The Hanani-Tutte Theorem and Non-separating Planar Graphs

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In the memory of my grandfather, Rahim Reisi Dehkordi.

Declaration

This thesis contains no material which has been accepted for the award of any other degree or diploma at any university or equivalent institution and that, to the best of my knowledge and belief, this thesis contains no material previously published or written by another person, except where due reference is made in the text of the thesis.

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Abstract

Graphs provide a useful data structure to represent sets of relationships among objects. Since drawings of graphs help humans to comprehend graphs better, one common approach to studying graphs is to construct drawings of them on surfaces. With this motivation, we study several types of drawings of graphs.

A drawing D of a graph is planar if no two edges in D cross each other. A graph is planar if it has a planar drawing. A graph G is a non-separating planar graph if it has a planar drawing D such that for any cycle C in D, any two vertices not in C are on the same side of C in D.

Non-separating planar graphs are closed under taking minors and are a subclass of planar graphs and a superclass of outerplanar graphs.

In this thesis, we first show that a graph is a non-separating planar graph if and only if it does not contain $K_1 \cup K_4$ or $K_1 \cup K_{2,3}$ or $K_{1,1,3}$ as a minor. Furthermore, we provide a structural characterisation of this class of graphs. More specifically, we show that any maximal non-separating planar graph is either an outerplanar graph or a subgraph of a wheel or it can be obtained by subdividing some of the side-edges of the skeleton of a triangular prism (two disjoint triangles linked by a perfect matching).

Moreover, to demonstrate an application of non-separating planar graphs, we use our structural characterisation of non-separating planar graphs to prove that, for all $n \geq 8$, there are maximal linkless graphs on n vertices with $3n-3$ edges. This provides a partial answer to a question that was asked by Horst Sachs about the number of edges of linkless graphs.

The Hanani-Tutte Theorem states that a graph is planar if it has a drawing D such that any two edges in D cross an even number of times. We prove a Hanani-Tutte type theorem for non-separating planar graphs and then use this theorem to prove a stronger version of the Hanani-Tutte Theorem, namely that a graph is planar if it has a drawing in which any two disjoint edges cross an even number of times or it has a chordless cycle that enables a suitable decomposition of the graph into smaller non-separating planar graphs.

A drawing of a graph is a superthrackle if any two edges in the drawing

cross exactly once. A graph that has a superthrackle drawing is superthracklable. A drawing of a graph on a disc is an outer-drawing if all the vertices of the drawing are on the boundary of the disc. An outer-drawing that is a superthrackle is an outersuperthrackle and a graph that has an outersuperthrackle drawing is an outer-superthracklable graph.

We characterise outersuperthrackles. Then we define variations of outersuperthracklable graphs such as generalised outersuperthracklables and weak outersuperthracklables and show that they are all equivalent to the class of outersuperthracklable graphs.

Next we show that the classes of superthracklable graphs and generalised superthracklable graphs are the same for all surfaces.

Lastly, we compare the Hanani-Tutte Theorem with our previous results in this paper and show that for any surface Σ, there is a relation between the class of graphs that are not embeddable on Σ and the class of graphs that are not superthracklable with respect to Σ. More specifically, we show that, for any forbidden minor G for embeddability of graphs on a surface Σ, there are two infinite families of graphs that we can construct from G that are not superthracklable with respect to Σ.

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1 Introduction

In this chapter we give a brief introduction to the field and a summary of our research. We define the scope of our research and provide context for our results about (1) local manipulations of embeddings of graphs, (2) characterisation of non-separating planar graphs, (3) a stronger version of the Hanani-Tutte Theorem, (4) outerthrackles and its variations and (5) the relation between the Hanani-Tutte Theorem and thrackles.

Graphs are used as conceptual tools to model objects and relationships between them. Applications of such models are ubiquitous through computer science, engineering, mathematics, chemistry, economics, biology and even social sciences as many practical systems can be modelled by graphs. For example, in computer science, graphs are used to represent computer networks [9], databases [119] and information systems [116]. In chemistry, they are used to model molecular structures [71, 122]. In biology, graphs represent evolutionary trees [39, 68, 95] and gene regulatory networks [82, 84] and in sociology graphs are used to model human social networks [81].

When trying to study and understand graphs, it is often useful to construct easy-to-read drawings of them. A *drawing* of a graph on a plane is a diagram consisting of a set of points representing vertices of the graph and a set of lines or curves between the points (and internally disjoint from the points)

drawing of a graph

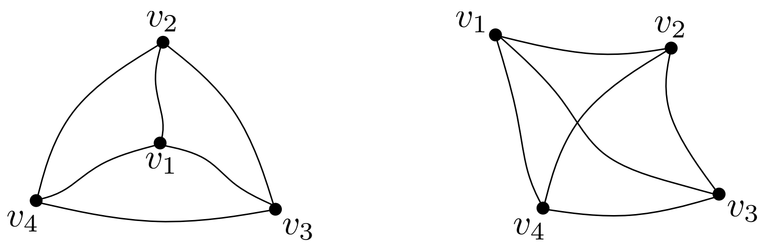


Fig. 1.1 Two drawings of the graph K_4

representing edges of the graph (see Figure 1.1). [1]

A graph has infinitely many drawings. To obtain a useful drawing of a graph for a specific application, we need to consider various criteria. For example, crossings or bends on edges may confuse whoever is reading a drawing. Therefore it is desirable to reduce the number of edge crossings or bends in a drawing [101, 102, 104, 131].

Constructing good drawings of graphs is so important that the field of *graph drawing*, as an area of mathematics and computer science, has developed and is the focus of the annual Graph Drawing Symposium. It combines methods from graph theory, geometry and information visualisation to devise graph drawing algorithms [8, 93]. *graph drawing*

Before we present our results, we briefly introduce three classes of graphs that are very closely related to our results: planar graphs, linkless graphs and thracklable graphs.

1.1 Planar Graphs

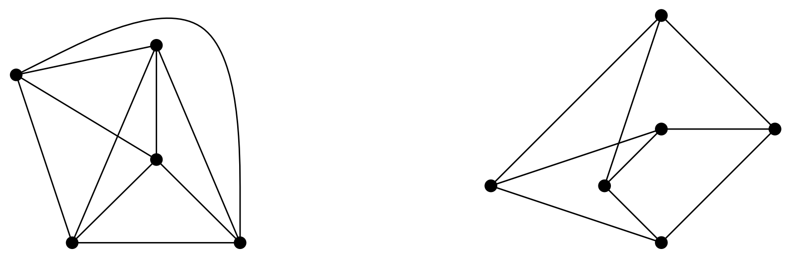
A *planar drawing* is a drawing in which no two edges cross. *Planar graphs* are the graphs that have a planar drawing. *planar drawing* *planar graph*

Planar graphs have a long history in graph theory [92]. Also, experiments of Purchase et al. [102, 104, 131] have shown that readability of a drawing is negatively correlated with the number of edge crossings. These two facts among others have motivated extensive investigations of drawings of graphs with no crossings (see [8, 71, 93, 116]).

An edge $e = (u, v)$ in a graph G is *subdivided* by replacing it with two edges, (u, w) and (w, v), where w is not a vertex of G. A subdivision of a graph G is a graph that can be obtained by subdividing an edge of the graph G or a subdivision of G. Any graph is a subdivision of itself. *subdivision*

In 1930, Kuratowski characterised planar graphs in terms of two forbidden

[1] For formal definitions of the terms used in this thesis refer to Chapter 2.



(a) K_5 (b) $K_{3,3}$

Fig. 1.2 Forbidden subdivisions/minors of planar graphs

subdivisions [75]. More specifically, he proved the following theorem:

Theorem 1 (Kuratowski's Theorem [75]). *A graph is planar if and only if it does not contain a subdivision of K_5 or a subdivision of $K_{3,3}$ as a subgraph (see Figure 1.2).*

Any graph G' that can be obtained from G by a series of edge deletions, vertex deletions and edge contractions is called a *minor* of G. We write $G' \preceq G$. Any graph G is a minor of itself. *minor*

Later on, Wagner proved the following theorem which implies Kuratowski's theorem:

Theorem 2 (Wagner [129]). *A graph is planar if and only if it does not contain K_5 or $K_{3,3}$ as a minor.*

Another famous characterisation of planar graphs is the Hanani-Tutte Theorem (sometimes known as the Strong Hanani-Tutte Theorem).

Theorem 3 (Strong Hanani-Tutte Theorem [124]). *In any drawing of a non-planar graph there exist two vertex-disjoint edges that cross each other an odd number of times.*

In other words, if we can draw a drawing of a graph G such that any two vertex-disjoint edges in D cross each other an even number of times then G is planar.

The Hanani-Tutte Theorem is often stated in the following weaker form:

Theorem 4 (Weak Hanani-Tutte Theorem). *If there is a drawing D of a graph G such that any two edges in D cross each other an even number of times, then G is planar.*

The Hanani-Tutte Theorem immediately holds in the reverse direction and therefore Hanani-Tutte Theorem provides us with a characterisation of planarity.

1.2 Linkless Graphs

A *realisation* of a graph $G = (V, E)$ is a representation $\mathcal{R}$ of G in $\mathbb{R}^3$, where each vertex is a distinct point and each edge is a closed continuous arc between the points representing its endpoints. Informally, realisations of graphs are drawings of graphs in $\mathbb{R}^3$. A *projection* of a graph G is a drawing of G with an over/under relation between the edges specified at each crossing.

realisation

projection

Kaufmann [70] and Yamada [136] independently proved that if $\mathcal{P}$ and $\mathcal{P}'$ are two (piecewise linear) projections of the same realisation of a graph in $\mathbb{R}^3$, then $\mathcal{P}$ and $\mathcal{P}'$ are related by a finite sequence of the local moves as is stated in the following theorem.

Theorem 5 (Kaufmann [70] and Yamada [136])**.** *Let $\mathcal{P}$ and $\mathcal{P}'$ be two projections of the same realisation of a graph. Then one can obtain $\mathcal{P}'$ from $\mathcal{P}$ by a finite sequence of the local moves given in Figure 1.3.*

Two vertex-disjoint cycles C_1 and C_2 that are embedded in $\mathbb{R}^3$ are *linked* if no topological sphere can be embedded in $\mathbb{R}^3$ separating C_1 from C_2. To put it in another way, two cycles C_1 and C_2 are not linked (unlinked) if they can be continuously deformed without ever intersecting each other until C_1 and C_2 end up on two different sides of a topological sphere embedded in $\mathbb{R}^3$. Informally, a link consists of two cycles that are embedded in three dimensions such that they cannot be separated unless we cut one of them (see Figure 1.4).

linked cycles

A realisation $\mathcal{R}$ of a graph is *linkless* if it does not contain two linked cycles. In other words, a realisation $\mathcal{R}$ of a graph is *linkless* if for any two disjoint cycles C_1 and C_2 in $\mathcal{R}$ one can embed a topological sphere in $\mathbb{R}^3$ that separates C_1 from C_2. A graph is *linkless* if it has a linkless realisation.

linkless realisation

linkless graph

1.3 Thracklable and Superthracklable Graphs

A *thrackle* is a drawing of a graph in which any two edges have exactly one point in common. In other words, in a thrackle, any two vertex-disjoint edges cross exactly once and incident edges do not cross (see for example, Figure 1.5). Any graph that has a thrackle drawing on a surface Σ is *thracklable* with respect to Σ.

thrackle

thracklable

Since readability of a drawing is negatively correlated with the number of edge crossings, we usually try to construct drawings of graphs with as few crossings as possible [103, 104, 132]. For this reason, in the literature, the following three constraints are usually enforced on graph drawings:

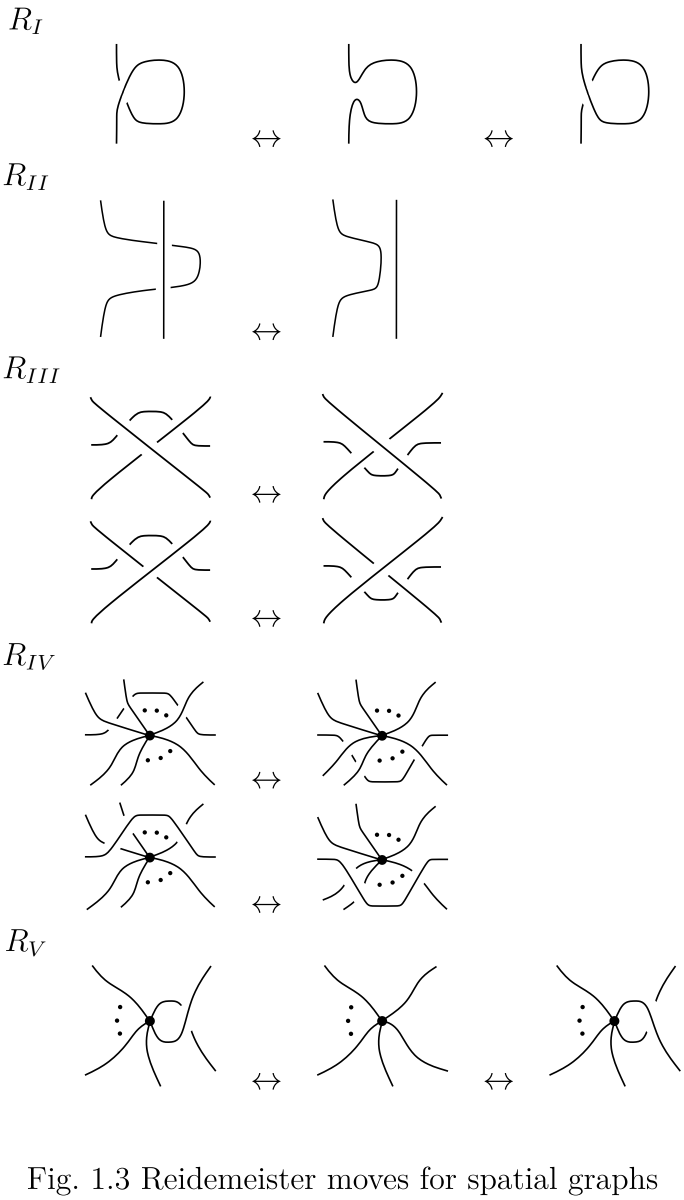


Fig. 1.3 Reidemeister moves for spatial graphs

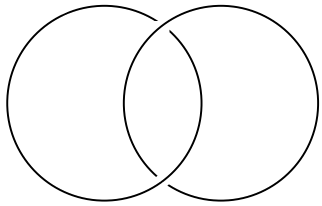


Fig. 1.4 A link is composed of two cycles in three dimensions that cannot be separated from each other.

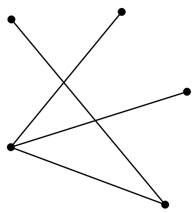


Fig. 1.5 A thrackle

- an edge does not cross itself.
- incident edges do not cross themselves.
- two edges do not cross more than once.

Having the above constraints in mind, a planar drawing is a best drawing that we hope we can construct for a graph and a thrackle is a worst drawing that we may be able to construct for a graph such that it meets all of the above constraints.

Many different variations of thrackles have been defined and studied during the past few years.

An *outerdrawing* of a graph G is a drawing of G on a disc such that all the edges are drawn on the disk and all the vertices of the drawing are on the boundary of the disc.

outerdrawing

An outerdrawing that is a thrackle is an *outerthrackle*. A graph that has a outerthrackle drawing is *outerthracklable*.

outerthrackle

outerthracklable

Cairns and Nikolayevsky characterised outerthracklable graphs as follows.

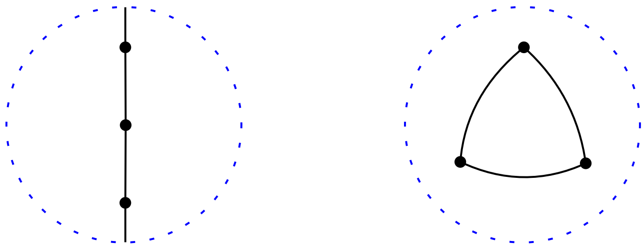
Theorem 6 ([17]). *Let G be an outerthracklable graph such that* $\deg(v) \geq 2$ *for any vertex v in G. Then G is an odd cycle.*

Any simple cycle on the projective plane is *2-sided* if it has a neighbourhood isomorphic to a cylinder. Any simple cycle on the projective plane is *1-sided* if it has a neighbourhood isomorphic to a Möbius strip (see for example, Figure 1.6). A *parity embedding* is an embedding of a graph in the projective plane in which a simple cycle C is 1-sided if and only if C is of odd length.

2-sided cycle

1-sided cycle

parity embedding



(a) A 1-sided cycle (b) A 2-sided cycle

Fig. 1.6 Examples of 1-sided and 2-sided cycles on the projective plane

A drawing D of a graph G is a *generalised thrackle* if any two edges in D have an odd number of points in common [16] (see for example, 2.17). Any graph with a generalised thrackle drawing is a *generalised thracklable* graph.

generalised thrackle

generalised thracklable

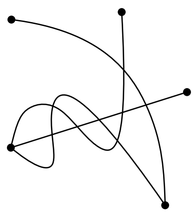


Fig. 1.7 A generalised thrackle

Cairns and Nikolayevsky characterised generalised thracklable graphs with respect to the plane as follows.

Theorem 7 ([16])**.** *A graph is generalised thracklable if and only if it has a parity embedding in the projective plane.*

Superthrackles are defined by Archdeacon and Stor [4]. A drawing D of a graph G on a surface Σ is a *superthrackle* if any two edges in D cross each other exactly once [4] (see for example, Figure 1.8). Any graph that has a superthrackle drawing on a surface Σ is *superthracklable* with respect to Σ.

superthrackle

superthracklable

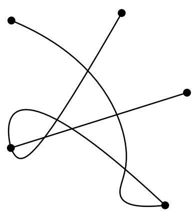


Fig. 1.8 A superthrackle

Archdeacon and Stor characterised superthrackles as follows.

Theorem 8 ([4])**.** *A graph is superthracklable with respect to the plane if and only if it has a parity embedding in the projective plane.*

An *even edge-subdivision* of an edge e in a graph replaces edge e with a path of an odd length. A *vertex subdivision* at u in a graph G subdivides every edge that is incident with u once.

even edge-subdivision

vertex subdivision

Two graphs G and G' are *parity homeomorphic* if there is a graph H that can be obtained from G and from G' by the operations of even edge-subdivision or vertex subdivision.

parity homeomorphic

Archdeacon and Stor also proved the following.

Theorem 9 ([4]). *The following two classes of graphs are equivalent:*

- *superthracklable graphs with respect to the plane,*
- *the graphs that does not have a subgraph that is parity homeomorphic to any graph in Figure 1.9.*

A drawing D of a graph G on a surface Σ is a *generalised superthrackle* if any two edges in η cross each other an odd number of times. Any graph with a generalised superthrackle drawing on a surface Σ is a *generalised superthracklable* graph with respect to Σ.

generalised superthrackle

generalised superthracklable

Cairns and Nikolayevsky characterised generalised thracklable graphs [16] and Archdeacon and Stor characterised superthracklable graphs [4] and the two characterisations are the same (see Theorem 31 and Theorem 8). That is, any generalised thracklable graph is a superthracklable graph.

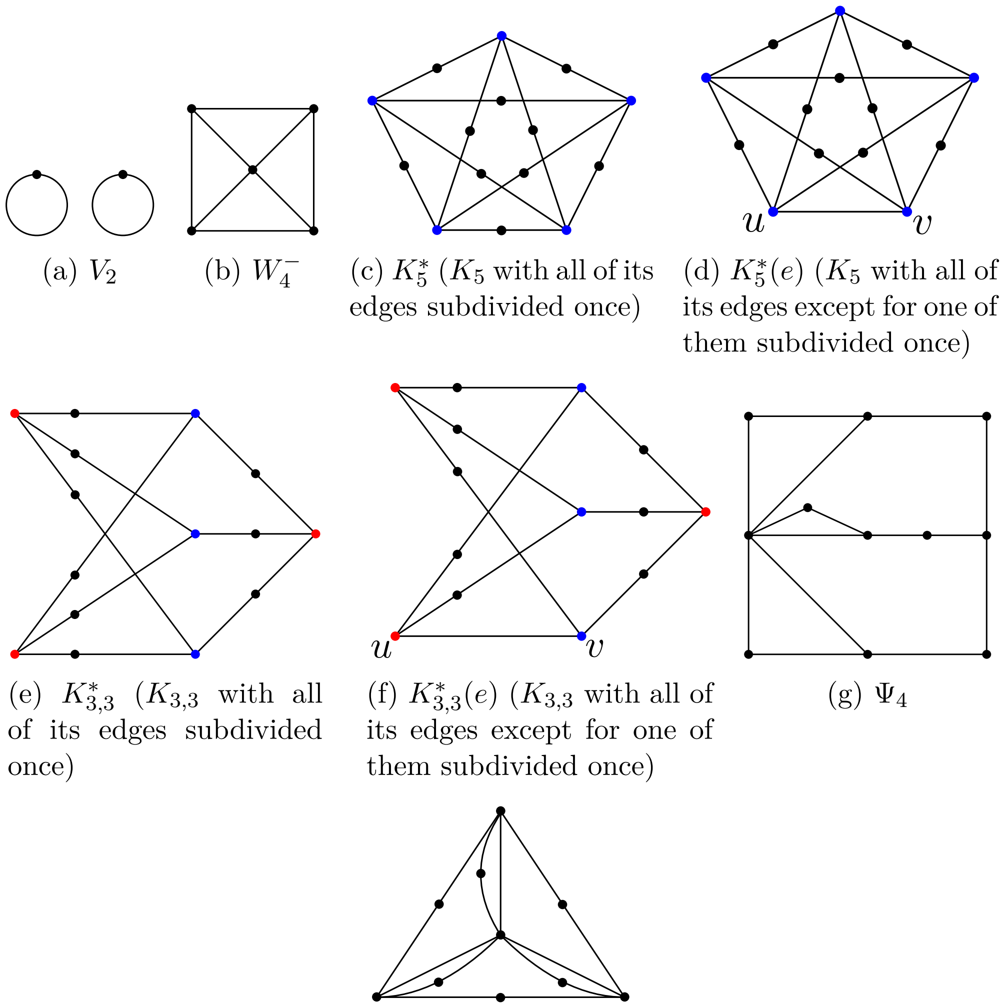
Theorem 10 ([4]). *A graph is superthracklable if and only if it is generalised superthracklable.*

1.4 Our Research

In this thesis we aim at studying graphs through their drawings. We define classes of graphs based on whether they have drawings that meet certain conditions and then we characterise those classes of graphs in various ways. Next we outline the structure of this thesis and provide a summary of our results in each chapter.

Chapter 2 contains the terminology and formal definitions that are frequently used in this thesis. We survey the main results in the field and provide the reader with the necessary background and literature review. This chapter also serves as a motivation for the results that are presented later in this thesis.

In Chapter 3 we present a number of moves for manipulating drawings in order to obtain different drawings of the same graph. We prove that any



(a) V_2

(b) W_4^-

(c) K_5^* (K_5 with all of its edges subdivided once)

(d) $K_5^*(e)$ (K_5 with all of its edges except for one of them subdivided once)

(e) $K_{3,3}^*$ ($K_{3,3}$ with all of its edges subdivided once)

(f) $K_{3,3}^*(e)$ ($K_{3,3}$ with all of its edges except for one of them subdivided once)

(g) Ψ_4

(h) Φ_4

Fig. 1.9 The obstruction set for superthrackles drawing D of a graph G can be manipulated to any other drawing D' of G using the aforementioned set of local moves. We then provide a set of local moves to change any drawing D of G to any other drawing D' of G on any given surface Σ. Results of this chapter will be used frequently throughout the rest of this thesis.

In Chapter 4 we prove two Hanani-Tutte type theorems for outerplanar graphs. The first Hanani-Tutte type theorem relates the parity of the number of crossings between edges of the graph and its cycles with the rotational order of the edges around the vertices of the graph in the drawing. The second Hanani-Tutte type theorem relates the parity of the number of crossings between vertex-disjoint edges to the rotational order of the edges around the outerface of the drawing.

Chapter 5 is dedicated to a class of graphs called non-separating planar graphs. We introduce this class of graphs since we encountered it multiple times during our research. Non-separating planar graphs, are a subclass of planar graphs and are closed under taking minors and therefore by the graph minor theorem they can be characterised by a finite family of minimal excluded minors. In this chapter, we determine the family of excluded minors for non-separating planar graphs, which consists of three graphs. We characterise non-separating planar graphs in terms of these three minimal excluded minors. Moreover, we provide a structural characterisation for non-separating planar graphs.

Chapter 6 provides us with a Hanani-Tutte type theorem for non-separating planar graphs. This characterisation will be used later to prove a stronger version of the Hanani-Tutte Theorem for planar graphs.

Chapter 7 is about two applications of non-separating planar graphs. First, it provides a stronger version of the Hanani-Tutte Theorem. As well as relying on the results of Chapter 5, we rely on the results of Chapter 4 to decompose planar graphs into smaller non-separating planar graphs and prove a stronger version of the Hanani-Tutte Theorem. As the second application of non-separating planar graphs, we construct a family of maximal linkless graphs that have at most $3|V| - 3$ edges, where $|V|$ is the number of vertices of the graph. This answer is related to a question asked by Sachs in 1983 [113].

In Chapter 8 we define different variations of outerthrackles and for any variation V of those variations, we characterise the class of graphs that have a V drawing. Lastly, we prove that all of these classes are equivalent.

Chapter 9 is dedicated to to finding a Hanani-Tutte type theorem for thrackles. In this chapter, we first prove a Hanani-Tutte Theorem for superthrackles. In fact we show that there is a theorem similar to the Weak Hanani-Tutte Theorem for superthrackles. Then we show that there is no theorem similar to the Strong Hanani-Tutte Theorem for superthrackles by providing a counterexample.

Lastly, Chapter 10 concludes the thesis by reflecting on the main theorems and results and presenting open problems that arise from our research as well as the directions for future work.

1.5 Publications

The following three papers are submitted as the results of the research conducted in this thesis.

- **Non-separating Planar Graphs** (with Graham Farr), submitted (see [31]).
- **On the Strong Hanani-Tutte Theorem** (with Graham Farr), submitted (see [32]).
- **Thrackles, Superthrackles and the Hanani-Tutte Theorem** (with Graham Farr), submitted (see [33]).

2

Literature Review

In this chapter we introduce the reader to the basic terminology of graph drawing such as graphs, drawings, realisations, embeddings, etc. We review the important results of the field and classify the graphs based on the surfaces that they can be embedded in.

We start this chapter by defining graphs and their drawings/realisations and work our way through to embeddings of graphs and other terms that we use troughout this thesis. For a more comprehensive discussion on these definitions and concepts we refer the reader to the following books:

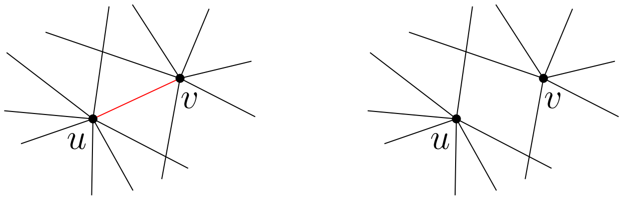
- For graph drawing concepts refer to [8, 71, 89, 93].
- Graph theory is described in [34, 62].
- For computational geometry see [29, 59].
- For complexity theory refer to [56].

2.1 Graphs

A *graph* $G = (V, E)$ consists of a finite set V of vertices and a finite multiset E of edges where each *edge* is an unordered pair of vertices.

graph

edge



(a) G (b) $G' = G \setminus (u, v)$

Fig. 2.1 Edge deletion

An edge that consists of vertex u and vertex v is denoted by (u, v). We usually denote the set of vertices of G by $V(G)$ and the set of edges of G by $E(G)$. The numbers of vertices and edges of a graph $G = (V, E)$ are $|V|$ and $|E|$ respectively.

loop
parallel edge
simple graph
multigraph

An edge (u, v) is a *loop* if $u = v$ and edges that are included more than once in the multiset E are called *parallel* edges. Two edges e and e' are *parallel* if they have the same endpoints. A graph that does not contain any loops or parallel edges is called *simple*. A graph in which loops and parallel edges are allowed is called a *multigraph*.

isomorphism
isomorphic

An *isomorphism* of graphs G and G' is a 1-1 mapping f of $V(G)$ onto $V(G')$ such that for any pair of vertices u and v: $(u, v) \in E(G)$ if and only if $(f(u), f(v)) \in E(G')$. Two graphs G and G' are *isomorphic* if there is an isomorphism between G and H.

For any vertex u in a graph G, we denote the set of neighbours of u by $N(u)$.

2.2 Graph Minor Theory

The theory of graph minors developed by Robertson and Seymour is one of the most important recent advancements in graph theory and even in mathematics. This substantial body of work is presented in a series of 23 papers (Graph Minors I–XXIII) over 20 years from the 1980s to 2004.

In order to state the main graph minor theorems, we first need to define some terms. Let $G = (V, E)$ be an arbitrary graph. Then one can obtain a graph $G' = (V', E')$ by:

edge deletion

1. *deleting an edge* (u, v) from G, where $V' = V$ and $E' = E \setminus (u, v)$. We denote G' by $G' \setminus (u, v)$ (see, for example, Figure 2.1).

vertex deletion

2. *deleting a vertex* u of G, where $V' = V \setminus u$ and $E' = E \setminus \{(u, v)|v \in V\}$ (see, for example, Figure 2.2).

3. *contracting an edge* (u, v) of G, where $u \neq v$ and $V' = V \setminus \{u, v\} \cup w$ and $E' = E \setminus \{(u, x), (v, x) | x \in V\} \cup \{(w, x) | x \in V, (u, x) \in E \vee (v, x) \in E\}$ (see, for example, Figure 2.3).

edge contraction

Any graph G' that can be obtained from G by a series of the above operations (edge deletion, vertex deletion, edge contraction) is called a *minor* of G. We denote this relation by $G' \preceq G$. Any graph G is a minor of itself ($G \preceq G$). A minor of G that is not isomorphic to G is called a *proper minor*.

minor

Any graph G'' that can be obtained from G by a series of the first two operation mentioned above (edge deletion, vertex deletion) is called a *subgraph* of G. The graph G'' is a *spanning* subgraph of G if it includes all the vertices of G.

subgraph

spanning

Now we state the Graph Minor Theorem (also known as the Robertson–Seymour Theorem or Wagner's Conjecture[1]).

Theorem 11 (Graph Minor Theorem [109]). *Let S be an infinite set of graphs. Then there exist two graphs G and G' in S such that G' is a minor of G.*

A set S of graphs is a *minor-closed set* or *minor-closed family* or *minor-closed class* of graphs if any minor of a graph $G \in S$ is also a member of S.

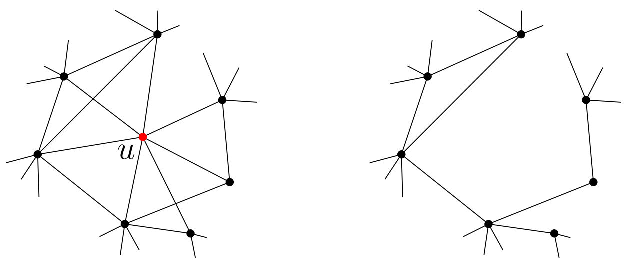
minor-closed set

Graph H is an *minimal excluded minor* for a family $\mathcal{G}$ of graphs, if no graph $G \in \mathcal{G}$ contains H as a minor and for any graph H^-, where $H^- \preceq H$, there is a graph $G' \in \mathcal{G}$ that contains H^- as a minor.

minimal excluded minor

The Graph Minor Theorem can also be formulated as follows:

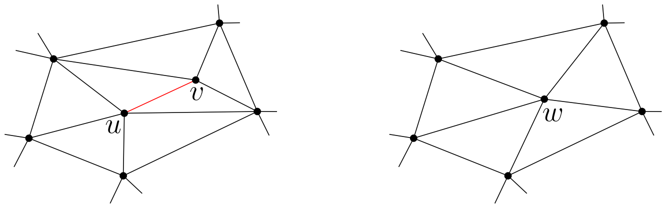
Corollary 1 ([109]). *For every minor-closed class of graphs $\mathcal{G}$, the set of minimal excluded minors for $\mathcal{G}$ is finite.*



(a) G (b) $G' = G \setminus u$

Fig. 2.2 Vertex deletion

[1] Wagner has posed this theorem in terms of a question [130] and although he has never conjectured this result it was known as Wagner's conjecture [34].



(a) G (b) $G' = G/(u, v)$

Fig. 2.3 Edge contraction

Another important result of the graph minor theory is about a polynomial time algorithm to verify whether a graph G' is a minor of a graph G.

Theorem 12 (Robertson and Seymour [108]). *For any fixed graph G', there is an algorithm to determine whether a given graph G with n vertices contains G' as a minor in $O(n^3)$ time*[2].

Theorem 12 and Corollary 1 together show that for every minor-closed set S of graphs, there is a polynomial time algorithm for testing whether a graph belongs to S or not. In other words, given a graph G and a minor-closed set S of graphs, one can check whether G is in S by checking whether G contains G' as a minor for each minimal excluded minor G' of S [11].

Theorem 13 ([12, 108]). *Let S be a minor-closed set of graphs. Then there is a polynomial time algorithm to test whether any graph G is in S.*

It is important to emphasise that Theorem 13 is non-constructive. That is, Theorem 13 only guarantees the existence of a polynomial time algorithm but does not provide us with the algorithm (which makes it an unusual result).

2.3 Embeddings of Graphs

A graph G is *embedded* in a topological space X if the vertices of G are distinct elements of X and every edge (u, v) of G is represented by a simple arc in X that joins u to v such that all such arcs are internally disjoint from each other and from the vertices of G. An *embedding* of a graph G in a topological space X is an isomorphism of G with a graph G' embedded in X [89]. embedding

A *surface* or a two-dimensional manifold Σ is a connected compact topological space which is locally homeomorphic to an open disk in the plane and for any two distinct points $x, y \in \Sigma$, there exists open neighbourhoods N_x of surface

[2] Recently, Kawarabayashi, Kobayashi and Reed improved the time complexity of this algorithm to $O(n^2)$ [72].

x and N_y of y such that $N_x \cap N_y = \emptyset$ [89]. To define a *surface with a boundary* we relax the definition of a surface and we allow the neighbourhood of the points to be homeomorphic to half-planes as well as disks.

surface with a boundary

Embeddings of graphs in two-dimensional manifolds are specified by the cyclic order of the edges around vertices [38, 106]. In this thesis, we denote the cyclic order of the edges at a vertex v of an embedding η by $\pi_\eta(v)$ and we denote the cyclic order of edges around all the vertices in an embedding η by Π_η ($\Pi_\eta := (\pi_\eta(v) : v \in V)$). In this thesis, embeddings are in the plane unless otherwise stated.

$\pi_\eta(v)$

Π_η

Embeddings of graphs in $\mathbb{R}^3$ are called *spatial embeddings* and are usually specified/depicted by a projection of the embedded graph in $\mathbb{R}^3$ on the plane and indicating the over/under relations between the edges (see [28, 64]).

spatial embedding

2.3.1 Planar Embeddings

A graph is *planar* if it can be embedded in the plane. A *plane graph* is a planar graph with a fixed embedding in the plane.

planar graph

plane graph

A plane graph divides the plane into connected regions called *faces*. Usually the unbounded face is called the *outer face*. Euler's formula establishes a relation between the numbers of vertices, edges and faces of a graph as is indicated in the following theorem.

face

outer face

Theorem 14 (Euler's formula, 1750)**.** *Let G be a connected plane graph with $|V|$ vertices, $|E|$ edges and $|F|$ faces. Then $|V| - |E| + |F| = 2$.*

Let $\mathcal{G}$ be a class of graphs and let G be a graph in $\mathcal{G}$. The graph G is *maximal* with respect to $\mathcal{G}$ if any graph G' that is obtained by adding an edge to G is not in $\mathcal{G}$.

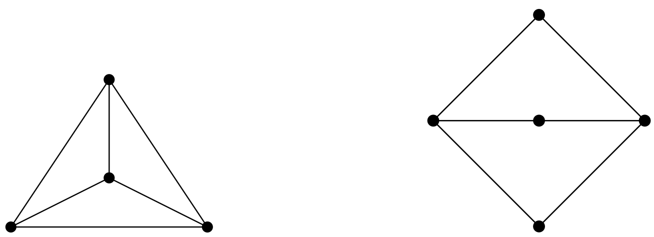
maximal

Using Euler's formula, one can prove a number of important theorems about plane graphs. One of these theorems is about the maximum number of edges of a plane graph.

Theorem 15 (see, for example, [34])**.** *Any maximal planar graph with $|V| \geq 3$ vertices has $3|V| - 6$ edges.*

Kuratowski and Wagner provide two elegant characterisations of planar graphs in terms of forbidden graphs (see Theorem 17 and Theorem 2).

However, neither Kuratowski's Theorem nor Theorem 2 leads to an efficient algorithm for testing whether or not a graph is planar. In 1974, Hopcroft and Tarjan managed to develop a linear-time algorithm for this problem [69].



(a) K_4 (b) $K_{2,3}$

Fig. 2.4 Minimal excluded minors of outerplanar graphs

Linear-time algorithms for finding planar embeddings of graphs were developed later by Chiba et al. [23] and Mehlhorn and Mutzel [87]. Later on, simpler linear-time algorithms were developed in order to perform planarity testing and find planar embeddings for planar graphs simultaneously (see [115]).

2.3.2 Outerplanar Embeddings

A graph is *outerplanar* if it has a planar embedding such that all the vertices lie on the same face; we normally assume this face to be the outer face. An alternative definition of an outerplanar graph is as follows. A graph is outerplanar if it can be embedded in a disk such that all the vertices lie on the boundary of the disk. An *outerplane* graph is an outerplanar graph with a fixed embedding in the disk. In this thesis, we denote the boundary of any disk d with $\partial(d)$.

outerplanar

outerplane

Some of the significance of this class of graphs is derived from the fact that some problems that are $\mathcal{NP}$-complete for planar graphs can be easily solved in polynomial time for outerplanar ones. Here we review some of the results on outerplanar graphs that are relevant to this thesis.

Chartrand and Harary characterised outerplanar graphs in terms of two minimal excluded minors as follows.

Theorem 16 (Chartrand and Harary [21]). *A graph G is outerplanar if and only if it does not contains K_4 or $K_{2,3}$ as a minor (see Figure 2.4).*

Two other important results about outerplanar graphs are as follows:

- A graph G is outerplanar if and only if it has an embedding η on the plane such that the dual embedding η^* of η has a vertex v such that $\eta^* - v$ contains no cycle [117].

- A bi-connected outerplanar graph has a unique Hamiltonian cycle [117].

Using Euler's formula, it is routine to see that an outerplanar graph G with $|V| \geq 2$ vertices has at most $2|V| - 3$ edges. Moreover, an outerplanar graph has at least two vertices of degree less than 3 and at least three vertices of degree less than four [117].

2.3.3 Book Embeddings

Book embeddings are, in a sense, a generalisation of outerplanar embeddings of graphs. Book embeddings are used in order to define several graph invariants such as book thickness [10] and pagewidth [67]. In addition to theoretical applications, book embeddings are used in designing VLSI layouts [25], graph drawing, knot theory [36], etc.

A *book* (also called a fan of half-planes [63]) is a topological space consisting of a single line l, called the *spine*, together with a collection of one or more half-planes, called the *pages* or *leaves* of the book, such that each half-plane has l as its boundary [35]. Clearly, books with a finite number of pages can be embedded into $\mathbb{R}^3$ by choosing l to be a line in $\mathbb{R}^3$ and by letting the k pages of the book correspond to k half-planes "rotating around" l, i.e., they are all disjoint except on l [10] (see Figure 2.5).

book
spine
page

A *book embedding* of a finite graph G into a book B is an embedding of G on B such that every vertex of G is drawn as a point on the spine of B, and every edge of G is an arc that lies within a single page of B such that no two edges cross. Clearly, every finite graph with $|E|$ edges has a book embedding into a book with $|E|$ pages. The *book thickness* (sometimes called *page number* or *stack number*) of G is the minimum number of pages required for a book embedding of G [10].

book embedding
book thickness
page number
stack number

It is straightforward to see that the book thickness of a graph is at most one if and only if it is outerplanar. The book thickness of a graph G is at most two if and only if G is a subgraph of a planar graph that has a Hamiltonian cycle [10]. Therefore, graphs with a page number of two are also known as subhamiltonian planar graphs [67]. All planar graphs have book thickness at most four [137, 138].

Finding the book thickness of a graph is $\mathcal{NP}$-hard. This follows from the fact that finding Hamiltonian cycles in maximal planar graphs is $\mathcal{NP}$-complete [25].

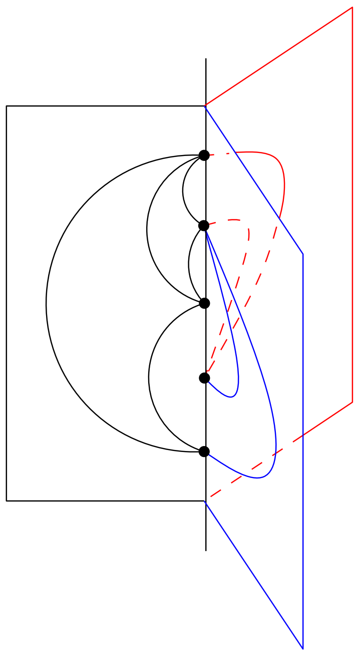


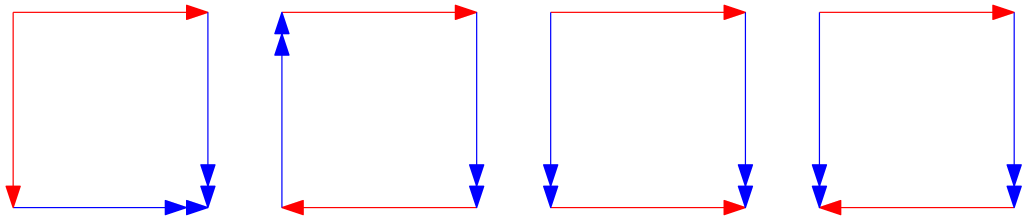
Fig. 2.5 A graph that is embedded in a book with 3 pages.

2.3.4 Embeddings on Surfaces

There are two infinite families of surfaces without boundaries up to homeomorphisms (see [37, 50, 89] for details and definitions). Kerékjártó for the first time gave a complete proof that every surface is homeomorphic to a space obtained from a sphere by adding handles and crosscaps [121, 128]. One can add a *handle* to a surface Σ by removing two disjoint open disks d_1 and d_2, with disjoint boundaries $\partial(d_1)$ and $\partial(d_1)$, from Σ and identifying $\partial(d_1)$ and $\partial(d_1)$ in such a way that the orientability of Σ is preserved [34]. To add a *crosscap* to a surface Σ, one can remove an open disk d with $\partial(d)$ as its boundary from Σ and then identify opposite points of $\partial(d)$ in pairs [34]. handle crosscap

Here, we use $\mathcal{T}_g$ and $\mathcal{N}_n$ to denote the orientable surface with genus g ($g \geq 0$) and the non-orientable surface which can be constructed by n cross caps, respectively.

Each surface can be constructed from an even-sided oriented polygon, called a *fundamental polygon*, by pairwise identification of its edges [43]. Figure 2.6 depicts four examples of surfaces that are constructed is this way. Precisely, any fundamental polygon can be represented symbolically as follows. Label each pair of the edges that are identified together with a distinct symbol (for example, a, b, etc.). Begin at any vertex, and proceed around the perimeter of the polygon in the clockwise direction until returning to the starting vertex. During this traversal, as you traverse each edge of the polygon, fundamental polygon



(a) Sphere (b) Real projective plane (c) Torus (d) Klein bottle

Fig. 2.6 Obtain the corresponding surface, by gluing the (iso-chromatic) sides with matching arrows of each polygon to each other so that the arrows point in the same direction.

record the label of that edge, say a, if the direction of the edge is clockwise, or record a^{-1} if the direction of the edge is counterclockwise.

For example, the fundamental polygon of the sphere (as shown in Figure 2.6(a)), when traversed clockwise starting at the upper left vertex, yields $abb^{-1}a^{-1}$.

Such a representation of a surface is called the *polygonal representation* of the surface. polygonal representation

Theorem 17 ([57, 133])**.** *Every surface can be represented by one the following three* standard forms of polygonal representation. *standard form*

1. aa^{-1} *(sphere; orientable)*
2. $a_1, a_1, a_2, a_2, \ldots, a_k, a_k$ *where* $k \geq 1$ *(sphere with* k *crosscaps; non-orientable)*
3. $a_1, b_1, a_1^{-1}, b_1^{-1}, a_2, b_2, a_2^{-1}, b_2^{-1}, \ldots, a_h, b_h, a_h^{-1}, b_h^{-1}$ *where* $h \geq 1$ *(sphere with* h *handles; orientable)*

We define the *Euler characteristic* $\chi(\Sigma)$ of a surface Σ as follows: Euler characteristic

$$\chi(\Sigma) = \begin{cases} 2 - 2g & \text{if } \Sigma = \mathcal{T}_g \\ 2 - n & \text{if } \Sigma = \mathcal{N}_n \end{cases}$$

Now we state the generalized Euler's formula for the surfaces [22, 50, 89].

A *2-cell embedding* is an embedding in which every face is homeomorphic to an open disk [76]. 2-cell embedding

Theorem 18 (Euler's formula)**.** *Let G be a connected 2-cell embedded graph on a surface Σ with $|V|$ vertices, $|E|$ edges and $|F|$ faces. Then*

$$|V| - |E| + |F| = \chi(\Sigma).$$

As an example, Figure 2.7 depicts an embedding of the Petersen graph on the projective plane $\mathcal{N}_1$. This embedded graph has 10 vertices, 15 edges and 6 faces. By substituting these values into the Euler's formula we can check that $\chi(\mathcal{N}_1) = 1$.

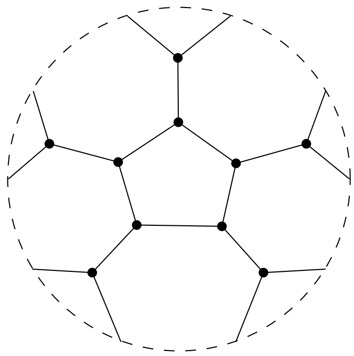


Fig. 2.7 An embedding of the Petersen graph on the real projective plane, $\mathcal{N}_1$.

We define the *genus* $\gamma(G)$ of a graph G as the minimum g such that G is embeddable in the surface $\mathcal{T}_g$. Similarly we define the *non-orientable genus* $\tilde{\gamma}(G)$ of a graph G as the minimum n, such that G is embeddable in the surface $\mathcal{N}_n$. The complexity of finding the genus of a graph was listed among the fundamental open problems of Garey and Johnson [56]. Later on Thomassen proved that this problem is $\mathcal{NP}$-complete [120].

genus

non-orientable genus

Ringel and Youngs proved the following theorem about the genus and the non-orientable genus of the complete graphs [106, 107].

Theorem 19 (Ringel and Youngs (1968) [107] [3])**.** *If $n \geq 3$ then*

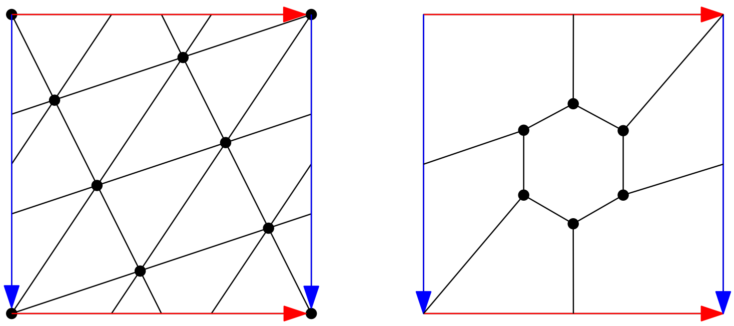
$$\gamma(K_n) = \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil.$$

If $n \geq 3$ and $n \neq 7$ then

$$\tilde{\gamma}(K_n) = \left\lceil \frac{(n-3)(n-4)}{6} \right\rceil.$$

Note that by the above theorem K_7 can be embedded on the torus $(\mathcal{T}_1)$. Figure 2.8 depicts embeddings of K_7 and $K_{3,3}$ on the torus $\mathcal{T}_1$.

[3] This theorem also completed the solution of an important problem in topological graph theory called the Heawood problem: finding the smallest number $H(\Sigma)$ such that any graph embedded in Σ can be coloured with $H(\Sigma)$ colours such that no two adjacent vertices have the same colour.



(a) An embedding of K_7 on torus

(b) An embedding of $K_{3,3}$ on torus

Fig. 2.8 Embeddings of two non-planar graphs on torus.

Embeddings of graphs on surfaces with boundaries have also been well studied. In particular, the disk dimension problem defined by Syslo and also Fellows and Langston is concerned with embeddings of graphs on surfaces with boundaries [40, 42, 118]. The disk dimension $d_g(G)$ of a graph G is the least positive integer k for which G embeds in $\mathcal{T}_g$ minus k disjoint open disks, with every vertex of G lying on the boundary of one of the forbidden disks.

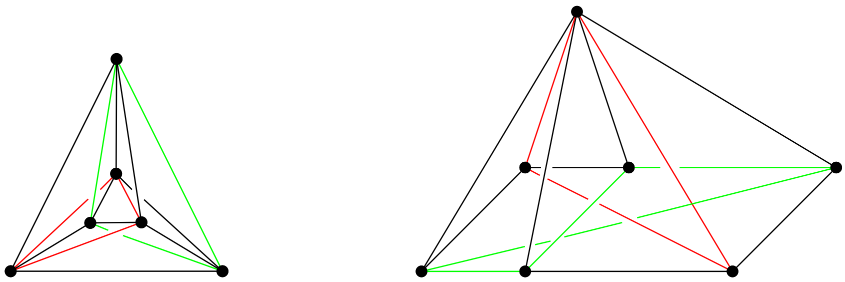
Let $G(g, d)$ (where $g, d \in \mathbb{N}$) denote the family of all graphs with $d_g \leq d$. Then $G(0, 1)$ denotes the family of outerplanar graphs, $G(0, \infty)$ denotes the family of planar graphs and $G(g, \infty)$ denotes the family of graphs with genus g.

The disk dimension problem is $\mathcal{NP}$-complete if d is not fixed, even for the case when $g = 0$ [56]. Moreover it is also $\mathcal{NP}$-complete for the case where $d = \infty$ and g is not fixed (this is equivalent to graph genus problem) [120]. For further details about the disk dimension problem, see [40, 42, 56, 118].

By the Graph Minor Theorem, the class of graphs that are embeddable on a fixed surface can be characterised in terms of a finite number of minimal excluded minors [5, 78, 88]. However apart from the cases of $G(0, 1)$, $G(0, 2)$, $G(0, \infty)$ and the graphs that are embeddable on the projective plane ($\mathcal{N}_1$), the complete lists of the minors are not known.

In fact, the graphs that are embeddable on $G(0, 1)$ are characterised in terms of 2 minimal excluded minors and the graphs that are embeddable on $G(0, 2)$, called outer-cylindrical graphs, are characterised in terms of 38 minimal excluded minors (see [5] for the complete list of these minimal excluded minors).

Graphs that are embeddable on the projective plane are characterised in terms of 35 minimal excluded minors (see [3, 89] for the complete list). Although there are over 16,000 minimal excluded minors known for toroidal



(a) The two cycles of a link, indicated in blue and red, in a realisation of K_6

(b) The two cycles of a link, indicated in blue and red, in a realisation of $K_{1,3,3}$

Fig. 2.9 Links in realisations of graphs

graphs (see [20, 55]) and over 4000 minimal excluded minors are known for graphs that are embeddable on the Klein bottle [30], the complete lists of minimal excluded minors for toroidal graphs and the graphs embeddable on the Klein bottle are not known yet.

2.3.5 Linked and Linkless Embeddings

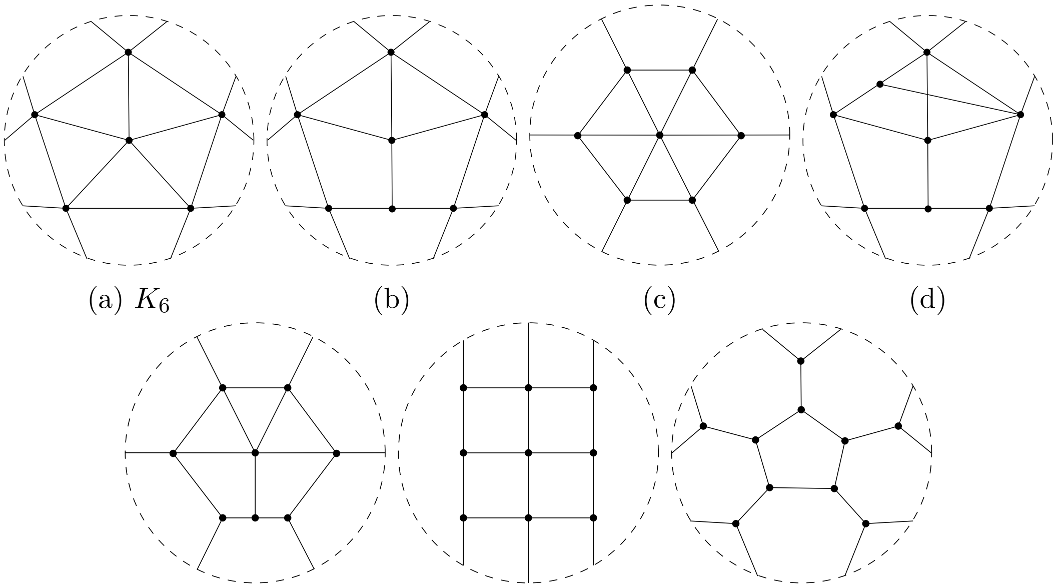
An embedding η of a graph in $\mathbb{R}^3$ is *linkless* if, for every pair of disjoint cycles C_1 and C_2, one can embed a topological sphere into $\mathbb{R}^3$ separating C_1 from C_2. Roughly speaking, if two cycles are linked in $\mathbb{R}^3$, we cannot contract one of them into a single point without cutting the other one (see Figure 2.9). A graph is *linkless*, if it has linkless embedding in $\mathbb{R}^3$. linkless

Sachs suggested the study of linkless embeddings for the first time [113]. He conjectured that these embeddings can be characterised by excluding the Petersen family of graphs.

Theorem 20 (Sachs Conjecture [113])**.** *A graph is linkless if and only if it does not contain any graph in the Petersen family as a minor.*

The *Petersen family* of graphs consists of K_6 and six other graphs, including the Petersen graph, as shown in Figure 2.10. All of these graphs can be obtained from each other by a series of $Y\Delta$ and ΔY exchanges which we will not define here (see, for example, [123, 139]). Petersen family of graphs

Conway, Gordon and Sachs proved that K_6 is not linkless [28]. Sachs also proved that the other members of the Petersen family of graphs are not linkless [113]. Moreover, in the same paper he showed that every minor of a linkless graph is linkless. Motwani, Raghunathan and Saran studied linkless embeddings from an algorithmic point of view [90]. They announced a proof for Conjecture 20 in [90] and [114] but the proof in [90] is not correct [111].



(a) K_6 (b) (c) (d)

(e) (f) (g) Petersen graph

Fig. 2.10 The Petersen family of graphs on the projective plane. (These drawings are drawn based on the drawings in [73]).

Robertson, Seymour and Thomas proved that G is linklessly embeddable in $\mathbb{R}^3$ if and only if it does not contain any graph in the Petersen family as a minor [110, 111]. They also show that a graph is linklessly embeddable in $\mathbb{R}^3$ if and only if it admits a flat embedding into $\mathbb{R}^3$. We shall discuss flat embeddings in Section 2.3.8. Since linkless graphs are closely related to flat graphs, some of the basic results about linkless graphs are not mentioned until Section 2.3.8.

Van der Holst presents a polynomial-time algorithm to compute a linkless embedding of a linkless graph [126]. Moreover he showed that, given an embedding of a graph, it can be decided whether this embedding is a linkless embedding in polynomial time [126].

2.3.6 3-linked and 3-Linkless Embeddings

A graph is *3-linkless* if there is a spatial embedding η of it such that for any three disjoint cycles in η one can embed a topological sphere into $\mathbb{R}^3$ separating one of the three cycles from the other two (see [46, 49]). Understanding this definition might be easier if we define 3-linked graphs and let the 3-linkless graphs to be the graphs that are not 3-linked. A graph is *3-linked* if every spatial embedding of it contains a non-split link of three components. That is, every cycle in a 3-linkless graph can be separated from any other two

3-linkless

3-linked

vertex-disjoint cycles in the graph via a topological sphere.

One might think that any linkless embedding is also a 3-linkless embedding but this is not true since, for example, the Borromean rings are linkless but not 3-linkless (see Figure 2.11).

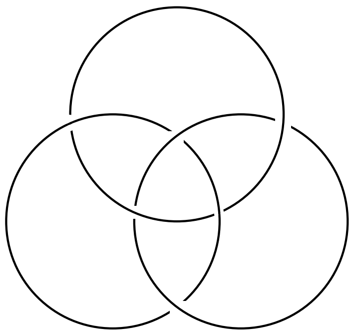


Fig. 2.11 The Borromean rings is embedded into $\mathbb{R}^3$ linklessly although it is not 3-linkless. Any pair of cycles of the Borromean rings are not linked but these three rings cannot be separated from each other unless one of them is cut.

Although a number of 3-linked graphs such as K_{10} are known, the complete set of excluded minors of this class of graphs is not known[4] [45].

2.3.7 Intrinsically Knotted, Knotted and Knotless Embeddings

A spatial embedding of a cycle is *knotted* if it is not contained in the surface of a topological sphere. A spatial embedding of a graph with no knotted cycles is called knotless. That is, every cycle in a *knotless* embedding is an un-knot. A graph is *knotless* if it has a knotless embedding. A graph is called *intrinsically knotted* (or *knotted*) if every spatial embedding of it contains a knotted cycle.

knotted

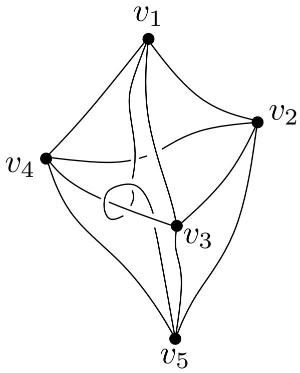
knotless

intrinsically knotted

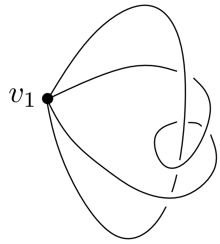
The class of intrinsically knotted graphs is closed under minors [41]. Conway and Gordon showed that K_7 is intrinsically knotted [28]. Motwani, Raghunathan and Saran showed that K_7 is a minor-minimal intrinsically knotted graph [90]. In fact 14 minor-minimal intrinsically knotted graphs are found among the Heawood family of graphs [65]. The *Heawood family* of graphs contains K_7 and 19 other graphs that are obtained from K_7 by a finite sequence of $Y\Delta$ and ΔY exchanges. Although additional minor-minimal intrinsically knotted graphs are known, intrinsically knotted graphs have not been completely characterised yet [47, 48, 74].

Heawood family of graphs

[4]Graph K_9 has a 3-linkless realisation in $\mathbb{R}^3$ [45].



(a) A realisation of K_5 that is linkless, knotted and not flat



(b) A linkless and knotless realisation which is not flat

Fig. 2.12 Linkless embeddings are not flat in general

Another interesting fact about intrinsically knotted graphs is that although ΔY exchanges preserve intrinsically knotted graphs, $Y\Delta$ exchanges may not preserve this property [44].

2.3.8 Flat Embedding

An embedding of a graph in $\mathbb{R}^3$ is *flat* if for every cycle C of G, one can embed a closed disk d with C as its boundary in $\mathbb{R}^3$ such that d is disjoint from the vertices and edges of G (i.e. the interior of d does not contain or intersect with any vertices or edges of G). Clearly every flat embedding is linkless, but the converse may not be true. Figure 2.12(a) depicts a linkless, knotted realisation of K_5 that is not flat. In fact every knotted cycle in $\mathbb{R}^3$ is linkless but it is not flat. As another example consider the borromean rings (Figure 2.11). Although the Borromean rings are linkless and knotless, they are not flat. One more simple example is provided in Figure 2.12(b). flat

Sachs proved that a linkless graph does not contain any minor in the Petersen family and conjectured that a graph is linkless if and only if it does not contain any of the graphs of the Petersen family as a minor. Böhme proposed the following conjecture which is a strengthening of Sachs' conjecture: a graph is flat if and only if it is linkless and a graph is linkless if and only if it has no minor in the Petersen family [13].

Since a flat graph is clearly linkless and Sachs showed that a linkless graph does not contain any minor in the Petersen family, in order to prove Sachs' conjecture it was required to prove that a graph with no minor in the Petersen family is flat. Finally Robertson, Seymour and Thomas proved Böhme's conjecture and showed that the classes of linkless and flat graphs are equivalent.

Theorem 21 (Robertson, Seymour and Thomas [110, 111])**.** *For a graph G, the following are equivalent:*

- *G is linkless*
- *G has a flat embedding*
- *G has no minor in the Petersen family.*

Flat embeddings in $\mathbb{R}^3$ are considered to be a generalisation of plane embeddings. For example, the famous theorem by Whitney which states that every 3-connected planar graph has a unique planar embedding, can be generalised to flat embeddings as follows: every 4-connected flat graph G has an "unique" flat embedding in $\mathbb{R}^3$ [110, 111]. Here, "unique" means equivalent up to ambient isotopy (which we shall define in the next chapter).

Kawarabayashi, Kreutzer and Mohar proved that there is a quadratic time algorithm that accepts a graph G as input and either detects one of the Petersen family of graphs as a minor in G, or returns a flat (and hence linkless) embedding of G in $\mathbb{R}^3$ [73].

2.4 Embeddings and the Colin de Vedière Invariant

The Colin de Verdière parameter $\mu(G)$ of a graph G is an interesting linear algebraic invariant introduced in [26] (see [27] for the English translation). Although this invariant is defined is in terms of matrices, it is closely related to topological embeddability properties of the graph.

The parameter $\mu(G)$ is minor-monotone; that is, if G' is a minor of G, then $\mu(G') \leq \mu(G)$ [26]. Hence by the graph minor theorem, the class of all graphs G with $\mu(G) \leq k$ can be described in terms of a finite collection of minimal excluded minors. Several well-know families of graphs can be characterised in terms of their Colin de Verdière invariants:

- $\mu(G) \leq 0$ if and only if G has no edges [27, 127]
- $\mu(G) \leq 1$ if and only if G is a disjoint union of paths [127]
- $\mu(G) \leq 2$ if and only if G is outerplanar [27, 127]
- $\mu(G) \leq 3$ if and only if G is planar [27, 127]

- $\mu(G) \leq 4$ if and only if G is linkless [79, 127]

The reason that Petersen family of graphs are minimal excluded minors for $\mu(G) \leq 4$ follows from the following three theorems [125].

Theorem 22 (Colin de Verdière [27])**.** *For any complete graph K_n,*

$$\mu(K_n) = n - 1.$$

In particular, $\mu(K_6) = 5$.

Theorem 23 (Bacher and Colin de Verdière [7])**.** *If G' is obtained from G by subdividing an edge, then $\mu(G') \geq \mu(G)$. If G is obtained from G' by suppressing a vertex of degree 2 and $\mu(G') \geq 4$ then $\mu(G) \geq \mu(G')$.*

Theorem 24 (Bacher and Colin de Verdière [7])**.** *If G' is obtained from G by a ΔY transformation, then $\mu(G') \geq \mu(G)$. If G is obtained from G' by a $Y\Delta$ transformation and $\mu(G') \geq 5$ then $\mu(G) \geq \mu(G')$.*

For more information and theorems on the Colin de Verdière parameter see [127].

2.5 Drawings, Realisations, Projections

A *drawing* D of a graph G on a surface Σ is a mapping f from vertices and edges of G that assigns: drawing of a graph

- to each vertex u of G, a distinct point $f(u)$ in Σ and
- to each edge (v_1, v_2) of G, a simple continuous arc $\zeta = f((v_1, v_2))$ in Σ connecting $f(v_1)$ to $f(v_2)$ such that ζ does not pass through the image under f of any vertex.

Two edges e and e' *cross* if they share a point other than their endpoints. The shared point is a *crossing* between e and e'. crossing

We sometimes refer to a point $f(u)$ in D that represents a vertex u in G as vertex u of D and similarly, we may refer to an arc ζ in D that represents an edge (u, v) in G as edge (u, v) of D. Moreover, we sometimes use D to denote the set of all points of Σ that represent vertices of G or belongs to arcs representing edges of G.

In this thesis, we assume that graph drawings are not *degenerate* [59], in that they satisfy the following conditions: degenerate drawing

- An edge does not contain a vertex other than its endpoints.
- Any two edges cross a finite number of times and the intersection of the arcs representing them must be a finite set of points.
- Edges must either properly cross or not cross at all (for example, they must not meet tangentially). More precisely, for a crossing point p on two edges e_1 and e_2, the cyclic order of the edges around p is e_1, e_2, e_1, e_2.
- No point represents more than one crossing. (It follows that a single edge cannot pass through the same crossing twice.)

Let D be a drawing of a graph G and let G^- be the subgraph of G induced by S, where $S \subseteq V(G)$ ($G^- = G[S]$). Then $D[G^-]$ is a drawing of G^- that is obtained as follows: $D[G]$

- for each vertex v in G^-, let v be represented by the same point that represents v in D, and
- for each edge e in G^-, let e be represented by the same arc that represents e in D.

A *realisation* of a graph is a drawing of it in $\mathbb{R}^3$ with no crossings. A *projection* of a graph G is a drawing of G with an over/under relation between the edges specified at each crossing (see for example Figure 2.13). realisation projection

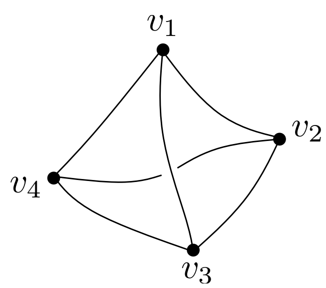
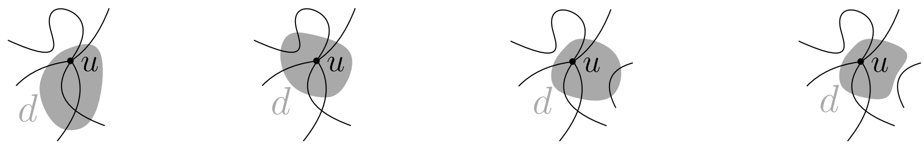


Fig. 2.13 A projection of K_4. Note that (v_1, v_3) is passing over (v_2, v_4).

For a vertex v in a drawing D, a *local disk* Σ_v at v is a sufficiently small neighbourhood homeomorphic to an open disk centred on v such that: local disk of a vertex

- Σ_v does not contain any vertex other than v,
- Σ_v does not contain any crossings,
- for any edge e incident with v, the intersection of the drawing of e with Σ_v is an arc homeomorphic to $[0, 1)$,

- every edge that is not incident with v is disjoint from Σ_v (see, for example, Figure 2.14).



(a) disk d contains a crossing

(b) disk d contains an edge segment that does not have u as its endpoint

(c) disk d contains an edge segment that does not have u as its endpoint

(d) disk d is a local disk for u

Fig. 2.14 (a)-(c) depicts three examples of disks that are not local disks of u. (d) depicts a local disk of u.

Similarly, for a non-self-intersecting edge $e = (u, v)$ in a drawing D, let a *local disk* Σ_e of the edge e be a sufficiently small disk that contains e in its interior such that:

local disk of an edge

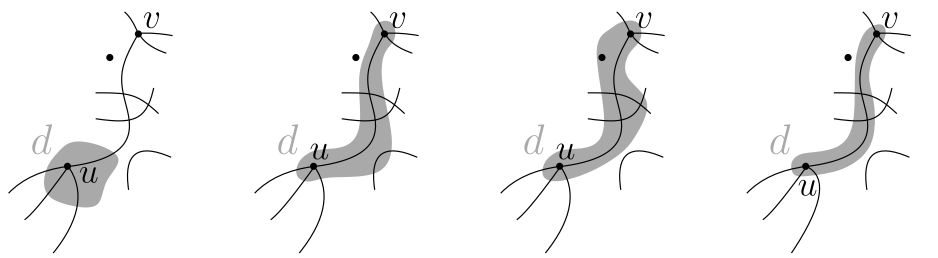
- Σ_e does not contain any vertex other than u or v,
- Σ_e does not contain any crossings other than the crossings on e,
- any maximal continuous segment of an edge f that intersects with Σ_e is either a arc homeomorphic to $(0, 1)$ that crosses e once or a arc homeomorphic to $[0, 1)$ that has u or v as one of its endpoints (see, for example, Figure 2.15).

2.6 The Hanani-Tutte Theorem

The Hanani-Tutte Theorem is another famous result in graph theory which can be thought of as a characterisation of planar graphs in terms of their drawings. Hanani proved the following in 1934.

Theorem 25 (Hanani [24]). *Every drawing of K_5 or $K_{3,3}$ contains two vertex-disjoint edges that cross an odd number of times.*

By Kuratowski's Theorem and Hanani's Theorem, we can easily see that in any drawing of a non-planar graph there are two vertex-disjoint paths that cross an odd number of times and hence there are two vertex-disjoint edges that cross an odd number of times. In other words, it is easy to derive the Strong Hanani-Tutte Theorem (Theorem 3), from Theorem 25.



(a) disk d does not contain the edge (u, v)

(b) disk d contains an edge segment that does not have u or v as its endpoint and does not cross (u, v)

(c) disk d contains a crossing that is not on (u, v)

(d) disk d is a local disk for (u, v)

Fig. 2.15 (a)-(c) depicts three examples of disks that are not local disks of (u, v). (d) depicts a local disk of (u, v).

The Strong Hanani-Tutte Theorem was first explicitly stated by Tutte [124]. This Theorem is usually used in the weaker form known as the Weak Hanani-Tutte Theorem (Theorem 4).

The difference between the strong version and the weak version of the Hanani-Tutte Theorem is that in the strong Hanani-Tutte we only require the *vertex-disjoint* pairs of edges in a drawing of the graph to cross each other an even number of times, whereas in the Weak Hanani-Tutte Theorem we require *any* pair of edges in a drawing of a graph to cross an even number of times.

Notice that both the strong and the Weak Hanani-Tutte Theorems provide us with a characterisation of planar graphs since the converses of both of these theorems are true. More specifically, by definition any planar graph has a drawing in which any two edges cross zero times.

The Hanani-Tutte Theorem in its weak form has been generalised to all 2-manifolds [98]. This is especially interesting as we do not yet have minimal excluded minor characterisations for the graphs that can be drawn without crossings on surfaces other than the sphere and projective plane.

Theorem 26 ([98])**.** *For every surface Σ, a graph G has a drawing on Σ with no crossings if and only if it has a drawing D on Σ such that any two edges cross an even number of times in D.*

However, the Strong Hanani-Tutte Theorem has only been generalised to the projective plane [97]. In fact we know that it cannot be generalised to the orientable surface of genus four [51].

Theorem 27 ([97])**.** *A graph G has a drawing on the projective plane with no crossings if and only if it has a drawing D on the projective plane such that any two vertex-disjoint edges cross an even number of times in D.*

Theorem 28 ([51])**.** *There exists a graph G with a drawing in the compact orientable surface Σ with 4 handles in which every pair of vertex-disjoint edges cross an even number of times such that G does not have a planar drawing in Σ.*

There are numerous other versions of the Hanani-Tutte Theorem. For example, see the following slightly stronger version of the Weak Hanani-Tutte Theorem:

Theorem 29 ([77, 98])**.** *Let D be a drawing of a graph G such that any two edges cross an even number of times in D. Then G is planar and has a planar drawing D' with a rotation system of edges around the vertices that is the same as the rotation system of edges around the vertices in D.*

Recently, Fulek, Kynčl and Pálvölgyi unified this version of the Weak Hanani-Tutte Theorem with the Strong Hanani-Tutte Theorem as follows.

Theorem 30 ([54])**.** *Let G be a graph and let $W \subseteq V(G)$. Let D be a drawing of G where every pair of edges that are vertex-disjoint or have a common endpoint in W cross an even number of times. Then G has a planar drawing where the cyclic orders of edges at vertices of W are the same as in D.*

If W is empty in the above theorem, then the theorem is equivalent to the strong version of the Hanani-Tutte Theorem. If $W = V(G)$ then the theorem is equivalent to the weak version of the Hanani-Tutte Theorem.

2.7 Thrackles and Superthrackles

A *thrackle* is a drawing of a graph in which any two edges have exactly one point in common [105]. In other words, in a thrackle, any two vertex-disjoint edges cross exactly once and incident edges do not cross (see for example, Figure 2.16). Any graph that has a thrackle drawing on a surface Σ is *thracklable* with respect to Σ. thrackle thracklable

The notion of thrackle was defined by John Conway as he conjectured the following:

Conjecture 1 (Conway's Thrackle Conjecture [14, 105])**.** *For a thracklable graph $G = (V, E)$, $|E| \leq |V|$.*

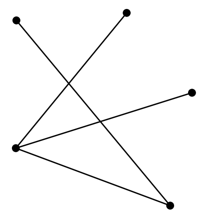


Fig. 2.16 A thrackle

Despite considerable efforts, Conway's thrackle conjecture is still open.

Lovász, Pach and Szegedy proved that every bipartite thracklable graph is planar [80] and hence the number of edges of a bipartite thrackle with n vertices cannot exceed $2n - 3$. This bound later was improved to $(3n - 3)/2$ by Cairns and Nikolayevsky [15] and then to $\frac{167}{117}n \approx 1.428n$ by Fulek and Pach [52].

Currently, the tightest upper bound on the number of edges of a thrackle is due to Fulek and Pach. They proved an upper bound of $1.3984n$ edges for a thrackle with n vertices [53]. There are numerous other papers trying to tighten the upper bound on the number of edges of thrackles (see for example, [18, 61, 99, 100]).

Assuming that Conway's Thrackle Conjecture is true, Woodall proved that the bound stated in the conjecure is tight since any cycle other than C_4 is a thrackle [135]. Moreover, with the same assumption Woodall characterised all thrackles as follows [135]: a graph is a thrackle if and only if

- it has at most one cycle of odd length, and
- it does not contain C_4, and
- each of its connected components contains at most one cycle.

With this theorem in mind, to prove Conway's thrackle conjecture it is enough to verify that a graph that consists of two even cycles with one vertex in common is not a thrackle [100, 135].

Many different variations of thrackles have been defined and studied during the past few years. Pach and Sterling defined x-monotone thrackles as the ones whose edges are arcs that meet every vertical line in at most one point. They also verified Conway's thrackle conjecture for this class of graphs.

Cairns and Nikolayevsky characterised outerthracklable graphs (see Theorem 6). Moreover, they proved that the number of edges of an outerthracklable graph does not exceed the number of vertices of the graph [17].

A drawing D of a graph G is a *generalised thrackle* if any two edges in D have an odd number of points in common [16] (see for example, 2.17). Any graph with a generalised thrackle drawing is a *generalised thracklable* graph.

generalised thrackle

generalised thracklable

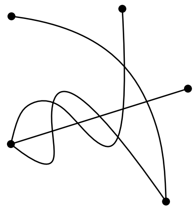


Fig. 2.17 A generalised thrackle

Cairns and Nikolayevsky characterised generalised thracklable graphs as follows.

Theorem 31 ([16]). *A graph is generalised thracklable on the plane if and only if it has a parity embedding in the projective plane.*

Superthrackles are defined by Archdeacon and Stor in [4]. They also characterised superthrackles (see Theorem 8).

A drawing of a graph on a surface Σ is a *1-point superthrackle* if it can be drawn as a superthrackle on Σ such that all the edge crossings occur at a common point[5]. Any graph that can be drawn as a 1-point superthrackle on surface Σ is *1-point superthracklable* with respect to Σ.

1-point superthrackle

1-point superthracklable

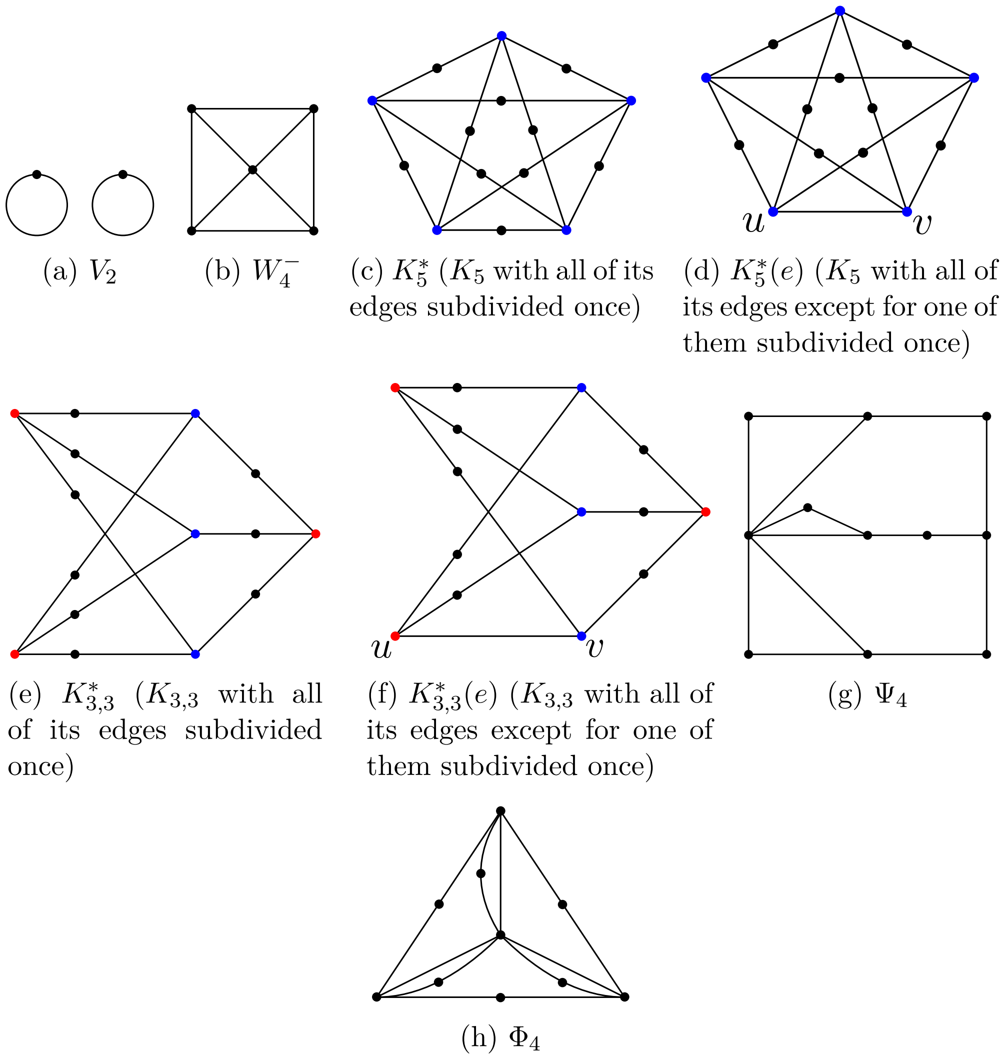
Archdeacon and Stor also proved the following.

Theorem 32 ([4]). *The following classes of graphs are equivalent:*

- *superthracklable graphs,*
- *1-point superthracklable graphs,*
- *graphs that have a parity embedding on the projective plane,*
- *graphs without a subgraph that is parity homeomorphic to any graph in Figure 2.18.*

Cairns and Nikolayevsky characterised generalised thracklable graphs [16] and Archdeacon and Stor characterised superthracklable graphs [4] and the two characterisations are the same (see Theorem 31 and Theorem 8). That is, any generalised thracklable graph is a superthracklable graph.

[5] For this specific definition we relax the definition of a drawing so that three or more edges can all cross at a common point.



(a) V_2

(b) W_4^-

(c) K_5^* (K_5 with all of its edges subdivided once)

(d) $K_5^*(e)$ (K_5 with all of its edges except for one of them subdivided once)

(e) $K_{3,3}^*$ ($K_{3,3}$ with all of its edges subdivided once)

(f) $K_{3,3}^*(e)$ ($K_{3,3}$ with all of its edges except for one of them subdivided once)

(g) Ψ_4

(h) Φ_4

Fig. 2.18 The obstruction set for superthrackles

Theorem 33 ([4])**.** *A graph is superthracklable if and only if it is generalised superthracklable.*

Tangles, tangled thrackles, and spherical thracles are some of the other variations on thrackles. For definitions and results see [19, 94, 112].

For applications of thrackles see [1, 60].

3

Local Manipulations of Drawings

In this chapter we develop a set of moves that enable us to transform any drawing of a graph G to any other drawing of G. We generalise these moves to be applicable to drawings on all surfaces. We will use the results of this chapter frequently in the rest of the thesis.

3.1 Local Manipulations of Drawings in the Plane

In this chapter we introduce a set of local moves that would help us to transform any drawing of a graph G to any other drawing of G. References to these moves can be found in the literature individually. However to the best of our knowledge we do not have a theorem in the literature that introduces a set of moves for manipulating any drawing of a graph to obtain any other drawing of it. For example, the R_{III}^{p} move, which we will introduce shortly, is used by Gioan under the name of triangle transformation or a triangle switch to manipulate some of the drawings of K_n to each other [58]. The same move is used by Arroyo et al. under the name of triangle-flip for manipulating drawings of complete graphs [6].

Recall that Kaufmann [70] and Yamada [136] independently proved that if two (piecewise linear) projections $\mathcal{P}$ and $\mathcal{P}'$ represent the same embedding of a graph in $\mathbb{R}^3$ then one can manipulate $\mathcal{P}$ and obtain $\mathcal{P}'$ by a finite sequence of local moves (Theorem 5). We will use Theorem 5 to prove that there is a set of five local moves that we can use to change any drawing of a graph to any other drawing of it. The main idea in this proof is that any embedding of a graph G in $\mathbb{R}^3$ can be transformed to any other embedding of G in $\mathbb{R}^3$ by allowing the edges to move freely in 3-space and also by allowing them to pass through each other.

Proposition 1. *Let D and D' be two drawings of a graph G in the plane. Then D and D' are related by a finite sequence of the local moves given in Figure 3.2.*

Proof. The main idea of this proof is that any embedding of a graph G in $\mathbb{R}^3$ can be converted to any other embedding of G in $\mathbb{R}^3$ if we allow the edges to deform and pass through each other (see, for example, [2], page 219). Roughly speaking, we convert D and D' to two projections $\mathcal{P}$ and $\mathcal{P}'$ of G and utilise Theorem 5 (the Reidemeister moves given in Figure 1.3) and the idea we mentioned above to obtain $\mathcal{P}'$ from $\mathcal{P}$. We then show than any of the Reidemeister moves that we use to obtain $\mathcal{P}'$ from $\mathcal{P}$ corresponds to a Reidemeister move in Figure 3.2 which we could have used to obtain D' from D. In the rest of this proof we will lay out these ideas in more detail.

Convert D and D' to two projections $\mathcal{P}$ and $\mathcal{P}'$ of G by changing the crossings in D and D' to overcrossings and undercrossings arbitrarily. $\mathcal{P}$ and $\mathcal{P}'$ are projections of two embeddings of G in $\mathbb{R}^3$, which may not be ambient isotopic. However, as it was mentioned earlier, any embedding of a graph G in $\mathbb{R}^3$ can be converted to any other embedding of G in $\mathbb{R}^3$ if we allow the edges to deform and pass through each other. That is, by Theorem 5 we can change $\mathcal{P}$ to $\mathcal{P}'$ by a finite sequence of Reidemeister moves from the set given in Figure 1.3 and changing some overcrossings to undercrossings and some undercrossings to overcrossings.

Obtain $\mathcal{P}'$ from $\mathcal{P}$ using a finite number k of steps, where each step is either a single Reidemeister move from Figure 1.3 or changing an overcrossing to an undercrossing or vice versa. Then change the undercrossings and overcrossings to simple crossings in order to obtain D' from $\mathcal{P}'$. Let $\mathcal{P}_0 = \mathcal{P}$. For each $i \in \{0, 1, ..., k\}$, let $\mathcal{P}_i$ represent the projection that is obtained by applying the first i steps to $\mathcal{P}$, and let D_i be the drawing of G in the plane that is obtained by changing overcrossings and undercrossings of $\mathcal{P}_i$ to crossings (see Figure 3.1).

$$\begin{tikzcd}
\mathcal{P}_0 = \mathcal{P} \arrow[r] \arrow[d] & \mathcal{P}_1 \arrow[r] \arrow[d] & \mathcal{P}_2 \arrow[d] & \cdots & \mathcal{P}_i \arrow[r] \arrow[d] & \mathcal{P}_{i+1} \arrow[d] & \cdots & \mathcal{P}_k = \mathcal{P}' \arrow[d] \\
D_0 = D & D_1 & D_2 & & D_i & D_{i+1} & & D_k = D'
\end{tikzcd}$$

Fig. 3.1 Transforming $\mathcal{P}$ to $\mathcal{P}'$ through a series of steps using Reidemeister moves from Figure 1.3 and changes between overcrossings and undercrossings. Any horizontal arrow in this picture represents either a Reidemeister move or a change of an undercrossing to overcrossing or vice versa. Any downward vertical arrow represents a transformation of a projection to a drawing by changing overcrossings or undercrossings to crossings.

The transformation of $\mathcal{P}$ to $\mathcal{P}'$ via the intermediate steps $\mathcal{P}_i \to \mathcal{P}_{i+1}$, together with the crossing removals that reduce $\mathcal{P}_i$ to D_i, yield a transformation of D to D' through a series of steps by transforming each D_i to D_{i+1}. In each step we can obtain D_{i+1} from D_i, for all $i \in \{0, \ldots, k-1\}$, by:

1. Obtaining the projection $\mathcal{P}_i$ from D_i by changing the crossings of D_i to overcrossings and undercrossings according to $\mathcal{P}_i$.

2. Using one of the Reidemeister moves in Figure 1.3, or changing an undercrossing to an overcrossing or vice versa, to change $\mathcal{P}_i$ to $\mathcal{P}_{i+1}$.

3. Obtaining a drawing D_{i+1} from $\mathcal{P}_{i+1}$ by changing overcrossings and undercrossings of $\mathcal{P}_{i+1}$ to crossings.

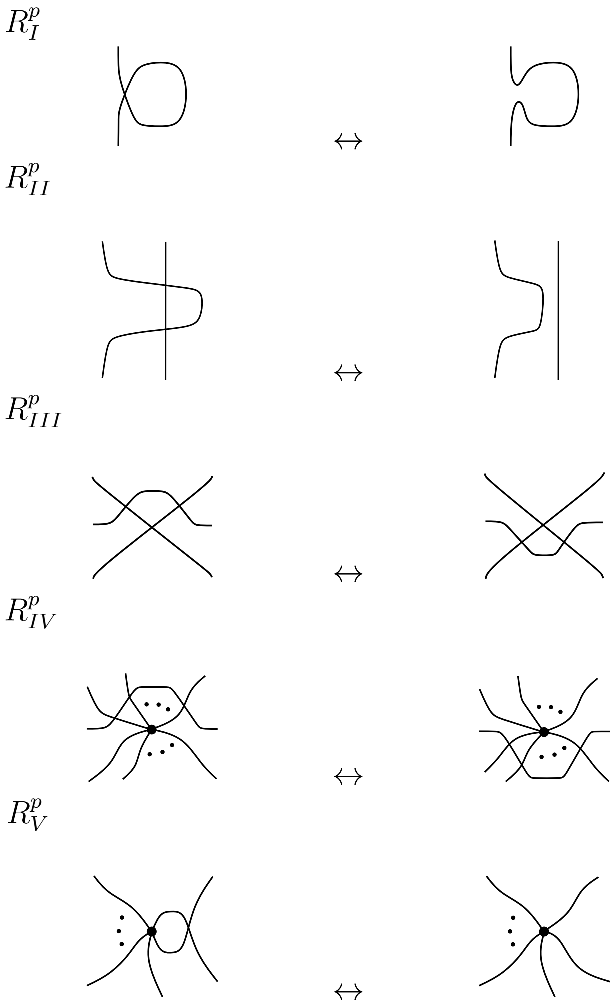
If the transformation of $\mathcal{P}_i$ to $\mathcal{P}_{i+1}$ is obtained by a change of an overcrossing to an undercrossing or vice versa, then $D_i = D_{i+1}$. If the transformation of $\mathcal{P}_i$ to $\mathcal{P}_{i+1}$ is obtained by a Reidemeister move from Figure 1.3, then the transformation of D_i to D_{i+1} is obtained by the corresponding Reidemeister move from Figure 3.2.

□

Throughout the rest of this thesis we refer to any of the moves in Figure 3.2 by the symbol indicated in the figure (for example we refer to the first move in this figure by R_I^p).

3.2 Local Manipulations of Drawings on Surfaces

In this section we generalise Proposition 1 to drawings on all surfaces.



R_I^p
R_{II}^p
R_{III}^p
R_{IV}^p
R_V^p

Fig. 3.2 Reidemeister moves for plane graphs where R_{IV}^p and R_V^p are shown for a vertex of degree 5.

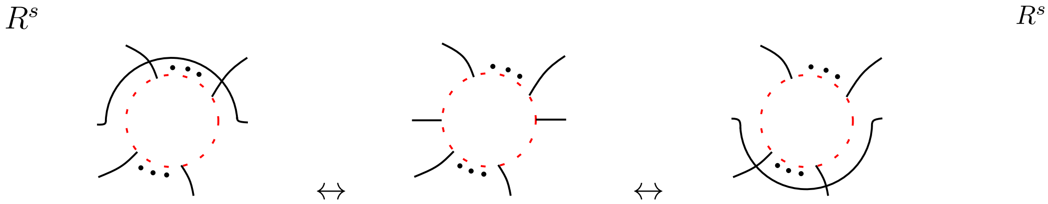


Fig. 3.3 Reidemeister move R^s that is useful in manipulation of drawings of graphs on a surface Σ other than the plane. The (red) dashed line represents $\partial'(\Sigma)$ (or the fundamental polygon of Σ). Note that only the local neighbourhood of the boundary of Σ is shown in this figure.

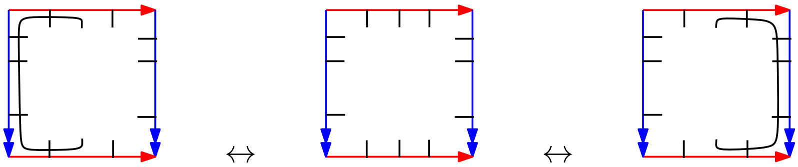


Fig. 3.4 An illustration of R^s for the torus.

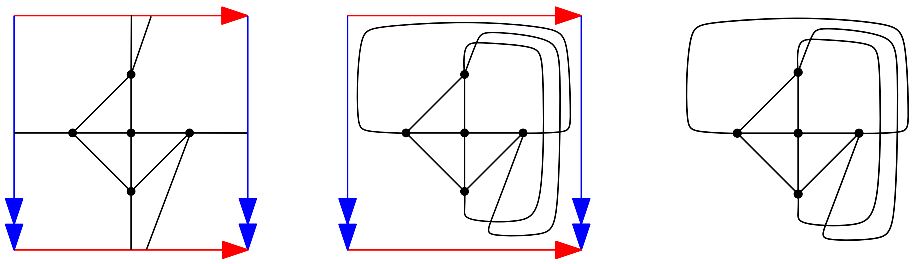
Let Σ be any surface. We use Proposition 1 to prove that one can change any drawing of a graph G on Σ to any other drawing of G on Σ using the moves in Figure 3.2 and the extra move shown in Figure 3.3. Figure 3.4 depicts an example of this move on the torus.

Throughout the rest of this thesis we refer to the move in Figure 3.3 by R^s.

Proposition 2. *Let η and η' be two drawings of a graph G on a surface Σ and let $\partial'(\Sigma)$ denote the boundary of the fundamental polygon*[1] *of Σ. Then η and η' are related by a finite sequence of the set of local moves depicted in Figures 3.2 and Figure 3.3.*

Proof. Let η_{temp} be a drawing of G obtained from η by removing all the crossings between edges of η and $\partial'(\Sigma)$ using a finite number of R^s moves (see Figures 3.5(a) and 3.5(b)). Let η^- be a drawing of G on the plane Σ^+ that is obtained from η_{temp} by gluing a disk to $\partial'(\Sigma)$ instead of identifying the points of $\partial'(\Sigma)$ pairwise in the usual way (see Figure 3.5(c)). Similarly, let η'_{temp} be a drawing of G obtained from η' by removing all the crossings between edges of η' and $\partial'(\Sigma)$ using a finite number of R^s moves and let η'^- be a drawing of G on the plane Σ^+ that is obtained from η'_{temp} by gluing a disk to $\partial'(\Sigma)$.

[1] For the definition of the fundamental polygon of a surface refer to Section 2.3.4.



(a) η (b) η_{temp} (c) η^-

Fig. 3.5 η is a drawing of G on Σ. η_{temp} is obtained from η by removing all the crossings between edges of η and $\partial'(\Sigma)$. η^- is a drawing of G on the plane Σ^+ that is obtained by gluing a disk to $\partial'(\Sigma)$.

By Proposition 1, there is a finite sequence of Reidemeister moves, each taken from Figure 3.2, that transforms η^- to η'^-. Obtain η'^- from η^- in this way. Now it is straightforward to remove a disk d from Σ^+ and identify the points on its boundary $\partial(d)$ in pairs, so as to convert Σ^+ back to Σ and then use R^s as necessary to obtain the drawing η' of G on Σ.

That is, we can obtain η' from η by using the moves in Figure 3.2 and the extra move shown in Figure 3.3. □

3.3 Local Manipulations of Drawings on Surfaces with Boundaries

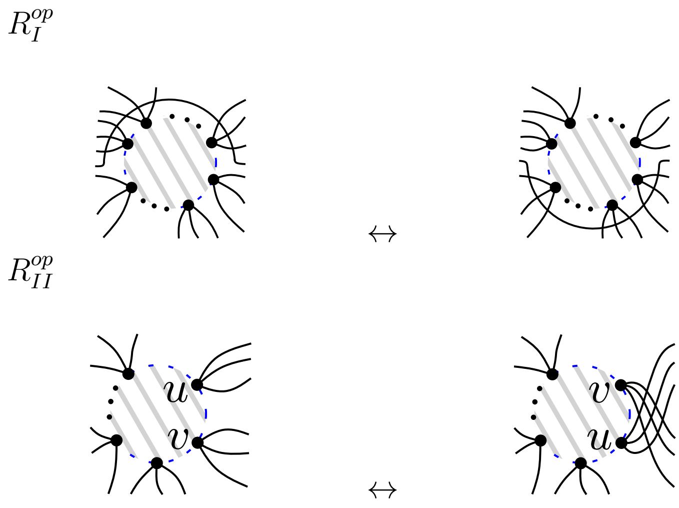
Let Σ be a surface with a boundary $\partial(\Sigma)$ that consists of a union of k disjoint closed curves $c_1, c_2, \ldots, c_k$. A drawing η of a graph G on a surface Σ is an *outer-k-drawing* if all the vertices of G are located on $\partial(\Sigma)$. In such a drawing, each vertex is located on a component c_i, $1 \leq i \leq k$, of the boundary. For any outer-k-drawing η on a surface Σ, we define the *vertex-to-boundary assignment* to be the function that maps each vertex v of G to the component c of $\partial(\Sigma)$ on which the vertex is located. Note that this function does not specify the cyclic order of the vertices along c. We denote the vertex-to-boundary assignment of η by $\dot{o}(\eta)$.

outer-k-drawing

vertex-to-boundary assignment

We use Theorem 2 to prove the following.

Proposition 3. *Let η and η' be two outer-k-drawings of a graph G on a surface Σ with the boundary $\partial(\Sigma)$ and let $\dot{o}(\eta) = \dot{o}(\eta')$. Then η can be changed to η' by a finite sequence of the local moves depicted in Figures 3.2, 3.3 and 3.6 except for R^p_{IV}.*



R_I^{op}

R_{II}^{op}

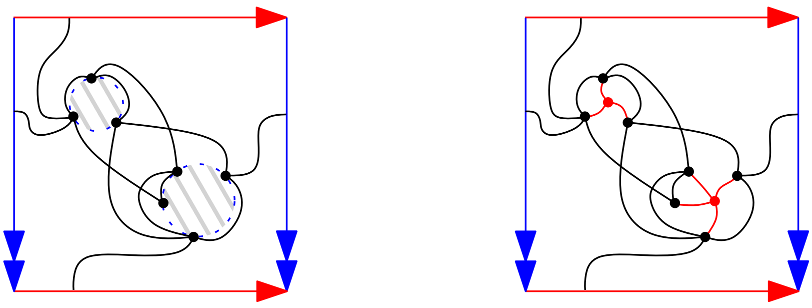
Fig. 3.6 Reidemeister moves for outerdrawings of graphs where R_I^{op} and R_{II}^{op} are shown for a cycle of the boundary on which all the vertices have three edges incident with them.

Proof. Let $c_1, c_2, \ldots, c_k$ be the components of $\partial(\Sigma)$. Let η_{temp} be a drawing of G on the surface Σ^+ obtained from η by gluing a disk d_i to component c_i, for each i, $1 \leq i \leq k$, of $\partial(\Sigma)$ so that, for each i, the boundary of d_i is identified with c_i. Let G^+ be a graph obtained by adding a vertex v_i to G for each component $c_i, 1 \leq i \leq k$, of the boundary $\partial(\Sigma)$ such that v_i is adjacent to all the vertices on c_i. We call these extra vertices *disk vertices*. Let η^+ be a drawing of G^+ on Σ^+ that is obtained from η_{temp} by placing v_i in the interior of d_i and making sure that the edges incident with v_i are all routed within d_i, meeting c_i only at their endpoints, and that they do not cross (see, e.g., Figure 3.7).

disk vertex

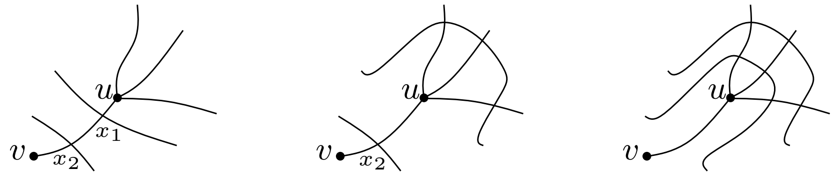
Similarly, let η'_{temp} be a drawing of G on the surface Σ^+ that is obtained from η' by gluing a disk d_i to each component $c_i, 1 \leq i \leq k$, of the boundary of Σ. Moreover, let η'^+ be a drawing of G^+ that is obtained from η'_{temp} by placing v_i on d_i and making sure that the edges incident with v_i are all routed within d_i, meeting c_i only at their endpoints, and that they do not cross (see, e.g., Figure 3.7).

Since η^+ and η'^+ are two drawings of G^+ on the surface Σ^+, by Proposition 2 η^+ and η'^+ are related by a finite sequence of the Reidemeister moves given in Figures 3.2 and 3.3. Obtain the drawing η'^+ from η^+ by a finite sequence S of j of the Reidemeister moves given in Figures 3.2 and 3.3. Let $\eta_l^+, 1 \leq l \leq j$ represent the drawing obtained by applying the first l Reidemeister moves in S to η^+. Now let η_l be the outer-k-drawing of G that is



(a) η (or η') (b) η^+ (or η'^+)

Fig. 3.7 Obtaining η^+ (or η'^+) from η (or η') by adding the red vertices and edges to it.

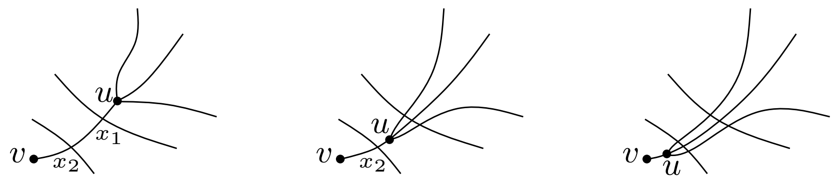


(a) η_l^+ (b) after removing x_1 (c) after removing x_2

Fig. 3.8 Step 1. Obtaining η_l from η_l^+

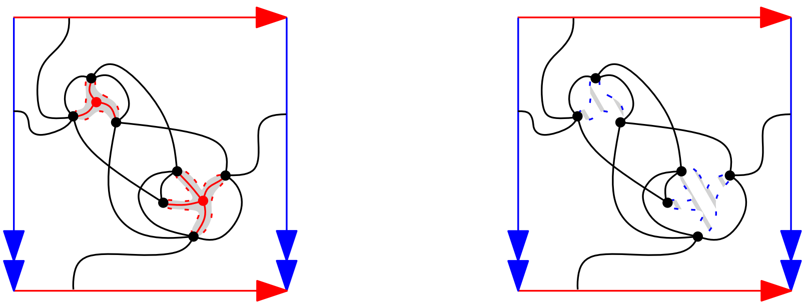
obtained from η_l^+ by:

1. For any edge (u, v) that is incident with a disk vertex v, let $x_1, x_2, \ldots, x_{i'}$ be the crossings on (u, v) ordered from u to v. Remove $x_1, x_2, \ldots, x_{i'}$ in this specific order, using R^p_{IV}, by moving the edge that crosses (u, v) over u. (For the first two steps in this process, see Figure 3.8.) Denote this new drawing by η_l^*. An alternate (and equivalent) way of drawing the removal of the crossings on (u, v) is given in Figure 3.9. Throughout the rest of this proof we shall use the latter way of drawing the modifications of the drawings as the edges near v remain straighter in appearance and it makes understanding of the proof easier.



(a) η_l^+ (b) after removing x_1 (c) after removing x_2

Fig. 3.9 An alternate way of achieving step 1 (where we obtain η_l from η_l^+).



(a) choosing a disk for any disk vertex

(b) removing the disks associated with the disk vertices from the surface

Fig. 3.10 Step 2 of obtaining η_l from η_l^+

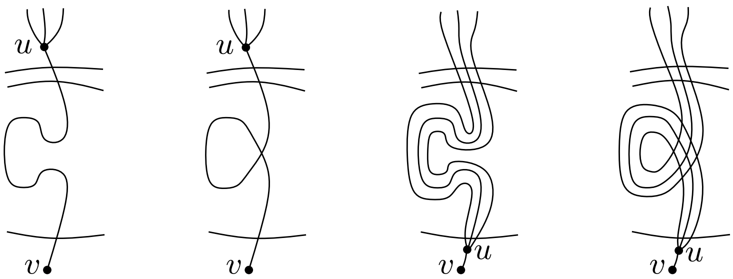
2. At this point there are no crossings on the edges that are incident with the disk vertices. For any disk vertex v, let d_v be a disk in Σ^+ with all the vertices in $N(v)$ on its boundary such that only v and all the curves representing edges incident with v, excluding their endpoints other than v, are located in the interior of d_v (see, e.g., Figure 3.10(a)). We call such disks *vertex disks*. Remove the interiors of all vertex disks along with vertex disk all the edges and vertices on them from Σ^+ (see, e.g., Figure 3.10(b)).

Transformation of η^+ to η'^+ happens through a series of steps by changing each η_i^+ to η_{i+1}^+. Similarly, transformation of η to η' happens through a series of steps by changing each η_i to η_{i+1}. Figure 3.11 depicts the process of transforming η to η' and its relation with the process of transforming η^+ to η'^+.

$$
\begin{tikzcd}
\eta^+ \arrow[r] \arrow[d] & \eta_1^+ \arrow[r] \arrow[d] & \eta_2^+ \arrow[d] & \cdots & \eta_l^+ \arrow[r] \arrow[d] & \eta_{l+1}^+ \arrow[d] & \cdots & \eta_j^+ = \eta'^+ \arrow[d] \\
\eta^* \arrow[d] & \eta_1^* \arrow[d] & \eta_2^* \arrow[d] & \cdots & \eta_l^* \arrow[d] & \eta_{l+1}^* \arrow[d] & \cdots & \eta_j^* = \eta' \arrow[d] \\
\eta & \eta_1 & \eta_2 & & \eta_l & \eta_{l+1} & & \eta_j = \eta'
\end{tikzcd}
$$

Fig. 3.11 Transformation of η to η' through a series of steps. Each horizontal arrow represents a Reidemeister move from Figure 3.2 or 3.3.

Next we show that any η_{l+1} can be obtained from η_l by a finite sequence of the local moves depicted in Figures 3.2, 3.3 and 3.6 except for R_{IV}^p. Let R be the Reidemeister move that changes η_l^+ to η_{l+1}^+. We have 5 cases:



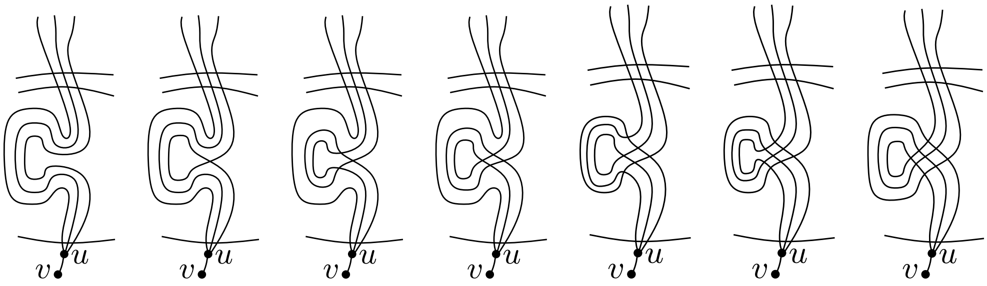
(a) η_l^+ (b) η_{l+1}^+ (c) η_l^* (d) η_{l+1}^*

Fig. 3.12 Comparing η_l^+ with η_{l+1}^+ and η_l^* with η_{l+1}^*, where $R = R_I^p$.

Case 1, $R = R_I^p$**.** Let e be the edge that is affected by R performed on η_l^+. If e is not incident with any disk vertex then all changes to the drawing occur outside the vertex disks, so η_l can be transformed to η_{l+1} using one R_I^p.

If e is incident with a disk vertex v then the difference between η_l^+ and η_{l+1}^+, and consequently η_l and η_{l+1}, would be in a finite number of edges (all the edges incident with u other than (u, v)) similar to what is depicted in Figure 3.12.

Then it is straightforward to transform η_l to η_{l+1} by tangling (or untangling) the edges of η_l one by one using a number of R_I^p and R_{II}^p as shown in Figure 3.13.

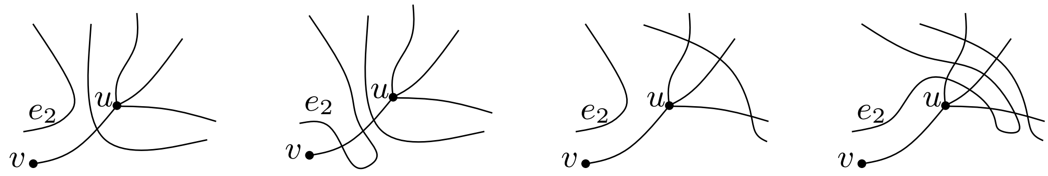


(a) (b) (c) (d) (e) (f) (g)

Fig. 3.13 Comparing η_l^* to η_{l+1}^*. η_l^* can be transformed to η_{l+1}^* using R_I^p and R_{II}^p. R_I^p is used to transform (a) to (b) and (c) to (d) and (e) to (f) and R_{II}^p is used to transform (b) to (c) and (d) to (e) and (f) to (g).

Case 2, $R = R_{II}^p$**.** Let e_1 and e_2 be the two edges that are affected by R performed on η_l^+. If e_1 and e_2 are not incident with any disk vertex then η_l can be transformed to η_{l+1} using R_{II}^p. If e_1 or e_2 or both are incident with disk vertices, then η_l can be transformed to η_{l+1} using R_{II}^{op} repeatedly for all the edges of G that are incident with that endpoint of e_1 or e_2 or both that is not a disk vertex (see, for example, Figure 3.14).

Case 3, $R = R_{III}^p$**.** Let e_1, e_2 and e_3 be the three edges that are affected

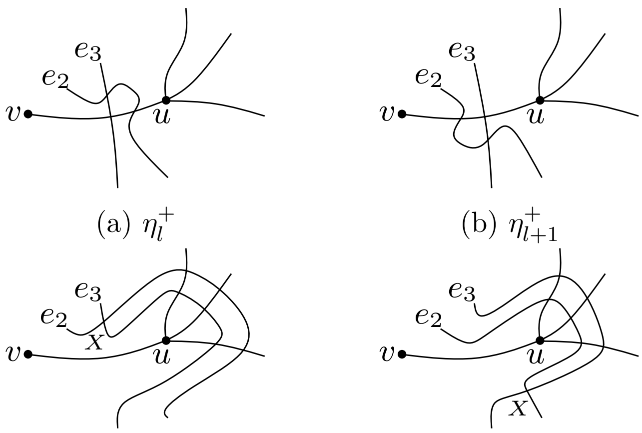


(a) η_l^+ (b) η_{l+1}^+ (c) η_l^* (d) η_{l+1}^*

Fig. 3.14 Comparing η_l^+ with η_{l+1}^+ and η_l^* with η_{l+1}^*, where $R = R_{II}^p$ and $e_1 = (u, v)$. η_l^* (c) can be changed to η_{l+1}^* (d) using R_{II}^p.

by R performed on η_l^+. If e_1, e_2 and e_3 are all edges of G in η_l^+, then η_l can be transformed to η_{l+1} using R_{III}^p.

If one, two or all three of the edges are incident with disk vertices then η_l can be transformed to η_{l+1} using R_{III}^{op} repeatedly for all the edges of G that are incident with those endpoints of e_1, e_2 or e_3 that are not disk vertices (see for example Figure 3.15).



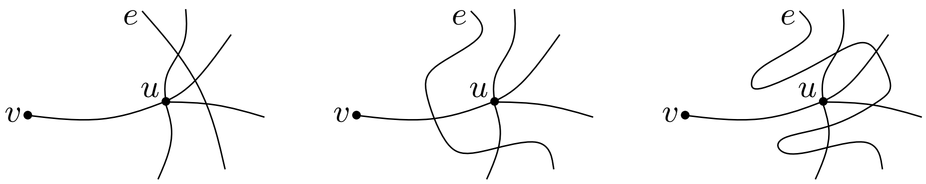
(a) η_l^+ (b) η_{l+1}^+ (c) η_l^* (d) η_{l+1}^*

Fig. 3.15 Comparing η_l^+ with η_{l+1}^+ and η_l^* with η_{l+1}^*, where $R = R_{III}^p$. Notice that η_l^*, (c), can be changed to η_{l+1}^*, (d), using R_{III}^p.

Case 4, $R = R_{IV}^p$**.** Let u be the vertex and e be the edge that are affected by R performed on η_l^+ as is shown in Figures 3.16(a) and 3.16(b). (Roughly speaking, let e be the edge that is pulled from one side of u to the other side of it.)

If u is a vertex of G in η_l^+, then η_l can be transformed to η_{l+1} by a number of R_{II}^p moves (see for example Figure 3.17). If v is a disk vertex then it is straightforward to see that η_l can be transformed to η_{l+1} using R_I^{op}.

Case 5, $R = R_V^p$**.** Let u be the vertex that is incident with both of the edges that are affected by R performed on η_l^+. If u is a vertex of G (u is not a disk vertex) then η_l can be transformed to η_{l+1} using R_V^p.



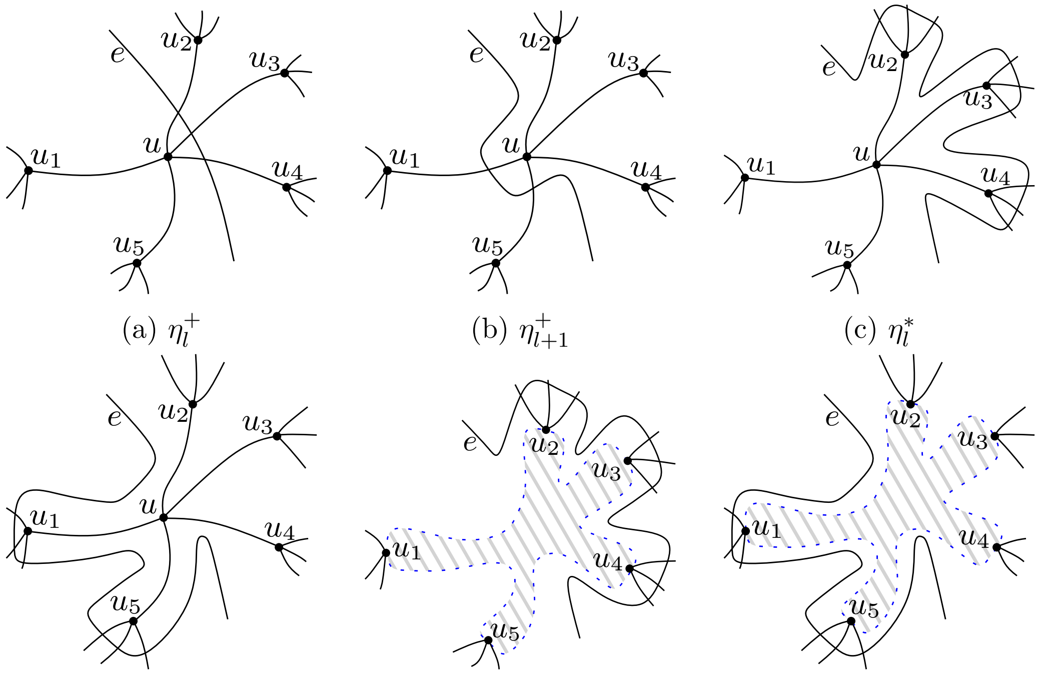
(a) η_l^+ (b) η_{l+1}^+ (c) η_{l+1}^*

Fig. 3.16 Comparing η_l^+ with η_{l+1}^+ and η_{l+1}^*, where $R = R_{IV}^p$ and v is the disk vertex. Notice that η_l^+ (b) can be changed to η_{l+1}^* (c) using R_{II}^p.

If u is a disk vertex then η_l can be transformed to η_{l+1} using R_{II}^{op} (see for example Figure 3.18).

That is, we can obtain η_{l+1} from η_l by a finite sequence of the local moves depicted in Figures 3.2, 3.3 and 3.6 except for R_{IV}^p. □

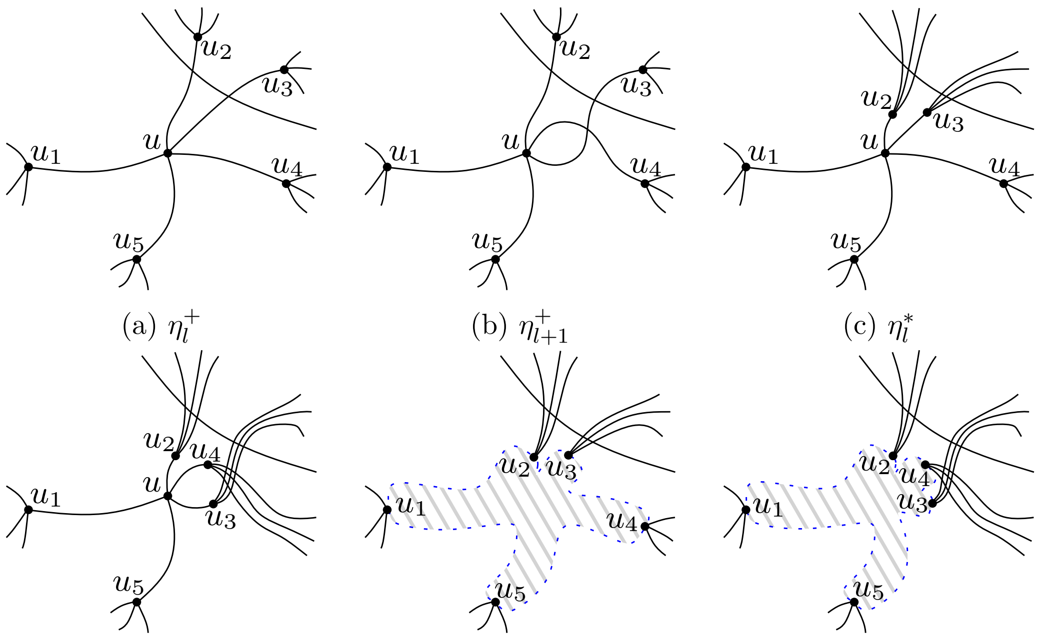
Throughout the rest of this thesis we refer to the moves in Figure 3.6 by their corresponding symbols (R_I^{op} and R_{II}^{op}).



(a) η_l^+ (b) η_{l+1}^+ (c) η_l^*

(d) η_{l+1}^* (e) η_l (f) η_{l+1}

Fig. 3.17 Comparing η_l^+ with η_{l+1}^+, where $R = R_{IV}^p$ and v is the disk vertex. Notice that η_l (Figure 3.17(e)) can be changed to η_{l+1} (Figure 3.17(f)) using R_I^{op}.



(a) η_l^+ (b) η_{l+1}^+ (c) η_l^*

(d) η_{l+1}^* (e) η_l (f) η_{l+1}

Fig. 3.18 Comparing η_l^+ with η_{l+1}^+ and η_l^* with η_{l+1}^*, where $R = R_V^p$ and u is the disk vertex. Notice that η_l (Figure 3.18(e)) can be changed to η_{l+1} (Figure 3.18(f)) using R_{II}^{op}.

4

The Hanani-Tutte Theorem and Outerplanar graphs

In this chapter we prove two Hanani-Tutte type theorems for outerplanar graphs. The first Hanani-Tutte type theorem is related to the rotational order of the edges around the vertices and the second one is related to the rotational order of the vertices around the boundary of the disk (or the outerface of the drawing). We finish this chapter with a couple of conjectures about outerplanar graphs.

In this chapter we develop two Hanani-Tutte type theorems for outerplanar graphs. The first one, which is similar to the Weak Hanani-Tutte Theorem, concerns drawings which preserve the rotational order of the edges around the vertices in the drawing. Let u be a vertex in an outerdrawing η. Let c be a circle centred at u that is located in the local neighbourhood of u. The rotational order of the edges around u in η (or $\pi_\eta(u)$), is the order in which we encounter all the edges adjacent to u as we traverse c without encountering the boundary of the disk on which η is drawn. For example, $\pi_\eta(u)$ in Figure 4.1(a) is $(e^u_1, e^u_2, \ldots, e^u_i, e, e^u_{i+1}, e^u_{i+2}, \ldots, e^u_j)$ rather than $(e^u_j, e^u_1, e^u_2, \ldots, e^u_i, e, e^u_{i+1}, e^u_{i+2}, \ldots, e^u_{j-1})$. Therefore in outerdrawings, rotational orders of edges around vertices are total orders and not cyclic orders.

The second one, which is similar to the Strong Hanani-Tutte Theorem, concerns the drawings which preserve the rotational order of the vertices around the boundary of the disk. We break this chapter in two sections accordingly.

4.1 Rotational Order of the Edges Around the Vertices

In this section we prove a Hanani-Tutte type theorem that is closely related to the rotational system of the edges around the vertices in outerdrawings of the graph. (Please refer to Section 2.3 for the definition of the rotational system of the edges around the vertices and its notation.)

Lemma 1. *Let η be an outerdrawing of a graph G such that any edge in η crosses any cycle in η an even number of times. Then there is an outerembedding η' of G such that $\Pi_{\eta'} = \Pi_\eta$.*

Proof. Let Σ be the disk (with the boundary $\partial(\Sigma)$ that consists of a cycle) on which η is drawn. We assume that G is connected.

We prove the lemma by induction on the number of vertices. In the base case η has a single vertex v and any edge (in this case, a loop) crosses any cycle (in this case, also a loop) an even number of times. For the proof of the base case we refer the reader to the elegant proof of the base case of Theorem 3.2 of [96] by Pelsmajer, Schaefer and Štefankovič.

We proceed to the inductive case. Let (u, v) be an edge of η, where $u \neq v$. Let

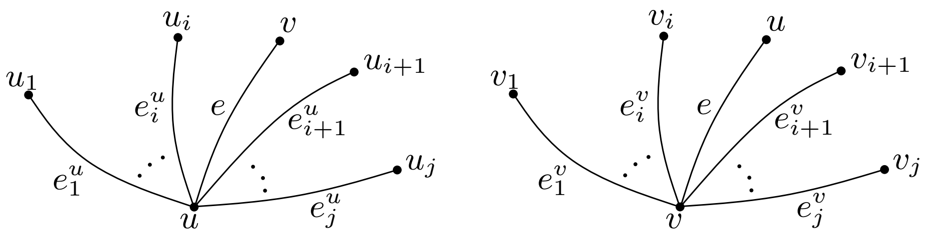
$$\pi_\eta(u) = (e_1^u, e_2^u, \ldots, e_i^u, e, e_{i+1}^u, e_{i+2}^u, \ldots, e_j^u)$$

where $1 \leq i \leq j$ and $e_k^u = (u, u_k)$ (see, for example, Figure 4.1(a)). Moreover, let

$$\pi_\eta(v) = (e_1^v, e_2^v, \ldots, e_{i'}^v, e, e_{i'+1}^v, e_{i'+2}^v, \ldots, e_{j'}^v)$$

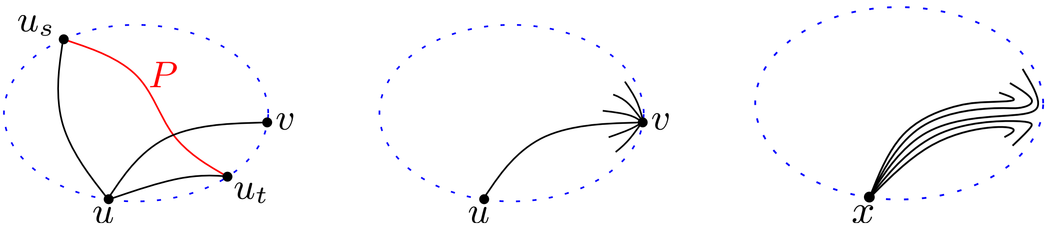
where $1 \leq i' \leq j'$ and $e_{k'}^v = (v, v_k')$ (see, for example, Figure 4.1(b)).

Since (u, v) crosses any cycle C an even number of times, there cannot be any path P from $u_s, 1 \leq s \leq i$, to $u_t, i+1 \leq t \leq j$, that is vertex-disjoint from (u, v). Otherwise, (u, v) crosses the cycle $(u, u_s) \cup P \cup (u_t, u)$ an odd number of times, which is a contradiction since η is an outerdrawing (see for example Figure 4.2(a)). Similarly, since (u, v) crosses any cycle C an even number of times, there cannot be any path from $v_s, 1 \leq s \leq i'$, to $v_t, i'+1 \leq t \leq j'$, that is vertex-disjoint from (u, v).



(a) edges incident with u in η (b) edges incident with v in η

Fig. 4.1 edges around u and v in η.



(a) (b) η (c) η^-

Fig. 4.2 Figure 4.2(a) depicts (u, v) crossing a cycle an odd number of times. Figures 4.2(b) and 4.2(c) depict the edge (u, v) and the edges incident with v as (u, v) is contracted to a vertex x while preserving the cyclic order of the edges around the vertices.

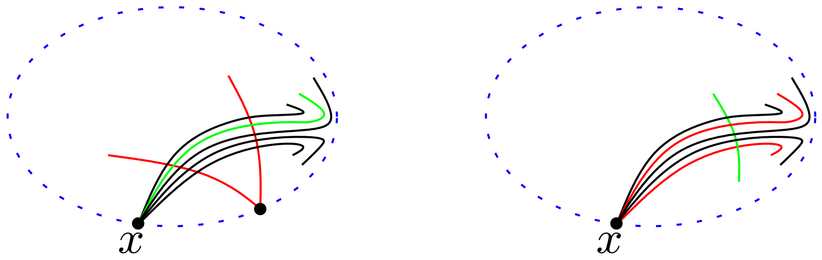
It follows that any path in η from any vertex $u_s, 1 \leq s \leq i$, to any vertex $u_t, i + 1 \leq t \leq j$, contains u or v, and any path in η from any vertex $v_s, 1 \leq s \leq i'$, to any vertex $v_t, i' + 1 \leq t \leq j'$, contains u or v.

Let G^- be the graph that is obtained by contracting (u, v) to x. Let η^- be a drawing of G^- that is obtained by contracting (u, v) in η while preserving the rotational order of the edges around each vertex such that the edges that are incident with v (in η) follow the path of (u, v) sufficiently closely and are joined to x in η^- (see, for example, Figures 4.2(b) and 4.2(c)). Clearly η^- is an outerdrawing. Moreover, any edge in η^- crosses any cycle in η^- an even number of times since:

1. (u, v) crosses any cycle in η an even number of times and therefore any edge that was incident with v in η and now follows the path of (u, v) in η^- crosses any cycle an even number of times (see Figure 4.3(a)).

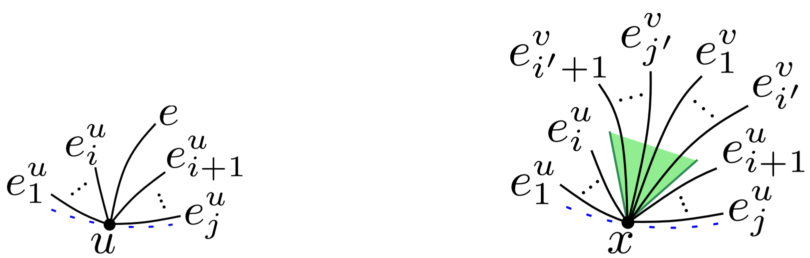
2. any cycle C containing v has two edges incident with v and therefore if these two edges each follow the path of (u, v) sufficiently closely, together they cross any edge along the way an even number of times (see Figure 4.3(b)).

By induction, there is an outerembedding η'^- of G^- such that $\Pi(\eta'^-) =$



(a) (b)

Fig. 4.3 Any edge e crosses any cycle C an even number of times in η^-. The edges of C are shown in red and e is shown in green.



(a) neighbourhood of u in η (b) neighbourhood of x in η'^-

Fig. 4.4 (a) order of the edges around u. (b) angle $\angle x_v$ in η'^- in green contains $e^v_1, e^v_2, \ldots, e^v_k$ and separates $e^u_1, e^u_2, \ldots, e^u_i$ from $e^u_{i+1}, e^u_{i+2}, \ldots, e^u_j$.

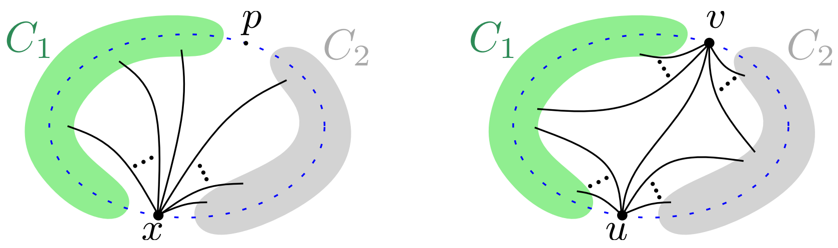
$\Pi(\eta^-)$. Since $\Pi(\eta'^-) = \Pi(\eta^-)$, there is an angle $\angle x_v$ in η'^- centred at x that contains all the intersections between the local disk Σ_x of x and $e^v_1, e^v_2, \ldots, e^v_{j'}$ and separates $e^u_1, e^u_2, \ldots, e^u_i$ from $e^u_{i+1}, e^u_{i+2}, \ldots, e^u_j$ (see Figure 4.4).

As mentioned before, any path in η from any vertex $u_s, 1 \leq s \leq i$, to any vertex $u_t, i+1 \leq t \leq j$, contains u or v, and any path in η from any vertex $v_s, 1 \leq s \leq i'$, to any vertex $v_t, i'+1 \leq t \leq j'$, contains u or v. Therefore any path from any of the vertices $u_1, u_2, \ldots, u_i$ or $v_{i'+1}, v_{i'+2}, \ldots, v_{j'}$ to any of the vertices $u_{i+1}, u_{i+2}, \ldots, u_j$ or $v_1, v_2, \ldots, v_{i'}$ passes through x.

Let $H = G^- \setminus x$. Let C_1 be the union of the connected components of H that contain at least one of the vertices $u_1, u_2, \ldots, u_i$ or $v_{i'+1}, v_{i'+2}, \ldots, v_{j'}$. Similarly, let C_2 be the union of the connected components of H that contain at least one of the vertices $u_{i+1}, u_{i+2}, \ldots, u_j$ or $v_1, v_2, \ldots, v_{i'}$.

Since any vertex in $\{u_1, u_2, \ldots, u_i\} \cup \{v_{i'+1}, v_{i'+2}, \ldots, v_{j'}\}$ is adjacent to u or v, x is adjacent to a vertex in C_1 and since any vertex in $\{u_{i+1}, u_{i+2}, \ldots, u_j\} \cup \{v_1, v_2, \ldots, v_{i'}\}$ is adjacent to u or v, x is adjacent to a vertex in C_2.

Let v_1 and v_2 be two vertices of C_1 and let v_3 be a vertex of C_2. Vertices v_1, v_3, v_2, x cannot appear in this order (or its reverse) on $\partial(\Sigma)$ in η'^- since then, considering $\pi(x)$, there would be at least one crossing in η'^- between two of the edges that are incident with x. In other words, the vertices of C_1



(a) η'^-; Point p separates vertices of C_1 from vertices of C_2.

(b) η'; v is located at the position of point p.

Fig. 4.5 Decontracting x to (u, v)

appear consecutively on $\partial(\Sigma)$ and the vertices of C_2 also appear consecutively on $\partial(\Sigma)$ (see Figure 4.5(a)).

Hence there is a point p on $\partial(\Sigma)$ such that the vertex in η'^- that appears before (or after) p on $\partial(\Sigma)$ belongs to C_1 and the vertex that appears after (or before) p on $\partial(\Sigma)$ belongs to C_2. Roughly speaking, p separates the vertices of C_1 form the vertices of C_2 on $\partial(\Sigma)$ (see Figure 4.5(a)) and since any path from any vertex of C_1 to any vertex of C_2 passes through x, it follows that p is on the same face of η'^- as is x (see Figure 4.5(a)).

Now it is straightforward to decontract x to (u, v) (without introducing any crossings), locating u at the position of x and v at the position of p while preserving $\Pi(\eta'^-)$, to obtain an outerembedding η' of G (see Figure 4.5(b)). Since there are no crossings in η'^- and $\Pi(\eta'^-) = \Pi(\eta^-)$, vertices $u_1, u_2, \ldots, u_i$ and $v_{i'+1}, v_{i'+2}, \ldots, v_{j'}$ (which are vertices of C_1) and $v_1, v_2, \ldots, v_{i'}$ and $u_{i+1}, u_{i+2}, \ldots, u_j$ (which are vertices of C_2) appear exactly in this order on $\partial(\Sigma)$ and therefore after the decontraction $\Pi_{\eta'} = \Pi_\eta$. □

The next two lemmas investigate the relationship between the Reidemeister moves we introduced in Chapter 3 and the parity of the number of crossings between an edge and a cycle in outerdrawings.

Lemma 2. *R_I^p, R_{II}^p, R_{III}^p and R_{II}^{op} preserve the parity of number of crossings between an edge and a cycle that are edge-disjoint in an outerdrawing.*

Proof. Clearly, R_{II}^p preserves the parity of the number of crossings between any two edge-disjoint cycles. Moreover, since the parity of the number of crossings between an edge and a cycle only changes by introducing or removing crossings between two distinct edges, it follows that R_I^p and R_{III}^p also preserve the parity of the number of crossings between two cycles.

Roughly speaking, in R_{II}^{op} we are switching the order of two vertices u and v on the boundary of a surface and we are introducing a crossing between

(a) before (or after) R_I^{op} (b) after (or before) R_I^{op}

Fig. 4.6 R_I^{op} preserves the parity of the number of crossings between an edge and a cycle in an outerdrawing.

any edge e_1 and cycle C where e is incident with u and C contains v (or e is incident with v and C contains u). Since any cycle has either 0 or 2 edges incident with u or v, we either increase the number of the crossings between any e and C by 0 or by 2. Therefore we do not change the parity of the number of crossings between the edges and cycles in a drawing by performing R_{II}^{op}. □

Lemma 3. *The R_I^{op} move preserves the parity of the number of crossings between two edges in an outerdrawing η.*

Proof. Let Σ be the disk on which η is drawn. Roughly speaking, in R_I^{op} we are pushing an edge e from one side of $\partial(\Sigma)$ to another side of it. Let $v_1, v_2, \ldots, v_i$ be the vertices on one side of $\partial(\Sigma)$ and let $v_{i+1}, v_{i+2}, \ldots, v_j$ be the vertices on the other side of $\partial(\Sigma)$ (see Figure 4.6).

Clearly, e is always involved in all the crossings that are removed or introduced. So if the parity of the number of crossings between two edges e_1 and e_2 changes then either $e = e_1$ or $e = e_2$. Without loss of generality let $e = e_1$ and let $e_2 = (u, v)$. We have two cases.

Case 1. $u \in \{v_1, v_2, \ldots, v_i\}$ **and** $v \in \{v_{i+1}, v_{i+2}, \ldots, v_j\}$ **(or vice versa).** In this case the number of crossings between the two edges does not change.

Case 2. $u, v \in \{v_1, v_2, \ldots, v_i\}$ **or** $u, v \in \{v_{i+1}, v_{i+2}, \ldots, v_j\}$**.** Therefore the R_I^{op} move either adds two crossings between the two edges or removes two crossings from the two edges. Therefore the parity of the number of crossings between e' and C remains the same. □

We are now prepared to prove the following lemma.

Lemma 4. *Let η be an outerembedding of a graph G and let η' be an outerdrawing of G such that $\Pi_\eta = \Pi_{\eta'}$. Then any edge crosses any cycle an even number of times in η'.*

Proof. Since there are no crossings in η, any edge crosses any cycle in η an even number of times. By Theorem 3, any outerdrawing η' of G can be obtained from η by R_I^p, R_{II}^p, R_{III}^p, R_V^p, R_I^{op} R_{II}^{op}. Moreover, by Lemma 2 and Lemma 3, all of these moves except for R_V^p preserve the parity of the number of crossings between an edge and a cycle in a drawing. So it is enough to show that we can obtain η' from η by R_I^p, R_{II}^p, R_{III}^p, R_V^p, R_I^{op}, R_{II}^{op} in such a way that the usage of R_V^p does not change the parity of the crossings between the edges and cycles.

Since the rotational order of edges around edges in outerdrawings is a total order and $\Pi_\eta = \Pi_{\eta'}$, any time that we use R_V^p to switch the place of an edge e_1 with an edge e_2 in the rotational order of edges around a vertex we need to use R_V^p to do the reverse and switch the place of e_2 with e_1 and therefore usage of R_V^p in the transformation of η to η' preserves the parity of the number of crossings between any cycle and edge in the drawing and the lemma follows. □

Lemma 1 and Lemma 4 provide us with the following theorem.

Theorem 34. *For any graph G, let Π be a rotational system of the edges around the vertices of G. The following statements are equivalent:*

1. *There is an outerembedding η of G where $\Pi_\eta = \Pi$.*

2. *There is an outerdrawing η' of G in which any edge crosses any cycle an even number of times and $\Pi_{\eta'} = \Pi$.*

3. *In any outerdrawing η'' of G where $\Pi_{\eta''} = \Pi$, every edge crosses every cycle an even number of times.*

One way of using the above theorem is as follows. To determine whether a rotational system Π of edges around the vertices of G is a rotational system of edges around the vertices of an outerembedding of G, we can draw an arbitrary outerdrawing η of G such that $\Pi_\eta = \Pi$ and check whether edges cross cycles an even number of times in η.

4.2 Rotational order of vertices around the boundary

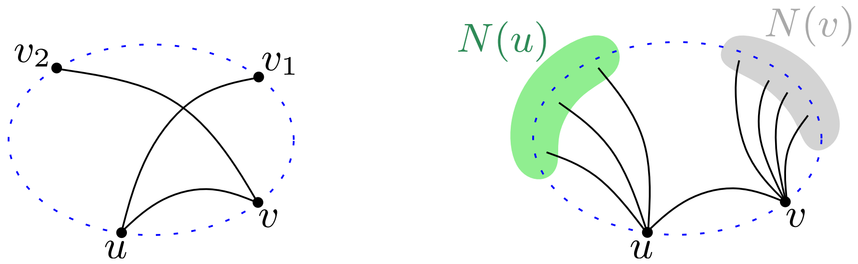
In this section we prove a Hanani-Tutte type theorem that is closely related to the rotational order of the vertices around the boundary of the disk in the drawings of the graph.

Lemma 5. *Let η be an outerdrawing of a graph G in which any two vertex-disjoint edges cross each other an even number of times. Then there is an outerembedding η' of G with $\rho(\eta') = \rho(\eta)$.*

We provide three different proofs for this lemma as in each of our method for proving this lemma is completely different.

Proof 1. We prove by induction on the number of the vertices. In the base case G has a single vertex and the lemma holds trivially.

Suppose that G has more than one vertex. Let Σ be the disk on which η is drawn. Let u and v be two consecutive vertices of η on $\partial(\Sigma)$. Let $v_1 \in N(u) \setminus v$ and $v_2 \in N(v) \setminus u$. Since any two vertex-disjoint edges in η cross an even number of times, u, v_2, v_1, v cannot appear in this order on $\partial(\Sigma)$, otherwise (u, v_1) crosses (v, v_2) an odd number of times (see, for example, Figure 4.9(a)). Therefore, vertices of $N(u)$ and vertices of $N(v)$ do not appear interleaved on $\partial(\Sigma)$. Roughly speaking, vertices of $N(u)$ appear on one of side of $\partial(\Sigma)$ and vertices of $N(v)$ appear on the other side of $\partial(\Sigma)$ (see Figure 4.9(b)).



(a) two edges crossing an odd number of times

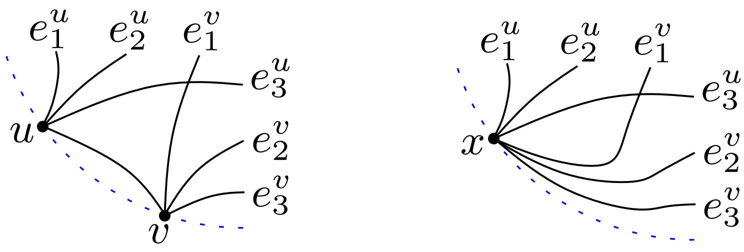
(b) vertices of $N(u)$ and $N(v)$ on $\partial(\Sigma)$

Fig. 4.7 Vertices of $N(u)$ and $N(v)$ cannot appear interleaved on $\partial(\Sigma)$

Obtain η_{temp} from η by deleting the edge (u, v) (if it exists) and adding an edge between u and v such that it follows $\partial(\Sigma)$ closely enough so that it does not cross any other edges. Since (u, v) is not crossed by any edges in η_{temp}, any two vertex-disjoint edges still cross each other an even number of times in η_{temp}. Moreover η_{temp} remains an outerdrawing.

Let G^- be the graph that is obtained by contracting (u, v) into a vertex x and let η^- be a drawing of G^- that is obtained by contracting (u, v) in η_{temp} while preserving the cyclic order of the edges around vertices as illustrated in Figure 4.8. Roughly speaking, the edges that are incident with v (in η) follow the path of (u, v) and are joined to x in η^-.

Since any two vertex-disjoint edges in η^- cross each other an even number of times and η^- is an outerdrawing, by induction there is an outerembedding

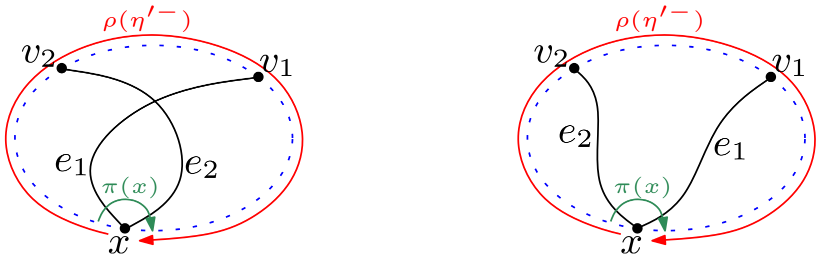


(a) before contraction (b) after contraction

Fig. 4.8 Contracting (u, v) to a vertex x while preserving the cyclic order of the edges around the vertices

η'^- of G such that $\rho(\eta'^-) = \rho(\eta^-)$. This implies that, vertices of $N(u)$ and vertices of $N(v)$ do not appear interleaved on $\partial(\Sigma)$.

Let $e_1 = (u, v_1)$ be an edge that is incident with u in η and let $e_2 = (v, v_2)$ be an edge that is incident with v in η. As there are no crossings in η'^-, e_1 and e_2 should appear in the same order in $\pi(x)$ as v_1 and v_2 appear in $\rho(\eta')$ (see Figure 4.9). Hence, if we let $e_1^u, e_2^u, \ldots, e_i^u$ be the edges that are incident with u in η and $e_1^v, e_2^v, \ldots, e_j^v$ be the edges that are incident with v in η, there is a straight-line ray r that starts at x and separates all the intersections between the local disk Σ_x of x and $e_1^u, e_2^u, \ldots, e_i^u$ from all the intersections between Σ_x and $e_1^v, e_2^v, \ldots, e_j^v$ (see Figure 4.10(a)).



(a) $\pi(x)$ and $\rho(\eta'^-)$ do not allow existence of a drawing η'^- without crossings

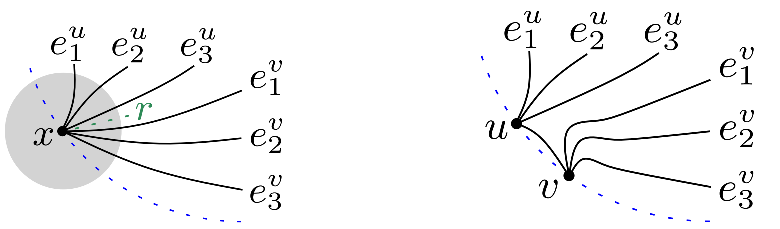
(b) $\pi(x)$ and $\rho(\eta'^-)$ in a η'^- without crossings

Fig. 4.9 Edges e_1 and e_2 should appear in the same order in $\pi(x)$ as v_1 and v_2 appear in $\rho(\eta'^-)$ so that η'^- does not have any crossings.

Now it is straightforward to decontract x to (u, v) and obtain an outerembedding η' of G with $\rho(\eta') = \rho(\eta)$ by locating u at the position of x and locating v at a point very close to x (see Figure 4.10).

□

Proof 2. Since R_I^p does not change the parity of the crossings between two independent edges, if an edge crosses itself we can use R_I^p to remove that



(a) before decontraction (b) after decontraction

Fig. 4.10 Decontracting x to (u, v)

crossing. Therefore, throughout this proof we assume that no edge crosses itself.

We prove this theorem by induction on the number of crossings. In the base case there is no crossing in η and therefore G is outerplanar. We proceed to the inductive cases.

Case 1. There are two independent edges $e = (v_1, v_2)$ and $e' = (v_3, v_4)$ in η such that e crosses e' an even number of times. Let x_1 and x_2 be two consecutive crossings between e and e' on e, so that crossings x_1 and x_2 divide e into three segments: the part from v_1 to x_1 or x_2, whichever appears first as we move from v_1 towards v_2 on (v_1, v_2) (say x_1), the part from x_1 to x_2 and the part from x_2 to v_2. Similarly x_1 and x_2 divide e' into three segments: the part from v_3 to x_1 or x_2, whichever appears first as we move from v_3 towards v_4 on (v_3, v_4) (say x_1), the part from x_1 to x_2 and the part from x_2 to v_4.

We shall reroute e or e' in η to obtain a drawing of G with a smaller number of crossings in which any two independent edges cross each other an even number of times. Let $\bar{l}$ denote the part of e from x_1 to x_2 and let $\overline{l'}$ denote the part of e' from x_1 to x_2. Let C denote the cycle that is formed by $\bar{l}$ and $\overline{l'}$. Since no edge crosses itself, $\bar{l}$ and $\overline{l'}$ do not have self-crossings. Moreover, since crossings x_1 and x_2 are two consecutive crossings between e and e' on e, $\bar{l}$ and $\overline{l'}$ do not cross each other. Therefore C is a simple cycle (C does not cross itself).

Since η is an outerdrawing, all the vertices of G are located outside C. Therefore the parity of the number of crossings of any arbitrary edge e'' and $\bar{l}$ is equal to the parity of the number of crossings of e'' and $\overline{l'}$. Therefore we can redraw e and e' near x_1 and x_2 to remove x_1 and x_2 (see Figure 4.11) such that:

- the redrawn e consists of the part of e from v_1 to x_1 and $\overline{l'}$ in η and the part of e from x_2 to v_2 and

(a) before (b) after

Fig. 4.11 Strengthened version of R_I^p.

(a) before (b) after

Fig. 4.12 Removing two crossings on two edges that cross each other an even number of times using the strengthened version of R_I^p.

- the redrawn e' consists of the part of e' from v_3 to x_1 and $\bar{l}$ in η and the part of e' from x_2 to v_4.

Then we can manipulate the drawing in the neighbourhood of x_1 and x_2 such that e and e' do not touch at x_1 and x_2. (See, e.g., Figure 4.12.)

Then we obtain a drawing η^- of G with a smaller number of crossings compared to η in which any two independent edges cross each other an even number of times. Moreover $\rho(\eta^-) = \rho(\eta)$. Therefore, by induction, G has an outerembedding η' such that $\rho(\eta') = \rho(\eta)$.

Case 2. No two independent edges in η cross an even number of times. In other words, if there is a crossing in η then it is formed by an intersection of two edges that are both incident with one vertex. Let x be a crossing in η that is formed by the intersection of $e = (u, v)$ and $e' = (v, w)$. Crossing x divides e into two segments: $\overline{ux}$, $\overline{xv}$ and it divides e' into two segments $\overline{vx}$, $\overline{xw}$. Let $\bar{l}$ denote the part of e from v to x and let $\overline{l'}$ denote the part of e' from v to x. Let C denote the simple cycle that is formed by $\bar{l}$ and $\overline{l'}$.

Since all the vertices of G, except v, are located outside C, the parity of the number of crossings of any arbitrary edge e'' (that is not incident with v) and $\bar{l}$ is equal to the parity of the number of crossings of e'' and $\overline{l'}$. Hence we can remove the crossing between e and e' using the strengthened version of the R_I^p move as shown in Figure 8.13, obtaining a drawing η^- of G with a smaller number of crossings compared to η in which any two independent edges cross each other an even number of times. Moreover $\rho(\eta^-) = \rho(\eta)$. Therefore, by induction, G has an embedding η' such that $\rho(\eta') = \rho(\eta)$.

□

Fig. 4.13 Removing a crossing from two adjacent edges using the strengthened version of R_I^p.

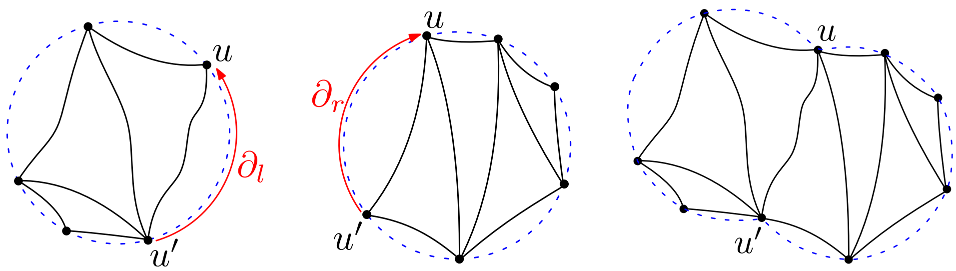
Proof 3. Define a *separating edge* to be an edge that joins two non-consecutive vertices in $\rho(\eta)$. We prove this lemma by induction on the number of separating edges. In the base case there are no separating edges in η. That is, the endpoints of any edge of G are consecutive in $\rho(\eta)$ and therefore G is a cycle and it is straightforward to draw an outerembedding of G. We proceed to the inductive case.

Let (u, u') be a separating edge in η. As we traverse the vertices in $\rho(\eta)$ in the clockwise direction, let U_r be the set of the vertices in $\rho(\eta)$ that appear after u and before u' and let U_l be the set of the vertices in $\rho(\eta)$ that appear after u' and before u. Both U_r and U_l are nonempty, by definition of the separating edge.

There is no edge (u_l, u_r) in η such that $u_l \in U_l$ and $u_r \in U_r$, otherwise it would cross (u, u') an odd number of times. Let $G_r = G[\{u, u'\} \cup U_r]$ and let η_r be the drawing of G_r obtained from η by deleting the vertices of U_l and their incident edges. Let $G_l = G[\{u, u'\} \cup U_l]$ and let η_l be the drawing of G_l obtained from η by deleting the vertices of U_r and their incident edges.

Any separating edge in either η_r and η_l is a separating edge in η. Moreover, (u, u') is a separating edge in η but it is not a separating edge in η_r or η_l. Therefore, the number of separating edges in G_r and G_l is fewer than the number of separating edges in G and η_r and η_l are two outerdrawings of G_r and G_l respectively in which any two vertex-disjoint edges cross an even number of times. Therefore by induction there is an outerembedding η'_r of G_r and an outerembedding η'_l of G_l such that $\rho(\eta'_r) = \rho(\eta_r)$ and $\rho(\eta'_l) = \rho(\eta_l)$.

Let d_r and d_l be the disks on which η_r and η_l are drawn. Since $\rho(\eta'_r) = \rho(\eta_r)$ and $\rho(\eta'_l) = \rho(\eta_l)$, u and u' appear consecutively on both $\partial(d_r)$ and $\partial(d_l)$. As we traverse $\partial(d_r)$ in the clockwise direction, let ∂_r be that part of $\partial(d_r)$ from u' to u (on which any point appears after u' and before u). Similarly, as we traverse $\partial(d_l)$ in the clockwise direction, let ∂_l be that part of $\partial(d_l)$ from u to u' (on which any point appears after u and before u'). Choose a direction for ∂_r and ∂_l such that both of them are directed from u' to u (see Figures 4.14(b) and 4.14(a)). Let η' be the embedding that is obtained from η'_r and η'_l by:



(a) η_l; ∂_l is depicted in red.

(b) η_r; ∂_r is depicted in red.

(c) η'

Fig. 4.14 Obtaining η' from η_l and η_r.

1. gluing d_r and d_r together such that ∂_r is identified with ∂_l in the forward direction.

2. deleting one of the two curves that represents (u, u') (see Figure 4.14(c)).

η' is an outerembedding of G and therefore the lemma follows. □

Let η be an outerdrawing of a graph G on a disk Σ. Let p_1 and p_2 be two points on $\partial(\Sigma)$. We say p_1 and p_2 *separate* vertices v_1 and v_2 of η if v_1 and v_2 are in two distinct components of $\partial(\Sigma) \setminus \{p_1, p_2\}$.

Proposition 4. *Let η be an outerdrawing of a graph G on a disk Σ and let p_1 and p_2 be two points on $\partial(\Sigma)$ such that:*

1. *the order of the vertices of G and the points p_1 and p_2 on $\partial(\Sigma)$ is as follows: $v_1, v_2, \ldots, v_i, p_1, v'_1, v'_2, \ldots, v'_{i'}, p_2$ (where $v_1, v_2, \ldots, v_i, v'_1, v'_2, \ldots, v'_{i'}$ are the vertices of G).*

2. *any pair of vertex-disjoint edges $(v_j, v'_k), 1 \leq j \leq i, 1 \leq k \leq i'$ cross each other an odd number of times.*

3. *any other pair of vertex-disjoint edges cross an even number of times.*

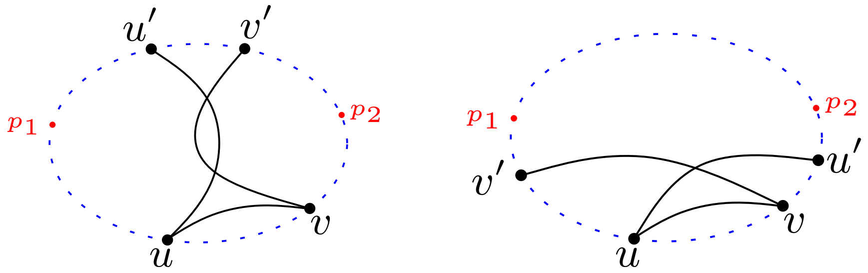
Then there is an outerembedding η' of G such that:

$$\rho(\eta') = v_1, v_2, \ldots, v_i, v'_{i'}, v'_{i'-1}, \ldots, v'_1.$$

Proof. We prove this by induction on the number of vertices. In the base case, G has two vertices that are separated by p_1 and p_2, and the lemma holds trivially. We proceed to the inductive case.

Let u and v be two consecutive vertices of η on $\partial(\Sigma)$ that are not separated by p_1 and p_2. Let $u' \in N(u) \setminus \{v\}$ and let $v' \in N(v) \setminus \{u\}$.

If p_1 and p_2 separate u' from u and v' from v, by the second condition of this lemma, u, u', v', v cannot appear in this order on $\partial(\Sigma)$, otherwise (u, u') crosses (v, v') an even number of times (see, for example, Figure 4.15(a)). Therefore, those vertices of $N(u)$ and $N(v)$ that are separated from u and v by p_1 and p_2 do not appear interleaved on $\partial(\Sigma)$ (see Figure 4.16(a)).



(a) Edges (u, u') and (v, v') crossing an even number of times, violating first condition of the lemma.

(b) Edges (u, u') and (v, v') crossing odd number of times, violating the second condition of the lemma.

Fig. 4.15 Order of vertices on $\partial(\Sigma)$

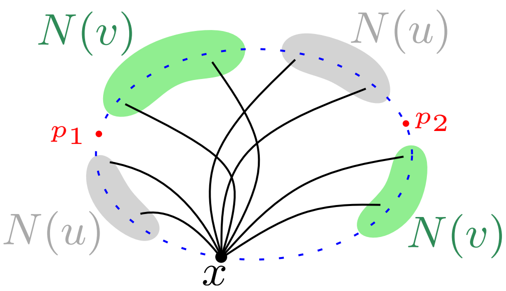
If p_1 and p_2 do not separate u' from u and v' from v, by the third condition of this lemma, u, v', u', v cannot appear in this order on $\partial(\Sigma)$, otherwise (u, u') crosses (v, v') an odd number of times (see for example, Figure 4.15(b)). Therefore, those vertices of $N(u)$ and $N(v)$ that are not separated from u and v by p_1 and p_2 do not appear interleaved on $\partial(\Sigma)$ (see Figure 4.16(b)).

Obtain a drawing η_{temp} from η by deleting the edge (u, v) from η (if it exists) and adding an edge between u and v such that it follows $\partial(\Sigma)$ closely enough that it does not meet any other edges. Since u and v are not separated by p_1 and p_2 in η_{temp} and (u, v) is not crossed by any edges in η_{temp}, both of the conditions of this lemma still hold in η_{temp}. Therefore η_{temp} remains an outerdrawing.

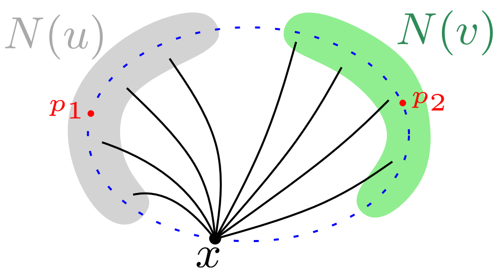
Obtain a graph G^- from G and a drawing η^- from η_{temp} by a similar process to that described in the first proof of Lemma 5 (see Figure 4.8).

It is straightforward to see that conditions 1 and 2 of this lemma hold for η^-. By induction there is an outerembedding η'^- of G such that if $\rho(\eta^-) = v_1, v_2, \ldots, x = v_j, \ldots, v_{i-1}, v'_1, v'_2, \ldots, v'_{i'}$ then $\rho(\eta'^-) = v_1, v_2, \ldots, x = v_j, \ldots, v_{i-1}, v'_{i'}, v'_{i'-1}, \ldots, v'_1$. Hence, vertices of $N(u)$ and vertices of $N(v)$ do not appear interleaved on $\partial(\Sigma)$. Roughly speaking, vertices of $N(u)$ appear on one of side of $\partial(\Sigma)$ and vertices of $N(v)$ appear on the other side of $\partial(\Sigma)$ (see Figure 4.16).

Therefore, by similar reasoning to that described in the first proof of



(a) η^-



(b) η'^-

Fig. 4.16 Order of vertices on $\partial(\Sigma)$ in η^- and η'^-

Lemma 5, it is possible to decontract x to (u, v) and obtain an outerembedding η' of G with $\rho(\eta'^-) = v_1, v_2, \ldots, v_{j-1}, v_j = u, v_{j+1} = v, \ldots, v_i, v'_{i'}, v'_{i'-1}, \ldots, v'_1$ (see Figure 4.10). □

The next lemma investigates the relation between the Reidemeister moves that we introduced in Chapter 3 and the parity of the number of crossings between two vertex-disjoint edges in outerdrawings.

Lemma 6. *R^p_I, R^p_{II}, R^p_{III}, R^p_V preserve the parity of the number of crossings between two vertex-disjoint edges in an outerdrawing.*

Proof. The proof is straightforward. □

Now we are prepared to prove the following lemma.

Lemma 7. *Let η be an outerembedding of a graph G on a disk Σ. Then any two vertex-disjoint edges cross an even number of times in any outerdrawing η' of G with $\Pi_{\eta'} = \Pi_\eta$.*

Proof. The proof of this lemma is very similar to the proof of Lemma 4. In the proof of Lemma 4 we used Lemma 2. Here we use Lemma 6 instead. □

Lemma 5 and Lemma 7 enable us to establish the following result.

Theorem 35. *Let ρ be a sequence of the vertices of a graph G. The following statements are equivalent:*

1. *There is an outerembedding η of G on a disk such that $\rho_\eta = \rho$.*

2. *There exists an outerdrawing η' of G such that $\rho_{\eta'} = \rho$ and any two vertex-disjoint edges in η' cross each other an even number of times.*

3. *In any outerdrawing η'' of G where $\rho_{\eta''} = \rho$, every two vertex-disjoint edges cross each other an even number of times.*

4.3 Characterisations of Outerplanar Graphs

Theorem 34 and Theorem 35 provide us with the following two characterisations of outerplanar graphs.

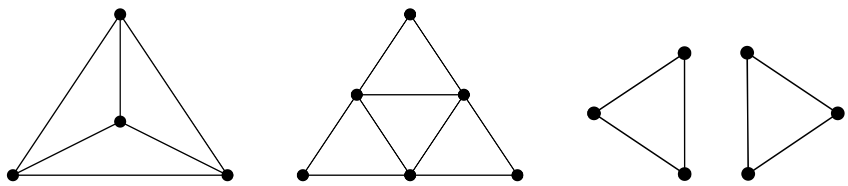
Theorem 36. *A graph G is outerplanar if and only if there is an outerdrawing η of G such that any edge in η crosses any cycle in η an even number of times.*

Theorem 37. *A graph G is outerplanar if and only if there is an outerdrawing η of G such that any two vertex-disjoint edges in η cross an even number of times.*

We propose the following two conjectures about outerplanar graphs.

Conjecture 2. *A graph G is outerplanar if and only if there is an outerdrawing η of G such that any two vertex-disjoint edges in any cycle in η cross each other an even number of times.*

Conjecture 3. *Let G^* be the dual of an outerembedding of a graph G. Then G^* does not contain the graphs shown in Figure 4.17 as minors.*



(a) K_4 (b) (c) $K_3 \cup K_3$

Fig. 4.17 Three minimal excluded minors, mentioned in Conjecture 3, for the graphs that are duals of outerembeddings.

5

Non-separating Planar Graphs

5.1 Non-separating Planar Graphs

In this chapter, we show that a graph is a non-separating planar graph if and only if it does not contain any of $K_1 \cup K_4$ or $K_1 \cup K_{2,3}$ or $K_{1,1,3}$ as a minor.
Furthermore, we show that any maximal non-separating planar graph is either an outerplanar graph or a wheel or it can be obtained by subdividing some of the side-edges of the 1-skeleton of a triangular prism (two disjoint triangles linked by a perfect matching).

Let C be a cycle in a planar drawing D of a graph G, then C is a *separating cycle* if there is at least one vertex in the interior of C and one vertex in the exterior of C. separating cycle

A *non-separating planar drawing* of a graph is a planar drawing of the graph that does not contain any separating cycles. A *non-separating planar graph* is a graph that has a non-separating planar drawing (see for example Figure 5.1). non-separating planar drawing non-separating planar graph

In this chapter we characterise non-separating planar graphs. Non-separating planar graphs are a subclass of planar graphs and a superclass of outerplanar

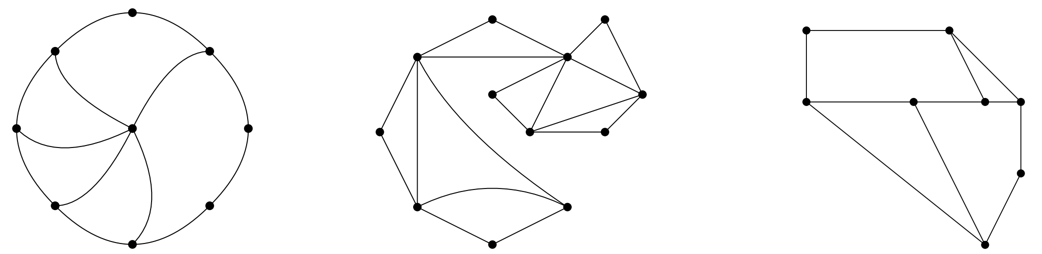
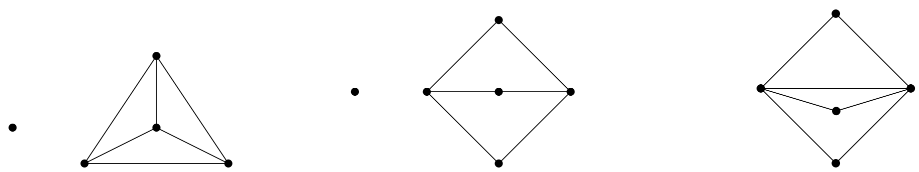


Fig. 5.1 Three examples of non-separating planar graphs

graphs and are closed under minors. To characterise non-separating planar graphs we prove Theorems 38 and 39 as follows.

Theorem 38. *A graph G is a non-separating planar graph if and only if it does not contain any of $K_1 \cup K_4$ or $K_1 \cup K_{2,3}$ or $K_{1,1,3}$ as a minor (see Figure 5.2).*[1]



(a) $K_1 \cup K_4$ (b) $K_1 \cup K_{2,3}$ (c) $K_{1,1,3}$

Fig. 5.2 Minimal excluded minors for non-separating planar graphs

triangular prism

elongated triangular prism

A graph is a *triangular prism* if it is isomorphic to the graph that is depicted in Figure 5.3(a). A graph is an *elongated triangular prism* if it is a triangular prism or if it is obtained by some sequence of subdivisions of the red dashed edges of the triangular prism depicted in Figure 5.3(b).

We also characterise non-separating planar graphs in terms of their structure as follows.

Theorem 39. *Any non-separating planar graph is one of the following:*

1. *an outerplanar graph,*
2. *a subgraph of a wheel,*
3. *a subgraph of an elongated triangular prism.*

Theorems 38 and 39 together provide us with Theorem 40:

Theorem 40. *The following are equivalent, for any graph G:*

[1] where $\cup$ denotes the disjoint union

(a) Triangular prism (b) Elongated triangular prism

Fig. 5.3 Triangular prism and elongated triangular prism

1. *G does not contain any of $K_1 \cup K_4$ or $K_1 \cup K_{2,3}$ or $K_{1,1,3}$ as a minor.*

2. *G is outerplanar or a subgraph of a wheel or a subgraph of an elongated triangular prism.*

3. *G is a non-separating planar graph.*

5.2 Preliminary Lemmas

A path P in a graph G is said to be *chordless* if there is no edge between any two non-consecutive vertices of P in G. A *uv-path* is a path from a vertex u to a vertex v.

chordless

uv-path

Vertices u and v, in a subdivision S of $K_{2,3}$, are called the *terminal vertices* of S if both u and v have degree 3 in S. Define the *terminal paths* in S as the three uv-paths in S.

terminal vertices

terminal paths

Next we will prove a couple of lemmas about the graphs that do not contain $K_{1,1,3}$ as a minor (see Figure 5.4).

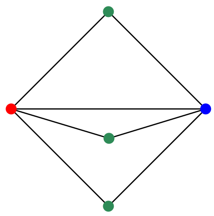


Fig. 5.4 $K_{1,1,3}$

Lemma 8. *Every terminal path in a spanning $K_{2,3}$-subdivision of a $K_{1,1,3}$-minor-free graph is chordless.*

Proof. Suppose that such a terminal path P has a chord e. Then it is easy to find a $K_{1,1,3}$ minor in the graph. $\square$

A vertex w of a uv-path P is an *inner vertex* of P if $w \neq u$ and $w \neq v$. An edge e of a path P is an *inner edge* of P if e is incident with two inner vertices of P.

inner vertex

inner edge

Given a set $\mathcal{P}$ of paths in a graph G, define a *middle path* $P \in \mathcal{P}$ to be a path such that for any other path $P' \in \mathcal{P}$ there is an edge in G that is incident with an inner vertex of P and an inner vertex of P'. In other words, for each path $P' \in \mathcal{P}$ other than P there is an inner vertex of P that is adjacent to an inner vertex of P' (see, e.g., Figure 5.5). Two vertices u and v are *co-path* with respect to $\mathcal{P}$ if u and v are on the same path in $\mathcal{P}$.

middle path

co-path

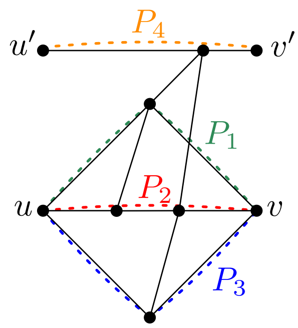


Fig. 5.5 P_2 is the only middle path among the four paths P_1, P_2, P_3, P_4, where P_1, P_2, P_3 are uv-paths and P_4 is a $u'v'$-path.

Any graph G that contains a $K_{2,3}$-subdivision is *middle-less* if there is no middle path among the terminal paths of any spanning subgraph of G that is a $K_{2,3}$-subdivision. Any graph G with a spanning $K_{2,3}$-subdivision is *middle-ful* if it is not middle-less.

middle-less

middle-ful

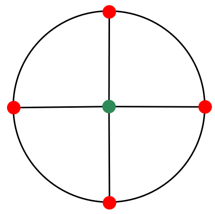
We divide the rest of lemmas in this section into two subsections. The first section is about the middle-less graphs and the second section is about the middle-ful ones.

5.2.1 Middle-less Graphs

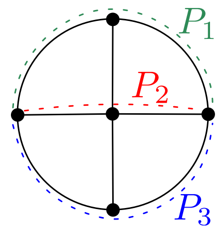
We start by proving that middle-less graphs do not contain W_4 as a minor.

Lemma 9. *If G is a middle-less graph then G does not contain W_4 as a minor (see Figure 5.6(a)).*

Proof. Suppose that there is a middle-less graph G that contains W_4 as a minor. Then it is straightforward to find a $K_{2,3}$-subdivision with a middle path in G. But this is a contradiction since G is middle-less (see, e.g., Figure 5.6(b)). $\square$



(a) W_4



(b) W_4 with P_2 as its middle path.

Fig. 5.6 Any graph with a W_4 minor is middle-ful.

Let U be a subset of the vertices of a graph G, then $G[U]$ denotes the subgraph of G induced by U. Similarly, for any subgraph H of the graph G, $G[H]$ denotes the subgraph of G that is induced by the vertices of H.

Lemma 10. *Let P_1, P_2, P_3 be the terminal paths in a spanning $K_{2,3}$-subdivision S of a middle-less graph G with neither a $K_{1,1,3}$-minor nor a $(K_1 \cup K_{2,3})$-minor where $G[P_1 \cup P_2]$ has an edge e that is not in P_1 or P_2. Then:*

- *either every edge of $G[P_2 \cup P_3]$ is an edge of $P_2 \cup P_3$ or every edge of $G[P_2 \cup P_1]$ is an edge of $P_1 \cup P_3$ and*
- *e is the only edge in $G[P_1 \cup P_2]$ that is not in P_1, P_2 and P_3.*

Proof. Let $G_1 = G[P_1 \cup P_2]$, $G_2 = G[P_2 \cup P_3]$ and $G_3 = G[P_3 \cup P_1]$ and let u and v be the two vertices of e. First we show that G_2 does not have any edge that is not an edge of P_2 or P_3. To reach a contradiction suppose that G_2 has an edge $e_1 = (u_1, v_1)$ that is not in $P_2 \cup P_3$. Moreover, by the assumptions of the lemma, there is an edge e in G_1 that is not in $P_1 \cup P_2$.

By Lemma 8, e and e_1 are not chords of P_1, P_2 or P_3 and therefore, without loss of generality, u is an inner vertex of P_1 and v is an inner vertex of P_2 and u_1 is an inner vertex of P_2 and v_1 is an inner vertex of P_3 (see, e.g., Figure 5.7). But this is a contradiction since then P_2 is a middle path and therefore G is not middle-less. Similarly we can show that G_3 does not have any edge that is not an edge of P_1 or P_3.

Now we show that there is at most one edge in G_1 that is not an edge of P_1 or P_2. To reach a contradiction suppose that G_1 has two edges $e_1 = (u_1, v_1)$ and $e_2 = (u_2, v_2)$ that are not among the edges of P_1 or P_2 (note that it is possible that either $u_1 = u_2$ or $v_1 = v_2$ or $u_1 = v_2$ or $v_1 = u_2$).

By Lemma 8, e_1 and e_2 are not chords of P_1 or P_2 and therefore, without loss of generality, let u_1 and u_2 be among the inner vertices of P_1 and v_1 and v_2 be among the inner vertices of P_2 (see, e.g., Figure 5.8).

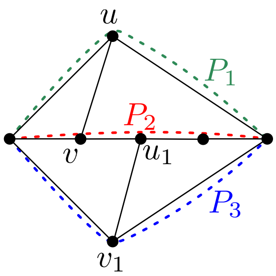


Fig. 5.7 u, v, u_1, v_1 in G

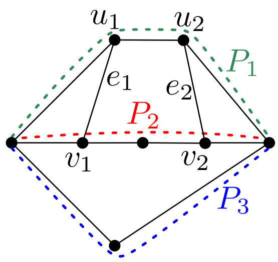
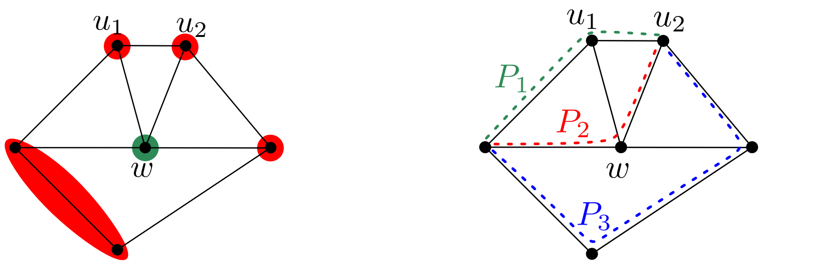


Fig. 5.8 e_1, e_2, P_1, P_2 and P_3 in G

Choose P to be either P_1 or P_2 so that the endpoints of e_1 and e_2 on the other path are distinct. Let G^- be the graph that is obtained by contracting all the edges of P except the ones that are incident with the terminal vertices of S into a single vertex w. It is easy to see that there is a W_4-minor in G^- (see, e.g., Figure 5.9). Then by Lemma 9, G is not middle-less, which is a contradiction.



(a) G^- (b) G^- with a middle path P_2

Fig. 5.9 Finding a middle path in G^-.

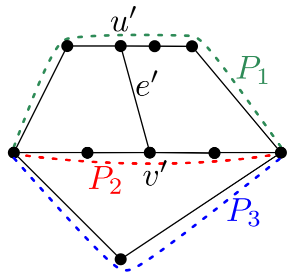
□

Lemma 11. *Let $\{u, v\}$ and $\{P_1, P_2, P_3\}$ be the sets of terminal vertices and terminal paths respectively in a spanning $K_{2,3}$-subdivision S of a middle-less graph G with no $K_{1,1,3}$-minor and no $(K_1 \cup K_{2,3})$-minor where the lengths of P_1 and P_2 are greater than 2 and $G[P_1 \cup P_2]$ has an edge $e' = (u', v')$ that is not in $P_1 \cup P_2$. Then either:*

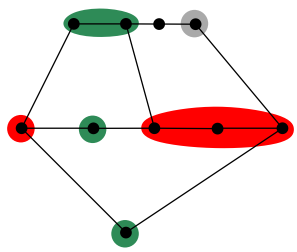
- *u' and v' are adjacent to u, or*
- *u' and v' are adjacent to v.*

Proof. By Lemma 8, e' is not a chord of P_1 or P_2 and therefore, without loss of generality, let u' be an inner vertex of P_1 and v' be an inner vertex of P_2. To reach a contradiction, suppose that u' and v' are not both adjacent to the same vertex u or v. We have two cases:

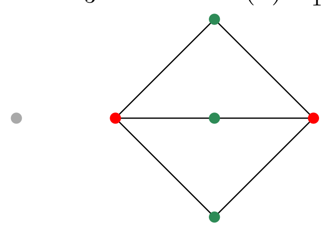
Case 1. Neither u' nor v' is adjacent to the terminal vertices. In this case it is easy to find a $K_1 \cup K_{2,3}$ minor in G (see, e.g., Figure 5.10 and Figure 5.18(a)).



(a) e', P_1, P_2 and P_3 in G



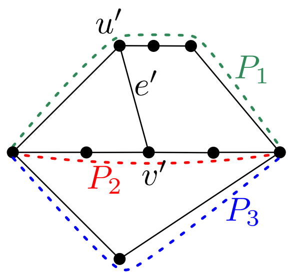
(b) P_1' and P_1'' in G



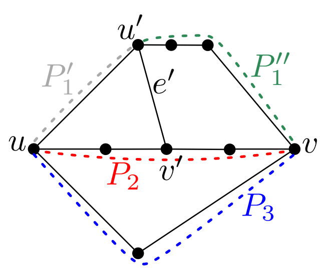
(c) $K_1 \cup K_{2,3}$

Fig. 5.10 e', P_1, P_2, P_3, P_1' and P_1'' in G. Compare the colouring scheme of Figure 5.10(b) with Figure 5.18(a) to see how $K_1 \cup K_{2,3}$ is a minor of G.

Case 2. One of the two vertices u' or v' is adjacent to u or v. Without loss of generality let u' be adjacent to u (see, e.g., Figure 5.11(a)).



(a) e', P_1, P_2 and P_3 in G



(b) P_1' and P_1'' in G

Fig. 5.11 e', P_1, P_2, P_3, P_1' and P_1'' in G.

The vertex u' splits P_1 into two shorter paths P_1' and P_1'', where P_1 contains the edge (u, u'). Without loss of generality, let P_1' be a shortest path among P_1' and P_1'' (see, e.g., Figure 5.11(b)). Then, since the lengths of P_1 and P_2 are greater than 2, it is easy to see that there is a $K_{2,3}$ minor in $P_1' \cup e' \cup P_2 \cup P_3$ and an inner vertex v'' on P_1'' such that $P_1' \cup e' \cup P_2 \cup P_3$ and v'' form a $K_1 \cup K_{2,3}$ minor in G (see, e.g., Figure 5.12). However, this is a contradiction since G is a $K_1 \cup K_{2,3}$-minor free graph.

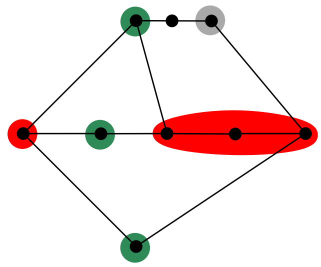
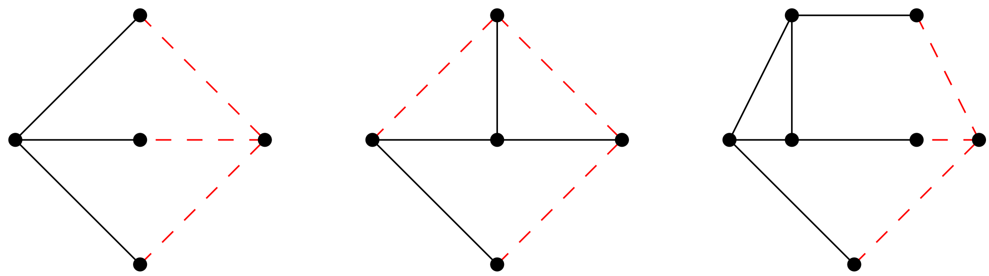


Fig. 5.12 Finding a $K_1 \cup K_{2,3}$-minor in G (compare with Figure 5.10(c)).

$\square$

Lemma 12. *Let $\mathcal{G}$ be the family of middle-less graphs with no $K_{1,1,3}$-minor, no $(K_1 \cup K_4)$-minor, no $(K_1 \cup K_{2,3})$-minor, and that contain a $K_{2,3}$-subdivision. Then any $G \in \mathcal{G}$ can be obtained by subdividing the dashed (red) edges of the graphs that are shown in Figure 5.13.*



(a) Type I (b) Type II (c) Type III

Fig. 5.13 Three types of middle-less non-separating planar graphs

Proof. Let P_1, P_2, P_3 be the terminal paths and u, v be the terminal vertices in a $K_{2,3}$-subdivision S of a graph $G \in \mathcal{G}$. Since G does not contain $K_1 \cup K_{2,3}$ as a minor, S is a spanning $K_{2,3}$-subdivision of G. If G does not have any edges other than the edges of P_1, P_2, P_3 then, clearly, G can be obtained by subdividing the dashed (red) edges of the graph depicted in Figure 5.13(a).

Now let us consider the case where G has an edge $e' = (u', v')$ that is not an edge of any of P_1, P_2, P_3. By Lemma 10, e' is the only edge in G that is not an edge of P_1, P_2 or P_3. By Lemma 8, e' is not a chord of P_1 or P_2 and therefore, without loss of generality, let u' be an inner vertex of P_1 and v' be an inner vertex of P_2. We have two cases:

Case 1. Either P_1 or P_2 has length 2. It is easy to verify that in this case G is a graph that can be obtained by subdividing the red dashed edges in Figure 5.13(b).

Case 2. The lengths of both P_1 and P_2 are more than 2. By Lemma 11, both u' and v' are adjacent to the same vertex u or v. Now it is easy to verify that in this case G is a graph that can be obtained by subdividing the red dashed edges in Figure 5.13(c). □

5.2.2 Middle-ful Graphs

Lemma 13. *There is at most one middle path in the set of terminal paths of a spanning $K_{2,3}$-subdivision of a $K_{1,1,3}$-minor-free graph.*

Proof. Let $\mathcal{P} = \{P_1, P_2, P_3\}$ be the set of terminal paths in a spanning $K_{2,3}$-subdivision S in a $K_{1,1,3}$-minor-free graph G. To reach a contradiction, suppose that there is more than one middle path in $\mathcal{P}$. Without loss of generality, let P_1 and P_2 both be middle paths. Since P_1 and P_2 are middle paths:

1. there is an edge incident with an inner vertex of P_1 and an inner vertex of P_2, and

2. there is an edge incident with an inner vertex of P_1 and an inner vertex of P_3, and

3. there is an edge incident with an inner vertex of P_2 and an inner vertex of P_3.

Now, it is easy to find a $K_{1,1,3}$ as a minor in G. see, e.g., Figure 5.14. □

Next we will prove a lemma about a class of graphs that does not contain $K_1 \cup K_4$ as a minor (see Figure 5.15).

Lemma 14. *Let P_1, P_2, P_3 be the terminal paths in a spanning $K_{2,3}$-subdivision S of a graph G with no $(K_1 \cup K_4)$-minor, where P_2 is a middle path. Then there is no pair of edges $e_1 = (u_1, v_1)$ and $e_2 = (u_1, v_2)$ in G such that u_1 is an inner vertex of P_1 or P_3 and v_1 and v_2 are two distinct inner vertices of P_2.*

Proof. To reach a contradiction suppose that there is an edge $e_1 = (u_1, v_1)$ and an edge $e_2 = (u_1, v_2)$ such that u_1 is an inner vertex of P_1 or P_3 and v_1

(a) G with P_1 and P_2 as middle paths.

(b) G contains $K_{1,1,3}$ as a minor.

Fig. 5.14 If P_1 and P_2 are middle paths then G contains $K_{1,1,3}$ as a minor. The colour scheme used here to colour the vertices of a $K_{1,1,3}$ minor is the same as the one used in Figure 5.4

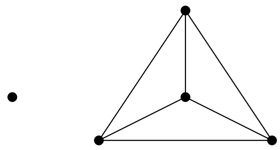
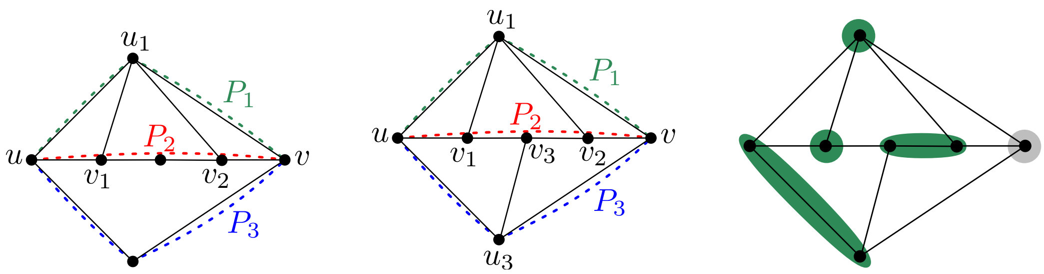


Fig. 5.15 $K_1 \cup K_4$

and v_2 are two inner vertices of P_2 (see, e.g., Figure 5.16(a)). Without loss of generality let u_1 be an inner vertex of P_1. Since P_2 is a middle path there is also an edge $e_3 = (u_3, v_3)$ in G, where v_3 can possibly be in $\{v_1, v_2\}$, such that u_3 is an inner vertex of P_3 and v_3 is an inner vertex of P_2 (see, e.g., Figure 5.16(b)). Now it is easy to find a $(K_1 \cup K_4)$-minor in G (see, e.g., Figure 5.16(c)).



(a) G with P_2 as a middle path.

(b) G contains $K_{1,1,3}$ as a minor.

(c) Finding a $K_1 \cup K_4$ minor in G.

Fig. 5.16 G contains $K_1 \cup K_4$ as a minor.

□

Let P be a path and h be a vertex that is not in P. Let G be the graph that is obtained from P and h by adding an edge (h, v) for every vertex v in fan
P. Then G is a *fan* graph and h is the *handle* of G. K_3 and K_4 minus an handle
edge are the only fan graphs that do not have a unique handle.

Let P be a uv-path. We define the *outer* inner vertices of P as those inner vertices of P that are adjacent to u and v on P. outer vertex

Lemma 15. *Let P_1, P_2, P_3 be the terminal paths in a spanning $K_{2,3}$-subdivision S of a graph G with no $K_{1,1,3}$-minor, no $(K_1 \cup K_4)$-minor and no $(K_1 \cup K_{2,3})$-minor, where P_2 is a middle path. Then, $G[P_1 \cup P_2]$ and $G[P_2 \cup P_3]$ are subgraphs of fan graphs whose handles are among the outer inner vertices of P_2.*

Proof. Let $G_1 = G[P_1 \cup P_2]$ and $G_2 = G[P_2 \cup P_3]$. First we show that G_1 and G_2 are subgraphs of fan graphs. To reach a contradiction suppose that either G_1 or G_2 is not a subgraph of a fan graph. Without loss of generality, suppose that G_1 is not a subgraph of a fan graph.

Since G_1 is not a subgraph of a fan graph, there are two edges $e_1 = (u_1, v_1)$ and $e_2 = (u_2, v_2)$ in G_1 that are neither an edge of P_1 nor an edge of P_2 and are vertex-disjoint. By Lemma 8, e_1 and e_2 are not chords of P_1 or P_2. In other words:

- u_1, v_1, u_2, v_2 are all inner vertices of P_1 and P_2.
- u_1 and v_1 are not co-path with respect to $\{P_1, P_2, P_3\}$.
- u_2 and v_2 are not co-path with respect to $\{P_1, P_2, P_3\}$.

Without loss of generality let u_1 and u_2 be the two endpoints of e_1 and e_2 on P_1 and let v_1 and v_2 be the other two endpoints of e_1 and e_2 on P_2. Let u and v be the terminal vertices of S. Contract all the edges of P_1 that are not incident with u and v into a single vertex w and let us denote the resulting minor of G by H. The graph H is a spanning $K_{2,3}$-subdivision, and satisfies all conditions in Lemma 14.

Since H is a minor of G, it does not contain a $K_1 \cup K_4$ minor. Moreover, P_2 is a middle path in H. Also, w is adjacent to v_1 and v_2 in H. Therefore, $e_1 = (w, v_1)$ and $e_2 = (w, v_2)$ are two edges of H that contradict Lemma 14 and therefore G_1 is a subgraph of a fan graph. We denote the corresponding fan graph by G_1^+.

Similarly, we conclude that G_2 is a subgraph of a fan graph and we denote the corresponding fan graph by G_2^+.

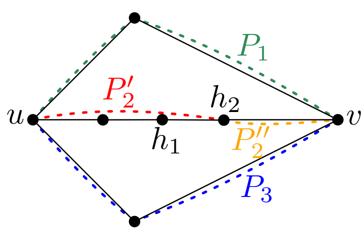
Next we show that the handles of fan graphs G_1^+ and G_2^+, which we denote by h_1 and h_2 respectively, are outer inner vertices of P_2. As the first step, we show that h_1 and h_2 are inner vertices of P_2 and then as the second step we show that both h_1 and h_2 are adjacent to either u or v on P_2 (i.e., h_1 and h_2 are outer inner vertices of P_2).

We use contradiction to prove the first step. To reach a contradiction suppose that either the handle of G_1^+ or the handle of G_2^+ is not an inner vertex of P_2. Without loss of generality, suppose that the handle of G_1^+ is not an inner vertex of P_2. Then it must be on P_1. So there are two edges $e_1 = (u_1', v')$ and $e_2 = (u_2', v')$ in G_1 that are not in $E(P_1) \cup E(P_2)$ and are incident with the same vertex v' on P_1.

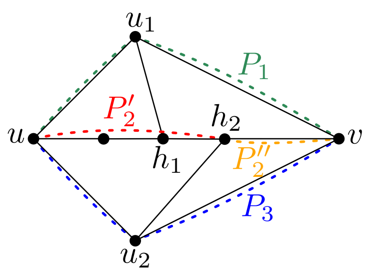
By Lemma 8, e_1 and e_2 are not chords of P_1 or P_2 and therefore v' is an inner vertex of P_1 and u_1' and u_2' are inner vertices of P_2. However, this is also in contradiction with Lemma 14.

We use contradiction to prove the second step as well. To reach a contradiction, without loss of generality, suppose that h_1 is not adjacent to u or v on P_2 and let h_2 be any vertex on P_2. The handle h_2 splits P_2 into two subpaths: P_2' from u to h_2 and P_2'' from h_2 to v. Without loss of generality, let h_1 be an inner vertex of P_2' or let $h_1 = h_2$ (see, e.g., Figure 5.17(a)).

Since P_2 is a middle path, there are two edges $e_1 = (u_1, x_1)$ and $e_2 = (u_2, x_2)$ such that u_1 is an inner vertex of P_1 and u_2 is an inner vertex of P_3. Since G_1 is a subgraph of a fan graph G_1^+ with handle h_1 we have $x_1 = h_1$ and since G_2 is a subgraph of fan graph G_2^+ with handle h_2 we have $x_2 = h_2$ (see, e.g., Figure 5.17(b)). Let P_1' be the part of P_1 from u to u_1 and let P_3' be the part of P_3 from u to u_2.



(a) h_1, h_2, P_2' and P_2'' in G.

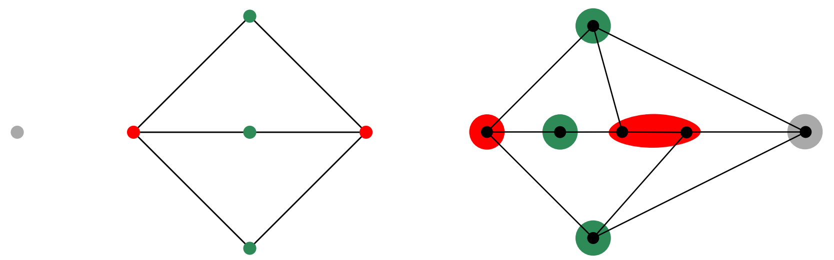


(b) u_1, u_2 in G.

Fig. 5.17 Finding $K_1 \cup K_{2,3}$ minor in G.

Now it is easy to see that v together with $P_1' \cup (u_1, h_1) \cup P_2' \cup (u_2, h_2) \cup P_3'$ contains a $K_1 \cup K_{2,3}$ minor, which is a contradiction (see, e.g., Figure 5.18). □

Lemma 16. *Let $G[P_1 \cup P_2]$ and $G[P_2 \cup P_3]$ be subgraphs of fan graphs G_1^+ and G_2^+ with the same handle h where P_1, P_2, P_3 are the terminal paths in a spanning $K_{2,3}$-subdivision S of a $K_{1,1,3}$-minor-free and $(K_1 \cup K_{2,3})$-minor-free graph G in which P_2 is a middle path. Then length of P_2 is 2.*



(a) $K_1 \cup K_{2,3}$

(b) G contains $K_1 \cup K_{2,3}$ as a minor.

Fig. 5.18 Finding $K_1 \cup K_{2,3}$ minor in G.

Proof. To reach a contradiction suppose that length of P_2 is greater than 2. Since P_2 is the middle path, by Lemma 15, h is an outer inner vertex of P_2. Now, it is easy to find a $K_1 \cup K_{2,3}$ minor in G which contradicts the assumptions of the lemma (see, e.g., Figure 5.19).

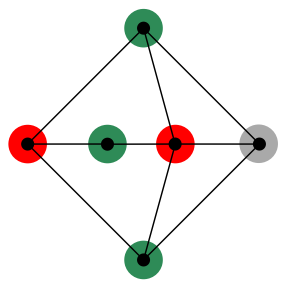


Fig. 5.19 Finding $K_1 \cup K_{2,3}$ in G

□

Lemma 17. *Let $G_1 = G[P_1 \cup P_2]$ and $G_2 = G[P_2 \cup P_3]$ be subgraphs of fan graphs G_1^+ with handle h_1 and G_2^+ with handle h_2 respectively such that $h_1 \neq h_2$, where u, v are the terminal vertices and P_1, P_2, P_3 are the terminal paths in a spanning $K_{2,3}$-subdivision S of a graph G with no $K_{1,1,3}$-minor, no $(K_1 \cup K_4)$-minor, no $(K_1 \cup K_{2,3})$-minor, and in which P_2 is a middle path.*

Then there is exactly one edge $e' = (h_1, v')$ in G_1 that is not in $P_1 \cup P_2$ and there is exactly one edge $e'' = (h_2, v'')$ in G_2 that is not in $P_2 \cup P_3$, where:

- *h_1 and v' are outer inner vertices of P_2 and P_1 respectively that are both adjacent to u or both adjacent to v and*

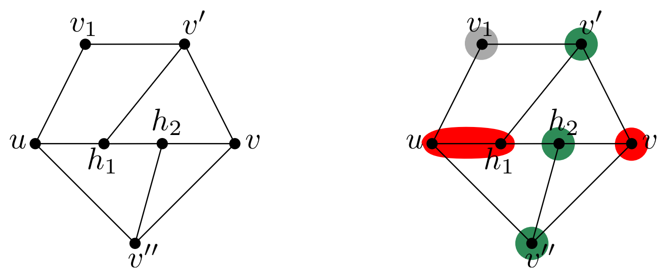
- *h_2 and v'' are outer inner vertices of P_2 and P_3 respectively that are both adjacent to u or both adjacent to v.*

Proof. Since P_2 is a middle path, there is an edge $e' = (h_1, v')$ in G_1 that is not in $P_1 \cup P_2$ and there is an edge $e'' = (h_2, v'')$ in G_2 that is not in $P_2 \cup P_3$. Moreover, by Lemma 15, h_1 and h_2 are outer inner vertices of P_2.

Now to reach a contradiction, without loss of generality, let h_1 be adjacent to u on P_2 but let v' be a vertex that is not adjacent to u on P_1. Let v_1 be the vertex that is adjacent to u on P_1.

Since h_1 and h_2 are inner vertices of the middle path P_2, by Lemma 8, v' is an inner vertex of P_1 and v'' is an inner vertex of P_2 (see, e.g., Figure 5.20(a)).

We know that v_1 appears before v' as we traverse P_1 from u towards v and h_1 appears before h_2 as we traverse P_2 from u towards v. Let P' be the part of P_1 that stretches from v' to v. Now it is easy to see that v_1 together with $(h_1, v') \cup P' \cup P_2 \cup P_3$ contains a $K_1 \cup K_{2,3}$ minor, which is a contradiction (see, e.g., Figure 5.20(b)).



(a) $K_1 \cup K_{2,3}$ (b) G contains $K_1 \cup K_{2,3}$ as a minor.

Fig. 5.20 Finding a $K_1 \cup K_{2,3}$ minor in G.

□

Lemma 18. *Let $\mathcal{G}$ be the family of middle-ful $K_{1,1,3}$-minor-free, $(K_1 \cup K_4)$-minor-free and $(K_1 \cup K_{2,3})$-minor-free graphs that contain a $K_{2,3}$-subdivision. Then any $G \in \mathcal{G}$ is either a subgraph of a wheel with at least 4 spokes or it is an elongated triangular prism.*

Proof. Let P_1, P_2, P_3 be the terminal paths and u, v be the terminal vertices in a $K_{2,3}$-subdivision S of a graph $G \in \mathcal{G}$ where P_2 is a middle path. Since G does not contain $K_1 \cup K_{2,3}$ as a minor, S is a spanning $K_{2,3}$-subdivision of G. Let $G_1 = G[P_1 \cup P_2]$ and $G_2 = G[P_2 \cup P_3]$. Since P_2 is a middle path, by Lemma 15, G_1 and G_2 are subgraphs of fan graph G_1^+ and G_2^+ with handles h_1 and h_2 where h_1 and h_2 are both among the outer inner vertices of P_2.

We break the rest of the proof into two cases:

Case 1. $h_1 = h_2$**.** By Lemma 16, the length of P_2 is 2 and therefore G is a subgraph of a wheel W. Moreover, since P_2 is a middle path, W has at least 4 spokes.

Case 2. $h_1 \neq h_2$**.** By Lemma 17, there is exactly one edge e_1 in G_1 that is not in $P_1 \cup P_2$ and exactly one edge e_2 in G_2 that is not in $P_2 \cup P_3$. Then, by Lemma 17, G is an elongated triangular prism. □

5.3 Proof of the Main Theorems

In this section we prove Theorems 38 and 39.

Lemma 19. *A graph G does not contain any of $K_1 \cup K_4$ or $K_1 \cup K_{2,3}$ or $K_{1,1,3}$ as a minor if and only if G is either an outerplanar graph or a subgraph of a wheel or an elongated triangular prism.*

Proof. It is straightforward to see that any outerplanar graph or a subgraph of a wheel or an elongated triangular prism does not contain any of $K_1 \cup K_4$ or $K_1 \cup K_{2,3}$ or $K_{1,1,3}$ as a minor. Next we prove the lemma in the other direction.

We break the proof into the following three cases:

1. G does not contain any of K_4 or $K_{2,3}$ as a minor.

2. G contains K_4 but does not contain $K_{2,3}$ as a minor.

3. G contains $K_{2,3}$ as a minor.

Case 1. G **does not contain any of** K_4 **or** $K_{2,3}$ **as a minor.** In this case, G is outerplanar.

Case 2. G **contains** K_4 **as a minor but it does not contain** $K_{2,3}$ **as a minor.** Since the degrees of the vertices in K_4 are less than 4, any subgraph contractible to K_4 is also a subdivision of K_4. Therefore, there is a subdivision S of K_4 in G.

Since G does not contain $K_1 \cup K_4$ as a minor, S is a spanning subgraph of G (any vertex of G is also a vertex of S). Moreover, since any proper subdivision of K_4 contains $K_{2,3}$ as a minor, K_4 is the only graph that contains K_4 as a minor but does not contain $K_{2,3}$ as a minor. So G is isomorphic to K_4 and is a subgraph of a wheel.

Case 3. G **contains** $K_{2,3}$ **as a minor.** Since the degrees of the vertices in $K_{2,3}$ are less than 4, any subgraph contractible to $K_{2,3}$ is also a subdivision of $K_{2,3}$. Therefore, there is a subdivision S of $K_{2,3}$ in G. Since G does not contain $K_1 \cup K_{2,3}$ as a minor, S is a spanning subgraph of G.

Here we have two cases:

Case 3a. G is middle-less. By Lemma 12, G can be obtained by subdividing the red dashed edges of one of the graphs shown in Figure 5.13. Now, any of the graphs shown in Figure 5.13 is either a subgraph of a wheel or an elongated triangular prism. Therefore G is either a subgraph of a wheel or a subgraph of an elongated triangular prism.

Case 3b. G is middle-ful. By Lemma 18, G is either a subgraph of a wheel or it is an elongated triangular prism. □

Now we are ready to prove Theorem 38.

Proof of Theorem 38. It is straightforward to verify that in any planar drawing of a graph that contains $K_1 \cup K_4$ or $K_1 \cup K_{2,3}$ or $K_{1,1,3}$ as a minor, there are two vertices that are separated by a cycle. Therefore, to prove this theorem, it is sufficient to show that any graph that does not contain any of $K_1 \cup K_4$ or $K_1 \cup K_{2,3}$ or $K_{1,1,3}$ as a minor is a non-separating planar graph.

By Lemma 19, any graph that does not contain any of $K_1 \cup K_4$ or $K_1 \cup K_{2,3}$ or $K_{1,1,3}$ as a minor is either an outerplanar graph or a subgraph of a wheel or an elongated triangular prism and it is easy to verify that any such graph is a non-separating planar graph. □

Now we prove Theorem 39

Proof of Theorem 39. Theorem 39 is a direct consequence of Lemma 19 and Theorem 38. □

6

A Hanani-Tutte Type Theorem for Non-separating Planar Graphs

In this chapter, we prove a Hanani-Tutte type theorem for non-separating planar graphs.

In the previous chapter, we characterised non-separating planar graphs in terms of minimal excluded minors. Moreover we gave a structural characterisation of them. In this chapter, we prove a Hanani-Tutte type theorem for non-separating planar graphs. To state the theorem that we want to prove, first we need to define some new terminology. Moreover, we will redefine the notion of a separating cycle so that it is applicable to drawings that are not planar.

A drawing D of a graph G in $\mathbb{R}^2$ partitions all the points of $\mathbb{R}^2 \setminus D$ (where by D we denote the set of all points in D) into a set of *regions*, denoted by *regions*(D), such that any two points p and q are in the same region $r \in$ *regions*(D) if there is a curve from p to q that does not cross any vertex or edge of D. Regions in a planar drawing are called *faces*. Two regions are *adjacent* if they share an edge or a segment of an edge on their boundaries.

region

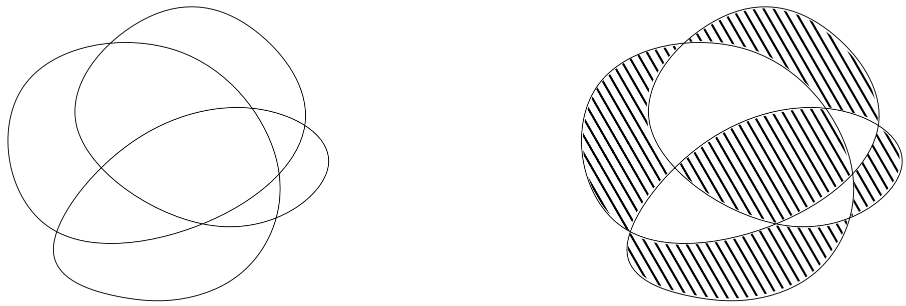
face

adjacent region

For any drawing D of a cycle, define a *black-and-white colouring* of the plane with respect to D to be a colouring of each region of *regions*(D) either

black-and-white colouring

black or white such that no two adjacent regions are coloured in the same colour (see, for example, Figure 6.1). Lemma 20 proves that such a colouring exists for any drawing of a cycle.



(a) A drawing D of cycle C

(b) Colouring regions of the plane in black and white based on D

Fig. 6.1

Let C be a cycle in a graph G. Let D be a drawing of G. Cycle C is a *separating cycle* in D if there is a pair of vertices u and v in G such that: separating cycle

- $u \notin V(C)$, $v \notin V(C)$ and
- u and v are located in regions with different colours in a black-white colouring of the plane with respect to $D[C]$.

Cycle C *separates* u from v in D. separates

A *non-separating* drawing of a graph is a drawing of the graph that does not contain any separating cycles. non-separating drawing

With this terminology we can redefine non-separating planar graph as follows. A *non-separating planar graph* is a graph that has a non-separating planar drawing. non-separating planar graph

Now we are ready to state the Hanani-Tutte type theorem for non-separating planar graphs.

Theorem 41. *Let D be a non-separating drawing of a graph G such that any two vertex-disjoint edges in D cross each other an even number of times. Then G is a non-separating planar graph.*

The reverse implication also holds, by the definition of non-separating planar graphs.

The rest of this chapter is dedicated to proving the above theorem.

6.1 Preliminary Results

Lemma 20. *Let D be a drawing of a cycle on the plane. Then there is a black-and-white colouring for D.*

Proof. Let G be a graph with a vertex for each region of D and an edge (u, v) for any two adjacent regions r_1 and r_2 of D where r_1 is represented by vertex u of G and r_2 is represented by vertex v of G.

The graph G is a dual of an Eulerian planar graph and therefore it is bipartite [134]. □

A *two-vertex-avoiding cycle* in a graph G is a cycle in G with $|V(G)| - 2$ vertices.

two-vertex-avoiding cycle

In any drawing D, define $\phi(D)$ to be the number of separating cycles in D and define $\chi(D)$ to be the number of pairs of vertex-disjoint edges that cross each other an odd number of times in D.

For an edge e and a vertex v in a graph $G = (V, E)$, we denote:

- the set of all the two-vertex-avoiding cycles that have e as an edge but do not have v as a vertex by $C(e - v)$ and

$C(e - v)$

- the set of all the edges that are incident with v but are vertex-disjoint from e by $E(v - e)$.

$E(v - e)$

Next we prove that the parity of $\phi + \chi$ is an invariant for drawings of some graphs that will be useful later.

Lemma 21. *Let G be a simple graph such that the shortest cycle in G is a two-vertex-avoiding cycle and for any edge e and any vertex v that is not an endpoint of e, $|E(v - e)|$ and $|C(e - v)|$ have the same parity. Let D and D' be two different drawings of G, then $\phi(D) + \chi(D)$ and $\phi(D') + \chi(D')$ have the same parity.*

Proof. By Proposition 1, any drawing D' of G can be obtained from D by performing a series of Reidemeister moves in $\{R^p_I, R^p_{II}, R^p_{III}, R^p_{IV}, R^p_V\}$. Therefore, in order to show that $\phi(D) + \chi(D)$ and $\phi(D') + \chi(D')$ have the same parity, it is enough to show that $\phi(D_1) + \chi(D_1)$ and $\phi(D_2) + \chi(D_2)$ have the same parity for any two drawings D_1 and D_2 of G where D_2 is obtained by performing a Reidemeister move in $\{R^p_I, R^p_{II}, R^p_{III}, R^p_{IV}, R^p_V\}$ on D_1.

Let D_2 be obtained from D_1 by performing any of the Reidemeister moves R^p_I, R^p_{II}, R^p_{III}, R^p_V. Since no edge is pushed over a vertex in any of these moves, the number of separating cycles does not change and therefore $\phi(D_1) =$

$\phi(D_2)$. Moreover, since the parity of the number of crossings of vertex-disjoint edges does not change in any of these moves, $\chi(D_1) = \chi(D_2)$. Therefore $\phi(D_1)+\chi(D_1)$ and $\phi(D_1)+\chi(D_1)$ have the same parity. So let D_2 be obtained from D_1 by performing a R^p_{IV} move, in which edge e is pushed over vertex v. We have two cases:

Case 1. Edge e is incident with v. In this case, all the edges involved in the move are incident with v. Therefore pushing e over v does not change the number of pairs of vertex-disjoint edges that cross each other and $\chi(D_1) = \chi(D_2)$. Moreover, as v is a vertex of any cycle C that contains e (since e is incident with v), this move does not change any separating cycle to a non-separating cycle or vice-versa and therefore $\phi(D_1) = \phi(D_2)$. Hence $\phi(D_1) + \chi(D_1)$ and $\phi(D_2) + \chi(D_2)$ have the same parity.

Case 2. Edge e is not incident with v. Let $C_{s\to n}$ be the set of cycles in G that are separating cycles in D_1 and are non-separating cycles in D_2 and let $C_{n\to s}$ be the set of cycles in G that are non-separating cycles in D_1 and are separating cycles in D_2. Note that, as e is pushed over v:

- any cycle in $C(e-v)$ is a two-vertex-avoiding cycle and hence it either changes from a separating cycle to a non-separating cycle or vice-versa and
- any cycle that is not in $C(e-v)$ is either a cycle that contains all vertices of G except for one of them or does not contain e and hence remains a separating cycle or a non-separating cycle.

Therefore we have $|C_{s\to n}| + |C_{n\to s}| = |C(e-v)|$.

Now let $E_{o\to e}$ be the set of all edges in G that cross e an odd number of times in D_1 and an even number of times in D_2 and let $E_{e\to o}$ be the set of all edges in G that cross e an even number of times in D_1 and an odd number of times in D_2. As e is pushed over v:

- the number of crossings between e and any edge e' which is in $E(v-e)$ (and is therefore vertex-disjoint from e) either increases or decreases by one (so the parity of the number of crossings between e and e' changes) and
- the number of crossings between any other two vertex-disjoint edges remains the same.

Therefore we have $|E_{o\to e}| + |E_{e\to o}| = |E(v-e)|$.

Since $|C_{s\to n}| + |C_{n\to s}|$ has the same parity as $|C(e-v)|$ and $\phi(D_2) = \phi(D_1) - |C_{s\to n}| + |C_{n\to s}|$, we find that $\phi(D_1) + \phi(D_2)$ has the same parity as $|C(e-v)|$. Moreover, since $|E_{o\to e}| + |E_{e\to o}|$ has the same parity as $|E(v-e)|$ and $\chi(D_2) = \chi(D_1) - |E_{o\to e}| + |E_{e\to o}|$, we find that $\chi(D_1) + \chi(D_2)$ has the same parity as $|E(v-e)|$. Therefore $\phi(D_1) + \phi(D_2) + \chi(D_1) + \chi(D_2)$ has the same parity as $|C(e-v)| + |E(v-e)|$.

By the Lemma's assumption, the parities of $|E(v-e)|$ and $|C(e-v)|$ are the same. Therefore the parities of $\phi(D_1) + \chi(D_1)$ and $\phi(D_2) + \chi(D_2)$ are the same. □

6.2 Proof of the Main Theorem

Lemma 22. *Any drawing of $K_1 \cup K_{2,3}$ or $K_1 \cup K_4$ or $K_{1,1,3}$ on the plane either contains a separating cycle or two vertex-disjoint edges that cross each other an odd number of times.*

Proof. To prove this lemma we show that $\phi(D) + \chi(D)$ is odd in any drawing of $K_1 \cup K_{2,3}$ or $K_1 \cup K_4$ or $K_{1,1,3}$.

Let G be $K_1 \cup K_{2,3}$ or $K_1 \cup K_4$ or $K_{1,1,3}$. First note that $K_1 \cup K_{2,3}$ and $K_1 \cup K_4$ and $K_{1,1,3}$ have six, five and five vertices respectively. Moreover the shortest cycle in $K_1 \cup K_{2,3}$ and $K_1 \cup K_4$ and $K_{1,1,3}$ has four, three and three vertices respectively. Therefore, the shortest cycle in G is a two-vertex-avoiding cycle.

Since any of $K_1 \cup K_{2,3}$ or $K_1 \cup K_4$ or $K_{1,1,3}$ have a unique combinatorial embedding up to isomorphism, we can verify, for any edge e in G and any vertex v in G that is not an endpoint of e, that $C(e-v)$ and $E(v-e)$ have the same parity. Therefore we can apply Lemma 21 to G.

Now let D be a planar drawing of G. Since D is planar, $\chi(D) = 0$ and since there is only one separating cycle in D, $\phi(D) = 1$ therefore $\phi(D)+\chi(D)$ is odd. Moreover by Lemma 21, for any two drawings D and D' of G, $\phi(D) + \chi(D)$ and $\phi(D')+\chi(D')$ have the same parity and hence $\phi+\chi$ is odd for any drawing of G. □

We use Lemma 22 to prove Theorem 41.

Proof of Theorem 41. By Lemma 22, we know that any drawing of $K_1 \cup K_4$, $K_1 \cup K_{2,3}$ or $K_{1,1,3}$ contains either a separating cycle or two vertex-disjoint edges that cross each other an odd number of times.

First we show that any drawing of a subdivision of $K_1 \cup K_4$, $K_1 \cup K_{2,3}$ or $K_{1,1,3}$ contains a separating cycle or two vertex-disjoint edges that cross each other an odd number of times. Let G be one of $K_1 \cup K_4$, $K_1 \cup K_{2,3}$ or $K_{1,1,3}$ and let $e_1, e_2, \ldots, e_i$ be all of the edges of G. Let G' be any subdivision of G. Let $P_1, P_2, \ldots, P_i$ be i internally disjoint paths in G' such that G' can be obtained from G by replacing $e_1, e_2, \ldots, e_i$ with the paths $P_1, P_2, \ldots, P_i$ respectively.

To reach a contradiction suppose that there is a drawing D' of G' with no separating cycles such that any two vertex-disjoint edges in D' cross each other an even number of times. Obtain a drawing D of G from D' as follows. For each $j, 1 \leq j \leq i$, replace path P_j, in D' with an edge e_j such that e_j is drawn along the curve representing P_j. Since any two vertex-disjoint edges in D' cross each other an even number of times, any two vertex-disjoint paths in D' cross each other an even number of times too. Therefore any two vertex-disjoint edges in D cross each other an even number of times. Moreover, since there are no separating cycles in D', there are no separating cycles in D. But this is a contradiction since, by Lemma 22, any drawing of any of $K_1 \cup K_4$, $K_1 \cup K_{2,3}$ or $K_{1,1,3}$ contains either a separating cycle or two vertex-disjoint edges that cross each other an odd number of times.

In other words, any non-separating drawing D in which any two edges cross an even number of times does not contain a subdivision of $K_1 \cup K_4$, $K_1 \cup K_{2,3}$ or $K_{1,1,3}$ and therefore, by Theorem 40, it is a non-separating planar graph. $\square$

7

Applications of Non-separating Planar Graphs

This chapter is about two applications of non-separating planar graphs. First, we use non-separating planar graphs and the Hanani-Tutte characterisation of them to prove a stronger version of the Strong Hanani-Tutte Theorem. Then we use them in a completely different context to prove that there are maximal linkless graphs with at most $3|V| - 3$ edges.

In Chapter 5, we characterised non-separating planar graphs in terms of three minimal excluded minors and we proved that any maximal non-separating planar graph is either a maximal outerplanar graph or a wheel or an elongated triangular prism. Then in Chapter 6, we gave a Hanani-Tutte type characterisation of non-separating planar graphs. In this chapter we will utilise these three theorems to first prove a stronger version of the Strong Hanani-Tutte Theorem and then to show that there are maximal linkless graphs with less than $4|V| - 10$, more specifically, $3|V| - 3$, edges.

To state the stronger version of the Strong Hanani-Tutte Theorem, first we need to define some new terms.

First recall that in Chapter 6, we redefined the notion of a separating cycle as follows. Let D be a drawing of a graph G and let C be a cycle in G. Cycle

C is a *separating cycle* in D if there is a pair of vertices u and v in G such that: separating cycle

- $u \notin V(C)$, $v \notin V(C)$ and
- u and v are located in regions with different colours in a black-white colouring of the plane with respect to $D[C]$.

Now we can define a decomposing cycle as follows. A separating cycle C in D is a *decomposing cycle* if any edge e that is vertex-disjoint from C crosses C an even number of times in D. For a black-and-white colouring of regions of the plane with respect to $D[C]$, where C is a decomposing cycle, let V_w be the set of vertices of G that are on a white region of the plane and let V_b be the set of vertices of G that are on a black region of the plane. decomposing cycle

Let G_w be the graph that is induced from G by the vertices of C and V_w ($G_w = G[V_w \cup V(C)]$) and let G_b be the graph that is induced from G by the vertices of C and V_b ($G_b = G[V_b \cup V(C)]$). Note that C is a part of both G_w and G_b. Then we say that C *decomposes* D into two drawings: $D[G_b]$ and $D[G_w]$. decompose

A drawing D is *evenly decomposable* if it complies with one of the following two conditions: evenly decomposable

- every two vertex-disjoint edges in D cross an even number of times;
- there is a chordless decomposing cycle that decomposes D into two evenly decomposable drawings.

Now we can state a stronger version of Theorem 3 (the Strong Hanani-Tutte Theorem).

Theorem 42. *Let D be an evenly decomposable drawing of a graph G on the plane. Then G is planar and there is a planar drawing D' of G such that any decomposing chordless cycle C in D' separates vertices u and v in D' if and only if C separates u and v in D.*

Since a drawing in which any two vertex-disjoint edges cross each other an even number of times is evenly decomposable by definition, Theorem 42 is at least as strong as the strong version of the Hanani-Tutte Theorem. Moreover since there are evenly decomposable drawings in which there is a pair of vertex-disjoint edges that cross each other an odd number of times, Theorem 42 is stronger than the strong version of the Hanani-Tutte Theorem.

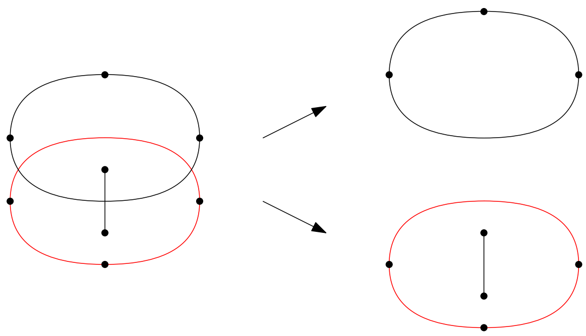


Fig. 7.1 An evenly decomposable drawing with two edges that cross each other an odd number of times. The cycle drawn in red is the decomposing cycle in this drawing.

Figure 7.1 depicts a simple example of an evenly decomposable drawing with a pair of vertex-disjoint edges that cross an odd number of times.

A class $\mathcal{G}$ of graphs is *pure* if $|E| = |E'|$ for any two maximal graphs $G = (V, E)$ and $G' = (V', E')$ in $\mathcal{G}$ where $|V| = |V'|$ [86]. The classes of outerplanar graphs and planar graphs are two well-known pure classes of graphs. The class of toroidal graphs is a well-known class of graphs that is not pure [66]. In fact by Theorem 39, it is easy to see that the class of non-separating planar graphs is not pure either. For more results on the number of edges of maximal graphs that are embeddable on surfaces or belong to minor-closed classes of graphs, see [85, 86]. pure

The second application of non-separating planar graphs is to linkless graphs. Recall that a *linkless* graph is a graph that has an embedding η in $\mathbb{R}^3$ such that no two cycles in η are linked (refer to Chapter 2 for the definition of linked cycles and others). linkless

Although linkless graphs are characterised in terms of a set of minimal excluded minors, there are a lot of unanswered questions about them. For example, since linkless graphs do not contain a K_6-minor, it follows that they have at most $4|V| - 10$ edges where $|V|$ is the number of vertices in G [83]. However, prior to the present work it was open whether all maximal linkless graphs have $4|V| - 10$ edges. In this chapter we show that there is a class $\mathcal{G}$ of maximal linkless graphs such that any graph $G \in \mathcal{G}$ has at most $3|V| - 3$ edges. More specifically, we prove the following theorem:

Theorem 43. *There exists an infinite family $\mathcal{G}$ of maximal linkless graphs such that any graph $G \in \mathcal{G}$ has at most $3|V(G)| - 3$ edges.*

Since there are maximal linkless graphs on n vertices with $4n-10$ edges, by Theorem 43, linkless graphs are not pure. Moreover, Theorem 43 is related to a question asked in 1983 by Horst Sachs about the number of edges of linkless graphs [113]. He asked the following question. What is the maximum number of edges of a linkless graph on $|V|$ vertices?

In Section 7.1 we prove Theorem 42 and provide an example for it. In Section 7.2 we prove Theorem 43 and demonstrate a connection between non-separating planar graphs and linkless graphs.

7.1 Hanani-Tutte Theorem, Stronger Version

Lemma 23. *Let D be a drawing of a graph G on the plane which contains a separating cycle. Then there exists a chordless separating cycle in D.*

Proof. Let $C = (v_1, v_2, \ldots, v_k)$ be the shortest separating cycle in D. We claim that C is chordless.

To the contrary, let $(v_i, v_j), 1 \leq i < j \leq k$, be a chord in C. Then we show that cycle $C_1 = (v_1, v_2, \ldots, v_i, v_j, v_{j+1}, \ldots, v_k)$ or cycle $C_2 = (v_i, v_{i+1}, \ldots, v_j)$ is a shorter separating cycle than C and hence we reach a contradiction.

Let a and b be two vertices that are separated by C and let ζ be a curve from a to b that does not pass through any vertices or crossings of D and is not tangent to any edge in D at any point. Since C is a separating cycle with respect to a and b, the curve ζ crosses C an odd number of times.

Now suppose that neither C_1 nor C_2 is separating. In other words, ζ crosses C_1 an even number of times and ζ crosses C_2 an even number of times.

Let α be the number of crossings between ζ and C_1, let β be the number of crossings between ζ and C_2, and let γ be the number of crossings between ζ and (v_i, v_j). Then the number of crossings between ζ and C is $\alpha + \beta - 2\gamma$ which is even, since α and β are even, and therefore C is not a separating cycle, which is a contradiction. □

Proof of Theorem 42. We prove this theorem by induction on the number of chordless decomposing cycles in the drawing of the graph.

In the base case, there are no chordless decomposing cycles in D. By the definition of evenly decomposable drawings, any two vertex-disjoint edges in D cross an even number of times.

First we show that D is a non-separating drawing. To reach a contradiction suppose that there is a separating cycle in D. By Lemma 23, there exists a chordless separating cycle C in D. Since any two vertex-disjoint edges in D cross an even number of times, C is a chordless decomposing cycle. This is a contradiction since according to the assumptions of the base case there are no chordless decomposing cycles in D.

Therefore D is a non-separating drawing. By Theorem 41, G is a non-separating planar graph, so it has a planar drawing in which there are no separating cycles. Therefore the requirements of the conclusion of this theorem are satisfied.

We proceed to the inductive case where there is a chordless decomposing cycle $C = (v_1, v_2, \ldots, v_k)$ in D. Cycle C decomposes G into two graphs $G_w = G[\{v_1, v_2, \ldots, v_k\} \cup V_w]$ and $G_b = G[\{v_1, v_2, \ldots, v_k\} \cup V_b]$ such that there is no edge between any vertex $v \in V_w$ of G_w and any vertex $v' \in V_b$ of G_b (otherwise there is an edge in D that crosses C an odd number of times). In other words, the only vertices and edges that are shared between G_w and G_b are the vertices and the edges of C.

Since $D[G_w]$ is contained entirely within D, any cycle that is not a separating cycle in D is not a separating cycle in $D[G_w]$ either. Moreover, since vertices of V_b are not present in $D[G_w]$, the cycle C — which was a separating cycle in D — is not a separating cycle in $D[G_w]$. Therefore, the number of separating cycles in $D[G_w]$ is less than the number of separating cycles in $D[G]$. Similarly, the number of separating cycles in $D[G_b]$ is less than the number of separating cycles in $D[G]$. Hence by induction it follows that there is a planar drawing D_w of G_w such that all the vertices of V_w are on the same side of C and there is a planar drawing D_b of G_b such that all the vertices of V_b are on the same side of C.

Since C is not a separating cycle in D_w and it does not have any chords, it is a face of D_w. Similarly, C is a face of D_b. Therefore it is easy to see that we can obtain a drawing D of G by gluing D_w and D_b on the vertices and edges of C. □

Figure 7.2 depicts an evenly decomposable drawing and its decomposition process. The drawing is decomposed a couple of times until there are no more decomposing cycles in it and any two vertex-disjoint edges cross each other an even number of times. Notice that in the first step of the decomposition we could have chosen the other cycle in the graph as the decomposing cycle.

Since Theorem 42 is stronger than the strong version of the Hanani-Tutte Theorem, we can state the Strong Hanani-Tutte Theorem as a corollary.

Corollary 2. *(Strong Hanani-Tutte Theorem) Let D be a drawing of a graph G such that any two vertex-disjoint edges in D cross an even number of times. Then G is planar.*

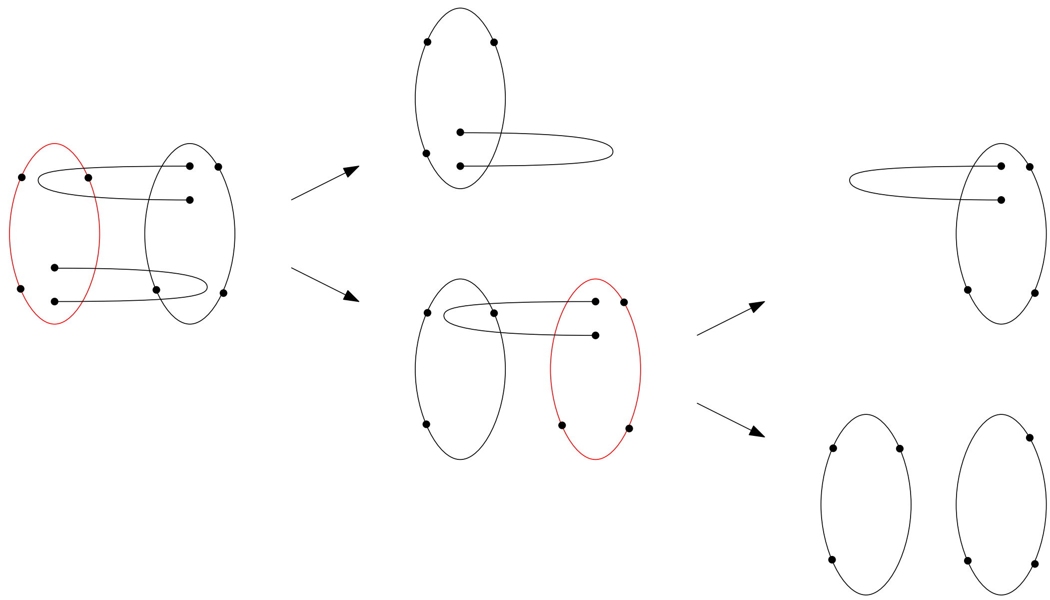


Fig. 7.2 An evenly decomposable drawings and its decomposition. The decomposing cycle is depicted in red in each step. Notice that some of the edges of the decomposing cycles are crossed an odd number of times by other vertex-disjoint edges and hence the Strong Hanani-Tutte Theorem is not applicable.

Lastly, note that, since any planar drawing is an evenly decomposable drawing, Theorem 42 can also be stated as a characterisation of planar graphs.

Theorem 44. *A graph G is planar if and only if G has an evenly decomposable drawing.*

7.2 Proof of Theorem 43

Consider two linked circles in three dimensions and a cross section of them that contains one of the two circles as depicted in Figure 7.3(a). Such a cross-section has a structure that resembles the structure of a separating cycle with a vertex inside it and another outside it.

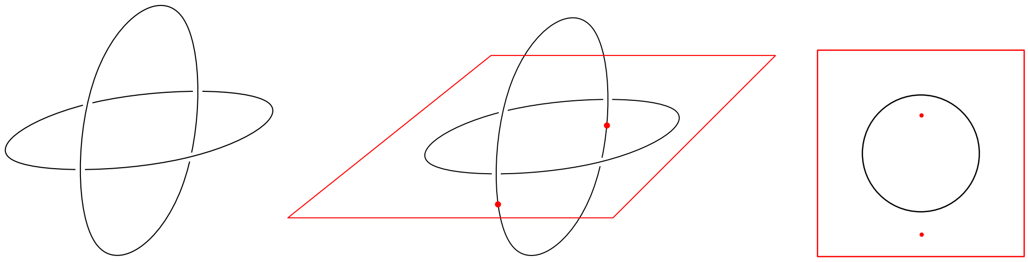
With this intuition in mind, we prove Theorem 43.

Lemma 24. *An elongated triangular prism has $|V(G)| + 3$ edges.*

Proof. The proof is straightforward. □

Lemma 25. *An elongated triangular prism contains both K_4 and $K_{2,3}$ as a minor.*

Proof. The proof is straightforward. □



(a) a link in 3D (b) intersection of a plane with a link (c) cross section of the link from the top

Fig. 7.3 A separating cycle in a cross section of a link with a plane

Proof of Theorem 43. Let G be an elongated prism. By Theorem 39, G is a maximal non-separating graph and by Lemma 24, G has $|V(G)| + 3$ edges. Moreover by Lemma 25, G contains both K_4 and $K_{2,3}$ as minors.

Let H be the graph that is obtained by adding two new vertices u and v to G such that u and v are each adjacent to all the vertices of G. Since G has $|V(G)| + 3$ edges, the graph H has at most $3|V(H)| - 3$ edges. There are no linked cycles in H, and hence it is a linkless graph. We claim that H is a maximal linkless graph.

To prove that H is a maximal linkless graph, we show that any graph H^+ that is obtained by adding an edge e to H is not a linkless graph. Since u and v are adjacent to all the vertices of G, edge e (in H^+) is either (u, v) or it is an edge between two vertices of G.

Let H^+ be the graph obtained by adding (u, v) to H. Since G contains both K_4 and $K_{2,3}$ as a minor, H^+ contains both K_6 and $K_{1,1,2,3}$ as a minor. The latter contains $K_{1,3,3}$. However, by Theorem 21, K_6 and $K_{1,3,3}$ are both minimal excluded minors for linkless graphs.

Now let H^+ be the graph that is obtained by adding an edge between two vertices of G in H. By Theorem 38, $G + e$ contains $K_4 \cup K_1$ or $K_{2,3} \cup K_1$ or $K_{1,1,3}$ as a minor.

If $G + e$ contains $K_4 \cup K_1$ as a minor, then H^+ contains K_6 as a minor and hence by Theorem 21 H^+ is not a linkless graph. If $G + e$ contains $K_{2,3} \cup K_1$ as a minor, then H^+ contains $K_{1,1,2,3}$ as a minor. But $K_{1,1,2,3}$ contains $K_{1,3,3}$ as a minor and therefore by Theorem 21 H^+ is not a linkless graph. If $G + e$ contains $K_{1,1,3}$ as a minor, then H^+ contains $K_{2,1,1,3}$ as a minor, which in turn contains $K_{1,3,3}$ as a minor. Therefore by Theorem 21 H^+ is not a linkless graph. □

8
Outerthrackles

In this chapter we characterise outersuperthrackles. Then we define variations of outersuperthracklable graphs such as generalised outersuperthracklable graphs and weak outersuperthracklable graphs and we show that these classes of graphs are all the same as the class of outersuperthracklable graphs.

A drawing D of a graph G on a surface Σ is a *weak thrackle* if any two vertex-disjoint edges in D cross each other exactly once (see for example, Figure 8.1(c)). In a weak thrackle two incident edges may or may not cross, compared with a thrackle in which such two edges do not cross. Any graph that has a weak thrackle drawing on a surface Σ is *weak thracklable* with respect to Σ.

weak thrackle

weak thracklable

A drawing D of a graph G on a surface Σ is a *generalised superthrackle* if any two edges in D cross each other an odd number of times (see for example, Figure 8.1(d)). Any graph with a generalised superthrackle drawing on a surface Σ is a *generalised superthracklable* graph with respect to Σ.

generalised superthrackle

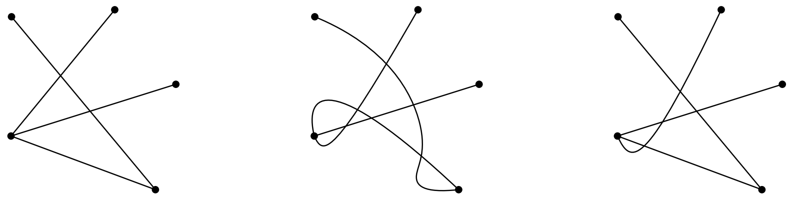
generalised superthracklable

A drawing D of a graph G on a surface Σ is a *weak generalised thrackle* if any two vertex-disjoint edges in D cross each other an odd number of times (see for example, Figure 8.1(e)). Any graph that can be drawn as a weak generalised thrackle on a surface Σ is a *weak generalised thracklable* graph

weak generalised thrackle

weak generalised thracklable

with respect to Σ.



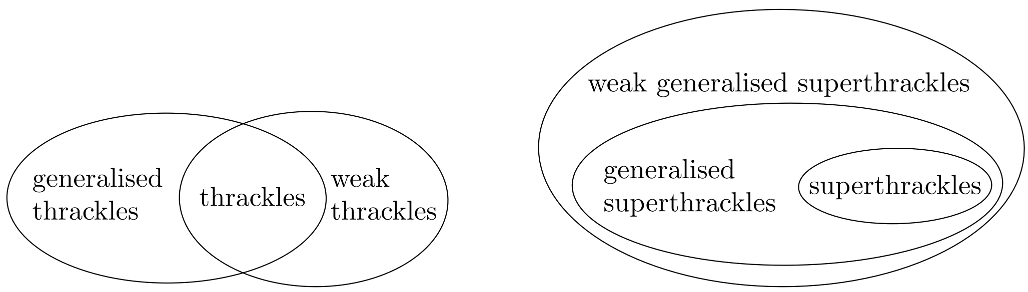
(a) thrackle (b) superthrackle (c) weak thrackle

(d) generalised superthrackle (e) weak generalised thrackle

Fig. 8.1 Examples of different variations of thrackles

From the above definitions we can immediately deduce that:

- any thrackle is both a generalised thrackle and a weak thrackle,
- any superthrackle is a generalised superthrackle and
- any generalised superthrackle is a weak generalised thrackle (see Figure 8.2).



(a) (b)

Fig. 8.2 Relationship between different variations of thrackles and superthrackles

A graph that has a thrackle outerdrawing is *outerthracklable*. A graph that has a superthrackle outerdrawing is *outersuperthracklable*. A graph that has a weak thrackle outerdrawing is *weak outerthracklable*. A graph that has a generalised superthrackle outerdrawing is *generalised outersuperthracklable*

outerthracklable

and a graph that has a weak generalised thrackle outerdrawing is called *weak generalised outerthracklable*.

generalised outersuperthracklable

In this chapter we characterise outersuperthracklable graphs. More specifically we prove the following theorem.

Theorem 45. *Any graph G is outersuperthracklable if and only if G does not contain any of:*

1. *$Star_{2,2,2}$ as a minor (see Figure 8.3(a)),*
2. *$K_2 \cup K_3$ as a minor (see Figure 8.3(b)),*
3. *any cycle of even length with four or more vertices.*

(a) $Star_{2,2,2}$ (b) $K_2 \cup K_3$

Fig. 8.3 Two minimal excluded minors for outersuperthracklable graphs.

We also show that the classes of outerthracklable graphs, outersuperthracklable graphs, weak outerthracklable graphs, generalised outersuperthracklable graphs and weak generalised outerthracklable graphs are all equivalent. In other words, we prove the following theorem.

Theorem 46. *The following five classes of graphs are equivalent:*

1. *outerthracklable graphs*
2. *outersuperthracklable graphs*
3. *weak outerthracklable graphs*
4. *generalised outersuperthracklable graphs*
5. *weak generalised outerthracklable graphs.*

The rest of this chapter is organised as follows. Section 8.1 is dedicated to characterisation of weak generalised outerthracklable graphs and Section 8.2 investigates the relation between outerthracklable graphs, outersuperthracklable graphs, weak outerthracklable graphs, generalised thracklable graphs and weak generalised thracklable graphs.

8.1 Weak Generalised Outerthracklable Graphs

In this section we characterise weak generalised outerthracklable graphs. We start by proving that weak generalised outerthracklable graphs cannot contain $Star_{2,2,2}$ as a minimal excluded minor.

Lemma 26. *None of the graphs C_4, $K_2 \cup K_3$ and $Star_{2,2,2}$ have a weak generalised outerthrackle drawing.*

Proof. Let G be any of the graphs C_4, $K_2 \cup K_3$ or $Star_{2,2,2}$. By just placing vertices of G in different cyclic orders around a disk, we observe that in any outerdrawing of G there are two edges (u, u') and (v, v') such that vertices u, u', v, v' appear in that order around the boundary of the disc and therefore (u, u') crosses (v, v') an even number of times. Therefore, none of C_4, $K_2 \cup K_3$ or $Star_{2,2,2}$ has a weak generalised outerthrackle drawing. □

Lemma 27. *Any graph G that contains $Star_{2,2,2}$ as a minor is not a weak generalised outerthracklable graph.*

Proof. Any graph that contains $Star_{2,2,2}$ as a minor has $Star_{2,2,2}$ as a subgraph. So by Lemma 26 any graph G that contains $Star_{2,2,2}$ as a minor does not have a weak generalised outerthrackle drawing and hence is not a weak generalised outerthracklable graph. □

Let $e = (u, v)$ be an edge of a graph G. Let G' be the graph that is obtained from G by replacing (u, v) with three edges (u, w), (w, x) and (x, v), where $w, x \notin V(G)$. Define the *double topological contraction* operation (or *double contraction* for short) to be the operation that is performed on G' to obtain G (see Figure 8.4).

double topological contraction

(a) before double contraction (G')

(b) after double contraction (G)

Fig. 8.4 Double topological contraction operation

Define a graph G^- to be a *double minor* of a graph G if we can obtain G^- from G by deleting vertices and edges and the double contraction operation.

double minor

Next we show that weak generalised outerthracklable graphs are closed under the double contraction operation (see Figure 8.4).

Lemma 28. *Weak generalised outerthracklable graphs are closed under the double contraction operation.*

Proof. Let η be a weak generalised outerthrackle drawing of G. Let (v_1, v_2), (v_2, v_3), (v_3, v_4) be three edges of G such that v_2 and v_3 have degree 2. Let G^- be the graph that is obtained from G by double contracting (v_1, v_2), (v_2, v_3), (v_3, v_4) to (v_1, v_4). Obtain a drawing η^- of G^- from η as follows:

1. remove (v_1, v_4) from η if (v_1, v_4) is an edge in G.
2. add (v_1, v_4) to η such that it follows the path of $(v_1, v_2) \cup (v_2, v_3) \cup (v_3, v_4)$ sufficiently closely so that (v_1, v_4) is drawn within the local disks of (v_1, v_2), (v_2, v_3) and (v_3, v_4) and for every crossing between an edge f and $(v_1, v_2) \cup (v_2, v_3) \cup (v_3, v_4)$ there is only one crossing between f and (v_1, v_4) (see Figure 8.5(a)).
3. remove vertices v_2, v_3 and edges $(v_1, v_2), (v_2, v_3), (v_3, v_4)$.
4. remove the self crossings of (v_1, v_4) by the R_I^p move that is shown in Figure 3.2 (see Figure 8.5(b)).

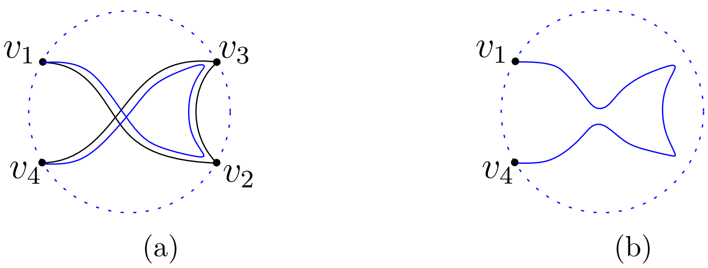


Fig. 8.5 Constructing η^- from η. (For simplicity, the rest of the edges of η or η' are not shown in this Figure.)

Since any two vertex-disjoint edges cross each other an odd number of times in η, any edge that is not incident with v_1, v_2, v_3, v_4 crosses $(v_1, v_2) \cup (v_2, v_3) \cup (v_3, v_4)$ an odd number of times in η. Therefore, any edge in η^- that is not incident with v_1 or v_4 crosses (v_1, v_4) an odd number of times and hence η^- is a drawing of G^- in which any two vertex-disjoint edges cross an odd number of times. $\square$

We use Lemma 28 to show that there cannot be a cycle C of odd size and an edge that is vertex-disjoint from C in any weak generalised outerthrackle.

Lemma 29. *Let G either be a cycle C with an even number of vertices or consist of a cycle C with an odd number of vertices and an edge e that is vertex-disjoint from C. Then G is not a weak generalised outerthracklable graph.*

Proof. We prove this lemma by contradiction. Let us assume that such a G is a weak generalised outerthracklable graph. Then by double contracting C (multiple times if necessary), we obtain a graph G' that is either C_4 or $K_2 \cup K_3$. If G is a weak generalised outerthracklable graph then, by Lemma 28, G' is a weak generalised outerthracklable graph. But this is a contradiction by Lemma 26. □

Now we prove Theorem 45.

Proof of Theorem 45. By Lemma 27 and Lemma 29 weak generalised outerthracklable graphs do not have $Star_{2,2,2}$ as minor or C_4 as a double minor. Moreover since $K_2 \cup K_3$ contains an edge and a cycle of odd length that are vertex-disjoint, by Lemma 29 weak generalised outerthracklable graphs do not contain $K_2 \cup K_3$ as a double minor either. Therefore we only need to show that if a graph G does not have $Star_{2,2,2}$ as minor, $K_2 \cup K_3$ as a double minor or C_4 as a double minor, then G is a weak generalised outerthracklable graph.

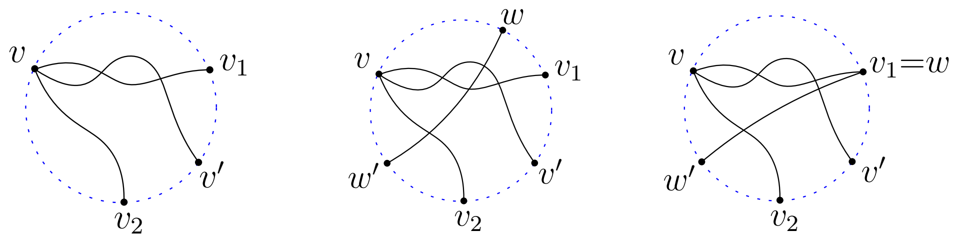
We prove this by induction on the number of vertices. In the base case G has one or two or three vertices and the lemma holds trivially. We proceed to the inductive case. We have two cases:

Case 1. There is a vertex v in G, with $\deg(v) \geq 3$. Since G contains neither C_4 nor $Star_{2,2,2}$, for any vertex v in G with $\deg(v) \geq 3$, there is a vertex v' adjacent to v in G such that $\deg(v') = 1$. Let v_1 and v_2 be two vertices (other than v') that are adjacent to v. Let G^- be the graph that is obtained from G by deleting v' and (v, v') from G.

By induction G^- has a weak generalised outerthrackle drawing η^-. Let d be the disc on which η^- is drawn. To obtain a drawing η of G from η^-, choose the location of v' on $\partial(d)$ such that the order of v, v', v_1, v_2 (clockwise or anticlockwise) on $\partial(d)$ is v, v_1, v', v_2 and let (v, v') be represented by an arbitrary curve from v to v' (see Figure 8.6(a)).

We need to show that (v, v') crosses any any other vertex-disjoint edge an odd number of times.

Let (w, w') be an arbitrary edge of G that is vertex-disjoint from (v, v'), if such an edge exists. If (w, w') is vertex-disjoint from both (v, v_1) and (v, v_2) then since any two vertex-disjoint edges in η^- cross an odd number of times, (w, w') crosses both (v, v_1) and (v, v_2) an odd number of times and



(a) $(v, v_1), (v, v')$ and (v, v_2) in η

(b) (w, w') crosses (v, v') an odd number of times

(c) (w, w') crosses (v, v') an odd number of times

Fig. 8.6 vertex-disjoint edges cross each other an odd number of times in η.

therefore the order of v, v', v_1, v_2, w, w' on $\partial(d)$ is either v, w, v_1, v', v_2, w' or v, w', v_1, v', v_2, w (see Figure 8.6(b)). In both of these cases, by the order of the vertices on $\partial(d)$, (w, w') crosses (v, v') an odd number of times.

Now let (w, w') be vertex-disjoint from either (v, v_1) or (v, v_2) (say (v, v_2)). That is either $w = v_1$ or $w' = v_1$. Without loss of generality let $w = v_1$. (v_1, w') (or in other words (w, w')) crosses (v, v_2) an odd number of times and therefore the order of $v, v', v_1 = w, v_2, w'$ on $\partial(d)$ (clockwise or anticlockwise) is $v, v_1 = w, v', v_2, w'$ (see Figure 8.6(c)). In this case, (w, w') crosses (v, v') an odd number of times as well. Therefore any edge that is vertex-disjoint from (v, v') crosses (v, v') an odd number of times and hence any two vertex-disjoint edges in η cross an odd number of times. That is, η is a weak generalised outerthrackle.

Case 2. There is no vertex v in G with $\deg(v) \geq 3$**.** Since the degree of any vertex in G is less than 3, G consists of a number of isolated vertices or paths and cycles. By condition 2, there is no edge in G that is vertex-disjoint from a cycle in G. Therefore G consists of a number of cycles, a number of paths and a number of isolated vertices.

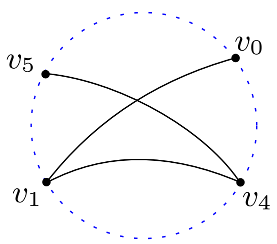
We ignore isolated vertices throughout the rest of this proof, as we can add isolated vertices to any outersuperthracklable graph and the result would be another outersuperthracklable graph.

Let G be a cycle. By condition 3, G cannot contain any cycle of even length. Therefore G is a cycle of odd length. Let $G = C_{2n+1}$. Let v_1, v_2, v_3, v_4 be four consecutive vertices of G. Let G^- be the graph that is obtained from G by double contracting $(v_1, v_2) \cup (v_2, v_3) \cup (v_3, v_4)$ to (v_1, v_4).

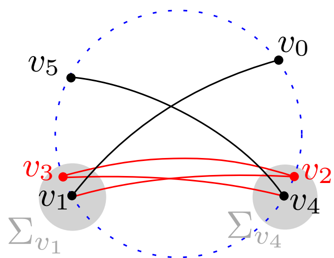
By induction, G^- has a weak generalised outerthrackle drawing η^-. Let d be the disc on which η^- is drawn. Let v_0 be the vertex of G^- that is in $N(v_1) \setminus v_4$ and let v_5 be the vertex of G^- that is in $N(v_4) \setminus v_1$ (v_0 may be equal to v_5).

The cyclic order in which vertices v_0, v_1, v_4, v_5 appear on $\partial(d)$ (clockwise or anticlockwise) is v_1, v_4, v_0, v_5 (see for example Figure 8.7(a)), else (v_0, v_1) crosses (v_4, v_5) an even number of times. Obtain a drawing η of G from η^- as follows:

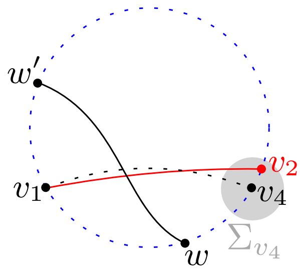
1. insert v_2 and v_3 on $\partial(d)$ such that the order of $v_0, v_1, v_2, v_3, v_4, v_5$ (clockwise or anticlockwise) on $\partial(d)$ is $v_0, v_5, v_3, v_1, v_4, v_2$ and v_2 is located in a local disk Σ_{v_4} of v_4 and v_3 is located in a local disk Σ_{v_1} of v_1.
2. represent $(v_1, v_2), (v_2, v_3)$ and (v_3, v_4) by three arbitrary curves between the corresponding vertices.
3. delete (v_1, v_4) (compare η^- and η in Figures 8.7(a) and 8.7(b)).



(a) $(v_0, v_1), (v_1, v_4)$ and (v_4, v_5) in η^-



(b) η, is obtained by adding the red vertices and edges to η^- and removing (v_1, v_4) from it. Notice that Σ_{v_1} and Σ_{v_4} are the local disks of v_1 and v_4 in η^- (before we add v_3 and v_2 to the drawing).



(c) order of w, w', v_1, v_2 and v_4 on $\partial(d)$. The dashed black line, represents the path of (v_1, v_4) in η^-.

Fig. 8.7 Vertex-disjoint edges cross each other an odd number of times in η.

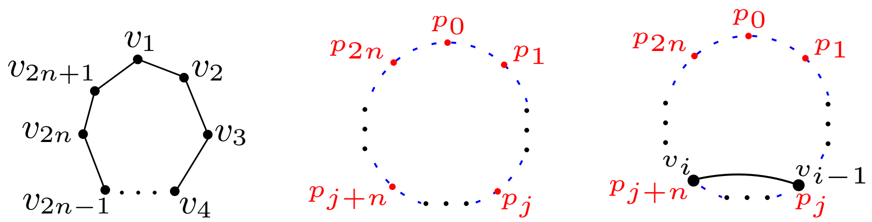
We claim that any two vertex-disjoint edges in η cross each other an odd number of times. Since any two vertex-disjoint edges in η^- cross each other an odd number of times, we only need to show that $(v_1, v_2), (v_2, v_3)$ and (v_3, v_4) cross any other edge that is vertex-disjoint from them an odd number of times.

Let E' be the set of edges in G^- that are vertex-disjoint from (v_1, v_4). The set of edges that are vertex-disjoint from (v_1, v_2) in η is $E' \cup (v_3, v_4)$. Let (w, w') be an arbitrary edge in E'. Since (w, w') crosses (v_1, v_4) in η^- an odd number of times, the order of w, w', v_1, v_4 on $\partial(d)$ (clockwise or anticlockwise) is either v_1, w, v_4, w' or v_1, w', v_4, w. Since we insert v_2 in Σ_{v_4} in η, it follows that the order of w, w', v_1, v_2 on $\partial(d)$ is either v_1, w, v_2, w' or v_1, w', v_2, w (see Figure 8.7(c)). Hence (w, w') crosses (v_1, v_2) an odd number of times. Moreover, as the order of v_1, v_2, v_3, v_4 (clockwise or anticlockwise) on $\partial(d)$ is v_3, v_1, v_4, v_2 (v_1 and v_2 are separated by v_3 and v_4 on the boundary), (v_1, v_2) crosses (v_3, v_4) an odd number of times. That is, any edge that is vertex-disjoint from (v_1, v_2) crosses (v_1, v_2) an odd number of times.

By a similar argument, any edge that is vertex-disjoint from (v_3, v_4) crosses (v_3, v_4) an odd number of times.

The set of edges that are vertex-disjoint from (v_2, v_3) in η is E'. Any edge in E' crosses (v_1, v_4) an odd number of times. Therefore, after we insert v_2 in Σ_{v_4} and v_3 in Σ_{v_1}, any edge in E' also crosses (v_2, v_3) an odd number of times. That is, η is a weak generalised outerthrackle.

If G is the union of a number of paths then we obtain a graph G^+ by adding a couple of edges (and maybe a vertex) to G until it becomes a cycle of odd length (so that we do not violate condition 3 in the theorem). We construct a weak generalised outerthrackle drawing of G^+ and then we delete the extra edges and vertices of G^+ to obtain a weak generalised outerthrackle drawing of G. □



(a) vertices of C_{2n+1}

(b) $p_1, p_2, \ldots, p_{2n+1}$ on $\partial(d)$.

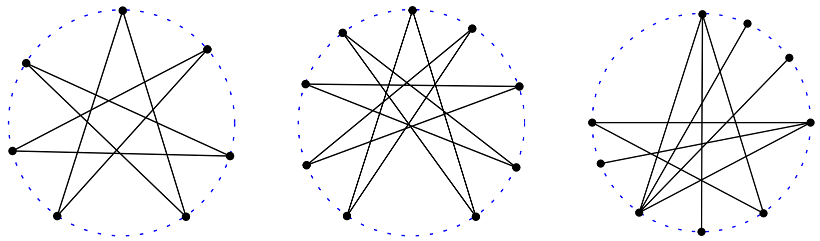
(c) position of v_{i-1} and v_i on $\partial(d)$.

Fig. 8.8 Constructing a weak generalised outerthrackle drawing of an odd cycle.

Figures 8.9(a) and 8.9(b) depict the thrackled outerdrawings of C_7 and C_9 respectively according to the algorithm that is embedded in the proof of Theorem 45 and Figure 8.9(c) depicts an outerdrawing of a graph G that has four edges more than a cycle drawn using the same algorithm.

8.2 Relationship between different types of super-thrackles

Let (v, v_1) and (v, v_2) be two incident edges of a graph G and let G' be the graph that is obtained from G by identifying v_1 and v_2 and then deleting



(a) C_7 (b) C_9 (c) C_5 with four extra edges.

Fig. 8.9 Three weak generalised outerthrackles

any loops formed. Define the *folding* operation to be the operation that is performed on G to obtain G'. More specifically, we say that we obtain G' from G by folding (v, v_1) onto (v, v_2). folding

Lemma 30. *Let $e = (v, v_1)$ and $e' = (v, v_2)$ be two incident edges that appear consecutively in $\pi_\eta(v)$ where η is a generalised superthrackle drawing of a graph. Then there is a non-self-intersecting curve ζ from v_1 to v_2 that crosses each edges of η an even number of times.*

Proof. Let ζ' be a curve from v_1 to v_2 such that:

1. ζ' and (v, v_1) are located consecutively in the cyclic order of the edges and ζ' around v_1 in η.

2. ζ' and (v, v_2) are located consecutively in the cyclic order of the edges and ζ' around v_1 in η.

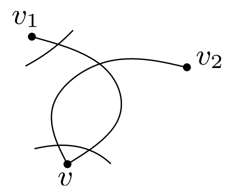
3. ζ' follows the paths of (v, v_1) and (v, v_2) sufficiently closely so that it is drawn within the local disks of (v, v_1) and (v, v_2) and for any crossing between an edge e and (v, v_1), there is only one crossing between e and ζ' and for any crossing between an edge e and (v, v_2), there is only one crossing between e and ζ' (see Figure 8.10).

Since η is a generalised superthrackle, any two edges cross an odd number of times in η and therefore ζ' crosses all the edges other than (v, v_1) and (v, v_2) an even number of times in η. Now obtain ζ from ζ' by:

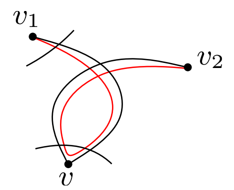
1. removing the self-crossings on ζ' using R_I^p.

2. using R_V^p, if necessary, to make sure that ζ crosses both (v, v_1) and (v, v_2) an even number of times.

Curve ζ crosses each edge of η an even number of times. $\square$



(a) Local area of (v, v_1) and (v, v_2) in η before adding ζ'



(b) Local area of (v, v_1) and (v, v_2) in η after adding ζ'

Fig. 8.10 Adding ζ' to η

Next we show that generalised superthracklable graphs are closed under the folding operation under certain circumstances.

Theorem 47. *Generalised superthracklable graphs are closed under folding of any two edges $e = (v, v_1)$ and $e' = (v, v_2)$ that appear consecutively at v in some generalised superthrackle drawing of the graph.*

Proof. Let η be a generalised superthrackle drawing of a graph G. Let (v, v_1) and (v, v_2) be two edges that appear consecutively in $\pi_\eta(v)$. By Lemma 30 there is a curve from v_1 to v_2 that crosses each edge of η an even number of times.

Let G^+ be the graph that is obtained by adding (v_1, v_2) to G (if it is not already in G). Let η^+ be a drawing of G^+ that is obtained from η by deleting the edge (v_1, v_2) from G (if (v_1, v_2) is already in η) and adding (v_1, v_2) back to η such that it is routed along ζ.

Remove the self-crossings of (v_1, v_2) using R_I^p and remove all the crossings on (v_1, v_2) by pushing the crossings over v_2 using R_{IV}^p. Then obtain a drawing η^- by contracting (v_1, v_2) to x such that:

- any edge f incident with v_2 (other than (v_1, v_2)) in η follows the path of (v_1, v_2) sufficiently closely in η^- so that, as it extended for any crossing between an edge f' and (v_1, v_2), there is only one crossing between f' and the new portion of f that follows the path of (v_1, v_2) and
- the rotational order of the edges around the vertices is preserved (as shown in Figure 8.11).

Let G^- be the graph that is represented by η^-. By definition, G^- is obtained from G by topological folding of (v, v_1) onto (v, v_2). Since any edge in η^+ crosses (v_1, v_2) an even number of times, R_{IV}^p does not change the parity of the number of crossings between the edges and the parity of the number

(a) η (b) η^-

Fig. 8.11 Obtaining η^- from η by contracting (v_1, v_2) to x

of crossings between the edges in η^- is the same as η. Therefore η^- is a generalised superthrackle drawing of G^-. The lemma follows. □

Now we are ready to prove that any weak generalised outerthrackle is an outersuperthrackle.

Theorem 48. *Let η be a weak generalised outerthrackle drawing of a graph G. Then there is an outersuperthrackle drawing η' of G with $\rho(\eta') = \rho(\eta)$.*

Proof. Throughout this proof we assume that no edge crosses itself. That is, since R_I^p does not change the parity of the number of crossings between vertex-disjoint edges, if an edge crosses itself we use R_I^p to remove the self crossing.

We prove this lemma in two steps. As the first step, we construct an outerthrackle drawing η_{temp} of G. Then, as the second step, we use η_{temp} to construct an outersuperthrackle drawing η' of G.

Step 1: Let η be a weak generalised outerthrackle drawing of a graph G. We need to show that there is an outerthrackle drawing η' of G with $\rho(\eta') = \rho(\eta)$.

We prove the first step by induction on the number of crossings in the drawing.

The minimum number of crossings in a weak generalised outerthrackle drawing occurs when any two vertex-disjoint edges cross once and there are no crossings between any two edges that are incident at the same vertex. Therefore, in the base case of the induction for step 1, vertex-disjoint edges cross once and edges that are incident at the same vertex do not cross. That is, η is an outerthrackle drawing of G and we are done. We proceed to the inductive cases.

Case 1, there are two edges $e = (v_1, v_2)$ and $e' = (v_3, v_4)$ in η that cross more than once. Without loss of generality let e and e' be vertex-disjoint. Let x_1 and x_2 be two crossings on e and e' that are consecutive on e.

(a) before (b) after

Fig. 8.12 Removing two crossings on two edges that cross each other an even number of times.

Crossings x_1 and x_2 divide e into three three segments: the part from v_1 to either x_1 or x_2 (whichever crossing that we reach first as we move along the path of (v_1, v_2) from v_1 to v_2), say x_1, the part from x_1 to x_2 and the part from v_2 to either x_1 or x_2 (whichever crossing that we reach first as we move along the path of (v_1, v_2) from v_2 to v_1). Similarly x_1 and x_2 divide e' into three line segments: the part from v_3 to x_1, the part from x_1 to x_2 and the part from x_2 to v_4.

We reroute e or e' in η to obtain a weak generalised outerthrackle drawing of G with a smaller number of crossings as follows. Let $\bar{l}$ denote the part of e that extends from x_1 to x_2 and let $\overline{l'}$ denote the part of e' that extends from x_1 to x_2. Let C denote the simple cycle (a cycle that does not cross itself) that is formed by $\bar{l}$ and $\overline{l'}$.

Since η is an outerdrawing, all the vertices of G are located outside C and therefore the parity of the number of crossings of any arbitrary edge e'' and $\bar{l}$ is equal to the parity of the number of crossings of e'' and $\overline{l'}$. Therefore if we remove x_1 and x_2 as is shown in Figure 8.12, we obtain a drawing η^- of G with a smaller number of crossings compared to η such that the parity of the number of times any two independent edges cross are the same. Therefore, η^- is a weak generalised outerthrackle drawing with a smaller number of crossings compared to η. Hence, by induction, G has a weak outerthrackle drawing η_{temp} in which the edges that are incident with a vertex do not cross.

Case 2, there are two incident edges e and e' in η that cross once. Obtain an outerdrawing η_2 of G from η by removing any crossings between two edges that are incident at the same vertex using the move that is shown in Figure 8.13.

By the same argument as in Case 1, this move does not change the parity of the number of crossings between vertex-disjoint edges in outerdrawings. Therefore, η_2 is an outerthrackle drawing of G with a smaller number of crossings compared to η and by induction, G has an outerthrackle drawing η_{temp} in which the edges that are adjacent to a vertex do not cross.

Step 2. In this step we use R_I^p to reverse the order of the edges adjacent

Fig. 8.13 Removing crossings on two adjacent edges (where (v_1, v_2) and (v_1, v_3) are the only two edges that are depicted here). The difference between R_I^p and this move is that here there might be other crossings between other edges and (v_1, v_2) or other edges and (v_1, v_3) that cross (v_1, v_2) or (v_1, v_3) between v_1 and the crossing.

to any vertex v of η_{temp} to obtain another drawing η' of G. That is, if $\pi(v)$ in η_{temp} is $e_1, e_2, \ldots, e_i$ then change $\pi(v)$ to $e_i, e_{i-1}, \ldots, e_1$ in η' through the following series of steps:

1. $\pi(v) = e_2, e_3, \ldots, e_i, e_1$ (see, for example, Figure 8.14(b)).

2. $\pi(v) = e_i, e_{i-1}, e_1, e_2, \ldots, e_{i-2}$ (see, for example, Figure 8.14(c)).

$\vdots$

$i-1$. $\pi_{\eta'}(v) = e_i, e_{i-1}, \ldots, e_1$ (see, for example, Figure 8.14(d)).

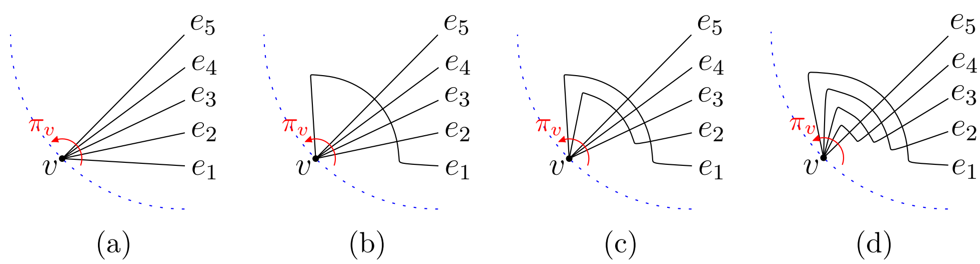


Fig. 8.14 Constructing η' from η_{temp} by reversing the order of the edges around the vertices.

R_I^p does not change the parity of the number of crossings between vertex-disjoint edges. However, it changes the parity of the number of crossings between the edges that are incident with v. Knowing that any two edges that are adjacent to one vertex cross each other an even number of times in η_{temp}, it is easy to see that any two edges that are adjacent to one vertex cross each other once in η'. Therefore, η' is a drawing of G in which any two edges cross once. The theorem follows. $\square$

We conclude this chapter by establishing the relationship between these different types of outerthracklable graphs.

Theorem 49. *The following four classes of graphs are equivalent:*

1. *outersuperthracklable graphs*
2. *weak outerthracklable graphs*
3. *generalised outersuperthracklable graphs*
4. *weak generalised outerthracklable graphs.*

Proof. By definition, any outersuperthrackle is both a weak outerthrackle and a generalised outersuperthrackle. Moreover any weak outerthrackle or generalised outersuperthrackle is a weak generalised outerthrackle. Therefore by definition, the class of weak generalised outerthracklable graphs includes all the weak outerthracklable graphs and all generalised outersuperthracklable graphs, and both of the classes of weak outerthracklable graphs and the generalised outersuperthracklable graphs include outersuperthracklable graphs (see Figure 8.15).

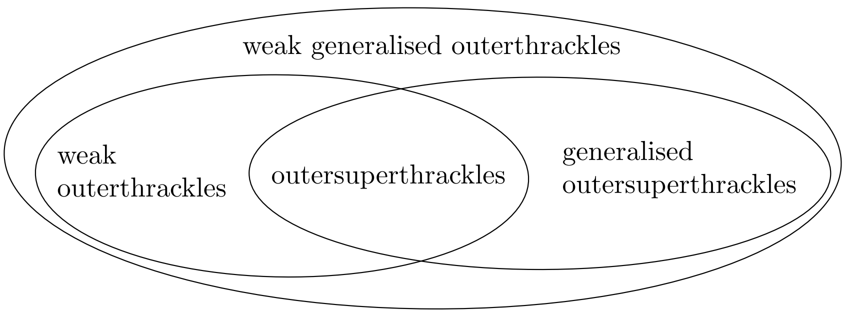


Fig. 8.15 Relationship between different types of outerthracklable graphs

Therefore, to prove that all the aforementioned classes of graphs are equal, we only need to show that any weak generalised outerthracklable graph is an outersuperthracklable graph. That is, the Theorem follows by Theorem 48. □

9

The Hanani-Tutthe Theorem and Superthrackles

In this chapter we prove that the classes of generalised superthracklable graphs and superthracklable graphs are equal.
Moreover, we show that there is a relationship between the class of graphs that are not embeddable on a surface Σ and the class of graphs that are not superthracklable with respect to Σ. More specifically, we show that given a minimal excluded minor G for embeddability of graphs on a surface Σ there are two infinite families of graphs that we can construct from G that are not superthracklable with respect to Σ.

Theorem 33 states that any generalised thracklable graph is a superthracklable graph. In this chapter, we provide a simple and direct proof for Theorem 33 and generalise it to all surfaces.

Theorem 50. *Any generalised superthracklable graph with respect to surface Σ is superthracklable with respect to Σ.*

Then we investigate the relationship between Theorem 33 and the Weak Hanani-Tutte Theorem. Note that there are similarities between Theorem 33 and the Weak Hanani-Tutte Theorem since:

- the Weak Hanani-Tutte Theorem can be rephrased as: every graph G with a drawing in which every two edges cross an even number of times has a drawing in which every two edges cross zero times, and
- Theorem 33 can be rephrased as: every graph G with a drawing in which every two edges cross an odd number of times has a drawing in which every two edges cross once.

For any graph $G = (V, E)$ and any subset E' of E, let $\mathcal{G}(G, E')$ be the family of all the graphs that are obtained from G as follows:

- Replace every edge $e = (u, v) \in E'$ with a (u, v)-path P of even length,
- Replace every edge $e = (u, v) \notin E'$ with any (u, v)-path P,
- such that all paths so introduced are internally disjoint from each other.

We use the weak Hanani-Tutte Theorem to prove Theorem 51.

Theorem 51. *Let Σ be a surface. Let G be a graph such that in any drawing of G on Σ there are two edges that cross each other an odd number of times. Let e be any edge of G and let G' be a graph in $\mathcal{G}(G, E \setminus \{e\})$. Then in any drawing of G' on Σ, there are two edges that cross each other an even number of times.*

Moreover, given that the Strong Hanani-Tutte Theorem holds for planar graphs one might think that the analogous statement holds for superthrackles, that is, that any weak generalised superthracklable graph is superthracklable. Lastly, we show that the latter statement is false.

The rest of this chapter is organised as follows. In Section 9.1 we prove equivalence of generalised superthracklable graphs and superthracklable graphs for all surfaces. Section 9.2 is dedicated to the relationship between the Hanani-Tutte Theorem and thrackles.

9.1 Generalised Superthrackles and Superthrackles

Lemma 31. *Let η be a generalised superthrackle drawing of a multigraph with only two vertices u and v and no loops. Then $\pi_\eta(u) = \pi_\eta(v)$.*

Proof. We prove this lemma by induction on the number m of edges in the drawing. In the base case η does not have any edges and the lemma holds trivially. We proceed to the inductive case.

Let $e_1, e_2, \ldots, e_m$ be the edges in η, named so that $\pi(u) = e_1, e_2, \ldots, e_m$. Let η^- be the drawing obtained by deleting e_m from η. Any two edges in η^- cross each other an odd number of times and therefore η^- is a generalised superthrackle as well. Hence, by induction, $\pi_{\eta^-}(u) = \pi_{\eta^-}(v) = e_1, e_2, \ldots, e_{m-1}$.

Now to reach a contradiction suppose that $\pi_\eta(u) \neq \pi_\eta(v)$. Since $\pi_{\eta^-}(u) = \pi_{\eta^-}(v)$, this means that e_m is not located between e_{m-1} and e_1 in $\pi_\eta(v)$.

Without loss of generality, let us assume that $\pi_\eta(v) = e_1, e_2, \ldots, e_i, e_m, e_{i+1}, e_{i+2}, \ldots, e_{m-1}$, where $1 \leq i \leq m-1$, as shown in Figure 9.1. Let C be the cycle that is defined by the two edges e_{m-1} and e_1.

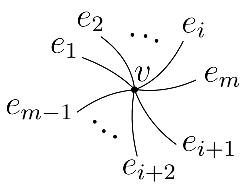


Fig. 9.1 $\pi_\eta(v)$

By Lemma 20, we can colour all the regions of the plane with respect to $\eta[C]$ either black or white, such that any curve that extends from a point p_1 to a point p_2 crosses C:

- an even number of times if p_1 and p_2 are located in regions with the same colour or
- an odd number of times if p_1 and p_2 are not located in regions with the same colour.

Now, let us consider the colouring of the regions in the neighbourhoods of vertices u and v. Such a colouring can be in the form of one of the four cases shown in Figure 9.2.

Since η is a generalised superthrackle, any edge of η other than e_1 and e_{m-1}, say e_2, crosses C an even number of times. Therefore the colouring of the regions of the plane in the neighbourhood of u and v cannot be similar to Figure 9.2(b) or Figure 9.2(c). (The edge e_2 should start and end in two isochromatic regions.) However, this would lead to a contradiction since then the initial and final portions of e_m, which crosses C an even number of times, lie in two regions that are not coloured the same. $\square$

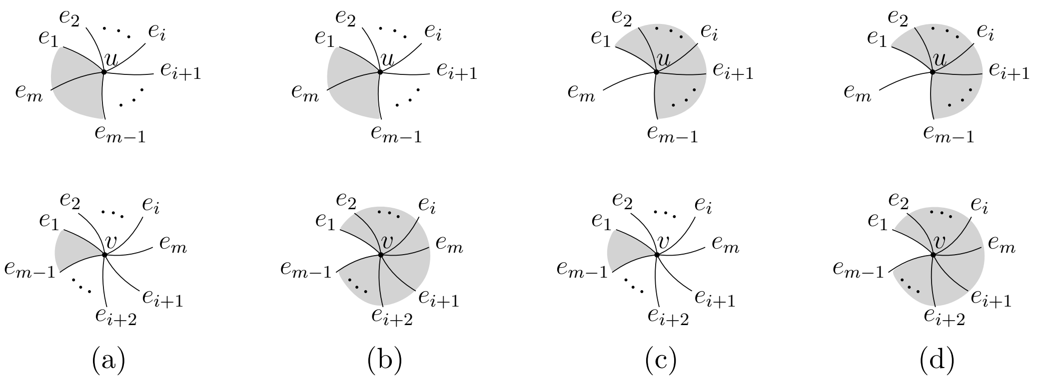


Fig. 9.2 Four different forms of black-and-white colouring of neighbourhood of u and v in η based on C

Now we deal with the case of a graph with two vertices that may have loops.

Lemma 32. *Let η be a generalised superthrackle drawing of a multigraph G with two vertices. There is a superthrackle drawing η' of G such that $\Pi(\eta') = \Pi(\eta)$.*

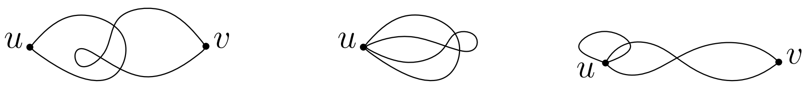
Proof. Let u and v be the vertices of G. Then the following three cases are forbidden in a generalised superthrackle drawing η of a generalised superthracklable G:

1. Let e_1 and e_2 be two loops in G where e_1 is incident with u and e_2 is incident with v. In this case, in any drawing of G, e_1 and e_2 cross each other an even number of times and therefore G is not a generalised superthracklable graph. (See, for example, Figure 9.3(a).)

2. Let e_1 and e_2 be two loops that are both incident with the same vertex, say u, then the restriction of $\pi_\eta(u)$ to e_1 and e_2 cannot be e_1, e_1, e_2, e_2 since in that case e_1 and e_2 cross each other an even number of times. (See, for example, Figure 9.3(b).)

3. Let e_1 and e_2 be two parallel edges between u and v, let e_3 be a loop that is incident with u, and let the two occurrences of e_3 in $\pi_\eta(u)$ be separated from each other by e_1 and e_2. (See, for example, Figure 9.3(c).) In this case, e_3 crosses the cycle defined by the two edges e_1 and e_2 an odd number of times since it starts and ends on opposite sides of the cycle. Therefore e_3 crosses either e_1 or e_2 an even number of times.

Using forbidden case 1, from this point on, we assume that if there is any loop in G it is incident with u and not with v.



(a) (b) (c)

Fig. 9.3 Three forbidden cases in a superthrackle drawing of a multigraph with two vertices.

We prove this lemma by induction on the number of edges of G. Let u and v be the two vertices of G. In the base case, there is at most one loop and at most one edge (u, v) in G. If G has has less than two edges then, by definition, any drawing of G is a superthrackle. So let us assume that there is a loop and an edge (u, v) in G. In this case, it is easy to see that there is a superthrackle drawing η' of G such that $\Pi(\eta') = \Pi(\eta)$. We proceed to the inductive case.

We have the following two cases:

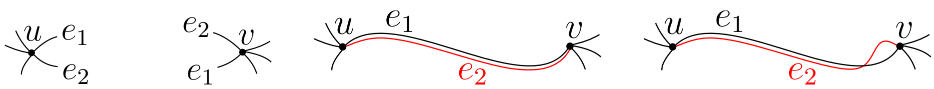
Case 1. There are at least two parallel edges between u and v in G. By the forbidden case 3, we know that all the endpoints of the edges that are not loops in G appear consecutively in $\pi_\eta(u)$. Therefore, by Lemma 31, there are two parallel edges $e_1 = (u, v)$ and $e_2 = (u, v)$ in G such that both of their endpoints appear consecutively and in the same order in both $\pi_\eta(u)$ and $\pi_\eta(v)$. (See, for example, Figure 9.4(a).)

Let G^- be the graph obtained by deleting e_2 from G and let η^- be the drawing obtained by deleting e_2 from η. By the inductive hypothesis G^- has a superthrackle drawing η_1^- such that $\Pi(\eta_1^-) = \Pi(\eta^-)$.

Now we obtain a drawing η_1 of G such that $\Pi(\eta_1) = \Pi(\eta)$ by adding the edge e_2 back to η_1^- as in the following two steps:

1. add e_2 to the drawing such that e_2 follows e_1 sufficiently closely so that it is drawn in the local disk of e_1 and does not meet e_1 and $\pi_{\eta_1}(u)$ is the same as $\pi_\eta(u)$. (See, for example, Figure 9.4(b).)

2. use the R_I^p move to switch the rotational order of e_1 and e_2 around v (See, for example, Figure 9.4(c).)



(a) order of e_1 and e_2 in $\pi_\eta(u)$ and $\pi_\eta(v)$ (b) e_2 closely follows e_1 (c) using R_1 to ensure that e_2 crosses e_1

Fig. 9.4 Obtaining η_1 by adding the edge e_2 to η_1^- in case 1.

Since η_1^- is a superthrackle, any pair of edges in η_1 that does not contain e_2 cross each other once. Moreover since e_2 follows e_1 sufficiently closely, e_2 also crosses any edge other than e_1 in η_1 once. Lastly, with the R_I^p move in step 2, we guarantee that e_2 crosses e_1 once as well. Hence any two edges in η_1 cross each other once and therefore η_1 is a superthrackle.

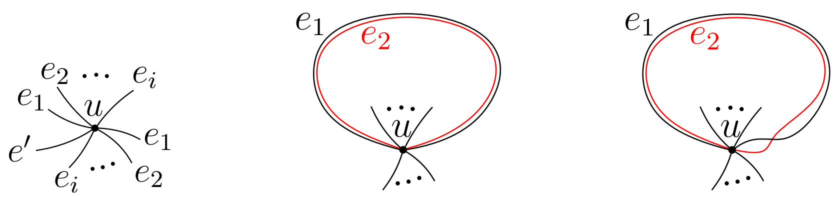
Case 2. There is at least two loops in G and there is at most one edge incident with both u and v. By forbidden case 1, all the loops in G are incident with one vertex. Let $e_1, e_2, \ldots, e_i$ be the loops that are incident with u and let e' be the edge that is incident with both u and v.

By forbidden cases 2 and 3, it is easy to see that the loops can be named so that $\pi_\eta(u) = e_1, e_2, \ldots, e_i,\ e_1, e_2, \ldots, e_i, e'$ (see Figure 9.5(a)). Let G^- be the graph obtained by deleting e_2 from G and let η^- be the drawing that is obtained by deleting e_2 from η. By the inductive hypothesis G^- has a superthrackle drawing η_1^- such that $\Pi(\eta_1^-) = \Pi(\eta^-)$.

Now we obtain a drawing η_1 of G such that $\Pi(\eta_1) = \Pi(\eta)$ by adding e_2 back to G as in the following two steps:

1. add e_2 to the drawing such that e_2 follows e_1 sufficiently closely and does not cross it (see for example Figure 9.5(b)).

2. use the R_I^p move to switch the rotational order of e_1 and e_2 around u such that $\Pi(\eta_1) = \Pi(\eta)$ (see for example Figure 9.5(c)).



(a) $\pi(u)$ (b) e_2 closely follows e_1 (c) using R_1 to ensure that e_2 crosses e_1

Fig. 9.5 Obtaining η_1 by adding the edge e_2 to η_1^- in case 2.

By a similar reasoning to case 1, any two edges in η_1 cross each other once and therefore η_1 is a superthrackle. □

We use the above lemma as the base case of the proof of the following theorem.

Theorem 52. *Let η be a generalised superthrackle drawing of a connected multigraph G. Then there is a superthrackle drawing η' of G such that $\Pi(\eta') = \Pi(\eta)$.*

Proof. It is easy to prove the lemma if G has only one vertex. So let us assume that G is connected and has at least two vertices.

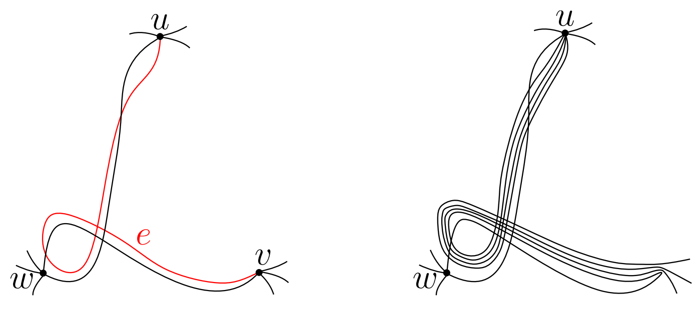
We prove this theorem by induction on the number of the vertices in G. In the base case, G has two vertices and by Lemma 32, we know that the theorem holds. We proceed to the inductive case where there are at least three vertices in G.

Let u and v be two distinct vertices of G such that there are two edges (u, w) and (v, w) in G where (u, w) and (v, w) appear consecutively in $\pi_\eta(w)$ and $w \neq u, v$. Since G is connected and G has at least three vertices, such u and v exist.

By Lemma 30, there is a curve c that extends from u to v and crosses each edge of G an even number of times. Let G^+ be the graph obtained by adding an edge $e = (u, v)$ to G and let η^+ be a drawing of G^+ obtained by adding e to η such that e is routed along the curve c.

Let G^- be the graph that is obtained by contracting e in G^+. G^- has one vertex fewer than G. Obtain a drawing η^- of G^- by contracting e in η^+ such that:

- any edge e' incident with v follows the route of e sufficiently closely until it reaches u without crossing any other edge incident with v;
- for any new crossing introduced between e' and another edge e'' in η^- there is a crossing between e and e'' on η^+ (see for example Figure 9.6).



(a) Edge e crosses all the other edges in η^+ an even number of times.

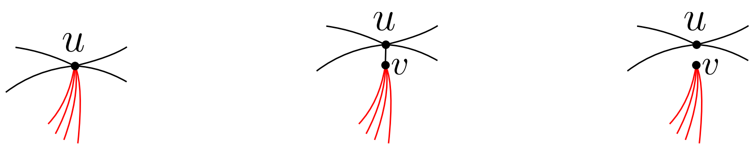
(b) Contracting e such that all the edges incident with v follow e sufficiently closely.

Fig. 9.6 Obtaining η by contracting e in η^+

The edge e crosses all other edges of η^+ an even number of times and since η is a generalised thrackle, all the edges in η^+ except e cross each other an odd number of times. Therefore, any two edges in η^- cross each other an odd number of times. In other words, η^- is a generalised thrackle as well.

Since G^- has one vertex fewer than G, by the inductive hypothesis, there is a superthrackle drawing η_1^- of G^- such that $\Pi(\eta_1^-) = \Pi(\eta^-)$.

All the edges that were incident with v in η^+ appear consecutively in $\pi_{\eta^-}(u)$. Therefore, since $\Pi(\eta_1^-) = \Pi(\eta^-)$, those edges appear consecutively in $\pi_{\eta_1^-}(u)$ as well (see for example Figure 9.7(a)). Hence it is easy to decontract e to obtain a drawing η_1^+ of G^+ such that all the edges of η_1^+ except for e cross each other once and e does not cross any other edges (see for example Figure 9.7(b)).



(a) $\pi_{\eta^-}(u)$. Edges that were incident with v in η^+ are depicted in red.

(b) Obtaining η_1^+ from η^- by decontracting e

(c) Obtaining η' from η_1^+ by deleting e

Fig. 9.7 Obtaining η'

Now we can delete e from η_1^+ to obtain a superthrackle drawing η' of G (see for example Figure 9.7(c)). □

9.2 The Hanani-Tutte Theorem and Superthrackles

In this section we examine the relationship between the Hanani-Tutte Theorem and superthrackles. The first subsection is about the connection between the Weak Hanani-Tutte Theorem and Superthrackles and the second subsection is about the connection between the Strong Hanani-Tutte Theorem and Superthrackles.

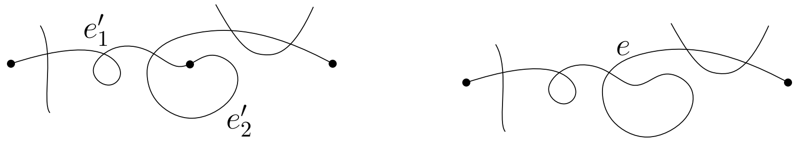
9.2.1 The Weak Hanani-Tutte Theorem and Superthrackles

Archdeacon and Stor characterised superthrackles in terms of eight forbidden configurations [4]. Four of these configurations are closely related to $K_{3,3}$ and K_5 which are the forbidden graphs in Kuratowski's Theorem. Next we will explain why is there such a close relation between these two theorems.

Now we are ready to prove Theorem 51.

Proof of Theorem 51. Every edge e in G is replaced by a path in G'. Let us denote that path by $P(e)$ and the length of that path by $l(e)$. Denote the edges of $P(e)$ by $e'_1, e'_2, \ldots, e'_{l(e)}$.

Let η' be a drawing of G'. Obtain a drawing η of G from η' by replacing edges $e'_1, e'_2, \ldots, e'_{l(e)}$ in η', for every edge e in G, with an edge e such that e is routed exactly on the curve along which the edges $e'_1, e'_2, \ldots, e'_{l(e)}$ are routed in Σ (see for example, Figure 9.8).



(a) Edges e'_1 and e'_2 of path $P(e)$ in drawing η' of graph G'

(b) Edge e in drawing η of graph G

Fig. 9.8 Replacing e^*_1 and e^*_2 with e, where $i(e)$ is 2

By the theorem's assumption, in any drawing η of G there are two edges that cross each other an odd number of times. Let e and f be two edges that cross an odd number of times in η. For every edge (u, v) in G with the exception of one of the edges, there is a (u, v) path with even length in G'. Therefore, e or f is replaced by a path of even length to obtain G' from G. Without loss of generality, let f be always an edge that is replaced by a path of even length as we obtain G' from G.

Let us denote the number of crossing between two paths P_1 and P_2 with $\chi(P_1, P_2)$. Since $\chi(e, f)$ is odd in η, $\chi(P(e), P(f))$ is odd in η'. But $\chi(P(e), P(f))$ is obtained by summing up $\chi(e_1, e_2)$ for all the pairs e_1, e_2 of edges where e_1 is an edge of $P(e)$ and e_2 is an edge of $P(f)$. Since $l(f)$ is even, there is an even number of such pairs of edges. To reach a contradiction, assume that all such pairs of edges cross an odd number of times. Then we have an even number of odd integers that sum up to an odd integer, which is a contradiction. Hence there is an edge e_1 in $P(e)$ and an edge f_2 in $P(f)$ such that e_1 crosses f_2 an even number of times (see for example, Figure 9.9).

$\square$

An implication of the Weak Hanani-Tutte Theorem along with the Kuratowski's Theorem is that, in any drawing of $K_{3,3}$ or K_5 or any subdivision of them, there are two edges that cross each other an odd number of times. This fact, along with Theorem 51, proves that K^*_5, $K^*_5(e)$, $K^*_{3,3}$ and $K^*_{3,3}(e)$

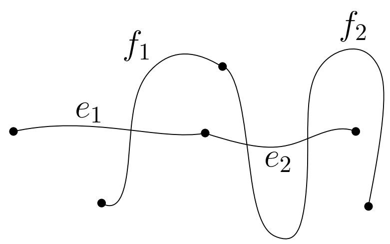


Fig. 9.9 $\pi_\eta(v)$

(depicted in Figure 2.18) have the property that, in any drawing of them in the plane, there are two edges that cross an even number of times. Therefore, by definition, these graphs are not generalised superthracklable and therefore not superthracklable either.

The Weak Hanani-Tutte Theorem can be generalised to all surfaces [96, 97]. That is, if a graph G does not have a drawing that can be drawn on a surface Σ without crossings, then there are two edges that cross each other an odd number of times in any drawing of G on Σ and hence, by Theorem 51, any graph in $\mathcal{G}(G, E \setminus \{e\})$ is not superthracklable on Σ.

Theorem 53. *Let G be a graph that is in the set of minimal excluded minors for embeddable graphs on a surface Σ. Let $H = (V, E)$ be a subdivision of G. Then any graph that contains a graph in $\mathcal{G}(H, E \setminus \{e\})$, where $e \in E$, is neither a generalised superthracklable graph nor a superthracklable graph with respect to Σ.*

Proof. Since H contains a subdivision G, it is not embeddable on Σ. Therefore, by the Weak Hanani Tutte Theorem for all surfaces [96, 97], in any drawing of H on Σ there are two edges that cross each other an odd number of times.

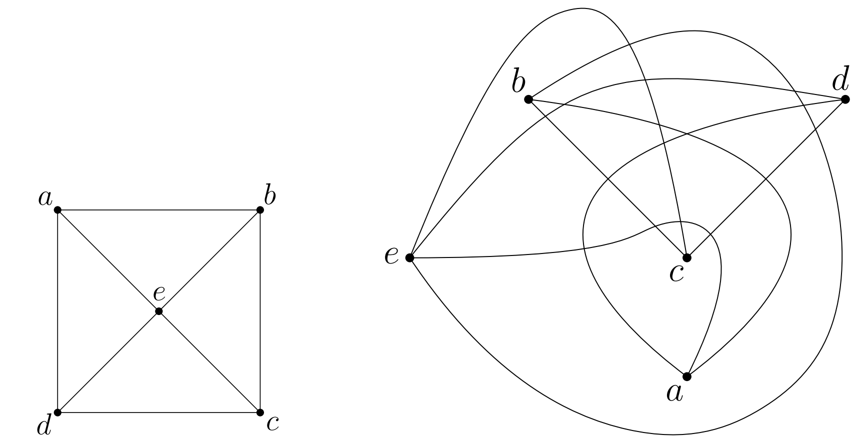
Therefore by Theorem 51 in any drawing of a graph H' in $\mathcal{G}(H, E \setminus \{e\})$ on Σ, there are two edges that cross each other an even number of times. Hence if a graph H'' contains H' it cannot be drawn on Σ such that any two edges of H'' cross an odd number of times. Hence, by Theorem 52, H'' is neither a generalised superthracklable graph nor a superthracklable graph. □

9.2.2 The Strong Hanani-Tutte Theorem and Superthrackles

In Section 9.1, we proved that any generalised superthracklable graph is a superthracklable graph. In other words, if there is a drawing of a graph G in which any two edges cross each other an odd number of times, then there is a drawing of G in which any two edges cross once. We also pointed out the similarities between this theorem and the Weak Hanani-Tutte Theorem.

A natural question that arises from the above observation is whether we can prove a theorem similar to the Strong Hanani-Tutte Theorem for superthrackles. Let G be a graph that has a drawing in which any two vertex-disjoint edges cross an odd number of times. Can we, with certainty, claim that G has a drawing in which any two edges cross once?

The answer to the above question is no. Figure 9.10(a) depicts a planar embedding of a graph G that is not superthracklable by Theorem 32. Figure 9.10(b) depicts a drawing of G in which any two vertex-disjoint edges cross each other an odd number of times.



(a) A planar embedding of a non-superthracklable graph G

(b) A drawing of G in which any two vertex-disjoint edges cross an odd number of times.

Fig. 9.10 A non-superthracklable graph G and a drawing of G in which any two vertex-disjoint edges cross an odd number of times.

10
Conclusion

In this chapter we summarise our main results and review the connections between them. We then discuss open questions and directions for further research.

In this thesis we investigated three main topics: the Hanani-Tutte Theorem, non-separating planar graphs and thrackles and some of their variations. These three topics may seem unrelated to each other at the first glance, but through the course of this thesis we have observed connections between them as follows.

We proved a stronger version of the Strong Hanani-Tutte Theorem. To do so, we first characterised non-separating planar graphs in terms of minimal excluded minors. We then used this characterisaion to prove a Hanani-Tutte type theorem for non-separating planar graphs which in turn was used in the proof of the stronger version of the Strong Hanani-Tutte Theorem. We also decomposed planar graphs into smaller non-separating planar graphs in the proof of the stronger version of the Strong Hanani-Tutte Theorem.

Motivated by the weak and the strong versions of the Hanani-Tutte Theorem, we defined a number of different types of outerthracklable graphs and showed that all of these classes of graphs are equal.

We also investigated the relationship of the Hanani-Tutte Theorem with thrackles and proved that any generalised superthracklable graph is a superthracklable graph, which is an analogue of the Weak Hanani-Tutte Theorem for superthrackles.

Then we showed that, given a minimal excluded minor G for the class of graphs that are embeddable on a surface Σ, we can use G to construct two infinite families of graphs that are not superthracklable with respect to Σ.

Moreover we found a counterexample for the analogue of the Strong Hanani-Tutte Theorem for superthrackles.

Lastly, we showed that there is a close relationship between non-separating planar graphs and linkless graphs by using a family of maximal non-separating planar graphs to construct a family of maximal linkless graphs on $|V|$ vertices with $3|V|-3$ edges.

10.1 Future Work

10.1.1 Non-separating Planar Graphs

We believe that non-separating planar graphs are an interesting class of graphs in their own right and that they deserve to be studied more thoroughly.

One can define a class of graphs similar to non-separating planar graphs with respect to surfaces other than the plane. For example, a graph G is a *non-separating toroidal* graph if it has a drawing D without crossings on the torus such that for any cycle C in D and any two vertices $u, v \in V(G) \setminus V(C)$, one can draw a curve from u to v without crossing any of the edges of C.

non-separating toroidal graph

Any such class of graphs is also closed under taking minors and hence it can be characterised using a finite set of minimal excluded minors. It would be especially interesting to know the set of minimal excluded minors for non-separating toroidal graphs since they are all minors of the minimal excluded minors for toroidal graphs and we do not yet know the complete set of minimal excluded minors for toroidal graphs (see [91]).

Chartrand and Harary proved that a graph is outerplanar if and only if it does not contains any of K_4 or $K_{2,3}$ as a minor [21]. We proved that a graph is a non-separating planar graph if and only if it does not contain any of $K_1 \cup K_4$ or $K_1 \cup K_{2,3}$ or $K_{1,1,3}$ as a minor (Theorem 38). Notice that for any minimal excluded minor f for outerplanar graphs, $f \cup K_1$ is an minimal excluded minor for non-separating planar graphs.

An interesting question to ask is whether or not such a relation exists for other surfaces. More specifically, let Σ be a surface other than the sphere and let $\mathcal{G}$ be the class of all graphs that can be embedded in Σ without a separating cycle. Moreover, let $\mathcal{F}$ be the class of all graphs that have a drawing on Σ such that all of its vertices are on the same face. Is it the case that for any minimal excluded minor f for $\mathcal{F}$, $f \cup K_1$ is an minimal excluded minor for $\mathcal{G}$? $\mathcal{G}$ $\mathcal{F}$

By Theorem 39, it is straightforward to see that, in any non-separating planar graph G, there is an edge e such that $G \setminus e$ is outerplanar. In fact, it is straightforward to see that in any non-separating planar drawing η of a graph G, there is an edge e such that the drawing that is obtained by removing e from η is outerplane.

It would be also interesting to know if there is such a relationship between similar classes of graphs on other surfaces. More specifically, is there an edge in any graph $G \in \mathcal{G}$ such that $G \setminus e \in \mathcal{F}$?

Sachs points out that for any integer $n \geq 4$, there is a maximal linkless graph with n vertices and $4n - 10$ edges. Moreover, since linkless graphs do not contain a K_6-minor, it follows that G has at most $4n - 10$ edges [83] and we showed that there are maximal linkless graphs with $3|V| - 3$ edges (Theorem 43). A natural question that arises as a result of this theorem is the following. Does every edge-maximal linkless graph with n vertices have at least $3n - 3$ edges?

Theorem 43 also showed that there is a connection between non-separating planar graphs and linkless graphs. It would be interesting to explore this connection further. In fact it was this connection that served as our first motivation for exploring the structure of non-separating planar graphs.

Lastly, in the view of the use we have made of non-separating planar graphs, it would be interesting to see if they have other applications or if they are related to other classes of graphs.

10.1.2 Hanani-Tutte Theorem

To prove a stronger version of the Strong Hanani-Tutte Theorem, namely Theorem 42, we used a Hanani-Tutte type characterisation of non-separating planar graphs, namely Theorem 41. Theorem 42 relies on the definition of evenly decomposable drawings and this definition is based on the requirements of Theorem 41. So Theorem 42 could be strengthened if one could provide a stronger Hanani-Tutte type characterisation for non-separating planar graphs.

In the statement of Lemma 21, we introduced a graph G as follows. Let G be a graph such that the shortest cycle in G is a two-vertex-avoiding cycle and for any edge e and any vertex v that is not an endpoint of e, $|E(v - e)|$ and

$|C(e-v)|$ have the same parity. Now let $\mathcal{H}$ be the family of all such graphs. $\mathcal{H}$ It would be interesting to know more about $\mathcal{H}$ and to know whether there are non-trivial graphs other than $K_1 \cup K_4$, $K_1 \cup K_{2,3}$ and $K_{1,1,3}$ that are included in this class of graphs.

Regarding the invariant $\phi + \chi$, which was introduced in Section 6.1, we conjecture the following.

Conjecture 4. *Let G be a minimal excluded minor for the class of graphs that have an embedding on a surface Σ with no separating cycle. For any drawing D of G on Σ, $\phi(D) + \chi(D)$ is odd.*

We also conjecture the following strengthening.

Conjecture 5. *Let D be a non-separating drawing of a graph G on a surface Σ such that any two vertex-disjoint edges in D cross an even number of times. Then G has a non-separating drawing with no crossings on Σ.*

10.1.3 Thrackles

Archdeacon and Stor proved that a graph is both superthracklable and generalised superthracklable if and only if it does not contain a subgraph that is parity homeomorphic to any graph in Figure 2.18 (see Theorem 32). Although superthracklable graphs and generalised superthracklable graphs are well studied, we still do not know whether for every surface there exists a finite set $\mathcal{S}$ of graphs such that we can characterise superthracklable graphs and generalised superthracklable graphs for that surface in terms of graphs without subgraphs that are parity homeomorphic to a graph in $\mathcal{S}$.

Moreover, we do not know about characterisations of superthracklable graphs on surfaces other than the plane. For example, what are the superthracklable graphs with respect to the projective plane?

References

[1] Eyal Ackerman, Jacob Fox, János Pach, and Andrew Suk. On grids in topological graphs. *Computational Geometry*, 47(7):710–723, 2014.

[2] Colin Conrad Adams. *The Knot Book: an Elementary Introduction to the Mathematical Theory of Knots.* American Mathematical Soc., 2004.

[3] Dan Archdeacon and Phil Huneke. A Kuratowski theorem for nonorientable surfaces. *Journal of Combinatorial Theory, Series B*, 46(2): 173–231, 1989.

[4] Dan Archdeacon and Kirsten Stor. Superthrackles. *Australasian Journal of Combinatorics*, 67(2):145–158, 2017.

[5] Dan Archdeacon, C Paul Bonnington, Nathaniel Dean, Nora Hartsfield, and Katherine Scott. Obstruction sets for outer-cylindrical graphs. *Journal of Graph Theory*, 38(1):42–64, 2001.

[6] Alan Arroyo, Dan McQuillan, R. Bruce Richter, and Gelasio Salazar. Drawings of K_n with the same rotation scheme are the same up to triangle-flips (Gioan's theorem). *Australas. J. Combin.*, 67:131–144, 2017.

[7] Roland Bacher and Yves Colin de Verdière. Multiplicités des valeurs propres et transformations étoile-triangle des graphes. *Bulletin de la Société Mathématique de France*, 123(4):517–533, 1995.

[8] Giuseppe Di Battista, Peter Eades, Roberto Tamassia, and Ioannis G Tollis. *Graph Drawing: Algorithms for the Visualization of Graphs.* Prentice Hall PTR, 1998.

[9] Richard Bellman. On a routing problem. *Quarterly of Applied Mathematics*, 16:87–90, 1958.

[10] Frank Bernhart and Paul C Kainen. The book thickness of a graph. *Journal of Combinatorial Theory, Series B*, 27(3):320–331, 1979.

[11] Daniel Bienstock and Michael A Langston. Algorithmic implications of the graph minor theorem. *Handbooks in Operations Research and Management Science*, 7:481–502, 1995.

[12] Daniel Bienstock and Michael A Langston. Algorithmic implications of the graph minor theorem. *Handbooks in Operations Research and Management Science*, 7:481–502, 1995.

[13] Thomas Böhme. In R. Bodendiek, editor, *Contemporary Methods in Graph Theory*, pages 151–167. BI-Wiss.-Verl. Mannheim, Wien/Zurich, 1990.

[14] Peter Brass, William OJ Moser, and János Pach. *Research Problems in Discrete Geometry*. Springer Science & Business Media, 2006.

[15] Grant Cairns and Yury Nikolayevsky. Bounds for generalized thrackles. *Discrete & Computational Geometry*, 23(2):191–206, 2000.

[16] Grant Cairns and Yury Nikolayevsky. Generalized thrackle drawings of non-bipartite graphs. *Discrete & Computational Geometry*, 41(1): 119–134, 2009.

[17] Grant Cairns and Yury Nikolayevsky. Outerplanar thrackles. *Graphs and Combinatorics*, 28(1):85–96, 2012.

[18] Grant Cairns, Margaret McIntyre, and Yury Nikolayevsky. The thrackle conjecture for K_5 and $K_{3,3}$. *Contemporary Mathematics*, 342:35–54, 2004.

[19] Grant Cairns, Timothy J Koussas, and Yuri Nikolayevsky. Great-circle spherical thrackles. *Discrete Mathematics*, 338(12):2507–2513, 2015.

[20] John Chambers. *Hunting for torus obstructions*. PhD thesis, University of Victoria, 2002.

[21] Gary Chartrand and Frank Harary. Planar permutation graphs. *Annales de l'I.H.P. Probabilités et statistiques*, 3(4):433–438, 1967.

[22] Gary Chartrand, Linda Lesniak, and Ping Zhang. *Graphs & digraphs*. CRC Press, 2010.

[23] Norishige Chiba, Kazunori Onoguchi, and Takao Nishizeki. Drawing plane graphs nicely. *Acta Informatica*, 22:187–201, 1985.

[24] Chaim Chojnacki. Über wesentlich unplättbare Kurven im dreidimensionalen Raume. *Fundamenta Mathematicae*, 23(1):135–142, 1934.

[25] Fan R K Chung, Frank Thomson Leighton, and Arnold L Rosenberg. Embedding graphs in books: a layout problem with applications to VLSI design. *SIAM Journal on Algebraic Discrete Methods*, 8(1):33–58, 1987.

[26] Yves Colin de Verdière. Sur un nouvel invariant des graphes et un critère de planarité. *Journal of Combinatorial Theory, Series B*, 50(1):11–21, 1990.

[27] Yves Colin de Verdière. On a new graph invariant and a criterion for planarity. In Neil Robertson and Paul D. Seymour, editors, *Graph Structure Theory*, volume 147 of *Contemporary Mathematics*, pages 137–148. American Mathematical Society, 1991.

[28] J. H. Conway and C. McA. Gordon. Knots and links in spatial graphs. *Journal of Graph Theory*, 7(4):445–453, 1983.

[29] Mark De Berg, Marc Van Kreveld, Mark Overmars, and Otfried Cheong Schwarzkopf. *Computational Geometry.* Springer, Berlin, 2000.

[30] R Decker. *On the orientable genus of a graph.* PhD thesis, Ohio State University, 1978.

[31] Hooman R Dehkordi and Graham Farr. Non-separating Planar Graphs. *arXiv preprint arXiv:1907.09817*, 2019.

[32] Hooman R Dehkordi and Graham Farr. On the Strong Hanani-Tutte Theorem. submitted, 2019.

[33] Hooman R Dehkordi and Graham Farr. Thrackles, Superthracles and the Hanani-Tutte Theorem. submitted, 2019.

[34] Reinhard Diestel. *Graph Theory.* Springer-Verlag Berlin and Heidelberg, 2000.

[35] Vida Dujmović and David R Wood. On linear layouts of graphs. *Discrete Mathematics & Theoretical Computer Science*, 6(2):339–358, 2004.

[36] Ivan A Dynnikov. Three-page approach to knot theory. encoding and local moves. *Functional Analysis and Its Applications*, 33(4):260–269, 1999.

[37] Herbert Edelsbrunner and John Harer. *Computational Topology: an Introduction.* American Mathematical Society, 2010.

[38] Jack Edmonds. A combinatorial representation of polyhedral surfaces. *Notices of the American Mathematical Society*, 7, 1960.

[39] Anton J Enright and Christos A Ouzounis. Biolayout—an automatic graph layout algorithm for similarity visualization. *Bioinformatics*, 17 (9):853–854, 2001.

[40] Michael R Fellows and Michael A Langston. Nonconstructive advances in polynomial-time complexity. *Information Processing Letters*, 26(3): 157–162, 1987.

[41] Michael R Fellows and Michael A Langston. Nonconstructive tools for proving polynomial-time decidability. *Journal of the ACM*, 35(3):727–739, 1988.

[42] Michael R Fellows, F. Hickling, and Maciej M Sysło. A topological parameterization and hard graph problems. *Congressius Numeratium*, 59(3):69–78, 1987.

[43] Peter A Firby and Cyril F Gardiner. *Surface Topology.* Elsevier, 2001.

[44] Erica Flapan and Ramin Naimi. The Y-triangle move does not preserve intrinsic knottedness. *Osaka Journal of Mathematics*, 45(1):107–111, 2008.

[45] Erica Flapan, Ramin Naimi, and James Pommersheim. Intrinsically triple linked complete graphs. *Topology and its Applications*, 115(2): 239–246, 2001.

[46] Erica Flapan, James Pommersheim, Joel Foisy, and Ramin Naimi. Intrinsically n-linked graphs. *Journal of Knot Theory and Its Ramifications*, 10(08):1143–1154, 2001.

[47] Joel Foisy. Intrinsically knotted graphs. *Journal of Graph Theory*, 39 (3):178–187, 2002.

[48] Joel Foisy. A newly recognized intrinsically knotted graph. *Journal of Graph Theory*, 43(3):199–209, 2003.

[49] Joel Foisy. Graphs with a knot or 3-component link in every spatial embedding. *Journal of Knot Theory and Its Ramifications*, 15(09):1113–1118, 2006.

[50] M Fréchet, K Fan, and H W Eves. *Initiation to Combinatorial Topology.* Prindle, Weber & Schmidt, 1967.

[51] Radoslav Fulek and Jan Kynčl. Counterexample to the Variant of the Hanani-Tutte Theorm on the Genus-4 Surface. *arXiv preprint arXiv:1709.00508*, 2017.

[52] Radoslav Fulek and János Pach. A computational approach to Conway's Thrackle Conjecture. *Computational Geometry*, 44(6-7):345–355, 2011.

[53] Radoslav Fulek and János Pach. Thrackles: An improved upper bound. *Discrete Applied Mathematics*, 2019.

[54] Radoslav Fulek, Jan Kynčl, and Dömötör Pálvölgyi. Unified Hanani-Tutte theorem. *Electronic Journal of Combinatorics*, 24, 12 2016.

[55] Andrei Gagarin, Wendy Myrvold, and John Chambers. Forbidden minors and subdivisions for toroidal graphs with no $K_{3,3}$'s. *Electronic Notes in Discrete Mathematics*, 22:151–156, 2005.

[56] Michael R Garey and David S Johnson. *Computers and Intractability: A Guide to the Theory of NP-Completeness.* WH Freeman & Co., San Francisco, 1979.

[57] Peter Giblin. *Graphs, Surfaces and Homology: An Introduction to Algebraic Topology.* Springer Science & Business Media, 2013.

[58] Emeric Gioan. Complete graph drawings up to triangle mutations. In *Graph-theoretic concepts in computer science*, volume 3787 of *Lecture Notes in Comput. Sci.*, pages 139–150. Springer, Berlin, 2005.

[59] Jacob E Goodman and Joseph O'Rourke. *Handbook of Discrete and Computational Geometry.* CRC press, 2010.

[60] Ronald L Graham. The largest small hexagon. *Journal of Combinatorial Theory, Series A*, 18(2):165–170, 1975.

[61] J E Green and Richard D Ringeisen. Combinatorial drawings and thrackle surfaces. In Y. Alavi, G. Chartrand, O.R. Oellermann, and A.J. Schwenk, editors, *Graph Theory, Combinatorics, and Algorithms*, pages 999–1009. Wiley, 1992.

[62] Jonathan L Gross and Jay Yellen. *Graph Theory and its Applications*. CRC press, 2005.

[63] Thomas C Hales. Sphere packings, II. *Discrete & Computational Geometry*, 18(2):135–149, 1997.

[64] Ryo Hanaki. Pseudo diagrams of knots, links and spatial graphs. *Osaka Journal of Mathematics*, 47(3):863–883, 2010.

[65] Ryo Hanaki, Ryo Nikkuni, Kouki Taniyama, and Akiko Yamazaki. On intrinsically knotted or completely 3-linked graphs. *Pacific Journal of Mathematics*, 252(2):407–425, 2011.

[66] F. Harary, P. C. Kainen, A. J. Schwenk, and A. T. White. A maximal toroidal graph which is not a triangulation. *Math. Scand.*, 33:108–112, 1973.

[67] Lenwood S Heath. Embedding outerplanar graphs in small books. *SIAM Journal on Algebraic Discrete Methods*, 8(2):198–218, 1987.

[68] Ivan Herman, Guy Melançon, and M Scott Marshall. Graph visualization and navigation in information visualization: A survey. *IEEE Transactions on Visualization and Computer Graphics*, 6(1):24–43, 2000.

[69] John Hopcroft and Robert Tarjan. Efficient planarity testing. *Journal of the ACM*, 21(4):549–568, 1974.

[70] Louis H Kauffman. Invariants of graphs in three-space. *Transactions of the American Mathematical Society*, 311(2):697–710, 1989.

[71] Michael Kaufmann and Dorothea Wagner. *Drawing Graphs: Methods and Models*. Springer, 2001.

[72] Ken-ichi Kawarabayashi, Yusuke Kobayashi, and Bruce Reed. The disjoint paths problem in quadratic time. *Journal of Combinatorial Theory, Series B*, 102(2):424–435, 2012.

[73] Ken-ichi Kawarabayashi, Stephan Kreutzer, and Bojan Mohar. Linkless and flat embeddings in 3-space. *Discrete & Computational Geometry*, 47(4):731–755, 2012.

[74] Takashi Kohara and Shin'ichi Suzuki. Some remarks on knots and links in spatial graphs. In *Knots*, volume 90, pages 435–445. de Gruyter, 1992.

[75] Casimir Kuratowski. Sur le problème des courbes gauches en topologie. *Fundamenta Mathematicae*, 15(1):271–283, 1930.

[76] Sergei K Lando and Alexander K Zvonkin. *Graphs on surfaces and their applications*, volume 141. Springer Science & Business Media, 2013.

[77] Martin Loebl and Gregor Masbaum. On the optimality of the Arf invariant formula for graph polynomials. *Advances in Mathematics*, 226 (1):332–349, 2011.

[78] László Lovász. Graph minor theory. *Bulletin of the American Mathematical Society*, 43(1):75–86, 2006.

[79] László Lovász and Alexander Schrijver. A Borsuk theorem for antipodal links and a spectral characterization of linklessly embeddable graphs. *Proceedings of the American Mathematical Society*, 126(5):1275–1285, 1998.

[80] László Lovász, János Pach, and Mario Szegedy. On Conway's thrackle conjecture. *Discrete & Computational Geometry*, 18(4):369–376, 1997.

[81] Dean Lusher, Johan Koskinen, and Garry Robins. *Exponential random graph models for social networks: Theory, methods, and applications.* Cambridge University Press, 2013.

[82] Avi Ma'ayan. Insights into the organization of biochemical regulatory networks using graph theory analyses. *Journal of Biological Chemistry*, 284(9):5451–5455, 2009.

[83] Wolfgang Mader. Homomorphiesätze für Graphen. *Mathematische Annalen*, 178(2):154–168, 1968.

[84] Oliver Mason and Mark Verwoerd. Graph theory and networks in biology. *IET Systems Biology*, 1(2):89–119, 2007.

[85] Colin McDiarmid and MichałPrzykucki. On the purity of minor-closed classes of graphs. *J. Combin. Theory Ser. B*, 135:295–318, 2019.

[86] Colin McDiarmid and David R. Wood. Edge-maximal graphs on surfaces. *Canad. J. Math.*, 70(4):925–942, 2018.

[87] Kurt Mehlhorn and Petra Mutzel. On the embedding phase of the Hopcroft and Tarjan planarity testing algorithm. *Algorithmica*, 16(2): 233–242, 1996.

[88] Bojan Mohar. Graph minors and graphs on surfaces. In J W P Hirschfeld, editor, *Survey in Cmbinatorics*, pages 145–163. Cambridge University Press, 2001.

[89] Bojan Mohar and Carsten Thomassen. *Graphs on Surfaces.* Johns Hopkins University Press, Baltimore, MD, 2001.

[90] Rajeev Motwani, Arvind Raghunathan, and Huzur Saran. Constructive results from graph minors: Linkless embeddings. In *29th Annual Symposium on Foundations of Computer Science, 1988*, pages 398–409. IEEE, 1988.

[91] Wendy Myrvold and Jennifer Woodcock. A large set of torus obstructions and how they were discovered. *The Electronic Journal of Combinatorics*, 25(1):1–16, 2018.

[92] Takao Nishizeki and Norishige Chiba. *Planar graphs: Theory and algorithms*, volume 32. Elsevier, 1988.

[93] Takao Nishizeki and Md S Rahman. *Planar Graph Drawing*. World Scientific, 2004.

[94] János Pach, Radoš Radoičić, and Géza Tóth. Tangled thrackles. In *Spanish Meeting on Computational Geometry*, pages 45–53. Springer, 2011.

[95] Georgios A Pavlopoulos, Anna-Lynn Wegener, and Reinhard Schneider. A survey of visualization tools for biological network analysis. *Biodata mining*, 1(1):1–11, 2008.

[96] Michael J. Pelsmajer, Marcus Schaefer, and Daniel Štefankovič. Removing even crossings. *Journal of Combinatorial Theory, Series B*, 97(4): 489 – 500, 2007.

[97] Michael J Pelsmajer, Marcus Schaefer, and Despina Stasi. Strong Hanani–Tutte on the projective plane. *SIAM Journal on Discrete Mathematics*, 23(3):1317–1323, 2009.

[98] Michael J Pelsmajer, Marcus Schaefer, and Daniel Štefankovič. Removing even crossings on surfaces. *European Journal of Combinatorics*, 30 (7):1704–1717, 2009.

[99] Amitai Perlstein and Rom Pinchasi. Generalized thrackles and geometric graphs in $\mathbb{R}^3$ with no pair of strongly avoiding edges. *Graphs and Combinatorics*, 24(4):373–389, 2008.

[100] Barry L Piazza, Richard D Ringeisen, and Sam K Stueckle. Subthrackleable graphs and four cycles. *Discrete Mathematics*, 127(1-3):265–276, 1994.

[101] Helen Purchase. Which aesthetic has the greatest effect on human understanding? In Giuseppe DiBattista, editor, *Graph Drawing*, pages 248–261. Springer, 1997.

[102] Helen Purchase. Effective information visualisation: a study of graph drawing aesthetics and algorithms. *Interacting with Computers*, 13(2): 147–162, 2000.

[103] Helen C Purchase. Effective information visualisation: a study of graph drawing aesthetics and algorithms. *Interacting with Computers*, 13(2): 147–162, 2000.

[104] Helen C. Purchase, David Carrington, and Jo-Anne Allder. Empirical evaluation of aesthetics-based graph layout. *Empirical Software Engineering*, 7(3):233–255, 2002.

[105] R D Ringeisen. Two old extremal graph drawing conjectures: Progress and perspectives. *Congressus Numerantium*, pages 91–104, 1996.

[106] Gerhard Ringel. The combinatorial map color theorem. *Journal of Graph Theory*, 1(2):141–155, 1977.

[107] Gerhard Ringel and John Williams Theodore Youngs. Solution of the Heawood map-coloring problem. *Proceedings of the National Academy of Sciences of the United States of America*, 60(2):438, 1968.

[108] Neil Robertson and Paul D Seymour. Graph minors. XIII. the disjoint paths problem. *Journal of Combinatorial Theory, Series B*, 63(1):65–110, 1995.

[109] Neil Robertson and Paul D Seymour. Graph minors. XX. Wagner's conjecture. *Journal of Combinatorial Theory, Series B*, 92(2):325–357, 2004.

[110] Neil Robertson, Paul D Seymour, and Robin Thomas. A survey of linkless embeddings. In Neil Robertson and Paul D. Seymour, editors, *Graph Structure Theory*, Contemporary Mathematics, pages 125–136. American Mathematical Society, 1991.

[111] Neil Robertson, Paul Seymour, and Robin Thomas. Sachs' linkless embedding conjecture. *Journal of Combinatorial Theory, Series B*, 64(2): 185–227, 1995.

[112] Andres J Ruiz-Vargas, Andrew Suk, and Csaba D Tóth. Disjoint edges in topological graphs and the tangled-thrackle conjecture. *European Journal of Combinatorics*, 51:398–406, 2016.

[113] Horst Sachs. On a spatial analogue of Kuratowski's theorem on planar graphs — an open problem. In M. Borowiecki, John W. Kennedy, and Maciej M. Sysło, editors, *Graph Theory*, pages 230–241, Berlin, Heidelberg, 1983. Springer Berlin Heidelberg.

[114] Huzur Saran. *Constructive results from graph minors: Linkless embeddings*. PhD thesis, Berkeley, 1989.

[115] Wei-Kuan Shih and Wen-Lian Hsu. A new planarity test. *Theoretical Computer Science*, 223(1-2):179–191, 1999.

[116] Kozo Sugiyama. *Graph Drawing and Applications for Software and Knowledge Engineers*, volume 11 of *Series on Software Engineering and Knowledge Engineering*. World Scientific, 2002.

[117] Maciej M. Sysło. Characterizations of outerplanar graphs. *Discrete Mathematics*, 26(1):47–53, 1979.

[118] Maciej M. Sysło. On some generalizations of outerplanar graphs: Results and open problems. In Gottfried Tinhofer and Gunther Schmidt, editors, *Graph-Theoretic Concepts in Computer Science*, pages 146–164. Springer Berlin, 1987.

[119] Roberto Tamassia. New layout techniques for entity-relationship diagrams. In Peter P Chen, editor, *Proceedings of the Fourth International Conference on Entity-Relationship Approach*, pages 304–311, Washington, DC, USA, 1985. IEEE Computer Society, IEEE Computer Society.

[120] Carsten Thomassen. The graph genus problem is NP-complete. *Journal of Algorithms*, 10(4):568–576, 1989.

[121] Carsten Thomassen. The Jordan-Schönflies theorem and the classification of surface. *American Mathematical Monthly*, pages 116–131, 1992.

[122] Ioannis G Tollis. Graph drawing and information visualization. *ACM Computing Surveys*, 28(4es):19, 1996.

[123] Klaus Truemper. On the delta-wye reduction for planar graphs. *Journal of Graph Theory*, 13(2):141–148, 1989.

[124] William T Tutte. Toward a theory of crossing numbers. *Journal of Combinatorial Theory*, 8(1):45–53, 1970.

[125] Hein van der Holst. Graphs and obstructions in four dimensions. *Journal of Combinatorial Theory, Series B*, 96(3):388–404, 2006.

[126] Hein van der Holst. A polynomial-time algorithm to find a linkless embedding of a graph. *Journal of Combinatorial Theory, Series B*, 99 (2):512–530, 2009.

[127] Hein van der Holst, László Lovász, and Alexander Schrijver. The Colin de Verdiere graph parameter. *Graph Theory and Computational Biology (Balatonlelle, 1996)*, pages 29–85, 1999.

[128] Béla von Kerékjártó. *Vorlesungen über Topologie*. Springer, 1923.

[129] Klaus Wagner. Bemerkungen zum Vierfarbenproblem. *Jahresbericht der Deutschen Mathematiker-Vereinigung*, 46:26–32, 1936.

[130] Klaus Wagner. *Graphentheorie*, volume 248 of *B. J. Hochschultaschenbücher*. Mannheim, 1970.

[131] Colin Ware, Helen Purchase, Linda Colpoys, and Matthew Mcgill. Cognitive measurements of graph aesthetics. *Information Visualization*, 1: 103–110, 2002.

[132] Colin Ware, Helen C Purchase, Linda Colpoys, and Matthew McGill. Cognitive measurements of graph aesthetics. *Information Visualization*, 1(2):103–110, 2002.

[133] Arthur T White. *Graphs of Groups on Surfaces: Interactions and Models*. Elsevier, 2001.

[134] Robin J Wilson. *Introduction to Graph Theory*. Longman, 1996.

[135] Douglas R Woodall. Thrackles and deadlock. In D J A Welsh, editor, *Combinatorial Mathematics and Its Applications*, volume 348, pages 335–348. Academic Press, New York, 1971.

[136] Shuji Yamada. An invariant of spatial graphs. *Journal of Graph Theory*, 13(5):537–551, 1989.

[137] Mihalis Yannakakis. Four pages are necessary and sufficient for planar graphs. In *Proceedings of the Eighteenth Annual ACM Symposium on Theory of Computing*, pages 104–108. ACM, 1986.

[138] Mihalis Yannakakis. Embedding planar graphs in four pages. *Journal of Computer and System Sciences*, 38(1):36–67, 1989.

[139] Yaming Yu. More forbidden minors for wye-delta-wye reducibility. *The Electronic Journal of Combinatorics*, 13(1):7, 2006.