{BS}}\left(S_0\xi_T, \int_0^T \sigma_t(1 - \rho_t^2)\mathrm{d}t\right)\right),$$

where

$$\mathrm{Put}_{\mathrm{BS}}(x, y) = Ke^{-\int_0^T r_t^d \mathrm{d}t}\mathcal{N}(-d_-) - xe^{-\int_0^T r_t^f \mathrm{d}t}\mathcal{N}(-d_+),$$
$$d_\pm(x, y) = d_\pm = \frac{\ln(x/K) + \int_0^T (r_t^d - r_t^f)\mathrm{d}t}{\sqrt{y}} \pm \frac{1}{2}\sqrt{y}.$$

Proof. By writing the driving Brownian motion of the spot as $W_t = \int_0^t \rho_u \mathrm{d}B_u + \int_0^t \sqrt{1 - \rho_u^2}\mathrm{d}Z_u$, where Z is a Brownian motion under $\mathbb{Q}$ which is independent of B, this gives the explicit strong solution of S as

$$S_T = S_0\xi_T \exp\left\{\int_0^T (r_t^d - r_t^f)\mathrm{d}t - \frac{1}{2}\int_0^T \sigma_t(1 - \rho_t^2)\mathrm{d}t + \int_0^T \sqrt{\sigma_t(1 - \rho_t^2)}\mathrm{d}Z_t\right\},$$
$$\xi_t := \exp\left\{\int_0^t \rho_u\sqrt{\sigma_u}\mathrm{d}B_u - \frac{1}{2}\int_0^t \rho_u^2\sigma_u\mathrm{d}u\right\}.$$

First, notice that both σ and ξ are adapted to the filtration $(\mathcal{F}_t^B)_{0 \le t \le T}$. Thus, it is evident that $S_T | \mathcal{F}_T^B$ will have a log-normal distribution

$$S_T | \mathcal{F}_T^B \sim \mathcal{LN}\left(\tilde{\mu}(T), \tilde{\sigma}^2(T)\right),$$

$$\tilde{\mu}(T) := \ln(S_0 \xi_T) + \int_0^T (r_t^d - r_t^f)\mathrm{d}t - \frac{1}{2}\int_0^T \sigma_t(1 - \rho_t^2)\mathrm{d}t,$$

$$\tilde{\sigma}^2(T) := \int_0^T \sigma_t(1 - \rho_t^2)\mathrm{d}t.$$

Hence, the calculation of $e^{-\int_0^T r_t^d \mathrm{d}t}\mathbb{E}((K - S_T)_+ | \mathcal{F}_T^B)$ will result in a Black-Scholes like formula.

$$e^{-\int_0^T r_t^d \mathrm{d}t}\mathbb{E}((K - S_T)_+ | \mathcal{F}_T^B)$$

$$= Ke^{-\int_0^T r_t^d \mathrm{d}t}\mathcal{N}\left(\frac{\ln(K) - \tilde{\mu}(T)}{\tilde{\sigma}(T)}\right) - e^{-\int_0^T r_t^d \mathrm{d}t}e^{\tilde{\mu}(T) + \frac{1}{2}\tilde{\sigma}^2(T)}\mathcal{N}\left(\frac{\ln(K) - \tilde{\mu}(T) - \tilde{\sigma}^2(T)}{\tilde{\sigma}(T)}\right)$$

$$= Ke^{-\int_0^T r_t^d \mathrm{d}t}\mathcal{N}\left(\frac{\ln(K) - \tilde{\mu}(T) - \frac{1}{2}\tilde{\sigma}^2(T)}{\tilde{\sigma}(T)} + \frac{1}{2}\tilde{\sigma}(T)\right)$$

$$- S_0\xi_T e^{-\int_0^T r_t^f \mathrm{d}t}\mathcal{N}\left(\frac{\ln(K) - \tilde{\mu}(T) - \frac{1}{2}\tilde{\sigma}^2(T)}{\tilde{\sigma}(T)} - \frac{1}{2}\tilde{\sigma}(T)\right)$$

$$= Ke^{-\int_0^T r_t^d \mathrm{d}t}\mathcal{N}\left(\frac{\ln(K/S_0\xi_T) - \int_0^T (r_t^d - r_t^f)\mathrm{d}t}{\tilde{\sigma}(T)} + \frac{1}{2}\tilde{\sigma}(T)\right)$$

$$- S_0\xi_T e^{-\int_0^T r_t^f \mathrm{d}t}\mathcal{N}\left(\frac{\ln(K/S_0\xi_T) - \int_0^T (r_t^d - r_t^f)\mathrm{d}t}{\tilde{\sigma}(T)} - \frac{1}{2}\tilde{\sigma}(T)\right).$$

It is immediate that $e^{-\int_0^T r_t^d \mathrm{d}t}\mathbb{E}((K - S_T)_+ | \mathcal{F}_T^B) = \mathrm{Put}_{\mathrm{BS}}\left(S_0\xi_T, \tilde{\sigma}^2(T)\right).$ $\qquad\square$

C.2 Mixing solution for the log-spot

Under the risk-neutral measure $\mathbb{Q}$, suppose that the spot S and volatility V follow the dynamics

$$\mathrm{d}S_t = S_t((r_t^d - r_t^f)\mathrm{d}t + V_t\mathrm{d}W_t), \quad S_0,$$
$$\mathrm{d}V_t = \alpha(t, V_t)\mathrm{d}t + \beta(t, V_t)\mathrm{d}B_t, \quad V_0,$$
$$\mathrm{d}\langle W, B\rangle_t = \rho_t\mathrm{d}t.$$

Let X denote the log-spot and k the log-strike. That is, $X_t = \ln S_t$ and $k = \ln K$. We give an outline of the result

$$\mathrm{Put} = e^{-\int_0^T r_t^d \mathrm{d}t}\mathbb{E}(e^k - e^{X_T})_+ = \mathbb{E}\left\{e^{-\int_0^T r_t^d \mathrm{d}t}\mathbb{E}\left[(e^k - e^{X_T})_+ | \mathcal{F}_T^B\right]\right\}$$

$$= \mathbb{E}\left(P_{\mathrm{BS}}\left(x_0 - \int_0^T \frac{1}{2}\rho_t^2 V_t^2\mathrm{d}t + \int_0^T \rho_t V_t\mathrm{d}B_t, \int_0^T V_t^2(1 - \rho_t^2)\mathrm{d}t\right)\right),$$

where

$$P_{\mathrm{BS}}(x, y) = e^k e^{-\int_0^T r_t^d \mathrm{d}t}\mathcal{N}(-d_-^{\ln}) - e^x e^{-\int_0^T r_t^f \mathrm{d}t}\mathcal{N}(-d_+^{\ln}),$$

$$d_\pm^{\ln}(x, y) = d_\pm^{\ln} = \frac{x - k + \int_0^T (r_t^d - r_t^f)\mathrm{d}t}{\sqrt{y}} \pm \frac{1}{2}\sqrt{y}.$$

Proof. By writing the driving Brownian motion of the spot as $W_t = \int_0^t \rho_u \mathrm{d}B_u + \int_0^t \sqrt{1 - \rho_u^2} \mathrm{d}Z_u$, where Z is a Brownian motion under $\mathbb{Q}$ which is independent of B, this gives the strong solution of X as

$$X_T = x_0 + \int_0^T \left(r_t^d - r_t^f - \frac{1}{2}V_t^2 \right) \mathrm{d}t + \int_0^T \rho_t V_t \mathrm{d}B_t + \int_0^T V_t \sqrt{1 - \rho_t^2} \mathrm{d}Z_t.$$

First, notice that V is adapted to the filtration $(\mathcal{F}_t^B)_{0 \leq t \leq T}$. Thus, it is evident that $X_T | \mathcal{F}_T^B$ will have a normal distribution. Then

$$X_T | \mathcal{F}_T^B \sim \mathcal{N} \left(\hat{\mu}(T), \tilde{\sigma}^2(T) \right),$$

$$\hat{\mu}(T) := x_0 + \int_0^T \left(r_t^d - r_t^f \right) \mathrm{d}t - \frac{1}{2} \int_0^T V_t^2 \mathrm{d}t + \int_0^T \rho_t V_t \mathrm{d}B_t,$$

$$\tilde{\sigma}^2(T) := \int_0^T V_t^2 (1 - \rho_t^2) \mathrm{d}t.$$

Also, let $\tilde{\mu}(T) := \hat{\mu}(T) - \int_0^T (r_t^d - r_t^f) \mathrm{d}t$. The calculation of $e^{-\int_0^T r_t^d \mathrm{d}t} \mathbb{E}((e^k - e^{X_T})_+ | \mathcal{F}_T^B)$ will result in a Black-Scholes like formula.

$$e^{-\int_0^T r_t^d \mathrm{d}t} \mathbb{E}((e^k - e^{X_T})_+ | \mathcal{F}_T^B)$$

$$= e^k e^{-\int_0^T r_t^d \mathrm{d}t} \mathcal{N} \left(\frac{k - \hat{\mu}(T)}{\tilde{\sigma}(T)} \right) - e^{-\int_0^T r_t^d \mathrm{d}t} e^{\hat{\mu}(T) + \frac{1}{2}\tilde{\sigma}^2(T)} \mathcal{N} \left(\frac{k - \hat{\mu}(T) - \tilde{\sigma}^2(T)}{\tilde{\sigma}(T)} \right)$$

$$= e^k e^{-\int_0^T r_t^d \mathrm{d}t} \mathcal{N} \left(\frac{k - \hat{\mu}(T) - \frac{1}{2}\tilde{\sigma}^2(T)}{\tilde{\sigma}(T)} + \frac{1}{2}\tilde{\sigma}(T) \right)$$

$$- e^{\tilde{\mu}(T) + \frac{1}{2}\tilde{\sigma}^2(T)} e^{-\int_0^T r_t^f \mathrm{d}t} \mathcal{N} \left(\frac{k - \hat{\mu}(T) - \frac{1}{2}\tilde{\sigma}^2(T)}{\tilde{\sigma}(T)} - \frac{1}{2}\tilde{\sigma}(T) \right)$$

$$= e^k e^{-\int_0^T r_t^d \mathrm{d}t} \mathcal{N} \left(\frac{k - (\tilde{\mu}(T) + \frac{1}{2}\tilde{\sigma}^2(T)) - \int_0^T (r_t^d - r_t^f) \mathrm{d}t}{\tilde{\sigma}(T)} + \frac{1}{2}\tilde{\sigma}(T) \right)$$

$$- e^{\tilde{\mu}(T) + \frac{1}{2}\tilde{\sigma}^2(T)} e^{-\int_0^T r_t^f \mathrm{d}t} \mathcal{N} \left(\frac{k - (\tilde{\mu}(T) + \frac{1}{2}\tilde{\sigma}^2(T)) - \int_0^T (r_t^d - r_t^f) \mathrm{d}t}{\tilde{\sigma}(T)} - \frac{1}{2}\tilde{\sigma}(T) \right).$$

It is now immediate that $e^{-\int_0^T r_t^d \mathrm{d}t} \mathbb{E}((e^k - e^{X_T})_+ | \mathcal{F}_T^B) = P_{\mathrm{BS}} \left(\tilde{\mu}(T) + \frac{1}{2}\tilde{\sigma}^2(T), \tilde{\sigma}^2(T) \right).$ $\qquad \square$