Local Brownian motions

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To my son, Maximilian

Abstract

Brownian motion is one of the most prominent examples of stochastic processes. Its application significantly changed the way we approach modelling time-continuous dynamical systems in the areas such as physics, finance and biology today.

The main idea of this thesis is to introduce stochastic processes that behave similarly to Brownian motion on a sufficiently small neighbourhood around any time, however might not do so on the whole time interval. This distinction will allow us to construct a large class of processes, Brownian motion included, that will be called *local Brownian motions*. In particular, we will see that according to our definition of local Brownian motion, the marginal distributions do not necessarily have to be Gaussian.

We will discover that some properties of Brownian motion will be transferred to the whole class of local Brownian motions, whereas others will remain true only for Brownian motion.

Further, we will be able to construct two rich families of local Brownian motions using different approaches. The first approach will be based on randomised scaled covariances between incremental processes of local Brownian motion. The second approach will use uncorrelated joint distributions to eliminate linear dependencies between the incremental processes of a local Brownian motion via copulas.

One of the difficulties in developing the stochastic calculus with respect to local Brownian motion is the fact that a proper local Brownian motion (Brownian motion excluded) is not a semimartingale. Hence the stochastic integral can not be defined in the usual way. However, the local properties of local Brownian motion will allow us to define a stochastic integral with respect to local Brownian motion, which can be seen as a generalisation of Itô integral. We will develop the theory of stochastic calculus with respect to local Brownian motion and give explicit solutions to some stochastic differential equations such as Black-Scholes and Ornstein-Uhlenbeck SDE's driven by a local Brownian motion.

Finally, we will discuss the strengths and weaknesses of using local Brownian motion for modelling in finance and other areas of stochastic modelling.

Declaration

This thesis is an original work of my research and contains no material which has been accepted for the award of any other degree or diploma at any university or equivalent institution and that, to the best of my knowledge and belief, this thesis contains no material previously published or written by another person, except where due reference is made in the text of the thesis.

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Werd ich zum Augenblicke sagen:
Verweile doch! du bist so schön!
Dann magst du mich in Fesseln schlagen,
Dann will ich gern zugrunde gehn!

Faust I (Faust)

Johann Wolfgang von Goethe

Contents

General notations

$\mathbb{N}$	set of natural numbers
$\mathbb{N}_0$	set of natural numbers with zero
$\mathbb{R}$	set of real numbers
$\mathbb{R}_+$	set of non-negative real numbers
$\overline{\mathbb{R}}_+$	set of extended non-negative real numbers by $+\infty$
i.i.d.	independent and identically distributed
w.p.	with probability
$\top$	transpose operator
$\wedge$	minimum
$\mathbb{1}_A(\cdot)$	indicator function of a set A
$\mathcal{B}(\mathbb{R})$	Borel σ-algebra on $\mathbb{R}$
$\mathcal{F}_t^X$	σ-algebra generated by the process X up to time t
$\mathbb{F}^X$	the natural filtration generated by X
$\rho_{X,Y}$	correlation between X and Y
$\sim$	distributed as
$\stackrel{d}{=}$	equal in distribution
$\gamma_X(\cdot,\cdot)$	covariance function of the process X
$\mathcal{N}(\mu,\sigma^2)$	normal (Gaussian) distribution with mean μ and variance σ^2
$\mathcal{N}_n(\overline{\mu},\Sigma)$	multivariate normal distribution with n-dimensional mean vector $\overline{\mu}$ and $n \times n$ covariance matrix Σ
$F_X(\cdot)$	cumulative distribution function of a random variable X
$\Phi(\cdot)$	cumulative distribution function of a standard normal random variable
$\mathrm{supp}(X)$	support of a random variable X
$\mathbb{E}^{\mathbb{P}}[\cdot]$	expectation with respect to the probability measure $\mathbb{P}$
L^2	the space of square integrable random variables
$\otimes$	tensor-product
$\oplus$	direct sum
$C^2(\mathbb{R};\mathbb{R})$	space of twice continuously differentiable, real-valued functions on $\mathbb{R}$
$C^{2,1}(\mathbb{R}^2;\mathbb{R})$	space of twice continuously differentiable in the first variable and continuously differentiable in the second variable, real-valued functions on $\mathbb{R}^2$

Chapter 1

Introduction

1.1 Brief historical background on Brownian motion

Brownian motion was discovered by a botanist Robert Brown (1773-1858) in 1827 while looking through a microscope at pollen of a plant immersed in water and observing a chaotic movement.

The phenomenon was physically and mathematically described about 80 years later in 1905 by Albert Einstein (1879-1955) in [17]. It was one of the four papers published in that year that contributed substantially to the foundation of modern physics.

In 1900, Louis Jean-Baptiste Alphonse Bachelier (1870-1946) applied Brownian motion to introduce a model for the movement of the stock prices in his doctoral thesis "Théorie de la Spéculation" [5]. This was a ground breaking work and the first application of Brownian motion to model the prices of financial products. Today, he is considered as the forefather of financial mathematics and a pioneer in the study of stochastic processes and their applications.

In 1923, Norbert Wiener (1894-1964) gave Brownian motion a solid mathematical foundation in [54], after some new ideas of Félix Édouard Justin Émile Borel (1871-1956), Henri Léon Lebesgue (1875-1941) and Percy John Daniell (1889-1946) emerged. It was also extensively studied by Paul Lévy (1886-1971), who contributed largely to the theory of probability in general.

In 1944, Kiyoshi Itô (1915-2008) published a paper [28] and later a book [29] where the definitions of stochastic integral and stochastic differential equation were introduced - two fundamental tools of stochastic calculus. Numerous mathematicians expanded stochastic calculus later, which resulted in Brownian motion to become the mostly used stochastic process for stochastic modelling in continuous time.

1.2 A brief survey on stochastic modelling

Although the work of Bachelier was of fundamental value and introduced the idea of modelling the asset price process using Brownian motion, it remained relatively unknown for many years until it was discovered by Paul Anthony Samuelson (1915-2009) [49] in 1965. Bachelier suggested to model the asset price process as a Brownian motion, which left the possibility for the asset price to take negative values. This was not desired due to practical reasons. Thus Samuelson extended the ideas from Bachelier work by proposing that instead of demanding that the asset price at any time is normally distributed, the log-returns of the asset price at any time should be normally distributed. This assumption is equivalent to the statement that the returns should be log-normally distributed, which eliminated the possibility of the asset price process to take negative values.

The paper by Fischer Black (1938-1995) and Scholes in [9] built upon Samuelson's ideas by assuming a geometric Brownian motion for the asset price process. In the paper they derived a formula to price a European option. [1] However, it was the way the formula was derived that was so significant to the field of financial mathematics and laid the foundation for the asset pricing theory we know today. Almost at the same time Robert Cox Merton [39] derived the formula using a slightly different argument. Scholes suggested that the formula was required for market efficiency, Merton that it had to be true due to non-arbitrage argument, and Black that it was required for market equilibrium. The main argument that Black and Scholes used in their paper was that using hedging it was possible to eliminate the systematic risk. Merton showed that assuming trading can be done continuously, the continuous hedging, also called dynamic hedging, would completely eliminate *any* risk. More precisely, he showed that it is possible to construct a portfolio that would replicate the payoff function of an option at any time and hence offset all the risk, i.e. the sensitivities of the replicating portfolio would perfectly match the sensitivities of the replicated option. These ideas lead to European option price being expressed as a solution to a famous Black-Scholes partial differential equation.

Black and Scholes made two significant assumptions: the interest rate and the volatility over the modeled period of time are constants. However, these assumptions were rejected numerous times by empirical studies (e.g. Mandelbrot & Hudson [38]). Also, it is widely accepted that for some particular time frame and frequency of the data points, the continuously compound returns are not normally distributed, but rather follow a so-called heavy tail distribution (e.g. Fallahgoul & Loeper [18]). This means that rare events are much more likely to occur than the normal distribution would suggest, and taking this into consideration is essential for proper risk management.

[1] The formula itself was derived a couple of years earlier by Edward Oakley Thorp [45], who decided to keep it secret.

Although the initial model can be extended to the case where the risk-free interest rate and volatility are time dependent, it still could not explain such a phenomenon as volatility smile [2].

The classical model paved the way to models that could address some of its drawbacks and hence being more accurate and realistic. Some of the suggested models include the local volatility model presented by Dupire [15] and Derman & Kani [13], in which the volatility of the asset price depends on time and asset price. These models are called local volatility models. A different approach which is due to Heston [26] is based on introducing another source of randomness into our system, which will drive the volatility process. The two noises can be correlated. Such models are called stochastic volatility models.

1.3 Motivation

The importance of Brownian motion in modelling stochastic dynamical systems comes from its properties. On one hand, Brownian motion is a martingale, and hence the complete machinery of stochastic analysis can be used as a tool to describe the behaviour of the system. On the other hand, it is a Markov process which simplifies many computational tasks in regards to a conditional distributions. Finally, its marginal distributions are normal, a distribution that approximates a properly normalised (finite or infinite) sum of i.i.d. random variables, due to a famous result in probability called Central Limit Theorem.

If we want to built processes that in some sense mimic the bahaviour of Brownian motion (but is different from it), we need to eliminate some properties and keep the others. In general, every stochastic process with independent increments (see Lévy processes in Sato [50]) and constant mean function is a martingale and a Markov process. Thus the property of independent increments is quite strong. In this thesis, we will explore the idea of processes that do not have independent increments but behave "locally" like a Brownian motion.

1.3.1 Theoretical perspective

Analysis of objects in continuous time starts sometimes with analysis of simplified, but in some sense similar, objects in discrete time. So was the case in this work. It is well known that one can construct a bivariate normal distribution (X, Y) with $X \sim \mathcal{N}(0, 1)$, $Y \sim \mathcal{N}(0, 1)$ and correlation ρ as $(X, Y) = (X, \rho X + \sqrt{1-\rho^2}\xi)$ with ξ being standard normal distribution independent of X. Now we could randomise the correlation between

[2]This is referred to the implied volatility being dependent on the strike price, i.e. the volatility that one would obtain if the market would price the options via the Black-Scholes model.

random variables $X \sim \mathcal{N}(0,1)$ and $Y \sim \mathcal{N}(0,1)$, and represent it by a random variable R as follows

$$\begin{pmatrix} X \\ Y \end{pmatrix}\Bigg|_{R=r} \sim \mathcal{N}_2\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & r \\ r & 1 \end{pmatrix}\right)$$

with $\text{supp}(R) = [-1, 1]$. This idea was used by Hamza and Klebaner in [23] to construct a family of non-Gaussian Markov martingales with Gaussian marginals. Although the construction yielded a rich family of processes, all processes (except the Brownian motion) were not continuous in time. Albin [1] was able to give a construction of a continuous-time process of such type. The existence of such processes introduced the notion of *fake Brownian motion*, i.e. a process that is a martingale with respect to its natural filtration, continuous in time with Gaussian marginals and is not a Brownian motion. Later, Oleszkiewicz [43] provided another simple construction of a fake Brownian motion.

It is easy to see that a fake Brownian motion cannot have independent increments. Suppose a fake Brownian motion has independent increments. Due to Cramér's Theorem (see Appendix), it would mean that any increment of the fake Brownian motion is Gaussian and stationary. Hence we obtain a Brownian motion, which is a contradiction to the assumption that the process is a fake Brownian motion.

If we want to construct a family of processes rich enough for different modelling purposes and keep some of the properties Brownian motion has, we need to loosen the property of independent increments as it is done in the case of fake Brownian motion. However, from here on, we will go down a different path. It is easy to show that any martingale has uncorrelated increments. In general, the converse is not true. That is, we can construct a process with uncorrelated increments that is not a martingale. However, we can also retain the property of uncorrelated increments for a process X by ensuring that X has zero mean function and covariance function $\gamma_X(s,t) = \min\{s,t\}$. So opposed to a fake Brownian motion, we don't demand our process to be a martingale but rather concentrate our attention on its mean and covariance function to ensure that the process has uncorrelated increments. Note that if a process is Gaussian with zero mean and covariance function $\min\{s,t\}$, then it has to be a Brownian motion and we didn't get any further.

Additionally, a fake Brownian motion preserves the Gaussian marginals. We don't want to limit our processes to this restriction. We would rather like to construct a family of processes that have possibly other marginal distribution than Gaussian but includes Gaussian marginals as well. With that motivation in mind, we chose to restrict our processes to have "sufficiently close" Gaussian increments (rather than Gaussian marginals). For a process that starts at zero, demanding that all increments are Gaussian will automatically result in Gaussian marginals. Hence, we only demand specific type of increments to be Gaussian. It will be more clear later what is meant by sufficiently close increments.

Let us construct a discrete-time process that has uncorrelated increments and does not have Gaussian marginals, but for each $i \in \{1,2,3\}$, the distribution of $X_i - X_{i-1}$ is

Gaussian. We define a process $\{X_n, n \in \{0,1,2,3\}\}$ by

$$\left.\begin{pmatrix} X_1 \\ X_2 - X_1 \\ X_3 - X_2 \end{pmatrix}\right|_{R=r} \sim \mathcal{N}_3\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & r \\ 0 & 1 & 0 \\ r & 0 & 1 \end{pmatrix}\right),$$

where R is independent of the family $\{X_1, X_2 - X_1, X_3 - X_2\}$ and can take values in the interval $[-1,1]$. Then, given $R = r$, the decomposition $(X_1, X_2 - X_1, X_3 - X_2) = AV$ exists such that a 3-dimensional random vector V has independent standard normal components, and

$$AA^\top = \begin{pmatrix} 1 & 0 & r \\ 0 & 1 & 0 \\ r & 0 & 1 \end{pmatrix}. \tag{1.1}$$

To find the matrix A, we set $V := (X_1, X_2 - X_1, \xi)$ with a standard normal random variable ξ that is independent of X_1 and $X_2 - X_1$. We are allowed to make this choice of V, since X_1 and $X_2 - X_1$ are independent. If we choose the representation of A to be

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \alpha & \beta & \gamma \end{pmatrix},$$

then from (1.1) we obtain $\alpha = r$, $\beta = 0$ and $\gamma = \sqrt{1-r^2}$. Hence the process X can be written as

$$X_n = \begin{cases} Z_1, & n = 1 \\ Z_1 + Z_2, & n = 2 \\ Z_1 + Z_2 + RZ_1 + \sqrt{1-R^2} Z_3, & n = 3 \end{cases}$$

where Z_1, Z_2 and Z_3 are standard normal random variables and mutually independent, and R is distributed on $[-1,1]$ and independent of the family $\{Z_1, Z_2, Z_3\}$. Then, the process X has zero mean function, and it is easy to see that the covariance function is $\gamma_X(s,t) = \min\{s,t\}$ if and only if $\mathbb{E}[R] = 0$. Additionally, we see that the increments $X_i - X_{i-1}$ are normally distributed for any $i \in \{1,2,3\}$. However, the distribution of X_3 given $R = r$ is $\mathcal{N}(0, 3 + 2r)$. Hence, the distribution of X_3 is Gaussian if and only if R is a constant, which combined with condition $\mathbb{E}[R] = 0$ implies that $R \equiv 0$. Hence, we constructed a discrete-time process that has uncorrelated increments, does not necessarily have Gaussian marginals with Gaussian "neighbouring" increments.

This easy construction paved the idea for the following question:

Question. Can we build a non-Gaussian continuous-time process $\{X_t, t \geq 0\}$ with zero mean function, covariance function $\min\{s,t\}$ and the property that for each $t > 0$, there exist an $\varepsilon > 0$ such that the process

$$\{X_s - X_u : s \in (t-\varepsilon, t+\varepsilon), u \in (t-\varepsilon, t+\varepsilon)\}$$

is Gaussian?

This question is the starting point of this work and while trying to find the answer to it, we will discover the processes that we will call *local Brownian motions*.

1.3.2 Practical perspective

The existence of such process can be useful for applications, since it can be used for stochastic modelling where the local behaviour of the system is observed (or assumed due to some theoretical reasons) to be driven by a Brownian motion.

One of the major reasons that Black-Scholes-Merton (B-S-M) model is popular among practitioners in finance is its tractability. It provides explicit analytical solutions for the large number of applications. However, it is also well known that by modelling, for example, the spot price of an asset using B-S-M model, one has to deal with some drawbacks.

Another significant feature observable in financial markets is the presence of quasi long range dependencies in financial time series (e.g. Boyarchenko in [10], page 4). The classical model, which assumes the autocorrelation function of log price differences to be zero, is generally consistent with the empirical analysis. However, the empirical analysis also shows that some non-linear dependencies between log price differences are not zero. For instance, the autocorrelation function of absolute values or the squares of the daily returns may stay positive for many lags. Hence the independent increment assumption, that is imposed by Brownian motion as a driving noise in classical model, appears to be too strong.

In this thesis we aim to develop a class of processes that can be used as a stochastic noise for modelling nondeterministic systems, the same way Brownian motion is used. Since we develop a stochastic calculus with respect to a class of processes, the restrictions mentioned above that arise in some applications in financial mathematics can be accounted for by choosing appropriate process from the class. In that sense, the stochastic calculus with respect to a local Brownian motion is a generalisation of the stochastic calculus with respect to the Brownian motion.

1.4 Roter Faden

In Chapter 2 we introduce the definition of locally Gaussian process and local Brownian motion. We show that a local Brownian motion is continuous, has quadratic variation T on interval $[0, T]$ and is not a semimartingale. Moreover, we give the first example showing that such processes exist.

In Chapter 3 we construct two families of local Brownian motions based on different approaches. Both families are rich enough for modelling purposes. We believe that especially the second family constructed via copulas has high potential to find applications in finance.

In Chapter 4 the general representation of a local Brownian motion is given. We will use some symmetric properties of joint distributions. Moreover, we show that under certain conditions, a local Brownian motion is not a Markov process.

In Chapter 5 the stochastic calculus with respect to a local Brownian motion is introduced and developed. We show that the stochastic integral with respect to a local Brownian motion can be seen in some sense as an extension of the Itô integral. The definition of a stochastic differential equation (SDE) with respect to a local Brownian motion will be given. In the same manner as for stochastic differential equations with Brownian motion, we define what it means for a process to be a solution of SDE with a local Brownian motion. The theory is consistent with the theory of stochastic calculus with respect to a Brownian motion. Moreover, we will see that the Itô Formula holds and that some prominent stochastic differential equations can be solved and the solutions can be derived from the usual stochastic calculus.

In Chapter 6 we summarize the results and mention some open questions and potential future research directions.

Finally, this thesis has an Appendix that reviews the basic material on Brownian motion, Hilbert spaces, copulas, basic extension techniques and stochastic calculus with respect to Brownian motion.

Chapter 2

Local Brownian motion

In this chapter we will introduce the notion of local Brownian motion by using locally Gaussian process. The first example of such process will be given and explored. It will be clear from the definition that there is a strong connection between a local Brownian motion and a Brownian motion. This will be rigorously formalised in the characterisation theorem. Moreover, we will obtain some properties of a local Brownian motion and show that, in general, it is not a semimartingale.

2.1 Definition of local Brownian motion

By default, all stochastic processes in this thesis should be assumed to be defined on a complete probability space.

Definition 2.1. Let $\mathbf{T}$ be an Euclidean space. A family of real-valued random variables, denoted by $\{\mathbb{X}(\mathbf{t}), \mathbf{t} \in \mathbf{T}\}$, is called *a real-valued random field on* $\mathbf{T}$. The mean and the covariance function of the random field $\{\mathbb{X}(\mathbf{t}), \mathbf{t} \in \mathbf{T}\}$ are defined as

$$\mu_{\mathbb{X}}(\mathbf{t}) = \mathbb{E}[\mathbb{X}(\mathbf{t})] \quad \text{and} \quad \gamma_{\mathbb{X}}(\mathbf{s}, \mathbf{t}) = \mathbb{E}[\mathbb{X}(\mathbf{s})\mathbb{X}(\mathbf{t})] - \mathbb{E}[\mathbb{X}(\mathbf{s})]\mathbb{E}[\mathbb{X}(\mathbf{t})],$$

respectively. If for any $n \in \mathbb{N}$ and for all $\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_n \in \mathbf{T}$, the finite dimensional distribution $(\mathbb{X}(\mathbf{t}_1), \mathbb{X}(\mathbf{t}_2), \ldots, \mathbb{X}(\mathbf{t}_n))$ is Gaussian, we call $\{\mathbb{X}(\mathbf{t}), \mathbf{t} \in \mathbf{T}\}$ a *Gaussian field.*

Definition 2.2. Let $\{X_t, t \geq 0\}$ be a process. A *difference field associated with* X is a field $\{\mathbb{X}(\mathbf{t}), \mathbf{t} \in [0, \infty)^2\}$ defined by $\mathbb{X}((t_1, t_2)) := X_{t_2} - X_{t_1}$ for all $(t_1, t_2) \in [0, \infty)^2$.

In this thesis we are interested in Gaussian difference fields, i.e. difference fields with Gaussian finite-dimensional distributions. From the property of Gaussian process, it can be easily seen that if a process is Gaussian, then the difference field associated with this process is Gaussian as well. In general, the converse of this statement does not hold, i.e. one can construct a Gaussian difference field associated to a process that is not Gaussian. Thus the class of processes that can be used to construct a Gaussian difference field is strictly bigger than the class of Gaussian processes. A trivial example of a Gaussian difference field associated with a non-Gaussian process is given below.

Example 2.3. Let $\{Y_t, t \geq 0\}$ be a Gaussian process. For any non-trivial random variable Z that is not Gaussian, the process $\{X_t := Y_t - Z, t \geq 0\}$ is not Gaussian. However, the difference field associated with X is Gaussian.

The next proposition shows that for a difference field $\{\mathbb{X}(\mathbf{t}), \mathbf{t} \in [0,\infty)^2\}$ to be Gaussian, it is enough to show that there exists an $s \in [0,\infty)$ such that the difference field $\{\mathbb{X}(\mathbf{t}), \mathbf{t} \in [0,\infty) \times \{s\}\}$ is Gaussian.

Proposition 2.4. Let $\{X_t, t \geq 0\}$ be a process and $\{\mathbb{X}(\mathbf{t}), \mathbf{t} \in [0,\infty)^2\}$ be the difference field associated with X. Then, the following statements are equivalent:

(i) The difference field $\{\mathbb{X}(\mathbf{t}), \mathbf{t} \in [0,\infty)^2\}$ is Gaussian.

(ii) For all $s \geq 0$, the process $\{X_t - X_s, t \geq 0\}$ is Gaussian.

(iii) There exists an $s \geq 0$ such that the process $\{X_t - X_s, t \geq 0\}$ is Gaussian.

Proof. By definition, (i) implies (ii) and (ii) implies (iii). It remains to show that (iii) implies (i). Suppose that for some $s \geq 0$, the process $\{X_t - X_s, t \geq 0\}$ is Gaussian. Then, for any $n \in \mathbb{N}$, $t_1, t_2, \dots, t_n, u_1, u_2, \dots, u_n \in [0,\infty)$ and a sequence of real numbers $\{\lambda_i, i \in \{1, 2, \dots, n\}\}$, any finite linear combination

$$\sum_{i=1}^{n} \lambda_i (X_{t_i} - X_{u_i}) = \sum_{i=1}^{n} \lambda_i \left[(X_{t_i} - X_s) - (X_{u_i} - X_s)\right]$$

is Gaussian. It follows that the random vector $(X_{t_1} - X_{u_1}, X_{t_2} - X_{u_2}, \dots, X_{t_n} - X_{u_n})$ is Gaussian and since it was chosen arbitrarily, any finite dimensional distribution of the field $\{\mathbb{X}(\mathbf{t}), \mathbf{t} \in [0,\infty)^2\}$ is Gaussian. Hence we have (i). □

Corollary 2.5. Let $\{X_t, t \geq 0\}$ be a process and $\{\mathbb{X}(\mathbf{t}), \mathbf{t} \in [0,\infty)^2\}$ be the difference field associated with X. Suppose that there exists a $k \geq 0$ such that X_k is either a constant or a normally distributed random variable independent of $\{X_t, t \in [0,\infty) \setminus \{k\}\}$. Then, $\{\mathbb{X}(\mathbf{t}), \mathbf{t} \in [0,\infty)^2\}$ is a Gaussian difference field if and only if $\{X_t, t \geq 0\}$ is a Gaussian process. In particular, the difference field $\{\mathbb{X}(\mathbf{t}), \mathbf{t} \in [0,\infty)^2\}$ with $X_0 = 0$ is Gaussian if and only if the process $\{X_t, t \geq 0\}$ is Gaussian.

Proof. The "if" statement follows immediately from the definition. The "only if" statement follows from Proposition 2.4, which yields that the process $\{X_t - X_k, t \geq 0\}$ is Gaussian. Due to assumption on X_k, we obtain that the process X is Gaussian. □

The next proposition states the direct link between the standard Brownian motion and the difference field associated with it.

Proposition 2.6. A process $\{X_t, t \geq 0\}$ is a standard Brownian motion if and only if the difference field associated with it, $\{\mathbb{X}(\mathbf{t}), \mathbf{t} \in [0,\infty)^2\}$, is Gaussian with $X_0 = 0$, $\mathbb{E}[\mathbb{X}(\mathbf{t})] = 0$ for any $\mathbf{t} \in [0,\infty)^2$ and for any $\mathbf{s} = (s_1, s_2), \mathbf{t} = (t_1, t_2)$, the covariance function is

$$\gamma_{\mathbb{X}}(\mathbf{s},\mathbf{t}) = \frac{1}{2}(|t_2 - s_1| + |t_1 - s_2| - |t_1 - s_1| - |t_2 - s_2|). \tag{2.1}$$

Proof. Let the process $\{X_t, t \geq 0\}$ be a standard Brownian motion. Then, the difference field $\{\mathbb{X}(\mathbf{t}), \mathbf{t} \in [0,\infty)^2\}$ is Gaussian with zero mean. Further, for $\mathbf{s} = (s_1, s_2)$ and $\mathbf{t} = (t_1, t_2)$,

$$\begin{aligned}
\gamma_{\mathbb{X}}(\mathbf{s},\mathbf{t}) &= \mathbb{E}\left[(X_{s_2} - X_{s_1})(X_{t_2} - X_{t_1})\right] - \mathbb{E}[X_{s_2} - X_{s_1}]\mathbb{E}[X_{t_2} - X_{t_1}] \\
&= \min\{s_2, t_2\} + \min\{s_1, t_1\} - \min\{s_2, t_1\} - \min\{s_1, t_2\} \\
&= \frac{1}{2}(|t_2 - s_1| + |t_1 - s_2| - |t_1 - s_1| - |t_2 - s_2|),
\end{aligned}$$

where $s = (s_1, s_2)$ and $t = (t_1, t_2)$. Conversely, if the difference field $\{\mathbb{X}(\mathbf{t}), \mathbf{t} \in [0,\infty)^2\}$ is Gaussian with $X_0 = 0$, $\mathbb{E}[\mathbb{X}(\mathbf{t})] = 0$ for any $\mathbf{t} \in [0,\infty)^2$ and covariance function (2.1), due to Corollary 2.5, the process $\{X_t, t \geq 0\}$ is Gaussian with zero mean. Moreover,

$$\mathbb{E}\left[X_s X_t\right] - \mathbb{E}[X_s]\mathbb{E}[X_t] = \gamma_{\mathbb{X}}((0,s),(0,t)) = \frac{1}{2}(t + s - |t - s|) = \min\{s,t\}$$

for $s, t \geq 0$. Therefore, the process $\{X_t, t \geq 0\}$ is a standard Brownian motion. □

Definition 2.7. We say that a process $\{X_t, t \geq 0\}$ *locally* Gaussian if for every $t > 0$ there exists $\varepsilon > 0$ such that the difference field associated with X on $(t - \varepsilon, t + \varepsilon)^2$, $\{\mathbb{X}(\mathbf{s}), \mathbf{s} \in (t - \varepsilon, t + \varepsilon)^2\}$, is Gaussian. We will refer to a locally Gaussian process that is not Gaussian as a *proper* locally Gaussian process.

Note that the definition of locally Gaussian process does not imply that for each $s > 0$ there exists an $\varepsilon > 0$ such that the process $\{X_t, t \in (s - \varepsilon, s + \varepsilon)\}$ is Gaussian. However, as we indicated in Proposition 2.4, the converse is true. An example of a locally Gaussian process is given below.

Example 2.8. For $i \in \{1,2,3\}$, let $\{X_t^{(i)}, t \geq 0\}$ with $X_0^{(i)} = 0$ be three independent Gaussian processes with zero mean functions and the same covariance functions. Further, let R be a zero-mean random variable taking values in $[-1,1]$ and independent of the family $\{X^{(1)}, X^{(2)}, X^{(3)}\}$. Then, the process

$$Y_t = \begin{cases} X_t^{(1)}, & 0 \leq t \leq 1 \\ Y_1 + X_{t-1}^{(2)}, & 1 < t \leq 2 \\ Y_2 + RX_{t-2}^{(1)} + \sqrt{1-R^2}X_{t-2}^{(3)}, & t > 2 \end{cases}$$

is locally Gaussian. It is clear that Y_t is Gaussian on $[0,2]$. Further, for any $t \geq 2$ and $0 < \varepsilon < 1$, the process $\{Y_s, s \in (t-\varepsilon, t+\varepsilon)\}$ is Gaussian as well. Hence Y is locally Gaussian. Furthermore, Y is not Gaussian unless $R = 0$: for instance,

$$Y_3|_{R=r} = (1+r)X_1^{(1)} + X_1^{(2)} + \sqrt{1-r^2}X_1^{(3)} \sim \mathcal{N}(0, (3+2r)\mathbb{E}[(X_1^{(1)})^2]),$$

thus Y_3 is Gaussian if and only if $R = 0$.

Finally, we can define local Brownian motion as a process that has the same covariance function as Brownian motion, but is “only” locally Gaussian. Since a Gaussian process is also locally Gaussian, consequentially, any Brownian motion is also a local Brownian motion.

Definition 2.9. We call a process $\{X_t, t \geq 0\}$ with $X_0 = 0$ a *local Brownian motion* if it is a locally Gaussian process with mean zero and covariance function $\gamma_X(s,t) = \min\{s,t\}$. A local Brownian motion that is not a Brownian motion is called a *proper local Brownian motion.*

Note that the covariance function and the zero mean property of a local Brownian motion ensure that the non-overlapping increments are uncorrelated. Consequentially, a local Brownian motion is a process with uncorrelated increments. However, in contrast to a Brownian motion, a proper local Brownian motion does not have independent increments and thus can have dependencies of higher order.

The following example shows that a proper local Brownian motion exists, and can be built by patching different standard Brownian motions with certain dependency structure among them. The example given below will be generalised and analysed in more detail in Chapter 3.

Example 2.10. For $i \in \{1,2,3\}$, let $\{W_t^{(i)}, t \in [0,T]\}$ be three independent standard Brownian motions and R be a zero-mean random variable taking values in $[-1,1]$. Suppose that R and $\{W^{(1)}, W^{(2)}, W^{(3)}\}$ are independent. Then, for $T > 2$, the process

$$X_t = \begin{cases} W_t^{(1)}, & 0 \leq t \leq 1 \\ X_1 + W_{t-1}^{(2)}, & 1 < t \leq 2 \\ X_2 + RW_{t-2}^{(1)} + \sqrt{1-R^2}W_{t-2}^{(3)}, & 2 < t \leq T \end{cases}$$

is a local Brownian motion.

First, let us show that the process $\{X_t, t \in [0,T]\}$ is locally Gaussian. Due to independence of $W^{(1)}$ and $W^{(2)}$, it can be seen that $\{X_t, t \in [0,2]\}$ is Gaussian and hence, by Proposition 2.4, the difference field associated with X, $\{\mathbb{X}(\mathbf{t}), \mathbf{t} \in [0,2]^2\}$, is Gaussian. Further, let $\varepsilon < 1$, $n \in \mathbb{N}$ and for each $i \in \{1, 2, \cdots, n\}$, $\{\mathbf{t}_i = (s_i, t_i)\} \in (2-\varepsilon, T]^2$. For any n_1, n_2 and n_3 with $n_1 + n_2 + n_3 = n$, and any family of real numbers $\{\alpha_i, i \in \{1, 2, \ldots, n_1\}\}$, $\{\beta_i, i \in \{1, 2, \ldots, n_2\}\}$ and $\{\gamma_i, i \in \{1, 2, \ldots, n_3\}\}$, any linear combination of $(\mathbb{X}_{\mathbf{t}_1}, \mathbb{X}_{\mathbf{t}_2}, \ldots, \mathbb{X}_{\mathbf{t}_n})$ can be written as

$$
\begin{aligned}
&\sum_{i=1}^{n_1} \alpha_i \left(R W^{(1)}_{t_i-2} + \sqrt{1-R^2} W^{(3)}_{t_i-2} - W^{(2)}_{s_i-1} + W^{(2)}_1 \right) \\
&\qquad + \sum_{i=1}^{n_2} \beta_i \left(R W^{(1)}_{t_i-2} + \sqrt{1-R^2} W^{(3)}_{t_i-2} - R W^{(1)}_{s_i-2} - \sqrt{1-R^2} W^{(3)}_{s_i-2} \right) \\
&\qquad + \sum_{i=1}^{n_3} \gamma_i \left(W^{(2)}_{t_i-1} - W^{(2)}_{s_i-1} \right) \\
&= R \left(\sum_{i=1}^{n_1} \alpha_i W^{(1)}_{t_i-2} + \sum_{i=1}^{n_2} \beta_i \left(W^{(1)}_{t_i-2} - W^{(1)}_{s_i-2} \right) \right) \\
&\qquad + \sqrt{1-R^2} \left(\sum_{i=1}^{n_1} \alpha_i W^{(3)}_{t_i-2} + \sum_{i=1}^{n_2} \beta_i \left(W^{(3)}_{t_i-2} - W^{(3)}_{s_i-2} \right) \right) \\
&\qquad - \sum_{i=1}^{n_1} \alpha_i \left(W^{(2)}_{s_i-1} - W^{(2)}_1 \right) + \sum_{i=1}^{n_3} \gamma_i \left(W^{(2)}_{t_i-1} - W^{(2)}_{s_i-1} \right).
\end{aligned}
$$

The sum of the last two terms is normally distributed and independent of the first two terms. Furthermore, using the fact that $W^{(1)}$ and $W^{(3)}$ are independent, we obtain that the distribution of the sum of the first two terms is Gaussian and does not depend on the distribution of R. Since all three Brownian motions are independent, every linear combination of n-dimensional vector $(\mathbb{X}_{\mathbf{t}_1}, \mathbb{X}_{\mathbf{t}_2}, \ldots, \mathbb{X}_{\mathbf{t}_n})$ is Gaussian. This shows that the difference field $\{\mathbb{X}(\mathbf{t}), \mathbf{t} \in (2-\varepsilon, T]^2\}$ is Gaussian. It follows that $\{X_t, t \in (2-\varepsilon, T]\}$ is a locally Gaussian process. Finally, if a process is locally Gaussian on two time intervals and the intersection of these time intervals is neither an empty set nor a point, the process is locally Gaussian on the union of both intervals. Hence the process $\{X_t, t \in [0,T]\}$ is locally Gaussian.

Note that the process X is not Gaussian for a non-constant R as was shown in Example 2.8. Further,

$$
\mathbb{E}\left[X_s X_t\right] = \begin{cases}
\mathbb{E}\left[W^{(1)}_s W^{(1)}_t\right] = \min\{s,t\}, & 0 \le s, t \le 1 \\
\mathbb{E}\left[W^{(1)}_s (W^{(1)}_1 + W^{(2)}_{t-1})\right] = s, & s \le 1 < t \le 2 \\
\mathbb{E}\left[W^{(1)}_s (W^{(1)}_1 + W^{(2)}_1 + R W^{(1)}_{t-2} + \sqrt{1-R^2} W^{(3)}_{t-2})\right] = s, & s \le 1, t > 2 \\
\mathbb{E}\left[(W^{(1)}_1 + W^{(2)}_{s-1})(W^{(1)}_1 + W^{(2)}_{t-1})\right] = \min\{s,t\}, & 1 < s, t \le 2,
\end{cases}
$$

for $1 < s \leq 2 < t$,

$$\mathbb{E}\left[(W_1^{(1)} + W_{s-1}^{(2)})(W_1^{(1)} + W_1^{(2)} + RW_{t-2}^{(1)} + \sqrt{1-R^2}W_{t-2}^{(3)})\right] = s,$$

and for $s, t \in [2, +\infty)$,

$$\begin{aligned}&\mathbb{E}\left[(W_1^{(1)} + W_1^{(2)} + RW_{s-2}^{(1)} + \sqrt{1-R^2}W_{s-2}^{(3)})(W_1^{(1)} + W_1^{(2)} + RW_{t-2}^{(1)} + \sqrt{1-R^2}W_{t-2}^{(3)})\right] \\ &= 1 + 1 + \mathbb{E}\left[R^2\right]\min\{s-2, t-2\} + (1 - \mathbb{E}\left[R^2\right])\min\{s-2, t-2\} = \min\{s, t\}.\end{aligned}$$

Since $\mathbb{E}[X_t] = 0$ for all $t \in [0, T]$, the covariance function is $\min\{s, t\}$.

2.2 Characterisation of local Brownian motions

Due to Proposition 2.4 a process $\{X_t, t \geq 0\}$ is a local Brownian motion if for any $t > 0$, there exists $\varepsilon > 0$ such that for any $u \in (t-\varepsilon, t+\varepsilon)$, the process $\{X_s - X_u, s \in [u, t+\varepsilon)\}$ is a Gaussian process. Direct computations yield the covariance function $\gamma_X(s_1, s_2) = \min\{s_1, s_2\} - u$ for $s_1, s_2 \geq u$. Thus $\{X_s - X_u, s \in [u, t+\varepsilon)\}$ is a Brownian motion shifted in time. Moreover, if we deal with a local Brownian motion on the time interval $[0, T]$, we can obtain the converse statement.

Theorem 2.11. *A process $\{X_t, t \in [0, T]\}$ with $X_0 = 0$ is a local Brownian motion if and only if there exists a finite partition $0 = t_0 < t_1 < \ldots < t_n = T$ such that for each $i \in \{1, 2, \ldots, n-1\}$, the process $\{X_t - X_{t_{i-1}}, t \in [t_{i-1}, t_{i+1}]\}$ is a Brownian motion and, with $\ell(t) := \max\{j : t > t_j\}$,*

$$Cov(X_u - X_{t_{\ell(u)}}, X_v - X_{t_{\ell(v)}}) = \begin{cases}\min\{u, v\} - t_{\ell(u)}, & \text{if } \ell(u) = \ell(v) \\ 0, & \text{otherwise}\end{cases}$$

for any $u, v \in [0, T]$.

Proof. First, consider the "only if" statement. Let process $\{X_t, t \in [0, T]\}$ with $X_0 = 0$ be a local Brownian motion. Then, there exists an open covering $\{A_j, j \in J\}$ of the interval $[0, T]$ such that the difference field associated with X, $\{\mathbb{X}(\mathbf{s}), \mathbf{s} \in A_j^2\}$, is Gaussian for any $j \in J$. Since $[0, T]$ is compact, we can extend a *finite* open covering $\bigcup_{j=1}^n A_{k_j}$ for $k_1, k_2, \cdots, k_n \in J$ such that the difference field $\{\mathbb{X}(\mathbf{s}), \mathbf{s} \in A_{k_j}^2\}$ is Gaussian for each $j \in \{1, 2, \ldots, n\}$. By taking the intersection points of the finite covering, we obtain a finite partition $0 = t_0 < t_1 < \ldots < t_n = T$ such that for each $i \in \{1, 2, \ldots, n-1\}$, the process $\{X_t - X_{t_{i-1}}, t \in [t_{i-1}, t_{i+1}]\}$ is Gaussian with zero mean. Then, for $u, v \in [0, T]$,

$$\begin{aligned}Cov(X_u - X_{t_{\ell(u)}}, &X_v - X_{t_{\ell(v)}}) \\ &= \min\{u, v\} - \min\{t_{\ell(u)}, v\} - \min\{u, t_{\ell(v)}\} + \min\{t_{\ell(u)}, t_{\ell(v)}\} \\ &= \begin{cases}\min\{u, v\} - t_{\ell(u)}, & \text{if } \ell(u) = \ell(v) \\ 0, & \text{otherwise.}\end{cases}\end{aligned}$$

Hence for each $i \in \{1, 2, \ldots, n\}$, the process $\{X_t - X_{t_{i-1}}, t \in [t_{i-1}, t_{i+1}]\}$ is a standard Brownian motion shifted in time.

Now the "if" statement. Suppose that $\{X_t, t \in [0, T]\}$ with $X_0 = 0$ satisfies the given assumptions. Then, the process $\{X_t, t \in [0, T]\}$ is locally Gaussian with mean zero. Note that for any $u \in [0, T]$, we can write $X_u = X_u - X_{t_{\ell(u)}} + \sum_{i=1}^{\ell(u)} (X_{t_i} - X_{t_{i-1}})$ and therefore

$$Cov(X_u, X_v) = Cov\Big(X_u - X_{t_{\ell(u)}} + \sum_{i=1}^{\ell(u)} (X_{t_i} - X_{t_{i-1}}), X_v - X_{t_{\ell(v)}} + \sum_{i=1}^{\ell(v)} (X_{t_i} - X_{t_{i-1}})\Big).$$

If u and v belong to the same interval of the partition, that is $\ell(u) = \ell(v)$, then

$$\gamma_X(u, v) = Cov(X_u, X_v) = \min\{u, v\} - t_{\ell(u)} + \sum_{i=1}^{\ell(u)} (t_i - t_{i-1}) = \min\{u, v\}.$$

For the case $\ell(u) < \ell(v)$, we get

$$\begin{aligned}
\gamma_X(u, v) &= Cov\Big(X_u - X_{t_{\ell(u)}} + \sum_{i=1}^{\ell(u)} (X_{t_i} - X_{t_{i-1}}), X_v - X_{t_{\ell(v)}} + \sum_{i=1}^{\ell(v)} (X_{t_i} - X_{t_{i-1}})\Big) \\
&= Cov\Big(X_u - X_{t_{\ell(u)}} + \sum_{i=1}^{\ell(u)} (X_{t_i} - X_{t_{i-1}}), X_v - X_{t_{\ell(u)}} + \sum_{i=1}^{\ell(u)} (X_{t_i} - X_{t_{i-1}})\Big) \\
&= \min\{u, v\} - t_{\ell(u)} + \sum_{i=1}^{\ell(u)} (t_i - t_{i-1}) = \min\{u, v\}.
\end{aligned}$$

Similarly for $\ell(u) > \ell(v)$,

$$\begin{aligned}
\gamma_X(u, v) &= Cov\Big(X_u - X_{t_{\ell(u)}} + \sum_{i=1}^{\ell(u)} (X_{t_i} - X_{t_{i-1}}), X_v - X_{t_{\ell(v)}} + \sum_{i=1}^{\ell(v)} (X_{t_i} - X_{t_{i-1}})\Big) \\
&= Cov\Big(X_u - X_{t_{\ell(v)}} + \sum_{i=1}^{\ell(v)} (X_{t_i} - X_{t_{i-1}}), X_v - X_{t_{\ell(v)}} + \sum_{i=1}^{\ell(v)} (X_{t_i} - X_{t_{i-1}})\Big) \\
&= \min\{u, v\} - t_{\ell(v)} + \sum_{i=1}^{\ell(v)} (t_i - t_{i-1}) = \min\{u, v\}.
\end{aligned}$$

It follows that $\gamma_X(u, v) = \min\{u, v\}$ for any $u, v \in [0, T]$. □

Definition 2.12. The finite partition $0 = t_0 < t_1 < \ldots < t_n = T$ from Theorem 2.11 will be called *a time partition of a local Brownian motion.*

Note that a local Brownian motion has infinitely many time partitions; by adding extra points to a time partition, we can always obtain a different, *finer* time partition. It is crucial to understand that if we consider an incremental process of a local Brownian motion $\{X_t - X_s, t > s\}$ for any $s > 0$, then it is not garanteed that this process is a Brownian motions (unless the local Brownian motion is a Brownian motion itself). Only the incremental processes that are lying within two neighbouring intervals in the time partition. We can also choose a length such that on any interval of that length an incremental process is a Brownian motion, i.e. for a local Brownian motion $\{X_t, t \in [0,T]\}$ with a time partition $0 = t_0 < t_1 < \ldots < t_n = T$, there exists $\delta \leq \min\{t_i - t_{i-1}, i \in \{1, 2, \ldots, n\}\}$ such that for any $s > 0$, the process $\{X_t - X_s, t \in [s, s+2\delta]\}$ is a Brownian motion.

2.3 Properties of local Brownian motion

As we showed in the last section, there is a direct connection between a local Brownian motion and a Brownian motion. Thus we would expect some basic local path properties of a Brownian motion to be inherited by a proper local Brownian motion and it comes as no surprise that this is, in fact, the case.

Proposition 2.13. Let $\{X_t, t \in [0,T]\}$ be a proper local Brownian motion. Then, it has a continuous path.

Proof. Let $0 = t_0 < t_1 < \ldots < t_n = T$ be a time partition of X. Then, by Theorem 2.11, the process $\{X_t - X_{t_{i-1}}, t \in [t_{i-1}, t_{i+1}]\}$ is a Brownian motion for each $i \in \{1, 2, \ldots, n-1\}$ and the assertion follows. □

Definition 2.14. For a local Brownian motion $\{X_t, t \in [0,T]\}$ with time partition $0 = t_0 < t_1 < \ldots < t_n = T$, the quadratic variation is defined as

$$[X,X]_T := [X,X]([0,T]) = \sum_{i=1}^{n} [X,X]([t_{i-1}, t_i]).$$

Since the quadratic variation of a Brownian motion exists on any finite interval, the quadratic variation of a local Brownian motion exists on any finite interval as well. Moreover, the quadratic variation defined above applied on a semimartingale is consistent with the definition of quadratic variation for semimartingales. Hence it can be seen as an extension of the latter.

Proposition 2.15. Let the process $\{X_t, t \in [0,T]\}$ be a proper local Brownian motion. Then, its quadratic variation over time interval $[0,T]$ is T.

Proof. Let $0 = t_0 < t_1 < \ldots < t_n = T$ be a time partition of X. Then, by Theorem 2.11, the process $\{X_t - X_{t_{i-1}}, t \in [t_{i-1}, t_{i+1}]\}$ is a Brownian motion for each $i \in \{1, 2, \ldots, n-1\}$ and using the definition of the quadratic variation for a local Brownian motion,

$$[X,X]_T = \sum_{i=1}^{n} [X,X]([t_{i-1}, t_i]) = \sum_{i=1}^{n} [X - X_{t_{i-1}}, X - X_{t_{i-1}}]([t_{i-1}, t_i]) = \sum_{i=1}^{n} (t_i - t_{i-1}) = T.$$

□

Corollary 2.16. Since a proper local Brownian motion is continuous and has non-zero quadratic variation on $[0, T]$, it has an infinite variation on $[0, T]$. Moreover, it is not monotone on $[0, T]$.

So far we could show that a proper local Brownian motion inherits some properties from Brownian motion. The next propositions will give us a significant difference between a proper local Brownian motion and a Brownian motion.

Proposition 2.17. A proper local Brownian motion is not a local martingale.

Proof. Assume that a proper local Brownian motion is a local martingale. Then, by applying Proposition 2.13, Proposition 2.15 and Levy's characterisation theorem of Brownian motion, it follows that the proper local Brownian motion is a Brownian motion. This leads to a contradiction. □

Proposition 2.18. A proper local Brownian motion is not a semimartingale.

Proof. Let the process $\{X_t, t \in [0, T]\}$ be a proper local Brownian motion and assume that it is a semimartingale. Due to Proposition 2.13, it is continuous. A continuous semimartingale can be written as a sum of a continuous local martingale $\{M_t, t \in [0, T]\}$ and a continuous process of finite variation $\{A_t, t \in [0, T]\}$. Fix $s \in [0, T)$ and consider the difference process $X_u - X_s = M_u - M_s + A_u - A_s$ for $u > s$. For a sufficiently small $\varepsilon > 0$, the process $\{X_u - X_s, u \in [s, s + \varepsilon)\}$ is a Brownian motion. Since the process $\{A_u - A_s, u \in [s, s + \varepsilon)\}$ is continuous and has finite variation, its quadratic variation on any time interval is zero. Therefore, by polarisation identity, the process $\{M_u - M_s, u \in [s, s + \varepsilon)\}$ has the same quadratic variation as the local Brownian motion and, by Levy's characterisation theorem, it is a Brownian motion. As a difference of two martingales, the process $\{A_u - A_s, u \in [s, s + \varepsilon)\}$ is a martingale with zero quadratic variation. Hence it does not change on the interval $[s, s+\varepsilon)$. Since s was chosen arbitrary, the process $\{A_t, t \in [0, T]\}$ does not change on the whole time space and hence $A_t \equiv 0$. This means that the proper local Brownian motion $\{X_t, t \in [0, T]\}$ is a local martingale, which contradicts Proposition 2.17. □

Chapter 3

Explicit constructions

In this chapter we will present two ways of constructing families of local Brownian motions. In the first section we will generalise the construction given in Example 2.10 via randomisation of the incremental processes. In the second section we will use the infinite series representation of Brownian motion and impose non-linear dependencies on joint distributions of incremental processes. We will give the conditions for associated copulas with these joint distributions and see that the convex sum of such copulas yields new copulas that can be used for constructions of local Brownian motions.

3.1 Local Brownian motion through randomised scaled covariances

3.1.1 Extension to a 4-step process

Let $0 = t_0 < t_1 < t_2 < t_3 < t_4 = T$, $\{W_t^{(i)}, t \in [0,\infty)\}$, $i \in \{1,2,3,4\}$ be a family of independent Brownian motions and R_1, R_2, R_3 be zero-mean random variables, taking values in the interval $[-1,1]$. Suppose that the family $\{R_1, R_2, R_3\}$ is independent of the family $\{W^{(1)}, W^{(2)}, W^{(3)}, W^{(4)}\}$. We define a process $\{X_t, t \in [0,T]\}$ that is an extension of the process given in Example 2.10 to four intervals as follows:

$$X_t = \begin{cases} W_t^{(1)}, & 0 \le t \le t_1 \\ X_{t_1} + W_{t-t_1}^{(2)}, & t_1 < t \le t_2 \\ X_{t_2} + R_1 W_{t-t_2}^{(1)} + \sqrt{1-R_1^2}\, W_{t-t_2}^{(3)}, & t_2 < t \le t_3 \\ X_{t_3} + R_2 W_{t-t_3}^{(1)} + R_3 W_{t-t_3}^{(2)} + a(R_1,R_2,R_3) W_{t-t_3}^{(3)} & \\ \qquad + \sqrt{1-R_2^2-R_3^2-a(R_1,R_2,R_3)^2}\, W_{t-t_3}^{(4)}, & t_3 < t \le t_4 \end{cases}$$

where $a : [-1,1]^3 \to \mathbb{R}$ is a continuous function. The random variables R_1, R_2 and R_3 can be seen as randomised cavariance of non-neighbouring incremental processes. It follows,

$$\begin{aligned} R_1 &= \frac{\mathbb{E}[X_s(X_t - X_{t_2})|R_1]}{s \wedge (t-t_2)}, \qquad \text{for } s \in (0,t_1], t \in (t_2,t_3] \\ R_2 &= \frac{\mathbb{E}[X_s(X_t - X_{t_3})|R_2]}{s \wedge (t-t_3)}, \qquad \text{for } s \in (0,t_1], t \in (t_3,t_4] \\ R_3 &= \frac{\mathbb{E}[(X_s - X_{t_1})(X_t - X_{t_3})|R_3]}{(s-t_1) \wedge (t-t_3)}, \qquad \text{for } s \in (t_1,t_2], t \in (t_3,t_4]. \end{aligned}$$

Note that the term $\sqrt{1 - R_2^2 - R_3^2 - a(R_1,R_2,R_3)^2}$ ensures that the distribution of the process $\{X_t - X_{t_3}, t \in [t_3,t_4]\}$ does not depend on R_1, R_2 and R_3.

Noxt, we obtain the condition on a, so that X is a local Brownian motion. Suppose that the process X is a local Brownian motion. Let $t_2 < l < t_3 < m < t_4$. By Theorem 2.11, the process $\{X_t - X_{t_2}, t \in [t_2,t_4]\}$ must be a Brownian motion and independent of the family $\{R_1, R_2, R_3\}$. Thus

$$\mathbb{E}[(X_{t_3} - X_l)(X_m - X_{t_3})|R_1,R_2,R_3] = \mathbb{E}[(X_{t_3} - X_l)(X_m - X_{t_3})] = 0. \tag{3.1}$$

On the other hand, by setting $C = (m-t_3) \wedge (t_3 - t_2) - (m - t_3) \wedge (l - t_2)$ from the construction of $\{X_t, t \in [0,t_4]\}$, we obtain

$$\mathbb{E}[(X_{t_3} - X_l)(X_m - X_{t_3})|R_1,R_2,R_3] = CR_1R_2 + Ca(R_1,R_2,R_3)\sqrt{1-R_1^2}. \tag{3.2}$$

From (3.1) and (3.2), we get unique representation of the function $a(R_1,R_2,R_3)$ as

$$a(R_1,R_2,R_3) = -\frac{R_1R_2}{\sqrt{1-R_1^2}}.$$

Another way we can obtain the function a is by looking at the distribution of $Y := (X_t - X_{t_3}) + (X_s - X_{t_2})$ for any $s \in [t_2,t_3]$ and $t = t_3 + (s - t_2)$. Due to Theorem 2.11, it must be normal with mean zero and variance $2(s-t_2)$. Since $t - t_3 = s - t_2$ and

$$\begin{aligned} Y = {} & R_1 W^{(1)}_{s-t_2} + \sqrt{1-R_1^2} W^{(3)}_{s-t_2} + R_2 W^{(1)}_{s-t_2} + R_3 W^{(2)}_{s-t_2} + a(R_1,R_2,R_3) W^{(3)}_{s-t_2} \\ & + \sqrt{1 - R_2^2 - R_3^2 - a(R_1,R_2,R_3)^2} W^{(4)}_{s-t_2}, \end{aligned}$$

we have $Y|_{R_1=r_1,R_2=r_2,R_3=r_3} \sim \mathcal{N}(0,(s-t_2)\sigma^2)$ with

$$\sigma^2 = (r_1+r_2)^2 + r_3^2 + \left(a(r_1,r_2,r_3) + \sqrt{1-r_1^2}\right)^2 + \left(1 - r_2^2 - r_3^2 - a(r_1,r_2,r_3)^2\right).$$

Therefore, Y is normally distributed with mean zero and variance $2(s-t_2)$ if and only if $\sigma^2 = 2$, which is the case if and only if

$$a(r_1,r_2,r_3) = -\frac{r_1r_2}{\sqrt{1-r_1^2}}.$$

Moreover, it is clear from the construction, that the condition

$$R_2^2 + R_3^2 + a(R_1,R_2,R_3)^2 \le 1$$

must hold for our process to be a local Brownian motion.

3.1.2 Extension to an n-step process

Our next goal is to generalise the procedure mentioned in the subsection above and extend the process to n time intervals. We will find a sufficient and necessary condition for the existence of such local Brownian motions.

Let $0 = t_0 < t_1 < \ldots < t_n = T$ be a partition on $[0,T]$. Let $\mathcal{R} = \{R_{ij} : 1 \leq i,j \leq n, j+2 \leq i\}$ be a family of zero-mean random variables on $[-1,1]$ with cardinality $|\mathcal{R}| = \binom{n}{2} - (n-1) = \frac{(n-1)(n-2)}{2}$. Further, let $\mathcal{W} = \{W^{(i)}, i \in \{1,2,\ldots,n\}\}$ be a family of independent Brownian motions. Suppose that two families $\mathcal{R}$ and $\mathcal{W}$ are independent. Additionally, let $\mathcal{A} = \{a_{ij} : 4 \leq i \leq n, 3 \leq j \leq i-1\}$ be a family of functions of the random variables from the family $\mathcal{R}$. The cardinality of this family is $|\mathcal{A}| = \frac{(n-3)(n-2)}{2}$. We define a process $\{X_t, t \in [0,T]\}$ such that for any $k \in \{3,4,\ldots,n\}$, we have

$$\boxed{X_t = \begin{cases} W_t^{(1)}, & 0 \leq t \leq t_1 \\ X_{t_1} + W_{t-t_1}^{(2)}, & t_1 < t \leq t_2 \\ X_{t_{k-1}} + \left(R_{k1}, R_{k2}, a_{k3}, \cdots, a_{kk-1}, \sqrt{1-\alpha_k}\right) \begin{pmatrix} W_{t-t_{k-1}}^{(1)} \\ W_{t-t_{k-1}}^{(2)} \\ \vdots \\ W_{t-t_{k-1}}^{(k)} \end{pmatrix}, & t_{k-1} < t \leq t_k \end{cases}} \tag{3.3}$$

where $\alpha_k = R_{k1}^2 + R_{k2}^2 + a_{k3}^2 + \ldots + a_{kk-1}^2$. Suppose that the process X is a local Brownian motion. Then, there exists a unique family $\mathcal{A}$ such that the process X is a local Brownian motion. In order to determine $\mathcal{A}$, we introduce a symmetric $n \times n$ matrix $\Sigma = (\sigma_{ij})_{1 \leq i,j \leq n}$ such that

$$\Sigma = \begin{pmatrix} 1 & 0 & R_{31} & R_{41} & R_{51} & \cdots & R_{n1} \\ 0 & 1 & 0 & R_{42} & R_{52} & \cdots & R_{n2} \\ R_{31} & 0 & 1 & 0 & R_{53} & \cdots & R_{n3} \\ R_{41} & R_{42} & 0 & 1 & 0 & \ddots & \vdots \\ R_{51} & R_{52} & R_{53} & 0 & 1 & \ddots & R_{nn-2} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ R_{n1} & R_{n2} & R_{n3} & \cdots & R_{nn-2} & 0 & 1 \end{pmatrix}.$$

The matrix Σ represents randomised scaled covariances of the incremental processes, i.e. for each $i,j \in \{1,2,\ldots,n\}$ and all $s \in [t_{i-1}, t_i]$, $t \in [t_{j-1}, t_j]$,

$$\sigma_{ij} = \frac{\mathbb{E}\left[(X_s - X_{t_{i-1}})(X_t - X_{t_{j-1}}) | \sigma_{ij}\right]}{(s - t_{i-1}) \wedge (t - t_{j-1})}.$$

Let us introduce a matrix $B = (b_{ij})_{1\le i,j\le n}$ such that

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\ R_{31} & 0 & \sqrt{1-R_{31}^2} & 0 & 0 & \cdots & 0 \\ R_{41} & R_{42} & a_{43} & \sqrt{1-\alpha_4} & 0 & \cdots & 0 \\ R_{51} & R_{52} & a_{53} & a_{54} & \sqrt{1-\alpha_5} & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ R_{n1} & R_{n2} & a_{n3} & a_{n4} & \cdots & a_{nn-1} & \sqrt{1-\alpha_n} \end{pmatrix}.$$

It follows that one can decompose Σ into a product of B and its transpose $B^\top$, $\Sigma = BB^\top$, if and only if for each $i \in \{4, 5, \ldots, n\}$,

$$\alpha_i \le 1. \tag{3.4}$$

In that case, the family $\mathcal{A}$ can be found recursively from a system of linear equations given by

$$\sum_{k=1}^{n} a_{ik}a_{jk} = \sigma_{ij}.$$

This family of local Brownian motions can be constructed using a different approach based on discrete-time version of a local Brownian motion.

Let $0 = t_0 < t_1 < \ldots < t_n = T$ be a partition on $[0,T]$ as above. Further, let the families $\mathcal{R}$ and Σ be defined as above. Then, we can define a time-discrete process $\{\widetilde{X}_{t_k}, k \in \{0,1,\ldots,n\}\}$ with $\tilde{X}_0 = 0$ such that

$$\left.\begin{pmatrix} \widetilde{X}_{t_1} \\ \widetilde{X}_{t_2} - \widetilde{X}_{t_1} \\ \vdots \\ \widetilde{X}_{t_n} - \widetilde{X}_{t_{n-1}} \end{pmatrix}\right|_{\mathcal{R}} \sim \mathcal{N}_n\left(0, D\Sigma D^\top\right) \text{ with } D = \begin{pmatrix} \sqrt{t_1} & 0 & \cdots & 0 \\ 0 & \sqrt{t_2-t_1} & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \sqrt{t_n - t_{n-1}} \end{pmatrix}$$

and Σ defined as in the previous construction. If the matrix $D\Sigma D^T$ is positive-semidefinite, which is the case if and only if Σ is positive semi-definite, then the Cholesky decomposition of Σ exists, i.e. there exists a lower triangular matrix $B = (b_{ij})_{1\le i,j\le n}$ with real non-negative diagonal entries such that $\Sigma = BB^T$. Then, we can write

$$\left.\begin{pmatrix} \widetilde{X}_{t_1} \\ \widetilde{X}_{t_2} - \widetilde{X}_{t_1} \\ \vdots \\ \widetilde{X}_{t_n} - \widetilde{X}_{t_{n-1}} \end{pmatrix}\right|_{\mathcal{R}} = DBZ \qquad \text{with} \qquad Z = \begin{pmatrix} Z_1 \\ Z_2 \\ \vdots \\ Z_n \end{pmatrix}$$

where $\mathcal{Z} = \{Z_i, i \in \{1,2,\ldots,n\}\}$ is the family of mutually independent standard normal random variables. The families $\mathcal{Z}$ and $\mathcal{R}$ are independent. Hence, if Σ is positive semi-definite, then we obtain a discrete-time process $\{\widetilde{X}_{t_k}, k \in \{0,1,\ldots,n\}\}$ with $\widetilde{X}_0 = 0$ and

$$\widetilde{X}_{t_k} = \widetilde{X}_{t_{k-1}} + \sqrt{t_j - t_{j-1}} \sum_{j=1}^{n} b_{kj} Z_j$$

for each $k \in \{1, 2, \ldots, n\}$.

In order to extend the process $\{\widetilde{X}_{t_k}, k \in \{0, 1, \ldots, n\}\}$ to a continuous-time process, we use a family of mutually independent standard Brownian motions from the previous construction $\mathcal{W}$ instead of the family $\mathcal{Z}$, i.e. for each $i \in \{1, 2, \ldots, n\}$ and any $t \in [t_{k-1}, t_k]$, we replace the random variable $\sqrt{t_i - t_{i-1}} Z_j$ with the process $W^{(j)}_{t-t_{k-1}}$ on interval $[t_{k-1}, t_k]$ for each $k \in \{1, 2, \ldots, n\}$. Finally, we obtain a process $\{X_t, t \in [0, T]\}$ with $X_0 = 0$ such that for any $k \in \{1, 2, \ldots, n\}$ and any $t_{k-1} \leq t \leq t_k$,

$$\boxed{X_t = X_{t_{k-1}} + \sum_{j=1}^{n} b_{kj} W^{(j)}_{t-t_{k-1}}}. \tag{3.5}$$

The matrix B can be determined from Cholesky decomposition, once it is established that the matrix Σ is positive semi-definite. Hence the sufficient and necessary condition for the constructed process to exist is that

$$\Sigma \text{ is positive semi-definite.} \tag{3.6}$$

Proposition 3.1. The condition (3.4) holds for the first construction if and only if the condition (3.6) holds for the second construction.

Proof. First the "if" statement. If a symmetric matrix Σ is positive semi-definite then it has a Cholesky decomposition, i.e. there exists a lower triangular matrix L with real and non-negative diagonal entries such that $\Sigma = LL^T$. Since B is a lower triangular matrix with $\Sigma = BB^T$, we can set $L = B$ and hence $\alpha_i \leq 1$ for $i \in \{4, 5, \ldots, n\}$.
Now the "only if" statement. Let B be a lower triangular matrix with real and non-negative diagonal entries such that the symmetric matrix Σ can be written as $\Sigma = BB^T$. Then, $\Sigma = BB^T$ is Cholesky decomposition and Σ is positive semi-definite. □

To summarise these two approaches: in the first construction we started off with a continuous-time process, determined by a randomised scaled covariance matrix of incremental processes Σ, and found the family $\mathcal{A}$ from decomposition $\Sigma = BB^T$ with B being a lower triangular matrix with real and non-negative diagonal entries. In the second construction we started with the randomised scaled covariance matrix Σ that represents the dependencies between the increments of the discrete-time process and applied the Cholesky decomposition to find the lower triangular matrix B from $\Sigma = BB^T$. By "filling in" independent Brownian motions between the time steps of discrete-time process, we obtain continuous-time process that is a local Brownian motion.

3.2 Local Brownian motion as an infinite series

A different approach to construct a local Brownian motion is to use the Paley-Wiener series representation of Brownian motion. It is well known that Brownian motion $\{B_t, t \in [0,\pi]\}$ has the series representation

$$B_t = \frac{t}{\sqrt{\pi}}\xi_0 + \sqrt{\frac{2}{\pi}} \sum_{k\geq 1} \frac{\sin(kt)}{k}\xi_k$$

with a family of independent standard normal random variables $\{\xi_k, k \in \mathbb{N}_0\}$. By rescaling the time interval, we obtain that for $a < b$, the process $\{B_t, t \in [a,b]\}$ given by

$$B_{t-a} = \frac{t-a}{\sqrt{b-a}}\xi_0 + \frac{\sqrt{2(b-a)}}{\pi} \sum_{k\geq 1} \frac{\sin(\pi k \frac{t-a}{b-a})}{k}\xi_k$$

for $t \in [a,b]$ is a Brownian motion.

Let $0 = t_0 < t_1 < \ldots < t_n = T$ be a partition of $[0,T]$. For each $i \in \{1,2,\ldots,n\}$, let $\{\xi_k^{(i)}, k \in \mathbb{N}_0\}$ be a family of mutually independent standard normal random variables and suppose that for each $i \neq j$, the families $\{\xi_k^{(i)}, k \in \mathbb{N}_0\}$ and $\{\xi_k^{(j)}, k \in \mathbb{N}_0\}$ are independent as well. By "gluing" together independent Brownian motions on each interval $[t_{i-1}, t_i]$, we obtain a Brownian motion $\{X_t, t \in [0,T]\}$ such that for any $t \in [t_{i-1}, t_i]$,

$$\boxed{X_t = X_{t_{i-1}} + \frac{t-t_{i-1}}{\sqrt{t_i - t_{i-1}}}\xi_0^{(i)} + \frac{\sqrt{2(t_i - t_{i-1})}}{\pi} \sum_{k\geq 1} \frac{\sin(\pi k \frac{t-t_{i-1}}{t_i - t_{i-1}})}{k}\xi_k^{(i)}}. \tag{3.7}$$

Suppose that the above conditions on families $\{\xi_k^{(i)}, k \in \mathbb{N}_0\}$ and $\{\xi_k^{(j)}, k \in \mathbb{N}_0\}$ hold with the exception that $\xi_0^{(1)}$ and $\xi_0^{(3)}$ being not independent. Direct calculation yields that the covariance function of $\{X_t, t \in [0,T]\}$ is $\gamma_X(s,t) = \min\{s,t\}$ if $\xi_0^{(1)}$ and $\xi_0^{(3)}$ are uncorrelated, i.e. $\mathbb{E}\left[\xi_0^{(1)}\xi_0^{(3)}\right] = 0$.

Remark 3.2. Note that this condition can be extended in the following way: The process $\{X_t, t \in [0,T]\}$ defined by (3.7) is a local Brownian motion if

$$\mathbb{E}\left[\xi_0^{(i)}\xi_0^{(j)}\right] = 0$$

for any $i \neq j$. Additionally, due to Theorem 2.11, for any $i \in \{1,2,\ldots,n-1\}$, the random variables $\xi_0^{(i)}$ and $\xi_0^{(i+1)}$ are independent. Since the conditions on the joint distributions $(\xi_0^{(i)}, \xi_0^{(j)})$ for any $i \neq j$, $|i-j| \neq 1$ are the same as on the joint distribution $(\xi_0^{(1)}, \xi_0^{(3)})$ (for the process X to be a local Brownian motion), we will just concentrate on the latter joint distribution being a pair of non-independent random variables.

Since the sufficient condition for the process (3.7) to be a local Brownian motion depends purely on the joint distribution $(\xi_0^{(1)}, \xi_0^{(3)})$, we need to immerse deeper into the area of uncorrelated bivariate distributions. Using copulas and Sklar's theorem, we can impose a non-linear dependence structure on the bivariate distribution $(\xi_0^{(1)}, \xi_0^{(3)})$ and thus obtain a proper local Brownian motion.

3.2.1 Uncorrelated copulas

There is a fairly simple way to construct a pair of uncorrelated random variables. We recall that an random variable X is symmetric about zero if $X \stackrel{d}{=} -X$.

Lemma 3.3. Let (X, Y) be a bivariate distribution with identical marginal distributions. If $(X, Y) \stackrel{d}{=} (-X, Y)$ or $(X, Y) \stackrel{d}{=} (X, -Y)$, then X and Y are uncorrelated.

Proof. Let $F_{X,Y}$ be the cumulative distribution function of the vector (X, Y). If $(X, Y) \stackrel{d}{=} (-X, Y)$, then direct calculation yields

$$\mathbb{E}[XY] = \mathbb{E}[(-X)Y] = -\mathbb{E}[XY]$$

and it follows that $\mathbb{E}[XY] = 0$. Clearly, $\mathbb{E}[X] = 0$, hence the the marginal distributions are uncorrelated. For $(X, Y) \stackrel{d}{=} (X, -Y)$ the calculation is identical. Hence the assertion follows. □

Let (U, V) be a pair of random variables that are symmetric about zero. We define (X, Y) as

$$(X, Y) = \begin{cases} (U, V), & \text{w.p. } \frac{1}{2} \\ (-U, V), & \text{w.p. } \frac{1}{2}, \end{cases} \tag{3.8}$$

or

$$(X, Y) = \begin{cases} (U, V), & \text{w.p. } \frac{1}{2} \\ (U, -V), & \text{w.p. } \frac{1}{2}. \end{cases} \tag{3.9}$$

Due to Lemma 3.3, the pair (X, Y) is uncorrelated for both cases. This method of constructing a bivariate distribution allows us to obtain uncorrelated pairs of random variables given any joint distribution. It is easy to see that if we construct a new joint distribution

$$(\widetilde{X}, \widetilde{Y}) = \begin{cases} (X, Y), & \text{w.p. } \frac{1}{2} \\ (-X, Y), & \text{w.p. } \frac{1}{2}, \end{cases} \tag{3.10}$$

where (X, Y) is obtained by (3.8), we will simply end up with $(\widetilde{X}, \widetilde{Y}) \stackrel{d}{=} (X, Y)$. Analogously, the same statement can be made for pairs of random variables being constructed iteratively by (3.9). Therefore a family of joint distributions constructed by either (3.8) or (3.9) is invariant with respect to the same construction, respectively.

For the purpose of applications, it is useful to consider a joint distribution in terms of its copula. We would like to give conditions on the copulas corresponding to the uncorrelated pairs of random variables given in (3.8) and (3.9).

Definition 3.4. A (2-dimensional) copula is said to be *uncorrelated* if its corresponding joint distribution with symmetric about zero marginals is uncorrelated.

Proposition 3.5. Let (U, V) be a pair of absolutely continuous random variables with U, V being symmetric about zero. Let (X', Y') be defined as in (3.8) and (X'', Y'') be defined as in (3.9). Then, $X' \stackrel{d}{=} U \stackrel{d}{=} X''$, $Y' \stackrel{d}{=} V \stackrel{d}{=} Y''$. Further, the copula of (X', Y') satisfies

$$C'(u, v) + C'(1-u, v) = v, \tag{3.11}$$

and the copula of (X'', Y'') satisfies

$$C''(u, v) + C''(u, 1-v) = u. \tag{3.12}$$

Moreover, $(X', Y') \stackrel{d}{=} (U, V)$ if and only if the copula of (U, V) satisfies (3.11) and $(X'', Y'') \stackrel{d}{=} (U, V)$ if and only if the copula of (U, V) satisfies (3.12).

Proof. Since the proof is identical for both joint distributions, we will prove the statement only for (X', Y').

The equalities in distribution of the marginals follow immediately. Further, note that

$$\mathbb{P}(X' \le x, Y' \le y) = \frac{1}{2}\mathbb{P}(U \le x, V \le y) + \frac{1}{2}\mathbb{P}(V \le y) - \frac{1}{2}\mathbb{P}(U < -x, V \le y)$$

for any $x \in \operatorname{supp}(U)$ and $y \in \operatorname{supp}(V)$. Let C be the copula of the joint distribution (U, V). Then, by symmetry of U and Sklar's theorem,

$$\begin{aligned} C'(F_U(x), F_V(y)) &= C'(F_{X'}(x), F_{Y'}(y)) \\ &= F_{X',Y'}(x, y) = \frac{1}{2}C(F_U(x), F_V(y)) + \frac{1}{2}F_V(y) - \frac{1}{2}C(1 - F_U(x), F_V(y)). \end{aligned}$$

For $u = F_U(x)$ and $v = F_V(y)$, we obtain

$$C'(u, v) = \frac{1}{2}C(u, v) + \frac{1}{2}v - \frac{1}{2}C(1-u, v), \tag{3.13}$$

and for $u = F_U(-x)$ and $v = F_V(y)$, we obtain

$$C'(1-u, v) = \frac{1}{2}C(1-u, v) + \frac{1}{2}v - \frac{1}{2}C(u, v).$$

Adding the above two equations yields $C'(u, v) + C'(1-u, v) = v$.

For the second assertion, suppose that $(X', Y') \stackrel{d}{=} (U, V)$. Then, $C = C'$ and hence C satisfies (3.11). Conversely, if C satisfies (3.11), then from (3.13) we have $C' = C$ and $(X', Y') \stackrel{d}{=} (U, V)$ follows. ◻

Corollary 3.6. If a copula C satisfies either

$$C(u, v) + C(1-u, v) = v \quad \text{or} \quad C(u, v) + C(u, 1-v) = u,$$

then it is uncorrelated.

Definition 3.7. The class of copulas satisfying (3.11) or (3.12) will be denoted by $\mathcal{C}^b$ or $\mathcal{C}^f$, respectively. The class of copulas satisfying both conditions will be denoted by $\mathcal{C}^j$.

Remark 3.8. The notations for $\mathcal{C}^b$ (backward conditional symmetric), $\mathcal{C}^f$ (forward conditional symmetric) and $\mathcal{C}^j$ (jointly symmetric) are chosen in a way that shows their direct relation to some symmetry properties of the associated joint distributions that will be introduced in Chapter 4.

Remark 3.9. Note that some prominent and widely used families of copulas, for example the Farlie-Gumbel-Morgenstein or Archimedean copulas, have only a single representation of an uncorrelated copula, which is simply the product copula.

The next natural step is to question if we can construct an uncorrelated copula from any copula such that the constructed copula satisfies (3.11) or (3.12). To answer this question, we will need the following lemma.

Lemma 3.10. Let C be a copula. Then, for any $u, v \in [0,1]$, the functions $\widehat{C}(u,v) := v - C(1-u,v)$ and $\widetilde{C}(u,v) := u - C(u,1-v)$ are also copulas.

Proof. Clearly, for any $u, v \in [0,1]$,

$$\widehat{C}(u,0) = 0 = \widehat{C}(0,v), \qquad \widetilde{C}(u,0) = 0 = \widetilde{C}(0,v)$$

and

$$\widehat{C}(u,1) = u = \widetilde{C}(u,1), \qquad \widehat{C}(1,v) = v = \widetilde{C}(1,v).$$

Further, for $u_1, u_2, v_1, v_2 \in [0,1]$ and $u_1 \leq u_2$, $v_1 \leq v_2$,

$$\begin{aligned}
&\widehat{C}(u_1,v_1) - \widehat{C}(u_1,v_2) - \widehat{C}(u_2,v_1) + \widehat{C}(u_2,v_2)\\
&= v_1 - C(1-u_1,v_1) - (v_2 - C(1-u_1,v_2)) - (v_1 - C(1-u_2,v_1)) + v_2 - C(1-u_2,v_2)\\
&= C(1-u_1,v_2) + C(1-u_2,v_1) - C(1-u_2,v_2) - C(1-u_1,v_1) \geq 0.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\widetilde{C}(u_1,v_1) - \widetilde{C}(u_1,v_2) - \widetilde{C}(u_2,v_1) + \widetilde{C}(u_2,v_2)\\
&= u_1 - C(u_1,1-v_1) - (u_1 - C(u_1,1-v_2)) - (u_2 - C(u_2,1-v_1)) + u_2 - C(u_2,1-v_2)\\
&= C(u_2,1-v_1) - C(u_1,1-v_1) - C(u_2,1-v_2) + C(u_1,1-v_2) \geq 0.
\end{aligned}$$

It follows that the functions are 2-increasing and the assertion follows. □

Proposition 3.11. Let C be a copula. Further, for $u, v \in [0,1]$, let $\widehat{C}(u,v) = v - C(1-u,v)$ and $\widetilde{C}(u,v) = u - C(u,1-v)$. Then,

$$C^b(u,v) := \frac{1}{2}C(u,v) + \frac{1}{2}\widehat{C}(u,v) \tag{3.14}$$

$$C^f(u,v) := \frac{1}{2}C(u,v) + \frac{1}{2}\widetilde{C}(u,v). \tag{3.15}$$

are uncorrelated copulas.

Proof. Due to Lemma 3.10, $\widehat{C}$ and $\widetilde{C}$ are copulas. As a convex sum of copulas, C^b and C^f are copulas. Direct calculation yields

$$\begin{aligned}&C^b(u,v)+C^b(1-u,v)\\&=\frac{1}{2}C(u,v)+\frac{1}{2}(v-C(1-u,v))+\frac{1}{2}C(1-u,v)+\frac{1}{2}(v-C(u,v))=v,\end{aligned}$$

and

$$\begin{aligned}&C^f(u,v)+C^f(u,1-v)\\&=\frac{1}{2}C(u,v)+\frac{1}{2}(u-C(u,1-v))+\frac{1}{2}C(u,1-v)+\frac{1}{2}(u-C(u,v))=u.\end{aligned}$$

Due to Corollary 3.6 the result follows. □

Let a pair of random variables (X,Y) be defined as

$$(X,Y)=\begin{cases}(U,V), & \text{w.p. } \frac{1}{4}\\(-U,V), & \text{w.p. } \frac{1}{4}\\(U,-V), & \text{w.p. } \frac{1}{4}\\(-U,-V), & \text{w.p. } \frac{1}{4}.\end{cases} \tag{3.16}$$

Clearly, this construction satisfies both constructions (3.8) and (3.9). We can construct a copula associated with bivariate distribution given in (3.16) from the copula of (U,V).

Proposition 3.12. Let C be a copula. Further, let $\widehat{C}(u,v)=v-C(1-u,v)$, $\widetilde{C}(u,v)=u-C(u,1-v)$ and $C^s(u,v)=u+v-1+C(1-u,1-v)$. Then,

$$C^j(u,v)=\frac{1}{4}C(u,v)+\frac{1}{4}\widehat{C}(u,v)+\frac{1}{4}\widetilde{C}(u,v)+\frac{1}{4}C^s(u,v) \tag{3.17}$$

is an uncorrelated copula.

Proof. Use Lemma 3.10 and the fact that a convex sum of copulas yields a copula. Direct calculations show

$$\begin{aligned}&C^j(u,v)+C^j(1-u,v)\\&=\frac{1}{4}\Big(C(u,v)+v-C(1-u,v)+u-C(u,1-v)+u+v-1+C(1-u,1-v)\Big)\\&\quad+\frac{1}{4}\Big(C(1-u,v)+v-C(u,v)+1-u-C(1-u,1-v)+1-u+v-1\Big)\\&\quad+\frac{1}{4}C(u,1-v)=v,\end{aligned}$$

and

$$\begin{aligned}&C^j(u,v)+C^j(u,1-v)\\&=\frac{1}{4}\Big(C(u,v)+v-C(1-u,v)+u-C(u,1-v)+u+v-1+C(1-u,1-v)\Big)\\&\quad+\frac{1}{4}\Big(C(u,1-v)+1-v-C(1-u,1-v)+u-C(u,v)+u+1-v-1\Big)\\&\quad+\frac{1}{4}C(1-u,v)=u.\end{aligned}$$

Due to Corollary 3.6 the result follows. □

Corollary 3.13. Let $C^b \in \mathcal{C}^b$. Let C be a copula constructed from C^b by using (3.15). Then, $C \in \mathcal{C}^j$. Similarly, let $C^f \in \mathcal{C}^f$ and C be a copula constructed from C^f by (3.14). Then, $C \in \mathcal{C}^j$.

Note that the copula C^s in Proposition 3.12 is called *survival copula*, since it represents the probability of each marginal distribution to "survive" beyond some threshold, i.e. if C is the copula of (X, Y), then

$$\mathbb{P}(X > F_X^{-1}(u), Y > F_Y^{-1}(v)) = C^s(1-u, 1-v) = 1 - u - v + C(u, v).$$

The family of joint distributions associated with copulas satisfying (3.17) is invariant with respect to the construction (3.8), (3.9) and (3.16).

Example 3.14 (Uncorrelated copulas built from Archimedean Copulas)**.** An Archimedean (2-dimensional) copula with parameter θ and generator ψ admits the representation

$$C(u, v; \theta) = \psi^{[-1]}(\psi(u; \theta) + \psi(v; \theta); \theta),$$

where $\psi : [0, 1] \times \Theta \to [0, \infty)$ is a continuous, strictly decreasing and convex function such that $\psi(1; \theta) = 0$, and $\psi^{[-1]}$ is its pseudo-inverse defined by

$$\psi^{[-1]}(t; \theta) = \begin{cases} \psi^{-1}(t; \theta), & 0 \le t \le \psi(0; \theta) \\ 0, & \textit{otherwise.} \end{cases}$$

Then, the functions

$$\begin{aligned} C^b(u, v) &= \frac{1}{2}C(u, v) + \frac{1}{2}(v - C(1-u, v)) \\ &= \frac{1}{2}\left(v + \psi^{[-1]}\left(\psi(u) + \psi(v)\right) - \psi^{[-1]}\left(\psi(1-u) + \psi(v)\right)\right), \\ C^f(u, v) &= \frac{1}{2}C(u, v) + \frac{1}{2}(u - C(u, 1-v)) \\ &= \frac{1}{2}\left(u + \psi^{[-1]}\left(\psi(u) + \psi(v)\right) - \psi^{[-1]}\left(\psi(u) + \psi(1-v)\right)\right) \end{aligned}$$

and

$$\begin{aligned} C^j(u, v) &= \frac{1}{2}(C^b(u, v) + C^f(u, v)) - \frac{1}{4}C(u, v) + \frac{1}{4}\left(u + v - 1 + C(1-u, 1-v)\right) \\ &= \frac{1}{4}\left(v + u + 2\psi^{[-1]}\left(\psi(u) + \psi(v)\right) - \psi^{[-1]}\left(\psi(1-u) + \psi(v)\right) - \psi^{[-1]}\left(\psi(u) + \psi(1-v)\right)\right) \\ &\quad - \frac{1}{4}\left(\psi^{[-1]}(\psi(u) + \psi(v)) - u - v + 1 - \psi^{[-1]}(\psi(1-u) + \psi(1-v))\right) \\ &= \frac{1}{4}\left(2u + 2v - 1 + \psi^{[-1]}(\psi(u) + \psi(v)) - \psi^{[-1]}(\psi(1-u) + \psi(v))\right) \\ &\quad - \frac{1}{4}\left(\psi^{[-1]}(\psi(u) + \psi(1-v)) - \psi^{[-1]}(\psi(1-u) + \psi(1-v))\right) \end{aligned}$$

are uncorrelated copulas for any pair of random variables symmetric about zero.

Note that due to Remark 3.9, the copulas constructed in the example above are not Archimedean.

Example 3.15 (Clayton and Gumbel uncorrelated copulas)**.** Clayton and Gumbel copulas are particular examples of Archimedean copulas with generators $\psi(t;\theta) = \frac{1}{\theta}(t^{-\theta} - 1)$ for $\theta \in [-1, \infty) \setminus \{0\}$ and $\psi(t;\theta) = (-\ln(t))^{\theta}$ for $\theta \in [1, \infty)$, respectively. Thus we obtain the following uncorrelated copulas:

$$
\begin{aligned}
C^{b}_{Cl}(u,v) &= \frac{1}{2}\left(v + \left[\max\left\{u^{-\theta} + v^{-\theta} - 1; 0\right\}\right]^{-1/\theta} - \left[\max\left\{(1-u)^{-\theta} + v^{-\theta} - 1; 0\right\}\right]^{-1/\theta}\right) \\
C^{f}_{Cl}(u,v) &= \frac{1}{2}\left(u + \left[\max\left\{u^{-\theta} + v^{-\theta} - 1; 0\right\}\right]^{-1/\theta} - \left[\max\left\{u^{-\theta} + (1-v)^{-\theta} - 1; 0\right\}\right]^{-1/\theta}\right) \\
C^{j}_{Cl}(u,v) &= \frac{1}{4}\left(2u + 2v - 1 + \left[\max\left\{u^{-\theta} + v^{-\theta} - 1; 0\right\}\right]^{-1/\theta}\right) \\
&\quad - \frac{1}{4}\left(\left[\max\left\{(1-u)^{-\theta} + v^{-\theta} - 1; 0\right\}\right]^{-1/\theta}\right) \\
&\quad - \frac{1}{4}\left(\left[\max\left\{u^{-\theta} + (1-v)^{-\theta} - 1; 0\right\}\right]^{-1/\theta}\right) \\
&\quad + \frac{1}{4}\left(\left[\max\left\{(1-u)^{-\theta} + (1-v)^{-\theta} - 1; 0\right\}\right]^{-1/\theta}\right) \\
C^{b}_{Gum}(u,v) &= \frac{1}{2}\left(v + \exp\left[-\left((-\log(u))^{\theta} + (-\log(v))^{\theta}\right)^{1/\theta}\right]\right) \\
&\quad - \frac{1}{2}\left(\exp\left[-\left((-\log(1-u))^{\theta} + (-\log(v))^{\theta}\right)^{1/\theta}\right]\right) \\
C^{f}_{Gum}(u,v) &= \frac{1}{2}\left(u + \exp\left[-\left((-\log(u))^{\theta} + (-\log(v))^{\theta}\right)^{1/\theta}\right]\right) \\
&\quad - \frac{1}{2}\left(\exp\left[-\left((-\log(u))^{\theta} + (-\log(1-v))^{\theta}\right)^{1/\theta}\right]\right) \\
C^{j}_{Gum}(u,v) &= \frac{1}{4}\left(2u + 2v - 1 + \exp\left[-\left((-\log(u))^{\theta} + (-\log(v))^{\theta}\right)^{1/\theta}\right]\right) \\
&\quad - \frac{1}{4}\left(\exp\left[-\left((-\log(1-u))^{\theta} + (-\log(v))^{\theta}\right)^{1/\theta}\right]\right) \\
&\quad - \frac{1}{4}\left(\exp\left[-\left((-\log(u))^{\theta} + (-\log(1-v))^{\theta}\right)^{1/\theta}\right]\right) \\
&\quad + \frac{1}{4}\left(\exp\left[-\left((-\log(1-u))^{\theta} + (-\log(v))^{\theta}\right)^{1/\theta}\right]\right).
\end{aligned}
$$

Let us recall that a convex linear combination of copulas is a finite linear combination of copulas with non-negative coefficients that add up to one. We can build new uncorrelated copulas from the ones introduced above by building convex linear combinations. The concept of convex linear combination of copulas can be extended to the infinite convex set of copulas indexed by a continuous parameter. This extension is called *convex sums*. The parameter set can be seen as the support of a continuous random variable and the coefficients are the values of the density. Since the probability density is non-negative and the integral of the density over all parameter space equals one, the convex property is automatically satisfied.

Proposition 3.16. Let $\{\theta_i, i \in \{1, 2, \dots, n\}\}$ be a family of non-negative real numbers such that $\sum_{i=1}^n \theta_i = 1$ and Θ be a continuous random variable. Further, let $\{C_{\theta_i}, i \in \{1, 2, \dots, n\}\}$ and $\{C_\theta, \theta \in \operatorname{supp}(\Theta)\}$ be two families of uncorrelated copulas. Then, both functions

$$C(u,v) = \sum_{i=1}^n \theta_i C_{\theta_i}(u,v) \quad \text{and} \quad C^\Theta(u,v) = \int_{\mathbb{R}} C_\theta(u,v) dF_\Theta(\theta)$$

are uncorrelated copulas. Moreover, the classes $\mathcal{C}^b$ and $\mathcal{C}^f$ are invariant with respect to convex linear combination and convex sums.

Proof. It is clear that any convex sum of copulas is a copula. Let $c(u,v)$ be the density of $C(u,v)$ and, for each $i \in \{1, 2, \dots, n\}$, $c_{\theta_i}(u,v)$ be the density of $C_{\theta_i}(u,v)$. Then,

$$\int_{\mathbb{R}} \int_{\mathbb{R}} xy f_X(x) f_Y(y) c(x,y) dx dy = \sum_{i=1}^n \theta_i \int_{\mathbb{R}} \int_{\mathbb{R}} xy f_X(x) f_Y(y) c_{\theta_i}(x,y) dx dy,$$

which equals to zero if each copula from the family $\{C_{\theta_i}, i \in \{1, 2, \dots, n\}\}$ is uncorrelated. Further, let c^Θ be the density of C^Θ and c_θ be the density of C_θ. We see immediately that

$$\int_{\mathbb{R}} \int_{\mathbb{R}} xy f_X(x) f_Y(y) c^\Theta(x,y) dx dy = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \int_{\mathbb{R}} xy f_X(x) f_Y(y) c_\theta(x,y) dx dy \right) f_\Theta(\theta) d\theta,$$

which equals to zero if each copula from the family $\{C_\theta, \theta \in \operatorname{supp}(\Theta)\}$ is uncorrelated.

Finally, for $\{C_{\theta_i}, i \in \{1, 2, \dots, n\}\} \in \mathcal{C}^b$, we have

$$C(u,v) + C(1-u,v) = \sum_{i=1}^n \theta_i C_{\theta_i}(u,v) + \sum_{i=1}^n \theta_i C_{\theta_i}(1-u,v) = v \sum_{i=1}^n \theta_i = v,$$

and, for $\{C_\theta, \theta \in \operatorname{supp}(\Theta)\} \in \mathcal{C}^b$,

$$C^\Theta(u,v) + C^\Theta(1-u,v) = \int_{\mathbb{R}} C_\theta(u,v) dF_\Theta(\theta) + \int_{\mathbb{R}} C_\theta(1-u,v) dF_\Theta(\theta) = v \int_{\mathbb{R}} dF_\Theta(\theta) = v.$$

The proof for $\mathcal{C}^f$ is analogous. □

The distribution of a random variable Θ is also called *the mixing distribution* of the parametrised family of copulas (see Nelsen [42], p. 72).

Chapter 4

General representation

In this chapter we will take a general approach towards the constructions of local Brownian motions. We will use symmetric classification of joint distributions that will impose pairwise dependencies between incremental processes of local Brownian motions. Further, we will explore the question whether a local Brownian motion is a Markov process and present several partial results.

4.1 Representation of local Brownian motion

Let $\{X_t, t \in [0,T]\}$ be a local Brownian motion with time partition $0 = t_0 < t_1 < \ldots < t_n = T$. Then, it has the following general representation

$$\boxed{X_t = \begin{cases} \widetilde{W}_t^{(1)}, & t \in [0,t_1] \\ X_{t_1} + \widetilde{W}_{t-t_1}^{(2)}, & t \in [t_1,t_2] \\ \cdots & \\ X_{t_{n-1}} + \widetilde{W}_{t-t_{n-1}}^{(n)}, & t \in [t_{n-1},T], \end{cases}} \tag{4.1}$$

where the process $\{\widetilde{W}_{t-t_{i-1}}^{(i)}, t \in [t_{i-1},t_i]\}$ is a standard Brownian motion for each $i \in \{1,2,\ldots,n\}$. This representation follows directly from the properties of local Brownian motion (continuity and appropriate quadratic variation) and Theorem 2.11, i.e. if the process $\{X_t, t \in [0,T]\}$ is a local Brownian motion, then there exists a finite time partition $0 = t_0 < t_1 < \ldots < t_n = T$ such that the process $\{X_t - X_{t_{i-1}}, t \in [t_{i-1},t_i]\}$ is a Brownian motion for each $i \in \{1,2,\ldots,n\}$ with $Cov(X_s - X_{t_{i-1}}, X_t - X_{t_{i-1}}) = \min\{s,t\} - t_{i-1}$. Due to Proposition 2.15,

$$[X_\cdot - X_{t_{i-1}}, X_\cdot - X_{t_{i-1}}]([t_{i-1},t]) = t - t_{i-1}.$$

It follows that $\{X_t - X_{t_{i-1}}, t \in [t_{i-1}, t_i]\}$ is a Brownian motion with quadratic variation $t - t_{i-1}$, which is represented by $\widetilde{W}^{(i)}_{t-t_{i-1}}$ in (4.1).

Remark 4.1. We chose the notation $\widetilde{W}^{(i)}$ for the Brownian motions in (4.1) to avoid the possibility of the confusion with Brownian motion $W^{(i)}$ from Example 2.10.

For $i \neq j$, $s \in [t_{i-1}, t_i]$ and $t \in [t_{j-1}, t_j]$, we write

$$F^{(i,j)}_{s-t_{i-1},t-t_{j-1}}(x, y) \quad \text{for} \quad F_{\widetilde{W}^{(i)}_{s-t_{i-1}},\widetilde{W}^{(j)}_{t-t_{j-1}}}(x, y),$$

for all $x, y \in \mathbb{R}$. It is clear that if for any $i \neq j$, $s \in [t_{i-1}, t_i]$ and $t \in [t_{j-1}, t_j]$, the joint distribution $F^{(i,j)}_{s-t_{i-1},t-t_{j-1}}$ is a product measure, then the local Brownian motion $\{X_t, t \in [0, T]\}$ is a Brownian motion. According to Theorem 2.11, for any $i \in \{1, 2, \ldots, n\}$, the neighbouring Brownian motions $\widetilde{W}^{(i-1)}$ and $\widetilde{W}^{(i)}$ are independent. The joint distributions of non-neighbouring Brownian motions are bound to ensure that the covariance function of the local Brownian motion is satisfied. Hence the necessary condition for the process X in (4.1) to be a local Brownian motion is

$$\mathbb{E}[(X_s - X_{t_{i-1}})(X_t - X_{t_{j-1}})] = \mathbb{E}[\widetilde{W}^{(i)}_{s-t_{i-1}}\widetilde{W}^{(j)}_{t-t_{j-1}}] = 0$$

for any $i \neq j$, $s \in [t_{i-1}, t_i]$ and $t \in [t_{j-1}, t_j]$.

Let us recollect a few definitions on the symmetries of joint distribution that were taken from Nelsen [42].

Definition 4.2. An absolutely continuous joint distribution (X, Y) is said to be *conditionally symmetric (around zero) in one variable* if for all $x \in \operatorname{supp}(X)$ and $y \in \operatorname{supp}(Y)$, either $f_{X|Y}(x|y) = f_{X|Y}(-x|y)$ or $f_{Y|X}(y|x) = f_{Y|X}(-y|x)$ holds. An absolutely continuous joint distribution (X, Y) is said to be *jointly symmetric* if for all $x, y \in \mathbb{R}$,

$$f_{X,Y}(x, y) = f_{X,Y}(-x, y) = f_{X,Y}(x, -y) = f_{X,Y}(-x, -y).$$

Proposition 4.3. An absolutely continuous joint distribution is jointly symmetric if and only if it is conditionally symmetric in both variables.

Proof. See Nelsen [41]. □

The next lemma provides us with a few sufficient conditions on the joint distributions of the Brownian motions $\widetilde{W}^{(i)}$ such that the covariance function for a local Brownian motion is satisfied.

Lemma 4.4. Let $\{\widetilde{W}^{(i)}_{t-t_{i-1}}, t \in [t_{i-1}, t_i]\}$ and $\{\widetilde{W}^{(j)}_{t-t_{j-1}}, t \in [t_{j-1}, t_j]\}$ be two Brownian motions from representation (4.1). For each $i \neq j$, $s \in [t_{i-1}, t_i]$ and $t \in [t_{j-1}, t_j]$, considere the following statements:

(i) $F^{(i,j)}_{s-t_{i-1},t-t_{j-1}}$ be jointly symmetric,

(ii) $F^{(i,j)}_{s-t_{i-1},t-t_{j-1}}$ be conditional symmetric in one variable,

(iii) $\mathbb{E}[\widetilde{W}^{(i)}_{s-t_{i-1}}\widetilde{W}^{(j)}_{t-t_{j-1}}] = 0$.

Then, (i) implies (ii), which in turn implies (iii).

Proof. It follows immediately from Proposition 4.3 and Lemma 3.3. □

Since we are dealing with the joint distributions of random variables of a process (which has time-ordering), we would like to distinguish between the conditional symmetries in the first and in the second variables depending on the time of the variable the joint distribution is conditioned on.

Definition 4.5. Consider the joint distribution $F^{(i,j)}_{s-t_{i-1},t-t_{j-1}}$ for $i < j$, $s \in [t_{i-1}, t_i]$, $t \in [t_{j-1}, t_j]$. The distribution is said to be *backward conditionally symmetric* if it is symmetric in the first variable (of the earlier time) conditioning on the second variable. It is said to be *forward conditionally symmetric* if it is symmetric in the second variable (of the later time) conditioning on the first variaable.

Following the definitions above for the joint distributions, we introduce corresponding definitions for the associated copulas.

Definition 4.6. A copula is said to be *jointly symmetric* or *backward/forward conditionally symmetric* if the associated joint distribution is jointly symmetric or backward/forward conditionally symmetric, respectively.

Definition 4.7. A local Brownian motion is said to be *jointly symmetric* or *backward/forward conditionally symmetric* if for all $i \neq j$, $s \in [t_{i-1}, t_i]$ and $t \in [t_{j-1}, t_j]$, the joint distribution $F^{(i,j)}_{s-t_{i-1},t-t_{j-1}}$ is jointly symmetric or backward/forward conditionally symmetric, respectively.

We recall that due to Proposition 3.5, for a backward (or forward) conditionally symmetric copula, the equation $C(u,v) + C(1-u,v) = v$ (or $C(u,v) + C(u,1-v) = u$) holds for all $u, v \in [0,1]$. Due to Proposition 4.3, for a jointly symmetric copula both equation must hold. In Chapter 3, we showed that it is possible to construct a backward/forward conditionally symmetric copula (and a jointly symmetric copula) from any copula. Hence the class of uncorrelated copulas is quite rich.

Remark 4.8. Note that in this thesis we narrow our approach of constructing a local Brownian motion on pairwise dependencies of incremental processes. Of course, one could go beyond that and look at the n-dimensional distribution of $(\widetilde{W}^{(1)}, \widetilde{W}^{(2)}, \dots, \widetilde{W}^{(n)})$, where the joint distributions of $(\widetilde{W}^{(i)}, \widetilde{W}^{(i+1)})$ are product measures for all $i \in \{1, 2, \dots, n-1\}$ and the joint distributions of $(\widetilde{W}^{(i)}, \widetilde{W}^{(j)})$ are uncorrelated for $i \neq j$.

4.2 More on randomised scaled covariances

Now we would like to know how the first construction via randomised schaled covariances between two non-neighbouring incremental processes fits into our general framework.

Lemma 4.9. Let R be a random variable distributed on $[-1,1]$. Then, for all $x, y \in \mathbb{R}$, any positive even function g and any non-constant function h, the following holds:

$$\mathbb{E}\left[h\left(\frac{x-Ry}{\sqrt{1-R^2}}\right)g(R)\right] = \mathbb{E}\left[h\left(\frac{x+Ry}{\sqrt{1-R^2}}\right)g(R)\right]$$

if and only if R is symmetric about zero.

Proof. Direct calculation yields

$$\begin{aligned}
&\int_{-1}^{1}\left(h\left(\frac{x-ry}{\sqrt{1-r^2}}\right)-h\left(\frac{x+ry}{\sqrt{1-r^2}}\right)\right)g(r)dF_R(r)\\
&=\int_{0}^{1}\left(h\left(\frac{x-ry}{\sqrt{1-r^2}}\right)-h\left(\frac{x+ry}{\sqrt{1-r^2}}\right)\right)g(r)dF_R(r)\\
&\quad-\int_{1}^{0}\left(h\left(\frac{x-ry}{\sqrt{1-r^2}}\right)-h\left(\frac{x+ry}{\sqrt{1-r^2}}\right)\right)g(-r)dF_R(-r)\\
&=\int_{0}^{1}\left(h\left(\frac{x-ry}{\sqrt{1-r^2}}\right)-h\left(\frac{x+ry}{\sqrt{1-r^2}}\right)\right)g(r)dF_R(r)\\
&\quad+\int_{0}^{1}\left(h\left(\frac{x-ry}{\sqrt{1-r^2}}\right)-h\left(\frac{x+ry}{\sqrt{1-r^2}}\right)\right)g(r)dF_R(-r)\\
&=\int_{0}^{1}\left(h\left(\frac{x-ry}{\sqrt{1-r^2}}\right)-h\left(\frac{x+ry}{\sqrt{1-r^2}}\right)\right)g(r)d\left(F_R(r)+F_R(-r)\right),
\end{aligned}$$

which equals to zero for all $x, y \in \mathbb{R}$ if and only if R is symmetric about zero. □

Proposition 4.10. Let $\{X_t, t \in [0,T]\}$ be the local Brownian motion from Example 2.10. Then, for $s \in (0,1]$ and $t > 2$, the joint distribution of $(X_s, X_t - X_2)$ is jointly symmetric if and only if the random variable R is symmetric about zero. In that case, the process $\{X_t, t \in [0,T]\}$ is a jointly symmetric local Brownian motion.

Proof. Let us compute the joint probability density function of $(X_s, X_t - X_2)$:

$$\begin{aligned}
f_{X_s,X_t-X_2}(x,y) &= f_{W_s^{(1)},RW_{t-2}^{(1)}+\sqrt{1-R^2}W_{t-2}^{(3)}}(x,y)\\
&=\int_{-1}^{1} f_{r(W_{t-2}^{(1)}-W_s^{(1)})+\sqrt{1-r^2}W_{t-2}^{(3)}+rx|R,W_s^{(1)}}(y|r,x)f_{W_s^{(1)}}(x)dF_R(r)\\
&=\int_{-1}^{1} f_{r(W_{t-2}^{(1)}-W_s^{(1)})+\sqrt{1-r^2}W_{t-2}^{(3)}+rx}(y)f_{W_s^{(1)}}(x)dF_R(r).
\end{aligned}$$

Let $\xi = W_{t-2}^{(1)} - W_s^{(1)}$ which has distribution $\mathcal{N}(0, |t-2-s|)$. Then, for $t \neq s+2$,

$$f_{X_s,X_t-X_2}(x,y) = \int_{\mathbb{R}} \mathbb{E}\left[f_{W_{t-2}^{(3)}}\left(\frac{y-R(x+z)}{\sqrt{1-R^2}}\right)\frac{1}{\sqrt{1-R^2}}\right] f_\xi(z)dz f_{W_s^{(1)}}(x), \tag{4.2}$$

and for $t = s + 2$,

$$\begin{aligned} f_{X_s, X_t - X_2}(x, y) &= \int_{-1}^{1} f_{W_{t-2}^{(3)}}\left(\frac{y - rx}{\sqrt{1-r^2}}\right) \frac{1}{\sqrt{1-r^2}} dF_R(r) f_{W_s^{(1)}}(x) \\ &= \mathbb{E}\left[f_{W_{t-2}^{(3)}}\left(\frac{y - Rx}{\sqrt{1-R^2}}\right) \frac{1}{\sqrt{1-R^2}}\right] f_{W_s^{(1)}}(x). \end{aligned}$$

Suppose that R is symmetric about zero, that is, for all $y, u \in \mathbb{R}$, the equation

$$\mathbb{E}\left[f_{W_{t-2}^{(3)}}\left(\frac{y - Ru}{\sqrt{1-R^2}}\right) \frac{1}{\sqrt{1-R^2}}\right] = \mathbb{E}\left[f_{W_{t-2}^{(3)}}\left(\frac{y + Ru}{\sqrt{1-R^2}}\right) \frac{1}{\sqrt{1-R^2}}\right] \tag{4.3}$$

holds. Then, for all $t \neq s + 2$ and $x, y \in \mathbb{R}$, the conditional density function from (4.2) is

$$\begin{aligned} f_{X_t - X_2 | X_s}(y|x) &= \int_{\mathbb{R}} \mathbb{E}\left[f_{W_{t-2}^{(3)}}\left(\frac{y - R(x+z)}{\sqrt{1-R^2}}\right) \frac{1}{\sqrt{1-R^2}}\right] f_\xi(z) dz \\ &= \int_{\mathbb{R}} \mathbb{E}\left[f_{W_{t-2}^{(3)}}\left(\frac{y + R(x+z)}{\sqrt{1-R^2}}\right) \frac{1}{\sqrt{1-R^2}}\right] f_\xi(z) dz, \text{ (due to (4.3))} \\ &= \int_{\mathbb{R}} \mathbb{E}\left[f_{W_{t-2}^{(3)}}\left(\frac{-y - R(x+z)}{\sqrt{1-R^2}}\right) \frac{1}{\sqrt{1-R^2}}\right] f_\xi(z) dz \\ &= f_{X_t - X_2 | X_s}(-y|x), \end{aligned}$$

where the third equality is due to the fact that the density $f_{W_{t-2}^{(3)}}$ is an even function. Thus, $(X_s, X_t - X_2)$ is forward conditionally symmetric.

Moreover, from (4.2) and (4.3), we also have, for all $x, y \in \mathbb{R}$,

$$\begin{aligned} f_{X_s | X_t - X_2}(x|y) &= \int_{\mathbb{R}} \mathbb{E}\left[f_{W_{t-2}^{(3)}}\left(\frac{y - R(x+z)}{\sqrt{1-R^2}}\right) \frac{1}{\sqrt{1-R^2}}\right] f_\xi(z) dz \frac{f_{W_s^{(1)}}(x)}{f_{X_t - X_2}(y)} \\ &= \int_{\mathbb{R}} \mathbb{E}\left[f_{W_{t-2}^{(3)}}\left(\frac{y + R(-x-z)}{\sqrt{1-R^2}}\right) \frac{1}{\sqrt{1-R^2}}\right] f_\xi(z) dz \frac{f_{W_s^{(1)}}(x)}{f_{X_t - X_2}(y)} \\ &= -\int_{\mathbb{R}} \mathbb{E}\left[f_{W_{t-2}^{(3)}}\left(\frac{y + R(-x+z)}{\sqrt{1-R^2}}\right) \frac{1}{\sqrt{1-R^2}}\right] f_\xi(-z) d(-z) \frac{f_{W_s^{(1)}}(x)}{f_{X_t - X_2}(y)} \\ &= \int_{\mathbb{R}} \mathbb{E}\left[f_{W_{t-2}^{(3)}}\left(\frac{y + R(-x+z)}{\sqrt{1-R^2}}\right) \frac{1}{\sqrt{1-R^2}}\right] f_\xi(z) dz \frac{f_{W_s^{(1)}}(x)}{f_{X_t - X_2}(y)} \\ &= \int_{\mathbb{R}} \mathbb{E}\left[f_{W_{t-2}^{(3)}}\left(\frac{y - R(-x+z)}{\sqrt{1-R^2}}\right) \frac{1}{\sqrt{1-R^2}}\right] f_\xi(z) dz \frac{f_{W_s^{(1)}}(x)}{f_{X_t - X_2}(y)} \text{ (due to (4.3))} \\ &= f_{X_s | X_t - X_2}(-x|y), \end{aligned}$$

where the forth equation is true, due to the fact that the density f_ξ is an even function. It follows that $(X_s, X_t - X_2)$ is backward conditional symmetric. Therefore, $(X_s, X_t - X_2)$ is jointly symmetric.

Furthermore, suppose that for all $x, y \in \mathbb{R}$, we have $f_{X_t - X_2 | X_s}(y|x) = f_{X_t - X_2 | X_s}(-y|x)$, i.e.

$$\int_{\mathbb{R}} \mathbb{E}\left[\left(f_{W_{t-2}^{(3)}}\left(\frac{y - R(x+z)}{\sqrt{1-R^2}}\right) - f_{W_{t-2}^{(3)}}\left(\frac{y + R(x+z)}{\sqrt{1-R^2}}\right)\right) \frac{1}{\sqrt{1-R^2}}\right] f_\xi(z) dz = 0.$$

Since the equality holds for all $x, y \in \mathbb{R}$, it follows immediately that (4.3) must be true.

Similarly, suppose that for any $x, y \in \mathbb{R}$, $f_{X_s|X_t-X_2}(x|y) = f_{X_s|X_t-X_2}(-x|y)$, i.e.

$$\int_{\mathbb{R}} \mathbb{E}\left[\left(f_{W^{(3)}_{t-2}}\left(\frac{y-R(x+z)}{\sqrt{1-R^2}}\right) - f_{W^{(3)}_{t-2}}\left(\frac{y-R(-x+z)}{\sqrt{1-R^2}}\right)\right)\frac{1}{\sqrt{1-R^2}}\right] f_\xi(z)dz = 0.$$

Since the equation above holds for any $x, y \in \mathbb{R}$, it follows that equation (4.3) holds. Finally, applying Lemma 4.9 with $g(R) = \frac{1}{\sqrt{1-R^2}}$ and $h = f_{W^{(3)}_{t-2}}$ the equation (4.3) holds if and only if R is symmetric about zero. The identical proof for $t = s+2$ is omitted. □

Remark 4.11. Note that any symmetric about zero random variable that is distributed on the interval $[-a, a]$ has zero mean.

The proposition above leads to the following statement:

Corollary 4.12. Let $\{X_t, t \in [0, T]\}$ be a local Brownian motion from Example 2.10. Then, the following statements hold:

(i) If R nonsymmetric, the process $\{X_t, t \in [0, T]\}$ is neither a backward nor a forward conditional symmetric local Brownian motion.

(ii) If R is symmetric, the process $\{X_t, t \in [0, T]\}$ is jointly symmetric local Brownian motion.

Remark 4.13. The corollary above can be extended to the general case of the local Brownian motions generated by randomised scaled covariances between non-neighbouring incremental processes, which were discussed in Chapter 3. That is, the family $\mathcal{R}$ consists only of symmetric random variables about zero in (3.3) if and only if the local Brownian motion in (3.3) is jointly symmetric.

4.3 On Markovianity of local Brownian motion

The relatively loose restriction on the joint distributions of non-neighbouring Brownian motions in (4.1) gives us enough freedom to construct rich enough family of local Brownian motions. The inconvenience may lie, however, in the inability to derive some theoretical statements for all local Brownian motions. One of the questions we could only answer partially is whether a proper local Brownian motion can be a Markov process. Although we were not able to find a general answer to this question, some partial results were obtained and will be presented below.

Proposition 4.14. Let $\{X_t, t \in [0, T]\}$ be a local Brownian motion with time partition $0 = t_0 < t_1 < \ldots < t_n = T$. Assume that for each $i \in \{1, 2, \ldots, n\}$, the process $\{X_t - X_{t_{i-1}}, t \in [t_{i-1}, t_i]\}$ is independent of $X_{t_{i-1}}$. Then, the process $\{X_t, t \in [0, T]\}$ is Markov if and only if it is a Brownian motion.

Proof. The "if " statement is clear.

Now the "only if" statement. Suppose that $\{X_t, t \in [0,T]\}$ is a Markov process. Then, for $0 < s < t_2 < t \le t_3$, the Chapman-Kolmogorov equation yields

$$\begin{aligned} f_{X_t|X_s}(y|x) &= \int_{\mathbb{R}} f_{X_{t_2}|X_s}(z|x) f_{X_t|X_{t_2}}(y|z)dz \\ &= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi(t_2-s)}} e^{-\frac{(z-x)^2}{2(t_2-s)}} \frac{1}{\sqrt{2\pi(t-t_2)}} e^{-\frac{(y-z)^2}{2(t-t_2)}} dz \\ &= \frac{1}{\sqrt{2\pi(t-s)}} e^{-\frac{(y-x)^2}{2(t-s)}}. \end{aligned}$$

This transition function of a Markov process shows that $\{X_t, t \in [0,t_3]\}$ is a Brownian motion. Applying the same arguments iteratively from interval $[0,t_j]$ to $[0,t_{j+1}]$ gives the assertion. □

Remark 4.15. Except for the Brownian motion we were not able to find another local Brownian motion $\{X_t, t \in [0,T]\}$ such that for each index $i \in \{1,2,\ldots,n\}$ of the time partition $0 = t_0 < t_1 < \ldots < t_n = T$, the process $\{X_t - X_{t_{i-1}}, t \in [t_{i-1}, t_i]\}$ is independent of $X_{t_{i-1}}$. The existence of such process remains an open question.

Notice that a local Brownian motion is a Brownian motion if and only if for any $t_1 \in (0,T)$, $0 = t_0 < t_1 < t_2 = T$ is the time partition of the local Brownian motion.

Proposition 4.16. Let $\{X_t, t \in [0,T]\}$ be a local Brownian motion. If for any time partition $0 = t_0 < t_1 < \ldots < t_n = T$ of X, the process $\{X_t - X_{t_{i-1}}, t \in [t_{i-1}, t_{i+2}]\}$ is a Brownian motion for each $i \in \{1,2,\ldots,n-2\}$, then the process $\{X_t, t \in [0,T]\}$ is a Brownian motion.

Proof. If the process $\{X_t - X_{t_{i-1}}, t \in [t_{i-1}, t_{i+2}]\}$ is a Brownian motion for each $i \in \{1,2,\ldots,n-2\}$, then the partition $0 = t_0 < t_1 < t_3 < t_4 < t_6 < t_7 < \ldots < t_n = T$ is also a time partition of $\{X_t, t \in [0,T]\}$. Applying the assumption for the second time, we can see that the partition $0 = t_0 < t_1 < t_4 < t_5 < t_8 < t_9 < \ldots < t_n = T$ is a time partition of $\{X_t, t \in [0,T]\}$ as well. Using the same argument iteratively, we obtain a time partition $0 = t_0 < t_1 < t_n = T$, which indicates that the process $\{X_t, t \in [0,T]\}$ is a Brownian motion. □

To obtain our second partial result, we will impose a condition on local Brownian motion that is satisfied by many local Brownian motions constructed in this thesis. We then derive a statement that a proper local Brownian motion is not a Markov process if it satisfies this extra condition.

Lemma 4.17. Let X, Y, Z be three absolutely continuous random variables with full support on the real line. Further, let Y be independent of (X, Z) and Z be independent of Y given $X + Y$. Then, Z is independent of X and $X + Y$.

Proof. Using the Bayes' formula for probability density functions, we obtain that for any $z, y, w \in \mathbb{R}$,

$$\begin{aligned} f_{Z,Y|X+Y}(z, y|w) &= \frac{f_{Z,Y,X+Y}(z, y, w)}{f_{X+Y}(w)} \\ &= \frac{f_{Z,X+Y|Y}(z, w|y) f_Y(y)}{f_{X+Y}(\xi)} = \frac{f_{Z,X}(z, w-y) f_Y(y)}{f_{X+Y}(w)} \end{aligned} \tag{4.4}$$

and furthermore,

$$f_{Y|X+Y}(y|w) = \frac{f_{X+Y,Y}(w, y)}{f_{X+Y}(w)} = \frac{f_X(w-y) f_Y(y)}{f_{X+Y}(w)}. \tag{4.5}$$

However, due to independence of Z and Y given $X + Y$, for any $x, z \in \mathbb{R}$, we also have

$$f_{Z,Y|X+Y}(z, y|w) = f_{Z|X+Y}(z|w) f_{Y|X+Y}(y|w). \tag{4.6}$$

Finally, from (4.4), (4.5) and (4.6), we obtain $f_{Z,X}(z, w-y) = f_{Z|X+Y}(z|w) f_X(w-y)$ and thus

$$f_{Z|X}(z|w-y) = f_{Z|X+Y}(z|w)$$

for any $z, y, w \in \mathbb{R}$. Since the last equation holds for any $y \in \mathbb{R}$, we deduce that the random variable Z is independent of X and hence independent of $X + Y$. □

Proposition 4.18. Let $\{X_t, t \in [0, T]\}$ be a local Brownian motion from (4.1). Let $\widetilde{W}^{(i)}$ be independent of $(\widetilde{W}^{(i-1)}, \widetilde{W}^{(i+1)})$ for each $i \in \{1, 2, \dots, n-1\}$. Then, the process $\{X_t, t \in [0, T]\}$ is Markov if and only if it is a Brownian motion.

Proof. Suppose the process $\{X_t, t \in [0, T]\}$ is Markov. In particular, for each $i \in \{1, 2, \dots, n\}$ and any $t \in [t_i, t_{i+1}]$,

$$f_{X_t - X_{t_i}, X_{t_i} - X_{t_{i-1}} | X_{t_i}}(z, y|w) = f_{X_t - X_{t_i} | X_{t_i}}(z|w) f_{X_{t_i} - X_{t_{i-1}} | X_{t_i}}(y|w)$$

for any $w, y, z \in \mathbb{R}$. Let us consider first the process $\{X_t, t \in [0, t_3]\}$. Set $Z = X_t - X_{t_2} = \widetilde{W}^{(3)}_{t-t_2}$, $X + Y = X_{t_2}$ and $Y = X_{t_2} - X_{t_1} = \widetilde{W}^{(2)}_{t_2-t_1}$. Then, by assumption Y is independent of (X, Z) and by the equality above for $i = 2$ and $t = t_3$, we obtain Z is independent of Y given $X + Y$. By Lemma 4.17, the process $\{X_t - X_{t_2}, t \in [t_2, t_3]\}$ is independent of X_{t_2}. Further, we use induction method. For any $i \in \{3, \dots, n-1\}$, we set $Z = X_t - X_{t_i} = \widetilde{W}^{(i+1)}_{t-t_i}$, $X + Y = X_{t_i}$ and $Y = X_{t_i} - X_{t_{i-1}} = \widetilde{W}^{(i)}_{t_i - t_{i-1}}$. Using induction hypothesis, we know that Y is independent of (X, Z). Then, by assumption and Lemma 4.17, the process $\{X_t - X_{t_i}, t \in [t_i, t_{i+1}]\}$ is independent of X_{t_i}. Hence by Proposition 4.14, the assertion follows. □

Example 4.19. In Example 2.10, $W^{(2)}$ is independent of $(W^{(1)}, W^{(3)})$ and due to Proposition 4.18, the process is not Markov for non-constant R.

For a random variable Y, let $\mathbb{P}^Y$ be a regular conditional probability distribution given Y. Suppose that if we enlarge the natural filtration of the incremental process by its starting point. Then the incremental process is a Brownian motion if and only if it is a Brownian motion given the starting point.

Proposition 4.20. Let $\{X_t, t \in [0,T]\}$ be a process with non-trivial X_0. Further, let $\{Y_t = X_t - X_0, t \in [0,T]\}$ be a $(\mathbb{F}^Y, \mathbb{P})$-Brownian motion and $\mathbb{G}$ be the augmented filtration by X_0 with $\mathcal{G}_t^Y = \sigma(X_0) \vee \mathcal{F}_t^Y$ for any $t \in [0,T]$. Then, the process $\{Y_t, t \in [0,T]\}$ is a $(\mathbb{F}^Y, \mathbb{P}^{X_0})$-Brownian motion if and only if it is $(\mathbb{G}^Y, \mathbb{P})$-Brownian motion.

Proof. First we will prove the "only if" statement. Let $0 < s < t < T$, $A \in \mathcal{F}_s^Y$ and $B \in \mathcal{B}(\mathbb{R})$. Further, let

$$F \in \{A \cap \{X_0 \in B\} : A \in \mathcal{F}_s^Y, B \in \mathcal{B}(\mathbb{R})\}.$$

Then,

$$\mathbb{E}[(Y_t - Y_s)\mathbb{1}_F] = \int_B \mathbb{E}^{X_0}[(Y_t - Y_s)\mathbb{1}_A] d\mathbb{P}_{X_0} = 0.$$

It is easy to see that $\sigma(F) = \mathcal{G}_s^Y$. By using the monotone class theorem for functions, we obtain that for any bounded $\mathcal{G}_s^Y$-measurable function H,

$$\mathbb{E}[(Y_t - Y_s)H] = 0.$$

Therefore the process $\{Y_t, t \in [0,T]\}$ is a $(\mathbb{G}, \mathbb{P})$-martingale. Further, assume that for $A \in \mathcal{F}_T^Y$, $\mathbb{E}^{X_0}[\mathbb{1}_A] = 0$. Using law of total probability we obtain

$$\mathbb{P}(A) = \int_R \mathbb{P}(A|X_0) d\mathbb{P}_{X_0} = 0.$$

Hence $\mathbb{P} \ll \mathbb{P}^{X_0}$ on $\mathcal{F}_T^Y$ and therefore $[Y,Y]_T = T$ under $\mathbb{P}$. The continuity is clear. Finally, due to Lévy's characterisation theorem, the process $\{Y_t, t \in [0,T]\}$ is a $(\mathbb{G}^Y, \mathbb{P})$-Brownian motion.

Now the "if" statement. Assume that $\{Y_t, t \in [0,T]\}$ is a $(\mathbb{G}^Y, \mathbb{P})$-Brownian motion. Since $\mathcal{G}_t^Y = \mathcal{F}_t^X$ for any $t \in [0,T]$, it follows that for all $0 < s < t < T$,

$$\mathbb{E}[Y_t | \mathcal{F}_s^X] = Y_s$$

and therefore $\mathbb{E}[X_t | \mathcal{F}_s^X] = X_s$. Hence the process $\{X_t, t \in [0,T]\}$ is a $(\mathbb{F}^X, \mathbb{P})$-martingale. The quadratic variation and continuity of $\{Y_t, t \in [0,T]\}$ are passed on to the process $\{X_t, t \in [0,T]\}$. Lévy's characterisation theorem yields that the process $\{X_t, t \in [0,T]\}$ is a $(\mathbb{F}^X, \mathbb{P})$-Brownian motion. Therefore Y_t is independent of X_0 for all $t \in [0,T]$. That means that $\{Y_t, t \in [0,T]\}$ is a $(\mathbb{F}^Y, \mathbb{P}^{X_0})$-Brownian motion and the assertion follows. □

Chapter 5
Stochastic calculus

The immediate implication of Proposition 2.18 is that we cannot refer to the usual settings from stochastic calculus, where a stochastic integral is defined with respect to a semimartingale and the integrand is adapted (and a bit more) to the augmented natural filtration generated by the semimartingale. However, by restricting the integrands to processes that are adapted to some sub-σ-algebras of the augmented natural filtration generated by a local Brownian motion, we are able to define a stochastic integral with respect to any local Brownian motion. In this chapter we will develop this idea. Moreover, we will see that "Itô Formula" among other tools from stochastic calculus can be used for local Brownian motions. It will provide us with solutions to some prominent stochastic differential equations driven by a local Brownian motion.

5.1 General Wiener integral

Let us consider the space of deterministic square integrable functions on $[0,T]$ with respect to Lebesgue measure λ denoted by $L^2([0,T]) := L^2([0,T],\lambda)$ with norm $\|f\|_{L^2([0,T])} = \left(\int_0^T |f(t)|^2 dt\right)^{\frac{1}{2}} < \infty$. We recall that L^2 is the Hilbert space of square integrable random variables $X : \Omega \to \mathbb{R}$ with the scalar product $\langle X, Y\rangle_{L^2} = \mathbb{E}[XY]$ and accordingly the norm $\|X\|_{L^2} = (\mathbb{E}[|X|^2])^{\frac{1}{2}} < \infty$.

Theorem 5.1. *Let $\{X_t, t \in [0,T]\}$ be a local Brownian motion with time partition $0 = t_0 < t_1 < \ldots < t_n = T$ and $f \in L^2([0,T])$. Then, there exists a unique linear map*

$$I^X : L^2([0,T]) \to L^2$$

such that $I^X(f)$ is a finite sum of normally distributed random variables with

$$\mathbb{E}[I^X(f)] = 0 \quad \textit{and} \quad \mathbb{E}[I^X(f)^2] = \int_0^T |f(t)|^2 dt.$$

Proof. We will first define a linear mapping $\bar{I}^X$ on the space of simple functions and show that both properties hold for $\bar{I}^X$. Then, we will extend $\bar{I}^X$ uniquely to I^X using Hahn-Banach Theorem and prove that the assertion follows.

Step 1: We denote by $\mathcal{L}^2([0,T])$ the space of simple functions on $[0,T]$, i.e. functions of the form $f_m(t) = \sum_{i=1}^m a_i \mathbb{1}_{(u_{i-1},u_i]}(t)$ for a partition $0 = u_0 < u_1 < \ldots < u_m = T$ with a constant $a_i \in \mathbb{R}$ for each $i \in \{1, 2, \ldots, m\}$. For each interval $[t_{i-1}, t_i]$, there exists a partition $t_{i-1} = s_0^{(i)} < s_1^{(i)} < \ldots < s_{m_i}^{(i)} = t_i$ such that any simple function on $[t_{i-1}, t_i]$ can be written for $t \in [t_{i-1}, t_i]$ as

$$f_m(t) = \sum_{i=1}^m a_i \mathbb{1}_{(u_{i-1},u_i]}(t) = \sum_{i=1}^n \sum_{j=1}^{m_i} a_j^{(i)} \mathbb{1}_{(s_{j-1}^{(i)}, s_j^{(i)}]}(t) \tag{5.1}$$

with a sequence of real-valued constants $\{a_j^{(i)}, j \in \{1, 2, \ldots, m_i\}\}$ and $m = \sum_{i=1}^n m_i$. We define a mapping

$$\bar{I}^X : \mathcal{L}^2([0,T]) \to L^2 \quad \text{with} \quad \bar{I}^X(f_m) := \sum_{i=1}^n \sum_{j=1}^{m_i} a_j^{(i)} (X_{s_j^{(i)}} - X_{s_{j-1}^{(i)}}).$$

It is easy to see that it is linear. Further,

$$\mathbb{E}[\bar{I}^X(f_m)] = \sum_{i=1}^n \sum_{j=1}^{m_i} a_j^{(i)} \mathbb{E}[X_{s_j^{(i)}} - X_{s_{j-1}^{(i)}}] = 0$$

and

$$\begin{aligned}
\mathbb{E}[\bar{I}^X(f_m)^2] =& \sum_{i=1}^n \sum_{k=1}^n \left(\mathbb{E}\left[\sum_{j=1}^{m_i} a_j^{(i)} (X_{s_j^{(i)}} - X_{s_{j-1}^{(i)}}) \sum_{l=1}^{m_k} a_l^{(k)} (X_{s_l^{(k)}} - X_{s_{l-1}^{(k)}}) \right] \right) \\
=& \sum_{i=1}^n \mathbb{E}\left[\left(\sum_{j=1}^{m_i} a_j^{(i)} (X_{s_j^{(i)}} - X_{s_{j-1}^{(i)}}) \right)^2 \right] \\
&+ 2 \sum_{i=1}^n \sum_{k=i+1}^n \left(\sum_{j=1}^{m_i} \sum_{l=1}^{m_k} a_j^{(i)} a_l^{(k)} \mathbb{E}\left[(X_{s_j^{(i)}} - X_{s_{j-1}^{(i)}})(X_{s_l^{(k)}} - X_{s_{l-1}^{(k)}}) \right] \right) \\
=& \sum_{i=1}^n \sum_{j=1}^{m_i} \left(a_j^{(i)} \right)^2 \mathbb{E}\left[\left(X_{s_j^{(i)}} - X_{s_{j-1}^{(i)}} \right)^2 \right] \\
&+ 2 \sum_{i=1}^n \left(\sum_{j=1}^{m_i} \sum_{l=j+1}^{m_i} a_j^{(i)} a_l^{(i)} \mathbb{E}\left[(X_{s_j^{(i)}} - X_{s_{j-1}^{(i)}})(X_{s_l^{(i)}} - X_{s_{l-1}^{(i)}}) \right] \right) \\
&+ 2 \sum_{i=1}^n \sum_{k=i+1}^n \left(\sum_{j=1}^{m_i} \sum_{l=1}^{m_k} a_j^{(i)} a_l^{(k)} \mathbb{E}\left[(X_{s_j^{(i)}} - X_{s_{j-1}^{(i)}})(X_{s_l^{(k)}} - X_{s_{l-1}^{(k)}}) \right] \right) \\
=& \sum_{i=1}^n \sum_{j=1}^{m_i} \left(a_j^{(i)} \right)^2 \left(s_j^{(i)} - s_{j-1}^{(i)} \right) \\
=& \int_0^T |f_m(t)|^2 dt.
\end{aligned}$$

The third equality holds since a local Brownian motion has uncorrelated increments. Furthermore, $X_{s_j^{(i)}} - X_{s_{j-1}^{(i)}} \sim \mathcal{N}(0, s_j^{(i)} - s_{j-1}^{(i)})$ for each i, j and therefore the term $\bar{I}^X(f_m)$ is a finite sum of normally distributed random variables.
Step 2: For any $f \in L^2([0,T])$, there exists a sequence of simple functions $(f_m)_{m\geq 1} \in \mathcal{L}^2([0,T])$ such that $\|f_m - f\|_{L^2([0,T])} \to 0$ as $m \to \infty$. For $m, l \geq 1$,

$$\begin{aligned}
\|\bar{I}^X(f_m) - \bar{I}^X(f_l)\|_{L^2}^2 &= \mathbb{E}\left[|\bar{I}^X(f_m) - \bar{I}^X(f_l))|^2\right] = \mathbb{E}\left[|\bar{I}^X(f_m - f_l)|^2\right] \\
&= \int_0^T (f_m(t) - f_l(t))^2 dt \\
&= \int_0^T (f_m(t) - f(t) - (f_l(t) - f(t))^2 dt \\
&\leq 2\left(\int_0^T (f_m(t) - f(t))^2 dt + \int_0^T (f_l(t) - f(t))^2 dt\right)
\end{aligned}$$

and therefore $\|\bar{I}^X(f_m) - \bar{I}^X(f_l)\|_{L^2}^2 \to 0$ as $m, l \to \infty$. Hence $(\bar{I}^X(f_m))_{m\geq 1}$ is a Cauchy sequence in L^2 and it converges to an element denoted by $I^X(f) \in L^2$. Assume there exists another sequence $(g_m)_{m\geq 1} \in \mathcal{L}^2([0,T])$ such that $\|g_m - f\|_{L^2([0,T])} \to 0$ as $m \to \infty$. Applying the triangular inequality for the norm $\|\cdot\|_{L^2}$, we obtain that for $m \geq 1$, the following holds:

$$\begin{aligned}
\|\bar{I}^X(g_m) - I^X(f)\|_{L^2} &\leq \|\bar{I}^X(g_m) - \bar{I}^X(f_m)\|_{L^2} + \|\bar{I}^X(f_m) - I^X(f)\|_{L^2} \\
&= \left(\int_0^T (g_m(t) - f_m(t))^2 dt\right)^{\frac{1}{2}} + \|\bar{I}^X(f_m) - I^X(f)\|_{L^2} \\
&\leq \left(2\left(\|g_m - f\|_{L^2([0,T])} + \|f_m - f\|_{L^2([0,T])}\right)\right)^{\frac{1}{2}} + \|\bar{I}^X(f_m) - I^X(f)\|_{L^2}.
\end{aligned}$$

which converges to zero. Therefore $\bar{I}^X(g_m) \to I^X(f)$ as $m \to \infty$ in L^2, and $I^X(f)$ is well defined.

Step 3: Since $\bar{I}^X(f_m)$ converges to $I^X(f)$ in L^2 and $\int_0^T |f(t)|^2 dt < \infty$, we obtain

$$\mathbb{E}[I^X(f)] = \lim_{m\to\infty} \mathbb{E}[I^X(f_m)] = 0 \quad \text{and} \quad \mathbb{E}[I^X(f)^2] = \lim_{m\to\infty} \mathbb{E}[I^X(f_m)^2] = \int_0^T |f(t)|^2 dt.$$

As a consequence of convergence in L^2, the sequence $\bar{I}^X(f_m)$ converges to $I^X(f)$ also in distribution. We observe that with a fixed partition $0 = t_0 < t_1 < \ldots < t_n = T$, any simple function $(f_m)_{m\geq 1} \in \mathcal{L}^2([0,T])$ converges to $f(t) \in L^2([0,T])$ as $m \to \infty$ if and only if $(f_m)_{m\geq 1}$ converges to $f(t)$ as $m_i \to \infty$ for all $i \in \{1, 2, \ldots, n\}$ in (5.1). Using the Lévy's continuity theorem, the characteristic function of $I^X(f)$ is

$$\begin{aligned}\varphi_{I^X(f)}(t) &= \lim_{m\to\infty} \varphi_{\bar{I}^X(f_m)}(t) = \lim_{m_1\to\infty} \dots \lim_{m_n\to\infty} \varphi_{\sum_{i=1}^n \sum_{j=1}^{m_i} a_j^{(i)}(X_{s_j^{(i)}} - X_{s_{j-1}^{(i)}})}(t) \\ &= \lim_{m_1\to\infty} \dots \lim_{m_n\to\infty} \mathbb{E}\left[\prod_{i=1}^n e^{it\left(\sum_{j=1}^{m_i} a_j^{(i)}(X_{s_j^{(i)}} - X_{s_{j-1}^{(i)}})\right)}\right] \\ &= \mathbb{E}\left[\lim_{m_1\to\infty} \dots \lim_{m_n\to\infty} \prod_{i=1}^n e^{it\left(\sum_{j=1}^{m_i} a_j^{(i)}(X_{s_j^{(i)}} - X_{s_{j-1}^{(i)}})\right)}\right] \\ &= \mathbb{E}\left[\prod_{i=1}^n e^{it\left(\sum_{j=1}^{\infty} a_j^{(i)}(X_{s_j^{(i)}} - X_{s_{j-1}^{(i)}})\right)}\right] = \varphi_{\sum_{i=1}^n \xi_i}(t)\end{aligned}$$

with $\xi_i = \sum_{j=1}^\infty a_j^{(i)}(X_{s_j^{(i)}} - X_{s_{j-1}^{(i)}})$ being a normally distributed random variable for each $i \in \{1, 2, \dots, n\}$. This is due to the fact that for each $i \in \{1, 2 \dots, n\}$, the process $\{X_t - X_{t_{i-1}}, t \in [t_{i-1}, t_i]\}$ is a Brownian motion and thus the infinite sum $\sum_{j=1}^\infty a_j^{(i)}(X_{s_j^{(i)}} - X_{s_{j-1}^{(i)}})$ is normally distributed.

Finally, using Hahn-Banach Theorem the linear map $I^X(f)$ is unique; therefore $I^X(f)$ can be written as $I^X(f) = \sum_{i=1}^n \int_{t_{i-1}}^{t_i} f(t)dX_t$ and the assertion follows. □

Definition 5.2. For any $f \in L^2([0,T])$ and any local Brownian motion $\{X_t, t \in [0,T]\}$ with time partition $0 = t_0 < t_1 < \dots < t_n = T$, we will denote the limit by

$$\int_0^T f(t)dX_t := \sum_{i=1}^n \int_{t_{i-1}}^{t_i} f(t)dX_t$$

and call it the *general Wiener integral with respect to a local Brownian motion* X.

Note that the first and the second moments of the general Wiener integral are independent of the time partition of the local Brownian motion. Moreover, in the next section we will see (for a more general stochastic integral with respect to local Brownian motion) that the general Wiener integral is defined in such a way that it also does not dependent on the choice of the time partition of the local Brownian motion. Hence the definition can be extended to *any* time partition of a local Brownian motion.

Remark 5.3. For any i, the integral $\int_{t_{i-1}}^{t_i} f(t)dX_t$ is a Wiener integral. Hence the general Wiener integral is a finite sum of Wiener integrals. In the case where X is a Brownian motion, the general Wiener integral coincides with a Wiener integral. Note that the general Wiener integral is not necessarily normally distributed. In fact, the integral $I^X(f)$ is normally distributed for any $f \in L^2([0,T])$ if and only if the process X is a Brownian motion; if $I^X(f)$ is normally distributed for any $f \in L^2([0,T])$, then the process $\{X_t, t \in [0,T]\}$ has to be Gaussian.

We define an *integral process* for $f \in L^2([0,T])$ and $t \in [0,T]$, as

$$\int_0^t f(s)dX_s := \int_0^T f(s)\mathbb{1}_{[0,t]}(s)dX_s,$$

where $\int_0^T f(s)dX_s$ is a general Wiener integral.

Proposition 5.4. Let $\{X_t, t \in [0,T]\}$ be a local Brownian motion and $f \in L^2([0,T])$. Then, the integral process $\int_0^t f(s)dX_s$ is a local Gaussian process on $[0,T]$ with zero mean.

Proof. Let $0 = t_0 < t_1 < \ldots < t_n = T$ be a time partition of $\{X_t, t \in [0,T]\}$. Define a process $Y_t = \int_0^t f(s)dX_s$. Then, for each $i \in \{1, 2, \ldots, n-1\}$, the process $\{Y_t - Y_{t_{i-1}}, t \in [t_{i-1}, t_{i+1}]\}$ is a Gaussian process with zero mean. Thus the assertion follows. □

Proposition 5.5. Let $\{X_t, t \in [0,T]\}$ be a local Brownian motion and $f, g \in L^2([0,T])$. Then,

$$\mathbb{E}[I^X(f)I^X(g)] = \int_0^T f(t)g(t)dt.$$

Proof. By applying the linearity of general Wiener integral, direct calculation yields

$$\begin{aligned}\mathbb{E}[I^X(f)I^X(g)] &= \frac{1}{2}(\mathbb{E}[(I^X(f) + I^X(g))^2] - \mathbb{E}[I^X(f)^2] - \mathbb{E}[I^X(g)^2]) \\ &= \frac{1}{2}(\mathbb{E}[I^X(f+g)^2] - \mathbb{E}[I^X(f)^2] - \mathbb{E}[I^X(g)^2]) \\ &= \frac{1}{2}\int_0^T \left((f(t)+g(t))^2 - f(t)^2 - g(t)^2\right) dt = \int_0^T f(t)g(t)dt\end{aligned}$$

and the assertion follows. □

5.2 General Itô integral

Our next goal is to define an integral with respect to the local Brownian motion for a bigger class of integrands. The definition of the stochastic integral with respect to a local Brownian motion was chosen in such a way that we were able to use the established and extensively developed tools from stochastic calculus with respect to a semimartingale despite the fact that a proper local Brownian motion is not a semimartingale. The price we needed to pay is that this integral is only defined for a restrictive class of the integrands that we will call *regular piecewise adapted processes.* These processes are only measurable with respect to the local sub-σ-algebras of the regular natural filtration generated by the integrator. However, as we will see later, the restriction is quite loose and will enable us to solve many knows SDE's that are used in applications.

In order to deal with this idea accordingly we will first introduce some notations that are essential for the later definition.

5.2.1 Preliminary definitions

Definition 5.6. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\mathcal{P} = \{t_i, 0 = t_0 < t_1 < \ldots < t_n = T\}$ be a partition on $[0, T]$. A *piecewise filtration generated by* $\mathcal{P}$, denoted by $\{\mathcal{F}_t^{(\mathcal{P})}, t \in [0, T]\}$, is a family of sub-$\sigma$-algebras of $\mathcal{F}$ such that for each $i \in \{0, 1, \ldots, n-1\}$, $s, t \in (t_i, t_{i+1})$ and $s < t$, the property $\mathcal{F}_s^{(\mathcal{P})} \subset \mathcal{F}_t^{(\mathcal{P})}$ holds. A *right-closed* piecewise filtration is a piecewise filtration that is left-continuous at the partition points, i.e. for each $i \in \{0, 1, \ldots, n-1\}$, $s, t \in (t_i, t_{i+1}]$ and $s < t$, the property $\mathcal{F}_s^{(\mathcal{P})} \subseteq \mathcal{F}_t^{(\mathcal{P})}$ holds.

In other words, a right-closed piecewise filtration generated by a partition $0 = t_0 < t_1 < \ldots < t_n = T$ is a filtration within each interval $(t_{i-1}, t_i]$, but not necessarily a filtration on the whole interval $[0, T]$.

Remark 5.7. One can define a left-closed piecewise filtration in the same manner by changing the half-open interval $(t_i, t_{i+1}]$ to $[t_i, t_{i+1})$. We are not interested in left-closed piecewise filtration in this work, hence the proper definition is omitted. However, note that a piecewise filtration is right-closed and left-closed if and only if it is a filtration.

Definition 5.8. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\mathcal{P} = \{t_i, 0 = t_0 < t_1 < \ldots < t_n = T\}$ be a partition on $[0, T]$. A *memoryless* piecewise filtration is a right-closed piecewise filtration such that the right limit of the piecewise filtration at partition points is a trivial sigma-algebra, i.e. for each $i \in \{1, 2, \ldots, n\}$,

$$\mathcal{F}_{t_i+}^{\mathcal{P}} := \bigcap_{t_i < t < t_{i+1}} \mathcal{F}_t^{\mathcal{P}} = \{\Omega, \emptyset\}.$$

The memoryless piecewise filtration is a piecewise filtration with the property that at each starting point of an interval in the partition, the piecewise filtration "forgets" the past and "resets" to become a new filtration as it makes an infinitesimally small change forward in time. In particular, a memoryless piecewise filtration is not increasing.

Definition 5.9. Let $(\Omega, \mathcal{F}, \mathbb{F}^{(\mathcal{P})}, \mathbb{P})$ be a probability space, $\mathcal{P} = \{t_i, 0 = t_0 < t_1 < \ldots < t_n = T\}$ be a partition on $[0, T]$ and $\mathbb{F}^{(\mathcal{P})} = \{\mathcal{F}_t^{(\mathcal{P})}, t \in [0, T]\}$ be a memoryless piecewise filtration generated by $\mathcal{P}$. A process $\{H_t, t \in [0, T]\}$ is said to be *piecewise progressively measurable with respect to* $\mathbb{F}^{(\mathcal{P})}$ if for each $i \in \{1, 2, \ldots, n\}$ and any $t \in (t_{i-1}, t_i]$, the map

$$(t_{i-1}, t] \times \Omega \to \mathbb{R} \qquad (s, \omega) \mapsto H_s(\omega)$$

is $\mathcal{B}((t_{i-1}, t]) \otimes \mathcal{F}_t^{(\mathcal{P})}$-measurable.

In particular, if a process is piecewise progressively measurable with respect to a memoryless piecewise filtration, it is adapted to the same memoryless piecewise filtration.

Definition 5.10. A time partition of a local Brownian motion is called a *minimal time partition* if it has the smallest number of intervals. It will be denoted by

$$0 = \mathfrak{t}_0 < \mathfrak{t}_1 < \ldots < \mathfrak{t}_\mathfrak{n} = T.$$

The existence of a minimal time partition can be seen by the fact that any partition with fewer endpoints will impose Gaussian behaviour over bigger time intervals than $[\mathfrak{t}_{i-1}, \mathfrak{t}_{i+1}]$ and hence not acceptable for a proper local Brownian motion. Unless the local Brownian motion is a Brownian motion itself. In that case, the minimal time partition is simply $0 = t_0 < t_1 < t_2 = T$ for any $t_1 \in (0, T)$. Thus, any proper local Brownian motion will have at least three time intervals in the time partition otherwise it is a Brownian motion. Note that a minimal time partition is not unique. For instance, for a non-constant R in *Example* 2.10 a minimal time partition is $0 < 1 < 2 < T$, with three intervals. On the other hand, the partition $0 < 1.5 < 2 < T$ is also a time partition of the same local Brownian motion. Since the number of intervals in the latter time partition is three, it is also a minimal time partition.

Definition 5.11. A *canonical* memoryless piecewise filtration of the local Brownian motion $\{X_t, t \in [0, T]\}$ with a canonical time partition $0 = \mathfrak{t}_0 < \mathfrak{t}_1 < \ldots < \mathfrak{t}_\mathfrak{n} = T$, denoted by $\vec{\mathbb{F}}^X$, is defined as

$$\vec{\mathcal{F}}_t^X := \begin{cases} \mathcal{F}^X_{[0,t]}, & 0 \le t \le \mathfrak{t}_1 \\ \mathcal{F}^X_{[\mathfrak{t}_1,t]}, & \mathfrak{t}_1 < t \le \mathfrak{t}_2 \\ \cdots & \\ \mathcal{F}^X_{[\mathfrak{t}_{\mathfrak{n}-1},t]}, & \mathfrak{t}_{\mathfrak{n}-1} < t \le T \end{cases}$$

with $\mathcal{F}^X_{[\mathfrak{t}_{i-1},t]} := \sigma(\{X_s - X_{\mathfrak{t}_{i-1}}, s \in [\mathfrak{t}_{i-1}, t]\})$ for any $i \in \{1, 2, \ldots, \mathfrak{n}\}$.

It is clear that the piecewise filtration defined above is memoryless piecewise, since $\mathcal{F}^X_{[\mathfrak{t}_{i-1},t]}$ is the augmented filtration of a Brownian motion for each $i \in \{1, 2, \ldots, \mathfrak{n}\}$.

Definition 5.12. A *regular piecewise adapted* process is piecewise progressively measurable with respect to canonical memoryless piecewise filtration generated by a local Brownian motion, and for each $i \in \{1, 2, \ldots, n\}$, a càdlàg process on $(\mathfrak{t}_{i-1}, \mathfrak{t}_i)$.

5.2.2 Definition of general Itô integral

Let $C([0, T])$ be a sample space representing the set of continuous functions on the interval $[0, T]$ and $\{X_t, t \in [0, T]\}$ be a local Brownian motion with a canonical time partition $0 = \mathfrak{t}_0 < \mathfrak{t}_1 < \ldots < \mathfrak{t}_\mathfrak{n} = T$. Then, let $(C([0, T]), \mathbb{F}^X, \mathbb{P})$ be a filtered probability space with regular natural filtration $\mathbb{F}^X = (\mathcal{F}_t^X)_{t \in [0,T]}$. We will denote by $L^2(C([0, T]), \vec{\mathbb{F}}^X, \mathbb{P})$

with $\vec{\mathbb{F}}^X = (\vec{\mathcal{F}}_t^X)_{t\in[0,T]}$ the set of processes $\{H_t, t \in [0,T]\}$ that are regular piecewise adapted with respect to X such that

$$\mathbb{E}\left[\int_0^T H_t^2 dt\right] < \infty.$$

Further, let us define by $\mathcal{L}$ a set of processes from $L^2(C([0,T]), \vec{\mathbb{F}}^X, \mathbb{P})$ that can be written as

$$H_t = \sum_{i=1}^{\mathfrak{n}} \sum_{j=1}^{m_i} H_{j-1}^{(i)} \mathbb{1}_{(s_{j-1}^{(i)}, s_j^{(i)}]}(t) \mathbb{1}_{[\mathfrak{t}_{i-1}, \mathfrak{t}_i]}(t),$$

where for each $i \in \{1, 2 \ldots, \mathfrak{n}\}$, there exists a sub-partition of $[\mathfrak{t}_{i-1}, \mathfrak{t}_i]$ as $\mathfrak{t}_{i-1} = s_0^{(i)} < s_1^{(i)} < s_2^{(i)} < \ldots < s_{m_i}^{(i)} = \mathfrak{t}_i$, and $H_{j-1}^{(i)}$ is $\mathcal{F}^X_{[\mathfrak{t}_{i-1}, s_{j-1}^{(i)}]}$-measurable square integrable random variable. Now we are ready to define the stochastic integral with respect to a local Brownian motion for regular piecewise adapted processes.

Theorem 5.13. *Let $\{X_t, t \in [0,T]\}$ be a proper local Brownian motion with a canonical time partition $0 = \mathfrak{t}_0 < \mathfrak{t}_1 < \ldots < \mathfrak{t}_\mathfrak{n} = T$. There exists a unique linear map*

$$I^X : L^2(C([0,T]), \vec{\mathbb{F}}^X, \mathbb{P}) \to L^2$$

such that

$$I^X(H) = \sum_{i=1}^{\mathfrak{n}} \int_{\mathfrak{t}_{i-1}}^{\mathfrak{t}_i} H_t dX_t.$$

Moreover, the zero mean property holds, but the isometry property does not hold, i.e. for $H \in L^2(C([0,T]), \vec{\mathbb{F}}^X, \mathbb{P})$ we have

$$\mathbb{E}[I^X(H)] = 0 \quad \text{and} \quad \mathbb{E}[I^X(H)^2] \neq \mathbb{E}\left[\int_0^T H_t^2 dt\right].$$

Proof. Step 1: We will show that the space $L^2(C([0,T]), \vec{\mathbb{F}}^X, \mathbb{P})$ is a Hilbert space.

Let $(H_n)_{n\geq 1}$ be a Cauchy sequence in $L^2(C([0,T]), \vec{\mathbb{F}}^X, \mathbb{P})$. For each $i \in \{1, 2, \ldots, \mathfrak{n}\}$, we define a space $L^2(C([\mathfrak{t}_{i-1}, \mathfrak{t}_i]), \mathcal{F}^X_{[\mathfrak{t}_{i-1}, \mathfrak{t}_i]}, \mathbb{P})$ of functions of $[0,T]$ that are continuous and square integrable on $[\mathfrak{t}_{i-1}, \mathfrak{t}_i]$, i.e. for any function $f(t) \in L^2(C([\mathfrak{t}_{i-1}, \mathfrak{t}_i]), \mathcal{F}^X_{[\mathfrak{t}_{i-1}, \mathfrak{t}_i]}, \mathbb{P})$, we have $\mathbb{E}\left[\int_{\mathfrak{t}_{i-1}}^{\mathfrak{t}_i} f(t)^2 dt\right] < \infty$. Then, for each $i \in \{1, 2, \ldots, \mathfrak{n}\}$, the sequence $(H_n(t)\mathbb{1}_{[\mathfrak{t}_{i-1}, \mathfrak{t}_i]}(t))_{n\geq 1}$ is a Cauchy sequence in $L^2(C([\mathfrak{t}_{i-1}, \mathfrak{t}_i]), \mathcal{F}^X_{[\mathfrak{t}_{i-1}, \mathfrak{t}_i]}, \mathbb{P})$. Hence for $l, m \geq 0$,

$$\mathbb{E}\left[\int_0^T (H_l(t) - H_m(t))^2 dt\right] = \sum_{i=1}^{\mathfrak{n}} \mathbb{E}\left[\int_{\mathfrak{t}_{i-1}}^{\mathfrak{t}_i} (H_l(t) - H_m(t))^2 dt\right] \to 0$$

as $l, m \to \infty$, since the process $\{X_t - X_{\mathfrak{t}_{i-1}}, t \in [\mathfrak{t}_{i-1}, \mathfrak{t}_i]\}$ is a Brownian motion for each $i \in \{1, 2, \ldots, \mathfrak{n}\}$ and thus the space $L^2(C([\mathfrak{t}_{i-1}, \mathfrak{t}_i]), \mathcal{F}^X_{[\mathfrak{t}_{i-1}, \mathfrak{t}_i]}, \mathbb{P})$ is complete. In fact, it is the space of regular adapted processes with respect to Brownian motion $\{X_t - X_{\mathfrak{t}_{i-1}}, t \in$

$[\mathfrak{t}_{i-1}, \mathfrak{t}_i]\}$. It follows that since every Cauchy sequence converges in $L^2(C([0,T]), \vec{\mathbb{F}}^X, \mathbb{P})$, it is a complete space. Moreover,

$$L^2(C([0,T]), \vec{\mathbb{F}}^X, \mathbb{P}) = \bigoplus_{i=1}^{\mathfrak{n}} L^2(C([\mathfrak{t}_{i-1}, \mathfrak{t}_i]), \mathcal{F}^X_{[\mathfrak{t}_{i-1},\mathfrak{t}_i]}, \mathbb{P})$$

is called the Hilbert space direct sum of spaces $\{L^2(C([\mathfrak{t}_{i-1}, \mathfrak{t}_i]), \mathcal{F}^X_{[\mathfrak{t}_{i-1},\mathfrak{t}_i]}, \mathbb{P}), i \in \{1, 2, \ldots, \mathfrak{n}\}\}$.

Step 2: Assume that $H_t = \sum_{i=1}^{\mathfrak{n}} \sum_{j=1}^{m_i} H^{(i)}_{j-1} \mathbb{1}_{(s^{(i)}_{j-1}, s^{(i)}_j]}(t) \mathbb{1}_{[\mathfrak{t}_{i-1},\mathfrak{t}_i]}(t) \in \mathcal{L}$ and $\tilde{I}^X(H) = \sum_{i=1}^{\mathfrak{n}} \sum_{j=1}^{m_i} H^{(i)}_{j-1}(X_{s^{(i)}_j} - X_{s^{(i)}_{j-1}})$. Then, $\tilde{I}^X$ is linear and

$$\begin{aligned}
\mathbb{E}[\tilde{I}^X(H)] &= \mathbb{E}\left[\sum_{i=1}^{\mathfrak{n}} \sum_{j=1}^{m_i} H^{(i)}_{j-1}(X_{s^{(i)}_j} - X_{s^{(i)}_{j-1}})\right] \\
&= \sum_{i=1}^{\mathfrak{n}} \sum_{j=1}^{m_i} \mathbb{E}\left[H^{(i)}_{j-1} \mathbb{E}\left[X_{s^{(i)}_j} - X_{s^{(i)}_{j-1}} | \vec{\mathfrak{F}}^X_{s^{(i)}_{j-1}}\right]\right] = 0.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\mathbb{E}\left[\tilde{I}^X(H)^2\right] &= \mathbb{E}\left[\left(\sum_{i=1}^{\mathfrak{n}} \sum_{j=1}^{m_i} H^{(i)}_{j-1}(X_{s^{(i)}_j} - X_{s^{(i)}_{j-1}})\right)^2\right] \\
&= \mathbb{E}\left[\sum_{i=1}^{\mathfrak{n}} \sum_{j=1}^{m_i} (H^{(i)}_{j-1})^2 (X_{s^{(i)}_j} - X_{s^{(i)}_{j-1}})^2\right] \\
&\quad + 2\mathbb{E}\left[\sum_{i=1}^{\mathfrak{n}} \sum_{j=1}^{m_i} \sum_{k=j+1}^{m_i} H^{(i)}_{j-1}(X_{s^{(i)}_j} - X_{s^{(i)}_{j-1}}) H^{(i)}_{k-1}(X_{s^{(i)}_k} - X_{s^{(i)}_{k-1}})\right] \\
&\quad + 2\mathbb{E}\left[\sum_{i=1}^{\mathfrak{n}} \sum_{k=i+1}^{\mathfrak{n}} \sum_{j=1}^{m_i} \sum_{l=1}^{m_k} H^{(i)}_{j-1}(X_{s^{(i)}_j} - X_{s^{(i)}_{j-1}} H^{(k)}_{l-1}(X_{s^{(k)}_l} - X_{s^{(k)}_{l-1}}))\right] \\
&= \mathbb{E}\left[\int_0^T H_t^2 dt\right] \\
&\quad + 2\sum_{i=1}^{\mathfrak{n}} \sum_{j=1}^{m_i} \sum_{k=j+1}^{m_i} \mathbb{E}\left[H^{(i)}_{j-1}(X_{s^{(i)}_j} - X_{s^{(i)}_{j-1}}) H^{(i)}_{k-1} \mathbb{E}\left[X_{s^{(i)}_k} - X_{s^{(i)}_{k-1}} | \vec{\mathfrak{F}}^X_{s^{(i)}_{k-1}}\right]\right] \\
&\quad + 2\sum_{i=1}^{\mathfrak{n}} \sum_{k=i+1}^{\mathfrak{n}} \sum_{j=1}^{m_i} \sum_{l=1}^{m_k} \mathbb{E}\left[H^{(i)}_{j-1}(X_{s^{(i)}_j} - X_{s^{(i)}_{j-1}}) H^{(k)}_{l-1}(X_{s^{(k)}_l} - X_{s^{(k)}_{l-1}})\right] \\
&= \mathbb{E}\left[\int_0^T H_t^2 dt\right] \\
&\quad + 2\sum_{i=1}^{\mathfrak{n}} \sum_{k=i+1}^{\mathfrak{n}} \sum_{j=1}^{m_i} \sum_{l=1}^{m_k} \mathbb{E}\left[H^{(i)}_{j-1}(X_{s^{(i)}_j} - X_{s^{(i)}_{j-1}}) H^{(k)}_{l-1}(X_{s^{(k)}_l} - X_{s^{(k)}_{l-1}})\right]
\end{aligned}$$

with the last term to be zero if and only if the process $\{X_t, t \in [0,T]\}$ is a martingale, which, due to Proposition 2.17, contradicts the assumption.

Step 3: For each $i \in \{1, 2 \ldots, \mathfrak{n}\}$, let $\mathcal{L}_i$ be a space of functions of the form

$$H_t = \sum_{j=1}^{m_i} H_{j-1}^{(i)} \mathbb{1}_{(s_{j-1}^{(i)}, s_j^{(i)}]}(t)$$

for $t \in [\mathfrak{t}_{i-1}, \mathfrak{t}_i]$, a partition $\mathfrak{t}_{i-1} = s_0^{(i)} < s_1^{(i)} < s_2^{(i)} < \ldots < s_{m_i}^{(i)} = \mathfrak{t}_i$, and a square integrable, $\mathcal{F}^X_{[\mathfrak{t}_{i-1}, s_{j-1}^{(i)}]}$-measurable random variable $H_{j-1}^{(i)}$. Then, it is known that $\mathcal{L}_i$ is dense in $L^2(C([\mathfrak{t}_{i-1}, \mathfrak{t}_i]), \mathcal{F}^X_{[\mathfrak{t}_{i-1}, \mathfrak{t}_i]}, \mathbb{P})$. Moreover, any function in $\mathcal{L}$ can be written as a sum of functions in $\mathcal{L}_i$. Since $L^2(C([0,T]), \vec{\mathbb{F}}^X, \mathbb{P}) = \bigoplus_{i=1}^{\mathfrak{n}} L^2(C([\mathfrak{t}_{i-1}, \mathfrak{t}_i]), \mathcal{F}^X_{[\mathfrak{t}_{i-1}, \mathfrak{t}_i]}, \mathbb{P})$, it follows that $\mathcal{L}$ is dense in $L^2(C([0,T]), \vec{\mathbb{F}}^X, \mathbb{P})$.

Step 4: Due to the Cauchy-Schwarz inequality,

$$\begin{aligned}
\mathbb{E}\left[\tilde{I}^X(H)^2\right] &= \mathbb{E}\left[\left(\sum_{i=1}^{\mathfrak{n}} \sum_{j=1}^{m_i} H_{j-1}^{(i)} (X_{s_j^{(i)}} - X_{s_{j-1}^{(i)}})\right)^2\right] \\
&\leq \mathfrak{n}\mathbb{E}\left[\left(\sum_{j=1}^{m_1} H_{j-1}^{(1)} \left(X_{s_j^{(1)}} - X_{s_{j-1}^{(1)}}\right)\right)^2 + \cdots + \left(\sum_{j=1}^{m_{\mathfrak{n}}} H_{j-1}^{(\mathfrak{n})} \left(X_{s_j^{(\mathfrak{n})}} - X_{s_{j-1}^{(\mathfrak{n})}}\right)\right)^2\right] \\
&= \mathfrak{n} \sum_{i=1}^{\mathfrak{n}} \sum_{j=1}^{m_i} \mathbb{E}\left[\left(H_{j-1}^{(i)}\right)^2 \mathbb{E}\left[\left(X_{s_j^{(i)}} - X_{s_{j-1}^{(i)}}\right)^2 | \vec{\mathcal{F}}^X_{s_{j-1}^{(i)}}\right]\right] \\
&= \mathfrak{n} \sum_{i=1}^{\mathfrak{n}} \sum_{j=1}^{m_i} \mathbb{E}\left[\left(H_{j-1}^{(i)}\right)^2\right] \left(s_j^{(i)} - s_{j-1}^{(i)}\right) \\
&= \mathfrak{n}\mathbb{E}\left[\int_0^T H_t^2 dt\right],
\end{aligned}$$

and thus the linear transformation $\tilde{I}^X$ is bounded. Further, as we showed above, $\mathcal{L}$ is dense in the Hilbert space $L^2(C([0,T]), \vec{\mathbb{F}}^X, \mathbb{P})$. Finally, using the Hahn-Banach theorem gives the assertion, i.e. there exists a unique linear extension of $\tilde{I}$ on $L^2(C([0,T]), \vec{\mathbb{F}}^X, \mathbb{P})$ denoted by I^X such that $\mathbb{E}[I^X(H)] = 0$ and $\mathbb{E}[I^X(H)^2] \neq \mathbb{E}\left[\int_0^T H_t^2 dt\right]$ for any $H \in L^2(C([0,T]), \vec{\mathbb{F}}^X, \mathbb{P})$. From the construction of the space $L^2(C([0,T]), \vec{\mathbb{F}}^X, \mathbb{P})$, we can deduce that $I^X(H) = \sum_{i=1}^{\mathfrak{n}} \int_{\mathfrak{t}_{i-1}}^{\mathfrak{t}_i} H_t dX_t$. □

Definition 5.14. Let $\{X_t, t \in [0,T]\}$ be a local Brownian motion with a canonical time partition $0 = \mathfrak{t}_0 < \mathfrak{t}_1 < \ldots < \mathfrak{t}_{\mathfrak{n}} = T$. Further, let $\{H_t, t \in [0,T]\}$ be a regular piecewise adapted process with respect to X. The stochastic integral of H with respect to the local Brownian motion X is defined as

$$\int_0^T H_t dX_t := \sum_{i=1}^{\mathfrak{n}} \int_{\mathfrak{t}_{i-1}}^{\mathfrak{t}_i} H_t dX_t,$$

and is called *general Itô integral*.

Example 5.15. Let $\{X_t, t \in [0,T]\}$ be a proper local Brownian motion with a canonical time partition $0 = \mathfrak{t}_0 < \mathfrak{t}_1 < \ldots < \mathfrak{t}_\mathfrak{n} = T$. A function of the type

$$H_t = \sum_{i=1}^{\mathfrak{n}} H_t^{(i)} \mathbb{1}_{[\mathfrak{t}_{i-1},\mathfrak{t}_i]}(t),$$

for a regular $\sigma(\{X_s - X_{\mathfrak{t}_{i-1}}, s \in [\mathfrak{t}_{i-1}, t]\})$-adapted process $\{H_t^{(i)}, t \in [\mathfrak{t}_{i-1}, \mathfrak{t}_i]\}$ with $H_{\mathfrak{t}_{i-1}}^{(i)} = 0$ for each $i \in \{1, 2, \ldots, n\}$, is a regular piecewise adapted process with respect to X.

It is easy to see that a process can be regular piecewise adapted with respect to two different canonical time partitions of a local Brownian motion. Let us consider Example 2.10. Define the process $\{H_t, t \in [0,T]\}$ such that

$$H_t = \begin{cases} X_t, & 0 \le t \le 1 \\ X_t - X_1, & 1 < t \le 1.5 \\ X_t - X_{1.5}, & 1.5 < t \le 2 \\ X_t - X_2, & 2 < t \le T \end{cases}$$

Then, the process $\{H_t, t \in [0,T]\}$ is regular piecewise adapted with respect to X for a canonical time partition $0 < 1 < 2 < T$ and a canonical time partition $0 < 1.5 < 2 < T$. Hence the problem we encounter at this stage is that a general Itô integral is defined for a particular canonical time partition, and might yield different results for different canonical time partitions, hence being inconsistent. However, the proposition below shows that not only the definition is independent of the choice of canonical time partition, but also of the choice of any partition. The general Itô integral is partition independent.

Proposition 5.16. Let $\{X_t, t \in [0,T]\}$ be a local Brownian motion with two different time partitions and $\{H_t, t \in [0,T]\}$ be a regular piecewise adapted process with respect to X for both partitions. The integral $\int_0^T H_t dX_t$ is independent of the choice of the partition and hence well defined.

Proof. Let $\mathcal{P}_1 = \{t_i, 0 = t_0 < t_1 < \ldots < t_n = T\}$ and $\mathcal{P}_2 = \{s_i, 0 = s_0 < s_1 < \cdots < s_m = T\}$ be two time partitions with at least one $i \le \min\{m, n\}$ such that $s_i \ne t_i$. We define third partition $\mathcal{P}_3 = \mathcal{P}_2 \setminus \mathcal{P}_1$ as the set of points of the second partition that are not included in the set of points of the first partition. Further, we define a set $R = \mathcal{P}_1 \cup \mathcal{P}_2$ such that $0 = r_0 < r_1 < \ldots < r_{n+|\mathcal{P}_3|} = T$ and thus it is a *finer* time partition of X.

If for any i, both partition points r_{i-1} and r_i are either in $\mathcal{P}_1$ or in $\mathcal{P}_2$, the integral $\int_{r_{i-1}}^{r_i} H_t dX_t$ is well defined as an Itô integral. If $r_{i-1} \in \mathcal{P}_1$ and $r_i \in \mathcal{P}_2$, then either $(r_{i-1}, r_i) \subset (t_{j-1}, t_j)$ for some $j \in \{1, \ldots, n\}$ or $(r_{i-1}, r_i) \subset (s_{k-1}, s_k)$ for some $k \in \{1, \ldots, m\}$. Therefore also in that case the integral $\int_{r_{i-1}}^{r_i} H_t dX_t$ is well defined. Finally, by gluing some of the intervals of the finer partition R, we can obtain the initial two partitions and consequentially

$$\sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} H_t dX_t = \sum_{i=1}^{|R|} \int_{r_{i-1}}^{r_i} H_t dX_t = \sum_{i=1}^{m} \int_{s_{i-1}}^{s_i} H_t dX_t.$$

□

5.3 Stochastic differential equations with local Brownian motion

Once the general Itô integral is defined, we are able to define the integral process as follows: Let $\{X_t, t \in [0,T]\}$ be a local Brownian motion and $\{H_t, t \in [0,T]\}$ be a regular piecewise adapted process with respect to X. Then,

$$\int_0^t H_s dX_s := \int_0^T H_s \mathbb{1}_{[0,t]}(s) ds.$$

Using this definition we can make sense of the following stochastic differential equation.

Definition 5.17. Let $\{X_t, t \in [0,T]\}$ be a local Brownian motion. Further, let $\{\mu(x,t) : x \in \mathbb{R}, t \in [0,T]\}$ and $\{\sigma(x,t) : x \in \mathbb{R}, t \in [0,T]\}$ be measurable functions. Consider the *stochastic differential equation (SDE)*

$$dY_t = \mu(Y_t, t)dt + \sigma(Y_t, t)dX_t \tag{5.2}$$

that must be interpreted as a *stochastic integral equation (SIE)*

$$Y_t = Y_0 + \int_0^t \mu(Y_s, s)ds + \int_0^t \sigma(Y_s, s)dX_s. \tag{5.3}$$

A stochastic process $\{Y_t, t \in [0,T]\}$ is said to be a *solution of SDE* (5.2) if it satisfies SIE (5.3) almost surely with the following conditions:

(i) $\int_0^T |\mu(Y_t, t)|dt < \infty$ almost surely,

(ii) $\sigma(Y_t, t)$ is a regular piecewise adapted process with respect to X and

$$\int_0^T \sigma(Y_t, t)^2 dt < \infty$$

almost surely.

The process $\{Y_t, t \in [0,T]\}$ is called *general Itô process* or *general diffusion process* with *drift* $\mu(x,t)$ and *diffusion* $\sigma(x,t)$.

Example 5.18. A general Itô process $\{Y_t, t \in [0,T]\}$ that starts at x and satisfies the stochastic differential equation $dY_t = \mu_t dt + dX_t$ can be seen as a local Brownian motion with drift.

Theorem 5.19 (Existence and uniqueness)**.** *Let $\{X_t, t \in [0,T]\}$ be a local Brownian motion with a canonical time partition $0 = \mathsf{t}_0 < \mathsf{t}_1 < \ldots < \mathsf{t}_\mathsf{n} = T$. If the following conditions are satisfied* *(i) for any $i \in \{1, 2, \ldots, \mathfrak{n}\}$, the functions $\{\mu(x,t) : x \in \mathbb{R}, t \in [\mathfrak{t}_{i-1}, \mathfrak{t}_i]\}$ and $\{\sigma(x,t) : x \in \mathbb{R}, t \in [\mathfrak{t}_{i-1}, \mathfrak{t}_i]\}$ are locally Lipschitz in x uniformly in t, i.e. for every T and N, there exists a constant $K(T, N) > 0$ such that for all $|x|, |y| \leq N$ and any $t \in [\mathfrak{t}_{i-1}, \mathfrak{t}_i]$,*

$$|\mu(x,t) - \mu(y,t)| + |\sigma(x,t) - \sigma(y,t)| < K|x-y|,$$

(ii) for any $i \in \{1, 2, \ldots, \mathfrak{n}\}$, the functions $\{\mu(x,t) : x \in \mathbb{R}, t \in [\mathfrak{t}_{i-1}, \mathfrak{t}_i]\}$ and $\{\sigma(x,t) : x \in \mathbb{R}, t \in [\mathfrak{t}_{i-1}, \mathfrak{t}_i]\}$ satisfy linear growth condition, i.e. there exists a constant $K > 0$ such that for all $x \in \mathbb{R}$ and any $t \in [\mathfrak{t}_{i-1}, \mathfrak{t}_i]$,

$$|\mu(x,t)| + |\sigma(x,t)| \leq K(1 + |x|),$$

(iii) Z is $\mathcal{F}_0^X$-measurable and $\mathbb{E}[Z^2] < \infty$,

then for a regular piecewise adapted process $\sigma(Y_t, t)$ with respect to X, the stochastic differential equation

$$dY_t = \mu(Y_t, t)dt + \sigma(Y_t, t)dX_t \quad \text{with} \quad Y_0 = Z \tag{5.4}$$

has a unique (strong) solution.

Proof. For a canonical time partition $0 = \mathfrak{t}_0 < \mathfrak{t}_1 < \ldots < \mathfrak{t}_\mathfrak{n} = T$, we can obtain the existence on each subinterval from the conventional SDE, i.e.

$$\begin{aligned} Y_T &= Z + \int_0^T \mu(Y_t, t)dt + \int_0^T \sigma(Y_t, t)dX_t \\ &= Z + \sum_{i=1}^{\mathfrak{n}} \left(\int_{\mathfrak{t}_{i-1}}^{\mathfrak{t}_i} \mu(Y_t, t)dt + \int_{\mathfrak{t}_{i-1}}^{\mathfrak{t}_i} \sigma(Y_t, t)dX_t \right) \\ &= Z + \sum_{i=1}^{\mathfrak{n}} Y_{\mathfrak{t}_i}^{(i)} \end{aligned}$$

with $Y_t^{(i)} = Y_t - Y_{\mathfrak{t}_{i-1}}$ on $[\mathfrak{t}_{i-1}, \mathfrak{t}_i]$, such that for any i and any $t \in [\mathfrak{t}_{i-1}, \mathfrak{t}_i]$, we have

$$dY_t^{(i)} = dY_t = \mu(Y_t, t)dt + \sigma(Y_t, t)dX_t$$

on $[\mathfrak{t}_{i-1}, \mathfrak{t}_i]$. The process $\{Y_t^{(i)}, t \in [\mathfrak{t}_{i-1}, \mathfrak{t}_i]\}$ exists and is unique for each i, hence the solution $\{Y_t, t \in [0, T]\}$ exists as well.

For uniqueness of the solution, let us assume there is another solution $\tilde{Y}$ that satisfies (5.4). Then, using Cauchy-Schwarz inequality

$$\begin{aligned} \mathbb{E}[(Y_T - \tilde{Y}_T)^2] &= \mathbb{E}\left[\left(\int_0^T \left(\mu(Y_t, t) - \mu(\tilde{Y}_t, t) \right) dt + \int_0^T \left(\sigma(Y_t, t) - \sigma(\tilde{Y}_t, t) \right) dX_t \right)^2 \right] \\ &\leq \mathfrak{n} \sum_{i=1}^{\mathfrak{n}} \mathbb{E}\left[\left(\int_{\mathfrak{t}_{i-1}}^{\mathfrak{t}_i} \left(\mu(Y_t, t) - \mu(\tilde{Y}_t, t) \right) dt + \int_{\mathfrak{t}_{i-1}}^{\mathfrak{t}_i} \left(\sigma(Y_t, t) - \sigma(\tilde{Y}_t, t) \right) dX_t \right)^2 \right] \\ &= \mathfrak{n} \sum_{i=1}^{\mathfrak{n}} \mathbb{E}\left[\left(Y_{\mathfrak{t}_i} - \tilde{Y}_{\mathfrak{t}_i} \right)^2 \right], \end{aligned}$$

with $d\tilde{Y}_t = \mu(\tilde{Y}_t, t)dt + \sigma(\tilde{Y}_t, t)dX_t$ on $[\mathfrak{t}_{i-1}, \mathfrak{t}_i]$. Due to the fact that the solution $Y_t^{(i)}$ is unique on $[\mathfrak{t}_{i-1}, \mathfrak{t}_i]$, it follows that $Y_t = \tilde{Y}_t$ for all $0 \leq t \leq T$ almost surely and the assertion follows. □

Let $\{X_t, t \in [0,T]\}$ be a local Brownian motion with a canonical time partition $0 = \mathfrak{t}_0 < \mathfrak{t}_1 < \ldots < \mathfrak{t}_\mathfrak{n} = T$. Further, let $\{Y_t, t \in [0,T]\}$ be a general Itô process with $dY_t = \mu_t dt + \sigma_t dX_t$ with $Y_0 = x$. Further, let $\{H_t, t \in [0,T]\}$ be a regular piecewise adapted process with respect to X satisfying $\int_0^T |H_s\mu_s| ds < \infty$ and $\int_0^T H_s^2\sigma_s^2 ds < \infty$. Then, for $t \in [0,T]$,

$$\int_0^t H_s dY_s = \int_0^t H_s\mu_s ds + \int_0^t H_s\sigma_s dX_s.$$

Due to the local "path resemblance" of local Brownian motion and Brownian motion, we also have the following theorem, analogous to the Itô Formula.

Theorem 5.20. *Let $\{X_t, t \in [0,T]\}$ be a local Brownian motion and*

$$Y_t = Y_0 + \int_0^t \mu_s ds + \int_0^t \sigma_s dX_s, \quad 0 \leq t \leq T.$$

For $f \in C^2(\mathbb{R};\mathbb{R})$ and $t \in [0,T]$,

$$\boxed{f(Y_t) = f(Y_0) + \int_0^t \left(f'(Y_s)\mu_s + \frac{1}{2}f''(Y_s)\sigma_s^2\right) ds + \int_0^t f'(Y_s)\sigma_s dX_s}. \tag{5.5}$$

Proof. For a canonical time partition $0 = \mathfrak{t}_0 < \mathfrak{t}_1 < \ldots < \mathfrak{t}_\mathfrak{n} = T$, we can apply Itô Formula on each subinterval and obtain

$$f(Y_t) = f(Y_{\mathfrak{t}_{i-1}}) + \int_{\mathfrak{t}_{i-1}}^t \left(f'(Y_s)\mu_s + \frac{1}{2}f''(Y_s)\sigma_s^2\right) ds + \int_{\mathfrak{t}_{i-1}}^t f'(Y_s)\sigma_s dX_s$$

for $t \in [\mathfrak{t}_{i-1}, \mathfrak{t}_i]$. The assertion then follows. □

We can also derive the analogous to Itô Formula for functions of two variables. But before that we need to define the quadratic variation and covariation of a stochastic integral with respect to a local Brownian motion. Recall the definition of quadratic variation of a process in Appendix A.1. Let $\{X_t, t \in [0,T]\}$ be a local Brownian motion with a canonical time partition $0 = \mathfrak{t}_0 < \mathfrak{t}_1 < \ldots < \mathfrak{t}_\mathfrak{n} = T$. For $Y_t = \int_0^t \sigma_s dX_s$, we have

$$[Y,Y]_T = [Y,Y]([0,T]) = \sum_{i=1}^{\mathfrak{n}} [Y,Y]([\mathfrak{t}_{i-1}, \mathfrak{t}_i]) = \sum_{i=1}^{\mathfrak{n}} \int_{\mathfrak{t}_{i-1}}^{\mathfrak{t}_i} \sigma_s^2 ds = \int_0^T \sigma_s^2 ds.$$

More generally, for a general Itô process $\{Y_t, t \in [0,T]\}$ such that $Y_t = Y_0 + \int_0^t \mu_s ds + \int_0^t \sigma_s dX_s$, the quadratic variation on $[0,T]$ is $[Y,Y]_T = \int_0^T \sigma_s^2 ds$.

The quadratic covariation of two general Itô processes $\{Y_t, t \in [0,T]\}$ and $\{Z_t, t \in [0,T]\}$ with a canonical time partition $0 = \mathfrak{t}_0 < \mathfrak{t}_1 < \ldots < \mathfrak{t}_\mathfrak{n} = T$ is defined as

$$[Y,Z]_T = \sum_{i=1}^{\mathfrak{n}} [Y,Z]([\mathfrak{t}_{i-1}, \mathfrak{t}_i]) = \lim_{\delta_{m_i} \to 0} \sum_{i=1}^{\mathfrak{n}} (Y_{s_j^{(i)}} - Y_{s_{j-1}^{(i)}})(Z_{s_j^{(i)}} - Z_{s_{j-1}^{(i)}}),$$

where the limit is taken over partitions $\mathfrak{t}_{i-1} = s_0^{(i)} < s_1^{(i)} < \ldots < s_{m_i}^{(i)} = \mathfrak{t}_i$ with $\delta_{m_i} = \max_{1 \le j \le m_i}(s_j^{(i)} - s_{j-1}^{(i)})$. For example, suppose $dY_t = \mu_t^Y dt + \sigma_t^Y dX_t$ and $dZ_t = \mu_t^Z dt + \sigma_t^Z dX_t$, where X is a local Brownian motion, then $[Y,Z]_T = \int_0^T \sigma_t^Y \sigma_t^Z dt$.

From the definition of the quadratic covariation, we can also deduce the product rule.

$$\boxed{d(Y_t Z_t) = Y_t dZ_t + Z_t dY_t + d[Y,Z]_t}.$$

Now we can state the Itô Formula for two general Itô processes.

Theorem 5.21. *Let $\{X_t, t \in [0,T]\}$ be a local Brownian motion and define Y and Z as general Itô processes satisfying the following SDE's*

$$dY_t = \mu_t^Y dt + \sigma_t^Y dX_t \quad \text{and} \quad dZ_t = \mu_t^Z dt + \sigma_t^Z dX_t.$$

Then, for $F \in C^{2,2}(\mathbb{R};\mathbb{R})$,

$$\begin{aligned} F(Y_t, Z_t) = F(Y_0, Z_0) &+ \int_0^t \left(\frac{\partial F}{\partial y}(Y_s, Z_s)\sigma_s^Y + \frac{\partial F}{\partial z}(Y_s, Z_s)\sigma_s^Z \right) dX_s \\ &+ \int_0^t \left(\frac{\partial F}{\partial z}(Y_s, Z_s)\mu_s^Z + \frac{\partial F}{\partial y}(Y_s, Z_s)\mu_s^Y + \frac{\partial^2 F}{\partial y \partial z}(Y_s, Z_s)\sigma_s^Y \sigma_s^Z \right) ds \\ &+ \frac{1}{2} \int_0^t \left(\frac{\partial^2 F}{\partial y^2}(Y_s, Z_s)(\sigma_s^Y)^2 + \frac{\partial^2 F}{\partial z^2}(Y_s, Z_s)(\sigma_s^Z)^2 \right) ds \end{aligned}$$

Proof. For a canonical time partition $0 = \mathfrak{t}_0 < \mathfrak{t}_1 < \ldots < \mathfrak{t}_\mathfrak{n} = T$, we can apply Itô Formula on each subinterval and the assertion follows. □

Corollary 5.22. Let $\{X_t, t \in [0,T]\}$ be a local Brownian motion. Then, for a general Itô processes $dY_t = \mu_t^Y dt + \sigma_t^Y dX_t$ and a function $F \in C^{2,1}(\mathbb{R};\mathbb{R})$, we have

$$\begin{aligned} F(Y_t, t) &= F(Y_0, 0) + \int_0^t \frac{\partial F}{\partial y}(Y_s, s) dY_s + \int_0^t \frac{\partial F}{\partial s}(Y_s, s) ds + \frac{1}{2} \int_0^t \frac{\partial^2 F}{\partial y^2}(Y_s, s) d[Y,Y]_s \\ &= F(Y_0, 0) + \int_0^t \left(\frac{\partial F}{\partial y}(Y_s, s)\mu_s^Y + \frac{\partial F}{\partial s}(Y_s, s) + \frac{1}{2}\frac{\partial^2 F}{\partial y^2}(Y_s, s)(\sigma_s^Y)^2 \right) ds \\ &+ \int_0^t \frac{\partial F}{\partial y}(Y_s, s)\sigma_s^Y dX_s. \end{aligned}$$

Next we will show that the notion of stochastic exponential remains valid with proper local Brownian motions. Let $\{Y_t, t \in [0,T]\}$ satisfy $dY_t = \mu_t dt + \sigma_t dX_t$ where $\{X_t, t \in [0,T]\}$ is a local Brownian motion with a canonical time partition $0 = \mathfrak{t}_0 < \mathfrak{t}_1 < \ldots <$

$\mathfrak{t}_\mathfrak{n} = T$ and $(\mu_t)_{t\in[0,T]}$ and $(\sigma_t)_{t\in[0,T]}$ are piecewise regularly adapted with respect to X with $\int_0^T |\mu_t|dt < \infty$ and $\int_0^T \sigma_t^2 dt < \infty$. The stochastic exponential of $\{Y_t, t \in [0,T]\}$ is the process $\{U_t, t \in [0,T]\}$ with $U_0 = 1$ that satisfies

$$dU_t = U_t dY_t,$$

which has the solution $U_T = e^{Y_T - Y_0 - \frac{1}{2}[Y,Y]_T}$. We need to be careful since the process $\{U_t, t \in [0,T]\}$ is not piecewise regularly adapted with respect to Y. However, the stochastic exponential on $[0,T]$ can be found by computing stochastic exponential on each subinterval $[\mathfrak{t}_{i-1}, \mathfrak{t}_i]$. For instance, we could start off from the last interval of the time partition $[\mathfrak{t}_{\mathfrak{n}-1}, T]$ and proceed backwards recursively, i.e on $[\mathfrak{t}_{\mathfrak{n}-1}, T]$ the stochastic differential equation $dU_t = U_t dY_t$ is well defined and has the solution

$$U_T = U_{\mathfrak{t}_{\mathfrak{n}-1}} e^{Y_T - Y_{\mathfrak{t}_{\mathfrak{n}-1}} - \frac{1}{2}[Y,Y]([T - \mathfrak{t}_{\mathfrak{n}-1}])}.$$

Note that $U_{\mathfrak{t}_{\mathfrak{n}-1}}$ does not depend on t on $[T - \mathfrak{t}_{\mathfrak{n}-1}]$ and the term $e^{Y_t - Y_{\mathfrak{t}_{\mathfrak{n}-1}} - \frac{1}{2}[Y,Y]([t - \mathfrak{t}_{\mathfrak{n}-1}])}$ is regularly adapted with respect to Y on $[\mathfrak{t}_{\mathfrak{n}-1}, T]$. The solution on each interval can be verified by using the Itô Formula given above. Moreover, the uniqueness can be shown by considering the process $\tilde{U}_t/U_t$, where $\{\tilde{U}_t, t \in [0,T]\}$ is another process satisfying the SDE of U, and show that $d(\tilde{U}_t/U_t) = 0$.

Example 5.23 (Black-Scholes model with local Brownian motion)**.** Let $\{X_t, t \in [0,T]\}$ be a local Brownian motion and $\{Y_t, t \in [0,T]\}$ satisfy

$$dY_t = \mu Y_t dt + \sigma Y_t dX_t,$$

where $\mu \in \mathbb{R}$ and $\sigma > 0$ are constants. Then, Y_t has the solution

$$Y_t = Y_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma X_t}, \quad 0 \leq t \leq T.$$

This can be shown by applying the Itô Formula with $f(x) = \ln x$.

Example 5.24 (Ornstein-Uhlenbeck model with local Brownian motion)**.** Let $\{X_t, t \in [0,T]\}$ be a local Brownian motion and let $\{Y_t, t \in [0,T]\}$ satisfy

$$dY_t = \alpha(\beta - Y_t)dt + \sigma dX_t,$$

where $\alpha, \beta \in \mathbb{R}$ and $\sigma > 0$ are constants. Then, Y_t has the solution

$$Y_t = Y_0 e^{-\alpha t} + \beta(1 - e^{-\alpha t}) + \sigma \int_0^t e^{-\alpha(t-s)} dX_s, \quad 0 \leq t \leq T.$$

This can be shown by applying the Itô Formula on $f(x,t) = xe^{\alpha t}$.

Since a proper local Brownian motion is not a semimartingale, there exists no risk-neutral measure, such that the discounted price will become a martingale, see Harrison und Pliska [24]. However, we are able to price a financial product with respect to the real-world probability measure.

Example 5.25 (Option price with respect to the real-world probability measure)**.** Let $\{X_t, t \in [0,T]\}$ be a local Brownian motion from Example 2.10. Let us consider the Black-Scholes SDE

$$dS_t = rS_t dt + \sigma S_t dX_t, \qquad S_0 = s$$

with $r \in \mathbb{R}^+$ and $\sigma > 0$. Then, for $t \in [0,T]$,

$$S_t = se^{(r-\frac{1}{2}\sigma^2)t+\sigma X_t}.$$

For $0 < t < 2$, the process X is a Brownian motion and hence the density is normal with mean 0 and variance t. For $0 \leq s \leq 1$ and $t > 2$, we obtain

$$\left.\begin{pmatrix} X_t - X_2 \\ X_2 - X_1 \\ X_1 - X_s \end{pmatrix}\right|_{R=r} \sim \mathcal{N}_3\left(0, \Sigma\right)$$

with

$$\Sigma = \begin{pmatrix} t-2 & 0 & r\min\{t-2, 1-s\} \\ 0 & 1 & 0 \\ r\min\{t-2,1-s\} & 0 & 1-s \end{pmatrix}.$$

Therefore

$$X_t - X_s | R = r \sim \mathcal{N}(0, t-s+2r\min\{t-2, 1-s\}),$$

which yields the density

$$f_{X_t - X_s}(x) = \mathbb{E}\left[\phi(0, t-s+2R\min\{t-2,1-s\}, x)\right]$$

such that $\phi(\mu, \sigma^2, x)$ is a density of a normal random variable with mean μ and variance σ^2 at point x. The expectation is taken with respect to R.

Let $f(x) = (x-K)^+$ be the payoff function of an European call option with strike K. By using Tonelli's theorem, we obtain the price of an European call option at time $1 < t < T$ for strike K, maturity T and the risk-free interest rate r

$$C(S_t, t; \sigma; K, T; r) = \mathbb{E}\left[S_t e^{\frac{\sigma^2}{2}(\psi(R)^2-(T-t))}\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2)\right]$$

with

$$\psi(R) = \sqrt{T-t+2R\min\{T-2, 1-t\}}$$

$$d_1 = \sigma\psi(R) + d_2$$

$$d_2 = \frac{\ln(\frac{S_t}{K}) + (r-\frac{1}{2}\sigma^2)(T-t)}{\sigma\psi(R)}.$$

Recall that the process $\{X_t, t \in [0,T]\}$ is a Brownian motion if and only if $R = 0$ almost surely. Therefore by inserting $R = 0$ in the above formula, it is not surprising that we will get the famous European call option formula obtained by Black and Scholes in [9]. Depending on the distribution of R, the price can be obtained either explicitly or by using Monte Carlo simulations.

Chapter 6

Conclusion, discussion and future outlook

In this thesis we introduced and developed the notion of a local Brownian motion, a process that behaves similarly to Brownian motion on sufficiently small neighbourhood around any point in time. If the behaviour of the system is chaotic on some small observation time interval, but has possibly some dependencies on the whole time interval, the class of local Brownian motion can be applied. The class of marginal distributions of any local Brownian motion is the class of distributions that can be represented by the finite sum of normal distributions and therefore can be fully described by the dependencies of those. In fact, these normal distributions represent the distributions of the increments of a local Brownian motion on small neighbourhood. In case that a local Brownian motion is a Brownian motion, the sum of independent normal increments is simply a normal random variable. Therefore, using a local Brownian motion instead of simply a Brownian motion as a driving noise enlarges the class of possible marginal distributions and includes the normal distribution. The framework of local Brownian motion being a driving noise in non-deterministic dynamical systems can be used for instance for applications where from empirical data analysis or other quantitative evidence, the marginal distributions of the driving noise is not fitted well by normal distribution. We have mentioned in introduction that it is the case for some time series taken from asset prices.

Furthermore, since our initial idea was to develop the local Brownian motion as a stochastic driving noise, we chose to define the stochastic integral with respect to a local Brownian motion is such a way that the tools from stochastic calculus remained valid and applicable. The main tool from stochastic calculus, the "Itô Formula", can be applied for functions of local Brownian motions and was extended to two dimensional case in the same fashion it was done in stochastic calculus for Brownian motion. We saw that many techniques can be appropriately established for dealing with stochastic calculus for the class of local Brownian motion. The disadvantage we had to face was the restriction on the integrand that had to be imposed. However, the restrictions turned out to be fairly loose. In particular, it was shown that many SDE's that can be explicitly solved with Brownian motion, can also be explicitly solved with a local Brownian motion as a driving noise.

Since a proper local Brownian motion is not a semimartingale, there exists no equivalent measure to the real-world measure such that the discounted stock price becomes a martingale. According to the first fundamental theorem in asset pricing theory, a proper local Brownian motion can not be used for pricing or hedging of tradable financial assets. However, for any non-tradable financial assets, such as weather derivatives, the finding of a price via modelling can be done. One would need to adjust the parameters taking either from empirical statistical analysis or from theoretical assumptions. We did not explore statistical techniques that could be useful for dealing with this task.

In this work two different families of local Brownian motions were presented with different families of parameters. In the first construction the family of parameters are the randomised scaled covariances between the incremental processes. In the second construction the parameters are uncorrelated copulas that represent the joint distribution of the incremental processes. In fact, we showed that it is possible to eliminate a linear dependency from any pair on incremental processes. That means that any copula can be used to construct a local Brownian motion. The proper choice of the copula should be made based on the information about the higher order dependencies of the incremental processes of local Brownian motion. With the financial applications in mind, we believe that especially the second family of local Brownian motion can lead to a wide variety of applications, where the linear dependencies do not exist but the increments are assumed to be not independent.

The future outlook for this work might lay in developing statistical methods for fitting the parameters of local Brownian motions to the empirical data. One will be required to estimate the higher order dependencies in order to do so. Moreover, there are two open questions that remained unanswered:

Open question 1: Based on the idea that the non-linear dependencies between the incremental processes of a proper local Brownian motion exist, we made a conjecture that a proper local Brownian motion is not a Markov process. In fact, we showed that with some restrictions on a proper local Brownian motion the statement is true. However, we were not able to show that the statement is true for all local Brownian motions. Hence the following question remains open: Can a proper local Brownian motion be a Markov process?

Open question 2: Another open question is whether a proper local Brownian motion with Gaussian marginals exists? We strongly suspect that it does, but did not have enough time to construct an example. In fact, this question can be answered positively if one can find a non-Markov process that satisfies the conditions mentioned in Proposition 4.14 but is not a Brownian motion. The existence of such process remains unknown as well.

Appendix A
Appendix

The definitions and statements given in this section can be found in many books on the general theory of stochastic analysis, Brownian motion or stochastic differential equations. My main reference source were the following books: Athreya und Lahiri [4], Klebaner [32], Williams und Rogers [56], Revuz und Yor [46], Klenke [33] and Karatzas und Shreve [31].

A.1 Brownian motion

Recall that any probability space can be completed by adding all null sets to its σ-algebra and extending the probability measure accordingly. Therefore by default, we will assume that any family of real-valued random variables in this thesis is defined on a *complete* probability space.

Definition. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A *filtration*, denoted by $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$, is a family of sub-$\sigma$-algebras of $\mathcal{F}$ such that for each $s < t$, the property $\mathcal{F}_s \subseteq \mathcal{F}_t$ holds.

We say a filtration satisfies the *usual conditions* or is *regular*, when we mean that the filtration is *complete and right-continuous*. Complete means that the trivial σ-algebra $\mathcal{F}_0$ is complete, i.e. if $B \in \mathcal{F}_0$ is a null set and $A \subset B$, then $A \in \mathcal{F}_0$. Right-continuous means that for an infinitesimally small move forward in time, the available information does not change, i.e.

$$\mathcal{F}_s = \mathcal{F}_{s+} := \bigcap_{t>s} \mathcal{F}_t$$

for any $s \geq 0$. We can build a regular filtration $\mathbb{F}^*$ from any filtration $\mathbb{F}$ by setting $\mathcal{F}^*_s = \mathcal{F}_{s+}$ and completing it. Throughout this thesis, we will assume that any filtered probability space is equipped with regular filtration, if nothing else stated.

Definition. A stochastic process $\{W_t, t \geq 0\}$ is called a *Brownian motion* if

(i) it has stationary increments, i.e. $W_t - W_s \sim \mathcal{N}(0, t-s)$ for any $s < t$,

(ii) it has independent increments, i.e. $W_t - W_s$ and $W_v - W_u$ are independent for any $s < t < u < v$,

(iii) it is continuous in t almost surely.

If a Brownian motion starts at zero, we will call it a *standard* Brownian motion.

Definition. A random vector $(X_1, X_2, \dots, X_n)^T$ is said to have *Gaussian* or *multivariate normal* distribution if any linear combination of its components is normally distributed, i.e. for any family $\{\lambda_i, i \in \{1, 2, \dots, n\}\} \in \mathbb{R}$, the sum $\sum_{i=1}^n \lambda_i X_i \sim \mathcal{N}(\mu, \sigma^2)$ for some parameters $\mu \in \mathbb{R}$ and $\sigma \geq 0$.

Definition. A stochastic process $\{X_t, t \geq 0\}$ is said to be *Gaussian* if all its finite dimensional distributions are Gaussian, i.e. for any $n \in \mathbb{N}$ and any $t_1, t_2, \dots, t_n \in \mathbb{R}_+$, the random vector $(X_{t_1}, X_{t_2}, \dots, X_{t_n})$ is Gaussian. A Gaussian process is defined by its mean and autocorrelation function.

Proposition. A stochastic process $\{W_t, t \geq 0\}$ is a standard Brownian motion if and only if it is a Gaussian process with covariance function

$$\gamma_W(s,t) = \mathbb{E}\left[(W_s - \mathbb{E}[W_s])(W_t - \mathbb{E}[W_t])\right] = \min\{s, t\}.$$

Definition. A stochastic process $\{M_t, t \geq 0\}$ defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is called an $(\mathbb{F}, \mathbb{P})$-*martingale* if

(i) M is integrable, i.e. $\mathbb{E}^{\mathbb{P}}[|M_t|] < \infty$ for any $t \geq 0$,

(ii) $\mathbb{E}^{\mathbb{P}}[M_t | \mathcal{F}_s] = M_s$ almost surely for any $s < t$.

An $(\mathbb{F}, \mathbb{P})$-martingale is called *uniformly integrable* if the condition

$$\lim_{n \to \infty} \sup_{t \geq 0} \mathbb{E}^{\mathbb{P}}[|M_t| \mathbb{1}_{\{|M_t| > n\}}] = 0$$

is satisfied.

The uniformly integrability is often used to show that a martingale converges in L^1 due to Doob's second martingale convergence theorem (see Doob [14]).

Definition. Let $\{X_t, t \in [0,T]\}$ be a process. A filtration $\mathbb{F}^X = \{\mathcal{F}_t^X, t \in [0,T]\}$ with $\mathcal{F}_t^X = \sigma(\{X_s, s \in [0,t]\})$ is called the *natural filtration* generated by X.

It is well known that the natural filtration generated by a Brownian motion is not right-continuous. However, if we complete it, it will satisfy usual conditions.

Definition. A stochastic process $\{X_t, t \geq 0\}$ defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}^X, \mathbb{P})$ is said to be *Markov* if for any $A \in \mathcal{B}(\mathbb{R})$ and any $t \geq 0$, $s \geq 0$,

$$\mathbb{P}(X_{t+s} \in A | \mathcal{F}_t^X) = \mathbb{P}(X_{t+s} \in A | X_t)$$

almost surely.

A Markov process has the property that conditional we know the value of the process at time t, the distribution of the process going forward in time does not depend on the distribution of the process before t. In other words, given the value of the process now, the future and the past of the process are independent.

Proposition. Let $\{W_t, t \geq 0\}$ be a Brownian motion defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}^W, \mathbb{P})$. Then, W is a $(\mathbb{F}^W, \mathbb{P})$-martingale and a Markov process.

It is not hard to see that any process with independent increments is a Markov process. The converse is not true.

Definition. An $\mathbb{F}$-*stopping time* $\tau : \Omega \to \overline{\mathbb{R}}_+$ is a non-negative random variable defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ such that for each $t \geq 0$,

$$\{\tau \leq t\} \in \mathcal{F}_t.$$

If we consider a filtration as a flow of information generated by a stochastic process, then a stopping time is a random variable such that its occurrence (or nonoccurrence) before time t depends only on the information generated by a process until time t.

Definition. A stochastic process $\{X_t, t \geq 0\}$ is called a $(\mathbb{F}, \mathbb{P})$-*local martingale* if it is adapted to $\mathbb{F}$, and there exists a sequence of monotone increasing $\mathbb{F}$-stopping times $(\tau_n)_{n \in \mathbb{N}_0}$ with $\tau_n \to \infty$ as $n \to \infty$ such that for each $n \in \mathbb{N}_0$, the stopped processes $\{X_{t \wedge \tau_n}, t \geq 0\}$ is a uniformly integrable $(\mathbb{F}, \mathbb{P})$-martingale.

Definition. The *variation* of a function $f : \mathbb{R} \to \mathbb{R}$ over an interval $[a, b]$ is defined as

$$V_f([a, b]) = \sup \sum_{i=1}^{n} |f(t_i^n) - f(t_{i-1}^n)|, \tag{A.1}$$

where the supremum is taken over partitions $a = t_0^n < t_1^n < \ldots < t_n^n = b$. The function f is said to be of *finite variation* on $[a, b]$, if $V_f([a, b]) < \infty$.

By triangular inequality, the sums in (A.1) increase if new points are added to the partition. Hence we can write the definition of variation of a function $f : \mathbb{R} \to \mathbb{R}$ over an interval $[a, b]$ as

$$V_f([a, b]) = \lim_{\delta_n \to 0} \sum_{i=1}^{n} |f(t_i^n) - f(t_{i-1}^n)|,$$

where $\delta_n = \max_{1 \leq i \leq n}(t_i^n - t_{i-1}^n)$.

Definition. The *quadratic variation* of a process $\{X_t, t \geq 0\}$ over an interval $[0, T]$ is defined as a limit in probability (if it exists)

$$[X, X]_T = [X, X]([0, T]) = \lim_{\delta_n \to 0} \sum_{i=1}^{n} (X_{t_i^n} - X_{t_{i-1}^n})^2,$$

where the limit is taken over all *shrinking*[1] partitions $0 = t_0^n < t_1^n < \ldots < t_n^n = T$ with $\delta_n = \max_{1 \leq i \leq n}(t_i^n - t_{i-1}^n)$.

Definition. The *quadratic covariation* of two processes $\{Y_t, t \geq 0\}$ and $\{Z_t, t \geq 0\}$ over an interval $[0, T]$ is defined as

$$[Y, Z]_T = \frac{1}{2} \left([Y + Z, Y + Z]_T - [Y, Y]_T - [Z, Z]_T \right).$$

It is easy to see that quadratic covariation is symmetric and linear. Furthermore, by using the definition of quadratic variation for processes Y and Z, the quadratic covariation can be written as a limit in probability (if it exists)

$$[Y, Z]_T = \lim_{\delta_n \to 0} \sum_{i=1}^{n} (Y_{t_i^n} - Y_{t_{i-1}^n})(Z_{t_i^n} - Z_{t_{i-1}^n}),$$

where the limit is taken over all *shrinking* partitions $0 = t_0^n < t_1^n < \ldots < t_n^n = T$ with $\delta_n = \max_{1 \leq i \leq n}(t_i^n - t_{i-1}^n)$.

Definition. A process is called a *semimartingale* if it can be written as a sum of local martingale and a process of finite variation.

In general, the decomposition of a semimartingale is not unique. However, a *continuous semimartingale* admits a unique decomposition as a sum of a continuous local martingale and an adapted continuous process of finite variation. The quadratic variation of a semimartingale exists on any finite interval. The sum of two semimartingales is a semimartingale. Therefore by using the polarisation identity, the quadratic covariation of two semimartingales on any finite interval exists.

Theorem (Lévy's characterisation theorem)**.** A process $\{W_t, t \geq 0\}$ is a standard Brownian motion if and only of it is a continuous $(\mathbb{F}^W, \mathbb{P})$-local martingale with quadratic variation process $[W, W]_t = t$ for any $t \geq 0$.

Proof. See Klebaner [32], Klenke [33] or Revuz und Yor [46]. □

[1] Note that if we take the limit over *all* finite partitions on $[0, T]$, the limit is called *2-variation*. For Brownian motion the quadratic variation on any finite interval is finite, whereas the 2-variation on any finite interval is infinite.

A.2 Hilbert spaces

Let us recollect some basic properties on Hilbert spaces.

Definition. A function $\|\cdot\| : V \to \mathbb{R}_+$ is called a *norm* if it satisfies

(i) triangle inequality, i.e. $\|v_1 + v_2\| \leq \|v_1\| + \|v_2\|$ for $v_1, v_2 \in V$,

(ii) scalar homogeneity, i.e. $\|\alpha v\| = |\alpha|\|v\|$ for $\alpha \in \mathbb{R}$ and $v \in V$,

(iii) $\|v\| = 0$ if and only if $v = \overline{0}$, where $\overline{0}$ is a zero vector in V.

A pair $(V, \|\cdot\|)$ of a vector space V and a norm $\|\cdot\|$ defined on it is called a *normed vector space.*

Definition. For $0 < p \leq \infty$, let $L^p(\Omega, \mathcal{F}, \mathbb{P})$ be the set of all random variables X on $(\Omega, \mathcal{F}, \mathbb{P})$ such that $\|X\|_{L^p} < \infty$, where for $0 < p < \infty$,

$$\|X\|_{L^p} = \left(\mathbb{E}\left[|X|^p\right]\right)^{\min\{\frac{1}{p},1\}}$$

and for $p = \infty$,

$$\|X\|_{L^\infty} = \inf\{k : \mathbb{P}(\{|X| > k\}) = 0\}$$

(called the *essential supremum* of X). Throughout this thesis we will use the notation L^p for $L^p(\Omega, \mathcal{F}, \mathbb{P})$.

For $0 < p < 1$, the power $\min\{\frac{1}{p}, 1\}$ makes sure that the space $L^p(\Omega, \mathcal{F}, \mathbb{P})$ is a vector space over $\mathbb{R}$ (see Athreya und Lahiri [4], p. 89-90).

Definition. A pair (S, d) of a set S and a function $d : S \times S \to \overline{\mathbb{R}}_+$ is called a *metric space* if d satisfies

(i) symmetry condition, i.e. $d(x, y) = d(y, x)$ for $x, y \in S$,

(ii) triangle inequality $d(x, y) \leq d(x, z) + d(z, y)$ for $x, y, z \in S$,

(iii) $d(x, y) = 0$ if and only if $x = y$.

The function d is called a metric on S.

Definition. Let (S, d) be a metric space. We say that $(x_n)_{n\geq 1}$ is a *Cauchy sequence* in (S, d) if for every $\varepsilon > 0$, there exists an N such that for all $n, m \geq N$, we have $d(x_n, x_m) \leq \varepsilon$.

Definition. A metric space (S, d) is said to be *complete* if every Cauchy sequence in (S, d) converges to an element in S. That is, for every Cauchy sequence $(x_n)_{n\geq 1}$ in (S, d), there exists an element $x \in S$ such that $\lim_{n\to\infty} d(x_n, x) = 0$.

In L^p spaces we will define the distance by $d(X,Y) = \|X - Y\|_{L^p}$ for any $X, Y \in L^p$.

Definition. A pair $(V, \langle\cdot,\cdot\rangle$ of a vector space V over $\mathbb{R}$ and a function $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{R}$ that satisfies

(i) symmetry condition, i.e. $\langle v, w\rangle = \langle w, v\rangle$ for any $v, w \in V$,

(ii) linearity condition, i.e. $\langle a_1 v_1 + a_2 v_2, w\rangle = a_1\langle v_1, w\rangle + a_2\langle v_2, w\rangle$ for all $a_1, a_2 \in \mathbb{R}$ and all $v_1, v_2, w \in V$,

(iii) nonnegativity condition, i.e. $\langle v, v\rangle \geq 0$ for any $v \in V$ and $\langle v, v\rangle = 0$ if and only if $v = 0$

is called a real *inner product space.* The function $\langle\cdot,\cdot\rangle$ is called the *inner product.*

Definition. Let $(V, \langle\cdot,\cdot\rangle)$ be a real inner product space. Define the norm on V as $\|\cdot\| := \langle\cdot,\cdot\rangle^{\frac{1}{2}}$ and for any $v, w \in V$, the metric $d(v,w) := \|v - w\|$. Then, if the metric space (V, d) is complete, the normed vector space $(V, \|\cdot\|)$ is called a *Hilbert space.*

Theorem. The normed vector space $(L^2, \|\cdot\|_{L^2})$ is a Hilbert space.

Proof. See Athreya und Lahiri [4]. □

A.3 Copulas

Copulas are widely used in applications to present the dependencies between marginal distributions of a random vector.

Definition. A function $f : [0,1]^2 \to [0,1]$ is called *2-increasing* if for any $u_1, u_2, v_1, v_2 \in [0,1]$ with $u_1 < u_2$ and $v_1 < v_2$,

$$f(u_2, v_2) - f(u_2, v_1) - f(u_1, v_2) + f(u_1, v_1) \geq 0.$$

Definition. A *copula (or a two-dimensional copula)* is a 2-increasing function $C : [0,1]^2 \to [0,1]$ such that for every $u, v \in [0,1]$,

$$C(u,0) = 0 = C(0,v), \quad C(u,1) = u \quad \text{and} \quad C(1,v) = v.$$

Theorem. (Sklar's theorem) Let X and Y be random variables with cumulative distribution functions F_X and F_Y, respectively. Let (X,Y) be a joint distribution of X and Y, with cumulative distribution function $F_{X,Y}$. Then, there exists a copula C such that

$$F_{X,Y}(x,y) = C(F_X(x), F_Y(y)) \tag{A.2}$$

for any $x, y \in \overline{\mathbb{R}}$. If F_X and F_Y are continuous, then C is unique; otherwise, C is uniquely determined on $\operatorname{supp}(X) \times \operatorname{supp}(Y)$. Conversely, suppose that C is a copula and F_X and F_Y are cumulative distribution functions of X and Y, respectively. Then, the function $F_{X,Y}$ defined in (A.2) is a joint distribution function with marginals F_X and F_Y.

Proof. See Nelsen [42]. □

A.4 Some extension techniques and results from probability

A useful theorem in probability is the monotone class theorem. It provides the recipe on how to extend a statement that is true on a smaller class of sets called a π-system, to a σ-algebra generated by it. We will need to use the analogous statement for functions.

Definition. A *π-system* P on a set Ω is a collection of subsets of Ω such that

(i) P is non-empty,

(ii) for any $A, B \in P$, we have $A \cap B \in P$.

Theorem (Monotone class theorem for functions)**.** Let Ω be a set and S be a π-system on Ω such that $\Omega \in S$. Let $\mathcal{H}$ be a vector space of real-valued functions on Ω satisfying:
i) For all $A \in S$, $\mathbb{1}_A \in \mathcal{H}$,
ii) For an increasing non-negative sequence of functions $(f_n)_{n\geq 1}$ in $\mathcal{H}$ such that $f = \lim_{n\to\infty} f_n$ is bounded, we have $f \in \mathcal{H}$.
Under these assumptions, $\mathcal{H}$ contains all real-valued, bounded and $\sigma(S)$ measurable functions on Ω.

Proof. See Chung [11]. □

In functional analysis, the continuous linear extension theorem[2] is used to extend a bounded linear transformation defined on a normed vector space to its completion. This theorem will be used to define the stochastic integral with respect to a local Brownian motion.

Theorem (Hahn-Banach Theorem)**.** Let U be a normed vector space with completion $\bar{U}$ and V be a complete normed vector space. Further, let $T : U \to V$ be a bounded linear transformation. Then, there exists a unique extension $\bar{T} : \bar{U} \to V$ with the same operator norm as T.

Proof. See Rynne und Youngson [48]. □

Definition. Let X be a random variable. The *characteristic function* of X is a function $\varphi_X : \mathbb{R} \to \mathbb{C}$ such that

$$\varphi_X(t) = \mathrm{E}\left[e^{itX}\right] = \int_{\mathbb{R}} e^{itx}\, dF_X(x).$$

[2]Also called bounded linear transformation theorem.

A characteristic function completely determines the distribution of a random variable. It is also a useful tool to show that a sequence of random variables converges to another random variable in distribution, as given by the next theorem due to Lévy.

Theorem (Lévy's continuity theorem)**.** Suppose a sequence of random variables $(X_n)_{n\geq 1}$ has a corresponding sequence of characteristic functions $(\varphi_{X_n})_{n\geq 1}$, that converges pointwise to a function φ, i.e.

$$\varphi_{X_n}(t) \to \varphi(t) \text{ for any } t \in \mathbb{R}$$

as $n \to \infty$. Then, the following statements are equivalent:

(i) X_n converges in distribution to some random variable X, i.e. the sequence of F_{X_n} converge at every continuity point to F_X,

(ii) φ is a characteristic function of X,

(iii) φ is continuous in t.

Proof. See Williams [55]. □

Note that the sequence of random variables $(X_n)_{n\geq 1}$ in Lévy's continuity theorem does not need to be defined on the same probability space.

Let two random variables be independent. It is easy to show that if each of them is normally distributed, then their sum is also normally distributed. The following theorem, which is due to Cramér, gives the converse statement.

Theorem (Cramér's Theorem)**.** Let X and Y be two independent random variables such that the sum $X + Y$ is normally distributed. Then, X and Y are normally distributed.

Proof. See Cramér [12]. □

A.5 Stochastic calculus

In this subsection we will only consider stochastic calculus with respect to Brownian motion. However, the general theory for stochastic calculus is defined for semimartingales as an integrator process. Moreover, some extensive work has been done in the area that defines the stochastic integral with respect to a Gaussian process (also in case the process is not a semimartingale) using Malliavin calculus. One of the prominent classes of such processes is called *a class of fractional Brownian motions* (see Biagini u. a. [7] and Alòs und Nualart [2]).

Definition. A function that has right and left limits at any point of the domain and have one-sided limits at the boundary is called *regular*. A right-continuous functions with left limits is called *càdlàg*.[3]

Regular functions are considered to be well behaved since the only discontinuities they might have are jumps. Stochastic calculus can easily deal with functions having jump discontinuities (e.g. see Klebaner [32], chapter 9 or Pascucci [44], chapter 14).

Definition. Let $([0,T]\times\Omega, \mathcal{B}([0,T])\otimes\mathcal{F}_T, \mathbb{P})$ be a probability space where $\mathcal{B}([0,T])\otimes\mathcal{F}_T$ is the product σ-algebra on the product space $([0,T]\times\Omega)$. A process $\{H_t, t\in[0,T]\}$ defined on this space is said to be *progressively measurable* if for any time t, the map

$$[0,t]\times\Omega\to\mathbb{R} \qquad (s,\omega)\mapsto H_s(\omega)$$

is $\mathcal{B}([0,t])\otimes\mathcal{F}_t$-measurable. This implies that $\{H_t, t\in[0,T]\}$ is $\mathcal{F}_t$-adapted.

Progressively measurable processes are important for definition of Itô integral. Every adapted right-continuous with left limits or left-continuous with right limits process is progressively measurable (see Klebaner [32], p.93).

Definition. Let $\{X_t, t\in[0,T]\}$ be a process. We will say that a process defined on $([0,T]\times\Omega, \mathcal{B}([0,T])\otimes\mathcal{F}^X_T, \mathbb{P})$ is *regular adapted with respect to* X, if it is càdlàg and progressively measurable.

The next theorem gives the definition of the Itô integral and its main properties.

Theorem (Itô Integral)**.** Let $\{W_t, t\in[0,T]\}$ be a Brownian motion. Further, let the process $\{H_t, t\in[0,T]\}$ be regular adapted with respect to W such that $\int_0^T H_t^2 dt<\infty$ almost surely. Then,

$$\int_0^T H_t dW_t$$

is called an *Itô integral* and has the following properties:

(i) zero-mean property, i.e. $\mathbb{E}\left[\int_0^T H_t dW_t\right]=0$,

(ii) Itô isometry property, i.e. if $\int_0^T \mathbb{E}[H_t^2]dt<\infty$, then

$$\mathbb{E}\left[\left(\int_0^T H_t dW_t\right)^2\right]=\int_0^T \mathbb{E}\left[H_t^2\right]dt.$$

Moreover, let us define an integral process as $\int_0^t H_s dW_s := \int_0^T H_s \mathbb{1}_{[0,t]}(s) dW_s$. Then, the process $\{\int_0^t H_s dW_s, t\in[0,T]\}$ is an $(\mathbb{F}^W,\mathbb{P})$-martingale if $\int_0^T \mathbb{E}[H_t^2]dt<\infty$.

[3] In French “continue à droite, limite à gauche”, which means “right continuous with left limits”.

Definition. For a deterministic square integrable processes on $[0, T]$, the Itô integral is also called the *Wiener integral*.

Theorem (Itô Formula)**.** Let $\{W_t, t \in [0, T]\}$ be a Brownian motion and $F \in C^2(\mathbb{R}; \mathbb{R})$. Then, for any $t \in [0, T]$,

$$F(W_t) = F(W_0) + \int_0^t F'(W_s)dW_s + \frac{1}{2}\int_0^t F''(W_s)ds$$

Definition. Let $\{W_t, t \in [0, T]\}$ be a Brownian motion. Further, let $\{\mu(x, t) : x \in \mathbb{R}, t \in [0, T]\}$ and $\{\sigma(x, t) : x \in \mathbb{R}, t \in [0, T]\}$ be measurable functions. Consider the *stochastic differential equation (SDE)*

$$dY_t = \mu(Y_t, t)dt + \sigma(Y_t, t)dW_t \tag{A.3}$$

that must be interpreted as a *stochastic integral equation (SIE)*

$$Y_t = Y_0 + \int_0^T \mu(Y_t, t)dt + \int_0^T \sigma(Y_t, t)dW_t. \tag{A.4}$$

A stochastic process $\{Y_t, t \in [0, T]\}$ is said to be a *solution of SDE* (A.3) if it satisfies SIE (A.4) almost surely with the following conditions:

(i) $\int_0^T |\mu(Y_t, t)|dt < \infty$ almost surely

(ii) $\sigma(Y_t, t)$ is a regular adapted process with respect to W and $\int_0^T \sigma(Y_t, t)^2 dt < \infty$ almost surely

The process $\{Y_t, t \in [0, T]\}$ is called *Itô process* or *diffusion process* with *drift* $\mu(x, t)$ and *diffusion* $\sigma(x, t)$.

The following theorem gives conditions on the parameters of the SDE for the existence and uniqueness of a strong solution. The adjective "strong" signifies that the solution of the stochastic differential equation exists on the same probability space as the underlying Brownian motion.

Definition. A measurable function $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is said to satisfy the *Lipschitz condition* in x if there exists a constant $K > 0$ such that

$$|f(t, x) - f(t, y)| \le K|x - y|$$

for any $t \in [0, T]$ and any $x, y \in \mathbb{R}$.

Definition. A measurable function $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is said to satisfy the *linear growth condition* in x if there exists a constant $K > 0$ such that

$$|f(t, x)| \le K(1 + |x|)$$

for any $t \in [0, T]$ and any $x \in \mathbb{R}$.

Theorem (Existence and uniqueness)**.** Let $\{W_t, t \in [0,T]\}$ be a Brownian motion. If the following conditions are satisfied

(i) the functions $\{\mu(x,t) : x \in \mathbb{R}, t \in [0,T]\}$ and $\{\sigma(x,t) : x \in \mathbb{R}, t \in [0,T]\}$ are locally Lipschitz in x uniformly in t, i.e. for every T and N, there exists a constant $K(T,N) > 0$ such that for all $|x|, |y| \leq N$ and any $t \in [0,T]$,

$$|\mu(x,t) - \mu(y,t)| + |\sigma(x,t) - \sigma(y,t)| < K|x-y|,$$

(ii) the functions $\{\mu(x,t) : x \in \mathbb{R}, t \in [0,T]\}$ and $\{\sigma(x,t) : x \in \mathbb{R}, t \in [0,T]\}$ satisfy linear growth condition, i.e. there exists a constant $K > 0$ such that for all $x \in \mathbb{R}$ and any $t \in [0,T]$,

$$|\mu(x,t)| + |\sigma(x,t)| \leq K(1+|x|),$$

(iii) Z is $\mathcal{F}_0^W$-measurable and $\mathbb{E}[Z^2] < \infty$,

then the SDE

$$dY_t = \mu(Y_t,t)dt + \sigma(Y_t,t)dW_t, \quad Y_0 = Z$$

has a unique strong solution.

Proof. See Rogers und Williams [47]. □

Theorem (Itô Formula for 2-dimensional functions)**.** Let $\{Y_t, t \in [0,T]\}$ and $\{Z_t, t \in [0,T]\}$ be two processes. Suppose that the differentials for Y and Z are defined, the quadratic variations for each process exists, and furthermore the quadratic covariation between two processes exists. Then, for $F \in C^{2,2}(\mathbb{R};\mathbb{R})$,

$$\begin{aligned} F(Y_t,Z_t) = F(Y_0,Z_0) &+ \int_0^t \frac{\partial F}{\partial y}(Y_s,Z_s)dY_s + \int_0^t \frac{\partial F}{\partial z}(Y_s,Z_s)dZ_s \\ &+ \frac{1}{2}\int_0^t \frac{\partial^2 F}{\partial y^2}(Y_s,Z_s)d[Y,Y]_s + \frac{1}{2}\int_0^t \frac{\partial^2 F}{\partial z^2}(Y_s,Z_s)d[Z,Z]_s \\ &+ \int_0^t \frac{\partial^2 F}{\partial y \partial z}(Y_s,Z_s)d[Y,Z]_s \end{aligned} \tag{A.5}$$

Proof. See Revuz und Yor [46] or Klebaner [32]. □

For the Itô process satisfying SDE (A.3), we have

$$[Y,Y]_t = \int_0^t \sigma_s^2 ds.$$

Furthermore, for Itô processes $dY_t = \mu_t^Y dt + \sigma_t^Y dX_t$ and $dZ_t = \mu_t^Z dt + \sigma_t^Z dX_t$, the quadratic covariation exists and is

$$[Y,Z]_t = \int_0^t \sigma_s^Y \sigma_s^Z ds$$

for any $t \in [0, T]$. Therefore the *product rule* for two Itô processes

$$d(Y_t Z_t) = Y_t dZ_t + Z_t dY_t + d[Y, Z]_t$$

for $t \in [0, T]$ holds.

Theorem (Itô Formula for two Itô processes)**.** Let $\{W_t, t \in [0, T]\}$ be a Brownian motion and define Y and Z as Itô processes satisfying the following SDE's

$$dY_t = \mu_t^Y dt + \sigma_t^Y dW_t \quad \text{and} \quad dZ_t = \mu_t^Z dt + \sigma_t^Z dW_t.$$

Then, for $F \in C^{2,2}(\mathbb{R}; \mathbb{R})$,

$$\begin{aligned} F(Y_t, Z_t) = F(Y_0, Z_0) &+ \int_0^t \left(\frac{\partial F}{\partial y}(Y_s, Z_s)\sigma_s^Y + \frac{\partial F}{\partial z}(Y_s, Z_s)\sigma_s^Z \right) dW_s \\ &+ \int_0^t \left(\frac{\partial F}{\partial z}(Y_s, Z_s)\mu_s^Z + \frac{\partial F}{\partial y}(Y_s, Z_s)\mu_s^Y + \frac{\partial^2 F}{\partial y \partial z}(Y_s, Z_s)\sigma_s^Y \sigma_s^Z \right) ds \\ &+ \frac{1}{2} \int_0^t \left(\frac{\partial^2 F}{\partial y^2}(Y_s, Z_s)(\sigma_s^Y)^2 + \frac{\partial^2 F}{\partial z^2}(Y_s, Z_s)(\sigma_s^Z)^2 \right) ds \end{aligned}$$

Proof. Application of (A.5) for two Itô processes yields the assertion. □

Corollary. Let $\{W_t, t \in [0, T]\}$ be a Brownian motion. Then, for an Itô processes $dY_t = \mu_t^Y dt + \sigma_t^Y dW_t$ and a function $F \in C^{2,1}(\mathbb{R}; \mathbb{R})$, we have

$$\begin{aligned} F(Y_t, t) &= F(Y_0, 0) + \int_0^t \frac{\partial F}{\partial y}(Y_s, s) dY_s + \int_0^t \frac{\partial F}{\partial s}(Y_s, s) ds + \frac{1}{2} \int_0^t \frac{\partial^2 F}{\partial y^2}(Y_s, s) d[Y, Y]_s \\ &= F(Y_0, 0) + \int_0^t \left(\frac{\partial F}{\partial y}(Y_s, s)\mu_s^Y + \frac{\partial F}{\partial s}(Y_s, s) + \frac{1}{2} \frac{\partial^2 F}{\partial y^2}(Y_s, s)(\sigma_s^Y)^2 \right) ds \\ &\quad + \int_0^t \frac{\partial F}{\partial y}(Y_s, s)\sigma_s^Y dW_s. \end{aligned}$$

Proof. Application of the theorem above with $\mu_t^Z = 1$ and $\sigma_t^Z = 0$ yields the assertion. □

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