Galois groups of chromatic polynomials

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December 7, 2009

Abstract

The chromatic polynomial $P(G, \lambda)$ gives the number of ways a graph G can be properly coloured in at most λ colours. In this article we give a summary of the Galois groups of all chromatic polynomials of strongly non-clique-separable graphs of order at most 10 and all chromatic polynomial of non-clique-separable θ-graphs of order at most 19. We then consider a number of operations on graphs and show that under some conditions these operations give a graph that has a chromatic polynomial with the same Galois group as the chromatic polynomial of the original graph. Now it is clear that if two polynomials have solvable Galois groups, then the product of these polynomials is also solvable. However, it is not usually the case that the sum of two polynomials with solvable Galois groups has a solvable Galois group. We give cases where $P(G', \lambda) = P(G, \lambda) + P(Q, \lambda)$ where $P(G, \lambda)$ and $P(G', \lambda)$ have the same solvable Galois group and $P(Q, \lambda)$ has the trivial Galois group. Although such a G and G' can easily be found when their Galois group is the trivial group, we give a family of graphs that satisfy this expression where the Galois group of $P(G, \lambda)$ and $P(G', \lambda)$ is S_2 and another family where the Galois group is $D(5)$. The latter family is particularly interesting as most of the chromatic polynomials of degree at most 10 have symmetric Galois groups and occurrences of nonsymmetric groups are quite rare.

Keywords: Chromatic polynomial; Galois group

Mathematics Subject Classification (2010): MSC 05C31; MSC 05C15; MSC 11R32

1 Introduction

The chromatic polynomial $P(G, \lambda)$ gives the number of proper colourings of a graph G in at most λ colours. It was originally introduced by Birkhoff [2] in an unsuccessful attempt to prove the four colour theorem by algebraic methods.

The study of *chromatic roots*, the roots of the chromatic polynomial, has attracted much attention not only in graph theory [23, 6], but also in the area of statistical mechanics where the zeros of the chromatic polynomial correspond to the location of possible phase transitions [17, 18, 19]. However, there has been little research into chromatic roots using algebraic means, except for the proof that the non-integer Beraha numbers $B_i = 2\cos(2\pi/i)$, $i \geq 5$, (excluding possibly B_{10}) are not chromatic roots [17, 21]. As a start of an algebraic study of the chromatic polynomial we considered the *chromatic factorisation* of a graph G, that is where the chromatic polynomial of G can be expressed as

$$P(G,\lambda) = \frac{P(H_1,\lambda)P(H_2,\lambda)}{P(K_r,\lambda)},$$

where $r \geq 0$ [12, 11, 10]. The graphs H_1 and H_2 are called the *chromatic factors* of G. Any *clique-separable* graph, that is a graph that can be obtained by identifying an r-clique in H_1 with an r-clique in H_2, has a chromatic factorisation. In [12, 11] we identified graphs that are neither clique-separable nor chromatically equivalent to any clique-separable graph that have a chromatic factorisation.

A basic algebraic property of any polynomial is the solvability of the polynomial. A polynomial in $\mathbb{Q}[\lambda]$ is *solvable* by radicals if its roots can be expressed in radicals over $\mathbb{Q}$. A polynomial is solvable if and only if its Galois group is solvable [8, Theorem 16.10 p. 150]. In this article we give some preliminary results on the Galois groups of chromatic polynomials.

First we present a summary of the Galois groups of all chromatic polynomials of all *strongly non-clique-separable graphs*, that is graphs that are neither clique-separable nor chromatically equivalent to any clique-separable graph, of order at most 10. Full details are available in [9].

A *θ-graph* is a graph that can be obtained from three paths, $(u_0, u_1, \ldots, u_a)$, $(v_0, v_1, \ldots, v_b)$ and $(w_0, w_1, \ldots, w_c)$, $a, b, c \geq 1$, by identifying vertices u_0, v_0 and w_0 and identifying vertices u_a, v_b and w_c. We call this graph $\theta_{a,b,c}$. It can be obtained by identifying the cycles C_{a+b} and C_{b+c} on a path of length b and in this sense θ-graphs can be considered to be close relatives of cycles. In Section 3.2 we show that the Galois group of the chromatic polynomial of the cycle on C_n is isomorphic to $(\mathbb{Z}/(n-1)\mathbb{Z})^*$. In order to compare the Galois groups of chromatic polynomials of θ-graphs to the chromatic polynomials of cycles, we found the Galois groups of chromatic polynomials of all θ-graphs of order at most 19 where $a, b, c \geq 2$. A summary of these is given in Section 2.1. We do not include the cases where any of the paths have length one, as these correspond to the chromatic polynomials of a 2-gluing of two cycles, or a single cycle, or K_2.

We then consider some basic operations that can be performed on a graph to produce a graph that has a chromatic polynomial with the same Galois group as the chromatic polynomial of the original graph. These include chromatic equivalence and graphs obtained by identifying an r-clique in a graph with an r-clique in a *quasi-chordal graph*, that is, a graph that has a chromatic polynomial with only integer roots.

Some of the most fundamental properties of the chromatic polynomial are the deletion/contraction and addition/identification relations. Using these relations, the chromatic polynomial can be expressed as a sum (or difference) of two other chromatic polynomials. In general the sum (or difference) of two solvable chromatic polynomials may not have the same Galois group as either of the summands. However, in some cases, particularly when one of the summands is a chromatic polynomial of a quasi-chordal graph, the resulting polynomial may have the same Galois group as the other summand. Trees, cycles and complete graphs are some trivial examples of graphs that have chromatic polynomials that can be expressed as the sum of two solvable chromatic polynomials. The structure of these graphs is well-defined and general expressions for their chromatic polynomials are known. Furthermore, the chromatic polynomials of complete graphs and trees have the trivial Galois group. We give two examples of families of graphs with solvable chromatic polynomials that have non-trivial Galois groups and can be expressed as the sum of two solvable chromatic polynomials where one of the summands is the chromatic polynomial of some quasichordal graph. These families have chromatic polynomials with Galois groups S_2 and $D(5)$ respectively. We also look at the cycle on six vertices which has Galois group $C(4)$ and demonstrate how we can find other graphs with chromatic polynomials that have this Galois group.

2 Galois group computation

Tables 1–10 give a list of all Galois groups of chromatic polynomials of strongly non-clique-separable graphs of order at most 10 and the number of times they occur. We give a list of all chromatic polynomials of strongly non-clique-separable graphs of order at most 8 with their Galois group in [9]. This list includes a list of graph numbers corresponding to the numbering in B. McKay's collection of simple connected graphs [7]. Any chromatic polynomial is solvable if and only if its chromatic factors are solvable. Thus, this list of chromatic polynomials and their associated Galois groups enables us to determine which chromatic polynomials of degree at most 10 are solvable.

PARI/GP 2.3.0 [1] was used to compute the Galois groups of all the irreducible non-linear factors of the chromatic polynomials. The tables in this paper give a list of the Galois groups of all irreducible non-linear factors of a given chromatic polynomial. The notation used is that given in Appendix A of [3]. In the case where the chromatic polynomial has a single irreducible non-linear factor, the Galois group of this factor is the Galois group of the entire polynomial. When the chromatic polynomial has more than one irreducible non-linear factor, we use Magma V2.14-11 [22] to compute its Galois group as Pari is not able to compute Galois groups of reducible polynomials. In this case we give the order and the generators of the groups found by Magma. When the chromatic polynomial has only one irreducible non-linear factor there is no entry given in the Generators column of the tables.

There are only three strongly non-clique-separable graphs of order at most 3: the complete graphs K_1, K_2 and K_3. The chromatic polynomials of these graphs all have the trivial Galois group.

Any chromatic polynomial that factorises into linear factors in $\mathbb{Z}[\lambda]$ has the trivial Galois group. The class of chordal graphs is a class of graphs that have only integer roots [16]. D'Atonna, Mereghetti and Zamparini [5] showed that there are 224 non-chordal graphs of at most 9 vertices that have no complex chromatic roots: 206 of these are quasi-chordal graphs.

Chromatic polynomials of degree at most 10 with Galois groups corresponding to each of the transitive permutation groups of even degree ≤ 6 and to each of the transitive permutation groups of odd degree ≤ 5, excluding A_3 and $C(5)$, were found. However, chromatic polynomials exist with Galois group A_3. An example of such a chromatic polynomial was found by Peter Cameron using the ring of cliques structure. We have also found an infinite family of graphs that have chromatic polynomials with Galois group A_3.

It is not suprising that some transitive permutation groups of degree ≥ 7 do not correspond to any Galois group of a chromatic polynomial of degree at most 10, as only chromatic polynomials of strongly non-clique-separable graphs of order ≥ 9 may have an irreducible septic factor.

The majority of chromatic polynomials of degree at most 10 have the symmetric group S_l, $l \in [1, n-2]$, as their Galois group. In Table 5 about 91% of chromatic polynomials of order 8 with a single nonlinear irreducible factor have Galois group S_l, $1 \leq l \leq 6$, and in the case of chromatic polynomials with more than one irreducible factor, the Galois groups of all but one of these factors are the symmetric groups. In Tables 6–7 over 94% of chromatic polynomials of order 9 with a single nonlinear irreducible factor have Galois group S_l, $1 \leq l \leq 7$, and in over 90% of the cases where the chromatic polynomial has more than one nonlinear factor, all its nonlinear factors have symmetric Galois groups. In Tables 8–10 almost 99% of chromatic polynomials of order 10 with a single nonlinear irreducible factor have Galois group S_l, $1 \leq l \leq 8$, and in about 90% of the cases where the chromatic polynomial has more than one nonlinear factor, all its nonlinear factors have symmetric Galois groups. Excluding the symmetric groups, the dihedral groups and cyclic groups appear to occur most frequently in these tables. In Section 3.3 we give some graphs that have chromatic polynomials with Galois group $D(5)$. Some of the cyclic groups are discussed in Section 3.2. First we give the Galois group of the cycle graph, C_n, and show that when $n = p + 1$ for some prime p, then the Galois group of the chromatic polynomial of the cycle graph is $C(p-1)$. We then consider graphs that are Galois equivalent to C_6. Although we have examples of chromatic polynomials with cyclic Galois groups of degree ≤ 4 and 6, we have no example of a chromatic polynomial with Galois group $C(5)$.

Galois group	# of chromatic polynomials	# of graphs
Trivial group	1	1
S_2	1	1

Table 1: Chromatic polynomials of strongly non-clique-separable graphs of order 4 and their Galois groups.

Galois group	# of chromatic polynomials	# of graphs
Trivial group	1	1
S_2	3	3
S_3	1	1

Table 2: Galois groups of chromatic polynomials of strongly non-clique-separable graphs of order 5.

Galois group	# of chromatic polynomials	# of graphs
Trivial group	1	1
S_2	6	7
S_3	9	10
$C(4) = 4$	1	1
S_4	3	3

Table 3: Galois groups of chromatic polynomials of strongly non-clique-separable graphs of order 6.

Galois group	Order	Generators	# of chromatic polynomials	# of graphs
Trivial group	1		1	1
S_2	2		16	23
S_3	6		30	41
$C(4) = 4$	4		3	6
$E(4) = 2[\times]2$	4		3	3
$D(4)$	8		14	20
S_4	24		31	46
S_5	120		6	6
S_2, S_2	2	$(1, 2)(3, 4)$	1	1
S_2, S_2	4	$(1, 2)$; $(3, 4)$	1	1
S_2, S_3	12	$(1, 2)$; $(3, 4, 5)$; $(3, 4)$	1	1

Table 4: Galois groups of chromatic polynomials of strongly non-clique-separable graphs of order 7.

Galois group	Order	Generators	# of chromatic polynomials	# of graphs
Trivial group	1		1	1
S_2	2		33	65
S_3	6		114	233
$C(4) = 4$	4		4	12
$E(4) = 2[\times]2$	4		8	36
$D(4)$	8		58	184
A_4	12		6	11
S_4	24		302	755
$D(5) = 5 : 2$	10		3	16
$F(5) = 5 : 4$	20		1	1
S_5	120		360	740
$C(6) = 6 = 3[\times]2$	6		1	1
$2S_4(6) = [2^3]S(3) = 2 \wr S(3)$	48		1	1
S_6	720		25	27
S_2, S_2	2	(1, 2)(3, 4)	3	5
S_2, S_2	4	(1, 2); (3, 4)	17	34
S_2, S_3	6	(3, 5, 4); (1, 2)(3, 4)	1	1
S_2, S_3	12	(1, 2); (3, 4, 5); (3, 4)	46	97
S_2, $D(4)$	16	(3, 4); (5, 7)(6, 8); (7, 8)	1	1
S_2, S_4	48	(1, 2); (3, 4, 5, 6); (3, 4)	2	2

Table 5: Galois groups of chromatic polynomials of strongly non-clique-separable graphs of order 8.

Galois group	Order	Generators	# of chromatic polynomials	# of graphs
Trivial Group	1		1	1
S_2	2		78	296
S_3	3		373	1069
$C(4) = 4$	4		10	49
$E(4) = 2[\times]2$	4		38	210
$D(4)$	8		319	1709
A_4	12		25	156
S_4	24		2152	10527
$D(5) = 5 : 2$	10		22	122
$F_5 = 5 : 4$	20		7	51
A_5	60		15	81
S_5	120		6385	29924
$C(6) = 6 = 3[\times]2$	6		3	19
$D_6(6) = [3]2$	6		2	7
$D(6) = S(3)[\times]2$	12		14	77
$F_{18}(6) = [3^2]2 = 3 \wr 2$	18		9	86
$2A_4(6) = [2^3]3 = 2 \wr 3$	24		2	7
$S_4(6d) = [2^2]S(3)$	24		2	2
$S_4(6c) = \frac{1}{2}[2^3]S(3)$	24		3	6
$2S_4(6) = [2^3]S(3) = 2 \wr S(3)$	48		114	591
$L(6) = PSL(2,5) = A_5(6)$	60		2	6
$F_{36}(6) : 2 = [S(3)^2]2 = S(3) \wr 2$	72		174	830
$L(6) : 2 = PGL(2,5) = S_5(6)$	120		1	3
S_6	720		5197	15895
S_7	5040		90	108

Table 6: Galois groups of chromatic polynomials of strongly non-clique-separable graphs of order 9 (continued in Table 7).

Galois group	Order	Generators	# of chromatic polynomials	# of graphs
S_2, S_2	2	(1, 2)	4	10
S_2, S_2	2	(1, 2)(3, 4)	8	126
S_2, S_2	4	(1, 2); (3, 4)	71	604
S_2, S_3	6	(3, 5, 4); (1, 2)(3, 4)	7	44
S_2, S_3	12	(1, 2); (3, 4, 5); (3, 4)	347	2157
S_2, $C(4) = 4$	8	(1, 2); (3, 4, 5, 6)	3	27
S_2, $E(4) = 2[\times]2$	4	(1, 2)(3, 4)(5, 6); (3, 5)(4, 6)	5	34
S_2, $E(4) = 2[\times]2$	8	(1, 2); (3, 4)(5, 6); (3, 5)(4, 6)	5	25
S_2, $D(4)$	8	(1, 2)(3, 4)(5, 6); (4, 5)	12	59
S_2, $D(4)$	8	(3, 4, 5, 6); (1, 2)(4, 5)	3	15
S_2, $D(4)$	8	(3, 5)(4, 6)	1	4
S_2, $D(4)$	16	(1, 2); (3, 4)(5, 6); (3, 5)	38	212
S_2, A_4	24	(1, 2); (3, 4)(5, 6); (3, 4, 5)	2	16
S_2, S_4	48	(1, 2); (3, 4, 5, 6); (3, 4)	218	951
S_2, S_5	240	(1, 2); (3, 4, 5, 6, 7); (3, 4)	2	2
S_3, S_3	6	(1, 2, 3); (1, 2)	2	2
S_3, S_3	36	(1, 2, 3); (1, 2); (4, 5, 6); (4, 5)	31	81
S_3, $C(4) = 4$	24	(1, 2, 3); (1, 2); (4, 5, 6, 7)	1	1
S_3, $D(4)$	48	(1, 2, 3); (1, 2); (4, 5)(6, 7); (5, 7)	1	1
S_3, S_4	144	(1, 2, 3); (1, 2); (4, 5, 6, 7); (4, 5)	3	3

Table 7: Galois groups of chromatic polynomials of strongly non-clique-separable graphs of order 9 (continued).

Galois group	Order	Generators	# of chromatic polynomials	# of graphs
Trivial Group	1		1	1
S_2	2		136	712
S_3	6		1309	6607
$C(4) = 4$	4		27	372
$E(4) = 2[\times]2$	4		113	1257
$D(4)$	8		1218	13076
A_4	12		75	932
S_4	24		12519	107635
$D(5) = 5 : 2$	10		129	1103
$F(5) = 5 : 4$	20		60	771
A_5	60		108	1273
S_5	120		79331	685931
$C(6) = 6 = 3[\times]2$	6		16	571
$D_6(6) = [3]2$	6		17	250
$D(6) = S(3)[\times]2$	12		143	2808
$A_4(6) = [2^2]3$	12		8	121
$F_{18}(6) = [3^2]2 = 3 \wr 2$	18		50	1171
$2A_4(6) = [2^3]3 = 2 \wr 3$	24		13	141
$S_4(6d) = [2^2]S(3)$	24		59	1041
$S_4(6c) = \frac{1}{2}[2^3]S(3)$	24		21	621
$F_{18}(6) : 2 = [\frac{1}{2}S(3)^2]2$	36		30	414
$F_{36}(6) = \frac{1}{2}[S(3)^2]2$	72		6	22
$2S_4(6) = [2^3]S(3) = 2 \wr S(3)$	48		1384	26642
$L(6) = PSL(2,5) = A_5(6)$	60		34	665
$F_{36}(6) : 2 = [S(3)^2]2 = S(3) \wr 2$	72		2683	40345
$L(6) : 2 = PGL(2,5) = S_5(6)$	120		57	662
A_6	360		26	211
S_6	720		257203	2395512
$D(7) = 7 : 2$	14		1	21
$F_{42}(7) = 7 : 6$	42		1	32
S_7	5040		138773	606609
$E(8) : D_6 = S(4)[\times]2$	48		1	1
$[2^4]S(4)$	384		3	5
$[S(4)^2]2$	1152		4	6
S_8	40320		554	697

Table 8: Galois groups of chromatic polynomials of strongly non-clique-separable graphs of order 10 (continued in Table 9).

Galois group	Order	Generators	# of chromatic polynomials	# of graphs
S_2, S_2	2	(1, 2)	7	41
S_2, S_2	2	(1, 2)(3, 4)	32	525
S_2, S_2	4	(1, 2); (3, 4)	368	4969
S_2, S_3	6	(3, 5, 4); (1, 2)(3, 4)	84	1042
S_2, S_3	12	(1, 2); (3, 4, 5); (3, 4)	2274	24878
S_2, $C(4) = 4$	8	(1, 2); (3, 4, 5, 6)	53	922
S_2, $E(4) = 2[\times]2$	4	(1, 2)(3, 5)(4, 6); (1, 2)(3, 4)(5, 6)	36	1048
S_2, $E(4) = 2[\times]2$	8	(1, 2); (3, 4)(5, 6); (3, 5)(4, 6)	88	950
S_2, $D(4)$	8	(1, 2)(3, 4)(5, 6); (4, 6)	86	1698
S_2, $D(4)$	8	(3, 4)(5, 6); (1, 2)(4, 6)	18	294
S_2, $D(4)$	8	(3, 5, 4, 6); (1, 2)(5, 6)	34	836
S_2, $D(4)$	16	(1, 2); (3, 4)(5, 6); (3, 6)	607	9189
S_2, A_4	24	(1, 2); (3, 4)(5, 6); (3, 4, 5)	49	514
S_2, S_4	24	(1, 2)(3, 4); (1, 2)(4, 5); (1, 2)(5, 6)	13	52
S_2, S_4	48	(1, 2); (3, 4, 5, 6); (3, 4)	4681	65110
S_2, $D(5) = 5 : 2$	20	(1, 2); (3, 4)(5, 6); (3, 5, 6, 4, 7)	30	350
S_2, $F(5) = 5 : 4$	40	(1, 2); (3, 5, 7, 6); (3, 7)(5, 6); (3, 7, 6, 4, 5)	3	20
S_2, A_5	120	(1, 2); (5, 6, 7); (3, 4, 5)	5	22
S_2, S_5	240	(1, 2); (3, 4, 5, 6, 7); (3, 4)	3283	18664
S_2, $C(6) = 6 = 3[\times]2$	6	(1, 2)(3, 5, 7, 6, 8, 4)	1	1
S_2, $D_6(6) = [3]2$	6	(1, 2)(3, 4)(5, 7)(6, 8); (1, 2)(3, 5)(4, 8)(6, 7)	1	1
S_2, $F_{36}(6) : 2 = [S(3)^2]2 = S(3) \wr 2$	144	(1, 2); (3, 4)(5, 6)(7, 8); (3, 5, 7); (3, 5)	1	2
S_2, $2S_4(6) = [2^3]S(3) = 2 \wr S(3)$	96	(1, 2); (3, 6, 4)(5, 7, 8); (3, 4)(5, 7); (4, 5)	1	1
S_2, S_6	1440	(1, 2); (3, 4, 5, 6, 7, 8); (3, 4)	21	26
S_2, S_2, S_2	2	(1, 2)(3, 4)	1	3
S_2, S_2, S_2	4	(1, 2); (3, 4)	13	91
S_2, S_2, S_2	4	(1, 2)(5, 6); (3, 4)	4	4
S_2, S_2, S_2	8	(1, 2); (3, 4); (5, 6)	14	30
S_2, S_2, S_3	6	(3, 5, 4); (1, 2)(3, 4)	2	2
S_2, S_2, S_3	12	(1, 2); (3, 4, 5); (3, 4)	37	161
S_2, S_2, S_3	12	(1, 2)(3, 4); (5, 7); (5, 6)	10	46
S_2, S_2, S_3	12	(1, 2); (3, 4)(5, 6); (5, 7, 6)	1	3
S_2, S_2, S_3	24	(1, 2); (3, 4); (5, 6, 7); (5, 6)	98	486

Table 9: Galois groups of chromatic polynomials of strongly non-clique-separable graphs of order 10 (continued in Table 10).

Galois group	Order	Generators	# of chromatic polynomials	# of graphs
S_3, S_3	6	(1, 2, 3); (1, 2)	4	33
S_3, S_3	6	(2, 3)(5, 6) and (1, 3, 2)(4, 6, 5)	12	76
S_3, S_3	36	(1, 2, 3); (1, 2); (4, 5, 6); (4, 5)	839	6119
S_3, $C(4) = 4$	24	(1, 2, 3); (1, 2); (4, 5, 6, 7)	39	423
S_3, $E(4) = 2[\times 2]$	12	(1, 2, 3); (2, 3)(4, 6)(5, 7); (2, 3)(4, 5)(6, 7)	3	4
S_3, $E(4) = 2[\times]2$	24	(1, 2, 3); (1, 2); (4, 5)(6, 7); (4, 6)(5, 7)	46	232
S_3, $D(4)$	24	(2, 3)(4, 5)(6, 7); (1, 2, 3); (5, 7)	4	7
S_3, $D(4)$	24	(4, 6, 5, 7); (1, 2)(6, 7); (2, 3)(6, 7)	2	8
S_3, $D(4)$	48	(1, 2, 3); (1, 2); (4, 6)(5, 7); (6, 7)	295	2508
S_3, A_4	72	(1, 2, 3); (1, 2); (4, 5)(6, 7); (4, 5, 6)	21	248
S_3, S_4	144	(1, 2, 3); (1, 2); (4, 5, 6, 7); (4, 5)	1080	7267
S_3, S_5	720	(1, 2, 3); (1, 2); (4, 5, 6, 7, 8); (4, 5)	1	1
S_4, $E(4) = 2[\times]2$	96	(1, 2, 3, 4); (1, 2); (5, 6)(7, 8); (5, 7)(6, 8)	1	1

Table 10: Galois groups of chromatic polynomials of strongly non-clique-separable graphs of order 10 (continued).

2.1 Galois groups of chromatic polynomials of θ-graphs.

We now look at the Galois groups of chromatic polynomials of θ-graphs of order at most 19. The chromatic polynomials of all but two of these graphs have an irreducible factor with a symmetric Galois group.

Tables 11 and 12 give a list of all Galois groups of chromatic polynomials of $\theta_{a,b,c}$-graphs of order at most 19 where $a, b, c \geq 2$. The Galois group of each irreducible non-linear factor of each chromatic polynomial was calculated using Magma.

Table 11 gives the number of θ-graphs with chromatic polynomials having a given Galois group where the chromatic polynomial has a single non-linear irreducible factor. These correspond to 75 of the 147 $\theta_{a,b,c}$-graphs, $a, b, c \geq 2$, of order $n \leq 19$. All but two of these chromatic polynomials have Galois group $S_{n-\chi(\theta_{a,b,c})}$ where $\chi(\theta_{a,b,c})$, the chromatic number of the graph, is 2 if a, b and c have the same parity and is 3 otherwise. There are two exceptions: the chromatic polynomials of $\theta_{2,3,3}$ and $\theta_{2,3,5}$ that have Galois group $D(4)$ and $2S_4(6)$ respectively.

Table 12 gives the number of θ-graphs with chromatic polynomials having more than one irreducible non-linear factor. These correspond to 72 of the 147 $\theta_{a,b,c}$-graphs, $a, b, c \geq 2$, of order $n \leq 19$. This table gives a list of the Galois groups of all the irreducible non-linear factors of each chromatic polynomial. In each case one of the Galois groups is the symmetric Galois group S_l for $3 \leq l \leq n - \chi(\theta_{a,b,c})$. The other non-linear irreducible factors are all factors of chromatic polynomials of cycle graphs. This is not surprising as by applying a single addition-identification relation to any θ-graph we can express the chromatic polynomial as

$$P(\theta_{a,b,c}, \lambda) = \frac{P(C_{a+1}, \lambda)P(C_{b+1}, \lambda)P(C_{c+1}, \lambda)}{P(K_2, \lambda)P(K_2, \lambda)} + \frac{P(C_a, \lambda)P(C_b, \lambda)P(C_c, \lambda)}{P(K_1, \lambda)P(K_1, \lambda)}. \tag{1}$$

We consider $C_2 \cong K_2$. It is clear from (1) that

Theorem 1. *The chromatic polynomial $P(C_{a+1}, \lambda)$ divides $P(\theta_{a,a+1,b}, \lambda)$ and $P(C_{a+1}, \lambda)P(C_{a+2}, \lambda)$ divides $P(C_{a,a+1,a+2}, \lambda)$.*

All graphs in the families $\theta_{a,a+1,b}$ and $\theta_{a,a+1,a+2}$ have chromatic polynomials that are divisible by chromatic polynomials of particular cycle graphs. In the remainder of this section we give some other families of θ-graphs where the chromatic polynomials of all graphs in the family are divisible by the chromatic polynomial of some cycle graph.

From (1) it is clear that if any of the following occur the chromatic polynomial of $P(\theta_{a,b,c}, \lambda)$ is divisible by the chromatic polynomial of some cycle graph:

- $P(C_{a+1}, \lambda)$ divides $P(C_b, \lambda)$ or $P(C_c, \lambda)$
- $P(C_{b+1}, \lambda)$ divides $P(C_a, \lambda)$ or $P(C_c, \lambda)$
- $P(C_{c+1}, \lambda)$ divides $P(C_a, \lambda)$ or $P(C_b, \lambda)$
- $P(C_a, \lambda)$ divides $P(C_{b+1}, \lambda)$ or $P(C_{c+1}, \lambda)$

- $P(C_b, \lambda)$ divides $P(C_{a+1}, \lambda)$ or $P(C_{c+1}, \lambda)$
- $P(C_c, \lambda)$ divides $P(C_{a+1}, \lambda)$ or $P(C_{b+1}, \lambda)$
- For some cycle C_d, the chromatic polynomial $P(C_d, \lambda)$ divides at least one of $P(C_a, \lambda)$, $P(C_b, \lambda)$ or $P(C_c, \lambda)$ and at least one of $P(C_{a+1}, \lambda)$, $P(C_{b+1}, \lambda)$ or $P(C_{c+1}, \lambda)$.

If more than one of these cases occur, $P(\theta_{a,b,c}, \lambda)$ is divisible by the chromatic polynomials of at least two cycle graphs.

In Theorems 3 and 5 we give families of cycle graphs that have chromatic polynomials that are divisible by chromatic polynomials of cycle graphs of lower degree. Many of the divisibility relations in Table 12 result from these. The results in these theorems can be shown using the following fact:

Fact 2. *The polynomial $x^s - 1$ divides $x^t - 1$ if and only if s divides t [15, Lemma, p.99], and the polynomial $x^s + 1$ divides $x^t + 1$ if and only if s divides t and t/s is odd [14, Lemma 2.1].*

However, it is interesting to demonstrate that these divisibility properties can be proved by applying some simple operations on a cycle graph.

Theorem 3. *The chromatic polynomial $P(C_n, \lambda)$ divides $P(C_{2^{k-1}(n-1)+1}, \lambda)$, $k \geq 1$.*

Proof The proof uses induction on the size of k.

If $k = 1$, we have $P(C_n, \lambda)$ divides $P(C_n, \lambda)$.

If $k = 2$,

$$\begin{aligned} P(C_{2n-1}, \lambda) &= \frac{P(C_n, \lambda)P(C_{n+1}, \lambda)}{P(K_2, \lambda)} + \frac{P(C_{n-1}, \lambda)P(C_n, \lambda)}{P(K_1, \lambda)} \\ &= P(C_n, \lambda)\left(\frac{P(C_{n+1}, \lambda)}{P(K_2, \lambda)} + \frac{P(C_{n-1}, \lambda)}{P(K_1, \lambda)}\right), \end{aligned} \tag{2}$$

so $P(C_n, \lambda)$ divides $P(C_{2n-1}, \lambda)$.

If $k > 2$, then

$$\begin{aligned} P(C_{2^{k-1}(n-1)+1}, \lambda) =& \frac{P(C_{2^{k-2}(n-1)+1}, \lambda)P(C_{2^{k-2}(n-1)+2}, \lambda)}{P(K_2, \lambda)} \\ &+ \frac{P(C_{2^{k-2}(n-1)}, \lambda)P(C_{2^{k-2}(n-1)+1}, \lambda)}{P(K_1, \lambda)} \\ =& P(C_{2^{k-2}(n-1)+1}, \lambda)\left(\frac{P(C_{2^{k-2}(n-1)+2}, \lambda)}{P(K_2, \lambda)} + \frac{P(C_{2^{k-2}(n-1)}, \lambda)}{P(K_1, \lambda)}\right) \end{aligned}$$

and so $P(C_{2^{k-2}(n-1)+1}, \lambda)$ divides $P(C_{2^{k-1}(n-1)+1}, \lambda)$. By induction $P(C_n, \lambda)$ divides $P(C_{2^{k-2}(n-1)+1}, \lambda)$ and thus $P(C_n, \lambda)$ divides $P(C_{2^{k-1}(n-1)+1}, \lambda)$. □

From this theorem we get the following result that lists some of the families of θ-graphs with chromatic polynomials that are divisible by chromatic polynomials of cycles.

Corollary 4. *The following chromatic polynomials of cycles divide these families of θ-graphs:*

- $P(C_{a+1}, \lambda)$ *divides* $P(\theta_{a,2^{k-1}a+1,b}, \lambda)$
- $P(C_a, \lambda)$ *divides* $P(\theta_{a,2^{k-1}(a-1),b}, \lambda)$

Theorem 5. *The chromatic polynomial* $P(C_n, \lambda)$ *divides* $P(C_{3n-2}, \lambda)$*.*

Proof The chromatic polynomial of the cycle C_{3n-2} can be expressed as

$$P(C_{3n-2}, \lambda) = \frac{P(C_{2n-1}, \lambda)P(C_{n+1}, \lambda)}{P(K_2, \lambda)} + \frac{P(C_{2n-2}, \lambda)P(C_n, \lambda)}{P(K_1, \lambda)} \tag{3}$$

and by Theorem 3 $P(C_n, \lambda)$ divides $P(C_{2n-1}, \lambda)$, so $P(C_n, \lambda)$ divides $P(C_{3n-2}, \lambda)$. □

Corollary 6. *The chromatic polynomial* $P(C_a, \lambda)$ *divides* $P(\theta_{a,3a+1,b}, \lambda)$ *and* $P(C_a, \lambda)$ *divides* $P(\theta_{a,3(a-1),b}, \lambda)$*.*

In Table 13 we give the Galois groups of the chromatic polynomials of cycles of order $4 \leq n \leq 9$ and the θ-graphs or order at most 19 (excluding $\theta_{2,7,9}$) that have chromatic polynomials that can be expressed as

$$P(C_n, \lambda)P(\lambda) \text{ or } P(C_n, \lambda)(\lambda - 2)P(\lambda) \tag{4}$$

where $P(\lambda)$ is an irreducible factor that has the symmetric Galois group. We give the expression for each of these θ-graphs given in Corollaries 1, 4 and 6. The triplet, a, b, c, is considered to be unordered. For example, $2, 4, 5$ corresponds to the graph $\theta_{a,a+1,b}$ and $2, 3, 7$ corresponds to the graph $\theta_{a,2^{k-1}a+1,b}$. The graph $\theta_{2,7,9}$ is not included in Table 13. The chromatic polynomial

$$\begin{aligned} P(\theta_{2,7,9}, \lambda) = & \frac{P(C_3, \lambda)P(C_8, \lambda)P(C_{10}, \lambda)}{P(K_2, \lambda)P(K_2, \lambda)} \\ & + \frac{P(C_2, \lambda)P(C_7, \lambda)P(C_9, \lambda)}{P(K_1, \lambda)P(K_1, \lambda)}. \end{aligned} \tag{5}$$

In this case $P(C_4, \lambda)$ divides $P(\theta_{2,7,9}, \lambda)$ as $P(C_4, \lambda)$ divides both $P(C_{10}, \lambda)$ (from Theorem 5) and $P(C_9, \lambda)$ (from Theorem 3). Cases where the chromatic polynomial of the θ-graph is divisible by the chromatic polynomials of two cycles, C_{n_1} and C_{n_2}, of order ≥ 4 are not included in Table 13. However, all of these cases can be explained by the principles presented in Theorems 1, 3 and 5. For example, the chromatic polynomial of the graph $\theta_{5,7,8}$ can be shown to be divisible by $P(C_8, \lambda)$ by Theorem 1 and can be shown to be divisible by $P(C_5, \lambda)$ by Theorem 3 and Corollary 4. More generally

Theorem 7. *The chromatic polynomials* $P(C_a, \lambda)$ *and* $P(C_{2^{k-1}(a-1)}, \lambda)$ *divide the* $P(\theta_{a,2^{k-1}(a-1)-1,2^{k-1}(a-1)}, \lambda)$*.*

Proof By Theorem 1 $P(C_a, \lambda)$ divides $P(\theta_{a,2^{k-1}(a-1)-1,2^{k-1}(a-1)}, \lambda)$ and by Corollary 4 $P(C_{2^{k-1}(a-1)}, \lambda)$ divides $P(\theta_{a,2^{k-1}(a-1)-1,2^{k-1}(a-1)}, \lambda)$. □

In a similar manner all other cases can be explained by a combination of these theorems.

In Section 3.2 we look at the Galois groups of cycles and show that the Galois group of the chromatic polynomial of the cycle C_n is isomorphic to $(\mathbb{Z}/(n-1)\mathbb{Z})^*$

and is thus solvable. Every chromatic polynomial in Table 12 has only one irreducible factor that is not the factor of some cycle graph. This factor has the symmetric Galois group, which leads to the conjecture:

Conjecture 8. *The chromatic polynomial $P(\theta_{a,b,c}, \lambda)$, excluding $P(\theta_{2,3,3}, \lambda)$ and $P(\theta_{2,3,5}, \lambda)$, can be expressed*

$$P(\theta_{a,b,c}, \lambda) = \prod_{i=0}^{j} \frac{P(C_{s_i}, \lambda)}{P(K_{r_i}, \lambda)} P(\lambda) \tag{6}$$

where $j > 0$, $s_i \geq 2$, $r_i > 0$ and $P(\lambda)$ is an irreducible polynomial with Galois group S_{n-l} where $\chi(\theta_{a,b,c}) \leq l < j$.

Galois groups	# of graphs ($\chi = 2$)	# of graphs ($\chi = 3$)	Total # of graphs	$\theta_{a,b,c}$
S_3	1	1	2	2, 2, 2; 2, 2, 3
S_4	0	0	0	
S_5	1	1	2	2, 2, 4; 2, 2, 5
S_6	1	0	1	3, 3, 3
S_7	2	2	4	2, 2, 6; 2, 4, 4; 2, 2, 7; 2, 3, 6
S_8	1	2	3	3, 3, 5; 2, 5, 5; 3, 3, 6
S_9	2	3	5	2, 2, 8; 4, 4, 4; 2, 2, 9; 2, 3, 8; 2, 4, 7
S_{10}	1	3	4	3, 5, 5; 2, 3, 9; 2, 5, 7; 3, 3, 8
S_{11}	3	3	6	2, 2, 10; 2, 4, 8; 2, 6, 6; 2, 2, 11; 3, 6, 6; 4, 4, 7
S_{12}	2	3	5	3, 3, 9; 5, 5, 5; 2, 3, 11; 2, 5, 9; 2, 7, 7
S_{13}	4	6	10	2, 2, 12; 2, 4, 10; 2, 6, 8; 4, 4, 8; 2, 2, 13; 2, 3, 12; 2, 4, 11; 2, 5, 10; 2, 6, 9; 3, 6, 8
S_{14}	3	3	6	3, 3, 11; 3, 5, 9; 5, 5, 7; 3, 3, 12; 3, 6, 9; 4, 7, 7
S_{15}	4	6	10	2, 2, 14; 2, 8, 8; 4, 4, 10; 6, 6, 6; 2, 2, 15; 2, 3, 14; 2, 6, 11; 2, 7, 10; 3, 8, 8; 4, 4, 11
S_{16}	2	7	9	5, 5, 9; 5, 7, 7; 2, 3, 15; 2, 5, 13; 2, 7, 11; 2, 9, 9; 3, 3, 14; 3, 6, 11; 5, 5, 10
S_{17}	6	0	6	2, 2, 16; 2, 4, 14; 2, 6, 12; 2, 8, 10; 4, 8, 8; 6, 6, 8
$D(4)$	0	1	1	2, 3, 3
$2S_4(6)$ $= [2^3]S(3)$ $= 2 \wr S(3)$	0	1	1	2, 3, 5

Table 11: Galois groups of chromatic polynomials of θ-graphs of order at most 19, where there is exactly one non-linear factor.

Galois groups	# of graphs ($\chi = 2$)	# of graphs ($\chi = 3$)	Total # of graphs	$\theta_{a,b,c}$
S_2, S_3	0	1	1	$2,3,4$
S_2, S_4	0	1	1	$3,3,4$
S_2, S_5	0	2	2	$2,4,5;\ 3,4,4$
S_2, S_6	0	1	1	$2,3,7$
S_2, S_7	1	2	3	$2,4,6;\ 3,4,6;\ 4,4,5$
S_2, S_8	1	2	3	$3,3,7;\ 3,4,7\ 4,5,5$
S_2, S_9	1	3	4	$4,4,6;\ 2,3,10;\ 2,5,8;\ 3,4,8$
S_2, S_{10}	1	3	4	$3,5,7;\ 3,3,10;\ 3,5,8;\ 4,5,7$
S_2, S_{11}	1	2	3	$4,6,6;\ 3,4,10;\ 4,5,8$
S_2, S_{12}	1	5	6	$3,7,7;\ 2,3,13;\ 2,7,9;\ 3,4,11;\ 3,5,10;\ 5,5,8$
S_2, S_{13}	2	4	6	$2,4,12;\ 4,6,8;\ 2,4,13;\ 2,5,12;\ 3,4,12;\ 4,5,10$
S_2, S_{14}	2	2	4	$3,3,13;\ 3,7,9;\ 3,5,12;\ 3,7,10$
S_2, S_{15}	1	0	1	$4,4,12$
S_2, S_2, S_4	0	1	1	$3,4,5$
S_2, S_2, S_7	0	2	2	$2,4,9;\ 2,6,7$
S_2, S_2, S_8	0	2	2	$3,4,9;\ 3,6,7$
S_2, S_2, S_9	0	2	2	$4,4,9;\ 4,6,7$
S_2, S_2, S_{10}	0	1	1	$4,5,9$
S_2, S_2, S_{11}	0	2	2	$4,6,9;\ 6,6,7$
S_2, S_2, S_{12}	0	3	3	$3,4,13;\ 4,7,9;\ 6,7,7$
$C(4), S_5$	0	1	1	$2,5,6$
$C(4), S_6$	0	1	1	$3,5,6$
$C(4), S_8$	0	1	1	$5,5,6$
$C(4), S_9$	0	1	1	$5,6,6$
$C(4), S_{10}$	0	1	1	$2,5,11$
$C(4), S_{12}$	1	1	2	$3,5,11;\ 5,6,9$
$C(4), S_2, S_9$	1	3	4	$2,6,10;\ 3,6,10;\ 5,6,8;\ 2,8,9$
$C(4), S_2, S_{10}$	0	1	1	$4,5,11$
$C(4), S_2, S_{11}$	1	0	1	$4,6,10$
$C(4), S_2, S_2, S_3$	0	1	1	$4,5,6$
$C(4), S_2, S_2, S_6$	0	1	1	$5,6,7$
$C(6), S_7$	0	1	1	$2,7,8$
$C(6), S_9$	0	1	1	$4,7,8$
$C(6), S_2, S_6$	0	1	1	$3,7,8$
$C(6), S_2, S_8$	0	1	1	$5,7,8$
$E(4) = 2[\times]2, S_2, S_{10}$	0	1	1	$3,8,9;$

Table 12: Galois groups of chromatic polynomials of θ-graphs of order at most 19, where there is more than one non-linear factor.

n	$P(C_n,\lambda)$	Galois group(s)	$\theta_{a,a+1,b}$	$\theta_{a,2^{k-1}a+1,b}$	$\theta_{a,2^{k-1}(a-1),b}$	$\theta_{a,3a+1,b}$	$\theta_{a,3(a-1),b}$
4	$\lambda(\lambda-1)(\lambda^2-3\lambda+4)$	S_2	2,3,4; 3,3,4; 3,4,4 3,4,6; 3,4,7; 3,4,8 3,4,10; 3,4,11	($k=2$): 2,3,7; 3,3,7 ;3,5,7; 3,7,7; 3,7,9; 3,7,10 ($k=3$): 2,3,13; 3,3,13	($k=2$): 2,4,6; 4,4,6 4,6,6; 4,6,8 ($k=3$): 2,4,12; 3,4,12 4,4,12	2,3,10 3,3,10 2,4,13; 3,5,10	2,4,9
5	$\lambda(\lambda-1)(\lambda-2)(\lambda^2-2\lambda+2)$	S_2	2,4,5; 4,4,5 4,5,5: 4,5,7 4,5,8; 4,5,10		($k=2$): 2,5,8; 3,5,8 5,5,8		2,5,12; 3,5,12
6	$\lambda(\lambda-1)(\lambda^4-5\lambda^3+10\lambda^2-10\lambda+5)$	$C(4)$	2,5,6; 3,5,6; 5,5,6 5,6,6; 5,6,9	($k=2$): 2,5,11; 3,5,11			
7	$\lambda(\lambda-1)(\lambda-2)(\lambda^2-3\lambda+3)(\lambda^2-\lambda+1)$	S_2, S_2	2,6,7; 3,6,7; 4,6,7 6,6,7; 6,7,7;				
8	$\lambda(\lambda-1)$ $\times(\lambda^6-7\lambda^5+21\lambda^4-35\lambda^3+35\lambda^2-21\lambda+7)$	$C(6)$	2,7,8; 4,7,8; 5,7,8				
9	$\lambda(\lambda-1)(\lambda-2)(\lambda^2-2\lambda+2)$ $\times(\lambda^4-4\lambda^3+6\lambda^2-4\lambda+2)$	S_2, $E(4)$	2,8,9; 3,8,9				

Table 13: Galois groups of chromatic polynomials of θ-graphs of order at most 19, where $P(C_n,\lambda)$, $n\geq 4$ divides $P(\theta_{a,b,c},\lambda)$.

3 Some Galois groups of chromatic polynomials

In this section we will consider some chromatic polynomials with Galois groups $C(4)$, S_2 and $D(5)$. In Section 3.1 we consider some ways that a graph H can be obtained from a graph G such that $P(G,\lambda)$ and $P(H,\lambda)$ have the same Galois group. If $P(G,\lambda)$ and $P(H,\lambda)$ have the same Galois group, we say G and H are *Galois equivalent* and denote this $G \sim_{\mathcal{G}} H$. In Section 3.2 we use some of these methods to obtain graphs that are Galois equivalent. These graphs have chromatic polynomials with Galois group $C(4)$. In Section 3.3 we give two examples of where the sum of two solvable chromatic polynomials is a solvable chromatic polynomial where only one chromatic polynomial, one of the summands, has the trivial Galois group. The first example gives us a family of graphs that are Galois equivalent with chromatic polynomials with Galois group S_2, and the second gives us a family of graphs that have chromatic polynomials with Galois group $D(5)$.

3.1 Operations on graphs

The simplest case of two graphs that are Galois equivalent, but not isomorphic, is where the graphs are chromatically equivalent but not isomorphic.

Fact 9. *If $G \sim H$, then $G \sim_{\mathcal{G}} H$. Furthermore, if $G \sim H$ and $P(G,\lambda)$ is solvable, then $P(H,\lambda)$ is solvable.*

The irreducible non-linear factors of a chromatic polynomial fully determine its Galois group. If each of these factors is solvable, then the Galois group of the chromatic polynomial is solvable. Any graph G that is an r-gluing of graphs H and J has the chromatic polynomial

$$P(G,\lambda) = \frac{P(H,\lambda)P(J,\lambda)}{P(K_r,\lambda)} \tag{7}$$

so $P(G,\lambda)$ can be considered the product of the irreducible factors of $P(H,\lambda)$ and $P(J,\lambda)$ (excluding some repeated linear factors). It follows that

Fact 10. *If graph G that is an r-gluing of graphs H and J, then $P(G,\lambda)$ is solvable if and only if $P(H,\lambda)$ and $P(J,\lambda)$ are solvable*

and

Fact 11. *If G is an r-gluing of graphs H and J where J is a quasi-chordal graph, then $G \sim_{\mathcal{G}} H$. Furthermore if $P(H,\lambda)$ is solvable, then $P(G,\lambda)$ is solvable.*

We say G is the *join* of H and K_r, denoted $G = H \otimes K_r$, if $V(G) = V(H) \cup V(K_r)$ and $E(G) = E(H) \cup E(K_r) \cup \{uv : u \in V(H) \text{ and } v \in V(K_r)\}$. Now,

$$P(H \otimes K_r, \lambda) = P(K_r,\lambda)P(H,\lambda - r) \tag{8}$$

so we have

Theorem 12. *If $G = H \otimes K_r$, $r \geq 1$, then $G \sim_{\mathcal{G}} H$.*

Proof The chromatic polynomial of G is

$$\begin{aligned} P(G,\lambda) &= P(K_r,\lambda)P(H,\lambda - r) \\ &= \lambda(\lambda-1)\dots(\lambda-r+1)\Pi_{i=1}^{n}(\lambda - r_i - r) \end{aligned}$$

where the r_i are the roots of $P(G,\lambda)$. Thus it is clear that $P(G,\lambda)$ and $P(H,\lambda)$ have the same Galois group. □

Corollary 13. *If $P(H,\lambda)$ is solvable, then $P(G,\lambda) = P(H \otimes K_r, \lambda)$, $r \geq 1$, is solvable.*

In Section 3.2 we will give examples of chromatic polynomials with Galois group $C(4)$ and show that these graphs are Galois equivalent by using some of the operations presented in this section.

3.2 Galois equivalent graphs: products and joins

First we will consider the chromatic polynomial of the cycle C_n on n vertices.

Lemma 14. *The chromatic polynomial of the cycle C_n on n vertices is solvable.*

Proof Now the graph C_n has chromatic polynomial

$$P(C_n,\lambda) = (\lambda-1)((\lambda-1)^{n-1} - (-1)^{n-1}). \tag{9}$$

Each of the roots of $((\lambda-1)^{n-1} - (-1)^{n-1})$ is 1+ some root of unity. As the roots of unity can be expressed in radicals [20, Corollary 39, p. 257], $P(C_n,\lambda)$ has a solvable Galois group. □

Lemma 15. *The Galois group of $P(C_n,\lambda)$ is isomorphic to $(\mathbb{Z}/(n-1)\mathbb{Z})^*$, the multiplicative group of units of $\mathbb{Z}/(n-1)\mathbb{Z}$, and the order of this group is $\phi(n-1)$, where ϕ is Euler's totient function.*

Proof. The Galois group of $\lambda^{n-1} - 1 \in \mathbb{Q}[\lambda]$ is isomorphic to $(\mathbb{Z}/(n-1)\mathbb{Z})^*$ and has order $\phi(n-1)$ [8, Corollary 7.8, p.75]. The splitting field of $P(C_n,\lambda)$ is isomorphic to the splitting field of $\lambda^{n-1} - 1$. □

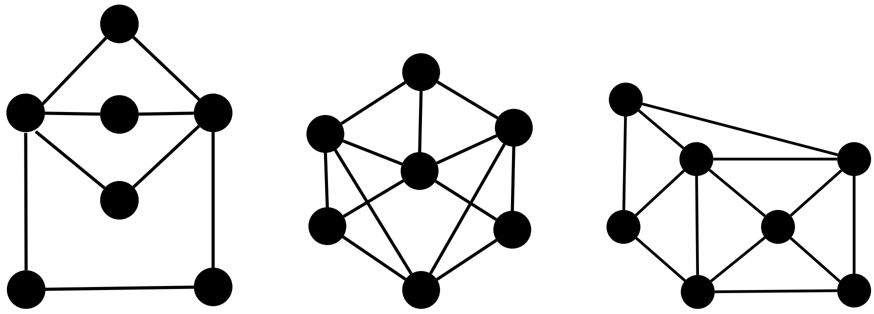
Corollary 16. *The chromatic polynomial of the cycle C_{p+1} where p is prime has Galois group $C(p-1)$.*

It is clear from Theorem 12 that any join of C_n and some complete graph is Galois equivalent to C_n, so

Theorem 17. *The Galois group of the chromatic polynomial of the graph $C_n \otimes K_r$, $r \geq 1$, is isomorphic to $(\mathbb{Z}/(n-1)\mathbb{Z})^*$. When $n = p+1$ for some p then the Galois group is $C(p-1)$.*

When $r = 1$ we have

Theorem 18. *The wheel with n spokes, W_n, is Galois equivalent to C_n and the Galois group of $P(W_n,\lambda)$ is $(\mathbb{Z}/(n-1)\mathbb{Z})^*$.*



(a) Graph H_1. (b) Graph H_2. (c) Graph H_3.

Figure 1: Galois equivalent graphs.

In particular we will consider the case when $n = 6$. The chromatic polynomial of C_6 is

$$\begin{aligned} P(C_6, \lambda) &= (\lambda - 1)((\lambda - 1)^5 + 1) \\ &= \lambda(\lambda - 1)(\lambda^4 - 5\lambda^3 + 10\lambda^2 - 10\lambda + 5) \end{aligned} \tag{10}$$

which by Corollary 16 has Galois group $C(4)$. By Theorem 18, C_6 is Galois equivalent to W_6. We now consider some graphs that cannot be obtained by gluing or joining some graph to the graph C_6.

The graphs given in Figure 1 have the following chromatic polynomials:

$$P(H_1, \lambda) = \lambda(\lambda - 1)(\lambda - 2)(\lambda^4 - 6\lambda^3 + 16\lambda^2 - 21\lambda + 11) \tag{11}$$

$$P(H_2, \lambda) = \lambda(\lambda - 1)(\lambda - 2)(\lambda^4 - 10\lambda^3 + 40\lambda^2 - 75\lambda + 55) \tag{12}$$

$$P(H_3, \lambda) = \lambda(\lambda - 1)(\lambda - 2)(\lambda^4 - 10\lambda^3 + 40\lambda^2 - 75\lambda + 55) \tag{13}$$

which all have Galois group $C(4)$, and so are Galois equivalent. Now it is clear from Fact 9 that as H_2 and H_3 are chromatically equivalent,

Theorem 19. *The graphs H_2 and H_3 given in Figure 1 are Galois equivalent.*

However, it is not obvious (other than by computation of the Galois groups) that the graphs H_1 and H_2 are Galois equivalent. But it can be shown using the operations presented in Section 3.1 that

Theorem 20. *The graphs H_1 and H_2 given in Figure 1 are Galois equivalent.*

Proof Now $P(H_1, \lambda)$ has Galois group $C(4)$. From (8) and (11) the graph $H_1 \otimes K_1$ has chromatic polynomial

$$\begin{aligned} P(H_1 \otimes K_1, \lambda) =& \lambda P(H_1, \lambda - 1) \\ =& \lambda(\lambda - 1)(\lambda - 2)(\lambda - 3) \\ & \times ((\lambda - 1)^4 - 6(\lambda - 1)^3 + 16(\lambda - 1)^2 - 21(\lambda - 1) + 11) \\ =& (\lambda - 1)(\lambda - 2)(\lambda - 3)(\lambda^4 - 10\lambda^3 + 40\lambda^2 - 75\lambda + 55) \end{aligned} \tag{14}$$

which by Theorem 12 has Galois group $C(4)$.

But this is chromatically equivalent to the graph G obtained by a 3-gluing of K_4 and graph H_2. This graph has chromatic polynomial

$$\begin{aligned} P(G,\lambda) &= \frac{P(H_2,\lambda)P(K_4,\lambda)}{P(K_3,\lambda)} \\ &= \lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda^4-10\lambda^3+40\lambda^2-75\lambda+55). \end{aligned} \tag{15}$$

Thus, by Fact 9, $H_1 \sim_{\mathcal{G}} G$ and thus $P(G,\lambda)$ has Galois group $C(4)$.

But G is a 3-gluing of K_4 and H_2, so by Fact 11, $G \sim_{\mathcal{G}} H_2$. Thus $P(H_2,\lambda)$ has Galois group $C(4)$: the same Galois group as $P(H_1,\lambda)$. □

Furthermore, as $H_2 \sim H_3$ it follows that

Corollary 21. *The graphs H_1 and H_3 given in Figure 1 are Galois equivalent.*

In summary, in Theorem 20 we gave a sequence of steps involving products, joins and chromatic equivalence to show the Galois equivalence of two graphs, H_1 and H_2. We showed that

$$H_1 \sim_{\mathcal{G}} (H_1 \otimes K_1) \sim_{\mathcal{G}} G \sim_{\mathcal{G}} H_2$$

and as $P(H_1,\lambda)$ has Galois group $C(4)$ so do the chromatic polynomials of these Galois equivalent graphs.

3.3 Galois equivalent graphs: sums and differences

As we have seen, the Galois group of the chromatic polynomial of a graph G is the same as that of the chromatic polynomial of a graph that is

- an r-gluing of G and a quasi-chordal graph, or
- a join of G and K_r, $r \geq 1$, or
- chromatically equivalent to G.

We now consider some cases where there exist Galois equivalent graphs, G and G', where $P(G',\lambda) = P(G,\lambda) \pm P(J,\lambda)$ for some graph J. This expression is easily satisfied if G belongs to some simple family of graphs such as trees, paths or complete graphs. However, in general we would not expect these graphs to be Galois equivalent. However, in this section we give an example of a sequence of graphs $G_0, G_1, \ldots, G_k$, $k \in \mathbb{N}$, where

$$P(G_i,\lambda) = P(G_{i-1},\lambda) \pm P(J,\lambda) \tag{16}$$

for some graph J where $G_i \sim_{\mathcal{G}} G_{i-1}$, $i > 0$. The graphs in this sequence have chromatic polynomials with Galois group S_2. We then give a second sequence of graphs satisfying (16) where each $P(G_i,\lambda)$ can be shown to have Galois group $D(5)$.

3.3.1 Chromatic polynomials with Galois group S_2

Let $H_{r,2}$ be the graph obtained by replacing an edge uv in K_r, $r \geq 3$, by a path u, w, v where w is a new vertex of degree 2. Now

$$\begin{aligned} P(H_{r,2},\lambda) &= P(H_{r,2}+uv,\lambda) + P(H_{r,2}/uv,\lambda) \\ &= \frac{P(K_r,\lambda)P(K_3,\lambda)}{P(K_2,\lambda)} + \frac{P(K_2,\lambda)P(K_{r-1},\lambda)}{P(K_1,\lambda)} \\ &= (\lambda-2)P(K_r,\lambda) + (\lambda-1)P(K_{r-1},\lambda) \\ &= P(K_{r-1},\lambda)((\lambda-2)(\lambda-r+1)+(\lambda-1)) \\ &= P(K_{r-1},\lambda)(\lambda^2 - r\lambda + (2r-3)). \end{aligned} \tag{17}$$

The Galois group of this polynomial is the Galois group of the quadratic $\lambda^2 - r\lambda + (2r-3)$. We use the following property of Pythagorean triples to show that the Galois group of this quadratic is S_2 when $r \neq 6$.

Fact 22. *There is no Pythagorean triple $(2, b, c)$, $b, c \in \mathbb{N}$.*

Corollary 23. *The Galois group of $P(H_{r,2},\lambda)$ is S_2 if $r \neq 6$. The Galois group of $P(H_{6,2},\lambda)$ is the trivial Galois group.*

Proof The Galois group of $P(H_{r,2},\lambda)$ is S_2 unless the discriminant $\Delta = r^2 - 8r + 12 = (r-4)^2 - 4$ is a square in $\mathbb{N}$ or $\Delta = 0$.

Suppose, in order to obtain a contradiction, that $\Delta \neq 0$ is a square in $\mathbb{N}$. Then $\Delta = (r-4)^2 - 4 = q^2$, $q \in \mathbb{N}$. But then $(2, q, (r-4))$ is a Pythagorean triple, a contradiction.

Now if the discriminant $\Delta = r^2 - 8r + 12 = (r-4)^2 - 4 = 0$, $r \geq 3$, then $r = 6$ and the Galois group is the trivial group.

If $r \neq 6$, the discriminant is neither 0 nor a square in $\mathbb{N}$ and the Galois group of $P(H_{r,2},\lambda)$ is S_2. $\square$

We now consider a family of graphs $\mathcal{G} = \{G_{r,k}\}$ where $G_{r,0} \cong H_{r,2}$ and $G_{r,k}$ is the graph with $V(G_{r,k}) = V(H_{r,2})$ and $E(G_{r,k}) = E(H_{r,2}) \cup \{(w, a_j) : 1 \leq j \leq k,$ and distinct $a_j \in V(H_{r,2}) \setminus \{u, v, w\}\}$ where $1 \leq k \leq r-2$ and $r \geq 3$.

Theorem 24. *The graph $G_{r,k} \in \mathcal{G}$ has chromatic polynomial $P(G_{r,k},\lambda) = P(K_{r-1},\lambda)(\lambda^2 - (r+k)\lambda + (2r-3+k(r-2)))$.*

Proof Now

$$\begin{aligned} P(G_{r,k},\lambda) &= P(G_{r,k}+uv,\lambda) + P(G_{r,k}/uv,\lambda) \\ &= \frac{P(K_{k+3},\lambda)P(K_r,\lambda)}{P(K_{k+2},\lambda)} + \frac{P(K_{k+2},\lambda)P(K_{r-1},\lambda)}{P(K_{k+1},\lambda)} \\ &= (\lambda-k-2)P(K_r,\lambda) + (\lambda-k-1)P(K_{r-1},\lambda) \\ &= (\lambda-k-2)(\lambda-r+1)P(K_{r-1},\lambda) + (\lambda-k-1)P(K_{r-1},\lambda) \\ &= P(K_{r-1},\lambda)\left((\lambda-k-2)(\lambda-r+1)+(\lambda-k-1)\right) \\ &= P(K_{r-1},\lambda)(\lambda^2-(r+k)\lambda+(2r-3+k(r-2))). \qquad \square \end{aligned}$$

Theorem 25. *If $G_{r,k} \in \mathcal{G}$, then $P(G_{r,k},\lambda)$ has Galois group S_2 if $r \notin \{k+2, k+6\}$. If $r \in \{k+2, k+6\}$, then $P(G_{r,k},\lambda)$ has the trivial Galois group.*

Proof From Theorem 24

$$P(G_{r,k},\lambda) = P(K_{r-1},\lambda)(\lambda^2 - (r+k)\lambda + (2r-3+k(r-2))). \qquad (18)$$

The Galois group of (18) is determined by the discriminant of $(\lambda^2 - (r+k)\lambda + (2r-3+k(r-2))$ which is

$$\begin{aligned}\Delta &= r^2 - 2(k+4)r + k^2 + 8k + 12 \\ &= r^2 - 2(k+4)r + (k+4)^2 - 4 \\ &= (r-(k+4))^2 - 4. \qquad (19)\end{aligned}$$

Now (18) has a reducible quadratic factor (and thus a trivial Galois group) if $\Delta = 0$ or Δ is a square in $\mathbb{N}$.

Now $\Delta = 0$ if and only if $r = k+2$ or $r = k+6$. So if $r \in \{k+2, k+6\}$ then $P(G_r, k, \lambda)$ has the trivial Galois group.

We claim that if $\Delta \neq 0$ then the Galois group is S_2. This is true since $\Delta \neq 0$ is not a square in $\mathbb{N}$, otherwise we would get a Pythagorean triple $(2, q, (r-(k+4))$. $\square$.

Corollary 26. *The graphs $G_{r,k}$ and $G_{r,k-s}$, $1 \leq s \leq k \leq r-2$, are Galois equivalent if one of the following sets of conditions are satisfied: (1) $s = 4$ and $k = r-2$, (2) $0 < k < r-6$, or (3) $r-6 < k < r-2$ and $s \neq k-r+6$.*

Proof The graphs $G_{r,k}$ and $G_{r,k-s}$ are Galois equivalent if $P(G_{r,k},\lambda)$ and $P(G_{r,k-s},\lambda)$ either both have an irreducible quadratic factor, in which case their Galois group is S_2, or both split completely, in which case their Galois group is the trivial group, in $\mathbb{Z}[\lambda]$.

Case 1: Suppose $P(G_{r,k},\lambda)$ and $P(G_{r,k-s},\lambda)$ both split completely in $\mathbb{Z}[\lambda]$. From Theorem 25 if $P(G_{r,k},\lambda)$ splits completely in $\mathbb{Z}[\lambda]$, then either

$$r = k+2 \qquad \text{or} \qquad (20)$$

$$r = k+6 \qquad (21)$$

and if $P(G_{r,k-s},\lambda)$ splits completely in $\mathbb{Z}[\lambda]$, then either

$$r = k+2-s \qquad \text{or} \qquad (22)$$

$$r = k+6-s. \qquad (23)$$

As $1 \leq s \leq k$, these conditions are only satisfied when $k = r-2$ and $s = 4$; that is, when $r = k+2 = k+6-s$.

Case 2: Suppose $P(G_{r,k},\lambda)$ and $P(G_{r,k-s},\lambda)$ both have an irreducible quadratic factor in $\mathbb{Z}[\lambda]$. From Theorem 25 if $P(G_{r,k},\lambda)$ has an irreducible quadratic factor then

$$k \notin \{r-2, r-6\} \qquad (24)$$

and if $P(G_{r,k-s},\lambda)$ has an irreducible quadratic factor in $\mathbb{Z}[\lambda]$, then

$$k \notin \{r-2+s, r-6+s\}. \qquad (25)$$

Now $k \leq r-2$, and $s > 0$ implies $k \neq r-2+s$. So the only condition in (25) required for $P(G_{r,k-s}, \lambda)$ to have Galois group S_2 is

$$k \neq r - 6 + s. \tag{26}$$

As $0 < k \leq r-2$ and given the conditions in (24), we need to consider the intervals $0 < k < r-6$ and $r-6 < k < r-2$. These are the intervals where $P(G_{r,k}, \lambda)$ has Galois group S_2.

Case 2A ($0 < k < r-6$): Now $0 < k < r-6$ and $s > 0$ implies (26), so the condition $0 < k < r-6$ implies that $G_{r,k}$ and $G_{r,k-s}$ both have Galois group S_2.

Case 2B ($r-6 < k < r-2$): From (26), $s \neq k-r+6$ must be satisfied if $G_{r,k-s}$ has Galois group S_2. Thus if $r-6 < k < r-2$ and $s \neq k-r+6$, then $P(G_{r,k}, \lambda) \sim_{\mathcal{G}} P(G_{r,k-s}, \lambda)$.

So, if $s = 4$ and $k = r-2$, then $P(G_{r,k}, \lambda)$ and $P(G_{r,k-s}, \lambda)$ both have Galois group S_1; and if either $0 < k < r-6$, or $r-6 < k < r-2$ and $s \neq k-r+6$, then $P(G_{r,k}, \lambda)$ and $P(G_{r,k-s}, \lambda)$ both have Galois group S_2. $\square$

For example, if $G_{r,0} \cong H_{10,2}$, then

- $P(G_{10,0}, \lambda) = P(K_9, \lambda)(\lambda^2 - 10\lambda + 17)$
- $P(G_{10,1}, \lambda) = P(K_9, \lambda)(\lambda^2 - 11\lambda + 25)$
- $P(G_{10,2}, \lambda) = P(K_9, \lambda)(\lambda^2 - 12\lambda + 33)$
- $P(G_{10,3}, \lambda) = P(K_9, \lambda)(\lambda^2 - 13\lambda + 41)$
- $P(G_{10,4}, \lambda) = P(K_9, \lambda)(\lambda - 7)^2$
- $P(G_{10,5}, \lambda) = P(K_9, \lambda)(\lambda^2 - 15\lambda + 57)$
- $P(G_{10,6}, \lambda) = P(K_9, \lambda)(\lambda^2 - 16\lambda + 65)$
- $P(G_{10,7}, \lambda) = P(K_9, \lambda)(\lambda^2 - 17\lambda + 73)$
- $P(G_{10,8}, \lambda) = P(K_9, \lambda)(\lambda - 9)^2$

When $k = 8$ and $s = 4$, we have $G_{10,4}$ and $G_{10,8}$ with chromatic polynomials that have Galois group S_1. The graphs $G_{10,k}$ and $G_{10,k-s}$ have Galois group S_2 if $0 < k < 4$ and $0 < s \leq k$, or if $4 < k < 8$ and $0 < s \leq k$, $s \neq k-4$.

It follows from Corollary 26 that

Corollary 27. *The graphs $G_{r,k}$ and $G_{r,k-1}$ are Galois equivalent if $0 < k < r-6$, or $r-4 \leq k < r-2$.*

Now using the addition/identification relation we have

$$\begin{aligned} P(G_{r,k-1}, \lambda) &= P(G_{r,k-1} + uv, \lambda) + P(G_{r,k-1}/uv, \lambda) \\ &= P(G_{r,k}, \lambda) + P(K_{r-1}, \lambda)(\lambda - (r-2)) \end{aligned} \tag{27}$$

where $uv \notin E(G_{r,k-1})$. Here the chromatic polynomial $P(K_{r-1}, \lambda)(\lambda-(r-2))$ is the chromatic polynomial of the graph obtained by an $(r-2)$-gluing of two copies of K_{r-1}. Thus we have an example of applying the the addition/identification relation to the graph $G_{r,k-1}$ (or by re-writing (27) an example of applying deletion/contraction relation to the graph $G_{r,k}$) in order to find a pair of Galois equivalent graphs (excluding the case where $G_{r,k}$ or $G_{r,k-1}$ has Galois group S_1).

3.3.2 Chromatic polynomials with Galois group $D(5)$

In the previous section we gave an example of infinitely many pairs of graphs, $G_{r,k}$ and $G_{r,k-1}$, that are Galois equivalent where $P(G_{r,k-1},\lambda)$ is the sum of $P(G_{r,k},\lambda)$ and the chromatic polynomial of the graph obtained by an $(r-2)$-gluing of two copies of K_{r-1}. In this example it was clear that the relationship between these graphs was a simple addition/identification relation. In this section we give an example of two graphs, G and G', that are Galois equivalent and satisfy

$$P(G',\lambda) = P(G,\lambda) + P(Q,\lambda) \tag{28}$$

for some graph Q. In this case the relationship between these graphs is not a simple addition/identification relation.

We will look at chromatic polynomials that have a quintic factor of the form

$$\lambda^5 - 11\lambda^4 + (44+k)\lambda^3 - (77+7k)\lambda^2 + (55+16k)\lambda - (11+12k). \tag{29}$$

where $k \in \mathbb{N}$. We begin by looking at properties of this quintic. We use the following theorem that Musser [13] describes as well-known in order to show this quintic is irreducible for all $k \in \mathbb{N}$.

Theorem 28. *Let $f(\lambda) \in \mathbb{Z}[\lambda]$ be a monic polynomial. If $f(\lambda)$ is irreducible modulo p, then $f(\lambda)$ is irreducible.*

Theorem 29. *The quintic $f(\lambda) = \lambda^5 - 11\lambda^4 + (44+k)\lambda^3 - (77+7k)\lambda^2 + (55+16k)\lambda - (11+12k)$ is irreducible in $\mathbb{Z}[\lambda]$ for $\forall k \in \mathbb{Z}$.*

Proof The polynomial $f(\lambda) \equiv \lambda^5 - \lambda^4 + k\lambda^3 - (k+1)\lambda^2 + \lambda - 1 \pmod 2$. The polynomials $\lambda^5 - \lambda^4 + \lambda^3 + \lambda - 1$ and $\lambda^5 - \lambda^4 - \lambda^2 + \lambda - 1$ can be seen to be irreducible mod 2. Thus $\lambda^5 - \lambda^4 + k\lambda^3 - (k+1)\lambda^2 + \lambda - 1$ is irreducible for all $k \in \mathbb{Z}$. Hence $f(\lambda)$ is irreducible modulo 2, so by Theorem 28, $f(\lambda)$ is irreducible in $\mathbb{Z}[\lambda]$ for all $k \in \mathbb{Z}$. □

Theorem 30. *The quintic $f(\lambda) = \lambda^5 - 11\lambda^4 + (44+k)\lambda^3 - (77+7k)\lambda^2 + (55+16k)\lambda - (11+12k)$ is solvable for all $k \in \mathbb{N}$ and has Galois group $D(5)$.*

Proof Let $\Delta(f)$ be the discriminant of $f(\lambda)$. Now,

$$\Delta(f) = (4k^3 - 56k^2 + 220k - 121)^2. \tag{30}$$

As the discriminant is a square, the Galois Group of $f(\lambda)$ is one of $C(5)$, $D(5)$ or A_5 [4, p. 375]. We first show that the Galois group of $f(\lambda)$ is not A_5.

The sextic resolvent of the quintic $f(\lambda)$ is

$$\theta_f(y) = (y^3 + b_2y^2 + b_4y + b_6)^2 - 2^{10}\Delta(f)y, \tag{31}$$

where the b_i are

$$\begin{aligned}
b_2 &= 8s_1s_3 - 3s_2^2 - 20s_4 \\
b_3 &= 3s_2^4 - 16s_1s_2^2s_3 + 16s_1^2s_3^2 + 16s_2s_3^2 + 16s_1^2s_2s_4 - 8s_2^2s_4 \\
&\quad - 112s_1s_3s_4 + 240s_4^2 - 64s_1^3s_5 + 240s_1s_2s_5 - 400s_3s_5 \\
b_6 &= 8s_1s_2^4s_3 - s_2^6 - 16s_1^2s_2^2s_3^2 - 16s_2^3s_3^2 + 64s_1s_2s_3^3 - 64s_3^4 \\
&\quad - 16s_1^2s_2^3s_4 + 28s_2^4s_4 + 64s_1^3s_2s_3s_4 - 112s_1s_2^2s_3s_4 \\
&\quad - 128s_1^2s_3^2s_4 + 224s_2s_3^2s_4 - 64s_1^4s_4^2 + 224s_1^2s_2s_4^2 \\
&\quad - 176s_2^2s_4^2 - 64s_1s_3s_4^2 + 320s_4^3 + 48s_1s_2^3s_5 \\
&\quad - 192s_1^2s_2s_3s_5 - 80s_2^2s_3s_5 + 640s_1s_3^2s_5 + 384s_1^3s_4s_5 \\
&\quad - 640s_1s_2s_4s_5 - 1600s_3s_4s_5 - 1600s_1^2s_5^2 + 4000s_2s_5^2 \qquad (32)
\end{aligned}$$

and the s_j are the coefficients of $f(\lambda)$, namely:

$$\begin{aligned}
s_1 &= 11 \\
s_2 &= 44 + k \\
s_3 &= 77 + 7k \\
s_4 &= 55 + 16k \\
s_5 &= 11 + 12k. \qquad (33)
\end{aligned}$$

Now, if the sextic resolvent has a root in $\mathbb{Q}$, then A_5 is not the Galois group of $f(\lambda)$ [4, pp. 373–375].

Using (30), (32) and (33), the resolvent in (31) becomes

$$\begin{aligned}
\theta_f(y) = (y - k^2)(&288y^2k^5 - 10y^2k^6 + 26976y^2k^3 - 3568y^2k^4 - 5y^4k^2 \\
&- 10912y^3k - 224y^3k^3 + 10y^3k^4 + 2144y^3k^2 + 64y^4k + 565312y^2k \\
&- 148720y^2k^2 - 8518400yk + 1703680yk^2 - 161920yk^3 + 35056yk^4 \\
&+ 2016yk^6 - 160yk^7 - 12000yk^5 + 5yk^8 - 14992384 + 14992384y \\
&- 149440k^5 + 24532992k + 22960k^6 + 32k^7 - k^{10} - 328k^8 + 32k^9 \\
&- 1022208y^2 + 3269376k^3 + 38400k^4 - 14372864k^2 + y^5 - 264y^4 \\
&+ 25168y^3). \qquad (34)
\end{aligned}$$

As k is an integer, (34) has root k^2 in $\mathbb{Z}$. Thus, the Galois group of $f(\lambda)$ is not A_5. Hence, the Galois group must be either $C(5)$ or $D(5)$.

We show that the Galois group is not $C(5)$ by showing that $f(\lambda)$ has a complex root that is not real. Now if $f(\lambda)$ has some non-real root, then the Galois group must contain an automorphism of order 2 mapping this root to its conjugate. The group $C(5)$ contains no element of order 2, so if $f(\lambda)$ has at least one non-real root, then $f(\lambda)$ has Galois group $D(5)$.

When $k = 1$ the quintic has a single real root, $\lambda \approx 2.302$ and thus has four non-real roots. Thus $f(\lambda)$ has Galois group $D(5)$ when $k = 1$. We now show that for $k > 1$, $f(\lambda)$ has at least one complex, non-real root.

The derivative of $f(\lambda)$ is

$$f'(\lambda) = 5\lambda^4 - 44\lambda^3 + 3(44 + k)\lambda^2 - 2(77 + 7k)\lambda + (55 + 16k) \qquad (35)$$

and the second derivative of $f(\lambda)$ is

$$f''(\lambda) = 20\lambda^3 - 132\lambda^2 + 6(44+k)\lambda - 2(77+7k) \tag{36}$$

which has discriminant

$$\Delta = -17280k^3 - 1653696k^2 + 2737152k - 387072. \tag{37}$$

As $\Delta < 0$ for $k > 1$, the second derivative has one real root [4, Theorem 1.3.1 (b), p.15], so $f(\lambda)$ has at most three real roots. Thus $f(\lambda)$ has at least two complex roots for $k \geq 2$. $\square$

Corollary 31. *Any polynomial of the form $q(\lambda)f(\lambda - s)$ where $q(\lambda)$ splits completely in $\mathbb{Z}[\lambda]$ and $f(\lambda) = \lambda^5 - 11\lambda^4 + (44+k)\lambda^3 - (77+7k)\lambda^2 + (55+16k)\lambda - (11+12k)$, $s \geq 0$ and $k \in \mathbb{N}$ has Galois group $D(5)$.*

Now $f(\lambda)$ can be re-written as follows:

$$\begin{aligned} f(\lambda) =&(\lambda^5 - 11\lambda^4 + (44+k)\lambda^3 - (77+7k)\lambda^2 + (55+16k)\lambda - (11+12k)) \\ =&(\lambda^5 - 11\lambda^4 + 44\lambda^3 - 77\lambda^2 + 55\lambda - 11) + k(\lambda-2)^2(\lambda-3) \end{aligned} \tag{38}$$

and thus

$$\begin{aligned} f(\lambda - s) =&((\lambda-s)^5 - 11(\lambda-s)^4 + 44(\lambda-s)^3 - 77(\lambda-s)^2 + 55(\lambda-s) - 11) \\ &+ k(\lambda - s - 2)^2(\lambda - s - 3) \qquad (39) \\ =&(\lambda^5 - (11+5s)\lambda^4 + (44 + 2s(5s+22))\lambda^3 \\ &- (77 + 2s(5s^2 + 33s + 66))\lambda^2 \\ &+ (55 + s(5s^3 + 44s^2 + 132s + 154))\lambda \\ &- (11 + s(s^4 + 11s^3 + 44s^2 + 77s + 55))) \\ &+ k(\lambda - s - 2)^2(\lambda - s - 3) \qquad (40) \end{aligned}$$

We now consider some graphs that have chromatic polynomials with one non-linear irreducible factor, a quintic $f(\lambda)$, that has the form given in (29). We show that if a graph G whose chromatic polynomial has this quintic as its only non-linear irreducible factor satisfies certain conditions, then a Galois equivalent graph G' can be obtained by applying a simple sequence of deletion/contraction and addition/identification relations to G where

$$P(G', \lambda) = P(G, \lambda) + q(\lambda)(\lambda - s - 2)^2(\lambda - s - 3) \tag{41}$$

where $q(\lambda) = P(G, \lambda)/f(\lambda)$. If G' also satisfies these conditions, then a sequence of Galois equivalent graphs can be obtained by this process.

Theorem 32. *If G is a graph with $P(G, \lambda) = q(\lambda)f(\lambda - s)$ where $q(\lambda)$ splits completely in $\mathbb{Z}[\lambda]$ and $f(\lambda)$ is the quintic in (29), and G' is a graph with $P(G', \lambda) = P(G, \lambda) + q(\lambda)(\lambda - s - 2)^2(\lambda - s - 3)$, then $P(G', \lambda)$ has a solvable Galois group is $D(5)$ and $G \sim_{\mathcal{G}} G'$.*

Proof Now

$$\begin{aligned} P(G', \lambda) =&P(G, \lambda) + q(\lambda)(\lambda - s - 2)^2(\lambda - s - 3) \\ =&q(\lambda)(f(\lambda - s) + (\lambda - s - 2)^2(\lambda - s - 3)) \end{aligned} \tag{42}$$

which using (39) becomes

$$\begin{aligned}P(G',\lambda) =&q(\lambda)((((\lambda-s)^5-11(\lambda-s)^4+44(\lambda-s)^3-77(\lambda-s)^2+55(\lambda-s)-11)\\&+k(\lambda-s-2)^2(\lambda-s-3)+(\lambda-s-2)^2(\lambda-s-3))\\=&q(\lambda)((((\lambda-s)^5-11(\lambda-s)^4+44(\lambda-s)^3-77(\lambda-s)^2+55(\lambda-s)-11)\\&+(k+1)(\lambda-s-2)^2(\lambda-s-3)). \qquad (43)\end{aligned}$$

Now the quintic

$$\begin{aligned}&((\lambda-s)^5-11(\lambda-s)^4+44(\lambda-s)^3-77(\lambda-s)^2+55(\lambda-s)-11)\\&+(k+1)(\lambda-s-2)^2(\lambda-s-3))\end{aligned}$$

is a quintic of the form of (39), so by Corollary 31 the polynomials $P(G',\lambda)$ and $P(G,\lambda)$ have Galois group $D(5)$ and thus $G\sim_{\mathcal{G}} G'$. □
.

Note that $q(\lambda)$ is the product of all irreducible linear factors of $P(G,\lambda)$ and thus

$$q(\lambda)=\Pi_{i=0}^{n-6}(\lambda-i)^{a_i} \qquad (44)$$

where $\Sigma_{i=0}^{n-6}a_i=n-5$ and $a_i=0\Rightarrow a_j=0$ for all $j\geq i$. Thus the term added to $P(G,\lambda)$ to obtain $P(G',\lambda)$ is $\Pi_{i=0}^{n-6}(\lambda-i)^{a_i}(\lambda-s-2)^2(\lambda-s-3)$, $s\geq 0$ which is the chromatic polynomial of some quasi-chordal graph Q and has the trivial Galois group. Most of the chromatic polynomials given in Section 2 have Galois group S_l, $l\geq 1$ which is not solvable for $l\geq 5$. But in Theorem 32 the chromatic polynomials of graphs G and G' have Galois group $D(5)$. Thus, Theorem 32 demonstrates that there exist cases where the sum of two solvable chromatic polynomials (and only one of these polynomials having the symmetric Galois group) is a solvable chromatic polynomial that is not the symmetric Galois group. Thus we have pairs (and even sequences) of Galois equivalent graphs $G_0,G_1,\ldots,G_k$ where

$$P(G_i,\lambda)=P(G_{i-1},\lambda)+P(Q,\lambda),\ i>0, \qquad (45)$$

$P(G_i,\lambda)$ has Galois Group $D(5)$ for all i and $P(Q,\lambda)$ has Galois group S_1.

Theorem 33. *Let G be a graph with chromatic polynomial $q(\lambda)f(\lambda-s)$, $s\geq 0$, where $q(\lambda)$ is given in (44) and $f(\lambda)=\lambda^5-11\lambda^4+(44+k)\lambda^3-(77+7k)\lambda^2+(55+16k)\lambda-(11+12k)$. If there exists $e\in E(G)$, $f=uv$ where u and v are non-adjacent vertices in G, and $g=wx$ where w and x are non-adjacent vertices in $G\setminus e/f$ such that $P(G\setminus e/f/g,\lambda)=q(\lambda)(\lambda-s-2)^2(\lambda-s-3)$ and $P(G\setminus e/f\setminus g,\lambda)=P(G/e,\lambda)$, then $P(G\setminus e+f,\lambda)=P(G,\lambda)+q(\lambda)(\lambda-s-2)^2(\lambda-s-3)$.*

Proof

$$\begin{aligned}P(G,\lambda)&=P(G\setminus e,\lambda)-P(G/e,\lambda)\\&=P(G\setminus e+f,\lambda)+P(G\setminus e/f,\lambda)-P(G/e,\lambda)\\&=P(G\setminus e+f,\lambda)-P(G\setminus e/f/g,\lambda)\\&=P(G\setminus e+f,\lambda)-q(\lambda)(\lambda-s-2)^2(\lambda-s-3). \qquad (46)\end{aligned}$$

Thus,

$$P(G \setminus e + f, \lambda) = P(G, \lambda) + q(\lambda)(\lambda - s - 2)^2(\lambda - s - 3). \qquad \square$$

When $s = 0$, the quintic in (40) becomes

$$f(\lambda) = \lambda^5 - 11\lambda^4 + 44\lambda^3 - 77\lambda^2 + 55\lambda - 11 + k(\lambda - 2)^2(\lambda - 3)$$

for some $k \in \mathbb{N}$. Chromatic polynomials of strongly non-clique-separable graphs of order at most 10 with this irreducible quintic factor are given in Table 14. In each case the irreducible quintic factor has Galois group $D(5)$. The first three chromatic polynomials in this table are the chromatic polynomials of the graphs 3537, 6050 and 6035 respectively (graph-numbering is as in [7]). Both Graph 3537 and Graph 6050 satisfy the conditions in Theorem 33 for $r = 0$. Figure 2 shows that Graph 6050 can be formed from Graph 3537, and that Graph 6035 can be formed from Graph 6050. In this figure we use the standard notation of representing the chromatic polynomial of a graph by the graph itself. The steps illustrated in Figure 2 correspond to those given in (46) in Theorem 33. All these graphs are Galois equivalent. The sequence of steps given in Figure 2 can be considered to give a *certificate of Galois equivalence* for these graphs. We defined certificates of factorisation and chromatic equivalence in [12]. Here we give a sequence of steps $P_0, P_1, \ldots, P_k$ with $P_0 = P(G, \lambda)$, $P_k = P(G', \lambda) + P(Q, \lambda)$ where each step P_i corresponds to a step given in (46), $r = 0$, $G' \cong G + \backslash e + f$ and $P(Q, \lambda) = \lambda(\lambda - 1)(\lambda - 2)^3(\lambda - 3)$.

When $s = 1$, the quintic in (40) becomes

$$\begin{aligned} f(\lambda - 1) =& (\lambda^5 - 16\lambda^4 + 98\lambda^3 - 285\lambda^2 + 390\lambda - 199) + k(\lambda - 3)^2(\lambda - 4) \\ =& \lambda^5 - 16\lambda^4 + (98 + k)\lambda^3 - (285 + 10k)\lambda^2 + (390 + 33k)\lambda - (199 + 36k) \end{aligned}$$

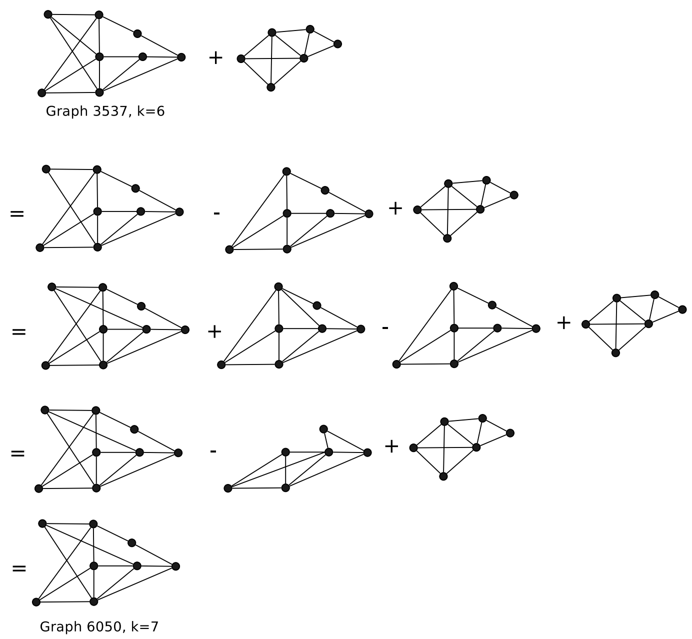
for $k \in \mathbb{N}$. Chromatic polynomials of strongly non-clique-separable graphs of order at most 10 with this irreducible quintic factor are given in Tables 15. By Theorem 32 all the chromatic polynomials in these tables have Galois group $D(5)$.

When $s = 2$, the quintic in (40) becomes

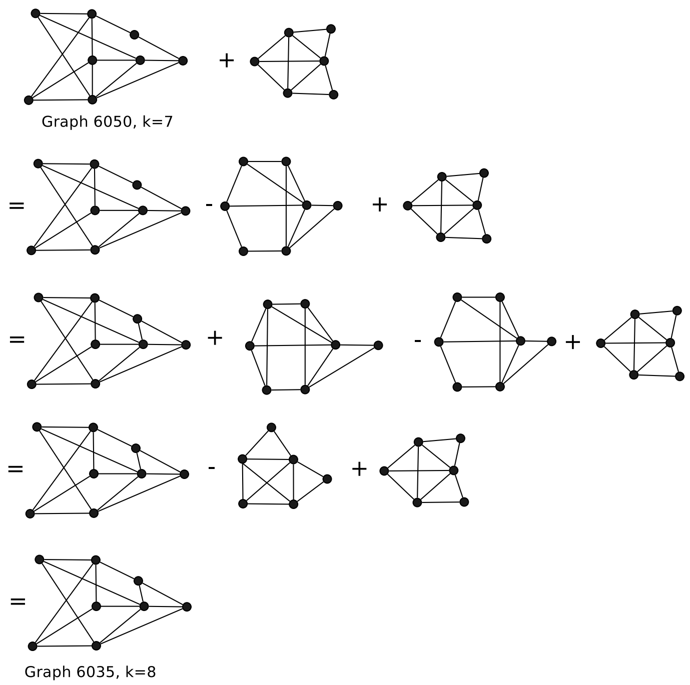
$$\begin{aligned} f(\lambda - 2) =& (\lambda^5 - 21\lambda^4 + 172\lambda^3 - 685\lambda^2 + 1323\lambda - 989) + k(\lambda - 4)^2(\lambda - 5) \\ =& \lambda^5 - 21\lambda^4 + (172 + k)\lambda^3 - (685 + 13k)\lambda^2 + (1323 + 56k)\lambda - (989 + 80k) \end{aligned}$$

for $k \in \mathbb{N}$. Chromatic polynomials of strongly non-clique-separable graphs of order at most 10 with this irreducible quintic factor are given in Table 16. By Theorem 32 all the chromatic polynomials in these tables have Galois group $D(5)$.

Tables 14, 15 and 16 list the chromatic polynomials of strongly non-clique-separable graphs with the quintic factor $f(\lambda - s)$ given in (40) for $s = 0, 1$ and 2 respectively. These tables do not include chromatic polynomials of clique-separable graphs. In Section 3.1 we saw that a graph G is Galois equivalent to any r-gluing of G and a quasichordal graph Q. Thus, for any chromatic polynomial $P(G, \lambda)$ in these tables there are infinitely many other chromatic polynomials with the same Galois group. These chromatic polynomials have the form $P(G, \lambda)P(Q, \lambda)/P(K_r, \lambda)$ where Q contains a clique of size at least $r \geq 1$.



(a) Graph 6050 is isomorphic to the graph obtained by removing a single edge from and adding another edge to Graph 3537.



(b) Graph 3537 is isomorphic to the graph obtained by removing a single edge from and adding another edge to Graph 6035.

Figure 2: Graph examples where $P(G,\lambda) + \lambda(\lambda-1)(\lambda-2)^3(\lambda-3) = P(G \setminus e + f, \lambda)$ and $G \sim_{\mathcal{G}} G \setminus e + f$.

n	k	$P(G,\lambda)$
8	6	$\lambda(\lambda-1)(\lambda-2)(\lambda^5-11\lambda^4+50\lambda^3-119\lambda^2+151\lambda-83)$
8	7	$\lambda(\lambda-1)(\lambda-2)(\lambda^5-11\lambda^4+51\lambda^3-126\lambda^2+167\lambda-95)$
8	8	$\lambda(\lambda-1)(\lambda-2)(\lambda^5-11\lambda^4+52\lambda^3-133\lambda^2+183\lambda-107)$
9	5	$\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda^5-11\lambda^4+49\lambda^3-112\lambda^2+135\lambda-71)$
9	9	$\lambda(\lambda-1)(\lambda-2)(\lambda-2)(\lambda^5-11\lambda^4+53\lambda^3-140\lambda^2+199\lambda-119)$
9	9	$\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda^5-11\lambda^4+53\lambda^3-140\lambda^2+199\lambda-119)$
10	10	$\lambda(\lambda-1)(\lambda-2)(\lambda-2)(\lambda-2)(\lambda^5-11\lambda^4+54\lambda^3-147\lambda^2+215\lambda-131)$
10	10	$\lambda(\lambda-1)(\lambda-2)(\lambda-2)(\lambda-3)(\lambda^5-11\lambda^4+54\lambda^3-147\lambda^2+215\lambda-131)$
10	5	$\lambda(\lambda-1)(\lambda-2)(\lambda^2-4\lambda+5)(\lambda^5-11\lambda^4+49\lambda^3-112\lambda^2+135\lambda-71)$
10	6	$\lambda(\lambda-1)(\lambda-2)(\lambda^2-3\lambda+4)(\lambda^5-11\lambda^4+50\lambda^3-119\lambda^2+151\lambda-83)$
10	6	$\lambda(\lambda-1)(\lambda-2)(\lambda^2-4\lambda+6)(\lambda^5-11\lambda^4+50\lambda^3-119\lambda^2+151\lambda-83)$
10	6	$\lambda(\lambda-1)(\lambda-2)(\lambda^2-5\lambda+8)(\lambda^5-11\lambda^4+50\lambda^3-119\lambda^2+151\lambda-83)$
10	6	$\lambda(\lambda-1)(\lambda-2)(\lambda^2-5\lambda+9)(\lambda^5-11\lambda^4+50\lambda^3-119\lambda^2+151\lambda-83)$
10	6	$\lambda(\lambda-1)(\lambda-2)(\lambda^2-5\lambda+10)(\lambda^5-11\lambda^4+50\lambda^3-119\lambda^2+151\lambda-83)$
10	7	$\lambda(\lambda-1)(\lambda-2)(\lambda^2-3\lambda+4)(\lambda^5-11\lambda^4+51\lambda^3-126\lambda^2+167\lambda-95)$
10	7	$\lambda(\lambda-1)(\lambda-2)(\lambda^2-3\lambda+5)(\lambda^5-11\lambda^4+51\lambda^3-126\lambda^2+167\lambda-95)$
10	7	$\lambda(\lambda-1)(\lambda-2)(\lambda^2-4\lambda+6)(\lambda^5-11\lambda^4+51\lambda^3-126\lambda^2+167\lambda-95)$
10	7	$\lambda(\lambda-1)(\lambda-2)(\lambda^2-4\lambda+7)(\lambda^5-11\lambda^4+51\lambda^3-126\lambda^2+167\lambda-95)$
10	7	$\lambda(\lambda-1)(\lambda-2)(\lambda^2-5\lambda+8)(\lambda^5-11\lambda^4+51\lambda^3-126\lambda^2+167\lambda-95)$
10	7	$\lambda(\lambda-1)(\lambda-2)(\lambda^2-6\lambda+10)(\lambda^5-11\lambda^4+51\lambda^3-126\lambda^2+167\lambda-95)$
10	8	$\lambda(\lambda-1)(\lambda-2)(\lambda^2-4\lambda+6)(\lambda^5-11\lambda^4+52\lambda^3-133\lambda^2+183\lambda-107)$
10	8	$\lambda(\lambda-1)(\lambda-2)(\lambda^2-5\lambda+8)(\lambda^5-11\lambda^4+52\lambda^3-133\lambda^2+183\lambda-107)$
10	8	$\lambda(\lambda-1)(\lambda-2)(\lambda^2-5\lambda+9)(\lambda^5-11\lambda^4+52\lambda^3-133\lambda^2+183\lambda-107)$
10	8	$\lambda(\lambda-1)(\lambda-2)(\lambda^2-6\lambda+10)(\lambda^5-11\lambda^4+52\lambda^3-133\lambda^2+183\lambda-107)$
10	9	$\lambda(\lambda-1)(\lambda-2)(\lambda^2-4\lambda+5)(\lambda^5-11\lambda^4+53\lambda^3-140\lambda^2+199\lambda-119)$
10	9	$\lambda(\lambda-1)(\lambda-2)(\lambda^2-5\lambda+7)(\lambda^5-11\lambda^4+53\lambda^3-140\lambda^2+199\lambda-119)$
10	9	$\lambda(\lambda-1)(\lambda-2)(\lambda^2-5\lambda+8)(\lambda^5-11\lambda^4+53\lambda^3-140\lambda^2+199\lambda-119)$

Table 14: Chromatic polynomials of strongly non-clique-separable graphs of order at most 10 with the irreducible quintic factor $f(\lambda)=(\lambda^5-11\lambda^4+44\lambda^3-77\lambda^2+55\lambda-11)+k(\lambda-2)^2(\lambda-3)$, $k\in\mathbb{N}$.

n	k	$P(G,\lambda)$
9	6	$\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda^5-16\lambda^4+104\lambda^3-345\lambda^2+588\lambda-415)$
9	7	$\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda^5-16\lambda^4+105\lambda^3-355\lambda^2+621\lambda-451)$
9	8	$\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda^5-16\lambda^4+106\lambda^3-365\lambda^2+654\lambda-487)$
9	9	$\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda^5-16\lambda^4+107\lambda^3-375\lambda^2+687\lambda-523)$
9	10	$\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda^5-16\lambda^4+108\lambda^3-385\lambda^2+720\lambda-559)$
10	5	$\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)(\lambda^5-16\lambda^4+103\lambda^3-335\lambda^2+555\lambda-379)$
10	11	$\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-3)(\lambda^5-16\lambda^4+109\lambda^3-395\lambda^2+753\lambda-595)$
10	11	$\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)(\lambda^5-16\lambda^4+109\lambda^3-395\lambda^2+753\lambda-595)$

Table 15: Chromatic polynomials of strongly non-clique-separable graphs of order n with solvable quintic factor $\lambda^5-16\lambda^4+98\lambda^3-285\lambda^2+390\lambda-199)+k(\lambda-3)^2(\lambda-4)$, $k\in\mathbb{N}$.

n	k	$P(G,\lambda)$
10	6	$\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)(\lambda^5-21\lambda^4+178\lambda^3-763\lambda^2+1659\lambda-1469)$
10	7	$\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)(\lambda^5-21\lambda^4+179\lambda^3-776\lambda^2+1715\lambda-1549)$
10	8	$\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)(\lambda^5-21\lambda^4+180\lambda^3-789\lambda^2+1771\lambda-1629)$
10	9	$\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)(\lambda^5-21\lambda^4+181\lambda^3-802\lambda^2+1827\lambda-1709)$
10	10	$\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)(\lambda^5-21\lambda^4+182\lambda^3-815\lambda^2+1883\lambda-1789)$

Table 16: Chromatic polynomials of strongly non-clique-separable graphs of order n with solvable quintic factor $(\lambda^5-21\lambda^4+172\lambda^3-685\lambda^2+1323\lambda-989)+k(\lambda-4)^2(\lambda-5)$, $k\in\mathbb{N}$.

In Section 3.1 we also saw that a graph G is Galois equivalent to any join of G and K_r, $r\geq 1$. Many of the chromatic polynomials in Tables 15 and 16 are the chromatic polynomial of $G\otimes K_r$, $r=1,2$, where G is a graph with a chromatic polynomial in Table 14. In these cases the quintic factor of $P(G\otimes K_r,\lambda)$ has the same k value as the quintic factor of $P(G,\lambda)$.

But not all chromatic polynomials in Tables 15 and 16 can be obtained in this way. For example, the chromatic polynomials of degree 8 in Table 14 have the irreducible quintic factor, $f(\lambda)$, for $k=6,7$ and 8. The chromatic polynomials, $P(G\otimes K_r,\lambda)$, $r=1,2$, where G is a graph of order 8 with a chromatic polynomial in this table correspond to the chromatic polynomials of degree 9 in Table 15 and the chromatic polynomials of degree 10 in Table 16 where $k=6,7$ and 8. However, these Tables 15 and 16 also include the $k=9,10$ cases which cannot be explained in this way. But graphs with these chromatic polynomials can be obtained by using the techniques discussed in Theorem 33.

So, if $P(G,\lambda)=q(\lambda)f(\lambda-r)$ where $q(\lambda)$ is the polynomial in (44) and $f(\lambda)$ is the quintic in (29), then if there exists a graph G' that satisfies at least one of the following three relations:

$$G' \text{ is an } r\text{-gluing of } G \text{ and } H, \text{ where } P(H,\lambda) \text{ splits in } \mathbb{Z}[\lambda] \quad (47)$$

$$G' = G\otimes K_r \text{ for some } r\geq 1 \quad (48)$$

$$P(G',\lambda)=P(G,\lambda)+q(\lambda)(\lambda-r-2)^2(\lambda-r-3) \quad (49)$$

then $P(G,\lambda)$ and $P(G',\lambda)$ both have Galois group $D(5)$.

4 Conclusion

In this paper we summarised the Galois groups of all chromatic polynomials of strongly non-clique-separable graphs of order at most 10 and all $\theta_{a,b,c}$-graphs of order at most 19 where $a,b,c\geq 2$. The majority of these chromatic polynomials have a symmetric Galois group. However, a number of these polynomials have the dihedral Galois group $D(5)$. We gave some results for families of graphs that have chromatic polynomials with this group.

Some basic operations were given that can be applied to any graph G in order to obtain a graph that is Galois equivalent to G. We then considered some cases where

$$P(G',\lambda)=P(G,\lambda)\pm P(Q,\lambda) \quad (50)$$

where Q is a quasi-chordal graph and the graphs G and G' both have solvable chromatic polynomials. In particular we considered cases where $G \sim_{\mathcal{G}} G'$.

It is easy to see that the chromatic polynomials of cycles, complete graphs, trees and paths satisfy (50). Complete graphs, trees and paths are all quasichordal graphs. In these cases, we can always find quasichordal graphs G and G' that satisfy (50). Here $P(G, \lambda)$, $P(G', \lambda)$ and $P(Q, \lambda)$ all have the trivial Galois group. In the case of cycles the chromatic polynomial can be expressed as

$$P(C_{n-1}, \lambda) = P(P_n, \lambda) - P(C_n, \lambda) \tag{51}$$

where P_n, the path on n vertices, is a chordal graph and the chromatic polynomials, $P(C_n, \lambda)$ and $P(C_{n-1}, \lambda)$ have Galois groups isomorphic to $(\mathbb{Z}/(n-1)\mathbb{Z})^*$ and $(\mathbb{Z}/(n-2)\mathbb{Z})^*$ respectively. In this case $C_n \not\sim_{\mathcal{G}} C_{n-1}$ (unless $n = 3$), but both C_n and C_{n-1} have solvable chromatic polynomials.

Although it is easy to find $G \sim_{\mathcal{G}} G'$ that satisfy (50) where $P(G, \lambda)$ and $P(G', \lambda)$ have the trivial Galois group, it is harder to identify cases where $G \sim_{\mathcal{G}} G'$ and $P(G, \lambda)$ and $P(G', \lambda)$ have a non-trivial Galois group. We give some sequences of graphs $\{G_i\}$ where ,given certain conditions, each $G_i, i > 0$, is Galois equivalent to G_{i-1}, and their Galois groups are non-trivial. The first sequence consists of graphs with Galois group S_2 and the second sequence consists of graphs with Galois group $D(5)$.

The latter sequence is particularly interesting as $D(5)$ is not a symmetric group and thus is the Galois group of a very small percentage of the chromatic polynomials of degree at most 10.

Further work in identifying the common structure of graphs G and G' that satisfy (50) may provide insights into the relationship between the structure of a graph and the Galois group of its chromatic polynomial. It would be interesting to identify other sequences of graphs that satisfy this expression. In particular, as the majority of chromatic polynomials of graphs of order at most 10 have a symmetric Galois group, it would be interesting to identify sequences of graphs with chromatic polynomials with non-symmetric Galois groups.

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