Completion and Embedding Problems for Combinatorial Designs

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Abstract

The topic of when a partial combinatorial design can be completed or embedded has attracted a great deal of interest over the years. In this thesis, we investigate four topics related to the completion or embedding of partial H-designs.

The first two chapters lay the foundation by introducing the background and context for our work. Chapter 3 deals with completions of partial K_k-designs on K_n. In this chapter, we determine exactly the minimum number of blocks in an uncompletable partial K_k-design on K_n for all sufficiently large n. This result is reminiscent of Evans' now-proved conjecture on completions of partial latin squares. We also prove some related results concerning edge decompositions of almost complete graphs into copies of K_k. In Chapter 4, we present some complexity results regarding embeddings of partial K_3-designs. For a given partial K_3-design on K_u it is known that an embedding of order $v \geqslant 2u + 1$ exists whenever v satisfies the obvious necessary conditions. Determining whether "small" embeddings of order $v < 2u+1$ exist is a more difficult task. We extend a result of Colbourn on the NP-completeness of these problems. We also exhibit a family of counterexamples to a conjecture concerning when small embeddings exist.

In Chapter 5, we consider the problem of when a partial $K_{1,k}$-design on K_n can be embedded in a $K_{1,k}$-design on K_{n+s} for a given integer s. We improve a result of Noble and Richardson, itself an improvement of a result of Hoffman and Roberts, by showing that any partial $K_{1,k}$-design on K_n can be embedded in a $K_{1,k}$-design on K_{n+s} for some s such that $s < \frac{9}{4}k$ when k is odd and $s < (6 - 2\sqrt{2})k$ when k is even. Moreover, we prove that for general k, these constants cannot be improved. We also obtain stronger results subject to placing a lower bound on n. Chapter 6 deals with completions of partial $K_{1,k}$-designs on K_n, adressing a problem analogous to the one considered in Chapter 3. We determine exactly the minimum number of stars in an uncompletable partial $K_{1,k}$-design on K_n. We conclude in Chapter 7 with an overview of some open problems arising from our work.

Declaration

This thesis contains no material which has been accepted for the award of any other degree or diploma at any university or equivalent institution and that, to the best of my knowledge and belief, this thesis contains no material previously published or written by another person, except where due reference is made in the text of the thesis.

Ajani De Vas Gunasekara
26 May 2022

Publications during enrolment

This thesis is a combination of several pieces of work (published, submitted and in preparation). Each of these works is joint work with other authors, as detailed below.

- Chapter 3 is based on a paper published in SIAM J. Disc. Math. [34]. This is joint work with Daniel Horsley.
- Chapter 4 is based on a paper published in J. Combin. Des. [15]. This is joint work with Darryn Bryant and Daniel Horsley.
- Chapter 5 is based on a submitted paper [33], which is joint work with Daniel Horsley.
- Chapter 6 is joint work with Daniel Horsley. A paper including content from this chapter is currently in preparation.

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Contents

List of Notations

$V(G)$	the vertex set of G
$E(G)$	the edge set of G
$\overline{G}$	the complement of G
$\delta(G)$	the minimum degree of G
$\Delta(G)$	the maximum degree of G
$G[S]$	the subgraph of G induced by S
$\chi'(G)$	the chromatic index of G
$\alpha(G)$	the independence number of G
$G \vee H$	the join graph of graphs G and H
$G - H$	the graph difference of graphs G and H
$\lfloor n \rfloor$	the floor function on real number n
$\lceil n \rceil$	the ceiling function on real number n
K_V	the complete graph with vertex set V
K_n	the complete graph of order n
$K_{m,n}$	the complete bipartite graph of order $m+n$
$K_{1,k}$	the k-star graph
$N_G(x)$	the neighbourhood of a vertex x in the graph G
$N_G(x,y)$	the mutual neighbourhood of vertices x and y in the graph G

Chapter 1

Introduction

> *" The White Rabbit put on his spectacles. 'Where shall I begin, please your Majesty?' he asked. 'Begin at the beginning,' the King said gravely, 'and go on till you come to the end: then stop.' "*

- Lewis Carroll, *Alice in Wonderland*

Combinatorics is the study of arrangements and combinations of discrete objects. According to Mirsky [70],

> "combinatorics is a range of linked studies which have something in common and yet diverge widely in their objectives, their methods, and the degree of coherence they have attained. Most are concerned with criteria for the existence of certain 'patterns' or 'arrangements' or 'configurations', where these terms need to be interpreted in a very broad sense."

This thesis examines combinatorial designs in two settings, namely completions of partial designs and embeddings of partial designs.

The origins of *combinatorial design theory* date back to the 18th century and are rooted in recreational mathematics (brain-teasers, mathematical puzzles) such as so called "36 officers problem" by Euler in 1782 [35]. Combinatorial design theoretic ideas were present in the work of Euler, Kirkman, Cayley, Hamilton, Sylvester, Moore and others. However, design theory rapidly developed in the 20th century as an independent branch of combinatorics due to applications in the design and analysis of statistical experiments [80]. Applications of designs are not only limited to analysis of experiments, but also useful in network analysis, cryptography and communication protocols, error correcting codes, mathematical biology, algorithm design, tournament scheduling, lotteries, etc [26, 27, 30, 93, 45]. In general, design theory studies the question of possible arrangements of elements of a finite set into subsets fulfilling certain "balance" properties [78, 80].

All the graphs considered in this thesis are undirected, unweighted simple graphs (no multi edges and loops) unless stated otherwise. We will provide the necessary definitions to understand this study as needed. Any graph theoretic terminology which is not defined in this thesis can be found in texts like Diestel [36] and West [89] for example. Let $V(G)$ denote the vertex set of a graph G and $E(G)$ denote the edge set of G. Let G and H be graphs. A *partial H-design on G* is a collection $\mathcal{D}$ of edge disjoint subgraphs of G, each isomorphic to H, whose edge sets partition a subset of the edge set of G. When the edge sets of copies of H partition the edge set of G itself, the object is known as an *H-design* *on* G. We also sometimes refer to (partial) H-designs on G as *(partial) H-decompositions on G*. Furthermore, if each vertex of G is in the same number of copies of H, then the H-design is said to be *balanced*.

Remark 1.0.1. *Let H and G be graphs. An H-design on G is also a partial H-design on G.*

Definition 1.0.2. For graphs G and H, the *order* of an H-design on G is the cardinality of $V(G)$.

In particular, H-designs are used to solve construction problems occurring in graph theory, database systems and many related areas [86]. The general definition of H-designs was introduced by Hell and Rosa in 1972 in their work on the generalised version of the famous handcuffed prisoners problem [54].

> " In a jail there were nine prisoners of a particularly dangerous character. Each morning they are allowed to walk handcuffed in the prison yard. Here is how they walked on Monday: $1-2-3, 4-5-6, 7-8-9$. Can they be arranged for Tuesday through Saturday so that no pair of prisoners is handcuffed together twice?"

The solution to this problem can be obtained by constructing a so-called resolvable 2-path design on K_9. One construction is as follows:

Monday	$1-2-3$	$4-5-6$	$7-8-9$
Tuesday	$1-3-5$	$2-4-8$	$6-9-7$
Wednesday	$2-5-7$	$4-3-8$	$9-1-6$
Thursday	$5-1-4$	$3-6-7$	$9-2-8$
Friday	$2-6-8$	$1-7-4$	$3-9-5$
Saturday	$1-8-5$	$2-7-3$	$9-4-6$

1.1 Existence of H-designs

The problem of determining whether an H-design exists has been thoroughly investigated. Numerous articles, books and surveys has been written on this subject (see [1], [7], [86]). For any graph H, there are three obvious necessary conditions for the existence of an H-design on G [86]. These are as follows:

Lemma 1.1.1. *Let G be a graph. If there exists an H-design on G, then*

(1) $|V(H)| \leqslant |V(G)|$ *or* $E(G) = \emptyset$,

(2) $|E(G)| \equiv 0 \pmod{|E(H)|}$,

(3) $\deg_G(x) \equiv 0 \pmod{d}$ *for each* $x \in V(G)$, *where d is the greatest common divisor of the degrees of the vertices in H.*

Proof. When $E(G) = \emptyset$, G is simply a collection of isolated vertices, and an H-design on G trivially exists, namely the empty design. Now suppose that an H-design on G exists, and assume for a contradiction that $|V(H)| > |V(G)|$. Then it is impossible to decompose G into copies of H as G must have at least many vertices as H to have subgraphs isomorphic to H. Therefore, (1) holds. Let b be the number of edge disjoint copies of H in the design. Then $b = |E(G)|/|E(H)|$, thus (2) holds. Let x be a vertex in $V(G)$ and let $H_1, \dots, H_s$ be the copies of H in the design that contain x. Then $\deg_G(x) = \sum_{i=1}^{s} \deg_{H_i}(x)$. Therefore, (3) holds. □

We use K_n to denote the complete graph of order n and, for a set V of vertices, we use K_V to denote the complete graph with vertex set V. Here, we are especially interested in H-designs on complete graphs, and so we often refer to these simply as *H-designs* and to an H-design on K_n as simply an *H-design of order n*. We also denote the complete bipartite graph with parts of sizes m and n by $K_{m,n}$.

When G is K_n Wilson [91] proved that the conditions in Lemma 1.1.1 are also sufficient provided that n is large enough.

Theorem 1.1.2 (Wilson, [91])**.** *Let H be a graph. There is an integer n_0 such that for all positive integers $n \geqslant n_0$ for which the congruences (2) and (3) in Lemma 1.1.1 hold, there exists an H-design on K_n.*

We refer to Theorem 1.1.2 as Wilson's theorem. This result is considered to be the key theorem of graph decomposition theory. In order to prove this, Wilson showed that there are infinitely many prime values n for which K_n has an H-decomposition and that there exists a positive integer $n_H \equiv 0 \pmod{|E(H)|}$ such that, if K_{n_0} has an H-decomposition for some n_0 then K_n has an H-decomposition for all sufficiently large integers $n \equiv n_0 \pmod{n_H}$.

Definition 1.1.3. We call a graph G satisfying second and third properties in Lemma 1.1.1, *H-divisible*. We call a positive integer n *H-admissible* if K_n is H-divisible.

Definition 1.1.4. The *spectrum* of a graph H, spec(H) is the set S of positive integers given by $n \in S$ if and only if an H-design on K_n exists.

Note that for a given graph H, the set of admissible integers and the spectrum may not be equal. Obviously, the spectrum is a subset of the set of H-admissible integers, but it may or may not be a proper subset. According to the Wilson's theorem, there can be at most finitely many admissible integers which are not in the spectrum of H. As one example of this behaviour, the set of K_6-admissible integers is $\{n \in \mathbb{Z}^+ : n \equiv 1 \text{ or } 6 \pmod{15}\}$ while the spectrum of K_6 is known to contain all K_6-admissible integers greater than 801, is known not to contain 16, 21, 36 or 46, and may or may not contain 29 other small values of n (see [30, §II.3.1]). However, there are some graphs H for which the set of H-admissible integers and the spectrum of H are equal. For example, the graph K_3.

Definition 1.1.5. The *leave* of a partial H-design of G is the graph L having vertex set $V(G)$ and edge set comprising all edges of G that are not in a copy of H in the partial design. It is worth noting that, in a complete design, leave is the empty graph.

1.2 Completions of partial H-designs

When we are given a partial H-design, a natural thing to do is to try to complete it: to add copies of H to it to make a complete design. A *completion* of a partial H-design $\mathcal{A}$ on G is an H-design $\mathcal{B}$ on G such that $\mathcal{A} \subseteq \mathcal{B}$. We call a partial design *completable* when it has a completion. For a partial H-design on K_n to have a completion it is necessary that $n \in \mathrm{spec}(H)$ and hence that n is H-admissible. But this is not sufficient in general. For example, it is not difficult to see that a partial K_3-design on K_7 consisting of two vertex-disjoint copies of K_3 is not completable.

Remark 1.2.1. *Completing a partial H-design is equivalent to finding an H-design of its leave.*

A *partial latin square* of order n is an $n \times n$ array consisting of elements from $\{1, 2, \dots, n\}$, each occurring at most once in each row and in each column. If each element occurs exactly once in each row and in each column, then the resulting object is a *latin square*. A (partial) latin square of order n can also be viewed as a (partial) K_3-design on the complete tripartite graph $K_{n,n,n}$, where the three partite sets correspond to the rows, columns and symbols of the square. Arguably the most famous question concerning completions is Evans' conjecture on completions of partial latin squares. In 1960, Evans [42] conjectured that a partial latin square of order n with at most $n-1$ entries can be completed to a latin square of order n. This bound is tight because there are partial latin squares of order n with n entries which cannot be completed for each $n \geqslant 2$. For example, consider a partial latin square of order n having entries only in the main diagonal such that the first $n-1$ cells in the main diagonal contain a 1 and the last contains a 2.

$$\begin{pmatrix} 1 & - & \cdots & - & - \\ - & 1 & \cdots & - & - \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ - & - & \cdots & 1 & - \\ - & - & \cdots & - & 2 \end{pmatrix}$$

Smetaniuk [76] and Anderson and Hilton [4] independently proved Evans' conjecture for all n. Unlike the completions considered by Evans' conjecture, in this thesis, we will be exclusively interested in completions of H-designs on K_n.

Now we know that not all partial designs are completable, even when the order is an admissible integer. Then it is obvious to ask whether we can embed a given partial design in a complete design of a larger order, or whether we can embed a given partial design in another partial design of a larger order than the order of the original partial design.

1.3 Embeddings of partial H-designs

An *embedding* of a (partial) H-design $\mathcal{A}$ on G is an H-design $\mathcal{B}$ on a graph G' such that $\mathcal{A} \subseteq \mathcal{B}$ and G is a subgraph of G'. We say that $\mathcal{A}$ is *embedded* in $\mathcal{B}$. Intuitively, an embedding is an H-design such that a partial H-design of a smaller order resides within it. Observe that every H-design can be embedded in itself, and that is known as the trivial embedding. In this thesis, we will be primarily interested in embeddings into H-designs on K_n. In particular, by an *embedding of order* n we mean an embedding into an H-design on K_n.

Definition 1.3.1. The set of all orders for which a partial H-design has an embedding is known as its *embedding spectrum*.

Obviously, to embed a partial H-design in an H-design on K_n we must first know that an H-design on K_n exists. In fact, the embedding spectrum of an empty H-design is exactly the spectrum of H.

Mathematical study of embeddings of designs began in 1971 due to the work of Treash on K_3-designs [84]. It follows from Wilson's theorem that any partial H-design has a finite non-trivial embedding. However, the order of the embedding is exponential with respect to the order of the partial design.

Theorem 1.3.2 ([91])**.** *Let H be a graph, v be a positive integer and $\mathcal{D}$ be a partial H-design on K_v. Then there exists a positive integer $n > v$ such that $\mathcal{D}$ can be embedded in an H-design on K_n.*

Proof. Let $\mathcal{D}$ be a partial H-design on K_v. Then we can consider $\mathcal{D}$ as an H-design on some subgraph G of K_v. From Wilson's theorem (Theorem 1.1.2), we have that the spectrum of G is infinite. Let $n \in \operatorname{spec}(G)$ such that $n > v$. That is, K_n has a G-decomposition. Then, since each copy of G has an H-decomposition, K_n has an H-decomposition which is an embedding of $\mathcal{D}$ in an H-design on K_n. □

Remark 1.3.3. *Note that a completion of a partial H-design $\mathcal{A}$ on G is equivalent to an embedding of $\mathcal{A}$ into an H-design $\mathcal{B}$ on G.*

Now the existence of an embedding of a (partial) H-design of order n into an H-design of order $n+s$ for some positive integer s has been established, it is obvious to ask what values of s can be obtained. In particular, we can try to find results that guarantee the existence of an embedding of any (partial) H-design of order n into an H-design of order n' for all H-admissible integers $n' \geqslant n+s$ for integer s perhaps depending on n. For example, for partial K_3-designs on K_n (known as *partial Steiner triple systems* of order n or partial $(n,3,1)$-designs), a best possible result along these lines has been obtained (see Chapter 2, Subsection 2.1.3).

A partial H-design $\mathcal{A}$ is said to be *maximal* if its leave contains no copy of H. To prove that every partial H-design on a graph G has an embedding into an H-design on some other graph G', it suffices to consider only maximal partial H-designs on G.

Definition 1.3.4. Let G and H be vertex-disjoint graphs, we let $G \vee H$ denote the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{xy : x \in V(G), y \in V(H)\}$.

Remark 1.3.5. *Let G be a graph of order n and suppose that G has a partial H-design $\mathcal{D}$. We can embed $\mathcal{D}$ in an H-design on K_{n+s} for some nonnegative integer s if and only if there is an H-design on $L \vee K_s$, where L is the leave of $\mathcal{D}$.*

We will discuss embeddings of partial K_k-designs and embeddings of partial $K_{1,k}$-designs, which are the two cases most relevant to us, in more detail in Sections 2.1.3 and 2.2.1 respectively. It is worth noting at this point, however, that there has also been a large amount of work concerning embeddings of partial latin squares. For some of the most famous results, see [5, 31, 42, 75].

1.4 A note on computational complexity

In this section, we will briefly talk about some basic concepts in computational complexity. We then briefly discuss the computational complexity of determining which graphs have H-decompositions for a given graph H. Algorithmic approaches are extremely important for many problems in combinatorial design theory. The discussion in this section will be somewhat informal, but the reader can refer to [23] and [47] for more details on the theory of computational complexity.

For our purpose, we will describe the concepts related to computational complexity in the context of decision problems. *Decision problems* are the kind of problems that have only two possible solutions, either "yes" or "no". The standard format of a decision problem consist of two parts. First we define the instances (inputs) of the problem in terms of various components such as sets, graphs, numbers, functions etc. In the second part, we present the yes - no question in terms of corresponding instances (inputs). An *algorithm* for solving a decision problem is a step-by-step process that accepts an instance of the problem and outputs "yes" or "no" correctly according to the given instance. We call a decision problem a *decidable* problem when there exists an algorithm for solving it that always stops with the correct answer. The *time complexity function* of an algorithm expresses the maximum amount of time that it takes to solve the decision problem for each possible input size n. When the time complexity function $f(n)$ is bounded by some polynomial function $p(n)$, that is $f(n) \leqslant p(n)$ for all values of $n \geqslant 0$, or in other words when the algorithm requires at most polynomial amount of time to solve the given input, we call that algorithm a *polynomial time algorithm.* The set of all decision problems which have polynomial time algorithms is denoted by class P.

There are many decidable problems for which no polynomial time algorithm is known. Many of these problems belong to a class known as NP (non-deterministic polynomial time). A decision problem is said to belong to the class NP if every instance (input) for which the correct answer is yes, has a "certificate" or a "characteristic" of being a yes instance whose validity can be verified with a polynomial amount of computation. That is, a decision problem L is in NP if and only if there exist polynomials p and q and an algorithm A such that,

- For all instances x of L and certificates y, the algorithm A runs in time $p(|x|)$ on input (x, y), where $|x|$ denotes the size of the input x.
- For all instances x of L for which the correct answer is yes, there exists a certificate y of size $q(|x|)$ such that A outputs a yes on input (x, y)
- For all instances x of L for which the correct answer is no, and for all certificates y of size $q(|x|)$, A outputs a no on input (x, y).

Polynomial time reduction is a method for solving one decision problem in terms of another decision problem. Let L_1 and L_2 be two decision problems. A *polynomial transformation* or *polynomial-time many-one reduction* from L_1 to L_2 is a polynomial time algorithm Φ for transforming inputs to L_1 into inputs to L_2, such that for any input x of L_1, the answer to L_1 on input x is the same as the answer to L_2 on input $\Phi(x)$. Intuitively, this means that L_2 is at least hard as L_1 because we can solve L_1 by solving L_2. A decision problem is NP-*complete* if it belongs to class NP and all problems in NP have polynomial transformations to it.

To show that a decision problem L_2 is NP-complete, we first need to show its membership to NP. Next, we need to find a polynomial transformation to L_2 from a known NP-complete problem, say L_1. Intuitively, we need to show that L_2 is at least as hard as a problem known to be NP-complete. In Chapter 4, we show that a decision problem we call F-EMBED is NP-complete by reducing (transforming) to F-EMBED from the problem of whether a cubic graph is properly 3-edge colourable, which is well known to be NP-complete [61].

We now briefly discuss one decision problem which is extremely relevant to the topic of H-designs. Let H be a fixed graph. Then the H-decomposition problem can be stated as follows:

H-DECOMPOSITION

Input: A graph G.

Question: Does G have an H-decomposition?

It is known that H-DECOMPOSITION is polynomial if H has at most 2 edges [60]. In 1981, Holyer [60] proved that H-DECOMPOSITION is NP-complete when $H = K_n$ for $n \geqslant 3$. Moreover, he conjectured that H-DECOMPOSITION is NP-complete when H consists of at least 3 edges. Then in 1997, Dor and Tarsi [37] completely proved the Holyer's conjecture. Theorem 1.1.2 deals with H-DECOMPOSITION when the input graph G is restricted to be a copy of K_n for sufficiently large orders n.

1.5 Thesis outline

This thesis investigates four topics related to the completion or embedding of partial H-designs. The first chapter has given a general overview of H-designs, completions and embeddings of partial H-designs, laying the groundwork of the concepts that we use throughout this thesis. In the second chapter, we discuss existence, completion and embedding related problems for K_k-designs and $K_{1,k}$-designs.

We begin our journey in the third chapter, which deals with completions of partial K_k-designs on K_n. In this chapter, we determine exactly the minimum number of copies of K_k in an uncompletable partial K_k-design on K_n for all sufficiently large n. This result is reminiscent of Evans' now-proved conjecture on completions of partial latin squares. We also prove some related results concerning edge decompositions of almost complete graphs into copies of K_k.

In the fourth chapter, we present some complexity results regarding embeddings of partial K_3-designs. For a given partial K_3-design on K_u it is known that an embedding of order $v \geqslant 2u+1$ exists whenever v satisfies the obvious necessary conditions. Determining whether "small" embeddings of order $v < 2u + 1$ exist is a more difficult task. Here we extend a result of Colbourn on the NP-completeness of these problems. We also exhibit a family of counterexamples to a conjecture concerning when small embeddings exist.

In the fifth chapter, we consider the problem of when a partial $K_{1,k}$-design on K_n can be embedded in a $K_{1,k}$-design on K_{n+s} for a given integer s. We improve a result of Noble and Richardson, itself an improvement of a result of Hoffman and Roberts, by showing that any partial $K_{1,k}$-design on K_n can be embedded in a $K_{1,k}$-design on K_{n+s} for some s such that $s < \frac{9}{4}k$ when k is odd and $s < (6 - 2\sqrt{2})k$ when k is even. Moreover, we prove that for general k, these constants cannot be improved. We also obtain stronger results subject to placing a lower bound on n.

In the sixth chapter, we discuss completions of partial $K_{1,k}$-designs. We determine exactly the minimum number of copies of $K_{1,k}$ in an uncompletable partial $K_{1,k}$-design on K_n. This result is analogous to our main result in the third chapter. We conclude our journey in the seventh chapter, which gives a summary of this thesis and provides an overview of possible open problems arising from our work.

We attempt to keep each of Chapters 3, 4, 5 and 6 as self-contained as possible. For this reason, some of the definitions and results discussed so far will be restated within these chapters.

Chapter 2

Background

“ The difficulty of literature is not to write, but to write what you mean; not to affect your reader, but to affect him precisely as you wish. ”

- Robert Louis Stevenson

In this chapter we will discuss existence, completion problems and embedding problems for H-designs in the context of block designs and star designs.

2.1 Block designs

A *(partial) (n, k, λ)-design* is a (partial) K_k-design on the complete λ-fold multigraph of order n, λK_n. These are sometimes referred to as *balanced incomplete block designs* or simply as *block designs*. When $n > k$, the design is a *non-trivial design* and when $n = k$, it is a *trivial design*. We often write a (partial) (n, k, λ)-design as a pair $(V, \mathcal{A})$ where V is the vertex set of λK_n and $\mathcal{A}$ is the collection of vertex sets of edge disjoint copies of K_k. We sometimes call the elements of V *points* and the elements of $\mathcal{A}$ *blocks*. In this thesis, we only focus on $(n, k, 1)$-designs. The study of block designs dates back to 1835 due to Plücker's work on algebraic curves [30]. He came across a $(9, 3, 1)$-design and claimed that an $(n, 3, 1)$-design exists only when $n \equiv 3 \pmod 6$. However, in 1839, he changed this condition to $n \equiv 1, 3 \pmod 6$.

Recall that, by Lemma 1.1.1, if an $(n, k, 1)$-design $(V, \mathcal{A})$ exists, then $\binom{n}{2} \equiv 0 \pmod{\binom{k}{2}}$ and $n - 1 \equiv 0 \pmod{k-1}$. This implies that the total number of blocks in $\mathcal{A}$ is equal to $\frac{n(n-1)}{k(k-1)}$ and each point in V occurs in precisely $\frac{n-1}{k-1}$ blocks of $\mathcal{A}$.

Remark 2.1.1. *An $(n, 2, 1)$-design exists trivially for each integer $n \geqslant 2$.*

Fisher [44] famously proved that any non-trivial (n, k, λ)-design must have at least n blocks. Below we give the special case of this corresponding to $\lambda = 1$.

Theorem 2.1.2 (Fisher [44])**.** *Let $(V, \mathcal{A})$ be a non-trivial $(n, k, 1)$-design. Then, $n \geqslant k(k-1) + 1$.*

Proof. Let $B = \{x_1, x_2, \ldots, x_k\} \in \mathcal{A}$. There exists $x_0 \in V$ such that $x_0 \notin B$ since $(V, \mathcal{A})$ is non-trivial. For each $i \in \{1, \ldots, k\}$, there exists a unique block $B_i \in \mathcal{A}$ such that $\{x_0, x_i\} \subseteq B_i$ because x_0 cannot be in a block with more than one point from B. Note that each such B_i contains $k-2$ different points from $V \setminus (B \cup \{x_0\})$ because no two blocks both containing x_0 can share any other point. Therefore, $n \geqslant k(k-2)+k+1 = k(k-1)+1$. $\square$

The following is an immediate consequence of Wilson's theorem (Theorem 1.1.2).

Corollary 2.1.3 (Wilson [90])**.** *Let $k \geqslant 3$ be an integer. Then $(n, k, 1)$-designs exist for all sufficiently large integers n for which the following congruences hold:*

(1) $n - 1 \equiv 0 \pmod{k-1}$*,*

(2) $n(n-1) \equiv 0 \pmod{k(k-1)}$*.*

Hanani [52, 53], proved that "for all sufficiently large integers n" in Corollary 2.1.3 can be replaced by "for all integers n" whenever $k \leqslant 5$.

Example 2.1.4. No $(16, 6, 1)$-design can exist even though it satisfies the two congruences in Corollary 2.1.3. The reason is $n < k(k-1)+1 = 31$ and hence this contradicts Fisher's inequality given in Theorem 2.1.2.

Gustavsson [50] proved that every sufficiently large graph satisfying K_k-divisibility conditions and having high minimum degree has a K_k-decomposition.

Theorem 2.1.5 ([50])**.** *For every integer $k \geqslant 3$, there exists $\gamma, n_0 > 0$ such that every graph G with $n \geqslant n_0$ vertices and minimum degree $\delta(G) \geqslant (1-\gamma)n$, satisfying $|E(G)| \equiv 0 \pmod{\binom{k}{2}}$ and $\deg_G(x) \equiv 0 \pmod{(k-1)}$ for every $x \in V(G)$ has a K_k-decomposition.*

Significant attention has been paid to how large the value of γ can be made in Theorem 2.1.5. Montgomery [71] showed that γ can be taken as $\frac{1}{100k} - \epsilon$ for any $\epsilon > 0$ for each $k \geqslant 4$. For the special case $k = 3$, a succession of results has been obtained by Yuster [94], Garaschuk [46], Dross [39] and Dukes and Horsley [40], with the state of the art being that γ can be taken as 0.1726, due to Delcourt and Postle [32]. Nash-Williams' conjecture [72] asserts that γ can be taken as $\frac{1}{4}$, which would be the best possible if true.

In Chapter 3, for each k and each sufficiently large n, we find the minimum m such that every K_k-divisible graph of order n with at least m edges has a K_k-decomposition (see Theorem 3.1.3).

2.1.1 Steiner triple systems

The most studied type of block designs are $(n, 3, 1)$-designs, known as *Steiner triple systems*. We sometimes denote a (partial) Steiner triple system of order n by (P)STS(n). Steiner triple systems were first defined by Woolhouse in 1844 in the 1733 prize question of *Lady's and Gentleman's Diary*.

> "Determine the number of combinations that can be made out of n symbols, p symbols in each; with this limitation, that no combination of q symbols, which may appear in any one of them shall be repeated in any other."

In 1847, Kirkman gave an important answer to the $p = 3$ and $q = 2$ case of this problem by proving that a Steiner triple system of order n exists if and only if $n \equiv 1, 3 \pmod 6$ [64, 67]. We outline this result here because of its importance.

Theorem 2.1.6. *A Steiner triple system of order n exists if and only if $n \equiv 1, 3 \pmod 6$.*

Suppose that a Steiner triple system of order n exists. Then by Lemma 1.1.1 it is necessary that n obeys the following two congruences: $n(n-1) \equiv 0 \pmod 6$ and $n-1 \equiv 0 \pmod 2$. This implies $n \equiv 1, 3, 5 \pmod 6$ because n needs to be odd. However, if $n \equiv 5 \pmod 6$, then $n(n-1) \not\equiv 0 \pmod 6$. Therefore, $n \equiv 1, 3 \pmod 6$. Establishing that a Steiner triple system of order n exists if $n \equiv 1, 3 \pmod 6$, can be accomplished by two well known constructions:

(i) $n \equiv 3 \pmod 6$: Bose construction [67, Section 1.2];

(ii) $n \equiv 1 \pmod 6$: Skolem construction [67, Section 1.3].

Another remarkable problem in the history of Steiner triple systems is Kirkman's schoolgirl problem (1850) [28].

> "Fifteen young ladies in a school walk out three abreast for seven days in succession: it is required to arrange them daily, so that no two shall walk twice abreast."

The answer to this problem is a $(15, 3, 1)$-design with an additional property known as resolvability. This recreational problem got the attention of many, and a number of mathematicians studied the problem and its generalizations.

Characterizing the leaves of partial Steiner triple systems

Recall that the leave of a partial Steiner triple system, $(U, \mathcal{A})$ is the graph L having vertex set U and edge set $E(L) = \{xy : \{x, y, z\} \notin \mathcal{A}$ for any $z \in U\}$. It is interesting to know that when a given graph G can be the leave of a PSTS(n), where n is the number of vertices of G.

Definition 2.1.7. For graphs G and H we define $G - H$ to be the graph with vertex set $V(G)$ and edge set $E(G) \setminus E(H)$.

Determining whether a graph G is the leave of some PSTS(n) is equivalent to asking whether $K_n - G$ has a K_3-decomposition. For an example, $K_{1,3}$ is the leave of the PSTS(4) having just one triple.

One important reason for studying partial Steiner triple systems is to determine what substructures arise in Steiner triple systems. This naturally leads to the investigation of completion and embedding of partial designs. Characterization of leaves is useful in the search for small order embeddings of partial Steiner triple systems (which we will define later) [25].

Some necessary divisibility conditions for a given graph to be a leave of a partial Steiner triple system, which are given in the following lemma, can be easily observed.

Definition 2.1.8. For a graph G, we denote the minimum and maximum degree of G by $\delta(G)$ and $\Delta(G)$ and denote the complement of G by $\overline{G}$.

Lemma 2.1.9 ([28])**.** *If G is a leave of a PSTS$(|V(G)|)$, then $\deg_G(x) \equiv |V(G)| - 1 \pmod 2$ for each $x \in V(G)$ and $\binom{|V(G)|}{2} \equiv |E(G)| \pmod 3$.*

Proof. Observe that, a complete graph on $|V(G)|$ vertices has degree at each vertex equal to $|V(G)| - 1$ and number of edges equal to $\binom{|V(G)|}{2}$. Since G is a leave of a PSTS$(|V(G)|)$, $\overline{G}$ is a union of edge disjoint copies of K_3. Thus, we have the result. □

However, the congruences in the above lemma are not sufficient for a graph G to be a leave of a PSTS($|V(G)|$). To illustrate this, consider the following example.

Example 2.1.10. Let G be a graph on 7 vertices consisting of two vertex disjoint copies of K_3 and an isolated vertex. Without loss of generality, suppose that $\{1,2,3\}$ and $\{4,5,6\}$ are the vertex sets of copies of K_3 and 7 is the isolated vertex. We have $\deg_G(x) \equiv |V(G)| - 1 \pmod 2$ for each $x \in V(G)$ and $\binom{|V(G)|}{2} \equiv |E(G)| \pmod 3$. If G is a leave of PSTS(7), then $\overline{G}$ has a K_3-decomposition. Thus, the four edges incident to the vertex 1 in $\overline{G}$ need to be used in exactly two copies of K_3, but that is a contradiction because the edges 45, 56 and 46 are used in G. Therefore, G cannot be a leave of a PSTS(7).

In 1970, Nash-Williams [72] gave a construction for an infinite family of graphs satisfying necessary conditions given in Lemma 2.1.9 but which are not leaves. Colbourn and Rosa gave a slightly generalized construction of such graphs [28]. Moreover, they obtained the necessary density conditions for a given graph to be a leave of a partial Steiner triple system.

Definition 2.1.11. Let G be a graph whose vertex set is partitioned into two sets A and B. We call an edge of G a *cross edge*, whenever it contains one vertex from each class, otherwise it is an *inside edge*.

Lemma 2.1.12 ([28]). *Let G be a graph with n vertices and e edges. If the vertices of G can be partitioned into sets of sizes s and $n-s$ so that G has c cross edges, then G is the leave of a partial Steiner triple system only if*

$$2\left(\binom{s}{2} + \binom{n-s}{2} - e + c\right) \geqslant s(n-s) - c.$$

Proof. Note that K_n has $\binom{s}{2} + \binom{n-s}{2}$ inside edges and $s(n-s)$ cross edges. Since G is a leave of a partial Steiner triple system, $\overline{G}$ has a K_3-decomposition. Then the number of inside edges of $\overline{G}$, $\binom{s}{2} + \binom{n-s}{2} - (e-c)$, must be at least the half of the number of cross edges of $\overline{G}$, $s(n-s) - c$, because each K_3 in $\overline{G}$ contains at least one inside edge. □

Observe that the inequality in above lemma is not sufficient for a graph G to be a leave of a PSTS($|V(G)|$). To illustrate this consider the following example.

Example 2.1.13. Let G be a graph on 15 vertices that is the union of six copies of K_3 whose vertex sets are as follows:

$$\{\{1,2,3\},\{1,4,5\},\{1,6,7\},\{1,8,9\},\{1,10,11\},\{12,13,14\}\}.$$

For instance, for the partition given by $A = \{1,2,\ldots,11\}$ and $B = \{12,13,14,15\}$, G has 18 edges in total and 0 cross edges. So we can see that G satisfies the inequality of Lemma 2.1.12 for this partition. In fact, it can be seen that the inequality holds when $n = 15$ and $e = 18$ for any $s \in \{0,\ldots,15\}$ and $c \geqslant 0$. So G satisfies the inequality for every partition. However, $\overline{G}$ does not have a K_3-decomposition because the pairs $\{1,12\}$, $\{1,13\}$ and $\{1,14\}$ must be in different triples and that is impossible.

Definition 2.1.14. An *(proper) edge colouring* of a graph G is an assignment of colours to the edges of G such that no pair of adjacent edges receive the same colour. We call a graph G, *k-edge colourable* if it has an edge colouring using k or fewer colours.

It is not so straightforward to establish sufficient conditions for a graph to be a leave of a partial Steiner triple system. In 1983, Colbourn showed that the problem of determining whether a given graph is a leave of a partial Steiner triple system is NP-complete (see [21]). He used the fact that deciding 3-edge colourability of cubic graphs is NP-complete due to Holyer ([61]), to prove this.

Recall that a partial Steiner triple system is said to be *maximal*, when there are no copies of K_3 in the leave. Leaves of maximal partial Steiner triple systems are studied in [24], [55], [79, §40.4] and [77]. In particular, the following result concerning graphs with maximum degree 2 has been obtained.

Theorem 2.1.15 ([24], [55])**.** *Let G be a triangle-free graph satisfying the congruences in Lemma 2.1.9 such that every vertex of G has degree 0 or 2. Then G is a leave of a maximal partial Steiner triple system if and only if G is not the vertex disjoint union of a 4-cycle and a 5-cycle (with no isolated vertices).*

We say a graph is *even (odd)* whenever all the vertex degrees are even (odd). In [25], Colbourn proved that any even graph with the number of edges being a multiple of 3 can be extended to a leave by adding polynomially many isolated vertices. Moreover, he proved that any odd graph having at most two copies of $K_{1,3}$ can be extended to a leave by adding polynomially many vertex disjoint edges.

Related to our concerns in Chapter 3, the possible sizes of K_3-*free* graphs whose complements are K_3-divisible but not K_3-decomposable are considered in [77]. Our results in Chapter 3, in particular Corollary 3.1.5 improve the lower bounds in that paper for sufficiently large order graphs.

2.1.2 Completions of partial block designs

Recall that, according to Theorem 1.3.2, we know that partial block designs have finite non-trivial embeddings. However, there are not many completion and embedding results available for partial block designs in general. Some of this lack is inevitable because the problem of existence for block designs with blocks of size at least 6 is not yet resolved for small orders. In fact, most of the investigation of questions concerning completions and embeddings of partial block designs has focussed on the case of partial Steiner triple systems, and consequently our discussion will also concentrate on this case.

A *completion* of a partial $(n, k, 1)$-design $(V, \mathcal{A})$ is a (complete) $(n, k, 1)$-design $(V, \mathcal{B})$ such that $\mathcal{A} \subseteq \mathcal{B}$. A partial $(n, k, 1)$-design is *completable* when it has a completion. Recall that the *leave* of a partial $(n, k, 1)$-design $(V, \mathcal{A})$ is the graph G having vertex set V and the edge set $E(G) = \{xy : x, y \in V \text{ such that } \{x, y\} \nsubseteq A \text{ for all } A \in \mathcal{A}\}$. Finding a completion of a partial $(n, k, 1)$-design is equivalent to finding a K_k-decomposition of its leave. However, not all partial $(n, k, 1)$-designs are completable; to be completable n must be K_k-admissible, but this is by no means sufficient. To illustrate this, consider the following toy example of a partial $(9, 3, 1)$-design which is not completable.

Example 2.1.16. Let $(V, \mathcal{A})$, where $V = \{1, \dots, 9\}$ and $\mathcal{A} = \{\{1, 2, 3\}, \{1, 4, 5\}, \{6, 7, 8\}\}$, be a partial $(9, 3, 1)$-design. Clearly $\mathcal{A}$ is not completable because the pairs $\{1, 6\}, \{1, 7\}$ and $\{1, 8\}$ must be in different blocks and that is impossible.

Since not all partial Steiner triple systems are completable, one can try to characterize which partial Steiner triple systems of K_3-admissible orders are completable. But Colbourn [21] proved that it is not an easy task by proving it is NP-complete to decide whether a given partial Steiner triple system can be completed.

Theorem 2.1.17 ([21]). *Deciding whether a partial Steiner triple system of a K_3-admissible order can be completed is* NP*-complete.*

In [22] it is observed that two families of partial Steiner triple systems are easily seen to be completable. These families are partial Steiner triple systems in which some fixed point is in every triple and partial Steiner triple systems consisting of an odd number of pairwise disjoint triples. Any Steiner triple system of the appropriate order will contain a copy of any partial Steiner triple system in the first family. A so-called Kirkman triple system when the order is 3 modulo 6 and a so-called Hanani triple system when the order is 1 modulo 6 will contain a copy of any partial Steiner triple system in the second family. Kirkman and Hanani triple systems are known to exist for the appropriate orders (see [28]).

Colbourn et al. showed that a partial Steiner triple system is completable if it has two points x and y such that one block contains both x and y and each other block contains either x or y. These partial Steiner triple systems are known as double star.

Theorem 2.1.18 ([22]). *Any double star can be completed to a Steiner triple system.*

In 2014, Horsley [62] made the following conjecture concerning completions of partial Steiner triple systems with very few blocks.

Conjecture 2.1.19 ([62]). *Every partial Steiner triple system of K_3-admissible order $n \geqslant 7$ with at most $\frac{1}{2}(n-5)$ blocks has a completion.*

In Chapter 3 we establish a generalization of Conjecture 2.1.19 for sufficiently large n.

Theorem 2.1.20. *Let $k \geqslant 3$ be a fixed integer. There is an integer n_0 such that for all K_k-admissible integers $n \geqslant n_0$, any partial $(n,k,1)$-design having at most $\frac{n-1}{k-1} - k + 1$ blocks is completable. Furthermore, for all K_k-admissible integers $n \geqslant (k-1)^2 + 1$ there is a partial $(n,k,1)$-design with $\frac{n-1}{k-1} - k + 2$ blocks that is not completable.*

Theorem 2.1.20 complements a recent result of Nenadov et al. [73], who showed that any partial $(n,k,1)$-design of large order with few blocks can be "almost completed". To be precise, for each $k \geqslant 3$, they showed that there exist $\epsilon, n_0 > 0$ such that we can add blocks to any partial $(n,k,1)$-design $(V,\mathcal{A})$ with $n > n_0$ and $|\mathcal{A}| \leqslant \epsilon n^2$ to obtain another partial $(n,k,1)$-design whose leave has at most $21k^3\sqrt{|\mathcal{A}|}\,n$ edges.

2.1.3 Embeddings of partial block designs

An *embedding* of a partial $(n,k,1)$-design $(V,\mathcal{A})$ is a (complete) $(n',k,1)$-design $(W,\mathcal{B})$ such that $V \subseteq W$ and $\mathcal{A} \subseteq \mathcal{B}$. Recall that Theorem 1.3.2 implies the existence of finite embeddings of partial $(n,k,1)$-designs. Aside from this, there are few embedding results for partial $(n,k,1)$-designs when $k > 3$. One such result was recently obtained by Nenadov et al., however.

Theorem 2.1.21 ([73]). *For every integer $k \geqslant 3$, there exist $\epsilon, n_0 > 0$ such that for any partial $(n,k,1)$-design $(V,\mathcal{A})$ with $n > n_0$ and $|\mathcal{A}| \leqslant \epsilon n^2$, there exist an embedding of $(V,\mathcal{A})$ of order at most $n + 7k^2\sqrt{|\mathcal{A}|}$.*

We now discuss embeddings in the context of Steiner triple systems. In 1973, Doyen and Wilson proved that any (complete) Steiner triple system of order v can be embedded in some Steiner triple system of order w if and only if $w \geqslant 2v + 1$ [38].

Theorem 2.1.22 ([38]). *Any Steiner triple system of order v can be embedded in a Steiner triple system of order w if and only if w is K_3-admissible and $w \geqslant 2v+1$.*

Determining when embeddings of partial Steiner triple systems exist is a more complicated question, however. In 1971 Treash [84] proved that every partial Steiner triple system has a finite embedding, but the embeddings she constructed were of some exponential order in the order of the original system. This result opened the doors for a large collection of embedding related results on Steiner triple systems and other combinatorial designs. The key ingredient of her proof is an inductive argument based on a construction that embeds a (complete) Steiner triple system of order u in a (complete) Steiner triple system of order $2u+1$.

Lemma 2.1.23 ($u \to 2u+1$ construction [84]). *Let $(U, \mathcal{A})$ be a given Steiner triple system where $U = \{1, 2, 3, \ldots, u\}$. Let $U' = U \cup \{x_0, x_1, x_2 \ldots, x_u\}$. Define $\mathcal{A}'$ the set of triples given as follows:*

- $\mathcal{A} \subseteq \mathcal{A}'$
- $\{x_i, x_j, k\} \in \mathcal{A}'$ *whenever* $\{i, j, k\} \in \mathcal{A}$
- $\{x_0, x_i, i\} \in \mathcal{A}'$ *for all* $i \in U$.

Then $(U', \mathcal{A}')$ is a Steiner triple system and moreover is an embedding of $(U, \mathcal{A})$.

Proof. It is only a routine task to check that $(U', \mathcal{A}')$ is actually a Steiner triple system. $\square$

Consider the following example of the $u \to 2u+1$ construction when $u = 3$.

Example 2.1.24. Let $(U, \mathcal{A})$ where $\mathcal{A} = \{\{1, 2, 3\}\}$ be a Steiner triple system of order 3. Define $U' = \{1, 2, 3, x_0, x_1, x_2, x_3\}$ and

$$\mathcal{A}' = \{\{1, 2, 3\}, \{x_1, x_2, 3\}, \{1, x_2, x_3\}, \{x_1, 2, x_3\}, \{x_0, x_1, 1\}, \{x_0, x_2, 2\}, \{x_0, x_3, 3\}\}.$$

Obviously, $(U', \mathcal{A}')$ is a Steiner triple system of order 7 and moreover is an embedding of $(U, \mathcal{A})$

Treash was able to iteratively apply a slight variation on this construction to build a Steiner triple system containing a copy of the partial Steiner triple system to be embedded.

Then in 1975 Lindner [67] was able to reduce the order of the embedding. He proved that a partial Steiner triple system of order u can be embedded in a Steiner triple system of order v for any $v \geqslant 6u+3$ and $v \equiv 1, 3 \pmod 6$. Moreover, he conjectured that the lower bound can be modified to $v \geqslant 2u+1$ for any $v \equiv 1, 3 \pmod 6$. In 1980, Anderson, Hilton and Mendelsohn [3] and in 2004, Bryant [11] further reduced the bound to $v \geqslant 4u+1$ and $v \geqslant 3u-2$ respectively. Finally, in 2009, Bryant and Horsley [14] were able to prove Lindner's conjecture.

Theorem 2.1.25 ([14]). *Any partial Steiner triple system of order u can be embedded in a Steiner triple system of order v if $v \equiv 1, 3 \pmod 6$ and $v \geqslant 2u+1$.*

The proof of Theorem 2.1.25 is based on using so-called *edge switching techniques* to progressively modify a partial embedding until it eventually becomes a complete embedding. Bryant and Horsley separately considered the cases when the partial Steiner triple system has few triples and when it has many triples relative to its order, and finally combined these results to get the desired outcome.

The bound of $v \geqslant 2u+1$ in Theorem 2.1.25 cannot be improved in general due to the fact that for each $u \geqslant 9$ there exists a PSTS(u) which cannot be embedded in an STS(v) for any $v < 2u+1$. It is known that for each odd $u \geqslant 9$, there exists a PSTS(u) whose leave L is a union of a cycle of length 4 or 6 and some isolated vertices (see Theorem 9.15, [28]). Suppose that this PSTS(u) is embedded in an STS($u+w$) on point set $V(L) \cup W$ where $|W| = w$ for some integer w. Clearly $w \neq 0$. Consider the partition $\{V(L), W\}$ of $V(L \vee K_W)$. Then $6 + \binom{w}{2} \geqslant \frac{1}{2}uw$ due to the fact that the number of inside edges of $L \vee K_W$ is least half the number of cross edges of $L \vee K_W$. This implies $w \geqslant u+1$, noting that $w \equiv u+1 \pmod 2$, that $w \neq 0$ and that $u \geqslant 9$. For each even $u \geqslant 10$, by deleting a point in the cycle and all the triples that contain it in one of the PSTSs just discussed, we can obtain PSTS(u) whose leave has either $\frac{u}{2}+1$ or $\frac{u}{2}+3$ edges. A similar argument shows that this PSTS does not have an embedding of order less than $2u+1$.

Many partial Steiner triple systems, however, do have embeddings of order less than $2u+1$. We call such embeddings *small* embeddings. To illustrate this, consider the following example.

Example 2.1.26. Let $(U, \mathcal{A})$ where $\mathcal{A} = \{\{1,2,3\},\{1,4,7\},\{2,6,7\},\{3,5,7\},\{4,5,6\}\}$ be a PSTS(7). It can be embedded in an STS(9), $(V, \mathcal{B})$ where, $V = U \cup \{8,9\}$ and

$$\mathcal{B} = \mathcal{A} \cup \{\{1,5,9\},\{1,6,8\},\{2,4,9\},\{2,5,8\},\{3,4,8\},\{3,6,9\},\{7,8,9\}\}.$$

Small embeddings of partial Steiner triple systems

Even though the bound $2u+1$ is sharp in general, it has been investigated whether this bound can be reduced for certain special partial systems. It turned out that the bound can be modified for some sparse partial Steiner triple systems (partial systems having few triples with respect to their order). Horsley [62] showed that every partial Steiner triple system of order $u \geqslant 62$ having at most $\frac{u^2}{50} - \frac{11u}{100} - \frac{116}{75}$ triples has an embedding of order v for each K_3-admissible integer $v \geqslant \frac{1}{5}(8u+17)$. The $k=3$ case of Theorem 2.1.21 states that for a real constant ϵ, a partial Steiner triple system of order u having $t \leqslant \epsilon u^2$ triples has an embedding of order at most $u + O(\sqrt{t})$.

Recall that we call the set of all orders for which a partial Steiner triple system has an embedding its embedding spectrum. In [12], Bryant et al. found the complete embedding spectrum of all PSTS(10) having cubic leaves. Moreover, in [13] Bryant and Horsley determined the embedding spectrum of partial Steiner triple systems whose leave is a complete bipartite graph.

Theorem 2.1.27 ([13]). *A PSTS($u+w$) with the leave being $K_{u,w}$ can be embedded in an STS(v) if and only if*

(1) u, v *and* w *are odd;*

(2) $\binom{v}{2} - \binom{u}{2} - \binom{w}{2} \equiv 0 \pmod 3$*; and*

(3) $v \geqslant u + w + \max\{u, w\}$.

Horsley [63], has used so-called edge switching techniques to establish the existence of embeddings of certain orders for partial Steiner triple systems with small leaves (having few edges relative to the order) of low maximum degree.

Determining whether a given partial Steiner triple system has a small embedding is hard in general. In [21] Colbourn proved that the problem of determining whether a given partial Steiner triple system has a small embedding is NP-complete. Formally, consider the following decision problem.

SMALL-EMBED
Instance: A partial Steiner triple system $(U, \mathcal{A})$.
Question: Does $(U, \mathcal{A})$ have an embedding of order less than $2|U| + 1$?

Theorem 2.1.28 ([21]). SMALL-EMBED *is* NP*-complete.*

To prove this, he used the following lemma, which is in turn proved by constructing specific partial Steiner triple systems with the help of so-called Latin backgrounds.

Lemma 2.1.29 ([21]). *For every cubic graph G there is a PSTS(u) $(U, \mathcal{A})$ such that $(U, \mathcal{A})$ has no embedding of order v for $u < v < 2u + 1$ and $(U, \mathcal{A})$ is completable if and only if G is 3-edge-colourable.*

This lemma allows Colbourn to reduce the problem of whether a cubic graph has a 3-edge colouring to SMALL-EMBED. Because the former problem is known to be NP-complete, this establishes his result.

Now we can see that there are reasonable questions about small embeddings that Colbourn's result (Theorem 2.1.28) does not cover. For example, we could ask: when does a given partial Steiner triple system have an embedding of order $u + 15$? Similarly, we could ask: when does a given partial Steiner triple system have an embedding of order between $\frac{6u}{5}$ and $\frac{7u}{5}$? Colbourn's result does not say whether either of these questions are NP-complete. We have obtained a result that shows questions of the kind we gave above are also hard (see Chapter 4).

In [10] Bryant made a conjecture about the existence of K_3-decompositions of $L \vee K_w$. Recall that a partial Steiner triple system of order u with a leave L can be embedded in a Steiner triple system of order $v = u + w$ if and only if there exists a K_3-decomposition of $L \vee K_w$. Bryant conjectured that certain conditions that can be seen to be necessary for the existence of a K_3-decomposition of $L \vee K_w$ are also sufficient.

Conjecture 2.1.30 ([10]). *Let L be a graph with u vertices, and let w be a nonnegative integer. Then there exists a K_3-decomposition of $L \vee K_w$ if and only if following conditions are satisfied.*

(1) $\deg_L(x) \equiv w \pmod 2$ *for each vertex x of L;*

(2) *$u + w$ is odd for $w > 0$;*

(3) $|E(L)| + uw + \binom{w}{2} \equiv 0 \pmod 3$*; and*

(4) *There exists a subgraph G of L such that*

  (i) *$L - G$ has a K_3-decomposition;*

  (ii) $w^2 - (u + 1)w + 2|E(G)| \geqslant 0$*;*

  (iii) *G is w-edge colourable.*

In the above conjecture, necessity can be easily observed. Let W be a set of w vertices disjoint from $V(L)$ and suppose that a K_3-decomposition $\mathcal{D}$ of $L \vee K_W$ exists. Then for each $x \in V(L)$, $\deg_{L\vee K_W}(x) = \deg_L(x) + w \equiv 0 \pmod 2$. Thus, (1) holds. Clearly $u + w$ is odd if $w > 0$ because $\deg_{L\vee K_W}(z) = u + w - 1 \equiv 0 \pmod 2$ for each $z \in V(K_W)$. Thus, (2) holds. Moreover, $|E(L \vee K_W)| = |E(L)| + uw + \binom{w}{2} \equiv 0 \pmod 3$, hence (3).

To prove (4) we define a graph H as follows: let H be a spanning subgraph of L such that whenever $\{x, y, z\} \in \mathcal{D}$ with $x, y, z \in V(L)$, we have $xy, xz, yz \in E(H)$. This implies H has a K_3-decomposition. Let $G = L - H$. Each copy of K_3 in $\mathcal{D}$ contains at least one inside edge with respect to the partition $\{V(L), W\}$ (recall the Definition 2.1.11) and therefore $|E(G)| + \binom{w}{2} \geqslant \frac{1}{2}uw$ or, equivalently, $w^2 - (u + 1)w + 2|E(G)| \geqslant 0$. By the definitions of H and G, the vertex set of each copy of K_3 in $\mathcal{D}$ containing an edge of G contains a vertex from W. We can define an edge colouring of G with colour set W as follows: colour each edge xy in G with the element z of W such that $\{x, y, z\} \in \mathcal{D}$. Thus G is w-edge colourable.

Bryant [10] proved that Conjecture 2.1.30 holds when $\Delta(L) \leqslant 2$. Furthermore, he gave the necessary and sufficient conditions for a maximal partial Steiner triple system of order u having a non-empty leave with degrees either 0 or some d to have an embedding of order $u + d$. Clearly, $u + d < 2u + 1$ as $d < u$. He also found the embedding spectrum when $d \leqslant 2$.

Conjecture 2.1.30 postulates a neat characterization of the existence of embeddings of small orders in terms of the well studied problems of K_3-decomposition and proper edge colouring of graphs. In Chapter 4 we have provided a family of counterexamples to Bryant's conjecture, suggesting that things may not be so simple.

Theorem 2.1.31. *For each even integer $w \geqslant 4$, there is a partial Steiner triple system whose leave is a counterexample to Conjecture 2.1.30.*

2.2 k-star designs

A *k-star* is a complete bipartite graph $K_{1,k}$. Let G be a graph, a (partial) k-star design on G is a (partial) $K_{1,k}$-design on G. The vertex of degree k in a k-star is called its *centre* and other k vertices are called *leaf vertices* or *tail vertices*. Star decompositions are widely used in optimisation problems such as resource allocation [43], parallel computing, computer networks, generating optimal binary-valued balanced file organizing schemes [83] etc.

The problem of decomposing a graph into k-stars has been thoroughly investigated since the 1970s. Before 1974, in unpublished work, Ae, Yamamoto and Yoshida showed that K_{3n} for $n > 1$ is 3-star decomposable [2]. Cain [16] proved that the necessary and sufficient conditions for K_{mk} having a k-star decomposition are m being even or k being odd. Moreover, she proved that K_{mk+1} has a k-star decomposition if K_{mk} has a k-star decomposition. The problem of when there exists a decomposition of a complete graph into stars of uniform size was independently resolved by Tarsi [81] and Yamamoto et al. [92]. Tarsi gave necessary and sufficient conditions for the existence of a decomposition of a complete multigraph into k-stars, while Yamamoto et al. proved the simple graph case along with an analogous statement for complete bipartite graphs. We state the main theorem and then briefly outline the main elements of their proofs.

Theorem 2.2.1. *[[81], [92]] A k-star design on K_n exists if and only if*

(1) $n \geqslant 2k$ and

(2) $n(n-1) \equiv 0 \pmod{2k}$.

In the above theorem, necessity can be easily seen. First, suppose that a k-star design of order n exists. Then obviously $\binom{n}{2} \equiv 0 \pmod{k}$, which is equivalent to $n(n-1) \equiv 0 \pmod{2k}$. If K_n has a k-star decomposition, then at most one vertex can have zero stars centred at it because for each edge of K_n, there must be at least one k-star centred on at least one end vertex. Therefore, $n-1 \leqslant \frac{n(n-1)}{2k}$, which is equivalent to $n \geqslant 2k$.

To show the sufficiency, Tarsi's proof carefully constructs an orientation of K_n such that the outdegree of each vertex is divisible by k. Then the edges directed out from each vertex can be partitioned into a number of k-stars centred at that vertex. Together these stars form a k-star decomposition of K_n.

Yamamoto et al. observed that the edge set of K_n can be identified with the triangular set $T = \{(i,j) : 1 \leqslant i < j \leqslant n\}$ of $\binom{n}{2}$ lattice points (i,j). A k-star can be identified with a subset of T composed of k lattice points such that, for some $i \in \{1,\ldots,n-1\}$, each of the points is in the i^{th} row or the i^{th} column. They call such a set of points a star-type subset of T. They completed the sufficiency part of the proof of Theorem 2.2.1 by giving an algorithm for decomposing the set T into $\binom{n}{2}/k$ mutually disjoint star-type subsets with k points.

Yamamoto et al. [92] showed that the necessary and sufficient conditions for existence of a k-star decomposition of the complete bipartite graph $K_{m,n}$ are $mn \equiv 0 \pmod{k}$ and if $k > m$ then $n \equiv 0 \pmod{k}$ or if $k > n$ then $m \equiv 0 \pmod{k}$. Obviously, there is no k-star decomposition if both m and n are strictly less than k. Observe that, the edge set of $K_{n,m}$ can be identified with the rectangular set $R = \{(i,j) : 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n\}$ of mn lattice points (i,j). A k-star can be identified with a subset of R composed of k lattice points lies in the same row or same column, and such a set is called a star-type subset of R. Yamamoto et al.'s proof involves decomposition of the rectangular set R of mn lattice points into the union of mn/k mutually disjoint star-type subsets with k points. Hoffman and Roberts [58] point out that this result can also be deduced from the main theorem of [56] which concerns $K_{a,b}$ decompositions of $K_{m,n}$ where a,b,m,n are positive integers.

An obvious necessary condition for an arbitrary graph to have a k-star decomposition is that its number of edges is divisible by k. Trivially, any graph has a decomposition into 1-stars. Let G be a connected graph. In 1980, Caro and Schönheim proved that the obvious necessary condition of number of edges of G being divisible by 2 is also sufficient for the existence of a 2-star design on G. If G is not connected, then we can consider its connected components. An edge in an connected graph is a *bridge* if removing it disconnects the graph.

Theorem 2.2.2 ([19]). *A 2-star design on a connected graph G exists if and only if $|E(G)| \equiv 0 \pmod{2}$.*

Proof. Throughout this proof, for a graph G, we let $G - xy$ be the graph obtained from G by deleting the edge xy (if $xy \in E(G)$) and let $G \cup xy$ be the graph obtained from G by adding the edge xy (if $xy \notin E(G)$). We also let $xy \cup xz$ denote the 2-star whose edges are xy and xz.

Let G be a connected graph. If G has a 2-star decomposition, then obviously its number of edges is divisible by 2. Now suppose that $|E(G)| \equiv 0 \pmod{2}$. We proceed by induction on $|E(G)|$. If $|E(G)| = 2$, then G must be a 2-star since G is connected. Therefore, we assume $|E(G)| \geqslant 4$.

Case 1: Suppose that G has a bridge, say xy. Then, since G is connected, $G - xy$ has two connected components: one, say A, having an odd number of edges and the other, say B, having an even number of edges. Without loss of generality, suppose that $x \in V(A)$ and $y \in V(B)$. Suppose that B has at least two edges, then by inductive hypothesis we can show that $A \cup xy$ and B have 2-star decompositions and hence G has a 2-star decomposition.

If B is the single vertex $\{y\}$, then we remove another edge, say xz in A (which is adjacent to xy). If $A - xz$ is still connected, then it has an even number of edges and $xy \cup xz$ is a 2-star, therefore again by inductive hypothesis G has a 2-star decomposition. Otherwise, $A - xz$ splits into two connected components A_1 and A_2 both having an even number of edges or odd number of edges. Without loss of generality, suppose that $x \in V(A_1)$ and $z \in V(A_2)$. If they both have an even number of edges, then we can apply induction separately to A_1, A_2 and $xy \cup xz$. If they have an odd number of edges, then we can consider $A_1 \cup xy$ and $A_2 \cup zx$ and then apply induction.

Case 2: Suppose that G has no bridge. Then for any $xy \in E(G)$, $G - xy$ is connected. Consider another edge which is adjacent to xy, without loss of generality, suppose that edge is xz. Next consider $G - xy - xz$. If $G - xy - xz$ is connected, then we can apply the induction hypothesis to $G - xy - xz$ and $xy \cup xz$. If $G - xy - xz$ is disconnected, then it has exactly two connected components, say A and B, both having an even number of edges or an odd number of edges. Without loss of generality, suppose that $x \in V(A)$ and $y \in V(B)$. If A and B both have an even number of edges, we can apply induction separately to A, B and $xy \cup xz$. If they both have an odd number of edges, then consider $A \cup xz$ and $B \cup xy$. Then we can apply induction to $A \cup xy$ and $B \cup xz$ and hence G has a 2-star decomposition. □

In 1981, Tarsi established some sufficient conditions for the decomposition of a graph into stars of specified sizes. One consequence of these results that will be useful in this thesis is that, if a graph G has moderately high vertex degrees and if its number of edges is divisible by k, then G has a k-star decomposition.

Theorem 2.2.3 ([82])**.** *Let G be a graph with n vertices such that $\deg_G(x) \geqslant \frac{1}{2}n + k - 1$ for every $x \in V(G)$. Then G has a k-star decomposition if $|E(G)| \equiv 0 \pmod{k}$.*

In addition, Bryant et al. [9] proved that the obvious necessary conditions are also sufficient for the existence of a k-star decomposition of n-cube graph Q_n.

In general, the problem of determining whether a given graph G has a k-star decomposition for $k \geqslant 3$ is known to be NP-complete due to the result of Dor and Tarsi [37] mentioned in Section 1.4, Chapter 1. If $k = 1$ we have noted that the problem is trivial. For $k = 2$, we only have to determine whether each component of G has an even number of edges due to Theorem 2.2.2. Another way to show that determining whether a given graph G has a 2-star decomposition is polynomial is as follows. Let G be any given graph, then construct its line graph $L(G)$. A 2-star decomposition of G is equivalent to a perfect matching of $L(G)$. Therefore, determining whether a given graph has a 2-star decomposition can be reduced to finding a perfect matching of its line graph and in [41], Edmonds has proved that determining whether a given graph has a perfect matching can be done in polynomial time.

For any $k \geqslant 1$ Hoffman proved that, if the number of k-stars centred at each vertex of G is specified, then we can determine whether G has a k-star decomposition with the given k-star distribution in polynomial time [57].

Definition 2.2.4. Let G be a graph. For a given k-star decomposition $\mathcal{D}$ of G, we can define a function $\gamma : V(G) \to \mathbb{Z}^{\geqslant 0}$ called the *central function*, where $\gamma(x)$ is the number of k-stars of $\mathcal{D}$ whose centre is x for each $x \in V(G)$. Moreover, γ satisfies the property, $k\sum_{x\in V(G)} \gamma(x) = |E(G)|$.

Consider the following decision problem:

CENTRAL k-STAR DESIGN

Input: A graph G, positive integer k and a function $\gamma : V(G) \to \mathbb{Z}^{\geqslant 0}$.

Question: Is there a k-star decomposition of G whose central function is γ?

Hoffman [57] proved that CENTRAL k-STAR DESIGN is in class P using a network flow argument on a network derived from γ and G.

Definition 2.2.5. Among all the possible partial k-star decompositions of K_n, a *maximum partial k-star decomposition of K_n* is one with greatest number of k-stars.

Let G be a graph. Recall that the leave of a partial k-star decomposition of G is the graph L having the vertex set $V(G)$ and the edge set comprising all edges of G that are not in a k-star in the decomposition. Hoffman and Roberts [59] have exactly determined the size of a maximum partial k-star decomposition of K_n and moreover they have characterized the possible leaves when $k < n < 2k$. It is obvious that, if $n \leqslant k$, then there will be zero stars in a partial k-star decomposition of K_n and hence the leave will be just K_n. By Theorem 2.2.1, if $n \geqslant 2k$, then K_n has a k-star decomposition if and only if $\binom{n}{2} \equiv 0 \pmod{k}$.

Theorem 2.2.6 ([59]). *Let n and k be positive integers such that $n \geqslant 2k$. Then there are $\lfloor \frac{1}{k}\binom{n}{2} \rfloor$ k-stars in a maximum partial k-star decomposition of K_n. Furthermore, one possible leave is an m-star and isolated vertices, where m is a positive integer strictly less than k.*

It is obvious that at most $\lfloor \frac{1}{k}\binom{n}{2} \rfloor$ stars can be in a partial k-star decomposition of K_n. Hoffman and Roberts observe that such decompositions can be found using a result of Lin and Shyu [66] that characterises when a complete graph can be decomposed into stars of various specified sizes.

Theorem 2.2.7 ([59]). *Let n and k be positive integers such that $k < n < 2k$. Then there are $2n - 2k - 1$ k-stars in a maximum partial k-star decomposition of K_n. Furthermore, the leave of such a decomposition must be a copy of K_{2k-n+1} and isolated vertices.*

We can see that when $n < 2k$, any vertex of K_n can have at most one star centred on it, because $\deg_{K_n}(x) < 2k - 1$ for each $x \in V(K_n)$. Suppose that N is the set of vertices of $V(K_n)$ having zero k-stars centred on them and S is the set of vertices of $V(K_n)$ having exactly one k-star centred on them. Note that, $V(K_n) = N \cup S$ and $N \cap S = \emptyset$. Then for each $x \in S$, the k-star centred on x has at most $|N| = n - |S|$ tail vertices in N and hence at least $k-(n-|S|)$ tail vertices in S. Therefore, $|S|(k-n+|S|) \leqslant \binom{|S|}{2}$ where $\binom{|S|}{2}$ is the number of edges having both end vertices in S. This is equivalent to $|S| \leqslant 2n - 2k - 1$. Therefore, a partial k-star decomposition of K_n when $k < n < 2k$ must have at most $2n - 2k - 1$ k-stars and if a decomposition with $2n - 2k - 1$ stars exists then its leave must be K_N where $|N| = 2k - n + 1$. Furthermore, Hoffman and Roberts [59] used the idea of a so-called regular tournament to construct such partial k-star decompositions.

2.2.1 Embeddings of partial k-star designs

Recall that an *embedding* of a partial k-star decomposition $\mathcal{A}$ of a graph G' is a partial k-star decomposition $\mathcal{B}$ of another graph G such that $\mathcal{A} \subseteq \mathcal{B}$ and G' is a subgraph of G. One can pose the problem of, for a given n, finding the smallest c such that every partial k-star decomposition of K_n has an embedding in a k-star decomposition of K_{n+s} for some $s \leqslant c$. In 2012, Hoffman and Roberts [58] proved a result along these lines.

Theorem 2.2.8 ([58])**.** *A partial k-star decomposition of K_n can be embedded into a k-star decomposition of K_{n+s} for some $s \leqslant 7k-4$ when k is odd and $s \leqslant 8k-4$ when k is even.*

The key elements of the proof are as follows. Let $\mathcal{D}$ be any partial k-star decomposition of K_V where $|V| = n$. First, the authors embed $\mathcal{D}$ in a partial k-star decomposition $\mathcal{D}'$ of $K_{V \cup M}$ where M is a set of $2k-1$ new vertices in such a way that each edge of the new leave L is between two vertices in M. To use up the edges in L they introduce a set T of t new vertices, where t is yet to be determined. Let G be the graph with $V(G) = X \cup T$ and $E(G) = (E(K_{X \cup T}) \setminus E(K_X)) \cup E(L)$ where $X \subseteq V \cup M$ such that $M \subseteq X$ and, moreover, $|X|$ is the smallest positive integer satisfying $|X| - n + 1 \equiv 0 \pmod{2k}$ when k is even or $|X| - n + 1 \equiv 0 \pmod{k}$ when k is odd. Next, they find a k-star decomposition $\mathcal{D}''$ of G and find a k-star decomposition $\mathcal{D}'''$ of the remaining complete bipartite graph (with parts $(V \cup M) \setminus X$ and T) using the result of Yamamoto et al. [92]. Then $\mathcal{D} \cup \mathcal{D}' \cup \mathcal{D}'' \cup \mathcal{D}'''$ is an embedding of $\mathcal{D}$. Finally, they show that an appropriate t can be chosen so that $t \leqslant 6k-3$ when k is even and $t \leqslant 5k-3$ when k is odd. Furthermore, the authors conjectured that the smallest possible upper bound on s is about $2k$.

In 2019 Noble and Richardson [74] improved the bounds on s to $s \leqslant 3k-2$ when k is odd and $s \leqslant 4k-2$ when k is even.

Theorem 2.2.9 ([74])**.** *A partial k-star decomposition of K_n can be embedded into a k-star decomposition of K_{n+s} for some $s \leqslant 3k-2$ when k is odd and $s \leqslant 4k-2$ when k is even.*

For an arbitrary maximal partial k-star decomposition of K_n with a leave L, the mechanics of Noble and Richardson's proof are as follows. First they choose a suitable s. They then consider a triangular "staircase diagram" with empty cells corresponding to positions of 1s below the lead diagonal in the adjacency matrix of $L \vee K_s$, where the first n rows and columns correspond to the vertices of L. Note that, in the first n rows of the diagram, each column has at most $k-1$ empty cells since the partial decomposition is maximal. They then colour the empty cells in this diagram in such a way that each colour class of cells is either k cells in a single row or k cells in a single column. It is not hard to see that such a colouring corresponds to a k-star decomposition of $L \vee K_s$. The colouring is constructed by first creating a "vertical" colour class in each of the first n columns of the diagram in such a way that all of the empty cells in the first n rows are coloured and the number of uncoloured cells left in each of the last s rows is congruent to 0 modulo k. They then complete the colouring by adding "horizontal" colour classes.

The bounds of Theorem 2.2.9 are not tight, however. In Chapter 5 we improve these bounds on s to $s < \frac{9}{4}k$ when k is odd and $s < (6-2\sqrt{2})k$ when k is even, which are best possible up to the order of k.

Chapter 3

Completing partial block designs

“ Because I longed
To comprehend the infinite
I drew a line
Between the known and unknown ”

- Elizabeth Bartlett, *Because I Longed*

3.1 Introduction

Recall that, for positive integers n, k and λ with $n \geqslant k$, an (n,k,λ)-*design* is a pair $(V, \mathcal{B})$ where V is a set of n *points* and $\mathcal{B}$ is a collection of k-subsets of V called *blocks* such that each pair of points occur together in exactly λ blocks. If we weaken this condition to demand only that each pair of points occur together in at most λ blocks, then the resulting object is a *partial* (n,k,λ)-*design*. In this chapter we are only concerned with $(n,k,1)$-designs and partial $(n,k,1)$-designs. A *completion* of a partial $(n,k,1)$-design $(V,\mathcal{A})$ is a (complete) $(n,k,1)$-design $(V,\mathcal{B})$ such that $\mathcal{A} \subseteq \mathcal{B}$. A partial $(n,k,1)$-design is *completable* when it has a completion. The *leave* of a partial $(n,k,1)$-design $(V,\mathcal{A})$ is the graph G having vertex set V and the edge set $E(G) = \{xy : x, y \in V \text{ such that } \{x,y\} \nsubseteq A$ for all $A \in \mathcal{A}\}$.

As previously mentioned, an $(n,2,1)$-design exists trivially for each integer $n \geqslant 2$. It is obvious that if an $(n,k,1)$-design exists then $n(n-1) \equiv 0 \pmod{k(k-1)}$ and $n \equiv 1 \pmod{(k-1)}$. We call integers n satisfying these restrictions k-*admissible*. Wilson [90] showed that, for each integer $k \geqslant 3$, there exists an $(n,k,1)$-design for each sufficiently large k-admissible value of n. Obviously, if a partial $(n,k,1)$-design is completable, then n is k-admissible. Our main result in this chapter is to show that, for each sufficiently large k-admissible order n, all partial $(n,k,1)$-designs with at most $\frac{n-1}{k-1} - k + 1$ blocks are completable and that this bound is tight.

Theorem 3.1.1. *Let $k \geqslant 3$ be a fixed integer. There is an integer n_0 such that for all k-admissible integers $n \geqslant n_0$, any partial $(n,k,1)$-design with at most $\frac{n-1}{k-1} - k + 1$ blocks is completable. Furthermore, for all k-admissible integers $n \geqslant (k-1)^2 + 1$ there is a partial $(n,k,1)$-design with $\frac{n-1}{k-1} - k + 2$ blocks that is not completable.*

The existence of the uncompletable partial designs claimed in Theorem 3.1.1 is easily proved (see Lemma 3.2.3(a)). For sufficiently large n, Theorem 3.1.1 establishes a generalisation of a conjecture of Horsley in [62] that any partial $(n,3,1)$-design having at most $\frac{n-5}{2}$ blocks is completable. Theorem 3.1.1 also nicely complements recent results of Nenadov, Sudakov and Wagner [73]. They show that there exist $\epsilon, n_0 > 0$ such that we can add blocks to any partial $(n,k,1)$-design $(V,\mathcal{A})$ with $n > n_0$ and $|\mathcal{A}| \leqslant \epsilon n^2$ to obtain another partial $(n,k,1)$-design whose leave has at most $21k^3\sqrt{|\mathcal{A}|}\,n$ edges. They also show that we can add points and blocks to such a design to obtain a (complete) $(n',k,1)$-design such that $n' \leqslant n + 7k^2\sqrt{|\mathcal{A}|}$.

Theorem 3.1.1 is also reminiscent of a well known conjecture of Evans. Recall that a partial latin square of order n is an $n \times n$ array in which each cell is either empty or contains an element of $\{1,\ldots,n\}$, and each element of $\{1,\ldots,n\}$ occurs at most once in each row and column. A latin square is a partial latin square with no empty cells. Evans [42] conjectured that every partial latin square of order n with at most $n-1$ filled cells can be completed to a latin square. This bound is tight because there is a partial latin square of order n with n filled cells that is not completable for each $n \geqslant 2$. Smetaniuk [76] and Anderson and Hilton [4] independently proved Evans' conjecture for all n.

There are few completion results available for partial (n,k,λ)-designs (refer to Section 2.1.2 for a detailed overview). Colbourn [21] has shown that it is NP-complete to decide whether a given partial $(n,3,1)$-design can be completed. In [22] it is observed that partial $(n,3,1)$-designs in which some fixed point is in every block and partial $(n,3,1)$-designs consisting of an odd number of pairwise disjoint blocks are easily seen to be completable. It is then shown that a partial $(n,3,1)$-design is completable if it has two points x and y such that one block contains both x and y and each other block contains either x or y.

Remember that a *K_k-decomposition* of a graph G is a set of copies of K_k in G whose edge sets partition $E(G)$. An $(n,k,1)$-design is equivalent to a K_k-decomposition of K_n and a partial $(n,k,1)$-design is equivalent to a K_k-decomposition of some subgraph of K_n. Finding a completion of a partial $(n,k,1)$-design is equivalent to finding a K_k-decomposition of its leave, and throughout the remainder of the chapter we will often view completions in this way. If a graph G has a K_k-decomposition, then we must have $|E(G)| \equiv 0 \pmod{\binom{k}{2}}$ and $\deg_G(x) \equiv 0 \pmod{k-1}$ for each $x \in V(G)$. We call graphs that obey these necessary conditions *K_k-divisible*. So Theorem 3.1.1 can be rephrased as saying that, for sufficiently large n, any graph G on n vertices that is the leave of a partial $(n,k,1)$-design and whose complement has at most $(\frac{n-1}{k-1} - k + 1)\binom{k}{2}$ edges, has a K_k-decomposition. It is natural to ask whether we can relax the condition that the graph is the leave of a partial design. We prove two subsidiary results which show that this can only be done at the expense of increasing the bound on the number of edges in G. Theorem 3.1.2 considers the case where G need not be a leave but must still have order congruent to 1 modulo $k-1$, and Theorem 3.1.3 considers the case where G can be any K_k-divisible graph.

Theorem 3.1.2. *Let $k \geqslant 3$ be a fixed integer. There is an integer n_0 such that for all integers $n \geqslant n_0$ with $n \equiv 1 \pmod{k-1}$, any K_k-divisible graph G of order n has a K_k-decomposition if*

$$|E(G)| > \binom{n}{2} - \left(\tfrac{n-1}{k-1} - \ell\right)\binom{k}{2} \quad \text{where} \quad \ell = \tfrac{1}{4}(k^2-k-2).$$

Furthermore, if $k = 3$ or $k \equiv 2 \pmod 4$, then for all k-admissible $n \geqslant \frac{1}{2}k(k-1)^2 + 1$

there is a K_k-divisible graph G of order n such that $|E(G)| = \binom{n}{2} - (\frac{n-1}{k-1} - \ell)\binom{k}{2}$ and G is not K_k-decomposable.

Theorem 3.1.3. *Let $k \geqslant 3$ be a fixed integer. There is an integer n_0 such that for all integers $n \geqslant n_0$, any K_k-divisible graph G of order n has a K_k-decomposition if*

$$|E(G)| > \begin{cases} \binom{n}{2} - n + \frac{1}{2}(k+1) & \text{if } k \geqslant 4 \\ \binom{n}{2} - n & \text{if } k = 3. \end{cases}$$

Furthermore, if k divides $s^2 - s - 1$ for some positive integer s, then for $n = s(k-1)+2$ there is a K_k-divisible graph G of order n such that $|E(G)| = \binom{n}{2} - n + \frac{1}{2}(k+1)$ and G is not K_k-decomposable. Finally, for each integer $n \geqslant 12$ with $n \equiv 0 \pmod 6$, there is a K_3-divisible graph G of order n such that $|E(G)| = \binom{n}{2} - n$ and G is not K_3-decomposable.

Remark 3.1.4. *The case division in Theorem 3.1.3 is due to the fact that we go to a little extra effort to obtain a tight bound for the special case $k = 3$.*

Note that there are infinitely many values of k, all of them odd, such that k divides $s^2 - s - 1$ for some positive integer s. From Theorems 3.1.2 and 3.1.3 it is not too difficult to determine the maximum number of edges in a graph of order n that is K_3-divisible but not K_3-decomposable for all sufficiently large n.

Corollary 3.1.5. *There is an integer n_0 such that for all integers $n \geqslant n_0$, any K_3-divisible graph G of order n has a K_3-decomposition if $|E(G)| > \binom{n}{2} - e(n)$, where*

$$e(n) = \begin{cases} \frac{1}{2}(3n-9) & \text{if } n \equiv 1, 3 \pmod 6 \\ \frac{1}{2}(3n-7) & \text{if } n \equiv 5 \pmod 6 \\ n+2 & \text{if } n \equiv 2, 4 \pmod 6 \\ n & \text{if } n \equiv 0 \pmod 6. \end{cases}$$

Furthermore, for each $n \geqslant 7$ there is a K_3-divisible graph G of order n such that $|E(G)| = \binom{n}{2} - e(n)$ and G is not K_3-decomposable.

Very recently, Gruslys and Letzter [49] have proved that any graph of order $n \geqslant 7$ with strictly more than $\binom{n}{2} - (n-3)$ edges has a *fractional* K_3-decomposition. This makes an interesting comparison with Theorem 3.1.3 and Corollary 3.1.5. Considering complements, Theorems 3.1.2 and 3.1.3 can be thought of as concerning which graphs are or are not the leaves of partial $(n, k, 1)$-designs. This question has received some attention: see [28, Chapter 9], [79, §40.4] and the references therein, for example. Perhaps closest to our concerns here, the possible sizes of *triangle-free* graphs whose complements are K_3-divisible but not K_3-decomposable are considered in [77]. Our results here improve the lower bounds in that paper.

3.2 Preliminaries

For a family $\mathcal{A}$ of subsets of a set V and an element $x \in V$, we let $\mathcal{A}_x = \{A \in \mathcal{A} : x \in A\}$. For a set A of vertices we use K_A to denote the complete graph with vertex set A. For a graph G and a subset S of $V(G)$, we denote by $G[S]$ the subgraph of G induced by S. We also denote the minimum and maximum degree of G by $\delta(G)$ and $\Delta(G)$ and the complement of G by $\overline{G}$. For graphs G and H we denote by $G \cup H$ the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$ and denote by $G - H$ the graph with vertex set $V(G)$ and edge set $E(G) \setminus E(H)$.

Definition 3.2.1. For a positive integer r, a *K_r-factor* of a graph G is a set of copies of K_r in G whose vertex sets partition $V(G)$.

Definition 3.2.2. For vertices x and y of a graph G, we use $N_G(x,y)$ to denote the mutual neighbourhood $N_G(x) \cap N_G(y)$ of x and y.

In Lemma 3.2.3(a), (b) and (c) below, we establish the tightness claims in Theorems 3.1.1 and 3.1.2 and in the $k \geqslant 4$ case of Theorem 3.1.3 respectively.

Lemma 3.2.3. *Let $k \geqslant 3$ be an integer.*

(a) *For all K_k-admissible integers $n \geqslant (k-1)^2+1$ there is a partial $(n,k,1)$-design with $\frac{n-1}{k-1} - k + 2$ blocks that is not completable.*

(b) *If $k = 3$ or $k \equiv 2 \pmod 4$ then, for all K_k-admissible integers $n \geqslant \frac{1}{2}k(k-1)^2+1$, there is a K_k-divisible graph G of order n such that*

$$|E(\overline{G})| = \left(\tfrac{n-1}{k-1} - \tfrac{1}{4}(k^2-k-2)\right)\tbinom{k}{2}$$

and G is not K_k-decomposable.

(c) *If k divides $s^2 - s - 1$ for some positive integer s then, for $n = s(k-1)+2$, there is a K_k-divisible graph G of order n with $|E(\overline{G})| = n - \frac{1}{2}(k+1)$ that is not K_k-decomposable.*

Proof. We first prove (a). Let $(V, \mathcal{A})$ be a partial $(n,k,1)$-design with $|\mathcal{A}| = \frac{n-1}{k-1} - k + 2$ such that $\frac{n-1}{k-1} - k + 1$ blocks each contain some fixed point $z \in V$ and the remaining block, say A_0, is disjoint from every other block in $\mathcal{A}$. So $|\mathcal{A}_z| = \frac{n-1}{k-1} - k + 1$. Suppose for a contradiction that $(V, \mathcal{B})$ is a completion of $(V, \mathcal{A})$. In $(V, \mathcal{B})$ each point lies in exactly $\frac{n-1}{k-1}$ blocks. Thus $|\mathcal{B}_z \setminus \mathcal{A}_z| = k-1$. But $|\mathcal{B}_z \setminus \mathcal{A}_z| \geqslant k$ because each pair in $\{\{x,z\} : x \in A_0\}$ must occur in a different block. This is a contradiction.

We now prove (b). If $k=3$ then the leave of the partial $(n,k,1)$-design defined in (a) has the required properties, so we may assume that $k \equiv 2 \pmod 4$. Let V be a set of n vertices and let $z \in V$. Let $t = \frac{n-1}{k-1} - \frac{k}{2}(k-1)$ and let $A_1, \ldots, A_t$ be k-subsets of V such that $A_i \cap A_j = \{z\}$ for all distinct $i, j \in \{1, \ldots, t\}$. Let A_0 be a $(\frac{k}{2}(k-1)+1)$-subset of V such that A_0 is disjoint from A_i for all $i \in \{1, \ldots, t\}$. Take G to be the graph $K_V - \bigcup_{i=0}^{t} K_{A_i}$ and note

$$|E(\overline{G})| = t\tbinom{k}{2} + \tbinom{k(k-1)/2+1}{2} = \left(\tfrac{n-1}{k-1} - \tfrac{1}{4}(k^2-k-2)\right)\tbinom{k}{2}.$$

Furthermore, $\deg_{\overline{G}}(x) \equiv 0 \pmod{k-1}$ for each $x \in V$ and hence, using the fact that K_V is K_k-divisible since n is K_k-admissible, we have that G is K_k-divisible. Now suppose for a contradiction there is a K_k-decomposition $\mathcal{D}$ of G. We have $\deg_G(z) = n-1-t(k-1) = \frac{k}{2}(k-1)^2$, so z is a vertex of exactly $\frac{k}{2}(k-1)$ copies of K_k in $\mathcal{D}$. But z must be a vertex of at least $|A_0| = \frac{k}{2}(k-1)+1$ copies of K_k in $\mathcal{D}$ because each edge in $\{xz : x \in A_0\}$ must occur in a different copy of K_k. This is a contradiction.

Finally, we prove (c). Let V be a set of n vertices, where $n = s(k-1)+2$ for some positive integer s with $s^2 - s - 1 \equiv 0 \pmod k$, and let $z \in V$. Observe that k is odd since $s^2 - s - 1$ is odd. Let G be a graph on vertex set V such that $\overline{G}$ is the vertex-disjoint union of a star with $n-k$ edges centred at z and a perfect matching on the remaining $k-1$ vertices. Note that $|E(G)| = \binom{n}{2} - n + \frac{1}{2}(k+1)$ and hence that $|E(G)| \equiv 0 \pmod{\binom{k}{2}}$

because $n = s(k-1)+2$ and $s^2(k-1)^2+s(k-1)+(k-1) \equiv 0 \pmod{k(k-1)}$. Furthermore, $\deg_G(z) = k-1$ and $\deg_G(x) = n-2 = s(k-1)$ for all $x \in V \setminus \{z\}$ and hence G is K_k-divisible. Let $U = N_G(z)$ and note that any K_k-decomposition of G must include a copy of K_k with vertex set $\{z\} \cup U$. But this is impossible because $\overline{G}[U]$ is a perfect matching on $k-1$ vertices. $\square$

Remark 3.2.4. *Note that the construction from the proof of Lemma 3.2.3(b) cannot be converted into a counterexample to Theorem 3.1.1 because, by Fisher's inequality (see Theorem 2.1.2), $K_{k(k-1)/2+1}$ is not K_k-decomposable.*

Observe that Theorem 3.1.1 is tight for almost all feasible values of k and n, while Theorems 3.1.2 and 3.1.3 are tight only for some values of k. So there remains the possibility that the bounds in Theorems 3.1.2 and 3.1.3 can be improved for particular values of k.

We also require some examples of graphs that are K_3-divisible but not K_3-decomposable to establish the tightness claims in the $k = 3$ case of Theorem 3.1.3 and in Corollary 3.1.5. Note that we have already shown that Corollary 3.1.5 is tight for $n \equiv 1, 3 \pmod 6$ in Lemma 3.2.3(b).

Lemma 3.2.5.

(a) *For each integer $n \geqslant 12$ with $n \equiv 0 \pmod 6$, there is a K_3-divisible graph G of order n with $|E(\overline{G})| = n$ that is not K_3-decomposable.*

(b) *For each integer $n \geqslant 11$ such that $n \equiv 5 \pmod 6$ there is a K_3-divisible graph G of order n with $|E(\overline{G})| = \frac{1}{2}(3n-7)$ that is not K_3-decomposable.*

(c) *For each integer $n \geqslant 8$ such that $n \equiv 2, 4 \pmod 6$ there is a K_3-divisible graph G of order n with $|E(\overline{G})| = n+2$ that is not K_3-decomposable.*

Proof. We first prove (a). Let V be a set of n vertices, where $n \geqslant 12$ and $n \equiv 0 \pmod 6$, and let $z \in V$. Let G be a graph on vertex set V such that $\overline{G}$ is the vertex-disjoint union of a star with $n-7$ edges centred at z, a copy of K_4 with some vertex set A, and a copy of K_2. Clearly $|E(\overline{G})| = n$ and G is K_3-divisible. A K_3-decomposition of G must contain exactly three copies of K_3 that have z as one of their vertices, but each of the four edges between z and a vertex in A must occur in a different copy of K_3. So G has no K_3-decomposition.

We now prove (b). Let V be a set of n vertices, where $n \geqslant 11$ and $n \equiv 5 \pmod 6$, and let $z \in V$. Let G be a graph on vertex set V such that $\overline{G}$ is the union of $\frac{1}{2}(n-9)$ edge-disjoint copies of K_3 whose vertex sets pairwise have intersection $\{z\}$, a copy of K_5 with some vertex set A that is disjoint from the vertex set of each copy of K_3, and three isolated vertices. It is easy to check that, $|E(\overline{G})| = \frac{1}{2}(3n-7)$ and G is K_3-divisible. A K_3-decomposition of G must contain exactly four copies of K_3 that have z as one of their vertices, but each of the five edges between z and a vertex in A must occur in a different copy of K_3. So G has no K_3-decomposition.

Finally we prove (c). Let V be a set of n vertices, where $n \geqslant 8$ and $n \equiv 2, 4 \pmod 6$. Let G be a graph on vertex set V such that $\overline{G}$ is the union of a star with $n-3$ edges centred at z and the graph with edge set $\{ux, uy, vx, vy, xy\}$, where u and v are distinct tail vertices of the star and x and y are the two vertices of V not in the star. It is easy to check that, $|E(\overline{G})| = n+2$ and G is K_3-divisible. A K_3-decomposition of G must contain a copy of K_3 with vertex set $\{x, y, z\}$ but this is impossible since $xy \in E(\overline{G})$. $\square$

The rest of the chapter is devoted to proving the first parts of the theorems and Corollary 3.1.5. Our approach is based on the fact that K_k-divisible graphs with large order and high minimum degree are known to be K_k-decomposable. For each integer $k \geqslant 3$, δ_{K_k} is defined to be the infimum of all positive real numbers δ that satisfy the following: there is a positive integer n_0 such that every K_k-divisible graph of order $n > n_0$ and minimum degree at least δn has a K_k-decomposition. Delcourt and Postle [32] have shown that $\delta_{K_3} \leqslant 0.82733$ and Montgomery [71] has shown that $\delta_{K_k} \leqslant 1 - \frac{1}{100k}$ for each $k \geqslant 4$. Both of these results rely on the work of Glock, Kühn, Lo, Montgomery and Osthus in [48]. For our purposes here, it is enough to know that $\delta_{K_k} < 1$ for each $k \geqslant 3$. Often, simply applying this fact to an almost complete graph will show it to be K_k-decomposable. However, this approach will not work if the graph contains vertices of low degree. In these situations we follow [73] in deleting copies of K_k from the graph until the vertices that began with low degree become isolated. We can then remove the isolated vertices and apply the fact that $\delta_{K_k} < 1$ to the resulting graph to show that the original graph is K_k-decomposable. We will make use of the following well known theorems of Turán and of Hajnal and Szemerédi.

Theorem 3.2.6 ([85])**.** *Let $r \geqslant 2$ be an integer. If a graph H has more than $\frac{r-2}{2r-2}|V(H)|^2$ edges, then it contains a copy of K_r.*

Theorem 3.2.7 ([51])**.** *Let r be a positive integer. If a graph H has $|V(H)| \equiv 0 \pmod{r}$ and $\delta(H) \geqslant \frac{r-1}{r}|V(H)|$, then it contains a K_r-factor.*

The following simple inductive argument encapsulates the basics of our approach. Given a graph G on an indexed vertex set $\{z_1, \dots, z_s\}$ and two edges $z_i z_j$ and $z_{i'} z_{j'}$ of G where $i < j$ and $i' < j'$, we say that $z_i z_j$ *lexicographically precedes* $z_{i'} z_{j'}$ if either $i < i'$ or $i = i'$ and $j < j'$. Recall that $N_G(x, y)$ is the mutual neighbourhood $N_G(x) \cap N_G(y)$ of x and y.

Lemma 3.2.8. *Let $k \geqslant 3$ be a fixed integer and let $\gamma < 1 - \delta_{K_k}$ be a positive constant. For all sufficiently large integers n the following holds. Let G be a K_k-divisible graph of order n, let $S = \{z_1, \dots, z_s\}$ be an indexed subset of $V(G)$, and suppose that*

(i) *$|N_G(x) \setminus S| \geqslant (1 - \gamma)n + (k-2)|N_G(x) \cap S|$ for each $x \in V(G) \setminus S$;*

(ii) *either $N_G(z) = \emptyset$ or $|N_G(z) \setminus S| > (k-1)\gamma n + (k-2)|N_G(z) \cap S|$ for each $z \in S$;*

(iii) *for any $i, j \in \{1, \dots, s\}$ such that $i < j$ and $z_i z_j \in E(G)$ we have*

$$|N_G(z_i, z_j) \setminus S| > (k-3)\gamma n + (k-2)\ell_G(z_i z_j)$$

where $\ell_G(z_i z_j) = |N_G(z_i) \cap \{z_1, \dots, z_{j-1}\}| + |N_G(z_j) \cap \{z_1, \dots, z_{i-1}\}|$ is the number of edges of $G[S]$ that are adjacent to $z_i z_j$ and lexicographically precede it.

Then G has a K_k-decomposition.

Proof. We prove the result by induction on the quantity $\sigma(G) = \sum_{z \in S} \deg_G(z)$. Let $s = |S|$. If $\sigma(G) = 0$, then the vertices in S are isolated and $\deg_G(x) \geqslant (1 - \gamma)n \geqslant (1 - \gamma)(n - s)$ for each $x \in V(G) \setminus S$ by (i). So the graph obtained from G by deleting the vertices in S is K_k-decomposable by the definition of δ_{K_k} since $\gamma < 1 - \delta_{K_k}$, and thus the result follows. So we may assume that $\sigma(G) > 0$.

We consider two cases according to whether $G[S]$ is empty. In each case we form a new graph G' from G by removing the edges of some number of copies of K_k in G and then complete the proof by showing that G' satisfies the inductive hypotheses. Note that G' will be K_k-divisible because G is K_k-divisible. In what follows it will be useful to observe that (i) implies that the vertex x is nonadjacent to at most γn vertices in G (including itself) for each $x \in V(G) \setminus S$.

Case 1: Suppose that $G[S]$ is not empty. Let $z_i z_j$, where $i < j$, be the lexicographically first edge in $G[S]$. Let H be the subgraph of G induced by $N_G(z_i, z_j) \setminus S$. Then $|V(H)| > (k-3)\gamma n$ by (iii). We claim that there is a subset X of $V(H)$ such that $H[X]$ is a copy of K_{k-2}. If $k = 3$, this is immediate because $|V(H)| > 0$. If $k \geqslant 4$, then $\deg_H(x) \geqslant |V(H)| - \gamma n > \frac{k-4}{k-3}|V(H)|$ for each $x \in V(H)$ where the first inequality follows by (i) and the second from $|V(H)| > (k-3)\gamma n$. So it follows from Theorem 3.2.6 that such an X exists. Let $G' = G - K_B$ where $B = X \cup \{z_i, z_j\}$. Note that $\sigma(G') < \sigma(G)$, so it suffices to show that G' satisfies (i), (ii) and (iii).

Observe that $|N_{G'}(x) \setminus S| = |N_G(x) \setminus S| - (k-3)$ and $|N_{G'}(x) \cap S| = |N_G(x) \cap S| - 2$ for each $x \in X$, and $N_{G'}(x) = N_G(x)$ for each $x \in V \setminus (S \cup X)$. Thus G' satisfies (i) because G satisfies (i). Also, $|N_{G'}(z) \setminus S| = |N_G(z) \setminus S| - (k-2)$ and $|N_{G'}(z) \cap S| = |N_G(z) \cap S| - 1$ for each $z \in \{z_i, z_j\}$, and $N_{G'}(z) = N_G(z)$ for each $z \in S \setminus \{z_i, z_j\}$. Thus G' satisfies (ii) because G satisfies (ii). If $G'[S]$ is empty, then G' satisfies (iii) trivially. Otherwise, let $z_{i'} z_{j'}$ be an arbitrary edge in $G'[S]$ where $i' < j'$. If $\{i', j'\} \cap \{i, j\} = \emptyset$, then $N_{G'}(z_{i'}, z_{j'}) \setminus S = N_G(z_{i'}, z_{j'}) \setminus S$ and $\ell_{G'}(z_{i'} z_{j'}) = \ell_G(z_{i'} z_{j'})$. Otherwise either $i' = i$ and $j' > j$ or $i' = j$ by our definition of $z_i z_j$. Then $|N_{G'}(z_{i'}, z_{j'}) \setminus S| \geqslant |N_G(z_{i'}, z_{j'}) \setminus S| - (k-2)$ and $\ell_{G'}(z_{i'} z_{j'}) = \ell_G(z_{i'} z_{j'}) - 1$. Thus G' satisfies (iii) because G satisfies (iii).

Case 2: Suppose that $G[S]$ is empty. Because $\sigma(G) > 0$, there is an $i \in \{1, \ldots, s\}$ such that $N_G(z_i) \neq \emptyset$. Let H be the subgraph of G induced by $N_G(z_i)$. By (ii), $|V(H)| > (k-1)\gamma n$ and, because G is K_k-divisible, $|V(H)| = t(k-1)$ for some integer t. By (i), for each $x \in V(H)$, we have $\deg_H(x) \geqslant |V(H)| - \gamma n > \frac{k-2}{k-1}|V(H)|$. So Theorem 3.2.7 implies that there is a partition $\{X_1, \ldots, X_t\}$ of $V(H)$ such that $H[X_j]$ is a copy of K_{k-1} for each $j \in \{1, \ldots, t\}$. Let $G' = G - \bigcup_{j=1}^{t} K_{B_j}$ where $B_j = X_j \cup \{z_i\}$ for each $j \in \{1, \ldots, t\}$.

Observe that $|N_{G'}(x) \setminus S| = |N_G(x) \setminus S| - (k-2)$ and $|N_{G'}(x) \cap S| = |N_G(x) \cap S| - 1$ for each $x \in V(H)$, and $N_{G'}(x) = N_G(x)$ for each $x \in V \setminus (S \cup V(H))$. Thus G' satisfies (i) because G satisfies (i). Also, $N_{G'}(z_i) = \emptyset$ and $N_{G'}(z) = N_G(z)$ for each $z \in S \setminus \{z_i\}$. Thus G' satisfies (ii) because G satisfies (ii). Furthermore, $G'[S]$ is empty and so G' satisfies (iii) trivially. □

Remark 3.2.9. *Note that $|N_G(x) \cap S|$ in conditions (i) and (ii) of Lemma 3.2.8 is at most s, and $\ell_G(z_i z_j)$ in condition (iii) is less than $2s$. This will be useful to remember when we apply Lemma 3.2.8 below.*

We only require Lemma 3.2.8 in order to prove our next result, Lemma 3.2.10, which may be of some independent interest. It shows that we can guarantee a K_k-divisible graph with a positive proportion of non-edges has a K_k-decomposition if we further require that each edge is in sufficiently many triangles.

Lemma 3.2.10. *Let $k \geqslant 3$ be a fixed integer, and let $\gamma < 1 - \delta_{K_k}$ be a positive constant. For any sufficiently large integer n, a K_k-divisible graph G of order n is K_k-decomposable if $|E(G)| \geqslant (1 - \frac{1}{4k}\gamma^2)\binom{n}{2}$ and $|N_G(x, y)| > k\gamma n$ for each $xy \in E(G)$.*

Proof. Let G be a K_k-divisible graph of order n with $|E(G)| \geqslant (1 - \frac{1}{4k}\gamma^2)\binom{n}{2}$ and $|N_G(x, y)| > k\gamma n$ for each $xy \in E(G)$. Note that $|E(\overline{G})| \leqslant \frac{1}{4k}\gamma^2\binom{n}{2}$. Let $S = \{x \in$

$V(G) : \deg_{\overline{G}}(x) \geqslant \frac{1}{2}\gamma n\}$ and $|S| = s$. So we have $\frac{1}{2}\gamma ns \leqslant 2|E(\overline{G})| \leqslant \frac{1}{2k}\gamma^2\binom{n}{2}$, and hence $s < \frac{1}{2k}\gamma n$. It suffices to show that G and S satisfy conditions (i), (ii) and (iii) of Lemma 3.2.8.

(i) Consider any vertex $x \in V(G) \setminus S$. We have $\deg_G(x) > (1 - \frac{1}{2}\gamma)n - 1$ by the definition of S. Therefore, $|N_G(x) \setminus S| > (1 - \frac{1}{2}\gamma)n - 1 - s > (1 - \frac{k+1}{2k}\gamma)n - 1$. Thus, condition (i) of Lemma 3.2.8 holds, noting that $(k-2)|N_G(x) \cap S| \leqslant (k-2)s < \frac{k-2}{2k}\gamma n$ in that condition.

(ii) Consider any vertex $x \in S$. If $N_G(x) = \emptyset$, then (ii) is satisfied for x. Otherwise, for any vertex $y \in V(G)$ such that $xy \in E(G)$, we have $|N_G(x,y)| > k\gamma n$ by our hypotheses, and hence

$$|N_G(x) \setminus S| \geqslant |N_G(x,y) \setminus S| > k\gamma n - s > (k - \tfrac{1}{2k})\gamma n. \tag{3.1}$$

Thus condition (ii) of Lemma 3.2.8 holds, noting that $(k-2)|N_G(x) \cap S| \leqslant (k-2)s < \frac{k-2}{2k}\gamma n$ in that condition.

(iii) Consider any edge $xy \in E(G[S])$. By (3.1), we have $|N_G(x,y) \setminus S| > (k - \frac{1}{2k})\gamma n$. Thus, condition (iii) of Lemma 3.2.8 holds, noting that $(k-2)\ell_G(xy) < 2(k-2)s < \frac{k-2}{k}\gamma n$ in that condition. □

3.3 Proof of Theorem 3.1.1

Suppose that $(V, \mathcal{A})$ is a partial $(n, k, 1)$-design with $|\mathcal{A}| = \frac{n-1}{k-1} - k + 1$ and that G is its leave. One important situation in which we cannot complete $(V, \mathcal{A})$ by applying Lemma 3.2.10 to G is when there is a point $z \in V$ which is in nearly every block in $\mathcal{A}$ (since then edges of G incident with z will not be in enough triangles). In this case, completing $(V, \mathcal{A})$ will necessarily involve finding a K_{k-1}-factor in $G[N_G(z)]$. Lemma 3.3.2 below allows us to accomplish this task. It is simpler and more natural to consider the complement and state the result in terms of a colouring of a union of cliques.

Definition 3.3.1. A *proper colouring of a graph H with colour set C* is an assignment $\varphi : V(H) \to C$ of colours from C to the vertices of H such that adjacent vertices receive different colours. The *colour class* of a colour $c \in C$ under φ is the set $\varphi^{-1}(c)$ of all vertices to which φ assigns colour c.

The basic strategy in the proof of Lemma 3.3.2 is the commonly-used one of colouring vertices greedily according to a degeneracy ordering. A *degeneracy ordering* $v_1, \ldots, v_n$ of the vertices of a graph H is one for which v_i is a vertex of minimum degree in $H[\{v_1, \ldots, v_i\}]$ for each $i \in \{1, \ldots, n\}$. Such an ordering is easily obtained by choosing a vertex of minimum degree in a graph, deleting it and placing it last in the ordering, and repeating this procedure recursively. Sometimes our greedy strategy will get stuck, however, and in these cases we will be forced to recolour an already-coloured vertex.

Lemma 3.3.2. *Let k and a be integers with $k \geqslant 3$ and $a \geqslant k-1$, let V be a set of $a(k-1)$ vertices, and let $\mathcal{A}$ be a set of subsets of V such that $|\mathcal{A}| \leqslant a - k + 1$, $|A| \leqslant k$ for all $A \in \mathcal{A}$ and $|A \cap A'| \leqslant 1$ for all distinct $A, A' \in \mathcal{A}$. The graph H with vertex set V and edge set $\bigcup_{A \in \mathcal{A}} E(K_A)$ has a proper colouring with a colours such that each colour class has order $k-1$.*

Proof. Let C be a set of a colours. For the duration of this proof we call a proper colouring *legal* if its colour set is (a subset of) C and each of its colour classes has order at most $k-1$. Let $v_1, \ldots, v_{a(k-1)}$ be a degeneracy ordering of the vertices in V. Let

$V_i = \{v_1, \ldots, v_i\}$ and $H_i = H[V_i]$ for each $i \in \{1, \ldots, a(k-1)\}$. Clearly H_a has a legal colouring as we may colour each vertex with a different colour. We assume that there is a legal colouring φ_{j-1} of H_{j-1} for some $j \in \{a+1, \ldots, a(k-1)\}$ and proceed to show that we can find a legal colouring of φ_j of H_j. Extending φ_{j-1} by assigning v_j a new colour c might fail to result in a legal colouring for two reasons: either c may already be assigned by φ_{j-1} to $k-1$ vertices or c may be assigned by φ_{j-1} to a vertex adjacent in H_j to v_j. Accordingly, let $C_{\mathrm{F}} = \{c \in C : |\varphi_{j-1}^{-1}(c)| = k-1\}$, let C_{N} be the set of colours in C that are assigned by φ_{j-1} to vertices adjacent in H_j to v_j, and let $a_{\mathrm{N}} = |C_{\mathrm{N}}|$. We think of colours in C_{F} as "full" and those in C_{N} as "neighbouring".

If $C \setminus (C_{\mathrm{F}} \cup C_{\mathrm{N}})$ is nonempty, then we can extend φ_{j-1} to a legal colouring φ_j of H_j by assigning any colour in $C \setminus (C_{\mathrm{F}} \cup C_{\mathrm{N}})$ to v_j. So we may assume that $C_{\mathrm{F}} \cup C_{\mathrm{N}} = C$. Since $j-1$, the number of vertices already coloured, is less than $a(k-1)$, it follows from the definition of C_{F} that $|C_{\mathrm{F}}| < a$ and hence that $C_{\mathrm{N}} \setminus C_{\mathrm{F}} \neq \emptyset$ and $a_{\mathrm{N}} \geqslant 1$. Let c' be a colour in $C_{\mathrm{N}} \setminus C_{\mathrm{F}}$ and let $V' = \varphi_{j-1}^{-1}(c')$. Let $V_{\mathrm{F}}^* = \bigcup_{c \in C_{\mathrm{F}} \setminus C_{\mathrm{N}}} \varphi_{j-1}^{-1}(c)$ be the set of vertices already assigned a colour in $C_{\mathrm{F}} \setminus C_{\mathrm{N}}$. We aim to proceed by colouring v_j with a colour in $C_{\mathrm{F}} \setminus C_{\mathrm{N}}$ but also recolouring a vertex of that colour with c'. We will be able to do this if the following claim holds.

Claim. There is a vertex in V_{F}^* that is not adjacent in H_j to any vertex in V'.

If this claim is true, we can let u be such a vertex in V_{F}^* and let φ_j be the colouring of H_j such that $\varphi_j(v_j) = \varphi_{j-1}(u)$, $\varphi_j(u) = c'$, and $\varphi_j(x) = \varphi_{j-1}(x)$ for each $x \in V_{j-1} \setminus \{u\}$. Since $\varphi_{j-1}(u) \notin C_{\mathrm{N}}$ and u is not adjacent in H_j to any vertex in V', it can be seen that φ_j is a proper colouring and since $c' \notin C_{\mathrm{F}}$ it can be seen that φ_j is a legal colouring. So it suffices to prove our claim.

Proof of claim. Suppose for a contradiction that each vertex in V_{F}^* is adjacent in H_j to some vertex in V'. Observe that V' and V_{F}^* are disjoint and that

$$|V'| \geqslant 1, \qquad |V_{\mathrm{F}}^*| = (k-1)(a - a_{\mathrm{N}}) \qquad \text{and} \qquad |V_j \setminus (V' \cup V_{\mathrm{F}}^*)| \geqslant a_{\mathrm{N}} \tag{3.2}$$

where the second of these follows because each of the $a - a_{\mathrm{N}}$ colours in $C_{\mathrm{F}} \setminus C_{\mathrm{N}}$ is assigned by φ_{j-1} to exactly $k-1$ vertices in $V_{j-1} \setminus V'$ and the third follows because $v_j \in V_j \setminus (V' \cup V_{\mathrm{F}}^*)$ and each of the $a_{\mathrm{N}} - 1$ colours in $C_{\mathrm{N}} \setminus \{c'\}$ is assigned by φ_{j-1} to at least one vertex in $V_{j-1} \setminus (V' \cup V_{\mathrm{F}}^*)$.

Let $\Phi = \sum_{x \in V_j} |\mathcal{A}_x| - k(a + k - 1)$. We will show that $\Phi > 0$ and hence obtain a contradiction to the hypothesis of the lemma that $\mathcal{A}$ contains at most $a - k + 1$ sets each of size at most k. We do this in two cases according to the value of a_{N}.

Case 1: Suppose that $a_{\mathrm{N}} \leqslant k - 1$. Observe that, for each $x \in V_j$, we have $|\mathcal{A}_x| \geqslant 1$ because v_j is adjacent in H_j to a vertex of colour c' and thus $\deg_{H_j}(x) \geqslant \deg_{H_j}(v_j) \geqslant 1$ by the properties of the degeneracy ordering. So we have $\sum_{x \in V_j \setminus V'} |\mathcal{A}_x| \geqslant |V_j \setminus V'| \geqslant (k-1)(a - a_{\mathrm{N}}) + a_{\mathrm{N}}$ by (3.2). Furthermore, each of the $|V_{\mathrm{F}}^*| + 1$ vertices in $V_{\mathrm{F}}^* \cup \{v_j\}$ is in a set in $\mathcal{A}$ that also contains a vertex in V' using our assumption that the claim fails and the fact that $c' \in C_{\mathrm{N}}$. Thus, because $|A| \leqslant k$ for each $A \in \mathcal{A}$, we have $\sum_{x \in V'} |\mathcal{A}_x| \geqslant \lceil \frac{1}{k-1}(|V_{\mathrm{F}}^*| + 1) \rceil = a - a_{\mathrm{N}} + 1$ where the equality follows by (3.2). Using these lower bounds on $\sum_{x \in V_j \setminus V'} |\mathcal{A}_x|$ and $\sum_{x \in V'} |\mathcal{A}_x|$,

$$\Phi \geqslant (k-1)(a - a_{\mathrm{N}}) + a + 1 - k(a - k + 1) = (k-1)(k - a_{\mathrm{N}}) + 1.$$

Thus, since $a_{\mathrm{N}} \leqslant k - 1$ by the conditions of this case, $\Phi > 0$ and we have the required contradiction.

Case 2: Suppose that $a_{\mathrm{N}} \geqslant k$. We show this case cannot arise by obtaining a contradiction without the need for our assumption that the claim is false. Observe that $\deg_{H_j}(v_j) \geqslant a_{\mathrm{N}}$ by the definition of C_{N} and hence $\deg_{H_j}(x) \geqslant a_{\mathrm{N}}$ for each $x \in V_j$ by the properties of the degeneracy ordering. Now we have $\deg_{H_j}(x) \leqslant |\mathcal{A}_x|(k-1)$ for each $x \in V_j$ and hence

$$|\mathcal{A}_x| \geqslant \tfrac{1}{k-1}\deg_{H_j}(x) \geqslant \tfrac{1}{k-1}a_{\mathrm{N}} \quad \text{for each } x \in V_j. \tag{3.3}$$

So we have $\sum_{x\in V_j}|\mathcal{A}_x| \geqslant \frac{1}{k-1}a_{\mathrm{N}}|V_j| \geqslant \frac{1}{k-1}a_{\mathrm{N}}((k-1)(a-a_{\mathrm{N}})+a_{\mathrm{N}}+1)$ by (3.2) and (3.3). Thus,

$$\Phi \geqslant \frac{a_{\mathrm{N}}((k-1)(a-a_{\mathrm{N}})+a_{\mathrm{N}}+1)}{k-1} - k(a-k+1) = a(a_{\mathrm{N}}-k)+k(k-1)-\frac{(k-2)a_{\mathrm{N}}^2-a_{\mathrm{N}}}{k-1}. \tag{3.4}$$

In order to show that $\Phi > 0$ using (3.4) we require a lower bound on a.

We first show that $C_{\mathrm{F}} \setminus C_{\mathrm{N}}$ is nonempty and then use this fact to obtain the required lower bound on a. Let $m = \max\{|A \cap V_j| : A \in \mathcal{A}\}$ and A_1 be a set in $\mathcal{A}$ such that $|A_1 \cap V_j| = m$. Using the definition of m and a similar argument to the one used to establish (3.3), we see that $|\mathcal{A}_x| \geqslant \frac{1}{m-1}\deg_{H_j}(x) \geqslant \frac{1}{m-1}a_{\mathrm{N}}$ for each $x \in V_j$. So each vertex in $A_1 \cap V_j$ is in at least $\frac{1}{m-1}a_{\mathrm{N}} - 1$ sets in $\mathcal{A} \setminus \{A_1\}$. Further, no set in $\mathcal{A} \setminus \{A_1\}$ can contain more than one vertex in $A_1 \cap V_j$. Thus $|\mathcal{A}| - 1 \geqslant m(\frac{1}{m-1}a_{\mathrm{N}} - 1)$ and hence, using $|\mathcal{A}| \leqslant a-k+1$, we have $a \geqslant \frac{m}{m-1}a_{\mathrm{N}} - m + k$. So we have that $a > a_{\mathrm{N}}$ since $m \leqslant k$ and hence that $C_{\mathrm{F}} \setminus C_{\mathrm{N}}$ is indeed nonempty.

Let c'' be a colour in $C_{\mathrm{F}} \setminus C_{\mathrm{N}}$, let $V'' = \varphi_{j-1}^{-1}(c'')$, and note that $|V''| = k-1$ because $c'' \in C_{\mathrm{F}}$. No set in $\mathcal{A}$ can contain more than one vertex in V'' because φ_{j-1} is a proper colouring, and each vertex in V'' is in at least $\frac{1}{k-1}a_{\mathrm{N}}$ sets in $\mathcal{A}$ by (3.3). Thus $a-k+1 \geqslant |\mathcal{A}| \geqslant \frac{1}{k-1}a_{\mathrm{N}}|V''| = a_{\mathrm{N}}$ and hence $a \geqslant a_{\mathrm{N}} + k - 1$. Substituting this into (3.4) and simplifying, remembering that $a_{\mathrm{N}} \geqslant k$ by the conditions of this case, we obtain

$$\Phi \geqslant \frac{a_{\mathrm{N}}(a_{\mathrm{N}}-k+2)}{k-1} > 0$$

and we have the required contradiction. □

We observed in Lemma 3.2.3(b) that, for each $k \geqslant 6$ with $k \equiv 2 \pmod 4$, to guarantee a K_k-decomposition of a graph G of K_k-admissible order whose complement has at most $(\frac{n-1}{k-1} - k + 1)\binom{k}{2}$ edges, we require more than simply G being K_k-divisible (note that $\frac{1}{4}(k^2-k-2) > k-1$ for each $k \geqslant 6$). It is through Lemma 3.3.2 that our proof uses the stronger assumption that G is the leave of a partial $(n,k,1)$-design. The conclusion of Lemma 3.3.2 does not hold if we merely require that G be a graph of order $a(k-1)$ with at most $(a-k+1)\binom{k}{2}$ edges, even if we further demand that G be K_k-divisible. For example, for any integer $k \geqslant 6$ such that $k \equiv 2 \pmod 4$, if we take $a = \frac{1}{4}(k^2+3k-2)$, then the graph of order $a(k-1)$ consisting of a copy of $K_{k(k-1)/2+1}$ and isolated vertices has exactly $(a-k+1)\binom{k}{2}$ edges and is K_k-divisible, but clearly does not have a proper colouring with a colours.

With Lemma 3.3.2 in hand we are now in a position to prove Theorem 3.1.1. We find the required K_k-decomposition of the leave G of the partial design by first applying Lemma 3.3.2 to obtain the copies of K_k containing a particular vertex of minimum degree in G, and then using Lemma 3.2.10 to obtain the rest of the decomposition.

Proof of Theorem 3.1.1. The second part of the theorem was proved as Lemma 3.2.3(a), so it remains to prove the first part. Let $(V, \mathcal{A})$ be a partial $(n, k, 1)$-design such that n is K_k-admissible and $|\mathcal{A}| \leqslant \frac{n-1}{k-1} - k + 1$. Throughout the proof we assume that n is large relative to k and employ asymptotic notation with respect to this regime. Let G be the leave of $(V, \mathcal{A})$ and note that G is K_k-divisible because n is K_k-admissible. Let z be a point such that $|\mathcal{A}_z| \geqslant |\mathcal{A}_x|$ for each $x \in V$ and let $\mathcal{A}' = \mathcal{A} \setminus \mathcal{A}_z$. Let a be the integer such that $|\mathcal{A}_z| = \frac{n-1}{k-1} - a$, and note that $a \geqslant k-1$ and $|\mathcal{A}'| \leqslant a - k + 1$.

Let $U = N_G(z)$ and observe that $|A| = k$ for each $A \in \mathcal{A}'$, $|A \cap A'| \leqslant 1$ for all distinct $A, A' \in \mathcal{A}'$ and $\overline{G}[U] = \bigcup_{A \in \mathcal{A}'} K_{A \cap U}$. Thus, since $|U| = \deg_G(z) = a(k-1)$, we can apply Lemma 3.3.2 to show there is a proper colouring of $\overline{G}[U]$ with a colours in which each colour class has order $k-1$. Thus, there is a partition $\mathcal{U}$ of U such that $|\mathcal{U}| = a$ and $G[X]$ is a copy of K_{k-1} for each $X \in \mathcal{U}$. Let $\mathcal{B} = \{X \cup \{z\} : X \in \mathcal{U}\}$.

Let G' be the graph obtained from G by deleting the edges in $\bigcup_{B \in \mathcal{B}} E(K_B)$ and the vertex z. It suffices to show that we can apply Lemma 3.2.10 to find a K_k-decomposition $\mathcal{D}'$ of G', because then to complete $(V, \mathcal{A})$ we can add the blocks in $\mathcal{B}$ along with blocks corresponding to the copies of K_k in $\mathcal{D}'$. So it remains to show that G' satisfies the hypotheses of Lemma 3.2.10. Since G is K_k-divisible, so is G'. Observe that

$$G' = K_{V \setminus \{z\}} - \bigcup_{A \in \mathcal{A}_z \cup \mathcal{B}} K_{A \setminus \{z\}} - \bigcup_{A \in \mathcal{A}'} K_A,$$

and that each element of $V \setminus \{z\}$ is in exactly one set in $\{A \setminus \{z\} : A \in \mathcal{A}_z \cup \mathcal{B}\}$. Thus, for each $x \in V \setminus \{z\}$,

$$\deg_{\overline{G'}}(x) = (k-1)|\mathcal{A}'_x| + k - 2. \tag{3.5}$$

Now

$$|E(G')| = \binom{n}{2} - (|\mathcal{A}| + |\mathcal{B}|)\binom{k}{2} > \binom{n}{2} - k(n-1) = \binom{n}{2} - O(n) \tag{3.6}$$

where the first inequality follows because $|\mathcal{A}| < \frac{n-1}{k-1}$ by supposition and $|\mathcal{B}| \leqslant \frac{n-1}{k-1}$ by definition. Now let uv be an arbitrary edge of G' and note that this implies $|\mathcal{A}'_u \cap \mathcal{A}'_v| = 0$. We have $|\mathcal{A}'_u| + |\mathcal{A}'_v| \leqslant \frac{2}{3}|\mathcal{A}|$ because $|\mathcal{A}'_u|, |\mathcal{A}'_v| \leqslant |\mathcal{A}_z|$ by the definition of z and $|\mathcal{A}'_u| + |\mathcal{A}'_v| \leqslant |\mathcal{A}| - |\mathcal{A}_z|$. Then, using (3.5),

$$|N_{G'}(u, v)| \geqslant n - 1 - (k-1)(|\mathcal{A}'_u| + |\mathcal{A}'_v|) - 2(k-2) \geqslant \tfrac{1}{3}n - O(1) \tag{3.7}$$

where the second inequality follows because $|\mathcal{A}'_u| + |\mathcal{A}'_v| \leqslant \frac{2}{3}|\mathcal{A}| < \frac{2(n-1)}{3(k-1)}$. In view of (3.6) and (3.7), we can apply Lemma 3.2.10, choosing $\gamma < \min\{1 - \delta_{K_k}, \frac{1}{3k}\}$, to find a K_k-decomposition $\mathcal{D}'$ of G' and hence complete the proof. □

3.4 Proof of Theorems 3.1.2 and 3.1.3

The proofs of Theorems 3.1.2 and 3.1.3 proceed along similar lines to the proof of Theorem 3.1.1, although the details vary significantly. In each case, we first require a lemma analogous to Lemma 3.3.2: this is Lemma 3.4.1 in the case of Theorem 3.1.2 and Lemma 3.4.2 in the case of Theorem 3.1.3. Like Lemma 3.3.2, these lemmas are proved by colouring with a greedy algorithm that may recolour already-coloured vertices when required.

Lemma 3.4.1. *Let k and a be integers such that $k \geqslant 3$ and $a > \ell$, where $\ell = \frac{1}{4}(k^2-k-2)$. Let H be a graph of order $a(k-1)$ such that $\sum_{x\in V(H)}\lceil \frac{1}{k-1}\deg_H(x)\rceil < k(a-\ell)$. Then H has a proper colouring with a colours such that each colour class contains $k-1$ vertices.*

Proof. Note that ℓ may not be an integer, but $2\ell = \binom{k}{2} - 1$ is an integer. The set-up of the proof proceeds identically to that of the proof of Lemma 3.3.2 up to and including the paragraph after the claim. So we adopt all the notation defined up to that point and see that it suffices to prove the claim there, which we restate below.

Claim. There is a vertex in V_{F}^* that is not adjacent in H_j to any vertex in V'.

Proof of claim. Recall that $v_1,\dots,v_{a(k-1)}$ is a degeneracy ordering of $V(H)$, $V_i = \{v_1,\dots,v_i\}$ and $H_i = H[V_i]$ for each $i \in \{1,\dots a(k-1)\}$ and φ_{j-1} is a legal colouring of H_{j-1} with a set C of a colours for some $j \in \{a+1,\dots,a(k-1)\}$. Further, $V' = \varphi_{j-1}^{-1}(c')$ and $V_{\mathrm{F}}^* = \bigcup_{c\in C_{\mathrm{F}}\setminus C_{\mathrm{N}}}\varphi_{j-1}^{-1}(c)$ where c' is a colour in $C_{\mathrm{N}}\setminus C_{\mathrm{F}}$, $C_{\mathrm{F}} = \{c \in C : |\varphi_{j-1}^{-1}(c)| = k-1\}$ and C_{N} is the set of $a_{\mathrm{N}} \geqslant 1$ colours in C that are assigned by φ_{j-1} to vertices adjacent in H_j to v_j.

Suppose for a contradiction that each vertex in V_{F}^* is adjacent in H_j to some vertex in V'. As in the proof of Lemma 3.3.2, observe that V' and V_{F}^* are disjoint and that

$$|V'| \geqslant 1, \qquad |V_{\mathrm{F}}^*| = (k-1)(a-a_{\mathrm{N}}) \qquad \text{and} \qquad |V_j \setminus (V' \cup V_{\mathrm{F}}^*)| \geqslant a_{\mathrm{N}}. \tag{3.8}$$

Let $r_x = \lceil \frac{1}{k-1}\deg_{H_j}(x)\rceil$ for each $x \in V$ and let $\Phi = \sum_{x\in V_j} r_x - k(a-\ell)$. We will complete the proof by showing that $\Phi \geqslant 0$ and hence obtaining a contradiction to the hypothesis of the lemma that $\sum_{x\in V(H)}\lceil\frac{1}{k-1}\deg_H(x)\rceil < k(a-\ell)$. We do this in two cases according to the value of a_{N}.

Case 1: Suppose that $a_{\mathrm{N}} \leqslant k-1$. Observe that, for each $x \in V_j$, we have $r_x \geqslant 1$ for all $x \in V_j$ because v_j is adjacent in H_j to a vertex of colour c' and thus $\deg_{H_j}(x) \geqslant \deg_{H_j}(v_j) \geqslant 1$ by the properties of the degeneracy ordering. So we have $\sum_{x\in V_j\setminus V'} r_x \geqslant |V_j \setminus V'| \geqslant (k-1)(a-a_{\mathrm{N}}) + a_{\mathrm{N}}$ by (3.8). Furthermore, each of the $|V_{\mathrm{F}}^*|+1$ vertices in $V_{\mathrm{F}}^* \cup \{v_j\}$ is adjacent in H_j to a vertex in V' using our assumption that the claim fails and the fact that $c' \in C_{\mathrm{N}}$. Thus, $\sum_{x\in V'}\deg_{H_j}(x) \geqslant |V_{\mathrm{F}}^*|+1$ and so $\sum_{x\in V'} r_x \geqslant \lceil\frac{1}{k-1}(|V_{\mathrm{F}}^*|+1)\rceil = a - a_{\mathrm{N}} + 1$ where the equality follows by (3.8). Using these lower bounds on $\sum_{x\in V_j\setminus V'} r_x$ and $\sum_{x\in V'} r_x$,

$$\Phi \geqslant (k-1)(a-a_{\mathrm{N}}) + a + 1 - k(a-\ell) = k\ell - a_{\mathrm{N}}(k-1) + 1 \geqslant k(\ell-k+2),$$

where the last inequality follows by using the condition of this case that $a_{\mathrm{N}} \leqslant k-1$ and simplifying. Thus $\Phi \geqslant 0$ and we have the required contradiction because it is easily checked that $\ell \geqslant k-2$ since $k \geqslant 3$.

Case 2: Suppose that $a_{\mathrm{N}} \geqslant k$. We show this case cannot arise by obtaining a contradiction without the need for our assumption that the claim is false. Observe that $\deg_{H_j}(v_j) \geqslant a_{\mathrm{N}}$ by the definition of C_{N} and hence $\deg_{H_j}(x) \geqslant a_{\mathrm{N}}$ for each $x \in V_j$ by the properties of the degeneracy ordering. Thus,

$$r_x \geqslant \tfrac{1}{k-1}a_{\mathrm{N}} \quad \text{for each } x \in V_j. \tag{3.9}$$

So we have $\sum_{x\in V_j} r_x \geqslant \frac{1}{k-1}a_{\mathrm{N}}|V_j| \geqslant \frac{1}{k-1}a_{\mathrm{N}}((k-1)(a-a_{\mathrm{N}}) + a_{\mathrm{N}} + 1)$ by (3.8) and (3.9). Thus,

$$\Phi \geqslant \frac{a_{\mathrm{N}}((k-1)(a-a_{\mathrm{N}})+a_{\mathrm{N}}+1)}{k-1} - k(a-\ell) = a(a_{\mathrm{N}}-k) + k\ell - \frac{(k-2)a_{\mathrm{N}}^2 - a_{\mathrm{N}}}{k-1}$$
$$\geqslant k\ell - \frac{a_{\mathrm{N}}(k^2-k-1-a_{\mathrm{N}})}{k-1}$$

where for the last inequality we substituted $a \geqslant a_{\mathrm{N}}$ in view of the condition of this case that $a_{\mathrm{N}} \geqslant k$. It is routine to check that $a_{\mathrm{N}}(k^2-k-1-a_{\mathrm{N}}) \leqslant k(k-1)\ell$ using the definition of ℓ and the fact that either $a_{\mathrm{N}} \leqslant \binom{k}{2}-1$ or $a_{\mathrm{N}} \geqslant \binom{k}{2}$ since a_{N} is an integer. Thus $\Phi \geqslant 0$ and we have the required contradiction. □

As suggested by the proof of Lemma 3.2.3(b), for any $k \equiv 2 \pmod 4$, the tightness of Lemma 3.4.1 can be seen by taking $a = \frac{1}{2}k(k-1)$ and considering the graph of order $a(k-1)$ consisting of a copy of K_{a+1} and isolated vertices.

Proof of Theorem 3.1.2. The second part of the theorem follows by Lemma 3.2.3(b), so it remains to prove the first part. Let G be a K_k-divisible graph of order n such that $n \equiv 1 \pmod{(k-1)}$ and $|E(\overline{G})| < (\frac{n-1}{k-1}-\ell)\binom{k}{2}$. Throughout the proof we assume that n is large relative to k.

Observe that $\deg_{\overline{G}}(x) \equiv 0 \pmod{k-1}$ for each $x \in V(G)$ since G is K_k-divisible and $n \equiv 1 \pmod{(k-1)}$. Let z be a vertex of minimum degree in G and let $U = N_G(z)$. Since G is K_k-divisible there is an integer a such that $|U| = \deg_G(z) = a(k-1)$. Now $\deg_{\overline{G}}(z) = n-1-a(k-1)$, and each of the $n-1-a(k-1)$ vertices in $N_{\overline{G}}(z)$ has positive degree in $\overline{G}$ and hence has degree at least $k-1$. Thus $\sum_{x\in V(G)\setminus U} \deg_{\overline{G}}(x) \geqslant k(n-1-a(k-1))$, so

$$k(n-1-a(k-1)) + \sum_{x\in U} \deg_{\overline{G}}(x) \leqslant 2|E(\overline{G})| < k(k-1)\left(\tfrac{n-1}{k-1}-\ell\right)$$

and hence $\sum_{x\in U} \deg_{\overline{G}}(x) < k(k-1)(a-\ell)$. Thus, again using $\deg_{\overline{G}}(x) \equiv 0 \pmod{k-1}$ for each $x \in V(G)$,

$$\sum_{x\in U} \lceil \tfrac{1}{k-1} \deg_{\overline{G}[U]}(x) \rceil \leqslant \sum_{x\in U} \lceil \tfrac{1}{k-1} \deg_{\overline{G}}(x) \rceil = \sum_{x\in U} \tfrac{1}{k-1} \deg_{\overline{G}}(x) < k(a-\ell).$$

So we can apply Lemma 3.4.1 to find a proper colouring of $\overline{G}[U]$ with a colours in which each colour class has order $k-1$. Thus, there is a partition $\mathcal{U}$ of U such that $|\mathcal{U}| = a$ and $G[X]$ is a copy of K_{k-1} for each $X \in \mathcal{U}$. Let $\mathcal{D} = \{K_{X\cup\{z\}} : X \in \mathcal{U}\}$.

Let G' be the graph obtained from G by removing the edges of each copy of K_k in $\mathcal{D}$ and then deleting the (now isolated) vertex z. It suffices to show that we can apply Lemma 3.2.10 to find a K_k-decomposition $\mathcal{D}'$ of G', for then $\mathcal{D} \cup \mathcal{D}'$ will be a K_k-decomposition of G. Since G is K_k-divisible, so is G'. Now,

$$|E(G')| = \binom{n}{2} - |E(\overline{G})| - |\mathcal{D}|\binom{k}{2} > \binom{n}{2} - k(n-1) = \binom{n}{2} - O(n) \tag{3.10}$$

where the first inequality follows because $|E(\overline{G})| < \frac{n-1}{k-1}\binom{k}{2}$ and $|\mathcal{D}| \leqslant \frac{n-1}{k-1}$. Let uv be an arbitrary edge of G', let $T = (N_{\overline{G}}(u) \cup N_{\overline{G}}(v)) \setminus \{z\}$, and note that $u, v \notin T$. Each vertex in T has positive degree in $\overline{G}$ and hence degree at least $k-1$. Also $\deg_{\overline{G}}(u) + \deg_{\overline{G}}(v) \geqslant |T|$ and hence $\deg_{\overline{G}}(z) \geqslant \frac{1}{2}|T|$ by the definition of z. Thus we have

$$\tfrac{3}{2}|T| + (k-1)|T| \leqslant \sum_{x\in\{u,v,z\}} \deg_{\overline{G}}(x) + \sum_{x\in T} \deg_{\overline{G}}(x) \leqslant 2|E(\overline{G})| < kn$$

and hence $|T| \leqslant \frac{2k}{2k+1}n$. So we have $|T'| \leqslant \frac{2k}{2k+1}n + O(1)$, where $T' = N_{\overline{G'}}(u) \cup N_{\overline{G'}}(v)$, because it follows from the definition of G' that T' can be obtained from T by adding at most $2(k-1)$ vertices. Thus $|N_{G'}(u,v)| = n-3-|T'| \geqslant \frac{1}{2k+1}n - O(1)$. By this fact and (3.10), we can apply Lemma 3.2.10, choosing $\gamma < \min\{1-\delta_{K_k}, \frac{1}{k(2k+1)}\}$, to find a K_k-decomposition $\mathcal{D}'$ of G' and hence complete the proof. $\square$

In Lemma 3.4.2, we are forced to prove a slightly stronger result for $k=3$ so as to eventually obtain a tight result for $k=3$ in Theorem 3.1.3.

Lemma 3.4.2. *Let k and a be integers such that $k \geqslant 3$ and $a \geqslant 1$. Let H be a graph of order $a(k-1)$ such that either*

(i) $\sum_{x\in V(H)} \lceil \frac{1}{k-1}(\deg_H(x)-1)\rceil \leqslant a-2$; *or*

(ii) $k=3$, $\Delta(H) \leqslant 2a-2$, *and* $\sum_{x\in V(H)} \lceil \frac{1}{k-1}(\deg_H(x)-1)\rceil \leqslant a$.

Then H has a proper colouring with a colours such that each colour class has order $k-1$.

Proof. The set-up of the proof proceeds identically to that of the proof of Lemma 3.3.2 up to and including the paragraph after the claim. So we adopt all the notation defined up to that point and see that it suffices to prove the claim there, which we restate below.

Claim. There is a vertex in V_{F}^* that is not adjacent in H_j to any vertex in V'.

Proof of claim. Recall that $v_1,\dots,v_{a(k-1)}$ is a degeneracy ordering of $V(H)$, $V_i = \{v_1,\dots,v_i\}$ and $H_i = H[V_i]$ for each $i \in \{1,\dots a(k-1)\}$ and φ_{j-1} is a legal colouring of H_{j-1} with a set C of a colours for some $j \in \{a+1,\dots,a(k-1)\}$. Further, $V' = \varphi_{j-1}^{-1}(c')$ and $V_{\mathrm{F}}^* = \bigcup_{c\in C_{\mathrm{F}}\setminus C_{\mathrm{N}}} \varphi_{j-1}^{-1}(c)$ where c' is a colour in $C_{\mathrm{N}}\setminus C_{\mathrm{F}}$, $C_{\mathrm{F}} = \{c \in C : |\varphi_{j-1}^{-1}(c)| = k-1\}$ and C_{N} is the set of $a_{\mathrm{N}} \geqslant 1$ colours in C that are assigned by φ_{j-1} to vertices adjacent in H_j to v_j.

Suppose for a contradiction that each vertex in V_{F}^* is adjacent in H_j to some vertex in V'. As in the proof of Lemma 3.3.2, and further noting that $|V'| \leqslant k-2$ because $c' \notin C_{\mathrm{F}}$, observe that V' and V_{F}^* are disjoint and that

$$k-2 \geqslant |V'| \geqslant 1 \qquad \text{and} \qquad |V_{\mathrm{F}}^*| = (k-1)(a-a_{\mathrm{N}}). \tag{3.11}$$

We consider two cases based on whether $a_{\mathrm{N}} = 1$.

Case 1: Suppose $a_{\mathrm{N}} = 1$. Then, since $c' \in C_{\mathrm{N}} \setminus C_{\mathrm{F}}$, it must be the case that $C_{\mathrm{N}} = C_{\mathrm{N}} \setminus C_{\mathrm{F}} = \{c'\}$. It follows that $C_{\mathrm{F}} = C \setminus \{c'\}$ because $C_{\mathrm{F}} \cup C_{\mathrm{N}} = C$. Now each vertex in V_{F}^* is adjacent in H_j to a vertex in V' using our assumption that the claim fails. Thus $\sum_{x\in V'} \deg_{H_j}(x) \geqslant |V_{\mathrm{F}}^*|$ and so

$$\sum_{x\in V'} \tfrac{1}{k-1}(\deg_{H_j}(x)-1) \geqslant \tfrac{1}{k-1}|V_{\mathrm{F}}^*| - \tfrac{1}{k-1}|V'| > a-2$$

where the last inequality follows because $|V'| \leqslant k-2$ and $V_{\mathrm{F}}^* = (k-1)(a-1)$ by (3.11) since $a_{\mathrm{N}} = 1$. This contradicts (i) of our hypotheses, so we may assume that (ii) holds and hence $k=3$ and $\Delta(H) \leqslant 2a-2$. Then $|V'| = \{y\}$ for some $y \in V_{j-1}$ because $1 \leqslant |V'| \leqslant k-2 = 1$ by (3.11). Thus y is adjacent in H_j to each of the $(k-1)(a-1) = 2a-2$ vertices in V_{F}^* by our assumption that the claim fails. Furthermore, y is adjacent in H_j to v_j since $c' \in C_{\mathrm{N}}$. Thus $\deg_{H_j}(y) \geqslant 2a-1$ in contradiction to our assumption that $\Delta(H) \leqslant 2a-2$.

Case 2: Suppose $a_{\mathrm{N}} \geqslant 2$. We show this case cannot arise by obtaining a contradiction without the need for our assumption that the claim is false. Then $\deg_{H_j}(v_j) \geqslant a_{\mathrm{N}} \geqslant 2$ by the definition of C_{N}. So, for each $x \in V_j$, we have $\deg_{H_j}(x) \geqslant 2$ by the properties of the degeneracy ordering and hence $\lceil \frac{1}{k-1}(\deg_{H_j}(x)-1)\rceil \geqslant 1$. But then we have $\sum_{x\in V_j} \lceil \frac{1}{k-1}(\deg_{H_j}(x)-1)\rceil \geqslant j$ which contradicts both (i) and (ii) of our hypotheses since $j \geqslant a+1$. $\square$

For each odd $k \geqslant 5$ and each $a \geqslant 2$, the tightness of the condition $\sum_{x \in V(H)} \lceil \frac{1}{k-1}(\deg_H(x) - 1) \rceil \leqslant a - 2$ in Lemma 3.4.2 is witnessed by the graph of order $a(k-1)$ that is the vertex disjoint union of a star with $(a-1)(k-1)+1$ edges and a perfect matching with $\frac{1}{2}(k-3)$ edges. In any proper colouring of such a graph, the colour assigned to the centre vertex of the star must be assigned to fewer than $k-1$ vertices. The proof of Theorem 3.1.3 differs from the proof of Theorems 3.1.1 and 3.1.2 in that it appears that the order, and hence the degrees, of G can belong to any congruence class modulo $k-1$. However we quickly see that the critical case is when the order of G is congruent to 2 modulo $k-1$.

Proof of Theorem 3.1.3. The second part of the theorem follows by Lemma 3.2.3(c) and Lemma 3.2.5(a), so it remains to prove the first part. Let G be a K_k-divisible graph of order n such that either $|E(\overline{G})| < n - \frac{1}{2}(k+1)$ or $k = 3$ and $|E(\overline{G})| < n$. Then, because G cannot be K_3-divisible if $|E(\overline{G})| = n-2$, in fact we have either

- $|E(\overline{G})| < n - \frac{1}{2}(k+1)$; or
- $k = 3$ and $|E(\overline{G})| = n - 1$.

We assume that n is large relative to k and consider three cases according to the congruence class of n modulo $k-1$.

Case 1: Suppose that $k \geqslant 4$ and $n - 1 \equiv j \pmod{k-1}$ for some $j \in \{2, \dots, k-2\}$. Then, because G is K_k-divisible, $\deg_{\overline{G}}(x) \equiv j \pmod{k-1}$ for each $x \in V(G)$. Therefore, $|E(\overline{G})| \geqslant \frac{1}{2}jn \geqslant n$, contradicting our assumption. So this case cannot arise.

Case 2: Suppose that $n - 1 \equiv 0 \pmod{k-1}$. Then, because G is K_k-divisible, $\deg_{\overline{G}}(x) \equiv 0 \pmod{k-1}$ for each $x \in V(G)$. Let uv be an arbitrary edge of G. Let $T = N_{\overline{G}}(u) \cup N_{\overline{G}}(v)$ and note that $u, v \notin T$ and $|T| \leqslant \deg_{\overline{G}}(u) + \deg_{\overline{G}}(v)$. Also, $\deg_{\overline{G}}(x)$ is positive for each $x \in T$ and hence at least $k-1$. We have

$$|T| + (k-1)|T| \leqslant \sum_{x \in \{u,v\}} \deg_{\overline{G}}(x) + \sum_{x \in T} \deg_{\overline{G}}(x) \leqslant 2|E(\overline{G})| < 2n.$$

So $|T| < \frac{2n}{k} \leqslant \frac{2n}{3}$ since $k \geqslant 3$. Therefore, $|N_G(u,v)| = n - 2 - |T| \geqslant \frac{n}{3} - O(1)$. We also have $|E(G)| > \binom{n}{2} - n$. In view of these two facts, we can apply Lemma 3.2.10, choosing $\gamma < \min\{1 - \delta_{K_k}, \frac{1}{3k}\}$, to find a K_k-decomposition $\mathcal{D}$ of G and hence complete the proof.

Case 3: Suppose that $n - 1 \equiv 1 \pmod{k-1}$. Then, because G is K_k-divisible, $\deg_{\overline{G}}(x) \equiv 1 \pmod{k-1}$ for each $x \in V(G)$. It will be convenient to define $\rho = 0$ if $|E(\overline{G})| < n - \frac{1}{2}(k+1)$ and $\rho = 2$ if $k = 3$ and $|E(\overline{G})| = n - 1$, so that we always have $|E(\overline{G})| < n - \frac{1}{2}(k+1) + \rho$.

Let z be a vertex of minimum degree in G and let $U = N_G(z)$. We will show that $\overline{G}[U]$ obeys the hypotheses of Lemma 3.4.2. Since G is K_k-divisible there is an integer a such that $|U| = \deg_G(z) = a(k-1)$. We may assume that $a \geqslant 1$ for otherwise $a = 0$, $k = 3$, $\overline{G}$ is a star with $n-1$ edges and hence G is K_3-decomposable because its edges form a copy of K_{n-1} and G is K_3-divisible by assumption. Now $\deg_{\overline{G}}(z) = n - 1 - a(k-1)$, and each of the $n - 1 - a(k-1)$ vertices in $N_{\overline{G}}(z)$ has degree at least 1 in $\overline{G}$. Thus $\sum_{x \in V(G) \setminus U} \deg_{\overline{G}}(x) \geqslant 2n - 2 - 2a(k-1)$, so

$$\sum_{x \in U} \deg_{\overline{G}}(x) \leqslant 2|E(\overline{G})| - (2n - 2 - 2a(k-1)) < (2a-1)(k-1) + 2\rho \tag{3.12}$$

where the last inequality follows because $|E(\overline{G})| < n - \frac{1}{2}(k+1) + \rho$. Thus,

$$\sum_{x \in U} \lceil \tfrac{1}{k-1}(\deg_{\overline{G}[U]}(x) - 1) \rceil \leqslant \sum_{x \in U} \lceil \tfrac{1}{k-1}(\deg_{\overline{G}}(x) - 1) \rceil = \tfrac{1}{k-1} \sum_{x \in U} \deg_{\overline{G}}(x) - a < a - 1 + \rho$$

where the equality holds because $|U| = a(k-1)$ and $\deg_{\overline{G}}(x) \equiv 1 \pmod{k-1}$ for each $x \in U$, and the last inequality follows using (3.12) and the fact that $\frac{2}{k-1}\rho = \rho$ in all cases. So we in fact have $\sum_{x\in U}\lceil \frac{1}{k-1}(\deg_{\overline{G}[U]}(x)-1)\rceil \leqslant a-2+\rho$ because the terms are all integers. So if $\rho = 0$, then H obeys (i) in the hypotheses of Lemma 3.4.2. If $\rho = 2$ and hence $k = 3$, $\deg_{\overline{G}}(z) = n-2a-1$ and $|E(\overline{G})| = n-1$, then $\Delta(\overline{G}[U]) \leqslant 2a-2$. To see this, observe that otherwise $\overline{G}[U]$ has a vertex of degree at least $2a-1$ and so contains a star with $2a-1$ edges none of whose vertices are adjacent to z in $\overline{G}$. Thus, since $\deg_{\overline{G}}(z) = n-2a-1$ and $|E(\overline{G})| = n-1$, $\overline{G}$ would be a graph obtained by adding exactly one edge to the vertex disjoint union of a star with $n-2a-1$ edges and a star with $2a-1$ edges, which contradicts the fact that each vertex of $\overline{G}$ has odd degree. So if $\rho = 2$, then H obeys (ii) in the hypotheses of Lemma 3.4.2. Thus, by Lemma 3.4.2 there exists a proper colouring of $\overline{G}[U]$ with a colours in which each colour class has order $k-1$. So there is a partition $\mathcal{U}$ of U such that $|\mathcal{U}| = a$ and $G[X]$ is a copy of K_{k-1} for each $X \in \mathcal{U}$. Let $\mathcal{D} = \{K_{X\cup\{z\}} : X \in \mathcal{U}\}$.

Let G' be the graph obtained from G by removing the edges of each copy of K_k in $\mathcal{D}$ and then deleting the (now isolated) vertex z. It suffices to show that we can apply Lemma 3.2.10 to find a K_k-decomposition $\mathcal{D}'$ of G', for then $\mathcal{D} \cup \mathcal{D}'$ will be a K_k-decomposition of G. Since G is K_k-divisible, so is G'.

Let uv be an arbitrary edge of G', let $T = (N_{\overline{G}}(u) \cup N_{\overline{G}}(v)) \setminus \{z\}$, and note that $u, v \notin T$. Furthermore $\deg_{\overline{G}}(u) + \deg_{\overline{G}}(v) \geqslant |T|$. At most two edges of $\overline{G}$ are incident with two vertices in $\{u, v, z\}$ and hence

$$\deg_{\overline{G}}(u) + \deg_{\overline{G}}(v) + \deg_{\overline{G}}(z) \leqslant |E(\overline{G})| + 2 \leqslant n+1.$$

Thus, because $\deg_{\overline{G}}(u), \deg_{\overline{G}}(v) \leqslant \deg_{\overline{G}}(z)$ by the definition of z, we have that $|T| \leqslant \deg_{\overline{G}}(u) + \deg_{\overline{G}}(v) \leqslant \frac{2}{3}n + O(1)$. So, considering the way in which G' is obtained from G, $N_{G'}(u,v) \geqslant n-3-|T|-2(k-1) > \frac{1}{3}n - O(1)$. We also have

$$|E(G')| = \binom{n}{2} - |E(\overline{G})| - |\mathcal{D}|\binom{k}{2} > \binom{n}{2} - n - \tfrac{1}{2}k(n-1) = \binom{n}{2} - O(n)$$

because $|E(\overline{G})| < n$ and $|\mathcal{D}| \leqslant \frac{n-1}{k-1}$. In view of these two facts, we can apply Lemma 3.2.10, choosing $\gamma < \min\{1-\delta_{K_k}, \frac{1}{3k}\}$, to find a K_k-decomposition $\mathcal{D}'$ of G' and so complete the proof. □

The proof of Corollary 3.1.5 follows easily from Theorems 3.1.2 and 3.1.3 and Lemma 3.2.5.

Proof of Corollary 3.1.5. For $n \equiv 1, 3 \pmod 6$ the result follows immediately from Theorem 3.1.2 and for $n \equiv 0 \pmod 6$ the result follows immediately from Theorem 3.1.3. For $n \equiv 5 \pmod 6$, Lemma 3.2.5(b) gives a K_3-divisible graph with $\binom{n}{2} - \frac{1}{2}(3n-7)$ edges that has no K_3-decomposition and, furthermore, any K_3-divisible graph of order n with more than $\binom{n}{2} - \frac{1}{2}(3n-7)$ edges has at least $\binom{n}{2} - \frac{1}{2}(3n-13)$ edges and hence is K_3-decomposable by Theorem 3.1.2 if n is sufficiently large. For $n \equiv 2, 4 \pmod 6$, Lemma 3.2.5(c) gives a K_3-divisible graph with $\binom{n}{2} - n - 2$ edges that has no K_3-decomposition and, furthermore, any K_3-divisible graph of order n with more than $\binom{n}{2} - n - 2$ edges has at least $\binom{n}{2} - n + 1$ edges and hence is K_3-decomposable by Theorem 3.1.3 if n is sufficiently large. □

In Chapter 7 we discuss some possibilities for further work motivated by the results in this chapter.

Chapter 4

Complexity results for embedding partial Steiner triple systems

> *" When you think you're simplifying you're usually just transferring the complexity to another place."*
>
> \- Bill Buxton, *Microsoft Research*

4.1 Introduction

Recall that a *partial Steiner triple system of order* u, or PSTS(u), is a pair $(U, \mathcal{A})$ where U is a set of u elements and $\mathcal{A}$ is a set of triples of elements of U with the property that any two elements of U occur together in at most one triple. If any two elements of U occur together in exactly one triple then $(U, \mathcal{A})$ is a *Steiner triple system of order* u, or STS(u). As previously mentioned, it is well known that a Steiner triple system of order u exists if and only if $u \equiv 1, 3 \pmod 6$ [28]. This was first proved by Kirkman in [64]. We call integers congruent to 1 or 3 modulo 6 *admissible* and denote the set of positive admissible integers by $\mathbb{N}^{\dagger}$.

Remember that a K_3*-decomposition* of a graph G is a set of triangles in G such that each edge of G is in exactly one triangle in the set. A Steiner triple system of order v is equivalent to a K_3-decomposition of K_v and a partial Steiner triple system of order u is equivalent to a K_3-decomposition of some subgraph of K_u. The *leave* of a partial Steiner triple system $(U, \mathcal{A})$ is the graph L having vertex set U and the edge set $E(L) = \{xy : \{x, y, z\} \notin \mathcal{A} \text{ for all } z \in U\}$. For a partial Steiner triple system $(U, \mathcal{A})$, we say that a (complete) Steiner triple system $(V, \mathcal{B})$ is an *embedding* of $(U, \mathcal{A})$ if $U \subseteq V$ and $\mathcal{A} \subseteq \mathcal{B}$. A (proper) c*-edge colouring* of a graph G is an assignment of colours, chosen from some set of c colours, to the edges of G in such a way that any two edges incident with the same vertex receive distinct colours. All edge colourings considered in this chapter will be proper.

It is known that any PSTS(u) has an embedding of order v for each admissible integer $v \geqslant 2u + 1$ [14]. Moreover, the bound of $v \geqslant 2u + 1$ cannot be improved in general due to the fact that for any $u \geqslant 9$ there exists a PSTS(u) which cannot be embedded in an STS(v) for any $v < 2u + 1$ (see [28, Lemma 11.3]). Of course, many partial Steiner triple systems do have embeddings of order less than $2u + 1$. We call such embeddings *small* embeddings.

This chapter concerns the problem of determining whether a given partial Steiner triple system has a small embedding of a specified order. Various aspects of this problem have been addressed in many papers (see [10, 14, 21, 62, 73] or Section 2.1.3 for example). In this chapter, we provide updates on two of these contributions, namely [21] and [10].

In [21] Colbourn showed the problem of determining whether a given partial Steiner triple system has a small embedding is NP-complete. As previously mentioned, there are sensible questions about small embeddings that this result does not cover. For example we could ask: does a given partial Steiner triple system of order u have an embedding of order $u+10$? Similarly, we could ask: does a given partial Steiner triple system of order u have an embedding of order between $\frac{3u}{2}$ and $\frac{5u}{3}$? Colbourn's result does not say whether either of these questions is NP-complete. Our main result will show that many questions of this kind are also hard.

In order to be more precise we make some definitions. Call a function $F : \mathbb{N} \to \mathcal{P}(\mathbb{N})$ *admissible* if $F(u) \subseteq \{x \in \mathbb{N}^\dagger : x \geqslant u\}$ for each $u \in \mathbb{N}$. For each admissible function F we define a decision problem as follows.

F-EMBED
Instance: A partial Steiner triple system $(U, \mathcal{A})$.
Question: Does $(U, \mathcal{A})$ have an embedding of order v for some $v \in F(|U|)$?

More formally, Colbourn's result in [21] is that F^*-EMBED is NP-complete, where F^* is the admissible function defined by $F^*(u) = \{x \in \mathbb{N}^\dagger : u \leqslant x < 2u+1\}$ for each $u \in \mathbb{N}$. Here we extend this result by proving the following theorem. For a subset S of $\mathbb{N}$, we say that integers in S *occur polynomially often* if there is a polynomial $P(x)$ such that, for each $n \in \mathbb{N}$, we have $\{s \in S : n \leqslant s \leqslant P(n)\} \neq \emptyset$.

Theorem 4.1.1. *Let F be an admissible function. The decision problem F-EMBED is NP-complete if there exists a real number $\epsilon > 0$ such that integers u for which $F(u) \neq \emptyset$ and $\max(F(u)) < (2-\epsilon)u$ occur polynomially often.*

Note that the answer to F-EMBED for any PSTS(u) is obviously negative if $F(u) = \emptyset$ and is affirmative if $\max(F(u)) \geqslant 2u$ (and hence at least $2u+1$) because embeddings are known to exist for all non-small admissible orders. Thus, Theorem 4.1.1 is best possible except for the ϵ term and the mild condition of being nontrivial polynomially often. This latter condition merely rules out choices of F that are pathological in the sense that there are exponentially long intervals of orders u for which F-EMBED is trivial for all Steiner triple systems of order u. We give two examples of the definition of 'polynomially often' in action. The integers in $\{2^i : i \in \mathbb{N}\}$ occur polynomially often because if we take $P(x)$ to be the polynomial $2x$, then for any $n \in \mathbb{N}$ there is always an element of this set between n and $P(n)$ inclusive. On the other hand, the integers in $S' = \{a_i : i \in \mathbb{N}\}$, where $a_0 = 1$ and $a_{i+1} = 2^{a_i+1}$ for each $i \in \mathbb{N}$, do not occur polynomially often. To see this, let $P(x)$ be an arbitrary polynomial and note there is an integer x_0 such that $P(x) < 2^x$ for all $x > x_0$. Then, if we take n to be an element of S' greater than x_0, it can be seen that there are no elements of S' between $n+1$ and $P(n+1)$ inclusive.

For vertex-disjoint graphs G and H, we let $G \vee H$ denote the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{xy : x \in V(G), y \in V(H)\}$. As discussed in Section 2.1.3 Bryant [10] made a conjecture about the existence of K_3-decompositions of $L \vee K_w$. It is obvious that a partial Steiner triple system of order u with a leave L can be embedded in a Steiner triple system of order $v = u + w$ if and only if there exists a K_3-decomposition of $L \vee K_w$. He conjectured that certain conditions that can be seen to be necessary for the existence of a K_3-decomposition of $L \vee K_w$ are also sufficient (see [10, Lemma 2.1] for a proof of their necessity). For graphs G and H we define $G - H$ to be the graph with vertex set $V(G)$ and edge set $E(G) \setminus E(H)$.

Conjecture 4.1.2 ([10])**.** *Let L be a graph with u vertices and let w be a nonnegative integer. Then there exists a K_3-decomposition of $L \vee K_w$ if and only if the following conditions are satisfied.*

(1) $\deg_L(x) \equiv w \pmod 2$ *for each vertex x of L;*

(2) *$u + w$ is odd for $w > 0$;*

(3) $|E(L)| + uw + \binom{w}{2} \equiv 0 \pmod 3$*; and*

(4) *There exists a subgraph G of L such that*

  (i) *$L - G$ has a K_3-decomposition;*

  (ii) $w^2 - (u+1)w + 2|E(G)| \geqslant 0$*;*

  (iii) *G is w-edge colourable.*

Theorem 4.1.1 and the main result of [21] suggest that there may be no efficient algorithm for determining which small orders a partial Steiner triple system has an embedding into. But Conjecture 4.1.2 postulates a neat characterization of these orders in terms of chromatic indices of graphs. Here we suggest that things may not be so simple by exhibiting a family of counterexamples to Conjecture 4.1.2.

Theorem 4.1.3. *For each even integer $w \geqslant 4$, there is a partial Steiner triple system whose leave is a counterexample to Conjecture 4.1.2.*

For $w = 4$ we explicitly exhibit a system of order 15 whose leave is a counterexample to Conjecture 4.1.2 (see Example 4.3.2). For $w \geqslant 6$, however, we merely establish the existence of appropriate systems with large (unspecified) orders.

4.2 Hardness of finding small embeddings of specified orders

We aim to prove Theorem 4.1.1 by reducing to F-EMBED from the problem of whether a cubic graph is properly 3-edge colourable, which is well known to be NP-complete [61]. Critical to this approach will be the construction of a class of partial Steiner triple systems which we now define.

Definition 4.2.1. For positive integers u and v and a cubic graph G, a (u, v, G)-*background* is a PSTS(u) that has no embedding of order less than v and, further, has an embedding of order v if and only if G is 3-edge colourable.

Lemma 4.2.2. *If G is a cubic graph of order $n \geqslant 74$ and u and v are integers such that $v \equiv 1, 3 \pmod 6$, $u \geqslant 4n + 43$ and $u \leqslant v \leqslant 2u - 2n - 13$, then there exists a (u, v, G)-background.*

Before proceeding to prove Lemma 4.2.2, we show how Theorem 4.1.1 can be proved from Lemma 4.2.2.

Proof that Lemma 4.2.2 implies Theorem 4.1.1. Let F be an admissible function and let $\epsilon > 0$ be a real number such that integers u for which $F(u) \neq \emptyset$ and $\max(F(u)) < (2-\epsilon)u$ occur polynomially often. So there is a nondecreasing polynomial $P(x)$ such that, for each $i \in \mathbb{N}$, there is an integer u such that $i \leqslant u \leqslant P(i)$, $F(u) \neq \emptyset$ and $\max(F(u)) < (2-\epsilon)u$. We reduce to F-EMBED from the problem of whether a cubic graph is 3-edge colourable, which is well known to be NP-complete [61]. Of course, this latter problem remains NP-complete if we exclude finitely many inputs by requiring that the graph have order at least 74.

Suppose we are given a cubic graph G of order $n \geqslant 74$. Let $u_0 = \max(4n+43, \frac{1}{\epsilon}(2n+13))$ and let u be the smallest integer such that $u \geqslant u_0$, $F(u) \neq \emptyset$ and $\max(F(u)) < (2-\epsilon)u$, noting that u exists by the properties of F and ϵ. Furthermore, $u \leqslant P(u_0)$ by the definition of $P(x)$ and hence u is polynomial in n because $u \leqslant Q(n)$ where $Q(x)$ is the polynomial $P(\frac{1}{\epsilon}(4x+43))$. Then, because F is admissible and $u \geqslant \frac{1}{\epsilon}(2n+13)$, we have $u \leqslant \max(F(u)) \leqslant 2u-2n-13$. Let $v = \max(F(u))$. Thus u and v satisfy the hypothesis of Lemma 4.2.2 and hence there exists a (u,v,G)-background $(U,\mathcal{A})$. Because $(U,\mathcal{A})$ is a (u,v,G)-background, the answer to F-EMBED for input $(U,\mathcal{A})$ will be affirmative if and only if G is 3-edge colourable. □

So our goal in the rest of this section will be to prove Lemma 4.2.2. We recall some further notation that we will require. For graphs G and H we define $G \cup H$ to be the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. For a set S we denote the complete graph with vertex set S by K_S and denote its complement, the graph with vertex set S and empty edge set, by $\overline{K_S}$. For disjoint sets S and T, we denote the complete bipartite graph with parts S and T by $K_{S,T}$. We say a graph is *even* if each of its vertices has even degree.

Definition 4.2.3. A K_3*-packing* of a graph G is a K_3-decomposition of some subgraph H of G and the *leave* of such a packing is the graph $G - H$.

It will be useful for us to blur the distinction between partial Steiner triple systems and K_3-packings by representing the latter as sets of vertex triples rather than as sets of triangles. We do this throughout the chapter.

Lemma 4.2.4. *Let G be a cubic graph and let Z be a vertex set such that $|Z| = 3$ and Z is disjoint from $V(G)$.*

(i) *If G is 3-edge colourable then there is a K_3-decomposition of $\overline{K_Z} \vee G$.*

(ii) *If G is not 3-edge colourable then the leave of any K_3-packing of $\overline{K_Z} \vee G$ contains an edge incident with a vertex in Z.*

Proof. Let n be the order of G.

(i) Assume G is 3-edge colourable. Let γ be a proper 3-edge colouring of G with colour set Z. Then
$$\mathcal{Q} = \{\{x, y, \gamma(xy)\} : xy \in E(G)\}$$
is a K_3-decomposition of $\overline{K_Z} \vee G$. Each edge of G is obviously in exactly one triangle in $\mathcal{Q}$, and the fact that γ is a proper 3-edge colouring implies that each edge in $K_{Z,V(G)}$ is in exactly one triangle in $\mathcal{Q}$.

(ii) Suppose for a contradiction that G is not 3-edge colourable and that there is a triangle packing $\mathcal{Q}$ of $\overline{K_Z} \vee G$ such that every edge incident with a vertex in Z is in some triangle of $\mathcal{Q}$. Then each vertex in Z is in $\frac{n}{2}$ triangles in $\mathcal{Q}$ and hence for every edge xy in $E(G)$ there is a triangle $\{x, y, z\}$ in $\mathcal{Q}$ for some $z \in Z$. Define an edge colouring γ of G with colour set Z by setting $\gamma(xy) = z$ for each $xy \in E(G)$, where z is the unique element of Z such that $\{x, y, z\} \in \mathcal{Q}$. Then γ is a proper 3-edge colouring of G, which is a contradiction. □

Lemma 4.2.5 is our first step toward constructing (u, v, G)-backgrounds.

Lemma 4.2.5. *Let G be a cubic graph of order $n \geqslant 74$. Let A be a vertex set such that $V(G) \subseteq A$, $|A| \geqslant 2n + 1$ and $|A| \equiv 1, 3 \pmod 6$, and let $Z \subseteq A \setminus V(G)$ such that $|Z| = 3$. Then there exists a partial Steiner triple system $(A, \mathcal{B}_0)$ whose leave L has edge set $E(\overline{K_Z} \vee G)$.*

Proof. By [62, Theorem 5.2], if G' is an even graph of order a such that $a \geqslant 103$, $|E(G')| \equiv 0 \pmod 3$, $|E(G')| \geqslant \binom{a}{2} - \frac{1}{128}(3a^2 - 54a - 409)$ and at least $\frac{1}{8}(3a + 17)$ vertices of G' have degree $a - 1$, then there is a K_3-decomposition of G'.

Let $a = |A|$ and let $G' = K_A - (\overline{K_Z} \vee G)$. We will complete the proof by showing that G' satisfies the conditions above. Note that K_A is even because $a \equiv 1, 3 \pmod 6$ and $\overline{K_Z} \vee G$ is even because n is even, so G' is even. Next, we have $|V(G')| = a > 103$ because $a \geqslant 2n + 1$ and $n \geqslant 74$. Now $|E(G')| = \binom{a}{2} - (3n + \frac{3n}{2}) = \binom{a}{2} - \frac{9n}{2}$ and hence $|E(G')| \equiv 0 \pmod 3$ because $a \equiv 1, 3 \pmod 6$. Also $|E(G')| \geqslant \binom{a}{2} - \frac{1}{128}(3a^2 - 54a - 409)$ because

$$\tfrac{1}{128}(3a^2 - 54a - 409) \geqslant \tfrac{1}{128}((6n - 51)(2n + 1) - 409) \geqslant \tfrac{393}{128}(2n + 1) - \tfrac{409}{128} > \tfrac{9n}{2}$$

where the first inequality holds because $a \geqslant 2n+1$, and the second and third hold because $n \geqslant 74$. Finally, $a - n - 3$ vertices in G' have degree $a - 1$ and $a - n - 3 > \frac{1}{8}(3a + 17)$ because $5a \geqslant 10n + 5 > 8n + 41$ where the first inequality holds because $a \geqslant 2n + 1$ and the second holds because $n \geqslant 74$. □

We are now able to construct some of the (u, v, G)-backgrounds we require. We do this in Lemma 4.2.6 and then prove that they are in fact (u, v, G)-backgrounds in Lemma 4.2.7.

Lemma 4.2.6. *Let G be a cubic graph of order $n \geqslant 74$ and let u and d be integers such that $d \geqslant n + 2$, $u \geqslant d + 2n + 3$, $d \equiv 0 \pmod 6$ and $u \equiv 1, 3 \pmod 6$. There exists a PSTS(u) $(U, \mathcal{A})$ whose leave has edge set $E((\overline{K_Z} \vee G) \cup K_{A',D})$ where*

- *$\{A', D, V(G) \cup \{x\}\}$ is a partition of U for some $x \in U \setminus V(G)$;*
- *$|D| = d$;*
- *$Z \subseteq A'$ with $|Z| = 3$.*

Proof. Let U be a set with $|U| = u$ and $V(G) \subseteq U$, and let $\{A', D, V(G) \cup \{x\}\}$ be a partition of U satisfying the conditions of the lemma. Let $A'' = V(G) \cup \{x\}$ and let $A = U \setminus D = A' \cup A''$.

Observe that $|A| = u - d \geqslant 2n + 3$ and $|A| \equiv 1, 3 \pmod 6$ because $u \equiv 1, 3 \pmod 6$ and $d \equiv 0 \pmod 6$. Thus by Lemma 4.2.5 there exists a partial Steiner triple system $(A, \mathcal{B}_0)$ whose leave has edge set $E(\overline{K_Z} \vee G)$. If there exists a K_3-decomposition $\mathcal{B}_1$ of $K_{A'' \cup D} - K_{A''}$, then $(U, \mathcal{B}_0 \cup \mathcal{B}_1)$ will indeed be a partial Steiner triple system whose leave has edge set $E((\overline{K_Z} \vee G) \cup K_{A',D})$ and we are finished. So it suffices to show that such a K_3-decomposition exists.

It is known (see [38, 69]) that a K_3-decomposition of $K_v - K_w$ exists if and only if v and w are odd, $v \geqslant 2w+1$, and $\binom{v}{2} - \binom{w}{2} \equiv 0 \pmod{3}$. Now $|A'' \cup D| = n+1+d$ and $|A''| = n+1$ are both odd because n and d are even, and $n+1+d \geqslant 2n+3$ because $d \geqslant n+2$. Finally, $\binom{d+n+1}{2} - \binom{n+1}{2} = \frac{1}{2}d(d+2n+1) \equiv 0 \pmod{3}$ because $d \equiv 0 \pmod{6}$. $\square$

Lemma 4.2.7. *Let G be a cubic graph of order $n \geqslant 74$ and let u and v be integers such that $u \geqslant 3n+5$, $\frac{1}{2}(3u-n-1) \leqslant v \leqslant 2u-2n-3$ and $u \equiv v \equiv 1, 3 \pmod{6}$. Then there exists a (u,v,G)-background.*

Proof. Let $d = v - u$ and note that $d \equiv 0 \pmod{6}$ because $u \equiv v \pmod{6}$. Let $(U, \mathcal{A})$ be a PSTS(u) whose leave has the edge set $E((\overline{K_Z} \vee G) \cup K_{A',D})$ where

- $\{A', D, V(G) \cup \{x\}\}$ is a partition of U for some $x \in U \setminus V(G)$;
- $|D| = d$;
- $Z \subseteq A'$ with $|Z| = 3$.

The existence of such a partial Steiner triple system has been proved in Lemma 4.2.6, noting that $v \leqslant 2u - 2n - 3$ implies that $u \geqslant d + 2n + 3$ and that $u \geqslant 3n+5$ and $\frac{1}{2}(3u-n-1) \leqslant v$ imply $d \geqslant n+2$. Let L be the leave of $(U, \mathcal{A})$. We claim that $(U, \mathcal{A})$ is a (u,v,G)-background.

We will first show that $(U, \mathcal{A})$ has no embedding of order less than $v = u+d$, and has no embedding of order $v = u+d$ if G is not 3-edge colourable. Suppose $(U, \mathcal{A})$ has an embedding $(U \cup W, \mathcal{A} \cup \mathcal{A}' \cup \mathcal{A}'')$ where W is disjoint from U, triples in $\mathcal{A}'$ are subsets of U and triples in $\mathcal{A}''$ each contain at least one vertex in W. Let L' be the leave of $(U, \mathcal{A} \cup \mathcal{A}')$. We show that $|W| \geqslant d$ and that $|W| \geqslant d+1$ if G is not 3-edge colourable.

Consider any vertex $y \in A' \setminus Z$. Because the subgraph of L induced by D is empty, no triple in $\mathcal{A}'$ can contain y and hence $\deg_{L'}(y) = \deg_L(y) = d$. Each of the d edges incident in L' with y is in a triple of $\mathcal{A}''$ whose third vertex is in W, and no two of these vertices in W may be the same. Therefore, $|W| \geqslant d$.

Now further assume G is not 3-edge colourable. The triples in $\mathcal{A}'$ form a K_3-packing of $\overline{K_Z} \vee G$, so by Lemma 4.2.4(ii) there exists a vertex $z \in Z$ such that $\deg_{L'}(z) \geqslant d+1$. Each of the at least $d+1$ edges incident in L' with z is in a triple of $\mathcal{A}''$ whose third vertex is in W, and no two of these vertices in W may be the same. Hence, $|W| \geqslant d+1$ if G is not 3-edge colourable.

Now, we will show that if G is 3-edge colourable then $(U, \mathcal{A})$ has an embedding of order $u+d$. Assume G is 3-edge colourable and let V be a vertex set with $|V| = u+d$ and $U \subseteq V$. Let $A = U \setminus D$ and let $a = |A| = u - d$. By Lemma 4.2.4(i), there is a K_3-decomposition $\mathcal{A}^\dagger$ of $\overline{K_Z} \vee G$. Then $(U, \mathcal{A} \cup \mathcal{A}^\dagger)$ is a PSTS(u) whose leave has edge set $E(K_{A',D})$. Equivalently, $\mathcal{A} \cup \mathcal{A}^\dagger$ is a K_3-decomposition of $K_A \cup K_{D \cup V(G) \cup \{x\}}$. Let $B = D \cup V(G) \cup \{x\}$. It suffices to show that there is a K_3-decomposition $\mathcal{A}^\ddagger$ of $K_V - (K_A \cup K_B)$ because then $(V, \mathcal{A} \cup \mathcal{A}^\dagger \cup \mathcal{A}^\ddagger)$ will be an embedding of $(U, \mathcal{A})$ of order $u+d$.

By [29, Theorem 3.1], there exists a K_3-decomposition of $K_V - (K_A \cup K_B)$ if

(i) $|B| \geqslant |A|$;

(ii) $|V| = 2|B| + |A| - 2|A \cap B|$;

(iii) $|A|$ and $|B|$ are odd;

(iv) $|A| \geqslant 2|A \cap B| + 1$; and

(v) $(|B| - |A \cap B|)(|A| - 2|A \cap B| - 1) \equiv 0 \pmod 3$.

So it suffices to show that (i) – (v) hold. Note $|V| = u+d = a+2d$, $|A| = a$, $|B| = d+n+1$, and $|A \cap B| = n + 1$. Because $\frac{1}{2}(3u - n - 1) \leqslant v$ we have that (i) holds, noting that $|B| = v - u + n + 1$ and $|A| = u - d = 2u - v$. Furthermore, (ii) and (iii) obviously hold, (iv) holds because $v \leqslant 2u - 2n - 3$, and (v) holds because $d \equiv 0 \pmod 6$. So there is a K_3-decomposition of $K_V - (K_A \cup K_B)$ and the proof is complete. □

We can now obtain all the (u, v, G)-backgrounds we require by simply adding new vertices to those we have already constructed.

Proof of Lemma 4.2.2. Let u and v be integers satisfying the hypotheses of the lemma. If, for some integer $u' \leqslant u$, we can find a (u', v, G)-background $(U', \mathcal{A})$, then the partial Steiner triple system $(U, \mathcal{A})$ obtained from $(U', \mathcal{A})$ by adding $u - u'$ new vertices will be a (u, v, G)-background. So it suffices to find an integer $u' \leqslant u$ such that u', v and G satisfy the hypotheses of Lemma 4.2.7. We choose u' to be the largest integer such that $u' \leqslant \min(u, \frac{1}{3}(2v + n + 1))$ and $u' \equiv v \pmod 6$. This implies u' must be odd.

Case 1. If $u \leqslant \frac{1}{3}(2v + n + 1)$, then $u - 5 \leqslant u' \leqslant u$. Thus $\frac{1}{2}(3u' - n - 1) \leqslant v$, because $u \leqslant \frac{1}{3}(2v + n + 1)$ by the conditions of this case and $u' \leqslant u$. Also, $v \leqslant 2u' - 2n - 3$ because $v \leqslant 2u - 2n - 13$ and $u \leqslant u' + 5$. Moreover $u' \geqslant 3n + 5$ because $u \geqslant 4n + 43$ and $u \leqslant u' + 5$. So u', v and G satisfy the hypotheses of Lemma 4.2.7.

Case 2. If $u > \frac{1}{3}(2v + n + 1)$, then $\frac{1}{3}(2v + n + 1) - 6 < u' \leqslant \frac{1}{3}(2v + n + 1)$. Thus $\frac{1}{2}(3u' - n - 1) \leqslant v$ because $u' \leqslant \frac{1}{3}(2v + n + 1)$. Also, $u' \geqslant 3n + 5$ because $u' > \frac{1}{3}(2v + n + 1) - 6$ and $v \geqslant 4n + 43$ imply that $u' > 3n + 23$. Finally $v \leqslant 2u' - 2n - 3$ because

$$2v < 3u' - n + 17 < 4u' - 4n - 6$$

where the first inequality holds because $\frac{1}{3}(2v + n + 1) - 6 < u'$ and the second holds because we have just seen that $u' > 3n + 23$. So, again, u', v and G satisfy the hypotheses of Lemma 4.2.7. □

In Chapter 7 we pose a natural question that is not answered by Theorem 4.1.1, see Question 7.0.1

4.3 Counterexamples to Conjecture 4.1.2

In this section we prove Theorem 4.1.3 by exhibiting, for each even $w \geqslant 4$, a Steiner triple system whose leave is a counterexample to Conjecture 4.1.2. We introduce some more notation. The maximum degree and minimum degree of a graph G are denoted by $\Delta(G)$ and $\delta(G)$ respectively. The smallest number of colours required to edge colour a graph G is the *chromatic index* of G, denoted $\chi'(G)$. Vizing's theorem [87] states that $\chi'(G) \in \{\Delta(G), \Delta(G) + 1\}$ for any graph G and König's theorem [65] states that $\chi'(G) = \Delta(G)$ for any bipartite graph G. A *matching* is a 1-regular graph. Note that the edges assigned a particular colour by an edge colouring always induce a matching. In an edge-colouring of a graph we say that a colour c *hits* a vertex u if there is an edge of colour c incident with u. Otherwise we say c *misses* u.

Our first lemma in this section encapsulates our strategy for finding graphs that form counterexamples to Conjecture 4.1.2.

Lemma 4.3.1. *An even graph L of odd order u is a counterexample to Conjecture 4.1.2 for a given even integer w if it satisfies*

(i) $|E(L)| = \frac{1}{2}w(u-w+1)$;

(ii) $\chi'(L) = w$;

(iii) *there are two vertices d_1 and d_2 of L such that, in any w-edge colouring of L, the set of colours that hit d_1 equals the set of colours that hit d_2.*

Proof. Let L be an even graph of odd order u that satisfies (i), (ii) and (iii) for a given w. We first prove that L satisfies the conditions in Conjecture 4.1.2. Obviously (1) and (2) of Conjecture 4.1.2 hold because L is an even graph, w is even and u is odd. Also, (3) of Conjecture 4.1.2 holds because $|E(L)| + uw + \binom{w}{2} = \frac{3}{2}uw \equiv 0 \pmod 3$ since $|E(L)| = \frac{1}{2}w(u-w+1)$. Moreover $|E(L)| = \frac{1}{2}w(u-w+1)$ implies $w^2-(u+1)w+2|E(L)| = 0$ and so (4) of Conjecture 4.1.2 holds with $G = L$, noting that $\chi'(L) = w$. Hence L satisfies all the conditions in Conjecture 4.1.2.

Now let $W = \{1, \ldots, w\}$ be a set disjoint from $V(L)$ and suppose for a contradiction that $L \vee K_W$ has a K_3-decomposition $\mathcal{D}$. Call the edges of $L \vee K_W$ with one endpoint in $V(L)$ and one endpoint in W *cross edges* and call the other edges *pure edges*. For $i \in \{0, 1, 2, 3\}$, call triangles in $\mathcal{D}$ that contain exactly i vertices in $V(L)$ *type-i triangles*. Now $L \vee K_W$ has uw cross edges and $|E(L)| + \binom{w}{2} = \frac{1}{2}uw$ pure edges. Thus, because each triangle in $\mathcal{D}$ contains at most two cross edges and at least one pure edge, $\mathcal{D}$ must consist of $|E(L)|$ type-2 triangles and $\binom{w}{2}$ type-1 triangles.

The $|E(L)|$ type-2 triangles in $\mathcal{D}$ induce a proper edge colouring γ of L with the colour set W defined by $\gamma(xy) = z$ for each $xy \in E(L)$, where z is the unique element of W such that $\{x, y, z\}$ is in $\mathcal{D}$. By (iii), in γ, the set of colours that hit d_1 equals the set of colours that hit d_2. Without loss of generality assume the set of colours that hit d_1 and d_2 is $\{3, 4, \ldots w\}$ and so colours 1 and 2 miss d_1 and d_2. Thus the only edges incident with d_1 and d_2 that do not occur in type-2 triangles in $\mathcal{D}$ are $\{d_1, 1\}$, $\{d_1, 2\}$, $\{d_2, 1\}$, $\{d_2, 2\}$. So these must occur in type-1 triangles in $\mathcal{D}$. However, this implies the contradiction that both the triangles $\{1, 2, d_1\}$ and $\{1, 2, d_2\}$ occur in $\mathcal{D}$. Therefore L is indeed a counterexample to Conjecture 4.1.2 for the given value of w. $\square$

We first exhibit a PSTS(15) whose leave forms a counterexample to Conjecture 4.1.2 for $w = 4$.

Example 4.3.2. Let $U = \{1, 2, \ldots, 15\}$ and let $\mathcal{A}$ be the set consisting of the following 27 triples.

$\{1,2,7\}$	$\{1,3,12\}$	$\{1,4,11\}$	$\{1,8,15\}$	$\{1,9,10\}$	$\{1,13,14\}$
$\{2,5,10\}$	$\{2,6,13\}$	$\{2,8,11\}$	$\{2,9,14\}$	$\{2,12,15\}$	$\{3,7,8\}$
$\{3,9,15\}$	$\{3,10,14\}$	$\{3,11,13\}$	$\{4,7,15\}$	$\{4,8,14\}$	$\{4,9,13\}$
$\{4,10,12\}$	$\{5,7,13\}$	$\{5,8,12\}$	$\{5,9,11\}$	$\{5,14,15\}$	$\{6,7,10\}$
$\{6,8,9\}$	$\{6,11,15\}$	$\{6,12,14\}$			

Then (U, A) is a PSTS(15) and the leave L of (U, A) has two components as shown in Figure 4.1.

We note that $|E(L)| = 24$ and that $\chi'(L) = 4$ because $\Delta(L) = 4$ and a 4-edge colouring of L is given by the different line styles in Figure 4.1. Further, in any 4-edge colouring of L, it is not difficult to see that the set of colours that hit vertex 1 equals the set of colours that hit vertex 2 (for a formal proof of this see Lemma 4.3.3). Thus L satisfies the conditions of Lemma 4.3.1 for $w = 4$ and so is a counterexample to Conjecture 4.1.2 for $w = 4$.

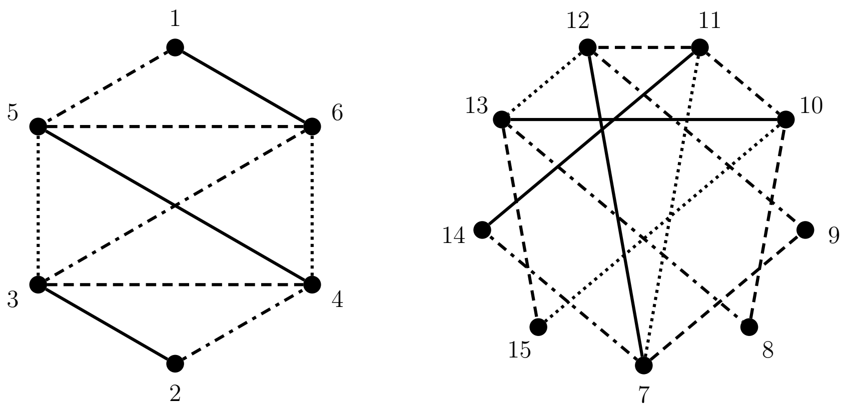

Figure 4.1: Leave of the PSTS(15) given in Example 4.3.2

We now generalise this small example to find counterexamples to Conjecture 4.1.2 (of much larger unspecified order) for all even $w \geqslant 6$. Our next lemma details how we generalise the component with six vertices in the leave L in Example 4.3.2. For an integer $n \geqslant 2$, let $\mathbb{Z}_n$ denote the additive group of integers modulo n.

Lemma 4.3.3. *Let $w \geqslant 4$ be an even integer and let L_1 be the complement of the graph with vertex set $\mathbb{Z}_{w+1} \cup \{\infty\}$ shown in Figure 4.2.*
Then

(i) *$\chi'(L_1) = w$; and*

(ii) *in any w-edge colouring of L_1, the set of colours that hit vertex 1 equals the set of colours that hit vertex 2.*

Proof. A proper w-edge colouring γ of L_1 with colour set $\mathbb{Z}_{w+1} \setminus \{0\}$ is given by

- $\gamma(xy) = x + y$ for each $xy \in E(L_1)$ with $x, y \in \mathbb{Z}_{w+1}$;
- $\gamma(x\infty) = 2x$ for each $x \in \{2, 3, \dots, w\}$;
- $\gamma(0\infty) = 2$.

Thus, since $\Delta(L_1) = w$, we have $\chi'(L_1) = w$ and (i) holds.

To prove (ii), consider an arbitrary w-edge colouring of L_1. Since vertices 1 and 2 have degree $w - 2$ and every other vertex has degree w, there are exactly two colours that miss vertex 1, exactly two colours that miss vertex 2, and each other vertex is hit by every colour. Since any colour that misses a vertex misses at least two vertices, it follows immediately that the two colours that miss vertex 1 are the same as the two colours that miss vertex 2. So (ii) holds. □

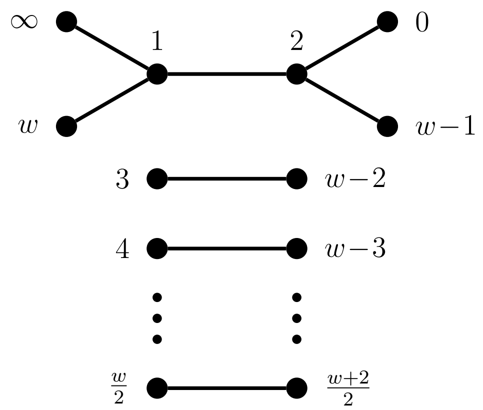

Figure 4.2: The complement of the graph L_1 in Lemma 4.3.3

We also require the following simple consequence of a theorem obtained by Dross [39] using a result of Barber et al. [6].

Lemma 4.3.4. *Let w be an even positive integer. There exists an integer u_0 such that for any even graph L with odd order $u \geqslant u_0$, $|E(L)| \equiv \binom{u}{2} \pmod 3$ and $\Delta(L) \leqslant w$, there is a partial Steiner triple system whose leave is L.*

Proof. Theorem 7 of [39] implies that there exists an integer n_0 such that any even graph G with $n \geqslant n_0$ vertices with $|E(G)| \equiv 0 \pmod 3$ and $\delta(G) \geqslant \frac{91}{100}n$ is K_3-decomposable. Take $u_0 = \max(n_0, \lceil \frac{100}{9}(w+1) \rceil)$ and suppose that L is an even graph with odd order $u \geqslant u_0$, $|E(L)| \equiv \binom{u}{2} \pmod 3$ and $\Delta(L) \leqslant w$. Let $\overline{L}$ be the complement of L. It suffices to show that there is a K_3-decomposition of $\overline{L}$.

Now, $\overline{L}$ is an even graph because L is an even graph of odd order, and $u \geqslant u_0 \geqslant n_0$. Furthermore $\delta(\overline{L}) \geqslant u - w - 1 \geqslant \frac{91}{100}u$ because $\Delta(L) \leqslant w$ and $u \geqslant u_0 \geqslant \frac{100}{9}(w+1)$. Finally, $|E(\overline{L})| \equiv \binom{u}{2} - |E(L)| \equiv 0 \pmod 3$. Thus we can apply [39, Theorem 7] to obtain a K_3-decomposition of $\overline{L}$. $\square$

Proof of Theorem 4.1.3. A partial Steiner triple system whose leave is a counterexample to Conjecture 4.1.2 for $w = 4$ was exhibited in Example 4.3.2. Let $w \geqslant 6$ be an even integer. We will show that there exists a partial Steiner triple system whose leave is a counterexample to Conjecture 4.1.2 for this value of w.

By Lemma 4.3.4 there exists an integer u_0 such that, for any even graph L with odd order $u \geqslant u_0$, $|E(L)| \equiv \binom{u}{2} \pmod 3$ and $\Delta(L) \leqslant w$, there is a partial Steiner triple system whose leave is L. Fix an odd $u \geqslant \max(u_0, 4w+1)$ such that $u+w \equiv 1, 3 \pmod 6$. We will find an even graph L of order u such that $|E(L)| \equiv \binom{u}{2} \pmod 3$ and L satisfies conditions (i), (ii) and (iii) of Lemma 4.3.1. This will suffice to complete the proof because L will have maximum degree at most w by (ii), and so by Lemma 4.3.4 there will be a partial Steiner triple system $(U, \mathcal{A})$ with leave L. We will take L to be a vertex-disjoint union of three graphs, L_1, L_2 and L_3, that we now define.

First we let L_1 be the graph of order $w+2$ given by Lemma 4.3.3. Note that $|E(L_1)| = \binom{w+2}{2} - \frac{w+6}{2} = \frac{1}{2}w^2 + w - 2$. Next let $t = \frac{1}{2}(u - 2w - 1)$, note that $t \geqslant w$ because $u \geqslant 4w+1$, and let L_2 be the bipartite graph with parts $\{a_0, \dots, a_{t-1}\}$ and $\{b_0, \dots, b_{t-1}\}$ and edge set

$$\big\{a_i b_j : i \in \{0, \dots, t-1\}, j \in \{i, \dots, i+w-1\}\big\} \setminus \{a_0b_1, a_0b_2, a_1b_1, a_1b_2\}$$

where the subscripts are considered modulo t. So L_2 is a graph obtained from a w-regular bipartite graph of order $2t$ by removing the edges of a 4-cycle. Hence $|E(L_2)| =$

$\frac{w}{2}(u-2w-1)-4$. Let L_3 be the graph with vertex set $\{c_1,\ldots,c_{w-1}\}$ and edge set $\{c_1c_2, c_1c_5, c_2c_5, c_3c_4, c_3c_5, c_4c_5\}$ (note $w \geqslant 6$). So L_3 is the union of $w-6$ isolated vertices and two copies of K_3 that share exactly one vertex, and hence $|E(L_3)| = 6$.

It only remains to show that L has the properties we desired. Clearly, L is an even graph of order u. Now $|E(L)| = \frac{1}{2}w(u-w+1)$ because $|E(L_1)| = \frac{1}{2}w^2+w-2$, $|E(L_2)| = \frac{1}{2}w(u-2w-1)-4$ and $|E(L_3)| = 6$. So L satisfies (i) of Lemma 4.3.1. Furthermore, $|E(L)| \equiv \binom{u}{2} \pmod 3$ because $\binom{u}{2} - |E(L)| = \frac{1}{2}((u+w)(u+w-1)-3uw)$ and $u+w \equiv 1,3 \pmod 6$. Also, L satisfies (ii) of Lemma 4.3.1 because $\chi'(L_1) = w$ by Lemma 4.3.3(i), $\chi'(L_2) = w$ since L_2 is bipartite and $\Delta(L_2) = w$, and $\chi'(L_3) \leqslant w$ since $\Delta(L_3) = 4$. Finally, L satisfies (iii) of Lemma 4.3.1 by Lemma 4.3.3(ii). $\square$

We discuss an open problem related to Theorem 4.1.3 in Chapter 7, (see Question 7.0.2).

Chapter 5

Embedding partial k-star designs

> *“ Because of the highly complex natures or structures of many beautiful objects, there will have to be a role for reason in their perception. But perceiving the nature or structure of an object is one thing. Perceiving its beauty is another.”*
>
> - The Stanford Encyclopedia of Philosophy, *The Concept of the Aesthetic*

5.1 Introduction

Remember that a *k-star decomposition* of a graph G is a collection of copies of $K_{1,k}$ in G such that each edge of G is in exactly one copy. If we weaken this condition to demand that each edge of G is in at most one copy, then the resulting object is a *partial k-star decomposition*. An *embedding* of a partial k-star decomposition $\mathcal{A}$ of a graph G is a k-star decomposition $\mathcal{B}$ of another graph H such that $\mathcal{A} \subseteq \mathcal{B}$ and G is a subgraph of H. The *leave* of a partial k-star decomposition of G is the graph L having vertex set $V(G)$ and edge set comprising all edges of G that are not in a k-star in the decomposition.

The problem of determining when a graph has a decomposition into k-stars has been thoroughly investigated. An obvious necessary condition for a graph to have a k-star decomposition is that its number of edges is divisible by k. Trivially, any graph has a decomposition into 1-stars. A simple inductive argument shows that any connected graph with an even number of edges has a 2-star decomposition (see [19, Theorem 1]). Tarsi [81] and Yamamoto et al. [92] independently proved that, for $n \geqslant 2$, a k-star decomposition of K_n exists if and only if $n \geqslant 2k$ and $\binom{n}{2} \equiv 0 \pmod{k}$. In fact, Tarsi gave necessary and sufficient conditions for the existence of a decomposition of a complete multigraph into k-stars while Yamamoto et al. also proved an analogous statement for complete bipartite graphs.

A result of Dor and Tarsi [37] implies that determining whether an arbitrary graph G has a k-star decomposition is NP-complete whenever $k \geqslant 3$. A result of Tarsi [82] gives a characterisation of when an arbitrary graph G has a k-star decomposition in which the number of k-stars that are centred on each vertex is specified. Other results in [82] imply various sufficient conditions for a graph to have a decomposition into k-stars. Hoffman and Roberts [59] exactly determined the maximum possible number of k-stars in a partial k-star decomposition of K_n and moreover characterised the possible leaves.

This chapter is concerned with the problem of when a partial k-star decomposition of K_n can be embedded in a k-star decomposition of K_{n+s}. In 2012, Hoffman and Roberts [58] proved that a partial k-star decomposition of K_n can be embedded in a k-star decomposition of K_{n+s} for some positive integer s such that $s \leqslant 7k-4$ when k is odd and $s \leqslant 8k-4$ when k is even. Furthermore, they conjectured that the smallest possible upper bound on s is around $2k$. In 2019, Noble and Richardson [74] improved the bounds on s to $s \leqslant 3k-2$ when k is odd and $s \leqslant 4k-2$ when k is even. As our first main result of the chapter we further improve these bounds.

Theorem 5.1.1. *Let $k \geqslant 2$ and $n \geqslant 1$ be integers. Any partial k-star decomposition of K_n can be embedded in a k-star decomposition of K_{n+s} for some s such that $s < \frac{9}{4}k$ when k is odd and $s < (6-2\sqrt{2})k$ when k is even.*

If either of the constants $\frac{9}{4}$ or $6-2\sqrt{2} \approx 3.17$ in the above result were decreased then the result would fail to hold for infinitely many k (see Lemmas 5.5.2 and 5.5.5). Our next main result shows, however, that these constants can be improved if we impose a lower bound on n.

Theorem 5.1.2. *Let $k \geqslant 2$ and $n > \frac{k(k-1)}{\sqrt{8k}-1}$ be integers. Any partial k-star decomposition of K_n can be embedded in a k-star decomposition of K_{n+s} for some s such that $s \leqslant 2k-2$ when k is odd and $s \leqslant 3k-2$ when k is even.*

Neither of the upper bounds on s in this result can be decreased, no matter what lower bound we place on n (see Lemmas 5.2.6(c) and 5.4.4(b)). We prove Theorem 5.1.2 as a consequence of the following result which shows that, when $s \geqslant k$ and n is large enough, the obvious necessary condition is also sufficient for the existence of an embedding of a partial k-star decomposition of K_n in a k-star decomposition of K_{n+s}.

Theorem 5.1.3. *Let $k \geqslant 2$ and $n > \frac{k(k-1)}{\sqrt{8k}-1}$ be integers. Any nonempty partial k-star decomposition of K_n can be embedded in a k-star decomposition of K_{n+s} for each $s \geqslant k$ such that $\binom{n+s}{2} \equiv 0 \pmod{k}$.*

The lower bound on s in this result cannot be decreased no matter what lower bound we place on n (see Lemma 5.2.6(b)). Moreover, the lower bound on n is asymptotically best possible as k becomes large (see Lemma 5.4.3).

5.2 Central functions and other preliminaries

We recall some more notation that we use throughout the chapter. Let G be a graph. Let $E(G)$, $V(G)$ and $\overline{G}$ denote the edge set, vertex set and complement of G respectively. For any $x \in V(G)$, $\deg_G(x)$ denotes the degree of x in G. The *neighbourhood* $N_G(x)$ of a vertex $x \in V(G)$ is the set of all vertices which are adjacent to x in G. For a subset U of $V(G)$ we use $G[U]$ to denote the subgraph of G induced by U.

For a set S of vertices we use K_S to denote the complete graph with vertex set S, and for disjoint sets S and T of vertices we use $K_{S,T}$ to denote the complete bipartite graph with parts S and T. For vertex-disjoint graphs G and H we use $G \vee H$ to denote the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup E(K_{V(G),V(H)})$. Our use of the notation $K_{S,T}$ will imply that S and T are disjoint and our use of the notation $G \vee H$ will imply that G and H are vertex-disjoint. As a special case, we take $G \vee K_\emptyset$ or $G \vee K_0$ to be simply the graph G. We can embed a partial k-star decomposition $\mathcal{D}$ of K_n

in a k-star decomposition of K_{n+s} for some nonnegative integer s if and only if there is a k-star decomposition of $L \vee K_s$, where L is the leave of $\mathcal{D}$.

We begin by emphasising the necessary and sufficient conditions for the existence of a k-star decomposition of K_n that we mentioned in the introduction and highlighting their effects in the special case where k is a prime power.

Theorem 5.2.1. [81, 92] *Let $k \geqslant 2$ and $n \geqslant 2$ be positive integers.*

(a) *A k-star decomposition of K_n exists if and only if $n \geqslant 2k$ and $\binom{n}{2} \equiv 0 \pmod{k}$.*

(b) *If k is a power of 2 then a k-star decomposition of K_n exists if and only if $n \geqslant 2k$ and $n \equiv 0 \pmod{2k}$ or $n \equiv 1 \pmod{2k}$.*

(c) *If k is a power of an odd prime then a k-star decomposition of K_n exists if and only if $n \geqslant 2k$ and $n \equiv 0 \pmod{k}$ or $n \equiv 1 \pmod{k}$.*

Parts (b) and (c) of Theorem 5.2.1 follow immediately from part (a) because $\binom{n}{2} \equiv 0 \pmod{k}$ is equivalent to $n \equiv 0 \pmod{2k}$ or $n \equiv 1 \pmod{2k}$ when k is a power of 2 and is equivalent to $n \equiv 0 \pmod{k}$ or $n \equiv 1 \pmod{k}$ when k is a power of an odd prime. We often exploit this limitation of the possible values of n when k is a prime power in our constructions of partial k-star decompositions without small embeddings.

As mentioned in the introduction, a simple inductive argument shows that any connected graph with an even number of edges has a 2-star decomposition (see [19, Theorem 1]). This immediately implies the following characterisation of when a graph $L \vee K_s$ has a 2-star decomposition.

Lemma 5.2.2. *Let L be a graph. There is a 2-star decomposition of $L \vee K_s$ if and only if*

- *$s = 0$ and each connected component of L has an even number of edges; or*
- *$s \geqslant 1$ and $|E(L \vee K_s)| \equiv 0 \pmod{2}$.*

Let $k \geqslant 2$ be an integer. In a k-star, the vertex of degree k is called the *centre*. For a given k-star decomposition $\mathcal{D}$ of G, we can define a function $\gamma : V(G) \to \mathbb{Z}^{\geqslant 0}$ called the *central function*, where $\gamma(x)$ is the number of k-stars of $\mathcal{D}$ whose centre is x for each $x \in V(G)$. It will be helpful to bear in mind the three following properties that must hold for any central function γ of a k-star decomposition of a graph G.

- $k \sum_{x \in V(G)} \gamma(x) = |E(G)|$.
- For each edge x_1x_2 of G, $\gamma(x_1) + \gamma(x_2) \geqslant 1$.
- For each vertex x of G, $k\gamma(x) \leqslant \deg_G(x)$ and if $k\gamma(x) = \deg_G(x)$ then each edge of G incident with x is in a k-star of $\mathcal{D}$ centred at x.

We call a function $\gamma : V(G) \to \mathbb{Z}^{\geqslant 0}$ such that $k \sum_{x \in V(G)} \gamma(x) = |E(G)|$ a *k-precentral function* for G. Crucial to our approach in this chapter is Lemma 5.2.3 below, which characterises when a k-star decomposition of a graph G with a specified central function exists. Lemma 5.2.3 is a simple consequence of a result of Tarsi [82, Theorem 2]. Because we will use Lemma 5.2.3 so extensively, we first introduce some notation that simplifies its statement and use.

Let $\mathcal{G}$ be a graph G equipped with a k-precentral function γ (note that G and γ determine the value of k). We call a k-star decomposition of G in which there are $\gamma(x)$ stars centred at x for each $x \in V(G)$ a *star $\mathcal{G}$-decomposition*. The notation we now define is implicitly dependent on $\mathcal{G}$, which will always be obvious from context. For any subset T of $V(G)$, let $\Delta_T = \Delta_T^+ - \Delta_T^-$ where $\Delta_T^- = k\sum_{x\in T}\gamma(x)$, $\Delta_T^+ = |E_T|$, and E_T is the set of edges of G that are incident to at least one vertex in T. Let Δ be the minimum of Δ_T over all subsets T of $V(G)$ and note that taking $T = \emptyset$ implies that $\Delta \leqslant 0$. Let $\mathcal{T}$ be the collection of subsets T of $V(G)$ for which $\Delta_T = \Delta$ and which, subject to this, have minimum cardinality.

Lemma 5.2.3. *Let $k \geqslant 2$ be an integer and let $\mathcal{G}$ be a graph G equipped with a k-precentral function γ.*

(i) *There exists a star $\mathcal{G}$-decomposition if and only if $\Delta = 0$.*

(ii) *For each $T \in \mathcal{T}$, $T \subseteq \{x \in V(G) : \gamma(x) \geqslant 1\}$.*

Proof. We first prove (i). It is clear that a star $\mathcal{G}$-decomposition exists if and only if there is an orientation of the edges of G such that exactly $k\gamma(x)$ edges are oriented out from x for each $x \in V(G)$. Remember that $k\sum_{x\in V(G)}\gamma(x) = |E(G)|$ because γ is a k-precentral function. Thus, by [82, Theorem 2] such an orientation exists if and only if $k\sum_{x\in S}\gamma(x) \geqslant |E(G[S])|$ for each subset S of $V(G)$. For a given subset S of $V(G)$, $k\sum_{x\in S}\gamma(x) = |E(G)| - \Delta_T^-$ and $E(G[S]) = E(G) \setminus E_T$, where $T = V(G) \setminus S$. Thus, such an orientation exists if and only if

$$\Delta_T \geqslant 0 \qquad \text{for each subset } T \text{ of } V(G). \tag{5.1}$$

Because $\Delta_\emptyset = 0$ and hence $\Delta \leqslant 0$, (5.1) is equivalent to $\Delta = 0$.

We now prove (ii). Let $T \in \mathcal{T}$ and suppose for a contradiction that $\gamma(x) = 0$ for some $x \in T$. We have that $\Delta_{T\setminus\{x\}} \leqslant \Delta_T$ because $\Delta_{T\setminus\{x\}}^- = \Delta_T^-$ and $\Delta_{T\setminus\{x\}}^+ \leqslant \Delta_T^+$ since $E_{T\setminus\{x\}} \subseteq E_T$. So, because $|T \setminus \{x\}| < |T|$, we have a contradiction to the definition of $\mathcal{T}$. □

Lemma 5.2.3 can also be obtained by specialising results in [57] or [17] concerning star decompositions of multigraphs. Through our notation Δ_T^+ and Δ_T^-, the condition of Lemma 5.2.3(i) is stated in the complement when compared to [82, Theorem 2], but this makes it consistent with the statements in [17, 57], which generalise more naturally to star packings of graphs.

We call a set U of vertices of a graph G *pairwise twin*, if $N_G(x) \setminus \{y\} = N_G(y) \setminus \{x\}$ for all $x, y \in U$. The next lemma aids us when applying Lemma 5.2.3 to graphs containing sets of pairwise twin vertices. Note that in a graph $G = L \vee K_S$, the vertices in S are pairwise twin and so we can apply the lemma with U chosen to be S.

Lemma 5.2.4. *Let $k \geqslant 2$ be an integer, let G be a graph and let U be a pairwise twin subset of $V(G)$. Let $\mathcal{G}$ be the graph G equipped with some k-precentral function γ and let $T \in \mathcal{T}$. For any $x_1 \in U \setminus T$ and $x_2 \in T \cap U$ we have $\gamma(x_1) < \gamma(x_2)$. In particular, if $\gamma(x) = \gamma(x')$ for all $x, x' \in U$ then, for each $T \in \mathcal{T}$, either $U \subseteq T$ or $T \cap U = \emptyset$.*

Proof. Suppose that $T \in \mathcal{T}$, $x_1 \in U \setminus T$ and $x_2 \in U \cap T$. Let $A = N_G(x_1) \setminus T$, and note that $A = N_G(x_2) \setminus (T \cup \{x_1\})$ because x_1 and x_2 are twin. Let $a = |A|$, $T_1 = T \cup \{x_1\}$ and $T_2 = T \setminus \{x_2\}$. Because $T \in \mathcal{T}$ and $|T_2| < |T|$ we have $\Delta_{T_1} \geqslant \Delta_T$ and $\Delta_{T_2} > \Delta_T$.

Observe that $\Delta^-_{T_1} = \Delta^-_T + k\gamma(x_1)$ and $\Delta^+_{T_1} = \Delta^+_T + a$ since $E_{T_1} = E_T \cup \{x_1 z : z \in A\}$. Therefore, $\Delta_{T_1} = \Delta_T + a - k\gamma(x_1)$ and so, because $\Delta_{T_1} \geqslant \Delta_T$, $k\gamma(x_1) \leqslant a$. Now, $\Delta^-_{T_2} = \Delta^-_T - k\gamma(x_2)$ and $\Delta^+_{T_2} \leqslant \Delta^+_T - a$ since $E_{T_2} = E_T \setminus (\{x_2 z : z \in A\} \cup X)$, where $X = \{x_1\}$ if $x_1x_2 \in E(G)$ and $X = \emptyset$ if $x_1x_2 \notin E(G)$. Therefore, $\Delta_{T_2} \leqslant \Delta_T - a + k\gamma(x_2)$ and so, because $\Delta_{T_2} > \Delta_T$, $a < k\gamma(x_2)$. Combining $k\gamma(x_1) \leqslant a$ and $a < k\gamma(x_2)$, we see we must have $\gamma(x_1) < \gamma(x_2)$.

Now suppose $\gamma(x) = \gamma(x')$ for all $x, x' \in U$. By what we have just proved, either $U \setminus T = \emptyset$ and hence $U \subseteq T$, or $T \cap U = \emptyset$. □

Many of the results in this chapter (including Theorem 5.1.3) effectively concern k-star decompositions of $L \vee K_s$ for some specified graph L and integer $s \geqslant k$. Lemma 5.2.6 below illustrates why we usually impose the condition that s be at least k in these results. First we state a special case of a result of Tarsi [82, Theorem 4] that we will often use to show that a certain graph is the leave of a partial k-star decomposition.

Theorem 5.2.5 ([82]). *Let G be a graph of order n such that $\deg_G(x) \geqslant \frac{1}{2}n + k - 1$ for each $x \in V(G)$. Then G has a k-star decomposition if $|E(G)| \equiv 0 \pmod{k}$.*

Lemma 5.2.6. *Let $k \geqslant 2$ and $n \geqslant 2$ be integers such that k is odd and $n \equiv 2 \pmod{2k}$. Let L be a graph of order n that has exactly one edge.*

(a) *There is a partial k-star decomposition of K_n whose leave is L.*

(b) *There is no k-star decomposition $L \vee K_{k-1}$, even though $|E(L \vee K_{k-1})| \equiv 0 \pmod{k}$.*

(c) *If k is a power of an odd prime, there is no k-star decomposition $L \vee K_s$ for any $s < 2k - 2$.*

Proof. We first prove (a) by showing that a k-star decomposition of $\overline{L}$ exists. This is trivial if $n = 2$. If $n \geqslant 2k + 2$, then $\deg_{\overline{L}}(y) \geqslant n - 2 \geqslant \frac{1}{2}n + k - 1$ for each $y \in V(L)$ and $|E(\overline{L})| = \binom{n}{2} - 1 \equiv 0 \pmod{k}$ since $n \equiv 2 \pmod{2k}$. Therefore, by Theorem 5.2.5, a k-star decomposition of $\overline{L}$ exists.

We now prove (b). Note that $|E(L \vee K_{k-1})| = 1 + n(k-1) + \binom{k-1}{2} \equiv 0 \pmod{k}$ because $n \equiv 2 \pmod{2k}$ and k is odd. Let r be the nonnegative integer such that $n = 2kr + 2$. Suppose for a contradiction that there is a k-star decomposition $\mathcal{D}$ of $L \vee K_S$, where $|S| = k - 1$, and let γ be the central function of $\mathcal{D}$. Now $|E(L \vee K_S)| = 1 + n(k-1) + \binom{k-1}{2}$ and so $\sum_{x \in V(L) \cup S} \gamma(x) = (2r + \frac{1}{2})(k-1) + 1$. Observe that $\deg_{L \vee K_S}(y_1) = \deg_{L \vee K_S}(y_2) = k$, where y_1y_2 is the only edge in L, and $\deg_{L \vee K_S}(y) = k - 1$ for each $y \in V(L) \setminus \{y_1, y_2\}$. So, without loss of generality, $\gamma(y_1) = 1$, every edge of $L \vee K_S$ incident with y_1 is in the star in $\mathcal{D}$ centred at y_1, and $\gamma(y) = 0$ for each $y \in V(L) \setminus \{y_1\}$. Thus $\sum_{z \in S} \gamma(z) = (2r + \frac{1}{2})(k-1)$. By the pigeonhole principle, it follows that $\gamma(z_1) \geqslant 2r + 1$ for some $z_1 \in S$ because $|S| = k - 1$. Now $\deg_{L \vee K_S}(z_1) = n + k - 2 = k(2r + 1)$ noting that $n = 2kr + 2$. So every edge incident with z_1 is in a star in $\mathcal{D}$ centred at z_1. But this contradicts the fact that the edge y_1z_1 is in the star in $\mathcal{D}$ centred at y_1.

We now prove (c). Suppose that k is a power of an odd prime. Assume for a contradiction that $\mathcal{D}$ is a k-star decomposition of $L \vee K_S$ where $|S| = s$ for some nonnegative integer $s < 2k - 2$. By Theorem 5.2.1(c), we have that $n + s \equiv 0 \pmod{k}$ or $n + s \equiv 1 \pmod{k}$ and hence, because $n \equiv 2 \pmod{2k}$, that $s \equiv k - 2 \pmod{k}$ or $s \equiv k - 1 \pmod{k}$. So $s \in \{k - 2, k - 1\}$ because $s < 2k - 2$. So then $s = k - 1$ because a k-star in $\mathcal{D}$ must be centred at an end vertex of the edge in L and these vertices have degree $s + 1$ in $L \vee K_S$. However, a k-star decomposition of $L \vee K_{k-1}$ does not exist by (b). □

5.3 Embedding maximal partial k-star decompositions

Recall that a partial k-star decomposition of a graph G is *maximal* if there is no star that can be added to it to produce a partial k-star decomposition of G containing more stars. Thus, a partial k-star decomposition of a graph G is maximal if and only if its leave has maximum degree at most $k-1$. In this section we prove results about embedding maximal partial k-star decompositions of K_n in k-star decompositions of K_{n+s} where $s \geqslant k$. These results will be crucial in proving the main theorems.

An *independent set* in a graph is a set of its vertices that are pairwise non-adjacent. The *independence number* $\alpha(G)$ of a graph G is the maximum cardinality of an independent set in G. In [20, Corollary 2], Caro and Roditty note that if a graph G has a decomposition into k-stars then $\alpha(G) \geqslant |V(G)| - \frac{1}{k}|E(G)|$. This can be seen by observing that any edge in G must have a star of the decomposition centred on at least one of its end-vertices. For the cases we are interested in, we formalise this observation in the following lemma.

Lemma 5.3.1. *Let $k \geqslant 2$, $n \geqslant 1$ and $s \geqslant 0$ be integers, and let L be a graph of order n. If there is k-star decomposition of $L \vee K_s$, then $\alpha(L) \geqslant n + s - \frac{1}{k}|E(L \vee K_s)|$.*

Proof. If there is a k-star decomposition of $L \vee K_s$, then $\alpha(L \vee K_s) \geqslant n+s-\frac{1}{k}|E(L \vee K_s)|$ by [20, Corollary 2]. Furthermore, it is easy to see that $\alpha(L \vee K_s) = \alpha(L)$. □

In this section we show that, for a maximal partial k-star decomposition $\mathcal{D}$ of K_n and an integer $s \geqslant k$ such that $\binom{n+s}{2} \equiv 0 \pmod{k}$, the obstacle described by Lemma 5.3.1 is the only thing that can prevent the existence of an embedding of $\mathcal{D}$ in a k-star decomposition of K_{n+s}. We do this in two lemmas: Lemma 5.3.2 deals with the case where the number of stars to be added is small and the obstacle may arise whereas Lemma 5.3.3 deals with the case where the number of stars to be added is large and the obstacle cannot arise.

Lemma 5.3.2. *Let k, n and s be integers with $s \geqslant k \geqslant 2$, and let L be a graph of order n with maximum degree at most $k-1$ and $|E(L \vee K_s)| \leqslant k(n+s)$. Then there is a k-star decomposition of $L \vee K_s$ if and only if $|E(L \vee K_s)| \equiv 0 \pmod{k}$ and $\alpha(L) \geqslant n + s - \frac{1}{k}|E(L \vee K_s)|$.*

Proof. The 'only if' direction follows from Lemma 5.3.1, so we only need to prove the 'if' direction.

Suppose that $|E(L \vee K_S)| \equiv 0 \pmod{k}$, where S is a set with $|S| = s$. Let $V = V(L \vee K_S)$ and $b = \frac{1}{k}|E(L \vee K_S)|$, and suppose that L has an independent set A containing $n+s-b$ vertices. Note that $n+s-b \geqslant 0$ because $|E(L \vee K_S)| \leqslant k(n+s)$ by our hypotheses. Define a k-precentral function γ for $L \vee K_S$ by $\gamma(x) = 0$ for each $x \in A$ and $\gamma(x) = 1$ for each $x \in V \setminus A$. This is indeed a k-precentral function for $L \vee K_S$ because $\sum_{x \in V} \gamma(x) = n + s - |A| = b$. Let $\mathcal{G}$ be the graph $L \vee K_S$ equipped with γ. We complete the proof by showing that $\Delta = 0$ and hence a star $\mathcal{G}$-decomposition exists by Lemma 5.2.3. Let $T \in \mathcal{T}$ and suppose for a contradiction that $\Delta_T < 0$. Since $\gamma(z) = 1$ for all $z \in S$, we can apply Lemma 5.2.4 with $U = S$ to conclude that either $T \cap S = \emptyset$ or $S \subseteq T$. We consider these cases separately, with the latter splitting into two subcases.

Case 1: Suppose that $T \cap S = \emptyset$. This implies $T \subseteq V(L)$. Then $\Delta_T^+ \geqslant s|T|$, because $E(K_{S,T}) \subseteq E_T$ and $\Delta_T^- = k|T|$ by the definition of γ and Lemma 5.2.3(ii). Therefore, we have $\Delta_T^- \leqslant \Delta_T^+$ as $s \geqslant k$. This contradicts $\Delta_T < 0$.

Case 2a: Suppose that $S \subseteq T$ but $T \neq V \setminus A$. Then there is a vertex $y \in V(L) \setminus (A \cup T)$ and, by the definition of γ, $\gamma(y) = 1$. Let $T_1 = T \cup \{y\}$. Then $\Delta^+_{T_1} \leqslant \Delta^+_T + k - 1$, noting that $\deg_L(y) \leqslant k-1$ and $\Delta^-_{T_1} = \Delta^-_T + k$. Therefore, $\Delta_{T_1} \leqslant \Delta_T - 1$ contradicting $T \in \mathcal{T}$.

Case 2b: Suppose that $T = V \setminus A$. Then $\Delta^+_T = |E(L \vee K_S)|$ because $E_T = E(L \vee K_S)$ since A is independent. Moreover, $\Delta^-_T = |E(L \vee K_S)|$ because γ is a k-precentral function for $L \vee K_S$. So $\Delta^+_T = \Delta^-_T$ contradicting $\Delta_T < 0$. □

Note that the condition $n \geqslant k$ in the following lemma will certainly hold whenever L is the leave of a nontrivial k-star decomposition.

Lemma 5.3.3. *Let k, n and s be positive integers with $s \geqslant k \geqslant 2$ and $n \geqslant k$, and let L be a graph of order n with maximum degree at most $k-1$ and $|E(L \vee K_s)| \geqslant k(n+s)$. Then there is a k-star decomposition of $L \vee K_s$ if and only if $|E(L \vee K_s)| \equiv 0 \pmod{k}$.*

Proof. If $L \vee K_s$ has a k-star decomposition, then obviously $|E(L \vee K_s)| \equiv 0 \pmod{k}$. So it suffices to prove the 'if' direction.

Assume that $|E(L \vee K_S)| \equiv 0 \pmod{k}$, where S is a set with $|S| = s$, let $b = \frac{1}{k}|E(L \vee K_S)|$ and note $b \geqslant n+s$ by the hypotheses of the lemma. Thus, we can define a k-precentral function γ on $L \vee K_S$ such that $\gamma(y) = 1$ for each $y \in V(L)$ and $\gamma(z) \in \{d, d+1\}$ for each $z \in S$, where $d = \lfloor \frac{b-n}{s} \rfloor$. Note that $d \geqslant 1$ since $b \geqslant n+s$ and let $S_0 = \{z \in S : \gamma(z) = d\}$. We will show there is a star $\mathcal{G}$-decomposition where $\mathcal{G}$ is $L \vee K_S$ equipped with γ.

Let $T \in \mathcal{T}$, $H = L[V(L) \setminus T]$, $h = |V(H)|$, and $e = |E(H)|$. By Lemma 5.2.3, it suffices to show that $\Delta_T \geqslant 0$. By Lemma 5.2.4 with $U = S$, we have that $T \cap S \in \{\emptyset, S \setminus S_0, S\}$. We separate the proof into three cases accordingly.

Case 1: Suppose that $T \cap S = \emptyset$. Then $T = V(L) \setminus V(H)$. Noting that $E_T = E(K_{S,V(L)\setminus V(H)}) \cup (E(L) \setminus E(H))$ and $\Delta^-_T = k(n-h)$, we have

$$\Delta_T = \big((n-h)s + |E(L)| - e\big) - k(n-h) = (n-h)(s-k) + |E(L)| - e.$$

This last expression is nonnegative because $n \geqslant h$, $s \geqslant k$ and $|E(L)| \geqslant e$.

Case 2: Suppose that $T \cap S = S$. Noting that $E_T = E(L \vee K_S) \setminus E(H)$, that $|E(L \vee K_S)| = bk$, and that $\Delta^-_T = k(b-h)$, we see that

$$\Delta_T = (bk - e) - k(b-h) = kh - e.$$

This last expression is nonnegative because $e \leqslant \frac{1}{2}h(k-1)$ since H has maximum degree at most $k-1$.

Case 3: Suppose that $T \cap S = S \setminus S_0$. Let $s_0 = |S_0|$. Noting that

$$E_T = E(L \vee K_S) \setminus \big(E(K_{S_0}) \cup E(K_{S_0,V(H)}) \cup E(H)\big),$$

that $|E(L \vee K_S)| = bk$, and that $\Delta^-_T = k(b - ds_0 - h)$, we see that

$$\Delta_T = \left(bk - \binom{s_0}{2} - hs_0 - e\right) - k(b - ds_0 - h) = \tfrac{s_0}{2}(2dk + 1 - s_0) + h(k - s_0) - e. \quad (5.2)$$

The remainder of the proof is a somewhat tedious verification that this last expression is nonnegative. We first observe the following three useful facts.

(F1) $2e \leqslant h(k-1)$

(F2) $e \leqslant k(n + s(d+1) - s_0) - ns - \binom{s}{2}$

(F3) $d \leqslant \frac{1}{2ks}(n(2s-k-1)+s(s-1)+2ks_0)-1$

Note that (F1) holds because H is a subgraph of L and thus has maximum degree at most $k-1$. Also, (F2) holds because $e \leqslant |E(L)| = bk - ns - \binom{s}{2}$ and $b = n + s(d+1) - s_0$ from the definition of γ. Further, (F3) holds because $b = n + s(d+1) - s_0$, $b = \frac{1}{k}(|E(L)| + ns + \binom{s}{2})$ and $|E(L)| \leqslant \frac{1}{2}n(k-1)$ since L has maximum degree at most $k-1$. We divide this case into subcases depending on the value of s_0.

Case 3a: Suppose that $s_0 \geqslant k$. Then substituting $h \leqslant n$ and (F2) into (5.2) we obtain

$$\Delta_T \geqslant \frac{s-s_0}{2}\big(s+s_0+2(n-k)-2dk-1\big). \tag{5.3}$$

Substituting (F3) into (5.3) and rearranging, we obtain

$$\Delta_T \geqslant \frac{s-s_0}{2s}\big((s_0-k)(s-k)+k(s+n-s_0-k)+n\big).$$

This last expression is nonnegative because $n \geqslant k$ and $s \geqslant s_0 \geqslant k$ using the conditions of this case.

Case 3b: Suppose that $s_0 \leqslant \frac{k+1}{2}$. Then substituting $e \leqslant \frac{1}{2}h(k-1)$ from (F1) into (5.2) we obtain

$$\Delta_T \geqslant \tfrac{s_0}{2}(2dk+1-s_0) + h(\tfrac{k+1}{2}-s_0).$$

This last expression can be seen to be nonnegative using $d \geqslant 1$ and $1 \leqslant s_0 \leqslant \frac{k+1}{2}$ from the conditions of this case.

Case 3c: Suppose that $\frac{k+2}{2} \leqslant s_0 \leqslant k-1$. Then substituting $h \geqslant \frac{2e}{k-1}$ from (F1) into (5.2) we obtain

$$\Delta_T \geqslant \tfrac{s_0}{2}(2dk+1-s_0) - \frac{e}{k-1}(2s_0-k-1). \tag{5.4}$$

Observing that $2s_0-k-1>0$ by the conditions of this case, substituting (F2) and rearranging, we obtain

$$\Delta_T \geqslant \frac{2s_0-k-1}{k-1}\left(\binom{s}{2}+n(s-k)-k(s-s_0)\right)-\binom{s_0}{2}+\frac{dk}{k-1}\big(s(k+1)-s_0(2s-k+1)\big). \tag{5.5}$$

We further divide this subcase according to the sign of the coefficient of d in (5.5).

Case 3c(i): Suppose that $s(k+1) < s_0(2s-k+1)$. Substituting (F3) into (5.5) and simplifying, we obtain

$$\Delta_T \geqslant \frac{s-s_0}{2s}\big(n+k(n-s_0)+s_0(s-k)\big). \tag{5.6}$$

We can easily see that Δ_T is nonnegative since $s \geqslant k$, $n \geqslant k$ and $s_0 \leqslant k-1$ by the conditions of Case 3c.

Case 3c(ii): Suppose that $s(k+1) \geqslant s_0(2s-k+1)$. Substituting $d \geqslant 1$ and $n \geqslant k$ in (5.5) and rearranging yields

$$\Delta_T \geqslant \frac{2s_0-k-1}{2(k-1)}\left(s^2-(2k+1)s-2k^2\right)+\frac{3k+1}{k-1}\binom{s_0}{2}. \tag{5.7}$$

Recall that $2s_0 > k-1$ by the conditions of Case 3c. Since $s \geqslant k$ is an integer, either $s=k$ or $s \geqslant k+1$, and hence $s^2-(2k+1)s \geqslant -k(k+1)$. Substituting this into (5.7) and rearranging, we obtain

$$\Delta_T \geqslant \frac{3k+1}{k-1}\binom{k-s_0+1}{2}. \tag{5.8}$$

This last expression is clearly nonnegative since $s_0 \leqslant k-1$ by the conditions of Case 3c. ∎

5.4 Proof of Theorems 5.1.2 and 5.1.3

Caro [18] and Wei [88] independently established the following lower bounds on the independence number of a graph.

Theorem 5.4.1 ([18], [88]). *For any graph G, the following hold.*

(a) $\alpha(G) \geqslant \sum_{x\in V(G)} \frac{1}{\deg_G(x)+1}$

(b) $\alpha(G) \geqslant \frac{|V(G)|^2}{2|E(G)|+|V(G)|}$

Part (b) of Theorem 5.4.1 follows immediately from part (a) because, by convexity,

$$\sum_{x\in V(G)} \frac{1}{\deg_G(x)+1} \geqslant \frac{|V(G)|}{d+1} \qquad \text{where} \qquad d = \frac{2|E(G)|}{|V(G)|}.$$

In Lemma 5.4.2 below we combine Theorem 5.4.1(b) with Lemmas 5.3.2 and 5.3.3 to show that, for any graph L, a k-star decomposition of $L \vee K_s$ must exist if $|E(L \vee K_s)| \equiv 0 \pmod{k}$ and s is greater than a certain function of k and $|V(L)|$. Theorem 5.1.3 then follows from Lemma 5.4.2 and, in turn, Theorem 5.1.2 follows from Theorem 5.1.3. For technical reasons we restrict Lemma 5.4.2 to $k \geqslant 3$. Lemma 5.2.2 covers the case when $k = 2$.

Lemma 5.4.2. *Let k, n and s be positive integers with $s \geqslant k \geqslant 3$ and $n \geqslant k$, and let L be a graph of order n such that $|E(L \vee K_s)| \equiv 0 \pmod{k}$. Then there is a k-star decomposition of $L \vee K_s$ if*

$$s > k - n + \frac{1}{2} + \sqrt{\left(n - \sqrt{2k}\right)^2 + k(k-3) + \tfrac{1}{4}}. \tag{5.9}$$

In particular, such a decomposition exists if $n > \frac{k(k-1)}{\sqrt{8k}-1}$.

Proof. Observe that the right hand side of (5.9) is real because $k \geqslant 3$. We first prove the first part of the lemma. Suppose that (5.9) holds. We may assume that L has maximum degree at most $k-1$ because otherwise we can greedily delete k-stars from L until this is the case, apply the proof, and finally add the deleted k-stars to the decomposition produced. Let $b = \frac{1}{k}|E(L \vee K_s)|$, note that b is an integer because $|E(L \vee K_s)| \equiv 0 \pmod{k}$, and let $e = |E(L)|$. If $b \geqslant n + s$, then a k-star decomposition of $L \vee K_s$ exists by Lemma 5.3.3, so we may assume that $b < n + s$. By Lemma 5.3.2 it suffices to show that $\alpha(L) \geqslant n + s - b$.

By Theorem 5.4.1 we have $\alpha(L) \geqslant \frac{n^2}{2e+n}$. So, because $\alpha(L)$ and $n + s - b$ are both integers, it is enough to show that $\frac{n^2}{2e+n} > n + s - b - 1$. Using $b = \frac{1}{k}(e + ns + \binom{s}{2})$ and multiplying through by $2k$, this is equivalent to showing that

$$s^2 + (2n - 2k - 1)s - 2kn + 2k + 2e + \frac{2kn^2}{(2e+n)} \tag{5.10}$$

is positive. Considered as a function of a real variable $e \geqslant 0$, (5.10) is minimised when $e = \frac{n}{2}(\sqrt{2k} - 1)$. Substituting this value for e and rearranging, we see that (5.10) is at least

$$s^2 + (2n - 2k - 1)s + 2k - \left(2k - 2\sqrt{2k} + 1\right) n.$$

Considering this last expression as a quadratic in s, it can be seen that it is positive when (5.9) holds. Thus, (5.10) is positive and $\alpha(L) \geqslant n+s-b$, as required.

We now prove the second part of the lemma. Suppose that $n > \frac{k(k-1)}{\sqrt{8k}-1}$. Since $s \geqslant k$, substituting $s = k$ into (5.9) and rearranging shows that (5.9) will hold if

$$n - \frac{1}{2} > \sqrt{\left(n-\sqrt{2k}\right)^2 + k(k-3) + \tfrac{1}{4}}.$$

By squaring both sides of this expression and rearranging, we see that it is equivalent to $n > \frac{k(k-1)}{\sqrt{8k}-1}$. Therefore, by the first part of the lemma, a k-star decomposition of $L \vee K_s$ exists. □

We can now prove Theorem 5.1.3 directly from Lemma 5.4.2.

Proof of Theorem 5.1.3. Let L be the leave of a nonempty partial k-star decomposition of K_n and note that this implies that $n > k$. Let s be an integer such that $s \geqslant k$ and $\binom{n+s}{2} \equiv 0 \pmod{k}$. Since L is the leave of a partial k-star decomposition and $\binom{n+s}{2} \equiv 0 \pmod{k}$, it follows that $|E(L \vee K_s)| \equiv 0 \pmod{k}$. So, by Lemma 5.4.2 if $k \geqslant 3$ and by Lemma 5.2.2 if $k = 2$, there is a k-star decomposition of $L \vee K_s$. □

Lemma 5.2.6(b) demonstrates that the lower bound on s in Theorem 5.1.3 cannot be decreased no matter what lower bound we place on n. Next, in Lemma 5.4.3, we show that in the case $s = k$ the lower bound on n in Theorem 5.1.3 is asymptotically best possible. To see that Lemma 5.4.3 implies this, note that $\frac{k(k-1)}{\sqrt{8k}-1} = (\frac{k}{2})^{3/2} + O(k)$ as k becomes large.

Lemma 5.4.3. *Let $k = 2^t$ for some odd integer $t \geqslant 7$, let $m = \sqrt{2k}$, and let $n = \frac{1}{4}km - k = (\frac{k}{2})^{3/2} - k$. Let L be a graph of order n that is a vertex disjoint union of $\frac{n}{m}$ copies of K_m. Then a partial k-star decomposition of K_n whose leave is L exists and furthermore it cannot be embedded in a k-star decomposition of K_{n+k}, even though $\binom{n+k}{2} \equiv 0 \pmod{k}$.*

Proof. Note that $m = 2^{(t+1)/2}$ is an integer divisible by 8 because t is odd and $t \geqslant 7$. Thus $n \equiv k \pmod{2k}$, $\frac{n}{m}$ is an integer and $\binom{n+k}{2} \equiv 0 \pmod{k}$. Note that $|E(L)| = \frac{n}{m}\binom{m}{2} = \frac{n}{2}(m-1)$. We first show that L is the leave of a partial k-star decomposition of K_n. Note that $\deg_{\overline{L}}(y) = n - m \geqslant \frac{1}{2}n + k - 1$ for each $y \in V(L)$ because $n = \frac{1}{4}km - k$ and $k \geqslant 128$. Furthermore, $E(\overline{L}) = \binom{n}{2} - \frac{n}{2}(m-1) = \frac{n}{2}(n-m) \equiv 0 \pmod{k}$ because $n \equiv 0 \pmod{k}$ and $n-m$ is even. Therefore, by Theorem 5.2.5, there is a k-star decomposition of $\overline{L}$.

We complete the proof by using Lemma 5.3.1 to show that there is no k-star decomposition of $L \vee K_k$. Observe that

$$n + k - \frac{1}{k}|E(L \vee K_k)| = n + k - \frac{1}{k}\left(\frac{n}{2}(m-1) + kn + \binom{k}{2}\right) = \frac{k}{4} + \frac{5\sqrt{2k}}{8}$$

where the first equality follows using $|E(L)| = \frac{n}{2}(m-1)$ and the second follows using $n = \frac{1}{4}km - k$ and $m = \sqrt{2k}$. On the other hand, $\alpha(L) = \frac{n}{m} = \frac{k}{4} - \frac{k}{m}$ because an independent set in L can contain at most one vertex from each copy of K_m. So we have $\alpha(L) < n + k - \frac{1}{k}|E(L \vee K_k)|$ and hence there is no k-star decomposition of $L \vee K_k$ by Lemma 5.3.1. □

Theorem 5.1.2 follows readily from Theorem 5.1.3.

Proof of Theorem 5.1.2. Let $\mathcal{D}$ be a partial k-star decomposition of K_n. If $\mathcal{D}$ is empty and n=1, then $\mathcal{D}$ is trivially its own embedding. If $\mathcal{D}$ is empty and $n \geqslant 2$, then there is an embedding of $\mathcal{D}$ in a k-star decomposition of K_{2k} by Theorem 5.2.1(a). So in either case the result holds, and hence we may assume that $\mathcal{D}$ is nonempty.

If k is even, let s be an element of $\{k, \ldots, 3k-2\}$ such that $n+s \equiv 0 \pmod{2k}$ or $n+s \equiv 1 \pmod{2k}$. If k is odd, let s be an element of $\{k, \ldots, 2k-2\}$ such that $n+s \equiv 0 \pmod{k}$ or $n+s \equiv 1 \pmod{k}$. In either case such an s exists because $\{k, \ldots, 3k-2\}$ contains $2k-1$ consecutive integers and $\{k, \ldots, 2k-2\}$ contains $k-1$ consecutive integers. Then $\binom{n+s}{2} \equiv 0 \pmod{k}$ by our definition of s. So by Theorem 5.1.3 there is an embedding of $\mathcal{D}$ in a k-star decomposition of K_{n+s} and hence the result is proved. □

Lemma 5.2.6(c) shows that the upper bound of $2k-2$ on s in the k odd case of Theorem 5.1.2 cannot be improved for any k that is a power of an odd prime. Next, in Lemma 5.4.4, we show that the upper bound of $3k-2$ on s in the k even case of Theorem 5.1.2 cannot be improved for any $k \geqslant 16$ that is a power of 4.

Lemma 5.4.4. *Let $k = 2^t$ for some even $t \geqslant 4$, and let $n \geqslant 3k+2$ be an integer such that $n \equiv k+2 \pmod{2k}$. Let L be a graph of order n that is a vertex disjoint union of one copy of $K_{\sqrt{k}}$, $\frac{1}{2}\sqrt{k}+1$ copies of K_2 and $n-2\sqrt{k}-2$ copies of K_1. A partial k-star decomposition of K_n whose leave is L exists and furthermore it cannot be embedded in a k-star decomposition of K_{n+s} for any $s < 3k-2$.*

Proof. A simple calculation shows that $|E(L)| = \frac{1}{2}(k+2)$. We first show that L is the leave of a partial k-star decomposition of K_n. Note that $\deg_{\overline{L}}(y) \geqslant n-\sqrt{k} \geqslant \frac{1}{2}n+k-1$ for each $y \in V(L)$ since $n \geqslant 3k+2$ and $k \geqslant 16$. Furthermore, $|E(\overline{L})| = \binom{n}{2} - \frac{1}{2}(k+2) \equiv 0 \pmod{k}$ since $n \equiv k+2 \pmod{2k}$. Therefore, a k-star decomposition of $\overline{L}$ exists by Theorem 5.2.5.

Now assume for a contradiction that $\mathcal{D}$ is a k-star decomposition of $L \vee K_S$ where $|S| = s$ for some nonnegative integer $s < 3k-2$ and let γ be the central function of $\mathcal{D}$. We must have that $n+s \equiv 0 \pmod{2k}$ or $n+s \equiv 1 \pmod{2k}$ by Theorem 5.2.1(b) and hence, because $n \equiv k+2 \pmod{2k}$, that $s \equiv k-2 \pmod{2k}$ or $s \equiv k-1 \pmod{2k}$. Therefore, $s \in \{k-2, k-1\}$ since $s < 3k-2$.

Let V_1 be the vertex set of the copy of $K_{\sqrt{k}}$ in L and let V_2 be the set of vertices in the $\frac{1}{2}\sqrt{k}+1$ copies of K_2 in L. If $s = k-2$, then $\deg_{L\vee K_S}(y) = k-1$ and hence $\gamma(y) = 0$ for each $y \in V_2$ which contradicts the fact that each edge in $L[V_2]$ is in a star in $\mathcal{D}$. Thus it must be that $s = k-1$ and $\mathcal{D}$ is a k-star decomposition of $L \vee K_{k-1}$. Let r be the positive integer such that $n = 2kr+k+2$. Observe the following.

- $\sum_{x\in V(L)\cup S} \gamma(x) = (2r+1)(k-1) + \frac{1}{2}k+1$ because $|E(L \vee K_{k-1})| = \frac{1}{2}(k+2)+n(k-1)+\binom{k-1}{2}$.
- $\sum_{y\in V_1} \gamma(y) \leqslant \sqrt{k}$ because $\deg_{L\vee K_S}(y) = k+\sqrt{k}-2 < 2k$ for each $y \in V_1$ and hence $\gamma(y) \leqslant 1$ for all $y \in V_1$.
- $\sum_{y\in V_2} \gamma(y) = \frac{1}{2}\sqrt{k}+1$ because $\deg_{L\vee K_S}(y) = k$ for each $y \in V_2$ and hence $\gamma(y_1)+\gamma(y_2) = 1$ for each edge y_1y_2 in $L[V_2]$.
- $\sum_{y\in V(L)\setminus(V_1\cup V_2)} \gamma(y) = 0$ because $\deg_{L\vee K_S}(y) = k-1$ for each $y \in V(L) \setminus (V_1 \cup V_2)$.

Using these four facts and simplifying we have

$$\sum_{z\in S}\gamma(z)=\sum_{x\in V(L)\cup S}\gamma(x)-\sum_{y\in V(L)}\gamma(y)\geqslant(2r+1)(k-1)+\tfrac{1}{2}k-\tfrac{3}{2}\sqrt{k}>(2r+1)(k-1)$$

where the last inequality follows because $k\geqslant 16$. So, by the pigeonhole principle, $\gamma(z_1)\geqslant 2r+2$ for some $z_1\in S$ because $s=k-1$. Now $\deg_{L\vee K_S}(z_1)=n+k-2=k(2r+2)$ noting that $n=2kr+k+2$, so it must be that $\gamma(z_1)=2r+2$ and that every edge incident with z_1 is in a star in $\mathcal{D}$ centred at z_1. But this contradicts the fact that, for any vertex $y_1\in V_2$ such that $\gamma(y_1)=1$, the edge y_1z_1 must be in a star in $\mathcal{D}$ centred at y_1. $\square$

5.5 Proof of Theorem 5.1.1

From Lemma 5.4.2, it is not too difficult to prove Theorem 5.1.1 in the case where k is even. Note that in fact the argument in the proof also applies when k is odd.

Lemma 5.5.1. *Let $k\geqslant 2$ and $n\geqslant 1$ be integers. Any partial k-star decomposition of K_n can be embedded in a k-star decomposition of K_{n+s} for some s such that $s<(6-2\sqrt{2})k$.*

Proof. Let $\mathcal{D}$ be a partial k-star decomposition of K_n and L be its leave. Note that we will have $|E(L\vee K_s)|\equiv 0 \pmod{k}$ for any integer s such that $n+s\equiv 0 \pmod{2k}$. If $k=2$ then we can choose $s\in\{1,2,3,4\}$ such that $n+s\equiv 0 \pmod{4}$ and $L\vee K_s$ will have a 2-star decomposition by Lemma 5.2.2, so we may assume $k\geqslant 3$. We consider three cases according to the value of n.

Case 1: Suppose that $n\geqslant 2\sqrt{2}k$. Let s be an integer such that $(4-2\sqrt{2})k\leqslant s<(6-2\sqrt{2})k$ and $n+s\equiv 0 \pmod{2k}$. By Lemma 5.4.2 there is a k-star decomposition of $L\vee K_s$ and hence the result is proved provided that (5.9) holds. The lower bound on s given by (5.9) can be seen to be decreasing in n, so it suffices to show that this bound is less than $(4-2\sqrt{2})k$ when $n=2\sqrt{2}k$. Substituting $n=2\sqrt{2}k$ into the bound gives

$$\left(1-2\sqrt{2}\right)k+\tfrac{1}{2}+\sqrt{9k^2-8k^{3/2}-k+\tfrac{1}{4}}$$

which is easily seen to be less than $(4-2\sqrt{2})k$ since the final term is less than $3k-\frac{1}{2}$.

Case 2: Suppose that $k+1\leqslant n<2\sqrt{2}k$. We show that we can embed $\mathcal{D}$ in a k-star decomposition of K_{4k}. Let $s=4k-n$ and note that $k\leqslant s<(6-2\sqrt{2})k$ since $k+1\leqslant n<2\sqrt{2}k$ and that $\binom{n+s}{2}\equiv 0 \pmod{k}$. By Lemma 5.4.2 there is a k-star decomposition of $L\vee K_s$ and hence the result is proved provided that (5.9) holds. Now (5.9) holds if and only if

$$\left(3k-\tfrac{1}{2}\right)^2>\left(n-\sqrt{2k}\right)^2+k(k-3)+\tfrac{1}{4}$$

and this can in turn be shown to hold using $n<2\sqrt{2}k$.

Case 3: Suppose that $1\leqslant n\leqslant k$. Then $\mathcal{D}$ is empty and hence a k-star decomposition of K_{2k}, which exists by Theorem 5.2.1(a), is an embedding of $\mathcal{D}$. $\square$

Lemma 5.5.2 below shows that if the constant $6-2\sqrt{2}$ in Theorem 5.1.1 were decreased then the result would fail to hold for each sufficiently large k that is 2 to some odd power. To see this, observe that the value of n in the statement of Lemma 5.5.2 is at most $2\sqrt{k(2k+1)}+2\sqrt{2k}$ and hence is $2\sqrt{2}k+O(\sqrt{k})$ as k becomes large.

Lemma 5.5.2. *Let $k = 2^t$ for some odd integer $t \geqslant 3$, let $m = \sqrt{2k}$, let n be the smallest integer such that $n \equiv 0 \pmod{m}$ and $n > 2\sqrt{k(2k+1)} + \sqrt{2k}$, and let L be a graph of order n that is a vertex disjoint union of $\frac{n}{m}$ copies of K_m. A partial k-star decomposition of K_n whose leave is L exists and furthermore it cannot be embedded in a k-star decomposition of K_{n+s} for any $s < 6k - n$.*

Proof. Observe that $|E(L)| = \frac{n}{m}\binom{m}{2} = \frac{n}{2}(m-1)$. We first show that L is the leave of a partial k-star decomposition of K_n. Note that $\deg_{\overline{L}}(y) = n - m \geqslant \frac{1}{2}n + k - 1$ for each $y \in V(L)$ since $n > 2\sqrt{k(2k+1)} + \sqrt{2k}$. Furthermore, $|E(\overline{L})| = \frac{n}{2}(n-m) \equiv 0 \pmod{k}$ because $n \equiv 0 \pmod{m}$. Therefore, by Theorem 5.2.5, a k-star decomposition of $\overline{L}$ exists.

Now suppose for a contradiction that a k-star decomposition of $L \vee K_S$ exists where $|S| = s$ for some nonnegative integer $s < 6k - n$. We must have $n + s \equiv 0 \pmod{2k}$ or $n + s \equiv 1 \pmod{2k}$ by Theorem 5.2.1(b). Therefore, because $0 \leqslant s < 6k - n$ and $n > 2k + 1$, we have $s \in \{4k - n, 4k - n + 1\}$.

Now $\alpha(L) = \frac{n}{m}$ because an independent set in L can contain at most one vertex from each copy of K_m. So we complete the proof by showing that $n+s-\frac{1}{k}(|E(L)|+ns+\binom{s}{2}) > \frac{n}{m}$ and hence concluding by Lemma 5.3.1 that there is no k-star decomposition of $L \vee K_S$. Using $|E(L)| = \frac{n}{2}(m-1)$ and $m = \sqrt{2k}$ and multiplying through by $2k$, this is equivalent to showing that

$$n\left(2k - 2\sqrt{2k} + 1\right) - s(s + 2n - 2k - 1) \tag{5.11}$$

is positive. Using $s \leqslant 4k - n + 1$, (5.11) is at least $n(n - 2\sqrt{2k}) - 2k(4k+1)$. In turn this can be shown to be positive using $n > 2\sqrt{k(2k+1)} + \sqrt{2k}$. □

In order to prove Theorem 5.1.1 when k is odd, we need to make a closer examination of leaves of partial k-star decompositions of K_n where $k < n \leqslant 2k$. It turns out that these leaves must contain a large clique and hence we can improve on the bound of Theorem 5.4.1(b) for their independence number using Theorem 5.4.1(a). Our first step is to improve on Theorem 5.4.1(b) in the case where the graph considered contains a large clique.

Lemma 5.5.3. *If L is a graph of order n such that L has a copy of K_r as a subgraph and $|E(L)| \leqslant \frac{1}{2}n(r-1)$, then*

$$\alpha(L) \geqslant 1 + \frac{(n-r)^2}{2|E(L)| + n - r^2}.$$

Proof. Let $V = V(L)$ and R be a subset of V such that $L[R]$ is a copy of K_r. Let $d = \frac{2|E(L)| - r(r-1)}{n-r}$ and note that $d \leqslant r - 1$ since $|E(L)| \leqslant \frac{1}{2}n(r-1)$. By Theorem 5.4.1(a) we have that

$$\alpha(L) \geqslant \sum_{x \in V} \frac{1}{\deg_L(x) + 1}. \tag{5.12}$$

Observe that $\deg_L(x) \geqslant r - 1$ for $x \in R$, that $|R| = r$, that $\sum_{x \in V} \deg_L(x) = 2|E(L)|$, and that $d \leqslant r - 1$. By convexity, the minimum value of $\sum_{i=1}^{n} \frac{1}{x_i+1}$, where the x_i are nonnegative reals subject to the constraints $x_i \geqslant r - 1$ for $i \in \{1, \ldots, r\}$ and $\sum_{i=1}^{n} x_i = 2|E(L)|$, occurs when $x_i = r - 1$ for each $i \in \{1, \ldots, r\}$ and $x_i = d$ for each $i \in \{r+1, \ldots, n\}$. Thus from (5.12) we have

$$\alpha(L) \geqslant \frac{r}{(r-1)+1} + \frac{n-r}{d+1} = 1 + \frac{(n-r)^2}{2|E(L)| + n - r^2}.$$ □

By combining Lemma 5.5.3 with Lemmas 5.3.2 and 5.3.3, we can improve on Lemma 5.4.2 in the special case where L is the leave of a partial k-star decomposition of K_n and $k < n \leqslant 2k$. Again, the $k = 2$ case is covered by Lemma 5.2.2.

Lemma 5.5.4. *Let k, n and s be integers such that $s \geqslant k \geqslant 3$, $2k \geqslant n > k$ and $\binom{n+s}{2} \equiv 0 \pmod{k}$. Any partial k-star decomposition of K_n can be embedded in a k-star decomposition of K_{n+s} if*

$$s > k - n + \frac{1}{2} + \sqrt{4k\left(\sqrt{n-k} - \tfrac{1}{\sqrt{2}}\right)^2 + k(k-3) + \tfrac{1}{4}} \qquad (5.13)$$

Proof. Observe that the right hand side of (5.13) is real because $k \geqslant 3$. Suppose that (5.13) holds. Let $\mathcal{D}$ be a partial k-star decomposition of K_n. We may assume that $\mathcal{D}$ is maximal for otherwise we can greedily add k-stars to $\mathcal{D}$ until it is maximal and then apply the proof. Let L be the leave of $\mathcal{D}$ and note that L has maximum degree at most $k-1$. Let $b = \frac{1}{k}|E(L \vee K_s)|$, note that b is an integer because $|E(L \vee K_s)| \equiv 0 \pmod{k}$ since $\binom{n+s}{2} \equiv 0 \pmod{k}$ and L is the leave of a partial k-star decomposition of K_n. If $b \geqslant n+s$, then a k-star decomposition of $L \vee K_s$ exists by Lemma 5.3.3, so we may assume that $b < n+s$. By Lemma 5.3.2 it suffices to show that $\alpha(L) \geqslant n+s-b$.

Let V_0 be the set of vertices in $V(L)$ that have no star in $\mathcal{D}$ centred at them. No star in $\mathcal{D}$ can contain an edge between a pair of vertices in V_0 and hence $L[V_0]$ must be a complete graph. Because $\mathcal{D}$ contains $\frac{1}{k}(\binom{n}{2} - |E(L)|)$ stars, $|V_0| \geqslant r$ where $r = n - \frac{1}{k}(\binom{n}{2} - e)$ and $e = |E(L)|$. Note that $r \geqslant 1$ since $k \geqslant \frac{n}{2}$ from our hypotheses. So L contains a copy of K_r as a subgraph. Also, it follows from the definition of r that $e = \binom{n}{2} - k(n-r)$ and hence, because $k \geqslant \frac{n}{2}$, that $e \leqslant \frac{1}{2}n(r-1)$. Thus, by Lemma 5.5.3 we have $\alpha(L) \geqslant 1 + \frac{(n-r)^2}{2e+n-r^2}$.

So, because $\alpha(L)$ and $n+s-b$ are both integers, it is enough to show that $1 + \frac{(n-r)^2}{2e+n-r^2} > n+s-b-1$. Using $b = \frac{1}{k}(e + ns + \binom{s}{2})$ and multiplying through by $2k$, this is equivalent to showing that

$$s^2 + (2n-2k-1)s - 2kn + 4k + 2e + \frac{2k(n-r)^2}{2e+n-r^2} \qquad (5.14)$$

is positive. Using $e = \binom{n}{2} - k(n-r)$, (5.14) is equal to

$$s^2 + (2n-2k-1)s - (4k-n)(n-1) + 2k\left(r + \frac{n-r}{n+r-2k}\right). \qquad (5.15)$$

Because L contains a copy of K_r as a subgraph, we have that $e \geqslant \binom{r}{2}$ or equivalently, using $e = \binom{n}{2} - k(n-r)$, that $\frac{1}{2}(n-r)(n+r-2k-1) \geqslant 0$. This implies that $2k+1-n \leqslant r \leqslant n$. Considered as a function of a real variable r where $2k+1-n \leqslant r \leqslant n$, (5.15) is minimised when $r = 2k - n + \sqrt{2n-2k}$ and, substituting this value for r and rearranging, we have that (5.15) is at least

$$s^2 + (2n-2k-1)s - (6k-n)(n-1) + 4k\left(k + \sqrt{2n-2k} - 1\right).$$

Considering this last expression as a quadratic in s, we can see that it is positive when (5.13) holds. Thus (5.14) is positive and $\alpha(L) \geqslant n+s-b$, as required. □

We now finish the proof of Theorem 5.1.1 by considering the case where k is odd.

Proof of Theorem 5.1.1. When k is even the result follows from Lemma 5.5.1, so we may assume that k is odd. Let $\mathcal{D}$ be a partial k-star decomposition of K_n and L be its leave. Note that we will have $|E(L \vee K_s)| \equiv 0 \pmod{k}$ for any integer s such that $n+s \equiv 0 \pmod{k}$. We consider four cases according to the value of n.

Case 1: Suppose that $n \geqslant 2\sqrt{2}k$. Let s be an integer such that $\frac{5}{4}k \leqslant s < \frac{9}{4}k$ and $n+s \equiv 0 \pmod{k}$. We saw in Case 1 of the proof of Lemma 5.5.1 that the right hand side of (5.9) is less than $(4-2\sqrt{2})k$ when $n \geqslant 2\sqrt{2}k$. So by Lemma 5.4.2 there is a k-star decomposition of $L \vee K_s$ and hence the result is proved, because $s \geqslant \frac{5}{4}k > (4-2\sqrt{2})k$.

Case 2: Suppose that $\frac{7}{4}k < n < 2\sqrt{2}k$. We show that we can embed $\mathcal{D}$ in a k-star decomposition of K_{4k}. Let $s = 4k-n$ and note that $k \leqslant s < \frac{9}{4}k$ since $\frac{7}{4}k < n < 2\sqrt{2}k$ and that $\binom{n+s}{2} \equiv 0 \pmod{k}$. We showed in Case 2 of the proof of Lemma 5.5.1 that (5.9) holds when $s = 4k-n$ and $n < 2\sqrt{2}k$. So by Lemma 5.4.2 there is a k-star decomposition of $L \vee K_s$.

Case 3: Suppose that $k+1 \leqslant n \leqslant \frac{7}{4}k$. We show that we can embed $\mathcal{D}$ in a k-star decomposition of K_{3k}. Let $s = 3k-n$ and note that $k \leqslant s < \frac{9}{4}k$ since $k+1 \leqslant n \leqslant \frac{7}{4}k$ and that $\binom{n+s}{2} \equiv 0 \pmod{k}$. Then (5.13) holds if and only if

$$\left(2k-\tfrac{1}{2}\right)^2 > 4k\left(n-\sqrt{2n-2k}\right) - k(3k+1) + \tfrac{1}{4}. \tag{5.16}$$

For $n \geqslant k+1$, the right hand side of (5.16) is increasing in n and hence (5.16) can be shown to hold for $k+1 \leqslant n \leqslant \frac{7}{4}k$ by substituting $n = \frac{7}{4}k$. So by Lemma 5.5.4 there is a k-star decomposition of $L \vee K_s$.

Case 4: Suppose that $1 \leqslant n \leqslant k$. Then $\mathcal{D}$ is empty and hence a k-star decomposition of K_{2k}, which exists by Theorem 5.2.1(a), is an embedding of $\mathcal{D}$. □

Finally, we prove Lemma 5.5.5, which shows that if the constant $\frac{9}{4}$ in Theorem 5.1.1 were decreased then the result would fail to hold for each sufficiently large k that is a power of an odd prime. To see this, observe that the definition of n in the statement of Lemma 5.5.5 can be rephrased as $n = \frac{1}{2}a + k$ where a is the smallest even perfect square that is greater than $\frac{3}{2}k + \sqrt{6k+6} + \frac{5}{2}$. Clearly then, $a = \frac{3}{2}k + O(\sqrt{k})$ and hence $n = \frac{7}{4}k + O(\sqrt{k})$ as k becomes large.

Lemma 5.5.5. *Let k be a sufficiently large integer that is a power of an odd prime and let n be the smallest integer such that $n > \frac{7}{4}k + \frac{1}{2}\sqrt{6k+6} + \frac{5}{4}$ and $\sqrt{2n-2k}$ is an integer. Let $m = \sqrt{2n-2k}$ and $r = 2k-n+m$, and let L be a graph of order n that is a vertex disjoint union of $m-1$ copies of K_m and a copy of K_r. A partial k-star decomposition of K_n whose leave is L exists and furthermore it has no embedding in a k-star decomposition of K_{n+s} for any $s < 4k-n$.*

Proof. Observe that, for sufficiently large k, $r = \frac{k}{4} + O(\sqrt{k})$ because $n = \frac{7}{4}k + O(\sqrt{k})$ as noted in the paragraph before the lemma. We first show that L is the leave of a partial k-star decomposition. Let V_0 be the vertex set of the copy of K_r in L and let $V_1, \ldots, V_{m-1}$ be the vertex sets of the copies of K_m in L. Let $\gamma : V(L) \to \mathbb{Z}^{\geqslant 0}$ be defined by $\gamma(x) = 0$ for each $x \in V_0$ and $\gamma(y) = 1$ for each $y \in V(L) \setminus V_0$. Then γ is a precentral function for $\overline{L}$, because we have $\frac{1}{k}(\binom{n}{2} - |E(L)|) = m(m-1)$ using $|E(L)| = \binom{r}{2} + (m-1)\binom{m}{2}$, the definition of r and $n = \frac{1}{2}m^2 + k$. Let $\mathcal{G}$ be $\overline{L}$ equipped with γ and let $T \in \mathcal{T}$. We will show that $\Delta_T = 0$ and hence that a k-star decomposition of $\overline{L}$ exists. For each $i \in \{1, \ldots, m-1\}$, we have $V_i \subseteq T$ or $T \cap V_i = \emptyset$ by Lemma 5.2.4 with $U = V_i$. So without loss of generality we can assume that $T = V_1 \cup \cdots \cup V_t$ for some $t \in \{0, \dots, m-1\}$. Then $\Delta_T^+ = \binom{t}{2}m^2 + mt(n - mt)$ and $\Delta_T^- = kmt$. Thus, using $n = \frac{1}{2}m^2 + k$ and simplifying,

$$\Delta_T = \tfrac{1}{2}tm^2(m-1-t)$$

which is nonnegative since $t \in \{0, \dots, m-1\}$. Thus $\Delta_T = 0$ and a k-star decomposition of $\overline{L}$ exists.

Now suppose for a contradiction that a k-star decomposition of $L \vee K_S$ exists where $|S| = s$ for some nonnegative integer $s < 4k - n$. We must have $n + s \equiv 0 \pmod{k}$ or $n+s \equiv 1 \pmod{k}$ by Theorem 5.2.1(c). Therefore, because $0 \leqslant s < 4k-n$ and $n > k+1$, we have $s \in \{2k-n, 2k-n+1, 3k-n, 3k-n+1\}$.

Now $\alpha(L) = m$ because an independent set in L can contain at most one vertex from the copy of K_r and at most one vertex from each copy of K_m. So we complete the proof by showing that $n + s - \frac{1}{k}(|E(L)| + ns + \binom{s}{2}) > m$ and hence concluding by Lemma 5.3.1 that there is no k-star decomposition of $L \vee K_s$. Using $|E(L)| = \binom{r}{2} + (m-1)\binom{m}{2}$, the definitions of r and m, and multiplying through by $2k$, this is equivalent to showing that

$$n(6k - n + 1) - 4k\left(k + \sqrt{2n - 2k}\right) - s(s + 2n - 2k - 1) \tag{5.17}$$

is positive. Using $s \leqslant 3k - n + 1$, (5.17) is at least $k(4n - 7k - 4\sqrt{2n-2k} - 1)$. In turn, this can be shown to be positive using $n > \frac{7}{4}k + \frac{1}{2}\sqrt{6k+6} + \frac{5}{4}$. □

In Chapter 7 we discuss some ways in which the results in this chapter might be improved. In particular, the constants in Theorem 5.1.1 are best possible for general k but improvements may be possible for specific values of k.

Chapter 6

Completing partial k-star designs

" In any sufficiently rich system statements are possible which can neither be proved nor refuted within the system, unless the system itself is inconsistent. You can describe your own language in your own language: but not quite. You can investigate your own brain by means of your own brain: but not quite..."

- Hans Magnus Enzensberger, *Homage to Gödel*

6.1 Introduction

This chapter is concerned with the problem of when a partial k-star decomposition of K_n can be completed. Here, when $n \geqslant 11k+20$ we determine exactly the minimum number of k-stars in an uncompletable partial k-star decomposition of K_n. Our result can be seen as an analogue of Theorem 3.1.1 for partial k-star decompositions. We will refer to Chapter 5 for most of the definitions. As mentioned in Definition 1.1.3, we call a positive integer n $K_{1,k}$-admissible if $\binom{n}{2} \equiv 0 \pmod{k}$. When $n \leqslant 2k-1$, it is straightforward to see that any partial k-star decomposition of K_n is uncompletable, even when n is $K_{1,k}$-admissible In fact, this follows from Theorem 5.2.1 (a) but we prove it here for the sake of completeness.

Lemma 6.1.1. *[82, 92] Let $k \geqslant 2$ be an integer. For all integers $n \leqslant 2k-1$, any partial k-star decomposition of K_n is not completable.*

Proof. If a partial k-star decomposition of K_n is completable, then it is essential that at most one vertex has zero k-stars centred at it. This implies $\frac{n(n-1)}{2k} \geqslant n-1$, which is equivalent to $n \geqslant 2k$. Therefore, when $n \leqslant 2k-1$, any partial k-star decomposition of K_n cannot be completed. $\square$

Let $k \geqslant 2$ be an integer. Our main result in this chapter exactly determines the minimum number of k-stars in an uncompletable partial k-star decomposition of K_n when $n \geqslant 11k+20$.

Theorem 6.1.2. *Let $k \geqslant 2$ be an integer. For each $K_{1,k}$-admissible integer n such that $n \geqslant 11k+20$, any partial k-star decomposition of K_n with at most u stars is completable, where*

$$u = \begin{cases} 2\lfloor \frac{n-2}{k} \rfloor - 1 & \text{if } n \not\equiv 1 \pmod{k} \\ \frac{2(n-1)}{k} - 2 & \text{if } n \equiv 1 \pmod{k}. \end{cases}$$

Furthermore, for each $K_{1,k}$-admissible integer $n > 1$, there is a partial k-star decomposition of K_n with $u + 1$ stars that is not completable.

The bound $n \geqslant 11k + 20$ in Theorem 6.1.2 is an artefact of our techniques, and it seems likely that it is not in fact required. It is worth noting that for $n = 2k$, the result of the theorem still holds since any partial k-star decomposition of K_{2k} with at most one star is trivially completable by Theorem 2.2.1.

It will be useful in what follows to note that if u is defined as in Theorem 6.1.2, then

$$u \leqslant \tfrac{2n-4}{k} - 1. \tag{6.1}$$

This is obvious when $n \not\equiv 1 \pmod{k}$ and can be confirmed using $k \geqslant 2$ when $n \equiv 1 \pmod{k}$.

6.2 Preliminaries

Let G be a graph. Let $N_G(x)$ denote the neighbourhood of a vertex x in G. Recall that, for a given k-star decomposition $\mathcal{D}$ of G, we can define a function $\gamma : V(G) \to \mathbb{Z}^{\geqslant 0}$ called the *central function*, where $\gamma(x)$ is the number of k-stars of $\mathcal{D}$ whose centre is x for each $x \in V(G)$. In Lemma 6.2.1(a) and (b) below, we establish the tightness claims in Theorems 6.1.2.

Lemma 6.2.1. *Let $k \geqslant 2$ be an integer.*

(a) *For all $K_{1,k}$-admissible integers $n > 1$ such that $n \not\equiv 1 \pmod{k}$ there is a partial k-star decomposition of K_n with $2\lfloor \frac{n-2}{k} \rfloor$ stars that is not completable.*

(b) *For all $K_{1,k}$-admissible integers $n > 1$ such that $n \equiv 1 \pmod{k}$ there is a partial k-star decomposition of K_n with $\frac{2(n-1)}{k} - 1$ stars that is not completable.*

Proof. We first prove (a). Let $\mathcal{D}$ be a partial k-star decomposition of K_n with exactly $2\lfloor \frac{n-2}{k} \rfloor$ stars and central function γ such that there exist distinct vertices $x_1, x_2 \in V(K_n)$ for which $\gamma(x_1) = \gamma(x_2) = \lfloor \frac{n-2}{k} \rfloor$ and x_1 and x_2 are adjacent in the leave L of $\mathcal{D}$. This implies $1 \leqslant \deg_L(x_i) \leqslant k - 1$ for each $i \in \{1, 2\}$ because $n \not\equiv 1 \pmod{k}$. If we want to complete $\mathcal{D}$, then we need to find a k-star decomposition of L and hence the edge x_1x_2 needs to be in a star centred at either x_1 or x_2. This is impossible because $\deg_L(x_1), \deg_L(x_2) \leqslant k - 1$. Hence, $\mathcal{D}$ cannot be completed.

We now prove (b). Let $\mathcal{D}$ be a partial k-star decomposition of K_n with exactly $\frac{2(n-1)}{k} - 1$ stars and central function γ such that there exist distinct vertices $x_1, x_2, x_3 \in V(K_n)$ for which $\gamma(x_1) = \gamma(x_2) = \frac{1}{k}(n-1) - 1$, $\gamma(x_3) = 1$, x_1 and x_2 are tail vertices of the star centred at x_3, and x_1 and x_2 are adjacent in the leave L of $\mathcal{D}$. For each $i \in \{1, 2\}$, this implies $\deg_L(x_i) = n - 1 - (n - 1 - k + 1) = k - 1$ because x_i is a tail vertex of one star and centre of $\frac{1}{k}(n-1) - 1$ stars. Thus, no more stars can be centred at either x_1 or x_2 and hence the edge x_1x_2 will not be included in a star. Therefore, $\mathcal{D}$ cannot be completed. □

The following lemma shows that Theorem 6.1.2 holds when $k = 2$. In fact, it is slightly stronger because it holds for all $K_{1,2}$-admissible integers $n > 1$.

Lemma 6.2.2. *For all $K_{1,2}$-admissible integers $n > 1$, any partial 2-star decomposition of K_n with at most $n - 3$ stars is completable. Furthermore, there is a partial 2-star decomposition of K_n with $n - 2$ stars that is not completable.*

Proof. Note that $n \geqslant 4$ since $n > 1$ and n is $K_{1,2}$-admissible. We can refer to Lemma 6.2.1 to construct the uncompletable designs. When $n \equiv 0 \pmod 2$ and $n \equiv 1 \pmod 2$, Lemma 6.2.1(a) and (b) respectively give constructions of uncompletable 2-star designs with $n-2$ stars.

Now we prove the first part. Let $\mathcal{D}$ be a partial 2-star decomposition of K_n such that $|\mathcal{D}| \leqslant n-3$ and let L be the leave of $\mathcal{D}$. Therefore, $|E(L)| \geqslant \binom{n}{2} - 2(n-3)$. Furthermore, since n is $K_{1,2}$-admissible, $|E(L)| \equiv 0 \pmod 2$. If each connected component of L has an even number of edges, then by Theorem 2.2.2, a 2-star decomposition of L exists and hence $\mathcal{D}$ is completable. So suppose for a contradiction that L has a connected component L_1 with an odd number of edges. Let $a = |V(L_1)|$. Note that $2 \leqslant a \leqslant n-2$ because $|E(L_1)| = 0$ if $a \leqslant 1$ and $|E(L_1)| = |E(L)| \equiv 0 \pmod 2$ if $a \geqslant n-1$. Therefore, suppose that $L = L_1 \cup L_2$ where $|V(L_2)| = n-a$. Then there is no edge of L between a vertex in L_1 and a vertex in L_2. Therefore, $|E(L)| \leqslant \binom{n}{2} - a(n-a)$. This implies $a(n-a) \leqslant 2(n-3)$, noting that $|E(L)| \geqslant \binom{n}{2} - 2(n-3)$, and this is equivalent to $a^2 - an + 2n - 6 \geqslant 0$. This contradicts $2 \leqslant a \leqslant n-2$. Thus, a 2-star decomposition of L exists and hence $\mathcal{D}$ is completable. $\square$

6.3 Proof of Theorem 6.1.2

We will complete the proof of our main theorem by finding a k-star decomposition of the leave of the partial k-star design. Our proof heavily relies on Tarsi's result on k-star decompositions of graphs having moderately high vertex degrees, which we will restate here.

Theorem 6.3.1 ([82])**.** *Let G be a graph with n vertices such that $\deg_G(x) \geqslant \frac{1}{2}n + k - 1$ for every $x \in V(G)$. Then G has a k-star decomposition if $|E(G)| \equiv 0 \pmod k$.*

However, we cannot directly apply Theorem 6.3.1 if the leave contains low degree vertices. In these situations, we first remove k-stars centred at low degree vertices until each of these vertices has degree at most $k-1$ (see Lemma 6.3.3). After that, we bring down the degrees of vertices in that set to zero by removing stars centred at adjacent vertices (see Lemma 6.3.4). Then we can apply Theorem 6.3.1 to the remaining graph. We use the following lemma to prove Lemma 6.3.3.

Lemma 6.3.2. *Let $k \geqslant 2$ and $n \geqslant k+1$ be integers and let $\mathcal{D}$ be a partial k-star decomposition of K_n with at most u stars, where*

$$u = \begin{cases} 2\lfloor \frac{n-2}{k} \rfloor - 1 & \text{if } n \not\equiv 1 \pmod k \\ \frac{2(n-1)}{k} - 2 & \text{if } n \equiv 1 \pmod k. \end{cases}$$

Let L be the leave of $\mathcal{D}$ and let S be a subset of $V(L)$ such that $L[S]$ has at least one edge. Then

(i) *if $|S| = 2$, then $\max\{\deg_L(x) : x \in S\} \geqslant k$,*

(ii) *if $|S| \geqslant 3$, then $\max\{\deg_L(x) : x \in S\} \geqslant \frac{1}{3}(n-1+k)$.*

Proof. Let γ be the central function of $\mathcal{D}$.

We first prove (i). Suppose that $|S| = 2$. Let y be a vertex in S such that $\gamma(y) \leqslant \gamma(x)$ for each $x \in S$. Then at most $u - 2\gamma(y)$ stars in $\mathcal{D}$ are centred on vertices in $V \setminus S$ and hence, because the vertices in S are not adjacent in $\overline{L}$,

$$\deg_{\overline{L}}(y) \leqslant k\gamma(y) + (u - 2\gamma(y)) = u + (k-2)\gamma(y) \leqslant \begin{cases} \frac{1}{2}k(u-1)+1 & \text{if } n \not\equiv 1 \pmod{k} \\ \frac{1}{2}ku & \text{if } n \equiv 1 \pmod{k}. \end{cases}$$

where the last inequality follows by observing that $\gamma(y) \leqslant \frac{1}{2}(u-1)$ if $n \not\equiv 1 \pmod{k}$ and $\gamma(y) \leqslant \frac{u}{2}$ if $n \equiv 1 \pmod{k}$ by the pigeonhole principle. Then in either case we can see that $\deg_{\overline{L}}(y) \leqslant n - 1 - k$ by applying the definition of u. Thus, we have $\deg_L(y) = n - 1 - \deg_{\overline{L}}(y) \geqslant k$.

Now we prove (ii) It obviously suffices to prove the result in the case $|S| = 3$, so suppose this is the case. Let $\sigma = \sum_{x \in S} \deg_{\overline{L}}(x)$, let $c = \sum_{x \in S} \gamma(x)$. Since at most 2 edges of K_S are in $\overline{L}$ and each of the $u - c$ stars centred at a vertex not in S can contribute at most 3 to σ, we have that

$$\sigma \leqslant ck + 2 + 3(u-c) = c(k-3) + 2 + 3u \leqslant uk + 2 \leqslant 2n - 2 - k.$$

where the second last inequality follows by noting that $c \leqslant u$ and the last inequality follows by substituting (6.1). Now, by the pigeonhole principle there is a vertex $y \in S$ such that $\deg_{\overline{L}}(y) \leqslant \frac{\sigma}{3} \leqslant \frac{2n-2-k}{3}$ and the result follows using $\deg_L(y) = n - 1 - \deg_{\overline{L}}(y)$. □

Lemma 6.3.3. *Let k and n be positive integers such that $k \geqslant 2$ and $n \geqslant 2k - 5$, let V be a set of n vertices, and let $\mathcal{D}$ be a partial k-star decomposition of K_V with at most u stars, where*

$$u = \begin{cases} 2\lfloor \frac{n-2}{k} \rfloor - 1 & \textit{if } n \not\equiv 1 \pmod{k} \\ \frac{2(n-1)}{k} - 2 & \textit{if } n \equiv 1 \pmod{k}. \end{cases}$$

Let A be a subset of V such that $|A| \leqslant \frac{1}{3}(n-2k+5)$. There is a partial k-star decomposition $\mathcal{D} \cup \mathcal{D}_A$ of K_n with leave L_A such that each star in $\mathcal{D}_A$ is centred at a vertex in A, $L_A[A]$ is empty and $\deg_{L_A}(z) < k$ for each $z \in A$.

Proof. We will prove the result by induction on $|A|$. If $A = \emptyset$, then the result holds trivially by taking $\mathcal{D}_A = \emptyset$. Now let L be the leave of $\mathcal{D}$ and A' be a nonempty subset of V such that $|A'| \leqslant \frac{1}{3}(n - 2k + 5)$ and suppose inductively that the result holds for all subsets A of V with $|A| < |A'|$. Let z' be an element of A' such that $\deg_L(z') \geqslant \deg_L(z)$ for each $z \in A'$, let $A = A' \setminus \{z'\}$ and let $a = |A|$. By our inductive hypothesis there is a partial k-star decomposition $\mathcal{D} \cup \mathcal{D}_A$ of K_n with leave L_A such that each star in $\mathcal{D}_A$ is centred at a vertex in A, $L_A[A]$ is empty and $\deg_{L_A}(z) < k$ for each $z \in A$. Let $e = |E(L_A[A'])|$ and note that each edge of $L_A[A']$ is incident with z' and hence $e \leqslant a$. Also, since each star in $\mathcal{D}_A$ is centred at a vertex in A and no edge in $E(L_A[A'])$ is in a star in $\mathcal{D}_A$, we have $\deg_{L_A}(z') \geqslant \deg_L(z') - a + e$.

We claim that there exists a set $\mathcal{D}'$ of $\lfloor \frac{1}{k} \deg_{L_A}(z') \rfloor$ edge-disjoint k-stars in L_A such that each star in $\mathcal{D}'$ is centred at z' and every edge of $L_A[A']$ is in a star in $\mathcal{D}'$. The claim will be true provided that

$$k \left\lfloor \tfrac{1}{k} \deg_{L_A}(z') \right\rfloor \geqslant e. \tag{6.2}$$

If $e = 0$, and so in particular if $a = 0$, then (6.2) holds. If $a = e = 1$, then (6.2) holds because, by Lemma 6.3.2(i) with $S = A'$, we have $\deg_{L_A}(z') = \deg_L(z') \geqslant k$. If $a \geqslant 2$ and $e \geqslant 1$, then

$$\deg_{L_A}(z') \geqslant \deg_L(z') - a + e \geqslant \tfrac{1}{3}(n - 1 + k) - a + e \geqslant e + k - 1$$

where the second inequality follows by Lemma 6.3.2(ii) with $S = A'$, and the last follows because $a \leqslant \frac{1}{3}(n-2k+2)$ since $|A'| \leqslant \frac{1}{3}(n-2k+5)$. From this we can see that (6.2) holds. Thus a suitable set $\mathcal{D}'$ of stars does indeed exist.

Let $\mathcal{D}_{A'} = \mathcal{D}_A \cup \mathcal{D}'$. Then each star in $\mathcal{D}_{A'}$ is centred at a vertex in A' and $\mathcal{D} \cup \mathcal{D}_{A'}$ is a partial k-star decomposition of K_n. Furthermore, $L_{A'}[A']$ is empty and $\deg_{L_{A'}}(z) < k$ for each $z \in A$. □

Lemma 6.3.4. *Let $k \geqslant 2$ be an integer, let L_0 be a graph with vertex set V such that $|E(L_0)| \equiv 0 \pmod{k}$, let $\{A, B\}$ be a partition of V such that $L_0[A]$ is empty, and let $b = |B|$. Then L_0 has a k-star decomposition if*

(i) $\deg_{L_0}(z) \leqslant k-1$ *for each* $z \in A$,

(ii) $\deg_{L_0}(x) \geqslant \lceil \frac{1}{2}b \rceil + 2k - 2$ *for each* $x \in B$, *and*

(iii) $|E(\overline{L_0}[B])| < \frac{1}{2}\left(\lceil \frac{1}{2}b \rceil + k\right)\left(\lfloor \frac{1}{2}b \rfloor - 2k + 1\right) - (b-1)(k-1)$.

Proof. Suppose that $B = \{x_1, \ldots, x_b\}$. For each $i \in \{1, \ldots, b\}$, in order, we will define a set $\mathcal{D}_i$ of $\lceil \frac{1}{k}|N_{L_0}(x_i) \cap A| \rceil$ stars centred on x_i such that $\mathcal{D}_1 \cup \cdots \cup \mathcal{D}_i$ is a partial k-star decomposition of L_0 with a leave L_i such that

- $N_{L_i}(x_j) \cap A = \emptyset$ and $|N_{L_i}(x_j) \cap B| \geqslant \lceil \frac{1}{2}b \rceil + k - 1$ for each $j \in \{1, \ldots, i\}$;
- $|N_{L_i}(x_j) \cap B| \geqslant \lceil \frac{1}{2}b \rceil + 2k - 2$ for each $j \in \{i+1, \ldots, b\}$.

To see that we can do this, suppose that we have successfully defined such sets $\mathcal{D}_1, \ldots, \mathcal{D}_{i-1}$ for some $i \in \{1, \ldots, b\}$. We will show we can define a suitable set $\mathcal{D}_i$.

Let

$$S_i = \big\{x \in \{x_1, \ldots, x_{i-1}\} : |N_{L_{i-1}}(x) \cap B| = \lceil \tfrac{1}{2}b \rceil + k - 1\big\} \cup \\ \big\{x \in \{x_{i+1}, \ldots, x_b\} : |N_{L_{i-1}}(x) \cap B| = \lceil \tfrac{1}{2}b \rceil + 2k - 2\big\}.$$

Intuitively, S_i is the set of vertices which cannot be tail vertices of a star in $\mathcal{D}_i$ without violating the conditions we require of L_i. We will choose $\mathcal{D}_i$ so that every vertex in A that is adjacent to x_i in L_{i-1} is a tail vertex of a star of $\mathcal{D}_i$ and no vertex in S_i is a tail vertex of a star in $\mathcal{D}_i$. It is clear we can do this if $|(N_{L_{i-1}}(x_i) \cap B) \setminus S_i| \geqslant k-1$. In turn this inequality will hold if $|S_i| \leqslant \lceil \frac{1}{2}b \rceil + k - 1$, using $|N_{L_{i-1}}(x_i) \cap B| \geqslant \lceil \frac{1}{2}b \rceil + 2k - 2$. Observe that, for each $x \in S_i$, $|N_{\overline{L_{i-1}}}(x) \cap B| \geqslant \lfloor \frac{1}{2}b \rfloor - 2k + 1$ since $|N_{\overline{L_{i-1}}}(x) \cap B| = b - 1 - |N_{L_{i-1}}(x) \cap B|$. Therefore, $|S_i| < \lceil \frac{1}{2}b \rceil + k$ since

$$\begin{aligned} \sum_{x \in B} |N_{\overline{L_{i-1}}}(x) \cap B| &\leqslant \sum_{x \in B} |N_{\overline{L_0}}(x) \cap B| + 2(i-1)(k-1) \\ &= 2\big|E\big(\overline{L_0}[B]\big)\big| + 2(i-1)(k-1) \\ &< \left(\lceil \tfrac{1}{2}b \rceil + k\right)\left(\lfloor \tfrac{1}{2}b \rfloor - 2k + 1\right) \end{aligned}$$

where the first inequality is due to the fact that in total the k-stars in $\mathcal{D}_j$ have at most $k-1$ tail vertices in $B \setminus \{x_j\}$ for each $j \in \{1, 2, \ldots, i-1\}$, and the last inequality is obtained by first using $i \leqslant b$ and then using (iii). Since $|S_i|$ is an integer strictly less than $\lceil \frac{1}{2}b \rceil + k$, we in fact have $|S_i| \leqslant \lceil \frac{1}{2}b \rceil + k - 1$ as desired.

So we can indeed choose $\mathcal{D}_i$ so that every vertex in A that is adjacent to x_i in L_{i-1} is a tail vertex of a star of $\mathcal{D}_i$ and no vertex in S_i is a tail vertex of a star in $\mathcal{D}_i$. From this, it is not too hard to see that L_i satisfies the required conditions by observing that $|N_{L_i}(x_i) \cap B| \geqslant |N_{L_{i-1}}(x_i) \cap B| - k + 1$, that $|N_{L_i}(x) \cap B| = |N_{L_{i-1}}(x) \cap B| - 1$ for all $x \in B \setminus \{x_i\}$ that are tail vertices of stars in $\mathcal{D}_i$, and that $|N_{L_i}(x) \cap B| = |N_{L_{i-1}}(x) \cap B|$ for all $x \in B \setminus \{x_i\}$ that are not tail vertices of stars in $\mathcal{D}_i$. Remember that no vertex in S_i is a tail vertex of a star in $\mathcal{D}_i$.

So we can construct a k-star decomposition $\mathcal{D}_1 \cup \cdots \cup \mathcal{D}_b$ of L_0 with a leave L_b such that $E(L_b) = E(L_b[B])$ and $|N_{L_b}(x) \cap B| \geqslant \lceil \frac{1}{2}b \rceil + k - 1$ for each $x \in B$. Thus we can apply Theorem 6.3.1 to find a k-star decomposition $\mathcal{D}'$ of L_b. Then $\mathcal{D}_1 \cup \cdots \cup \mathcal{D}_b \cup \mathcal{D}'$ is a k-star decomposition of L_0. □

With Lemma 6.3.3 and Lemma 6.3.4 in hand, we are now in a position to prove the first part of Theorem 6.1.2 when $k \geqslant 3$.

Proof of Theorem 6.1.2. The case where $k = 2$ is covered by Lemma 6.2.2, so we may assume $k \geqslant 3$. The second part of the theorem has been proved in Lemma 6.2.1, so it only remains to prove the first part.

Let V be a set of n vertices, let $\mathcal{D}$ be a partial k-star decomposition of K_V having at most u stars and let L be its leave. Note that, since n is $K_{1,k}$-admissible, $|E(L)| \equiv 0 \pmod{k}$. We have, using (6.1),

$$\sum_{y \in V} \deg_{\overline{L}}(y) = 2|E(\overline{L})| \leqslant 2ku \leqslant 4n - 2k - 8. \tag{6.3}$$

Let $A = \{z \in V : \deg_{\overline{L}}(z) \geqslant \lfloor \frac{n}{2} \rfloor - 2k - 4\}$ and let $a = |A|$. Then, by (6.3),

$$a \leqslant \tfrac{4n-2k-8}{(\lfloor n/2 \rfloor - 2k - 4)} < 13$$

where the last inequality follows because $n \geqslant 11k + 20$ and $k \geqslant 3$. So we have $a \leqslant 12$ since a is an integer and hence $a \leqslant \frac{1}{3}(n - 2k + 5)$ because $n \geqslant 11k + 20$ and $k \geqslant 3$. If $A = \emptyset$, then $\deg_L(x) \geqslant n - 1 - (\lfloor \frac{n}{2} \rfloor - 2k - 5) > \frac{n}{2} + k - 1$ for all $x \in V$. Hence, by Theorem 6.3.1 a k-star decomposition of L exists. Therefore, assume that $A \neq \emptyset$.

By Lemma 6.3.3, because $a \leqslant \frac{1}{3}(n - 2k + 5)$, we can find a partial k-star decomposition $\mathcal{D} \cup \mathcal{D}_0$ of K_n with leave L_0 such that each star in $\mathcal{D}_0$ is centred at a vertex in A, $L_0[A]$ is empty and $\deg_{L_0}(z) < k$ for each $z \in A$. Note that $|E(L_0)| \equiv 0 \pmod{k}$ because n is $K_{1,k}$-admissible. It suffices to show that L_0 obeys conditions (i), (ii), (iii) of Lemma 6.3.4 because then we can apply Lemma 6.3.4 to obtain a k-star decomposition $\mathcal{D}_1$ of L_0 and $\mathcal{D} \cup \mathcal{D}_0 \cup \mathcal{D}_1$ will be a completion of $\mathcal{D}$.

We have seen that L_0 obeys (i) of Lemma 6.3.4. For all $x \in V \setminus A$,

$$\deg_{L_0}(x) \geqslant \deg_L(x) - a \geqslant n - 1 - \left(\left\lfloor \tfrac{n}{2} \right\rfloor - 2k - 5\right) - a = \left\lceil \tfrac{n}{2} \right\rceil + 2k + 4 - a$$

where the first inequality follows because a vertex in $V(L) \setminus A$ can be a tail vertex of at most a stars in $\mathcal{D}_0$, and the second follows by the definition of A. Therefore, we have $\deg_{L_0}(x) \geqslant \lceil \frac{1}{2}(n-a) \rceil + 2k - 2$ for each $x \in V \setminus A$ because $a \leqslant 12$ and $n \equiv n - a \pmod{2}$ if $a = 12$. So L_0 obeys (ii) of Lemma 6.3.4. Now, observe

$$|E(\overline{L_0}[V \setminus A])| = |E(\overline{L}[V \setminus A])| \leqslant |E(\overline{L})| \leqslant ku \leqslant 2n - k - 4$$

where the first equality holds because each k-star in $\mathcal{D}_0$ is centred at a vertex in A and the last inequality follows by (6.1). So to show that L_0 obeys (iii) of Lemma 6.3.4 and complete the proof, it is enough to show that Φ is positive, where

$$\Phi = \tfrac{1}{2}(\lceil \tfrac{1}{2}(n-a) \rceil + k)(\lfloor \tfrac{1}{2}(n-a) \rfloor - 2k + 1) - (n - a - 1)(k - 1) - (2n - k - 4).$$

Since $k > -2k+1$, a lower bound on Φ can be obtained by substituting $\frac{1}{2}(n-a+1)$ for $\lceil \frac{1}{2}(n-a) \rceil$ and $\frac{1}{2}(n-a-1)$ for $\lfloor \frac{1}{2}(n-a) \rfloor$. Doing this and then simplifying, we obtain

$$\begin{aligned} 8\Phi &\geqslant a(a-2n+10k-10)+n^2-10kn-6n-8k^2+14k+25 \\ &\geqslant n(n-10k-30)-8k^2+134k+49 \\ &\geqslant 3k^2+44k-151>0 \end{aligned}$$

where the second inequality is obtained using $a \leqslant 12$ (noting that 12 is less than the quadratic's stationary point of $n-5k+5$), the third is obtained using $n \geqslant 11k+20$, and the last is obtained using $k \geqslant 3$. So L_0 obeys (i), (ii) and (iii) of Lemma 6.3.4 and the proof is complete. $\square$

Chapter 7

Conclusion and future work

" It's delightful when your imaginations come true, isn't it?"

- L. M. Montgomery

This thesis makes progress on completion and embedding problems related to combinatorial designs; namely, completing partial $(n, k, 1)$-designs with very few blocks, the complexity of embedding of partial Steiner triple systems into systems of specified order, finding small order embeddings of partial k-star designs and completing partial k-star designs with few k-stars. The work here leaves open many avenues for further investigation and interesting unanswered problems. We conclude by suggesting some examples of these.

Our main result in Chapter 3 determines the minimum number of blocks in an uncompletable partial $(n, k, 1)$-design when n is sufficiently large. The work here leaves many avenues for further investigation. It would of course be desirable to establish results similar to ours for all n rather than simply for sufficiently large n. However, for general k, even the existence problem for $(n, k, 1)$-designs is only resolved for large n. Even for values of k where the existence problem is completely solved, such an improvement of our results would not be achievable using the techniques we have employed, due to their reliance on the decomposition results in [48].

One could also ask for results similar to Theorem 3.1.1 for partial (n, k, λ)-designs for $\lambda \geqslant 2$. It may be that the techniques used in Chapter 3 could be adapted to prove such results. As mentioned in Section 3.2, Theorems 3.1.2 and 3.1.3 are not necessarily tight for all k, and so there is the possibility of improving them for specific values of k. Further, one could attempt to prove results analogous to Corollary 3.1.5 for values of k other than 3. These last two possible goals may involve significant case analysis, however. Finally, Lemma 3.2.10 suggests the problem of investigating what conditions on the number of edges and number of triangles per edge of a graph are sufficient to guarantee that it has a K_k-decomposition.

The main result of Chapter 4 concerns embeddings of partial $(n, 3, 1)$-designs; more specifically the complexity of determining the existence of embeddings of small orders. Our result, Theorem 4.1.1 does not answer the following natural question.

Question 7.0.1. Is the problem of determining whether a given partial Steiner triple system of order u has an embedding of order $2u - 1$ NP-complete?

Removing the ϵ term from Theorem 4.1.1 would necessarily entail answering this question.

We then provide a family of counterexamples (Theorem 4.1.3) to Bryant's conjecture, Conjecture 4.1.2. Theorem 4.1.3 shows that the conditions of Conjecture 4.1.2 do not suffice for the existence of a K_3-decomposition of $L \vee K_w$. It remains possible, however, that a slightly strengthened set of conditions does suffice.

Question 7.0.2. Let L be a graph with u vertices and let w be a nonnegative integer. Do the conditions of Conjecture 4.1.2 with (4)(iii) replaced by $\Delta(G) \leqslant w-1$ guarantee the existence of a K_3-decomposition of $L \vee K_w$?

Of course, these new conditions are not necessary for the existence of a K_3-decomposition of $L \vee K_w$.

Chapter 5 considers the problem of when a partial k-star decomposition of K_n can be embedded in a k-star decomposition of K_{n+s} for a given integer s. The constants $\frac{9}{4}$ or $6-2\sqrt{2}$ in Theorem 5.1.1 are best possible for general k, but it may be possible to improve them for certain specific values of k. Moreover, in Theorem 5.1.2 we did not show that the lower bound on n is best possible, and there might be a possibility of slightly improving it. Furthermore, the lower bound on n in Theorem 5.1.3 is best possible when k is a large odd power of 2, but it may be worth investigating whether it can be reduced for other values of k.

Chapter 6 determines exactly the minimum number of k-stars in an uncompletable partial k-star decomposition of K_n when $n \geqslant 11k+20$. It is worth investigating how to further improve the bound on n. Maybe one could define a suitable k-precentral function and then use an approach based on Lemma 5.2.3. Moreover, it would be desirable to determine the maximum number of edges in a $K_{1,k}$-divisible graph that is not k-star decomposable.

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