From the Imaginary to the Real: the Triumph of Rome

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When writing we sometimes make errors and when we try to decipher what is written by someone else we may also occasionally make errors, imputing to the author something he did not write, or else we may misconstrue what the scribe has actually written. Secondly, there is an interesting drift from abbreviations to using letters for 'unknowns' or 'variables'. Thirdly, the symbols on the page sometimes acquire a life of their own which is only slowly understood, even though the machinery the symbols provide may work well.

This paper gives examples of all of these, drawing on medieval manuscripts and on the history of mathematics. The third aspect is found in the work of Bombelli who was attempting to drain marshes near Rome in the mid-sixteenth century.

Let us begin with the first item. It is often easy to find errors but not always as easy to correct them. Consider the following:

The writer contributed histakes

The quotation clearly contains a mistake. It is more difficult to say what the correct version might be. It could be *The writer contributed mistakes*, but another possibility is *The writer contributed his takes*, meaning that the writer (of the words), who might not have been the author, was putting a spin on it. There is a third possibility: *The writer contributed hi stakes*. For this the context would probably have to be modern America, more specifically the USA. We shall consider the interpretations in order. First: The writer contributed mistakes.

In general, when a message is sent from A to B there are usually the following processes.[1] First, A puts the idea into a transmissible form. Then the message is conveyed to B by some means and finally B decodes the message. There are dangers at every stage. In the last world war, spies sending messages encoded them and errors could literally cost them their lives. (See for instance the fascinating book [16].) In the study of medieval history we are luckier: only careers may depend on erroneous messages being received.

In the case of a manuscript from, say, the thirteenth century the three processes listed above can become more complicated. Usually the message would be

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[1] This is taught in classes for Communication Arts as well as for Information Theory in the style of Shannon [26].

written by a scribe, who might, or might not, have received it directly from the author. In any case, there is a question of how he would decipher it.

Consider the following from the thirteenth century treatise by Grocheio, *De Musice*, in Harley 281 (GB-Lbl Harl. 281):[2]

Tonus autem multipliciter dicitur velud aux in motibus.

In another version of the text in Darmstadt (D-DS 2663) this is rendered:

Tonus autem multipliciter dicitur uelud aux in montibus.

and on the web site `http://www.uga.edu/~thema/` (accessed 11.x.2005) the transcription reads:

Tonus autem multipliciter dicitur uelud aurum in motibus.

The writing in the two manuscripts is clear and neat. There is no question (in this case) as to what the letters are. The problem is that 'aux' is an uncommon Latin word. Indeed it only occurs once in this text and nowhere else in the manuscript volumes in which these two texts reside.[3] It was therefore natural to try to decide what the correct word should be, assuming, as Rohloff [23] and our group[4] (on first encounter) did, that the word 'aux' was incorrect. 'Nix in montibus' made a certain amount of sense since the snowline does go up and down over the seasons. However it was Charles Burnett who identified the word as coming from the Arabic 'awj', meaning apogee. Ironically 'awj' seems to be an Arab version of the Latin word 'apsis'.

Grocheio was living in the thirteenth century in Paris and writing about music theory in a style informed by much science, and in particular, astronomy.[5] *Aux* and *motus* are the two words that Pedersen does not translate in his version of *Theorica Planetarum*, a text thought by many to be written by Gerard of Cremona (1114–187) (see [15] but also [19]). This is why we conclude that the first version is the correct one. In English it could be translated as

> Now a tone is spoken of in many ways, just like the apogee in [planetary] motions.

The context in which the message was conceived is therefore important in deciphering what is written. Two scribes wrote the first two versions, both of them

[2] The two transcriptions in the *Thesaurus Musicarum Latinarum*, the one by Wolf from [33] at `http://www.music.indiana.edu/tml/14th/GROTHE_TEXT.html` and the one from Rohloff [23] at `http://www.music.indiana.edu/tml/14th/GRODEM_TEXT.html` both read *Tonus autem multipliciter dicitur uelud nix in montibus.* (Both were accessed 30.x.2005.)

[3] 'Motus' occurs five other times.

[4] The author is part of group working on a new translation of Grocheio's work at Monash university, Australia. the other members of the group are Constant Mews, Carol Williams, Catherine Jeffreys and Leigh McKinnon.

[5] For a full discussion of Grocheio's background see [11].

probably in the thirteenth century or at the latest, about 1300. The third quotation is from the transcription by Sandra Pinegar of the Harley manuscript. She has a 1991 PhD in Historical Musicology, not Astronomy or any other science. The context of the reader is therefore also important. Why should a musicologist of today know about the planetary motions, or (also mentioned by Grocheio) the distance of the sun? Here the situation would seem to be that the intellectual of modern times has greater depth, but less breadth, of knowledge than in centuries past.

The writer contributed his takes

This brings us to the use of abbreviations and other, possibly new, symbols. We shall see that there is an interesting slide from abbreviations to the special use of letters, especially as found in logic and mathematics.

Besides the problem of misreading what is correct text, another source of confusion is that the thirteenth century scribe used abbreviations in profusion. These could cause difficulties. On driving into downtown San Diego on the freeway one used to see a sign: 'Cruise Ships USE AIRPORT EXIT' which unfortunately has now been replaced. I have never actually seen a cruise ship on the freeway, but the sign is useful to some automobile drivers. Such simple abbreviations, in this case abbreviating the grammar, are generally easy to decipher, though we depend on the context very much. Modern advertising has developed the art of abbreviation of sentences to a high level. Likewise medieval calligraphy developed a collection of abbreviations of individual syllables, as documented by A. Cappelli in his compendious book [6]. Sometimes, however, there is a question as to whether a mark made by the writing instrument actually represents a letter. Nowadays we are familiar with (at least) simple quadratic equations such as $x^2 + 4x = 4$ and everyone refers to x as the 'unknown'. In considering such an equation one regards the equation as communicating information about some number. This number is unknown but definite. The problem is to work out what the number is. In other equations, such as $x + y = y + x$, then the 'variables' x and y play a different rôle. It is not a question of x or y being unknown, but rather that one can choose any (numerical) value for these two and the two sides of the equation will work out to the same result. In many simple cases we do not even need to use variables. We can say, in the above case, 'The order in which we add two numbers does not matter. We get the same result in either order.' The same applies to the way that we treat syllogisms. Consider the most familiar example:

$$\frac{\begin{array}{c}\text{All men are mortal}\\ \text{Socrates is a man}\end{array}}{\text{[Therefore] Socrates is mortal}}$$

No variables are needed here although in modern logic it will often be symbolized by

$$\frac{\begin{array}{c}\forall x(\mathit{Man}\,(x) \rightarrow \mathit{Mortal}\,(x))\\ \mathit{Man}\,(S)\end{array}}{\mathit{Mortal}\,(S)}$$

where the arrow is read 'implies' and ∀ is read 'for all'.

Thus symbols are *not essential* in such a context. Indeed symbols in the mathematical area did not come into common use until the sixteenth century (see below). We find a similar situation in ancient Chinese mathematics. In Chapter 8 of the *Nine Chapters*, an ancient Chinese book dating from no later than about 100 AD, we find a problem ([14], p.399.):

> Now given 3 bundles of top grade paddy [rice], 2 bundles of medium grade paddy, [and] 1 bundle of low grade paddy. Yield: 39 *dou*.[6] 2 bundles of top grade paddy, 3 bundles of medium grade paddy, [and] 1 bundle of low grade paddy. Yield: 34 *dou*. 1 bundles of top grade paddy, 2 bundles of medium grade paddy, [and] 3 bundle of low grade paddy. Yield: 26 *dou*. Tell: how much paddy does one bundle of each grade yield.

A modern approach would set this up as three equations:

$$\begin{aligned} 3x + 2y + z &= 39 \\ 2x + 3y + z &= 34 \\ x + 2y + 3z &= 26. \end{aligned}$$

and the mathematician would proudly announce the values of x, y and z. Nevertheless the letters x, y and z are unnecessary. First of all the Chinese laid out the problem on their counting boards (the Chinese predecessor of the abacus) in this way:

Top	*Med.*	*Low*	*Yield*
3	2	1	39
2	3	1	34
1	2	3	26

The answer can be stated as follows without using variables: top grade yields $9\frac{1}{4}$, medium yields $4\frac{1}{4}$ and low yields $2\frac{3}{4}$. It could, of course, be argued that we have actually used the expressions 'top grade', 'medium grade' and 'low grade' as names for the letters x, y and z. With our hindsight we can interpret the Chinese solution in that way. The ancient Chinese, however, could not.

A perhaps related situation occurs in the works of Diophantos who lived, we believe, in the third century AD. (See, for example, [12], [21] or [22].) Translators of his work (such as Tannery [29] have not always resisted the temptation to couch their translation in modern (and in this case, mathematical) terms. For Diophantos there was no concept of equation, indeed the concept of an equation is more recent. It was developed in China in the thirteenth century by Zhu Shijie (see [34], p.135 or [14], p. 184) and in the West in the sixteenth century (see e.g. [8]). However we shall not pursue the question of the concept of equation here but see, for example, [13]. Nevertheless, the problems that Diophantos solved are very easy to translate into the language of quadratic and other equations. So,

[6] A measure of about 2 litres.

with this caveat, we shall use phrases such as ‘quadratic equation’ when writing about his work.

Diophantos was interested in solving not only quadratic equations but other polynomial equations too. He gave his name to the problem of solving equations in whole (positive) numbers: Diophantine equations. The most familiar one corresponds to Pythagoras’ theorem: $x^2 + y^2 = z^2$ for which we know not only many solutions such as $3, 4, 5$ but also general solutions. However, in stating equations involving one unknown he did not use a letter of the (Greek) alphabet. He used a squiggle. Some authorities have suggested that he used a form of the Greek final sigma, a kind of s, for the unknown. Others have suggested that this was an abbreviation for ‘arithmos’ meaning ‘number’. However, Diophantos was looking for solutions to his problems. Therefore he was using his ‘s’ in the way that mathematicians today still use ‘unknowns’: they are quite definite quantities, but we simply we do not, at the start, what the number is (or what the numbers are). So in the case of Diophantos it seems plausible to argue that it is modern interpreters who are saying that Diophantos used symbols for variables.

It is in the later part of the thirteenth century that we first find the use of letters as variables. As far as the author can ascertain, this was the earliest occurrence of such use. This can be found in the work of Raymond Llull (1235–316).[7] In [10] it is argued that Llull used the letters B, C, D, E, F, G, H, I, K as variables. The basis for this is the fact that each of these letters can take on each of six possible values. For B these are: goodness, difference, whether? God, justice, avarice. Llull is interested in combinations of these letters (subject to certain restrictions that prohibit some, but not all, repetitions). In these combinations the values assigned to two different occurrences of a letter may be different, unlike in modern mathematics where one must substitute the same object for the same letter everywhere. Thus these letters are *not* being used as abbreviations.

It was only in the sixteenth century, in the work of Viète (1540–1603) [31] that unknowns, represented by letters, became fully integrated into mathematics in what is essentially the modern way. For Viète, symbols now have the universal meanings that we mentioned earlier in the context of the modern representation of a syllogism.

The signs or characters that are used in the typography of books today form a much wider collection than they did in the thirteenth century. At that time, for Latin authors, there was the Roman alphabet and musical notation had begun to be developed. As van Deusen puts it in [30], p.104, “The music historian would quickly notice that during the period of the assimilation of [Aristotle’s] *Physica*, musical notation underwent a revolutionary change, that is, from the abstraction of motion from shape[8] to the representation of (rhythmic) motion by shape.” In addition, from the twelfth century on, but only being taken up

[7] Here I disagree with Bonner [4] who says ‘the letters don’t represent variables but constants’.

[8] That is to say, representing, *inter alia*, the rising and falling of the pitch by going up and down the stave.

very slowly, or so it would appear, there was the use of Hindu-Arabic numerals. These are clearly distinct from the letters of the Roman alphabet. There shapes are different. They do not, however, have special names. Arabic names are not used, instead the Latin words that are used for Roman numerals are used for, not just individual Hindu-Arabic digits, but also strings of Hindu-Arabic numerals. Thus 1000 is read 'mille', 1234 is read the same as MCCXXXIV although its formation is entirely different: rearranging the Hindu-Arabic numerals gives a reading which is, in general, not a rearrangement of the Roman ones. Thus 4321 is read the same as MMMMCCCXXI, which contains more Roman numerals than before, while there are the same number of Hindu-Arabic numerals in the two numbers.

Thus these new symbols have different qualities from both the letters of the Roman alphabet and the Roman numerals. So what is their nature? The present author and Charles Burnett share similar, independently obtained, views, though Burnett's work will appear in print earlier (see [5, 11]). He elegantly writes:

> The retention of the visual order of the Indian numerals, I think, adds to the evidence for their being conceived as different from written text, ... Numerals were *hors de texte*, and like pictures, or geometrical diagrams, they kept the same directionality as they passed from one language context to another.

The alphabet, or the scribe's font, taken quite literally, i.e. meaning 'source', is significantly increased. Letters, which had clear rôles in making words, and sometimes numbers, are now supplemented by other signs by which numbers and music, *inter alia*, are notated.

Such notation developed steadily in mathematics. These new symbols were often significantly different from (Roman) letters. Examples are the signs for equality, =, and for addition, +, that are so familiar today. Thus the language of mathematics moved from being purely verbal – 'rhetorical algebra' was the way that algebra was presented before about 1500 – into a symbolic style. Some of the symbols are simply abbreviations, but abbreviations in the sense of a symbol representing a word. This is the case for = and +. In other cases they are contractions, though, as was the case with medieval scribes, the contractions may not simply be letters of the alphabet. For example, a medieval scribe might end a word with m̃ representing 'em', and although the ˜ might have originally represented an 'e' it is subsequently hard to identify the two inscriptions visually.

The new symbols, or old ones used in new ways, sometimes developed a life of their own and this was the case in the work on solving cubic equations. To see this we need to look very briefly at the history (but not the detailed technicalities) of the solution and the lack of clarity that there was.

Le scribe faisait des sautes

Algebra was the principal driver of this expansion of the language but, even in the sixteenth century, algebra was not part of the quadrivium. In her thesis, Cifoletti [8] describes the slow entry of algebra into the university at that time.

Already algorism, the ways of calculating using the (then new) Hindu-Arabic numerals, had entered the syllabus, as part of Arithmetica, about 1200 and its uptake was very slow. Texts on medieval palaeography simply say that they were in common use by 1400.

There was a sharp distinction between the academic study of Arithmetica in the universities and the practical use of algorism in the abacus schools. However, in these schools there were problems to be solved and algebra provided a means of attack. On the other hand Arithmetica, as taught in the universities, did not include algebra and it was not until the sixteenth century that it began to be taught there (see [8], chapter II). *L'Arithmétique* [20] by Peletier (1517–1582) presents the integration of a learned and a "vulgar" tradition (*op.cit.*, p.73).

It is worth remembering that negative and other kinds of numbers had an unresolved status in the sixteenth century. Negative numbers were not in common use and could usually be avoided as we do in ledgers, by putting them on the other side, in the mathematical case, on the other side of the equation (where they become positive numbers). In the *Algebra* [25] of al-Khwarizmi (*c.* 780–*c.*850) this is called 'restitution' (*almucabala*): by adding the same (positive) quantity to each side of an equation, the deficiency on one side is removed and, of course, we have an additional amount on the other side. Thus if we restore b to the equation $a - b = c$ we obtain $a - b + b = c + b$, that is $a = c + b$.

Irrational numbers were also a cause for suspicion. Irrational numbers are numbers that cannot be expressed as a fraction where one whole number is divided by another, for example, the square root of two. These had been a thorn in the side of the Pythagoreans when they were first discovered, for they meant that it was impossible to measure the diagonal of a square as such a fraction of the side of the square (see e.g. [9], chapter V). By the end of the sixteenth century such styles of number were beginning to be accepted as mathematically respectable entities. Stevin (1548–1620), writing in 1585 (after the death of Bombelli whom we treat below), has a section (see [28], vol. II, p.532):

> QU'IL N'Y A AUCUNS NOMBRES ABSURDES, IRRATIONELS, IRREGULIERS, INEXPLICABLES, OU SOURDS.
> C'est chose tres vulgaire entre les Autheurs d'Arith., de traicter de nombres comme$\sqrt{8}$ & semblables, qu'il appelent absurds, irrationels, irreguliers, inexplicables, sourds, & c. Ce que nous nions, à quelque nombre auenir: Mais par quelle raison l'adversaire le prouuera ilesprouuer?

The method of solving quadratic equations was by applying an algorithm, or recipe.[9] The algorithm in the case of quadratic equations is easily applied but requires mathematical understanding to see why it works. The algorithm as such was known (at least for certain kinds of these equations) in the Mesopotamian

[9] 'Algorism' comprises the algorithms for working with Hindu-Arabic numerals, as noted above; 'algorithms' are general procedures for producing results. The word is most commonly used today in the context of computer programs. Such programs encode the algorithms, the algorithms themselves are abstract procedures. The word 'algorithm' developed through association with the Greek word 'arithmos' (see [27]).

era about 1500 BC as is evidenced by cuneiform texts (See [17, 18]).[10] The necessary steps can be performed on an unsophisticated calculator today, requiring only the presence of a square root facility on the calculator. Now some quadratic equations, for example $x^2 = 4$, have two solutions (in this case $x = 2$ and $x = -2$), but some quadratic equations, for example, $x^2 + 1 = x$ do not have solutions in the ordinary, so-called 'real', numbers, for example $x^2 = -1$. The latter equations require the use of imaginary and complex numbers.[11] Surprisingly it was not in the context of solving quadratic equations, but of solving cubic equations that imaginary numbers first came into evidence.

Scipio dal Ferro (1465–1526) is generally accepted to be the first person to give a general method for solving cubic equations.[12] His dispute with Girolamo Cardano (1501–1576) about the priority is well-known, see for example, the classic text of Paul Rose [24] or the author's [9]. However, Cardano claimed that he had produced a mathematical proof (of the solution) whereas dal Ferro had only produced the solution. It seems worthwhile to distinguish between the *technique* for solving a cubic equation, which is in fact an algorithm, and the mathematical *theory* behind the solution.

In the case of cubic equations the equation is first transformed into a special form and then the problem of solving it becomes one of solving a different, this time quadratic, equation. After that cube roots have to be extracted. What happens here is that a different kind of number is inevitably involved. These new numbers are what we now call *complex numbers*. The reason for this is quite complicated technically and we shall not give the details here.[13] In the same way that a quadratic equation has up to two roots, we expect a cubic equation to have up to three. For example, consider $x^3 - 6x^2 + 7x = 6$. A little work will show that this is equivalent to $(x-1)(x-2)(x-3) = 0$ which clearly has the solutions $x = 1, 2, 3$. Ironically in the case where there are three such real roots the algorithm of dal Ferro inevitably involves taking the square roots of negative quantities (see Birkhoff and MacLane, *op.cit.*). Since all numbers, positive and negative, always have positive squares, the idea of taking the square root of a negative number appeared impossible: Cardano uses the word 'subtilitas' in dealing with such numbers. His calculations lead him to $5 + \sqrt{25-40}$ and $5 - \sqrt{25-40}$ which he has to multiply together. He says: Dismissing mental tortures, and multiplying $5 + \sqrt{-15}$ by $5 - \sqrt{-15}$ we obtain $25 - (-15)$. Therefore the product is 40. But he concludes:[14]

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[10] The problems were not expressed in terms of equations at that time but it is easy to translate them into equations.

[11] Complex numbers are sums of real and imaginary numbers. See below.

[12] There is a question as to whether Scipio dal Ferro could solve all kinds of cubic equations but this is not relevant to the present discussion.

[13] The technical reasons as to why these new (complex) numbers are required is explained in, for example, Birkhoff and MacLane's [2].

[14] '. . . and thus far does arithmetical subtlety go, of which this, the extreme, is, as I have said, so subtle that it is useless.' See [9], p.189 and [7], p.220.

> ... et huiusque progreditur Arithmetica subtilitas, cuius hoc extremum ut dixi, adeo est subtile, ut sit inutile.

It is in the work of Rafael Bombelli (1526–1572) that we find symbols, in this case, simply p. which we would now write as + and m. which we would write as $-$, working in well-established ways but thereby developing what I have termed a life of their own. Bombelli was the first to give a consistent exposition of how to manipulate complex numbers. However it should be noted that he did not provide a theory and such a theory had to wait until much later (as we note below). The publication of Bombelli's work was long delayed and it did not appear until 1572, although the work had been achieved in the mid-1550s, so its publication was long after the work of dal Ferro and Cardano in the 1540s. Bombelli was an engineer, at one time engaged in trying to drain the Val di Chiana marshes, which he did successfully. Subsequently he was also engaged to drain the Pontine marshes at Rome but this was unsuccessful and the task was not completed until last century. His *L'Algebra* draws heavily on the newly discovered Vatican manuscript of Diophantos for its problems but the technical treatment is Bombelli's alone. He systematically treats ordinary numbers, educing the rules for the manipulation of what we now write as p and $-$ signs, though he used 'p.' for 'plus (piu)' and 'm.' for 'minus (meno)'. He also used 'L' and its reverse as brackets: $\lfloor \quad \rfloor$. For the square root sign $\surd$ he used an R with a line through the tail as we do for a prescription. It is therefore very easy to read his notation and transcribe it in a thoroughly modern way.

In treating cubic equations he describes the complex numbers that arise in the computation of solutions as follows:[15]

> Ho trovato un'altra sorte di R.c. legate molto differenti dall'altre, la qual nasce dal Capitolo di cubo eguale a tanti e numero ....

With these he is then able to present *formal* rules for the manipulation of quantities such as $\sqrt{-1}$ or $\sqrt{-15}$. Now there are two square roots of any number, even a negative one, thus the two square roots of 3 are $+\sqrt{3}$ and $\sqrt{3}$ which Bombelli notates as 'p. di m. 3' and 'm.di m.3' ('piu di meno 3' and 'meno di meno 3').[16] He then went on to show, by examples, that these manipulations do actually

[15] 'I have found another kind of tied cube root very different from the others ....' [3], p.133. 'Tied cube root' refers to taking the cube root of a square root which is a complex number. The details are not important here but may be found in Bombelli's work and in précis in the author's [9], chapter IV, the point being that he has new expressions to deal with.

[16] The rules for addition and subtraction are simple: like goes with like, and his rules for multiplication are as follows;
Plus of minus times plus of minus makes minus $[+\sqrt{-}. + \sqrt{-} = -]$
Plus of minus times minus of minus makes plus $[+\sqrt{-}. - \sqrt{-} = +]$
Minus of minus times plus of minus makes plus $[-\sqrt{-}. + \sqrt{-} = +]$
Minus of minus times minus of minus makes minus $[-\sqrt{-}. - \sqrt{-} = -]$

produce numbers that are solutions of the given cubic equation. After producing examples of such solutions he adds:[17]

> Et benchè a molti parerà questa cosa stravagante, perchè di questa opinione fui ancho già un tempo, parendomi più tosto fosse sofistica che vera, nondimeno tanto cercai che trovai li dimostratione, la quale sarà qui sotto notata, sì che questa ancora si può mostrare in linea, che pur nelle operationi serve senza difficultade alcuna, et assai volte si trova la valuta del Tanto per numero (come si è trovato in questo esempio). Però ben vi applichi l'animo il lettore; che anco egli si trovarà ingannato.

There is some question about how justifiable Bombelli's 'mostra in linea' is from a *geometric* point of view, though algebraically there is no problem at all given his rules listed in footnote 16). It therefore seems reasonable to conclude that in this case we can say of the writer, namely Bombelli:

Le scribe faisait des sauts

rather than 'fautes'. And indeed, Bombelli's masterly exposition was for this author at least one place where

The writer contributed hi stakes.

The full formal treatment of complex numbers was accomplished somewhat later by giving them a geometric representation in the work of Argand (1768–822) in 1806 [1] though earlier found by Wessel (1745–818) some years earlier [32]. Nevertheless Bombelli's treatment of complex numbers, and in particular of imaginary numbers took mathematics out of the dark swamp in which Cardano had struggled into a clear arena. Bombelli made imaginary numbers real. In doing so he had used a purely formal approach, manipulating, according to what have now become familiar rules, symbols and combinations of symbols which, although we now regard them as designating a special kind of number and including signed numbers, were mystifying at that time. The approach was very much that of the practical engineer; the benefits were enormous for mathematics.

Thus the blind but observant pursuit of the standard rules for operating with plus and minus led Bombelli to a formal procedure which solved *all* cubic equations.

We have therefore seen how the inscriptions on vellum or paper can be read (or perhaps misread), in a way that leads one astray. We have also seen how abbreviations led, or developed, to new uses for letters, and finally how the careful but formal use of letters or other symbols can lead to new and powerful discoveries.

[17] *Op.cit.* p.225. 'And although to many this will appear an extravagant thing, because even I held this opinion some time ago, since it appeared to me to be more sophistic than true, nevertheless I searched hard and found the demonstration, which will be noted below. So then even this can be shown by geometry, which indeed works for these operations without any difficulty, and on many occasions one can find the nature of the unknown as a number. But let the reader apply all his strength of mind, for even he will find himself deceived [otherwise].'

Postscript.

The author hopes that it is not the case that

Interdum qui litteras faciebat terrores creabat,

but he admits that it may be true that

Interdum qui litteras faciebat errores creabat.

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