Problems Related to Sperner Set Systems

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Abstract

In 1928, Sperner proved his well known result bounding the maximum size of a family of subsets of an n-set, where no set in the family is a subset of another. Such a family is now referred to as a Sperner set system. Since then, many families with similar structure have been considered in the literature.

In 2005, Meagher, Moura and Stevens investigated Sperner partition systems, where an (n, k)-Sperner partition system is a collection of partitions of an n-set into k nonempty parts with the property that no part in any partition is a subset of a part in a different partition. In this thesis, we present a number of constructions for (n, k)-Sperner partition systems for various regimes of n and k. These constructions produce systems with a number of partitions asymptotically close to an upper bound due to Meagher, Moura and Stevens. We also exactly determine the maximum number of partitions when $n = 3k - 6$ and for a number of small values of n and k.

We then move on to look at cross-Sperner families. A pair of families of subsets of an n-set is cross-Sperner if no set in either family is a subset of a set in the other family. We begin by examining self intersecting cross-Sperner pairs of families, where we add the condition that any pair of sets from the same family have nonempty intersection. Using a result of Erdős, Herzog and Schönheim, we exactly determine the maximum size of such a pair of families.

Finally, we consider the problem of determining the maximum size of a family in a pair of cross-Sperner families after fixing the size of the other family. Using a generalised form of the classical technique of shifting, we give a complete, albeit recursive, solution to this problem. This is accomplished through determining, for a family $\mathcal{F}$ of m subsets of an n-set, the minimum total number of subsets and supersets of sets in $\mathcal{F}$.

Declaration

This thesis contains no material which has been accepted for the award of any other degree or diploma at any university or equivalent institution and that, to the best of my knowledge and belief, this thesis contains no material previously published or written by another person, except where due reference is made in the text of the thesis.

Adam Gowty
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Publications During Enrolment

This thesis is a combination of several pieces of work (published, submitted and in preparation). Each of these works is joint work with various authors.

- Chapter 3 is based on a paper published in European Journal of Combinatorics [8]. This is joint work with Yanxun Chang, Charles J. Colbourn, Daniel Horsley and Junling Zhou.
- Chapter 4 is based on a paper published in Journal of Combinatorial Designs [17]. This is joint work with Daniel Horsley.
- Chapter 5 is based on joint work with Daniel Horsley, which is in preparation for publication.
- Chapter 6 is based on ongoing joint work with Daniel Horsley and Adam Mammoliti.

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Contents

Chapter 1

Introduction

Sperner's theorem is a simple result that has spawned a wide range of research. Since its proof, much work has been published that either strengthens the result, see [18, 34, 9] for examples, or generalises it to other families, such as in [25]. This is still an active area of research, primarily due to the fact that the requirement of nothing being "covered" or "superseded" is a very fundamental one that can be applied to a wide range of scenarios. In this thesis, we investigate two "Sperner-like" families from the literature, with the goal of finding bounds on the sizes of these families similar to Sperner's original result, and to find constructions for families that meet these bounds.

Chapter 2 provides the necessary background to our work. It begins with a brief discussion of Sperner set systems in Section 2.1 and then, in Section 2.2, it gives a discussion of existing work on Sperner partition systems which grounds our contributions in Chapters 3 and 4. Sections 2.3 and 2.5 then give context for our work on cross-Sperner families which appears in Chapters 5 and 6, while Section 2.4 introduces a number of well known results that will be used throughout the thesis.

We start our original work in Chapter 3, where we begin our investigation into Sperner partition systems. We focus on presenting a construction for Sperner partition systems that, in most cases, produces systems with size asymptotic to an upper bound due to Meagher, Moura and Stevens. We also find an improvement on this known upper bound, but we then go on to show that the two bounds are asymptotically equal. In Chapter 4 we continue our work related to Sperner partition systems and make progress in treating regimes of parameters not covered by our main result from Chapter 3. We also present a number of results for the special case where the average size of a part in a partition is between 2 and 3.

In Chapters 5 and 6, we move on to investigating cross-Sperner families. Chapter 5 focuses on pairs of cross-Sperner families where each family is, individually, an intersecting family. We find upper bounds on the size of these pairs of families under both an additive and multiplicative metric. We also give examples of constructions that show that these bounds are tight in all cases. In Chapter 6, we go on to consider the problem of maximising the size of a family in a cross-Sperner pair when the size of the other family is fixed. We find a recursive function that determines this maximum size in every case.

Chapter 2

Background

Throughout this thesis, for a positive integer n, we will use $[n]$ to denote the set $\{1, \dots, n\}$. For some finite set X, we let 2^X denote the power set of X (i.e. $2^X = \{S : S \subseteq X\}$). Furthermore, for an integer k, we will use $\binom{X}{k}$ to denote the family $\{S : S \subseteq X \text{ and } |S| = k\}$.

2.1 Sperner set systems

We begin by introducing the concept upon which this thesis is built: Sperner set systems, which were first introduced by Sperner in 1928 [40].

Definition 2.1.1. *A* Sperner set system $\mathcal{F} \subseteq 2^X$*, is a family of subsets of a finite ground set X for which no set in the family is a subset of any other.*

Throughout this thesis, whenever we require a ground set with n points in it, we will always use the set $[n]$ unless we state otherwise. We now provide the following examples of Sperner set systems.

Note that when Sperner set systems are considered from the perspective of hypergraphs, they are sometimes known as clutters. It is also common for them to be referred to as antichains.

Looking at Figure 2.1, it is natural to ask if it is possible to add new sets to any of these families to obtain a larger Sperner set system. It is clear that nothing can be added to the first example as $\emptyset$ is a subset of every set. For the second example, sets such as $\{3, 4\}$ could be added to achieve a larger Sperner set system. In his work, Sperner was able to determine the size of the largest possible Sperner set system on a ground set of size n.

Theorem 2.1.2 (Sperner's Theorem [40])**.** *Let $\mathcal{F} \subseteq 2^{[n]}$ be a Sperner set system for some positive integer n. Then $|\mathcal{F}| \leqslant \binom{n}{\lfloor \frac{n}{2} \rfloor}$ and we have equality if and only if $\mathcal{F} = \binom{[n]}{\frac{n}{2}}$ when n is even and $\mathcal{F} = \binom{[n]}{\frac{n-1}{2}}$ or $\mathcal{F} = \binom{[n]}{\frac{n+1}{2}}$ when n is odd.*

$\{\emptyset\}$	$\{1, 2, 3\}$	$\{1, 2, 3\}$
	$\{4, 5\}$	$\{1, 2, 4\}$
		$\{1, 2, 5\}$
		$\{1, 3, 4\}$
		$\{1, 3, 5\}$
		$\{1, 4, 5\}$
		$\{2, 3, 4\}$
		$\{2, 3, 5\}$
		$\{2, 4, 5\}$
		$\{3, 4, 5\}$

Figure 2.1: Three examples of Sperner set systems on 5 points.

Observe that this result shows that the third system given in Figure 2.1 is in fact one of the two Sperner set systems of maximum size on 5 points. While this result gives us a tight bound on the maximum size of a Sperner set system, it gives us no information about the possible structure of smaller systems. Instead we must look to a result that was proven independently by Bollobás([5]), Lubell ([30]), Meshalkin ([34]) and Yamamoto ([42]), which takes into account the size of each set in the system. This result is normally referred to as the LYM inequality, but is also known as the BLYM inequality.

Theorem 2.1.3 (LYM inequality [30, 34, 42])**.** *Suppose $\mathcal{F} \subseteq 2^{[n]}$ is a Sperner set system for some positive integer n. Then*

$$\sum_{F \in \mathcal{F}} \frac{1}{\binom{n}{|F|}} \leqslant 1.$$

In essence, this result shows that the more sets of small size or large size that occur in the Sperner set system, the fewer sets the system can have in total. We shall present proofs of both Theorem 2.1.2 and Theorem 2.1.3 in Section 2.4 once some more concepts related to set systems have been introduced.

2.2 Sperner partition systems

After seeing Sperner's theorem it is natural to question if analogous results can be obtained for objects similar to Sperner set systems. Of particular interest for this thesis are Sperner partition systems, which were introduced by Meagher, Moura and Stevens in 2005 ([33]) and then further investigated by Li and Meagher in 2013 ([28]). Before we define Sperner partition systems, we need to introduce a handful of definitions relating to partitions.

Definition 2.2.1. *A k-*partition *of $[n]$ is a collection of k pairwise disjoint, nonempty subsets of $[n]$, which we will refer to as classes, whose union is $[n]$.*

A k-partition of $[n]$ is called *uniform* if every class of the partition is of the same size. Note that a k-partition of $[n]$ can only be uniform if k divides n, in which case every class must have size n/k. When k does not divide n, we consider the concept of almost uniform k-partitions. A k-partition of $[n]$ is called *almost uniform* if every class of the partition has a size of $\lfloor n/k \rfloor$ or $\lceil n/k \rceil$.

Definition 2.2.2. *An (n, k)-*partition system *is a collection of distinct k-partitions of $[n]$.*

Let $\mathcal{P}_k^n$ denote the collection of all k-partitions of $[n]$. An (n, k)-partition system is called *almost uniform* if every partition in the system is almost uniform.

$1234 \mid 5$	$123 \mid 45$
	$124 \mid 35$
	$125 \mid 34$
	$134 \mid 25$
	$135 \mid 24$
	$145 \mid 23$
	$234 \mid 15$
	$235 \mid 14$
	$245 \mid 13$
	$345 \mid 12$

Figure 2.2: Examples of a 2-partition of $\{1, 2, 3, 4, 5\}$ (left) and of a $(5, 2)$-partition system (right). Observe that the 2-partition is not almost uniform, whereas each partition in the $(5, 2)$-partition system is. This means that the $(5, 2)$-partition system is also an almost uniform system.

We are now able to formally introduce the concept of a Sperner partition system as first defined in [33].

Definition 2.2.3. *An (n, k)-*Sperner partition system *$\mathcal{P} \subseteq \mathcal{P}_k^n$, is an (n, k)-partition system with the property that for any distinct $P, Q \in \mathcal{P}$, where $P = \{P_1, \ldots, P_k\}$ and $Q = \{Q_1, \ldots, Q_k\}$, $P_i \nsubseteq Q_j$ and $Q_i \nsubseteq P_j$ for all $i, j \in [k]$.*

Besides Sperner set systems, Sperner partition systems also have connections to other combinatorial objects. For example, in design theory, extensive research has been perfomed on resolvable block designs, which are block designs where it is possible group the blocks into resolution classes, where each resolution class is a partition of the point set of the design. Refer to [22] for good a good overview of the topic. It is natural to try to determine the maximum number of partitions in an (n, k)-Sperner partition system and thus produce an analogue of Sperner's theorem. Let $\mathrm{SP}(n, k)$ denote this maximum number.

As part of their work in [28], Li and Meagher presented two constructions for Sperner partition systems that take a Sperner partition system and extend it to one on a larger ground set. The first construction is one that uses an (n, k)-Sperner partition system to build an $(n + 1, k)$-Sperner partition system.

Lemma 2.2.4 ([28]). *For integers $n > k > 0$,*

$$\mathrm{SP}(n, k) \leqslant \mathrm{SP}(n + 1, k).$$

Proof. Begin with an (n, k)-Sperner partition system with $\mathrm{SP}(n, k)$ partitions, and augment it by adding a new element to an arbirary class in each partition in the system. The resulting system will be an $(n+1, k)$-Sperner partition system with $\mathrm{SP}(n, k)$ partitions. □

123|45|67	123|45|678
124|57|36	124|578|36
134|56|27	134|568|27
17|246|35	178|246|35

Figure 2.3: An example of using Lemma 2.2.4 on a $(7, 3)$-Sperner partition system (left) to construct an $(8, 3)$-Sperner partition system with the same number of partitions (right).

The other construction that they presented allows for the construction of an $(n + k, k)$-Sperner partition system from an (n, k)-Sperner partition system.

Theorem 2.2.5 ([28]). *For all integers $n > k \geqslant 1$,*

$$k \times \mathrm{SP}(n, k) \leqslant \mathrm{SP}(n + k, k).$$

Proof. It is possible to take k different orderings on the set $\{n+1, \ldots, n+k\}$ such that for any pair of orderings, no position has the same element in both orderings (such orderings can be obtained from an abitrary $k \times k$ Latin square for example). Take an (n, k)-Sperner partition system $\mathcal{P}$ that has $\mathrm{SP}(n, k)$ partitions, and order the classes of each partition. For each partition in $\mathcal{P}$ and each of our k chosen orderings of $\{n + 1, \ldots, n + k\}$, add the ith element of the ordering to the ith class of the partition. Since the added elements are not in $[n]$, it can be seen that this results in an $(n + k, k)$-Sperner partition system with $k \times \mathrm{SP}(n, k)$ partitions. □

From here on in this chapter, as well as in Chapters 3 and 4, we will simplify some of our notation by letting $c(n, k)$ and $r(n, k)$ be the unique integers such that, for integers $n \geqslant k > 1$, $n = ck + r$ and $r \in \{0, \ldots, k - 1\}$. Furthermore, we will generally only refer to them as c and r respectively when n and k are obvious.

123|45|67	8	9	A	1238|459|67A
	9	A	8	1239|45A|678
	A	8	9	123A|458|679
124|57|36	8	9	A	1248|579|36A
	9	A	8	1249|57A|368
	A	8	9	124A|578|369
134|56|27	8	9	A	1348|569|27A
	9	A	8	1349|56A|278
	A	8	9	134A|568|279
17|246|35	8	9	A	178|2469|35A
	9	A	8	179|246A|358
	A	8	9	17A|2468|359

Figure 2.4: An example of using a $(7,3)$-Sperner partition system with 4 partitions (left) to construct an $(10,3)$-Sperner partition system with the 12 partitions (right) using the construction in Theorem 2.2.5. The middle column denotes the ordering of the set $\{8,9,A\}$ as called for in the construction, where A is used in place of the number 10.

After these constructive lower bounds on $\mathrm{SP}(n,k)$, upper bounds are also of great interest. In [33], Meagher, Moura and Stevens proved the following upper bound on the size of a Sperner partition system through the use of the LYM inequatity.

Theorem 2.2.6 ([33]). *For integers n,k,c,r such that $n > k > r \geqslant 0$, $k \geqslant 2$ and $n = ck+r$, $\mathrm{SP}(n,k) \leqslant \mathrm{MMS}(n,k)$ where*

$$\mathrm{MMS}(n,k) = \frac{\binom{n}{c}}{k - r + \frac{r(c+1)}{n-c}}.$$

Proof. Assume $k \geqslant 2$ and let $\mathcal{P} \subseteq \mathcal{P}_k^n$ be an (n,k)-Sperner partition system. Let $\mathcal{A}$ be the Sperner set system formed by taking the collection of all the classes in all of the partitions in $\mathcal{P}$. Note that $|\mathcal{A}| = k|\mathcal{P}|$. By the LYM Inequality (see Theorem 2.1.3), we have that

$$\sum_{S\in\mathcal{A}} \frac{1}{\binom{n}{|S|}} \leqslant 1.$$

Let p_i, for $i \in [n]$, denote the number of sets in $\mathcal{A}$ with size i. It follows that

$$\sum_{i=1}^{n} \frac{p_i}{\binom{n}{i}} \leqslant 1. \tag{2.1}$$

By defining the function $f(i) = \binom{n}{i}^{-1}$ for $i \in [n]$, as well as dividing equation (2.1) through by $|\mathcal{A}|$, we obtain

$$\sum_{i=1}^{n} \frac{p_i}{|\mathcal{A}|} f(i) \leqslant \frac{1}{|\mathcal{A}|}. \tag{2.2}$$

Observe that since $\mathcal{P}$ is a collection of k-partitions on an n-set and $\mathcal{A}$ is formed by taking the collection of all classes of the partitions in $\mathcal{P}$,

$$\sum_{i=1}^{n} ip_i = \sum_{S\in\mathcal{A}} |S| = n|\mathcal{P}| = n\frac{|\mathcal{A}|}{k}. \tag{2.3}$$

It is also possible to extend $f(i)$ from the integers $i \in [n]$ to the real numbers $1 \leqslant x \leqslant n$ by letting

$$f(x) = \begin{cases} f(n), & \text{if } x = n \\ (1-u)f(i) + uf(i+1), & \text{where } x = i+u, \text{ for } i \in [n-1] \text{ and } 0 \leqslant u < 1. \end{cases}$$

Furthermore, as proven in [32], it is known that this extension of $f(i)$ is convex. Thus we can see that

$$f\left(\frac{n}{k}\right) = f\left(\sum_{i=1}^{n} i\frac{p_i}{|\mathcal{A}|}\right) \leqslant \sum_{i=1}^{n} f(i)\frac{p_i}{|\mathcal{A}|} \leqslant \frac{1}{|\mathcal{A}|}, \tag{2.4}$$

with the equality coming from (2.3), the first inequality from the fact that right hand side is a convex combination of convex functions due to the fact that $\sum_{i=1}^{n} p_i/|\mathcal{A}| = 1$, and the final inequality from (2.2).

From the definition of $f(i)$ and its extension to the real numbers, we have that

$$f\left(\frac{n}{k}\right) = f\left(\frac{ck+r}{k}\right) = f\left(c+\frac{r}{k}\right) = \left(1-\frac{r}{k}\right)\binom{n}{c}^{-1} + \frac{r}{k}\binom{n}{c+1}^{-1}.$$

Thus from (2.4), we have that

$$|\mathcal{A}| \leqslant \frac{1}{\left(1-\frac{r}{k}\right)\binom{n}{c}^{-1} + \frac{r}{k}\binom{n}{c+1}^{-1}} = \frac{k}{k-r+\frac{r(c+1)}{n-c}}\binom{n}{c},$$

and therefore

$$|\mathcal{P}| \leqslant \frac{\binom{n}{c}}{k-r+\frac{r(c+1)}{n-c}}.$$

□

When considering the form of the bound $MMS(n,k)$ it is worth bearing in mind that an almost uniform (n,k)-Sperner partition system can have at most $\binom{n}{c}/(k-r)$ partitions because each partition must contain $k-r$ classes of size c. In fact Meagher, Moura and Stevens showed that in the case of $n = ck$, in other words when k divides n, it is actually possible to meet this bound. This is proved by combining Baranyai's theorem (see [2]) with the bound $\mathrm{MMS}(n,k)$.

Theorem 2.2.7 ([33])**.**

$$\mathrm{SP}(ck,k) = \mathrm{MMS}(ck,k) = \frac{1}{k}\binom{ck}{c}.$$

This leads to the question of whether $\mathrm{SP}(n,k)$ can be exactly determined for other values of n and k, which was investigated in the work done by Li and Meagher in [28]. In particular, the exact value of $\mathrm{SP}(n,k)$ has been determined in the following cases:

Lemma 2.2.8 ([28])**.** *For nonnegative integers n, c, k, r such that $n = ck + r$, and $0 \leqslant r \leqslant k-1$.*

(i) If $k = 1$ or $c = 1$, then $\mathrm{SP}(n,k) = 1$.

(ii) If $k = 2$, then $\mathrm{SP}(n,k) = \binom{2c+r-1}{c-1}$.

(iii) If k is even, $c = 2$, and $r = 1$, then $\mathrm{SP}(n,k) = 2k$.

Note that Lemma 2.2.8(i) comes as a direct result of the fact that requiring $k = 1$ or $c = 1$ results in partitions with either size n or 1 respectively, and Lemma 2.2.8(ii) follows from the Erdős-Ko-Rado theorem, which we introduce in Theorem 2.4.3 (observing that since $k = 2$, $r \in \{0,1\}$).

In some other cases, specifically some in which $c = 2$, Li and Meagher improved the previously known bounds on $\mathrm{SP}(n,k)$. These bounds are as follows:

Lemma 2.2.9 ([28])**.** *For nonnegative integers n, c, k, r such that $n = ck + r$, and $0 \leqslant r \leqslant k-1$.*

(i) If $r = 1$ and $c = 2$, $\mathrm{SP}(n,k) \leqslant 2k$.

(ii) If $k \geqslant 3$, $r = 2$ and $c = 2$, $2k+1 \leqslant \mathrm{SP}(n,k) \leqslant 2k+3$.

(iii) If $r = k-1$ and $c = 2$, $3k-1 \leqslant \mathrm{SP}(n,k)$.

2.3 Pairs of Families with Cross Properties

There is also a significant literature on pairs of families of sets with certain restrictions on how sets from one family relate to sets in the other. In this section we discuss the aspects of this literature that are most relevant to our work in Chapters 5 and 6. A pair $(\mathcal{F},\mathcal{G})$ of families, where $\mathcal{F},\mathcal{G} \subseteq 2^{[n]}$, is said to be *cross-intersecting* if $F \cap G \neq \emptyset$ for all $F \in \mathcal{F}$ and $G \in \mathcal{G}$, and it is said to be *cross-Sperner* if $F \nsubseteq G$ and $G \nsubseteq F$ for all $F \in \mathcal{F}$ and $G \in \mathcal{G}$. Our interest here is mostly focussed on cross-Sperner pairs of families and on cross-intersecting pairs of Sperner set systems. However we first briefly discuss the literature on cross-intersecting pairs of families, which were considered much earlier.

In 1967, Hilton and Milner [21] showed that if $(\mathcal{F}, \mathcal{G})$ is a cross-intersecting pair of nonempty families of k-subsets of $[n]$, then $|\mathcal{F}| + |\mathcal{G}| \leqslant \binom{n}{k} - \binom{n-k}{k} + 1$. That this bound is tight can be seen by taking $\mathcal{F}$ to contain a single k-set and $\mathcal{G}$ to be the family of all k-sets that intersect this k-set. In 1968 Kleitman [26] showed that if $\mathcal{F}$ is a family of k-subsets of $[n]$, $\mathcal{G}$ is a family of ℓ-subsets of $[n]$, $(\mathcal{F}, \mathcal{G})$ is a cross-intersecting, $k+\ell \leqslant n$, and $|\mathcal{F}| \geqslant \binom{n-1}{k-1}$, then $|\mathcal{G}| \leqslant \binom{n-1}{\ell-1}$. This result can be seen to be tight by fixing a point and taking $\mathcal{F}$ and $\mathcal{G}$ to be the families of all k-sets and ℓ-sets, respectively, containing the fixed point. In 2017, Frankl and Kupavskii extended both of these results. In [14] they proved an analogue of the result of [21] for the case where it is required that $|F \cap G| \geqslant s > 0$ for each $F \in \mathcal{F}$ and $G \in \mathcal{G}$. In this case, they showed that the largest value of $|\mathcal{F}| + |\mathcal{G}|$ is achieved by taking $\mathcal{F}$ to contain a single k-set and $\mathcal{G}$ to be the family of all k-sets that intersect this k-set in at least s elements. In [13], they extended the result of [26] in the special case where all sets in both families have the same size k, showing that if $|\mathcal{F}|$ is required to be larger than $\binom{n-1}{k-1} + \binom{n-i}{k-i+1}$ for some $3 \leqslant i \leqslant k+1$, then $|\mathcal{G}| \leqslant \binom{n-1}{k-1} - \binom{n-i}{k-1}$.

More directly relevant to our interests here is the work of Gerbner et al. in [16] which considers cross-Sperner families. They begin by bounding the maximum size of $|\mathcal{F}| + |\mathcal{G}|$ for a cross-Sperner pair. They first note that the problem is trivial if one of the families is allowed to be empty, as then the other family can be taken to be $2^{[n]}$. Considering the nontrivial case where neither family is empty, they were able to prove the following result.

Theorem 2.3.1 ([16])**.** *There exists an integer n_0 such that if $n \geqslant n_0$ and $\mathcal{F}, \mathcal{G} \subseteq 2^{[n]}$ are non empty and form a cross-Sperner pair, then $|\mathcal{F}| + |\mathcal{G}| \leqslant 2^n - 2^{\lfloor n/2 \rfloor} - 2^{\lceil n/2 \rceil} + 2$.*

They also proved that there is a unique way to attain equality in this bound, which is for either $\mathcal{F}$ or $\mathcal{G}$ to consist of exactly one set S of size $\lfloor n/2 \rfloor$ or $\lceil n/2 \rceil$ and the other family to consist of all subsets of $[n]$ that are neither supersets nor subsets of S. Of particular relevance to us here is that they observe that for a fixed $\mathcal{F}$, to maximise the size of $\mathcal{G}$, one should take all sets that are not supersets or subsets of sets in $\mathcal{F}$. This leads to a direct equivalence to the isoperimetric problem on the Boolean lattice graph, which we will introduce properly in Section 2.5.

Gerbner et al. then went on to give an upper bound on $|\mathcal{F}||\mathcal{G}|$. Their proof relies on the following lemma.

Lemma 2.3.2 ([16])**.** *Let n be a positive integer and let $\mathcal{F}, \mathcal{G} \subseteq 2^{[n]}$. If $(\mathcal{F}, \mathcal{G})$ is a cross-Sperner pair, then the families $\mathcal{F}$, $\mathcal{G}$, $\mathcal{F} \wedge \mathcal{G}$, and $\mathcal{F} \vee \mathcal{G}$ are pairwise disjoint, where $\mathcal{F} \wedge \mathcal{G} = \{F \cap G : F \in \mathcal{F}$ and $G \in \mathcal{G}\}$ and $\mathcal{F} \vee \mathcal{G} = \{F \cup G : F \in \mathcal{F}$ and $G \in \mathcal{G}\}$.*

This lemma allows for the easy application of the following corollary of the Ahlswede and Daykin inequality, also commonly known as the four functions theorem, which is a powerful result that applies to all distributive lattices (see [1] for the original work by Ahlswede and Daykin).

Lemma 2.3.3 ([1]). *Let $\mathcal{F}, \mathcal{G} \subseteq 2^{[n]}$ for some integer n. Then*

$$|\mathcal{F}||\mathcal{G}| \leqslant |\mathcal{F} \wedge \mathcal{G}||\mathcal{F} \vee \mathcal{G}|.$$

With these two results, Gerbner et al. proved the following upper bound on $|\mathcal{F}||\mathcal{G}|$.

Lemma 2.3.4 ([16]). *Let $n \geqslant 2$ be an integer and let $\mathcal{F}, \mathcal{G} \subseteq 2^{[n]}$. If $(\mathcal{F}, \mathcal{G})$ is a cross-Sperner pair, then $|\mathcal{F}||\mathcal{G}| \leqslant 2^{2n-4}$.*

An extremal construction that meets this bound is if $\mathcal{F}$ is taken to be the collection of all sets containing 1 and not n, and $\mathcal{G}$ is taken to be the collection of all sets containing n and not 1.

Also of interest to us are cross-intersecting pairs of Sperner set systems. Pyber first examined these families in 1986 in [37], where he focused on bounding $|\mathcal{F}||\mathcal{G}|$ when upper bounds are placed on both the size of sets in $\mathcal{F}$ and the size of sets in $\mathcal{G}$.

Theorem 2.3.5 ([37]). *Let n, k and ℓ be positive integers and let $\mathcal{F} \subseteq 2^{[n]}$ and $\mathcal{G} \subseteq 2^{[n]}$ be Sperner set systems such that $|F \cap G| > 0$, $|F| \leqslant k$ and $|G| \leqslant \ell$ for all $F \in \mathcal{F}$ and $G \in \mathcal{G}$.*

1 *If $k > \ell$ and $2k + \ell - 2 \leqslant n$, then $|\mathcal{F}||\mathcal{G}| \leqslant \binom{n-1}{k-1}\binom{n-1}{\ell-1}$.*
2 *If $k = \ell$ and $2k \leqslant n$, then $|\mathcal{F}||\mathcal{G}| \leqslant \binom{n-1}{k-1}^2$.*

He also showed that this bound is tight in both cases by fixing a point $x \in [n]$ and letting $\mathcal{F}$ be the family of all k-sets that contain x and $\mathcal{G}$ be the family of all ℓ-sets that contain x.

Wong and Tay later considered the problem in [41], focusing on bounding $|\mathcal{F}| + |\mathcal{G}|$ when no restriction is placed on on the sizes of sets.

Theorem 2.3.6 ([41]). *If $n \geqslant 3$ is an integer and $\mathcal{F}, \mathcal{G} \subseteq 2^{[n]}$ are Sperner set systems such that $|F \cap G| > 0$ for all $F \in \mathcal{F}$ and $G \in \mathcal{G}$, then*

$$|\mathcal{F}| + |\mathcal{G}| \leqslant \begin{cases} 2\binom{n}{\lceil n/2 \rceil} & \text{if } n \text{ is odd} \\ \binom{n}{n/2} + \binom{n}{n/2+1} & \text{if } n \text{ is even.} \end{cases} \tag{2.5}$$

Furthermore there are Sperner set systems $\mathcal{F}, \mathcal{G} \subseteq 2^{[n]}$ such that $|F \cap G| > 0$ for all $F \in \mathcal{F}$ and $G \in \mathcal{G}$ and for which equality holds in (2.5).

Like Pyber, they were also able to show that their bounds were tight. When n is odd, this is be achieved by letting $\mathcal{F}$ and $\mathcal{G}$ both be the collection of all $\lceil n/2 \rceil$-sets. When n is even, this is achieved by letting $\mathcal{F}$ be the collection of all $(n/2)$-sets and $\mathcal{G}$ be the collection of all $(n/2+1)$-sets.

2.4 Fundamental Results for Finite Set Systems

We now take a step back and introduce a number of concepts and results related to set systems which will be of use throughout the thesis. We begin by introducing circular permutations, which will allow us to give a proof for two fundamental results: the Erdős-Ko-Rado theorem and the LYM inequality. The latter of these two results will then in turn lead to a proof of Sperner's Theorem.

Definition 2.4.1. *A* circular permutation *of* $[n]$ *is an n-tuple of the form* $(a_1, \dots, a_n)$*, such that* $\{a_1, \dots, a_n\} = [n]$*. All subscripts will be treated modulo n, and furthermore, we will consider two circular permutations of $[n]$ to be identical if for some integer s, $a_i = b_{i+s}$ for all $i \in [n]$.*

Note that, for example, $(1,2,3,4,5)$, $(2,3,4,5,1)$ and $(5,1,2,3,4)$ are all the same circular permutation of $\{1,2,3,4,5\}$. Also note that there are $(n-1)!$ distinct circular permutations of $[n]$.

Definition 2.4.2. *An* arc *of length k of a circular permutation $\pi = (a_1, \dots, a_n)$, where $k < n$, is a set of k elements of $[n]$ which appear consecutively in π.*

So an arc of length k can be written uniquely as $\{a_r, ..., a_{r+k-1}\}$ for some $r \in [n]$. We call a_r the *starting point* of the arc and a_{r+k-1} the *ending point* of the arc. As an example, for the circular permutation $\pi = (2,4,1,5,3)$, the set $\{5,3,2\}$ is an arc of length 3 in π, and we say that it has the starting point 5 and the ending point 2. For a circular permutation of $[n]$, π, let $\mathcal{C}(\pi)$ denote the family of all arcs of π.

We shall now state and prove the Erdős-Ko-Rado theorem, which is a result that bounds the maximum size of a family of pairwise intersecting sets that are all the same size. We have already mentioned that this theorem was used to establish a result on Sperner partition systems, and it will see further use later in this thesis.

Theorem 2.4.3 (Erdős-Ko-Rado Theorem [10])**.** *Let n and k be positive integers such that $0 < k \leqslant n/2$, and let $\mathcal{F} \subseteq \binom{[n]}{k}$ where for all $F, G \in \mathcal{F}$, $|F \cap G| > 0$. Then $|\mathcal{F}| \leqslant \binom{n-1}{k-1}$.*

Proof. Consider an arbitrary cyclic permutation π of $[n]$. Observe that at most k arcs of length k in π can be in $\mathcal{F}$. This can be seen by considering an arbitrary arc $F =$

$\{a_r, a_{r+1}, \ldots, a_{r+k-1}\} \in C(\pi)$ such that $F \in \mathcal{F}$. Observe that for any arc of length k in $\mathcal{C}(\pi)$ that intersects F, exactly one of its starting or ending points is in F. Furthermore, since $k \leqslant n/2$, for each $i \in \{r, \ldots, r+k-1\}$ it cannot be the case that a_i is the ending point of an arc in $\mathcal{F}$ and a_{i+1} is the starting point of another arc in $\mathcal{F}$ as the two sets do not intersect. So for each $i \in \{r, \ldots, r+k-1\}$, either a_{i+1} is the starting point of an arc in $\mathcal{F}$, a_i is the ending point of an arc in $\mathcal{F}$, or neither is the case. Also note that F will be the unique arc in $C(\pi)$ with starting point a_r and ending point a_{r+k-1}. Observing this, we see that there are at most k arcs of length k in $C(\pi)$ that are also in $\mathcal{F}$.

We now proceed to count the number of pairs (π, F) where π is a cyclic permutation on $[n]$ of which F is an arc that appears in $\mathcal{F}$. To begin, for each $F \in \mathcal{F}$, there are $k!$ ways to order the elements in F, and $(n-k)!$ ways to order the elements of $[n] \setminus \mathcal{F}$. This means that there are $k!(n-k)!$ choices for π and thus $|\mathcal{F}|k!(n-k)!$ possible (π, F) pairs. If we instead first consider an arbitrary π, of which there are $(n-1)!$, it is shown above that each one can contain at most k sets from $\mathcal{F}$ which could be chosen as F. So it follows that there are at most $k(n-1)!$ pairs of the form (π, F). Thus we see that,

$$\begin{aligned} |\mathcal{F}|k!(n-k)! &\leqslant k(n-1)! \\ |\mathcal{F}| &\leqslant \binom{n-1}{k-1}. \end{aligned} \qquad \square$$

It is of interest to note that equality in the bound can be achieved for each $k \leqslant n/2$ by taking all k-sets that contain a fixed element. When $k = n/2$, equality can also be achieved by taking $\mathcal{F}$ to consist of all k-sets that do not contain a fixed element.

Figure 2.5: Examples of constructions meeting the bound in the Erdős-Ko-Rado theorem for $n = 4$ and $k = 2$.

It is important to note that while cases where $k > n/2$ are not covered by the Erdős-Ko-Rado theorem, they are trivially dealt with due to the fact that any pair of k-sets are guaranteed to intersect, and thus any subfamily of $\binom{[n]}{k}$ is guaranteed to be an intersecting family. So in these cases, the family $\binom{[n]}{k}$ is the extremal example.

We now look how Sperner set systems can be embedded into circular permutations, which is the final piece we need to prove the LYM inequality.

Lemma 2.4.4. *Consider a circular permutation $\pi = (a_1, \ldots, a_n)$ of $[n]$, and a Sperner set system $\mathcal{F} \subseteq \mathcal{C}(\pi)$. Then $|\mathcal{F}| \leqslant n$. If $|\mathcal{F}| = n$, then all sets in $\mathcal{F}$ are the same size.*

Proof. Consider an arbitrary a_i in π. If a_i is the ending point of two arcs in $\mathcal{F}$, then it is clear that the shorter arc must be a subset of the longer arc. As such, every point can be the ending point of at most one arc in $\mathcal{F}$ and hence $|\mathcal{F}| \leqslant n$.

Now assume $|\mathcal{F}| = n$, but not all arcs in $\mathcal{F}$ are of the same size. For this to be true, each a_i must be the ending point of a distinct arc. Consider an i such that the arc ending in a_{i+1} is longer than the arc ending in a_i. Clearly the second arc is a subset of the first, causing a contradiction. $\square$

With this lemma, we now have everything we need for a proof of the LYM inequality. This proof is originally due to Lubell [30].

Lubell's Proof of Theorem 2.1.3. ([30]) Let $\mathcal{F} \subseteq 2^{[n]}$ be a Sperner set system for some positive integer n. We proceed to prove the result by counting the number of pairs (F, π), where $F \in \mathcal{F}$ and π is a circular permutation of $[n]$ such that F appears in $\mathcal{C}(\pi)$.

For each $F \in \mathcal{F}$, we want to count the number of cyclic permutations that have F as an arc. There are $|F|!$ ways to order F, and for each of these, there are $(n - |F|)!$ ways to build the remainder of the cyclic permutation such that $\mathcal{F}$ appears as an arc. As such, there are $\sum_{F \in \mathcal{F}} |F|!(n - |F|)!$ pairs of the form (F, π). On the other hand, there are $(n-1)!$ circular permutations of $[n]$, and by Lemma 2.4.4, each can have at most n arcs in $\mathcal{F}$. As such there are at most $n!$ pairs of the form (F, π).

Combining these two facts, we see that

$$\sum_{F \in \mathcal{F}} |F|!(n - |F|)! \leqslant n!,$$

and as such

$$1 \geqslant \sum_{F \in \mathcal{F}} \frac{|F|!(n - |F|)!}{n!} = \sum_{F \in \mathcal{F}} \frac{1}{\binom{n}{|F|}}.$$

$\square$

As a consequence of using Lemma 2.4.4, we further note that equality in the LYM inequality can only be achieved by having all sets in $\mathcal{F}$ be the same size. Using the LYM inequality, we are able to give a simple proof of the bound given by Sperner's theorem.

Sperner's theorem as a consequence of the LYM inequality. Let $\mathcal{F} \subseteq 2^{[n]}$ be a Sperner set system. Due to the fact that $\binom{n}{k}$ is maximised when $k = \lfloor n/2 \rfloor$ or $\lceil n/2 \rceil$, it is clear that, by the LYM inequality,

$$\frac{|\mathcal{F}|}{\binom{n}{\lfloor \frac{n}{2} \rfloor}} = \sum_{S \in \mathcal{F}} \frac{1}{\binom{n}{\lfloor \frac{n}{2} \rfloor}} \leqslant \sum_{S \in \mathcal{F}} \frac{1}{\binom{n}{|S|}} \leqslant 1.$$

As such $|\mathcal{F}| \leqslant \binom{n}{\lfloor n/2 \rfloor}$, with equality only being reached if $|S| = \lfloor n/2 \rfloor$ or $\lceil n/2 \rceil$ for all $S \in \mathcal{F}$. When n is even, this gives us that $\mathcal{F} = \binom{[n]}{n/2}$. When n is odd, we observe that equality can only occur if there is equality in the LYM inequality and therefore, by our comment after the proof of the LYM inequality, can only be achieved when all sets are the same size, and thus $\mathcal{F} = \binom{[n]}{\lfloor n/2 \rfloor}$ or $\binom{[n]}{\lceil n/2 \rceil}$. □

Note that this is not the original method for proving Sperner's theorem; the original is a more complex proof that relies heavily on the concept of shadows and shades. While we won't present this more complex proof in this thesis (interested readers can find it in [40]), it is still useful to introduce both shadows and shades as they inform some of our work in Chapters 5 and 6.

Definition 2.4.5. *The* shadow *of a family of sets $\mathcal{F} \subseteq 2^{[n]}$, denoted by $\Delta(\mathcal{F})$ is the family of sets $\{G \in 2^{[n]} :$ for some $F \in \mathcal{F}, G \subseteq F$ and $|F \setminus G| = 1\}$.*

Definition 2.4.6. *The* shade *of a family of sets $\mathcal{F} \subseteq 2^{[n]}$, denoted by $\nabla(\mathcal{F})$ is the family of sets $\{G \in 2^{[n]} :$ for some $F \in \mathcal{F}, F \subseteq G$ and $|G \setminus F| = 1\}$.*

Note that unlike with shadows, the shade of a family implicitly depends on the ground set that the family is defined on. Also note that shadows and shades are also referred to as down-shadows and up-shadows in the literature.

$\mathcal{F}$	$\Delta\mathcal{F}$	$\nabla\mathcal{F}$
1	$\emptyset$	$12, 13, 14, 15$
$12, 34$	$1, 2, 3, 4$	$123, 124, 125, 234, 345, 134$
$123, 124, 345$	$12, 23, 13, 14, 24, 34, 35, 45$	$1234, 1235, 1245, 1345, 2345$

Table 2.1: Examples of shades and shadows for different families on the ground set $\{1, 2, 3, 4, 5\}$.

Thinking back to Sperner set systems, it is apparent that no set that appears in the shadow or shade of a family can be added to the family while maintaining the Sperner property. So in this context, it is useful to ask how to minimise the size of the shade or the shadow. To begin this work, we introduce the concept of shifting. For a family $\mathcal{F} \subseteq 2^{[n]}$, and a set $F \in \mathcal{F}$ we define $S_{ij}(F)$, for integers $i, j \in [n]$, as follows:

$$S_{ij}(F) = \begin{cases} (F \setminus \{j\}) \cup \{i\}, & \text{if } i \notin F, j \in F \text{ and } (F \setminus \{j\}) \cup \{i\} \notin \mathcal{F}, \\ F, & \text{otherwise.} \end{cases} \tag{2.6}$$

Using this, we then define the shift of the family, $S_{ij}(\mathcal{F})$ to be

$$S_{ij}(\mathcal{F}) = \{S_{ij}(F) : F \in \mathcal{F}\}. \tag{2.7}$$

When performing the shift of a family $\mathcal{F}$, for a set $F \in \mathcal{F}$, we say that F shifts if $S_{ij}(F) \neq F$ and that F is fixed if $S_{ij}(F) = F$. For an example of shifting, consider $S_{1,3}(\{\{1\}, \{2,3,4\}, \{2,3\}, \{1,2\}\})$. The sets $\{1\}$ and $\{1,2\}$ do not shift as they do not contain 3, the set $\{2,3,4\}$ shifts to $\{1,2,4\}$, and the set $\{2,3\}$ cannot shift, as it would shift to $\{1,2\}$ which is already in the family. As such, $S_{1,3}(\{\{1\}, \{2,3,4\}, \{2,3\}, \{1,2\}\}) = \{\{1\}, \{1,2,4\}, \{2,3\}, \{1,2\}\}$.

If $i < j$, we refer to $S_{ij}(\mathcal{F})$ as a *left shift*. If $i > j$, we refer to $S_{ij}(\mathcal{F})$ as a *right shift*. One fundamental fact about shifts is that taking the shift of any family will result in a family of the same size, a fact which we will now prove.

Proposition 2.4.7. *For $\mathcal{F} \subseteq 2^{[n]}$ and any $i, j \in [n]$, $|S_{ij}(\mathcal{F})| = |\mathcal{F}|$.*

Proof. Suppose for a contradiction that there exist $F_1, F_2 \in \mathcal{F}$ such that $F_1 \neq F_2$ but $S_{ij}(F_1) = S_{ij}(F_2)$. For this to occur, either one set shifts onto the other which remains fixed, or both sets shift to the same set. Without loss of generality, assume that F_1 shifts, i.e. that $i \notin F_1, j \in F_1$ and $(F_1 \setminus \{j\}) \cup \{i\} \notin \mathcal{F}$, and thus $(F_1 \setminus \{j\}) \cup \{i\} \in S_{ij}(\mathcal{F})$.

It cannot be that F_2 remains fixed, as if $S_{ij}(F_1) = S_{ij}(F_2) = F_2$, then $F_2 = (F_1 \setminus \{j\}) \cup \{i\}$, which contradicts the necessary conditions for F_1 being shifted. If instead both F_1 and F_2 were shifted, then $i \notin F_1, j \in F_1, i \notin F_2, j \in F_2$ and $(F_1 \setminus \{j\}) \cup \{i\} = (F_2 \setminus \{j\}) \cup \{i\}$. So $F_1 = F_2$, which is a contradiction. Thus it is apparent that the map $S_{ij} : \mathcal{F} \to S_{ij}(\mathcal{F})$ is a bijection, and so $|\mathcal{F}| = |S_{ij}(\mathcal{F})|$. □

We will refer to a family $\mathcal{F} \subseteq 2^{[n]}$ as being *left shifted* if for all $1 \leqslant i < j \leqslant n$, $S_{ij}(\mathcal{F}) = \mathcal{F}$. Furthermore, we often abbreviate left shifted to just *shifted*.

Shifting, and in particular shifted families, are an invaluable tool when dealing with problems related to shades and shadows. This is primarily due to the following two results (see [15] for a proof), which we will not be proving here, instead proving a more general result (Lemma 6.3.2) in Chapter 6.

Lemma 2.4.8. *For $\mathcal{F} \subseteq 2^{[n]}$, $\Delta(S_{ij}(\mathcal{F})) \subseteq S_{ij}(\Delta(\mathcal{F}))$ for all $i, j \in [n]$.*

Lemma 2.4.9. *For $\mathcal{F} \subseteq 2^{[n]}$, $\nabla(S_{ij}(\mathcal{F})) \subseteq S_{ij}(\nabla(\mathcal{F}))$ for all $i, j \in [n]$.*

These two results mean that if are considering an arbitrary family that has minimum shadow or shade, then we can repeatedly left shift it until we have a shifted family with the same size of shadow or shade. Knowing that a family is shifted provides a lot of insight into its structure. For example, if a family is shifted and has a set of size k in it, we know that the set $\{1, \ldots, k\}$ is in the family. Similarly, if there is a set that doesn't contain the element 1 in the family, then it must have been prevented from shifting by one that does. It is this particular fact that will be of great use in proving both the Kruskal-Katona theorem, which provides a tight lower bound on the size of the shadow of a family of sets of the same size, and an approximation of it due to Lovász.

Theorem 2.4.10 (Kruskal-Katona theorem [24, 27])**.** *Let $\mathcal{F}$ be a nonempty subset of $\binom{[n]}{k}$ for integers k and n where $0 < k \leqslant n$. Let $x_k, x_{k-1}, \ldots, x_j$ be the unique integers such that $x_k > x_{k-1} > \cdots > x_j \geqslant j \geqslant 1$ and*

$$|\mathcal{F}| = \binom{x_k}{k} + \binom{x_{k-1}}{k-1} + \cdots + \binom{x_j}{j}.$$

Then

$$|\Delta(\mathcal{F})| \geqslant \binom{x_k}{k-1} + \binom{x_{k-1}}{k-2} + \cdots + \binom{x_j}{j-1}.$$

Moreover the family of the first $|\mathcal{F}|$ k-sets in colexicographical order has a shadow that has size equal to the bound. The *colexicographical order* on $\binom{[n]}{k}$ is given by $X <_{\text{colex}} Y$ if and only if, for some $i \in [n]$, we have $i \in Y \setminus X$ and $X \cap \{i+1, \ldots, n\} = Y \cap \{i+1, \ldots, n\}$. The colexicographical order is sometimes also referred to as the *squashed* order.

However, there are circumstances where the form of this result makes it difficult to use. Instead, it can sometimes be more useful to look at a slightly weaker result that was proven by Lovász in [29]. To introduce this result, we require the following relaxation of binomial coefficients: for any real number $q \geqslant 1$ and a positive integer $k \leqslant q$, let $\binom{q}{k} = \frac{1}{k!}(q)(q-1)(q-2)\cdots(q-k+1)$, noting that when q is an integer, this does not change the usual definition.

Theorem 2.4.11 ([29], p. 95)**.** *Let $\mathcal{F}$ be a nonempty subset of $\binom{[n]}{k}$ for integers k and n where $0 < k \leqslant n$. Then $|\Delta(\mathcal{F})| \geqslant \binom{q}{k-1}$, where q is the unique real number such that $q \geqslant k$ and $\binom{q}{k} = |\mathcal{F}|$.*

Before proving Theorems 2.4.10 and 2.4.11, we will make some observations and introduce some notation that is shared between both results. By Lemma 2.4.8 we may assume that $\mathcal{F}$ is shifted. We proceed by induction on k and on $|\mathcal{F}|$ for both proofs. First, note that both results hold trivially for arbitrary $\mathcal{F}$ when $k = 1$ and for all k when $|\mathcal{F}| = 1$.

Let $\mathcal{F}_1 = \{F : F \in \mathcal{F} \text{ and } 1 \in F\}$ and $\mathcal{F}_1' = \{F \setminus \{1\} : F \in \mathcal{F} \text{ and } 1 \in F\}$. Observe that $|\Delta(\mathcal{F})| \geqslant |\mathcal{F}_1'| + |\Delta(\mathcal{F}_1')|$, as $\mathcal{F}_1'$ and $\{\{1\} \cup F : F \in \Delta(\mathcal{F}_1')\}$ are both subsets of $\Delta(\mathcal{F})$ and are clearly disjoint. Also note that $|\mathcal{F}_1| = |\mathcal{F}_1'|$.

We now proceed to prove both of these theorems at once, following the proof given in [12].

Proof of Theorem 2.4.10. Assume that $\mathcal{F}$ is a nonempty subset of $\binom{[n]}{k}$ for some $k \geqslant 2$ and that Theorem 2.4.10 holds true for all possible families of $(k-1)$-sets and for families of k-sets with size strictly less than $|\mathcal{F}|$. Let $x_k, x_{k-1}, \dots, x_j$ be the integers defined in Theorem 2.4.10.

If

$$|\mathcal{F}_1| \geqslant \binom{x_k - 1}{k-1} + \binom{x_{k-1} - 1}{k-2} + \cdots + \binom{x_j - 1}{j-1},$$

then by induction,

$$|\Delta(\mathcal{F}_1')| \geqslant \binom{x_k - 1}{k-2} + \binom{x_{k-1} - 1}{k-3} + \cdots + \binom{x_j - 1}{j-2}.$$

Thus we have that

$$\begin{aligned} |\Delta(\mathcal{F})| &\geqslant \left(\binom{x_k - 1}{k-1} + \binom{x_k - 1}{k-2}\right) + \cdots + \left(\binom{x_j - 1}{j-1} + \binom{x_j - 1}{j-2}\right) \\ &= \binom{x_k}{k-1} + \binom{x_{k-1}}{k-2} + \cdots + \binom{x_j}{j-1} \end{aligned}$$

as required. Now suppose

$$|\mathcal{F}_1| < \binom{x_k - 1}{k-1} + \binom{x_{k-1} - 1}{k-2} + \cdots + \binom{x_j - 1}{j-1}.$$

Thus, as $|\mathcal{F} \setminus \mathcal{F}_1| = |\mathcal{F}| - |\mathcal{F}_1|$, we have that

$$\begin{aligned} |\mathcal{F} \setminus \mathcal{F}_1| &> \left(\binom{x_k}{k} + \binom{x_{k-1}}{k-1} + \cdots + \binom{x_j}{j}\right) - \left(\binom{x_k - 1}{k-1} + \binom{x_{k-1} - 1}{k-2} + \cdots + \binom{x_j - 1}{j-1}\right) \\ &= \binom{x_k - 1}{k} + \binom{x_{k-1} - 1}{k-1} + \cdots + \binom{x_j - 1}{j}. \end{aligned}$$

So by induction, recalling that $\mathcal{F}_1$ is not empty and that $|\mathcal{F}_1| \geqslant |\Delta(\mathcal{F} \setminus \mathcal{F}_1)|$, we have that

$$|\mathcal{F}_1| \geqslant |\Delta(\mathcal{F} \setminus \mathcal{F}_1)| \geqslant \binom{x_k - 1}{k-1} + \binom{x_{k-1} - 1}{k-2} + \cdots + \binom{x_j - 1}{j-1},$$

which contradicts our supposition. □

Proof of Theorem 2.4.11. Assume that $\mathcal{F}$ is a nonempty subset of $\binom{[n]}{k}$ for some $k \geqslant 2$ and that Lemma 2.4.11 holds true for all possible families of $(k-1)$-sets and for families of k-sets with size strictly less than $|\mathcal{F}|$. Let q be the unique positive real number such that $q \geqslant k$ and $|\mathcal{F}| = \binom{x}{k}$. If $|\mathcal{F}_1| \geqslant \binom{q-1}{k-1}$, then by induction, since $\mathcal{F}_1'$ is a collection of $(k-1)$-sets, we have that $\Delta(\mathcal{F}_1') \geqslant \binom{q-1}{k-2}$, and thus

$$|\Delta(\mathcal{F})| \geqslant \binom{q-1}{k-1} + \binom{q-1}{k-2} = \binom{q}{k-1},$$

as required.

Supposed that instead $|\mathcal{F}_1'| < \binom{q-1}{k-1}$. Then $|\mathcal{F} \setminus \mathcal{F}_1| = |\mathcal{F}| - |\mathcal{F}_1| > \binom{q}{k} - \binom{q-1}{k-1} = \binom{q-1}{k}$. By induction, since $|F \setminus \mathcal{F}_1| < |\mathcal{F}|$ as $\mathcal{F}_1$ is nonempty due to $\mathcal{F}$ being shifted, we know that $|\Delta(\mathcal{F} \setminus \mathcal{F}_1)| \geqslant \binom{q-1}{k-1}$. Due to the fact that $\mathcal{F}$ is shifted, we also know that for any $E \in \Delta(\mathcal{F} \setminus \mathcal{F}_1)$, we have $(E \cup 1) \in \mathcal{F}$ and hence $E \in \mathcal{F}_1'$. Thus we see that $|\mathcal{F}_1'| \geqslant |\Delta(\mathcal{F} \setminus \mathcal{F}_1)| \geqslant \binom{q-1}{k-1}$, which is a contradiction.

□

2.5 Isoperimetric Problem

The isoperimetric problem is a classical problem in mathematics which asks a simple question: given a fixed "volume", what is the minimum "perimeter" of an object with that volume? The prototypical example of this problem is on $\mathbb{R}^n$, where the goal is to find the object with minimal surface area given a fixed volume. This question has also been investigated for many other mathematical objects, with different measures of "volume" and "perimeter" (see [23, 31, 36] for various different examples). Of particular interest to this thesis is the question of minimising the boundary of a collection of vertices in a given graph when the size of the collection is fixed.

Definition 2.5.1. *Let G be a graph and let X be a set of verices of F. The neighbourhood $N_G(X)$ of X is the set of all vertices of G that are adjacent to at least one vertex in X. The boundary $\partial_G(X)$ of X is the set of vertices $N_G(X) \setminus X$. We sometimes omit the subscript G when it is obvious from context.*

One well investigated example of the isoperimetric problem on graphs looks at minimising the boundary of collection of vertices in the hypercube.

Definition 2.5.2. *Let n be a nonnegative integer, then the n-*dimensional hypercube*, denoted by Q_n, is the graph (V, E), where $V = 2^{[n]}$ and $E = \{\{X, Y\} : X, Y \in V,\ X \in \Delta(Y)\}$, where $\Delta(Y)$ is the shadow of Y as introduced in Definition 2.4.5.*

Note that a set of vertices of Q_n is simply a family of subsets of $[n]$.

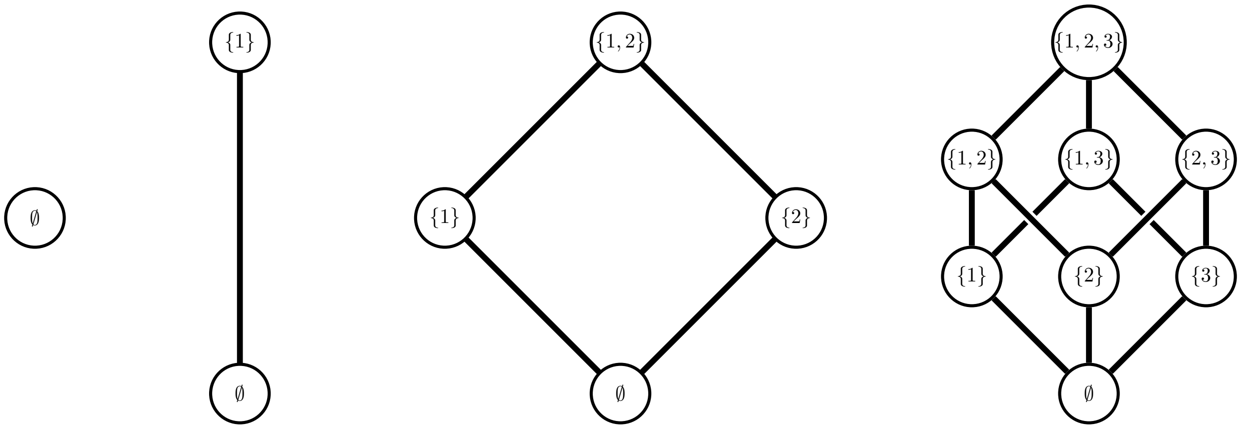

Figure 2.6: The first four hypercubes, $\mathcal{Q}_0$, $\mathcal{Q}_1$, $\mathcal{Q}_2$ and $\mathcal{Q}_3$

The most well known work related to this topic was done by Harper in [20], where he gave a construction that, for any specified size, gives a family with a minimal boundary among all families of this size. What makes this construction of particular interest is that it simply takes the initial segment of the vertices of $\mathcal{Q}_n$ in simplicial order.

Definition 2.5.3. *The* simplicial order *on the vertices of $\mathcal{Q}_n$ for some integer $n > 0$ is given by $X <_{sim} Y$ if $|X| < |Y|$ or if $|X| = |Y|$ and $X <_{lex} Y$. Here $<_{lex}$ denotes the* lexicographical order *on $\binom{[n]}{|X|}$ given by $X <_{lex} Y$ if, for some $i \in [n]$, we have $i \in X \setminus Y$ and $X \cap 1, ..., i = Y \cap 1, ..., i$.*

Thus we now have the tools to properly express Harper's theorem.

Theorem 2.5.4 (Harper's theorem [20])**.** *Let $\mathcal{F} \subseteq \mathcal{Q}_n$ and $\mathcal{G}$ be the initial segment of the simplicial order of $\mathcal{Q}_n$ such that $|\mathcal{F}| = |\mathcal{G}|$. Then $N(\mathcal{F}) \geqslant N(\mathcal{G})$.*

It is important to note that the initial segment of the simplicial order is not the unique minimising family, and more recent works such as [39] have made much progress in classifying the others.

In the papers [3] and [4], Bashov examined a variation of this problem, where he instead required all the sets in the family $\mathcal{F}$ to be the same size. Specifically, he asked how does one minimise the family $\Delta(\mathcal{F}) \cup \nabla(\mathcal{F})$, which he referred to as the *double sided shadow* of $\mathcal{F}$, when $\mathcal{F} \subseteq \binom{[n]}{k}$ and $|\mathcal{F}|$ is specified, for integers $n \geqslant k > 0$.

As previously mentioned, it is known that equality in the Kruskal-Katona theorem can be attained by taking the family $\mathcal{F}$ to be the first $|\mathcal{F}|$ k-sets in colexicographical order, and we have just seen that Harper's theorem states that the size of the boundary is minimised by taking the first $|\mathcal{F}|$ vertices in simplicial order. Bashov showed that his problem is unlike this in the sense that there is no ordering of $\binom{[n]}{k}$ such that every initial segment of the ordering produces a minimal boundary size. Specifically, he proved the following result.

Theorem 2.5.5 ([3, 4])**.** *Let n and k be integers such that $3 \leqslant k \leqslant n/2$ and $n \geqslant 10$. For every total order on $\binom{[n]}{k}$, there exists an integer $m \geqslant m'$ such that the initial segment of the order that is of length m is not minimal in terms of the double sided shadow, where*

- $m' = 4n - 14$ *if* $k = 3$*;*
- $m' = 1 + k(n-k) + (k-1)(2n-2k-3)$ *if* $4 \leqslant k < n/2$*; or*
- $m' = 1 + k^2 + (2k-3)(k-2) + \frac{k(k-1)^2}{2}$ *if* $k = n/2$.

In spite of this, Bashov was still able to find a number of extremal families for small values of n in [3]. In particular, he showed that for $|\mathcal{F}| = 1 + k(n-k)$, the extremal family up to isomorphism is the family $\mathcal{C}(n,k) = \{X \in \binom{n}{k} : |X \cap [k]| = k-1\}$, and for $|\mathcal{F}| < 1 + k(n-k)$ there exists an ordering on $\mathcal{C}(n,k)$ (which depends on k) such that the initial segment is minimal in terms of the double sided shadow.

In Chapter 6, we look at the isoperimetric problem on a graph closely related to the hypercube.

Chapter 3

Sperner partition systems

3.1 Introduction

As established in Section 2.2, $\mathrm{SP}(n,k)$ has only been exactly determined for a small number of families of n and k. In the unsolved cases, bounds are known on $\mathrm{SP}(n,k)$. In this chapter we introduce a new construction for Sperner partition systems using a result of Bryant [6]. With this we are able to establish that an upper bound, denoted by $\mathrm{MMS}(n,k)$ (see below), is asymptotically correct in many situations when c is suitably large. We also establish a new, tighter, upper bound on $\mathrm{SP}(n,k)$ and present a summary of the new best lower and upper bounds on $\mathrm{SP}(n,k)$ for a selection of small n and k.

Recall that for positive integers n and k such that $n \geqslant k$, with c and r being the unique integers such that $n = ck + r$ and $r \in \{0, \ldots, k-1\}$, Meagher, Moura and Stevens showed that $\mathrm{SP}(n,k) \leqslant \mathrm{MMS}(n,k)$ where

$$\mathrm{MMS}(n,k) = \frac{\binom{n}{c}}{k - r + \frac{r(c+1)}{n-c}}.$$

Note that $0 \leqslant \frac{r(c+1)}{n-c} \leqslant 1$ because $0 \leqslant r \leqslant k-1$. Using this upper bound together with Baranyai's theorem [2], they also established that $\mathrm{SP}(n,k) = \mathrm{MMS}(n,k) = \binom{n-1}{c-1}$ when k divides n, as stated above. Finally, they noted that $\mathrm{SP}(n+1,k) \geqslant \mathrm{SP}(n,k)$ because it is easy to augment an (n,k)-Sperner partition system to obtain an $(n+1,k)$-Sperner partition system with the same number of partitions. Thus they establish a naive lower bound $\mathrm{SP}(n,k) \geqslant \mathrm{NLB}(n,k)$ where

$$\mathrm{NLB}(n,k) = \frac{1}{k}\binom{n-r}{c}.$$

Despite its naivety, $\mathrm{NLB}(n,k)$ has hitherto been the best lower bound known on $\mathrm{SP}(n,k)$

for general n and k. In [28], Li and Meagher show that $\mathrm{SP}(2k+1,k) \in \{2k-1, 2k\}$, $\mathrm{SP}(2k+2,k) \in \{2k+1, 2k+2, 2k+3\}$ and $\mathrm{SP}(3k-1,k) \geqslant 3k-1$. They also establish an inductive lower bound by showing that $\mathrm{SP}(n+k,k) \geqslant k \cdot \mathrm{SP}(n,k)$ for $n \geqslant k \geqslant 2$.

As mentioned earlier, in this chapter we introduce a new construction for Sperner partition systems using a result of Bryant [6]. With this we are able to establish that the upper bound $\mathrm{MMS}(n,k)$ is asymptotically correct in many situations where c is large.

Theorem 3.1.1. *Let n and k be integers with $n \to \infty$, $k = o(n)$ and $k \geqslant 3$, and let c and r be the integers such that $n = ck + r$ and $r \in \{0, \ldots, k-1\}$. Then $\mathrm{SP}(n,k) \sim \mathrm{MMS}(n,k)$ if*

- *n is even and $r \notin \{1, k-1\}$; or*
- *$k - r \to \infty$.*

Note that the lower bound $\mathrm{NLB}(n,k)$ only implies the result of Theorem 3.1.1 when r is very small compared to k, and the result of [28] that $\mathrm{SP}(n+k,k) \geqslant k \cdot \mathrm{SP}(n,k)$ never implies Theorem 3.1.1 (see Lemmas 3.2.3 and 3.2.4). It is also worth noting that the Sperner partition systems we construct to prove Theorem 3.1.1 are almost uniform (see Lemmas 3.3.3 and 3.4.1, and note that it is easy to augment an almost uniform (n,k)-Sperner partition system to obtain an almost uniform $(n+1,k)$-Sperner partition system with the same number of partitions).

We also prove a result which provides an implicit upper bound on $\mathrm{SP}(n,k)$ for $k \geqslant 4$. In order to state it we require some definitions. For any nonnegative integer i and real number $y \geqslant i$, let $\binom{y}{i}$ represent $\frac{1}{i!}y(y-1)\cdots(y-i+1)$. Define, for each integer $c \geqslant 2$, a function $\mathrm{LL}_c : \{0\} \cup \mathbb{R}^{\geqslant 1} \to \mathbb{R}^{\geqslant 0}$ by $\mathrm{LL}_c(0) = 0$ and, for $x \geqslant 1$, $\mathrm{LL}_c(x) = \binom{q}{c-1}$ where q is the unique real number such that $q \geqslant c$ and $\binom{q}{c} = x$. An equivalent definition for $x \geqslant 1$ is $\mathrm{LL}_c(x) = \frac{c}{q-c+1}x$ where q is as before.

Theorem 3.1.2. *If n and k are integers such that $n \geqslant 2k+2$ and $k \geqslant 4$, then*

$$\left\lceil (1 - \tfrac{r(c+1)}{n}) \cdot \mathrm{SP}(n,k) \right\rceil + \mathrm{LL}_c\left(\left\lfloor \tfrac{r(c+1)}{n} \cdot \mathrm{SP}(n,k) \right\rfloor \right) \leqslant \binom{n-1}{c-1},$$

where c and r are the integers such that $n = ck + r$ and $r \in \{0, \ldots, k-1\}$.

For fixed n and k, the left hand side of the inequality

$$\left\lceil (1 - \tfrac{r(c+1)}{n})p \right\rceil + \mathrm{LL}_c\left(\left\lfloor \tfrac{r(c+1)}{n}p \right\rfloor \right) \leqslant \binom{n-1}{c-1} \tag{3.1}$$

is nondecreasing in p and hence there is a unique nonnegative integer p' such that (3.1) holds for each $p \in \{0, \ldots, p'\}$ and fails for each integer $p > p'$. This p' is an upper bound for

SP(n,k). We will see in Corollary 3.5.4 that p' is always at most MMS(n,k). In practice p' can be found via a binary search, beginning with NLB$(n,k) \leqslant p' \leqslant$ MMS(n,k).

This chapter is organised as follows. In the next section we introduce some of the notation and results we require. In Section 3.3 we detail the main construction we use to prove Theorem 3.1.1 and establish that it asymptotically matches the upper bound of MMS(n,k) when c is large and $r \neq k-1$. The proof of Theorem 3.1.1 is completed in Section 3.4 using a variant of our main construction. We then move on to prove Theorem 3.1.2 in Section 3.5. Finally, in Section 3.6, we conclude by examining the performance of our bounds for small parameter sets.

3.2 Preliminaries

For integers n and k with $n \geqslant k \geqslant 1$ we define $c = c(n,k)$ and $r = r(n,k)$ as the unique integers such that $n = ck + r$ and $r \in \{0, \dots, k-1\}$. We use these definitions of $c(n,k)$ and $r(n,k)$ throughout this chapter (as well as in Chapter 4) and abbreviate to simply c and r where there is no danger of confusion. We also use $n = ck + r$ frequently and tacitly in our calculations.

Recall that an (n,k)-Sperner partition system is said to be *almost uniform* if each class of each of its partitions has cardinality in $\{\lfloor \frac{n}{k} \rfloor, \lceil \frac{n}{k} \rceil\}$ and hence each partition has $k-r$ classes of cardinality c and r classes of cardinality $c+1$. For nonnegative integers x and i, we denote the ith falling factorial x by $(x)_i$.

A *hypergraph* H consists of a vertex set $V(H)$ together with a set $\mathcal{E}(H)$ of edges, each of which is a nonempty subset of $V(H)$. We do not allow loops or multiple edges. A *clutter* is a hypergraph none of whose edges is a subset of another. A clutter is exactly a Sperner family, but we use the term clutter when we wish to consider the object through a hypergraph-theoretic lens. A set of edges of a hypergraph is said to be *i-uniform* if each edge in it has cardinality i, and a hypergraph is said to be i-uniform if its entire edge set is i-uniform.

A *partial edge colouring* of a hypergraph is simply an assignment of colours to some or all of its edges with no further conditions imposed. If every edge is assigned a colour, it is an *edge colouring*. Let γ be a partial edge colouring of a hypergraph H with colour set C. For each $z \in C$, the set $\gamma^{-1}(z)$ of edges of H assigned colour z is called a *colour class* of γ. For each $z \in C$ and $x \in V(H)$, let the number of edges of H that are assigned the colour z by γ and contain the vertex x be denoted $\deg_z^\gamma(x)$. Further, for a subset Y of $V(H)$, we say that γ is *almost regular on Y* if $|\deg_z^\gamma(x) - \deg_z^\gamma(y)| \leqslant 1$ for all $z \in C$ and $x, y \in Y$. We will make use of the following result of Bryant from [6].

Theorem 3.2.1 ([6])**.** *Let H be a hypergraph, γ be an edge colouring of H with colour set C, and Y be a subset of $V(H)$ such that any permutation of Y is an automorphism of H. There exists a permutation θ of $\mathcal{E}(H)$ such that $|\theta(E)| = |E|$ and $\theta(E) \setminus Y = E \setminus Y$ for each $E \in \mathcal{E}(H)$, and such that the edge colouring γ' of H given by $\gamma'(E) = \gamma(\theta^{-1}(E))$ for each $E \in \mathcal{E}(H)$ is almost regular on Y.*

In fact, we will only require the following special case of Theorem 3.2.1.

Lemma 3.2.2. *Let n and k be integers with $n \geqslant k \geqslant 1$, let H be a clutter with $|V(H)| = n$, and let $\{X_1, \ldots, X_t\}$ be a partition of $V(H)$ such that any permutation of X_w is an automorphism of H for each $w \in \{1, \ldots, t\}$. Suppose there is a partial edge colouring γ of H with colour set C such that, for each $z \in C$, $|\gamma^{-1}(z)| = k$ and $\sum_{x \in X_w} \deg_z^{\gamma}(x) = |X_w|$ for each $w \in \{1, \ldots, t\}$. Then there is an (n,k)-Sperner partition system with $|C|$ partitions such that the classes of the partitions form a subset of $\mathcal{E}(H)$.*

Proof. Throughout this proof we will treat partial edge colourings of H with colour set C as edge colourings of H with colour set $C \cup \{\text{black}\}$ (where C does not contain black) by considering all uncoloured edges to be coloured black. This will allow us to apply Theorem 3.2.1 to them.

Let $X = V(H)$. Roughly speaking, we will perform t applications of Theorem 3.2.1, where on the ith application we "correct" the colouring on X_i. Formally, we will construct a sequence of partial edge colourings $\gamma_0, \ldots, \gamma_t$ of H with colour set C such that, for each $s \in \{0, \ldots, t\}$ and $c \in C$, $|\gamma_s^{-1}(c)| = k$, $\deg_c^{\gamma_s}(x) = 1$ for each $x \in \bigcup_{i=1}^{s} X_i$, and $\sum_{x \in X_i} \deg_c^{\gamma_s}(x) = |X_i|$ for each $i \in \{s+1, \ldots, t\}$. Let $\gamma_0 = \gamma$ and note that γ_0 satisfies the claimed conditions. Furthermore, it suffices to find a partial edge colouring γ_t satisfying the required conditions. To see this note that, for each $c \in C$, the edges assigned colour c by γ_t form a partition of X into k nonempty classes because the properties of γ_t guarantee that $|\gamma_t^{-1}(c)| = k$ and $\deg_c^{\gamma_t}(x) = 1$ for each $x \in X$. Thus the colour classes of γ_t will induce an (n,k)-Sperner partition system with the desired properties (any edges that are not coloured are not used as partition classes of the system).

Suppose inductively that a partial edge colouring γ_s satisfying the required conditions exists for some $s \in \{0, \ldots, t-1\}$. Now apply Theorem 3.2.1 with $Y = X_{s+1}$ to γ_s, to obtain a partial edge colouring γ_{s+1} of H. For each $c \in C$, $|\gamma_{s+1}^{-1}(c)| = |\gamma_s^{-1}(c)| = k$ and $\deg_c^{\gamma_{s+1}}(x) = \deg_c^{\gamma_s}(x)$ for each $x \in X \setminus X_{s+1}$. Furthermore, $\deg_c^{\gamma_{s+1}}(x) = 1$ for each $c \in C$ and $x \in X_{s+1}$, because $\sum_{x \in X_{s+1}} \deg_c^{\gamma_{s+1}}(x) = |X_{s+1}|$ and γ_{s+1} is almost regular on X_{s+1}. Thus γ_{s+1} satisfies the required conditions and the result follows. □

For functions $f(n), g(n)$, we say that $f(n) \not\sim g(n)$ if $\lim_{n \to \infty} \frac{f(n)}{g(n)} \neq 1$. The next two lemmas show that existing results in [28, 33] do not suffice to establish Theorem 3.1.1. Lemma 3.2.3

shows that the lower bound of $\mathrm{NLB}(n,k)$ only implies the conclusion of Theorem 3.1.1 when r is very small compared to k, and Lemma 3.2.4 shows that $\mathrm{SP}(n+k,k) \geqslant k \cdot \mathrm{SP}(n,k)$ never implies the conclusion of Theorem 3.1.1.

Lemma 3.2.3. *For integers n and k with $n > 2k$, $k \geqslant 3$, and $n \to \infty$, we have*

$$\mathrm{NLB}(n,k) \not\sim \mathrm{MMS}(n,k)$$

unless $k \to \infty$ and $r = o(k)$.

Proof. Note that

$$\frac{\mathrm{NLB}(n,k)}{\mathrm{MMS}(n,k)} = \frac{k - r + \frac{r(c+1)}{n-c}}{k}\left(\frac{(n-r)_c}{(n)_c}\right) < \frac{k - r + \frac{r(c+1)}{n-c}}{k}.$$

If $k \to \infty$, then the result follows because $r \neq o(k)$ and $\frac{r(c+1)}{n-c} \leqslant 1$. If $k \not\to \infty$ and $r \geqslant 2$, the result follows because $\frac{r(c+1)}{n-c} \leqslant 1$. If $k \not\to \infty$ and $r = 1$, then $\frac{r(c+1)}{n-c} \leqslant \frac{2}{3}$ because $k \geqslant 3$ and $c \geqslant 1$, and again the result follows. □

Lemma 3.2.4. *For integers n and k with $n \geqslant k$, $k \geqslant 3$ and $n \to \infty$, we have*

$$k \cdot \mathrm{MMS}(n,k) \not\sim \mathrm{MMS}(n+k,k).$$

Proof. Let $c = c(n,k)$ and $r = r(n,k)$. Note that

$$\frac{k \cdot \mathrm{MMS}(n,k)}{\mathrm{MMS}(n+k,k)} = \frac{k(c+1)(k - r + \frac{r(c+2)}{n+k-c-1})(n)_c}{\left(k - r + \frac{r(c+1)}{n-c}\right)(n+k)_{c+1}} \leqslant \frac{k(c+1)}{n+k}\left(\frac{(n)_c}{(n+k-1)_c}\right) \leqslant \left(1 - \frac{k-1}{k(c+2)}\right)^c,$$

where we used the fact that $\frac{r(c+2)}{n+k-c-1} \leqslant \frac{r(c+1)}{n-c}$ in the first inequality and the fact that $\frac{k(c+1)}{n+k} \leqslant 1$ in the second. Because $\frac{k-1}{k} \geqslant \frac{2}{3}$, the last expression can be seen to be decreasing in c for $c \geqslant 2$ and hence at most $\frac{25}{36}$. □

We conclude this section with a product construction for Sperner partition systems which generalises the inductive result of Li and Meagher mentioned in the introduction.

Lemma 3.2.5. *If m, n and k are positive integers such that $m \geqslant k$ and $n \geqslant k$, then*

$$\mathrm{SP}(m+n,k) \geqslant k \cdot \mathrm{SP}(m,k) \cdot \mathrm{SP}(n,k).$$

Proof. Let X and Y be disjoint sets with $|X| = m$ and $|Y| = n$. Let $p = \mathrm{SP}(m,k)$ and let $\mathcal{P} = \{\pi_1, \ldots, \pi_p\}$ be an (m,k)-Sperner partition system on X with p partitions, where $\pi_i = \{\pi_{i,1}, \dots, \pi_{i,k}\}$ for $i \in \{1, \dots, p\}$. Let $q = \mathrm{SP}(n, k)$ and let $\mathcal{Q} = \{\rho_1, \dots, \rho_q\}$ be an (n, k)-Sperner partition system on Y with q partitions, where $\rho_j = \{\rho_{j,1}, \dots, \rho_{j,k}\}$ for $j \in \{1, \dots, q\}$. We claim that

$$\big\{\sigma_{i,j,y} : i \in \{1, \dots, p\}, j \in \{1, \dots, q\}, y \in \{1, \dots, k\}\big\}$$

where

$$\sigma_{i,j,y} = \big\{\pi_{i,z} \cup \rho_{j,z+y} : z \in \{1, \dots, k\}\big\}$$

(with the second component of the subscripts treated modulo k) is an $(m+n, k)$-Sperner partition system with kpq partitions. To see that this claim is true, suppose that $\pi_{i,z} \cup \rho_{j,z+y} \subseteq \pi_{i',z'} \cup \rho_{j',z'+y'}$ for some $i, i' \in \{1, \dots, p\}$, $j, j' \in \{1, \dots, q\}$ and $y, z, y', z' \in \{1, \dots, k\}$. Because X and Y are disjoint, $\pi_{i,z} \subseteq \pi_{i',z'}$ and $\rho_{j,z+y} \subseteq \rho_{j',z'+y'}$. So, because $\mathcal{P}$ and $\mathcal{Q}$ are Sperner partition systems, $i = i'$, $z = z'$, $j = j'$ and, because $z = z'$, $y = y'$. This establishes the claim and hence the theorem. □

3.3 Main construction

The following technical lemma will be useful in our constructions. It enables us to partition the edges of certain uniform hypergraphs into triples that are "balanced" in some sense.

Lemma 3.3.1. *Let t be a positive integer, let H be a nonempty $(2t)$-uniform hypergraph with $V(H) = X$, and let Y be a subset of X. Suppose that there are nonnegative integers $e_0, \dots, e_t$ such that*

(i) $|\{E \in \mathcal{E}(H) : |E \cap Y| = t + i\}| = |\{E \in \mathcal{E}(H) : |E \cap Y| = t - i\}| = e_i$ *for each* $i \in \{0, \dots, t\}$;

(ii) $e_i \geqslant e_{i+1} + s$ *for each* $i \in \{0, \dots, s - 1\}$ *where s is the largest element of* $\{0, \dots, t\}$ *such that* $e_s > 0$.

For any $p \in \{0, \dots, \lfloor \frac{1}{3}|\mathcal{E}(H)|\rfloor\}$, we can partition some subset $\mathcal{E}^*$ of $\mathcal{E}(H)$ into p (unordered) triples such that

- $\sum_{i=1}^{3} |E_i \cap Y| = 3t$ *for each triple* $\{E_1, E_2, E_3\}$*; and*
- $|\mathcal{E}^*_i| = |\mathcal{E}^*_{-i}|$ *for each* $i \in \{1, \dots, t\}$*, where* $\mathcal{E}^*_i = \{E \in \mathcal{E}^* : |E \cap Y| = t + i\}$.

Proof. We prove the result by induction on $|\mathcal{E}(H)|$. In fact, we prove a slightly stronger result in which we do not require the full strength of (ii) when $p = 1$ but only that $e_0 \geqslant 1$ (note $|\mathcal{E}(H)| \geqslant 3$ when $p = 1$). Let s be the largest element of $\{0, \dots, t\}$ such that $e_s > 0$. Let the *type* of an edge E of H be $|E \cap Y| - t$ and the *type* of a triple be the multiset

$[x_1, x_2, x_3]$ where x_1, x_2, x_3 are the types of the three edges in the triple. If $p = 0$ the result is trivial. If $p = 1$, we can take a single triple of type $[-s, 0, s]$, because $|\mathcal{E}(H)| \geqslant 3$ and $e_0 > 0$. So we may assume $p \geqslant 2$. In each of a number of cases below we first choose some initial triples of specified types and then add the remaining triples (if any are required) by applying our inductive hypothesis to the hypergraph H' formed by the unassigned edges. The edges in the initial triples can be chosen arbitrarily subject to their specified type.

case	initial triples
$s = 0$	$[0, 0, 0]$
$s = 1$	$[-1, 0, 1]$
$s = 2$, ($e_2 = 1$ or $p = 2$)	$[-2, 1, 1]$ and $[2, -1, -1]$
$s = 2$, $e_2 \geqslant 2$, $p \geqslant 3$	$[-2, 0, 2]$, $[-2, 1, 1]$ and $[2, -1, -1]$
$s \geqslant 3$ odd	$[-s, i, s-i]$ and $[s, -i, i-s]$ for $i \in \{1, \dots, \min(e_s, \lfloor \frac{p}{2} \rfloor, \frac{s-1}{2})\}$
$s \geqslant 4$ even	$[-s, i, s-i]$ and $[s, -i, i-s]$ for $i \in \{1, \dots, \min(e_s, \lfloor \frac{p}{2} \rfloor, s-1)\}$

If $s \in \{0, 1, 2\}$, then using (i) and (ii) it is easy to confirm that we can choose triples of the types listed and then apply our inductive hypothesis to find the rest of the triples, so assume $s \geqslant 3$. For each $i \in \{-s, \dots, s\}$, let d_i be the number of edges of type i that are in the initial triples. Let $b = \frac{s-1}{2}$ if s is odd, let $b = s-1$ if s is even, and let $b' = \min(e_s, \lfloor \frac{p}{2} \rfloor)$.

- If $b' > b$, then $d_0 = 0$, $d_{-s} = d_s = b$ and $d_i = \frac{2b}{s-1}$ for each $i \in \{-s+1, \dots, s-1\} \setminus \{0\}$. Using this fact, along with (i) and (ii), it can be confirmed that we can choose triples of the types listed and then apply our inductive hypothesis to find the rest of the triples.
- If $b' \leqslant b$, then $d_0 = 0$, $d_{-s} = d_s = b'$ and $d_i \in \{\lfloor \frac{2b'}{s-1} \rfloor, \lceil \frac{2b'}{s-1} \rceil\}$ for each $i \in \{-s + 1, \dots, s-1\} \setminus \{0\}$. Using this fact, along with (i) and (ii), it can be confirmed that we can choose triples of the types listed and then apply our inductive hypothesis to find the rest of the triples. To see this, note the following.
  - If $e_s \leqslant \lfloor \frac{p}{2} \rfloor$, then H' contains no edges of type s or $-s$, so the condition (ii) required to apply our inductive hypothesis is weaker. Because of this, the fact that $|d_i - d_j| \leqslant 1$ for $i, j \in \{0, \dots, s-1\}$ is sufficient to establish this condition.
  - If $\lfloor \frac{p}{2} \rfloor < e_s$, then we only require one further triple and so the fact that $e_0 \geqslant 1$ suffices to establish our inductive hypothesis. □

The next, very simple, lemma will be used to show that condition (ii) of Lemma 3.3.1 holds in the situations in which it is applied.

Lemma 3.3.2. *Let n and t be positive integers such that $n \geqslant 6t-2$ is even, and let $e_i = \binom{n/2}{t-i}\binom{n/2}{t+i}$ for each $i \in \{0,\dots,t\}$. Then $e_i > e_{i+1} + t$ for each $i \in \{0,\dots,t-1\}$.*

Proof. The result holds when $t=1$, so assume that $t \geqslant 2$. Let $i \in \{0,\dots,t-1\}$. By routine calculation

$$e_i = \tfrac{(t+i+1)(n-2t+2i+2)}{(t-i)(n-2t-2i)} e_{i+1} \geqslant \tfrac{(t+1)(n-2t+2)}{t(n-2t)} e_{i+1} = \left(1 + \tfrac{n+2}{t(n-2t)}\right) e_{i+1}.$$

Thus it suffices to show that $e_{i+1} \geqslant t^2$ because then $\frac{n+2}{t(n-2t)} e_{i+1} > t$. If $i \in \{0,\dots,t-2\}$, then $e_{i+1} \geqslant t^2$ because $\binom{n/2}{t-i-1}\binom{n/2}{t+i+1} \geqslant \frac{n}{2}\cdot\frac{n}{2} \geqslant t^2$. Also, $e_t = \binom{n/2}{2t} \geqslant \binom{n/2}{2} \geqslant t^2$ because $n \geqslant 6t-2$. $\square$

The following lemma encapsulates the main construction used in our proof of Theorem 3.1.1. Recall that $c = c(n,k)$ and $r = r(n,k)$ are the integers such that $n = ck + r$ and $r \in \{0,\dots,k-1\}$.

Lemma 3.3.3. *Let n and k be integers such that $n \geqslant 2k$, $k \geqslant 3$, $r \neq 0$, and n and ck are both even. Let $u \in \{1,\dots,\lfloor \frac{c}{2} \rfloor\}$ such that $u = \frac{c}{2}$ if $r = k-1$. There exists an almost uniform (n,k)-Sperner partition system with p partitions where*

$$p = \min\left(\left\lfloor \frac{a(u)}{k-r} \right\rfloor, \left\lfloor \frac{b(u)}{r} \right\rfloor\right), \quad a(u) = \sum_{i=u}^{c-u} \binom{n/2}{i}\binom{n/2}{c-i}, \quad b(u) = 2\sum_{i=0}^{u-1} \binom{n/2}{i}\binom{n/2}{c+1-i}.$$

Proof. Note that r is even because n and ck are both even. Fix $u \in \{1,\dots,\lfloor \frac{c}{2} \rfloor\}$ and let $a = a(u)$ and $b = b(u)$. Let X_1 and X_2 be disjoint sets such that $|X_1| = |X_2| = \frac{n}{2}$, and let $X = X_1 \cup X_2$. For each $(i,j) \in \mathbb{N} \times \mathbb{N}$, let

$$\mathcal{E}_{(i,j)} = \{E \subseteq X : |E \cap X_1| = i, |E \cap X_2| = j\}$$

and note $|\mathcal{E}_{(i,j)}| = \binom{n/2}{i}\binom{n/2}{j}$. Let

$$\mathcal{A} = \bigcup_{(i,j)\in I'} \mathcal{E}_{(i,j)}, \qquad \text{where } I' = \{(i,j) \in \mathbb{N} \times \mathbb{N} : i+j = c, \min(i,j) \geqslant u\}$$
$$\mathcal{B} = \bigcup_{(i,j)\in I''} \mathcal{E}_{(i,j)}, \qquad \text{where } I'' = \{(i,j) \in \mathbb{N} \times \mathbb{N} : i+j = c+1, \min(i,j) \leqslant u-1\}.$$

Note that $|\mathcal{A}| = a$ and $|\mathcal{B}| = b$. Furthermore, no set in $\mathcal{A}$ is a subset of a set in $\mathcal{B}$ because, for any $A \in \mathcal{A}$ and $B \in \mathcal{B}$, $|A \cap X_i| \geqslant u > |B \cap X_i|$ for some $i \in \{1,2\}$. So the hypergraph H with vertex set X and edge set $\mathcal{A} \cup \mathcal{B}$ is a clutter. Let C be a set of p colours. Observe that, for each $i \in \{1,2\}$, any permutation of X_i is an automorphism of H. Thus, by Lemma 3.2.2, it suffices to find a partial edge colouring γ of H with colour set C such that, for each $c \in C$, $|\gamma^{-1}(c)| = k$ and $\sum_{x \in X_i} \deg_c^\gamma(x) = \frac{n}{2}$ for each $i \in \{1,2\}$. Note that the resulting Sperner partition system will be almost uniform because each edge in H has size c or $c+1$. Call a set of edges $\mathcal{E}' \subseteq \mathcal{E}(H)$ *compatible* if $\sum_{E \in \mathcal{E}'} |E \cap X_1| = \sum_{E \in \mathcal{E}'} |E \cap X_2|$.

Case 1. Suppose that k is even. Then each partition in an almost uniform (n,k)-Sperner partition system contains an even number, $k-r$, of classes of cardinality c and an even number, r, of classes of cardinality $c+1$.

Because $|\mathcal{E}_{(i,j)}| = |\mathcal{E}_{(j,i)}|$ for each $(i,j) \in I''$, we can partition $\mathcal{B}$ into $\frac{b}{2}$ compatible pairs. Also, $|\mathcal{E}_{(i,j)}| = |\mathcal{E}_{(j,i)}|$ for each $(i,j) \in I'$ and, if c is even, a pair of edges from $\mathcal{E}_{c/2,c/2}$ is compatible. Thus, we can find $\lfloor \frac{a}{2} \rfloor$ disjoint compatible pairs in $\mathcal{A}$ (one edge in $\mathcal{E}_{c/2,c/2}$ will be unpaired in the case where c is even and $|\mathcal{E}_{c/2,c/2}|$ is odd, and all edges will be paired otherwise).

Take a partial edge colouring γ of H with colour set C such that each colour class contains r edges in $\mathcal{B}$ that form $\frac{r}{2}$ compatible pairs and $k-r$ edges in $\mathcal{A}$ that form $\frac{k-r}{2}$ compatible pairs, and all remaining edges are uncoloured. This can be accomplished because $\frac{r}{2}p \leqslant \frac{r}{2}\lfloor \frac{b}{r} \rfloor \leqslant \frac{b}{2}$ and $\frac{k-r}{2}p \leqslant \frac{k-r}{2}\lfloor \frac{a}{k-r} \rfloor \leqslant \lfloor \frac{a}{2} \rfloor$. Observe that for each $c \in C$ we have that $\sum_{x \in X} \deg_c^\gamma(x) = r(c+1) + (k-r)c = n$ and, because the colour class can be partitioned into compatible pairs, $\sum_{x \in X_1} \deg_c^\gamma(x) = \sum_{x \in X_2} \deg_c^\gamma(x)$. Thus, as desired, we have that $\sum_{x \in X_i} \deg_c^\gamma(x) = \frac{n}{2}$ for each $c \in C$ and $i \in \{1,2\}$.

Case 2. Suppose that k is odd, c is even, and $r \neq k-1$. Then each partition in an almost uniform (n,k)-Sperner partition system contains an odd number, $k-r$, of classes of cardinality c and an even number, r, of classes of cardinality $c+1$. Apply Lemma 3.3.1 with $Y = X_1$, $t = \frac{c}{2}$, and $e_i = |\mathcal{E}_{(t-i,t+i)}| = |\mathcal{E}_{(t+i,t-i)}|$ for each $i \in \{0,\ldots,t\}$ to find p disjoint triples of edges in $\mathcal{A}$. The hypotheses of Lemma 3.3.1 can be seen to be satisfied using Lemma 3.3.2 and because $p \leqslant \lfloor \frac{a}{k-r} \rfloor \leqslant \lfloor \frac{a}{3} \rfloor$ since $k-r \geqslant 3$. Note that each triple given by Lemma 3.3.1 is compatible, and that the number of edges in $\mathcal{E}_{(i,j)}$ assigned to triples is equal to the number of edges in $\mathcal{E}_{(j,i)}$ assigned to triples for each $(i,j) \in I'$. Thus we can partition all, or all but one, of the unassigned edges in $\mathcal{A}$ into $\lfloor \frac{a-3p}{2} \rfloor$ compatible pairs. Take a partial edge colouring γ of H with colour set C such that each colour class contains r edges in $\mathcal{B}$ that form $\frac{r}{2}$ compatible pairs and $k-r$ edges in $\mathcal{A}$ that form one compatible triple and $\frac{k-r-3}{2}$ compatible pairs, and all remaining edges are uncoloured. This can be accomplished because $\frac{r}{2}p \leqslant \frac{r}{2}\lfloor \frac{b}{r} \rfloor \leqslant \frac{b}{2}$ and $\frac{k-r-3}{2}p \leqslant \lfloor \frac{a-3p}{2} \rfloor$ (since $\frac{k-r-3}{2}p \leqslant \frac{k-r}{2}\lfloor \frac{a}{k-r} \rfloor - \frac{3p}{2} \leqslant \frac{a-3p}{2}$ and $\frac{k-r-3}{2}p$ is an integer). Then γ has the properties we desire.

Case 3. Suppose that k is odd, c is even, and $r = k-1$. Then $u = \frac{c}{2}$ by our hypotheses and $\mathcal{A} = \mathcal{E}_{c/2,c/2}$. Let γ be a partial edge colouring of H with colour set C such that each colour class contains $k-1$ edges in $\mathcal{B}$ that form $\frac{k-1}{2}$ compatible pairs and one edge in $\mathcal{A}$.

Again γ has the properties we desire. $\square$

To extend the approach of Lemma 3.3.3 to cases where n is even and ck is odd would involve finding complementary triples of edges in $\mathcal{B}$. This can be difficult because the edges in $\mathcal{B}$ are "unbalanced" in terms of the sizes of their intersections with X_1 and X_2. To circumvent this problem we will introduce, in Section 3.4, a variation on our construction in which the edges in $\mathcal{B}$ are "balanced". First, however, we show that, when c is large and $r \neq k-1$, the lower bound implied by Lemma 3.3.3 asymptotically matches the $\mathrm{MMS}(n,k)$ upper bound, recalling that

$$\mathrm{MMS}(n,k) = \frac{\binom{n}{c}}{k - r + \frac{r(c+1)}{n-c}}.$$

Proof of Theorem 3.1.1 when n and ck are even. By our hypotheses, $r \neq k-1$. Furthermore, $\mathrm{SP}(n,k) = \mathrm{MMS}(n,k)$ when $r = 0$, so we may assume $2 \leqslant r < k-1$. Let $a(j)$ and $b(j)$ be as defined in Lemma 3.3.3 for each $j \in \{1,\ldots,\lfloor\frac{c}{2}\rfloor\}$, and additionally define $a(0) = \binom{n}{c}$, $b(0) = 0$, $a(\lfloor\frac{c}{2}\rfloor+1) = 0$, and $b(\lfloor\frac{c}{2}\rfloor+1) = \binom{n}{c+1}$. For each $j \in \{0,\ldots,\lfloor\frac{c}{2}\rfloor+1\}$, let $a_j = \lfloor\frac{a(j)}{k-r}\rfloor$ and $b_j = \lfloor\frac{b(j)}{r}\rfloor$. Note that $a_0 \geqslant \cdots \geqslant a_{\lfloor c/2\rfloor+1} = 0$, $0 = b_0 \leqslant \cdots \leqslant b_{\lfloor c/2\rfloor+1}$, $a_0 > b_0$ and $a_{\lfloor c/2\rfloor+1} < b_{\lfloor c/2\rfloor+1}$. Let w be the unique integer in $\{0,\ldots,\lfloor\frac{c}{2}\rfloor\}$ such that $a_{w+1} \leqslant b_{w+1}$ and $a_w > b_w$. By applying Lemma 3.3.3 with $u = w+1$ (or trivially if $w = \lfloor\frac{c}{2}\rfloor$) we have $\mathrm{SP}(n,k) \geqslant a_{w+1}$, and by applying Lemma 3.3.3 with $u = w$ (or trivially if $w = 0$) we have $\mathrm{SP}(n,k) \geqslant b_w$. Furthermore, one of these bounds is the best bound achievable via Lemma 3.3.3 because $a_{w+1} \geqslant \cdots \geqslant a_{\lfloor c/2\rfloor+1}$ and $b_0 \leqslant \cdots \leqslant b_w$. By definition of the function a, we have $a(w+1) = a(w) - \delta\binom{n/2}{w}\binom{n/2}{c-w}$, where $\delta = 2$ if $w < \frac{c}{2}$ and $\delta = 1$ if $w = \frac{c}{2}$. Hence

$$\mathrm{SP}(n,k) \geqslant a_{w+1} = \left\lfloor \frac{a(w) - \delta\binom{n/2}{w}\binom{n/2}{c-w}}{k-r} \right\rfloor \geqslant a_w - \frac{\delta\binom{n/2}{w}\binom{n/2}{c-w}}{k-r} - 1. \tag{3.2}$$

We will bound a_w and then apply (3.2). We now show that

$$(c+1)b(w) = (n-c)\left(\binom{n}{c} - a(w)\right) - \delta'(n-2w+2)\binom{n/2}{w-1}\binom{n/2}{c-w+1}, \tag{3.3}$$

where $\delta' = 1$ if $w \geqslant 1$ and $\delta' = 0$ if $w = 0$. We may assume $w \geqslant 1$, for otherwise $w = 0$, $b(w) = 0$, $a(w) = \binom{n}{c}$ and (3.3) holds. Now apply Lemma 3.3.3 with $u = w$, let $\mathcal{A}$ and $\mathcal{B}$ be as defined in its proof, and let $\mathcal{A}^c = \binom{X}{c} \setminus \mathcal{A}$. Note that $|\mathcal{A}| = a(w)$, $|\mathcal{B}| = b(w)$ and $|\mathcal{A}^c| = \binom{n}{c} - a(w)$. We now count, in two ways, the number of pairs (S,B) such that $S \in \mathcal{A}^c$, $B \in \mathcal{B}$ and $S \subseteq B$.

- Each of the $b(w)$ sets in $\mathcal{B}$ has exactly $c+1$ subsets in $\binom{X}{c}$ and each of these is in $\mathcal{A}^c$, because no set in $\mathcal{A}$ is a subset of a set in $\mathcal{B}$.
- By the definition of $\mathcal{A}$, $\min(|S \cap X_1|, |S \cap X_2|) \leqslant w-1$ for each $S \in \mathcal{A}^c$. Each of the $\binom{n}{c} - a(w)$ sets in $\mathcal{A}^c$ has $n-c$ supersets in $\binom{X}{c+1}$. For each $S \in \mathcal{A}^c$ such that $\min(|S \cap X_1|, |S \cap X_2|) \leqslant w-2$, all of these supersets of S are in $\mathcal{B}$. For each of the $2\binom{n/2}{w-1}\binom{n/2}{c-w+1}$ sets $S \in \mathcal{A}^c$ such that $\min(|S \cap X_1|, |S \cap X_2|) = w-1$, exactly $\frac{n}{2} - w + 1$ of these supersets of S are not in $\mathcal{B}$.

Equating our two counts, we see that (3.3) does indeed hold.

Because $a_w > b_w$, we have $\lfloor \frac{a(w)}{k-r} \rfloor > \lfloor \frac{b(w)}{r} \rfloor$ which implies $\frac{a(w)}{k-r} > \frac{b(w)}{r}$ or equivalently $b(w) < \frac{r}{k-r} a(w)$. Substituting this into (3.3) and solving for $a(w)$ we see

$$a(w) > \frac{(k-r)\left(\binom{n}{c} - \frac{\delta'(n-2w+2)}{n-c}\binom{n/2}{w-1}\binom{n/2}{c-w+1}\right)}{k-r+\frac{r(c+1)}{n-c}}. \tag{3.4}$$

Using $a_w > \frac{a(w)}{k-r} - 1$ and (3.4) in (3.2) we obtain

$$\mathrm{SP}(n,k) > \frac{\binom{n}{c} - \frac{\delta'(n-2w+2)}{n-c}\binom{n/2}{w-1}\binom{n/2}{c-w+1}}{k-r+\frac{r(c+1)}{n-c}} - \frac{\delta\binom{n/2}{w}\binom{n/2}{c-w}}{k-r} - 2,$$

or, equivalently,

$$\mathrm{SP}(n,k) > \frac{\binom{n}{c} - \frac{\delta'(n-2w+2)}{n-c}\binom{n/2}{w-1}\binom{n/2}{c-w+1} - \delta\left(1+\frac{r(c+1)}{(n-c)(k-r)}\right)\binom{n/2}{w}\binom{n/2}{c-w}}{k-r+\frac{r(c+1)}{n-c}} - 2. \tag{3.5}$$

In the above, note that $\delta' \leqslant 1$, $\delta \leqslant 2$, $\frac{n-2w+2}{n-c} \leqslant \frac{3}{2}$ when $w \geqslant 1$ because $k \geqslant 3$, and $\frac{r(c+1)}{(n-c)(k-r)} \leqslant 1$ because $r \leqslant k-1$.

Note that $\binom{n/2}{x}\binom{n/2}{c-x} \leqslant \binom{n/2}{\lfloor c/2 \rfloor}\binom{n/2}{\lceil c/2 \rceil}$ for any $x \in \{0,\dots,c\}$. By using this fact and then applying Stirling's approximation we have, for $n \to \infty$ with $k = o(n)$ and any $x \in \{0,\dots,c\}$,

$$\binom{n/2}{x}\binom{n/2}{c-x} \Big/ \binom{n}{c} \leqslant \sqrt{\frac{2n}{\pi c(n-c)}}(1+o(1)) \leqslant \sqrt{\frac{2k}{\pi c(k-1)}}(1+o(1)) = o(1)$$

(note that $n \to \infty$ with $k = o(n)$ implies $c \to \infty$). Applying this fact twice in (3.5) yields $\mathrm{SP}(n,k) > \mathrm{MMS}(n,k)(1-o(1))$. Combined with the fact that $\mathrm{SP}(n,k) \leqslant \mathrm{MMS}(n,k)$, this establishes the result. □

3.4 Completing the Proof of Theorem 3.1.1

As discussed after Lemma 3.3.3, we require a variation on our main construction in order to complete the proof of Theorem 3.1.1.

Lemma 3.4.1. *Let n and k be integers such that $n \geqslant 2k$, $k \geqslant 3$, n is even and ck is odd. Let $u \in \{\frac{c+1}{2}, \dots, c-1\}$ be such that $u = \frac{c+1}{2}$ if $r = 1$. There exists an almost-uniform (n,k)-Sperner partition system with p partitions where*

$$p = \min\left(\left\lfloor \frac{a(u)}{k-r} \right\rfloor, \left\lfloor \frac{b(u)}{r} \right\rfloor\right), \quad a(u) = 2\sum_{i=u+1}^{c} \binom{n/2}{i}\binom{n/2}{c-i}, \quad b(u) = \sum_{i=c+1-u}^{u} \binom{n/2}{i}\binom{n/2}{c+1-i}.$$

Proof. Note that each partition in an almost uniform (n,k)-Sperner partition system contains an even number, $k-r$, of classes of cardinality c and an odd number, r, of classes of cardinality $c+1$.

Fix $u \in \{\frac{c+1}{2}, \dots, c-1\}$ and let $a = a(u)$ and $b = b(u)$. Let X_1 and X_2 be disjoint sets such that $|X_1| = |X_2| = \frac{n}{2}$, and let $X = X_1 \cup X_2$. As in the proof of Lemma 3.3.3, for each $(i,j) \in \mathbb{N} \times \mathbb{N}$, let

$$\mathcal{E}_{(i,j)} = \{E \subseteq X : |E \cap X_1| = i, |E \cap X_2| = j\}.$$

Unlike the proof of Lemma 3.3.3, let

$$\mathcal{A} = \bigcup_{(i,j)\in I'} \mathcal{E}_{(i,j)}, \qquad \text{where } I' = \{(i,j) \in \mathbb{N}\times\mathbb{N} : i+j = c, \max(i,j) \geqslant u+1\}$$
$$\mathcal{B} = \bigcup_{(i,j)\in I''} \mathcal{E}_{(i,j)}, \qquad \text{where } I'' = \{(i,j) \in \mathbb{N}\times\mathbb{N} : i+j = c+1, \max(i,j) \leqslant u\}.$$

Note that $|\mathcal{A}| = a$ and $|\mathcal{B}| = b$. Furthermore, no set in $\mathcal{A}$ is a subset of a set in $\mathcal{B}$ because, for any $A \in \mathcal{A}$ and $B \in \mathcal{B}$, $|A \cap X_i| > u \geqslant |B \cap X_i|$ for some $i \in \{1,2\}$. Thus the hypergraph H with vertex set X and edge set $\mathcal{A} \cup \mathcal{B}$ is a clutter. Observe that, for each $i \in \{1,2\}$, any permutation of X_i is an automorphism of H. Let C be a set of p colours. By Lemma 3.2.2, it suffices to find a partial edge colouring γ of H with colour set C such that, for each $c \in C$, $|\gamma^{-1}(c)| = k$ and $\sum_{x\in X_i} \deg_c^\gamma(x) = \frac{n}{2}$ for each $i \in \{1,2\}$. Note that the resulting Sperner partition system will be almost uniform because each edge in H has size c or $c+1$. Again, call a set of edges $\mathcal{E}' \subseteq \mathcal{E}(H)$ *compatible* if $\sum_{E\in\mathcal{E}'} |E \cap X_1| = \sum_{E\in\mathcal{E}'} |E \cap X_2|$.

Case 1. Suppose that $r \neq 1$. Apply Lemma 3.3.1 with $Y = X_1$, $t = \frac{c+1}{2}$, and $e_i = |\mathcal{E}_{(t-i,t+i)}| = |\mathcal{E}_{(t+i,t-i)}|$ for each $i \in \{0,\dots,t\}$ to find p disjoint triples of edges in $\mathcal{B}$. The hypotheses of Lemma 3.3.1 can be seen to be satisfied using Lemma 3.3.2 and because $p \leqslant \lfloor \frac{b}{r} \rfloor \leqslant \lfloor \frac{b}{3} \rfloor$ since $r \geqslant 3$. Note that each triple given by Lemma 3.3.1 is compatible, and that the number of edges in $\mathcal{E}_{(i,j)}$ assigned to triples is equal to the number of edges in

$\mathcal{E}_{(j,i)}$ assigned to triples for each $(i,j) \in I''$. Thus we can partition all, or all but one, of the unassigned edges in $\mathcal{B}$ into $\lfloor \frac{b-3p}{2} \rfloor$ compatible pairs. Take a partial edge colouring γ of H with colour set C such that each colour class contains r edges in $\mathcal{B}$ that form one compatible triple and $\frac{r-3}{2}$ compatible pairs and $k-r$ edges in $\mathcal{A}$ that form $\frac{k-r}{2}$ compatible pairs, and all remaining edges are uncoloured. This can be accomplished because $\frac{k-r}{2}p \leqslant \frac{k-r}{2}\lfloor \frac{a}{k-r} \rfloor \leqslant \frac{a}{2}$ and $\frac{r-3}{2}p \leqslant \lfloor \frac{b-3p}{2} \rfloor$ (note that $\frac{r-3}{2}p$ is an integer less than or equal to $\frac{r}{2}\lfloor \frac{b}{r} \rfloor - \frac{3p}{2}$). Then γ has the properties we desire.

Case 2. Suppose that $r = 1$. Then $u = \frac{c+1}{2}$ by our hypotheses and $\mathcal{B} = \mathcal{E}_{(c+1)/2,(c+1)/2}$. Take a partial edge colouring γ of H with colour set C such that each colour class contains one edge in $\mathcal{B}$ and $k-1$ edges in $\mathcal{A}$ that form $\frac{k-1}{2}$ compatible pairs. Again γ has the properties we desire. □

The approach of Lemma 3.4.1 can also be applied when n and k are both even. However, computational evidence indicates that this approach almost always underperforms Lemma 3.3.3. We can now prove the remainder of Theorem 3.1.1.

Proof of Theorem 3.1.1. We saw in Section 3.3 that Theorem 3.1.1 holds when n and ck are both even. Here, we first use Lemma 3.4.1 to deal with almost all of the remaining cases where n is even, and then use the monotonicity of $\mathrm{SP}(n,k)$ in n to complete the rest of the proof.

Case 1. Suppose that n is even, ck is odd, and $r \neq 1$. The proof is very similar to the proof in the case where n and ck are even, but we highlight the differences.

Let $a(j)$ and $b(j)$ be as defined in Lemma 3.4.1 for each $j \in \{\frac{c+1}{2},\dots,c-1\}$, and additionally define $a(\frac{c-1}{2}) = \binom{n}{c}$, $b(\frac{c-1}{2}) = 0$, $a(c) = 0$, and $b(c) = \binom{n}{c+1} - 2\binom{n/2}{c+1}$. For each $j \in \{\frac{c-1}{2},\dots,c\}$, let $a_j = \lfloor \frac{a(j)}{k-r} \rfloor$ and $b_j = \lfloor \frac{b(j)}{r} \rfloor$. Note that $a_{(c-1)/2} \geqslant \cdots \geqslant a_c = 0$, $0 = b_{(c-1)/2} \leqslant \cdots \leqslant b_c$, $a_{(c-1)/2} > b_{(c-1)/2}$ and $a_c < b_c$. Let w be the unique integer in $\{\frac{c-1}{2},\dots,c-1\}$ such that $a_w > b_w$ and $a_{w+1} \leqslant b_{w+1}$. By applying Lemma 3.4.1 with $u = w+1$ (or trivially if $w = c-1$) we have $\mathrm{SP}(n,k) \geqslant a_{w+1}$. By definition of a, we have $a(w+1) = a(w) - 2\binom{n/2}{w+1}\binom{n/2}{c-w-1}$ and hence

$$\mathrm{SP}(n,k) \geqslant a_{w+1} = \left\lfloor \frac{a(w) - 2\binom{n/2}{w+1}\binom{n/2}{c-w-1}}{k-r} \right\rfloor \geqslant a_w - \frac{2\binom{n/2}{w+1}\binom{n/2}{c-w-1}}{k-r} - 1. \tag{3.6}$$

We now show that

$$(c+1)b(w) = (n-c)\left(\binom{n}{c} - a(w)\right) - \delta'(n-2w)\binom{n/2}{w}\binom{n/2}{c-w}, \tag{3.7}$$

where $\delta' = 1$ if $w \geqslant \frac{c+1}{2}$ and $\delta' = 0$ if $w = \frac{c-1}{2}$. We may assume $w \geqslant \frac{c+1}{2}$, for otherwise

$w = \frac{c-1}{2}$, $b(w) = 0$, $a(w) = \binom{n}{c}$ and (3.7) holds. Consider applying Lemma 3.4.1 with $u = w$, let $\mathcal{A}$ and $\mathcal{B}$ be as defined in its proof, and let $\mathcal{A}^c = \binom{X}{c} \setminus \mathcal{A}$. Note that $|\mathcal{A}| = a(w)$, $|\mathcal{B}| = b(w)$ and $|\mathcal{A}^c| = \binom{n}{c} - a(w)$. We now count, in two ways, the number of pairs (S, B) such that $S \in \mathcal{A}^c$, $B \in \mathcal{B}$ and $S \subseteq B$.

- Each of the $b(w)$ sets in $\mathcal{B}$ has exactly $c+1$ subsets in $\binom{X}{c}$ and each of these is in $\mathcal{A}^c$, because no set in $\mathcal{A}$ is a subset of a set in $\mathcal{B}$.
- By the definition of $\mathcal{A}$, $\max(|S \cap X_1|, |S \cap X_2|) \leqslant w$ for each $S \in \mathcal{A}^c$. Each of the $\binom{n}{c} - a(w)$ sets in $\mathcal{A}^c$ has $n - c$ supersets in $\binom{X}{c+1}$. For each $S \in \mathcal{A}^c$ such that $\max(|S \cap X_1|, |S \cap X_2|) \leqslant w - 1$, all of these supersets of S are in $\mathcal{B}$. For each of the $2\binom{n/2}{w}\binom{n/2}{c-w}$ sets $S \in \mathcal{A}^c$ such that $\max(|S \cap X_1|, |S \cap X_2|) = w$, exactly $\frac{n}{2} - w$ of these supersets of S are not in $\mathcal{B}$.

By equating our two counts, (3.7) holds.

Using (3.6) and (3.7) in place of (3.2) and (3.3), it is now routine to obtain the desired conclusion by following the argument from the case of the proof where n and ck are even.

Case 2. Suppose that n is odd, or that n is even and $r = 1$. By our hypotheses, $k - r \to \infty$. Note that

$$\frac{\mathrm{MMS}(n-1,k)}{\mathrm{MMS}(n,k)} = \frac{k - r + \frac{r(c+1)}{n-c}}{k - r + 1 + \frac{(r-1)(c+1)}{n-c-1}} \cdot \frac{n-c}{n} \geqslant \frac{k-r}{k-r+1} \cdot \frac{k-1}{k} = 1 - o(1),$$

where the first inequality follows because $\frac{r(c+1)}{n-c} \geqslant \frac{(r-1)(c+1)}{n-c-1}$ and $\frac{n-c}{n} \geqslant \frac{k-1}{k}$ and the second equality follows because $k - r \to \infty$. Hence we have $\mathrm{MMS}(n-1,k) = \mathrm{MMS}(n,k)(1 - o(1))$. Thus, if $\mathrm{SP}(n-1,k) = \mathrm{MMS}(n-1,k)(1-o(1))$, we have

$$\mathrm{SP}(n,k) \geqslant \mathrm{SP}(n-1,k) = \mathrm{MMS}(n-1,k)(1-o(1)) = \mathrm{MMS}(n,k)(1-o(1)). \tag{3.8}$$

If n is even and $r = 1$, we have $\mathrm{SP}(n-1,k) = \mathrm{MMS}(n-1,k)$ from [33] and thus (3.8) definitely holds. So the theorem holds in all the cases where n is even. But, having established this, we may assume that n is odd and we know that $\mathrm{SP}(n-1,k) = \mathrm{MMS}(n-1,k)(1-o(1))$. Hence the proof is complete, using (3.8). □

3.5 Proof of Theorem 3.1.2

Let X be a ground set, let $\mathcal{S}$ be a family of subsets of X, and let i be an integer. If each set in $\mathcal{S}$ has cardinality at least i, we extend the definitions of shadow and shade in the following fashion. Define $\Delta^i(\mathcal{S})$ to be the family of all sets in $\binom{X}{i}$ that are subsets of some set in $\mathcal{S}$. Similarly, if each set in $\mathcal{S}$ has cardinality at most i, then we define the $\nabla^i(\mathcal{S})$ to be the family of all sets in $\binom{X}{i}$ that are supersets of some set in $\mathcal{S}$. In particular, note that $\Delta^i(\mathcal{S})$ and $\nabla^i(\mathcal{S})$ allow for $\mathcal{S}$ to be a family of sets of different sizes, unlike the conventional definitions for shadow and shade.

The following theorem, due to Lovász [29, p. 95], gives a convenient approximation to the Kruskal-Katona theorem (see [24, 27] for the original theorem, and Section 2.4 for discussion).

Theorem 3.5.1. *If $i \geqslant 2$ is an integer, X is a set and $\mathcal{S} \subseteq \binom{X}{i}$, then $|\Delta^{i-1}(\mathcal{S})| \geqslant \mathrm{LL}_i(|\mathcal{S}|)$.*

Recall that the function LL_i was defined just prior to the statement of Theorem 3.1.2 in Section 3.1. It will be important for our purposes that, for a fixed integer $i \geqslant 2$, $\mathrm{LL}_i(x)$ is monotonically increasing and concave in x for $x \geqslant 1$ (see [7, Lemma 4]). We will make use of the following simple consequence of Theorem 3.5.1.

Lemma 3.5.2. *Let H be a clutter with edge set $\mathcal{E}$, and c be a positive integer such that $|E| \geqslant c$ for each $E \in \mathcal{E}$. Then $|\Delta^c(\mathcal{E})| \geqslant \min(|\mathcal{E}|, \binom{2c+1}{c} + 1)$.*

Proof. If each edge in $\mathcal{E}$ has cardinality c, then $\Delta^c(\mathcal{E}) = \mathcal{E}$ and the result holds trivially. So we may suppose inductively that the maximum cardinality of an edge in $\mathcal{E}$ is $j \geqslant c+1$ and that the result holds if the maximum cardinality of an edge in $\mathcal{E}$ is $j-1$.

Let $\mathcal{E}_i = \{E \in \mathcal{E} : |E| = i\}$ for each $i \in \{c, \dots, j\}$, and let H^* be a hypergraph with vertex set $V(H)$ and edge set $\mathcal{E}^* = (\mathcal{E} \setminus \mathcal{E}_j) \cup \Delta^{j-1}(\mathcal{E}_j)$. Because H is a clutter, H^* is a clutter and $\Delta^{j-1}(\mathcal{E}_j)$ is disjoint from $\mathcal{E}_{j-1}$.

- If $|\mathcal{E}_j| \leqslant \binom{2j-1}{j}$, then $|\mathcal{E}_j| = \binom{y}{j}$ for some real $y \leqslant 2j-1$ and hence $|\Delta^{j-1}(\mathcal{E}_j)| \geqslant \binom{y}{j-1} \geqslant \binom{y}{j} = |\mathcal{E}_j|$ using Theorem 3.5.1. Thus $|\mathcal{E}^*| \geqslant |\mathcal{E}|$.
- If $|\mathcal{E}_j| > \binom{2j-1}{j}$, then $|\Delta^{j-1}(\mathcal{E}_j)| > \binom{2j-1}{j-1} \geqslant \binom{2c+1}{c}$ by Theorem 3.5.1 and so $|\mathcal{E}^*| \geqslant \binom{2c+1}{c} + 1$.

So in either case $|\mathcal{E}^*| \geqslant \min(|\mathcal{E}|, \binom{2c+1}{c}+1)$. The result now follows by applying our inductive hypothesis to $\mathcal{E}^*$ and noting that $\Delta^c(\mathcal{E}^*) = \Delta^c(\mathcal{E})$. □

The bulk of the work of proving Theorem 3.1.2 is accomplished in the following lemma. It establishes that (3.1) holds subject to the existence of a clutter with desirable properties. It then only remains to show that, given an (n,k)-Sperner partition system with p partitions, a clutter satisfying the hypotheses of Lemma 3.5.3 can be obtained by considering the partition classes containing a particular element. (In fact, we must also do some tedious checking to ensure that (3.1) holds for "small" values of p not covered by Lemma 3.5.3.) In the proof of Theorem 3.1.2, this special element is chosen as one that, according to a certain metric, tends overall to appear in smaller partition classes.

Lemma 3.5.3. *Let n and k be integers such that $n \geqslant 2k+2$ and $k \geqslant 3$, and let p be an integer such that*
$$p \geqslant \max\left(\tfrac{2n}{c(2k-r)}\left(\tbinom{n-1}{c-1} - \tbinom{2c+1}{c-1}\right), \tbinom{n-1}{c-1}+1\right).$$
If there is a clutter with $n-1$ vertices and edge set $\mathcal{E}$ such that $|\mathcal{E}| \geqslant p$ and $\sum_{E\in\mathcal{E}} \frac{c-|E|}{|E|+1} \geqslant \frac{p(k-r)}{n}$, then
$$\left\lceil(1-\tfrac{r(c+1)}{n})p\right\rceil + \mathrm{LL}_c\left(\left\lfloor\tfrac{r(c+1)}{n}p\right\rfloor\right) \leqslant \binom{n-1}{c-1}.$$

Proof. Let H be a clutter satisfying the hypotheses of the lemma and let $X' = V(H)$. For each $i \in \{0,\dots,n-1\}$, let $\mathcal{E}_i = \{E \in \mathcal{E} : |E| = i\}$ and let $\mathcal{E}_{>c} = \mathcal{E}_{c+1} \cup \dots \cup \mathcal{E}_{n-1}$. We abbreviate $\lceil\frac{pc(k-r)}{n}\rceil$ to a_0. Note that $a_0 = \lceil(1-\frac{r(c+1)}{n})p\rceil$ using $n = ck+r$. We consider two cases according to minimum cardinality of an edge in $\mathcal{E}$.

Case 1. Suppose that $|E| \geqslant c-1$ for each $E \in \mathcal{E}$. Then the only edges in $\mathcal{E}$ that make a positive contribution toward $\sum_{E\in\mathcal{E}} \frac{c-|E|}{|E|+1}$ are those in $\mathcal{E}_{c-1}$ and so by our hypotheses we must have $\frac{1}{c}|\mathcal{E}_{c-1}| \geqslant \frac{p(k-r)}{n}$ and hence $|\mathcal{E}_{c-1}| \geqslant a_0$. Also, because $p \geqslant \binom{n-1}{c-1}+1$, we have $\mathcal{E} \nsubseteq \binom{X'}{c-1}$ and hence $|\mathcal{E}_c| + |\mathcal{E}_{>c}| \geqslant 1$. Let H^* be the hypergraph with vertex set X' and edge set $\mathcal{E}^* = (\mathcal{E} \setminus \mathcal{E}_{>c}) \cup \Delta^c(\mathcal{E}_{>c})$. Then $\mathcal{E}^* = \mathcal{E}_{c-1} \cup \mathcal{E}_c \cup \Delta^c(\mathcal{E}_{>c})$ and, because H is a clutter, H^* is a clutter and $\Delta^c(\mathcal{E}_{>c})$ is disjoint from $\mathcal{E}_c$.

There are $\binom{n-1}{c-1}$ sets in $\binom{X'}{c-1}$, and because H^* is a clutter each of these can be in at most one of $\mathcal{E}_{c-1}$ and $\Delta^{c-1}(\mathcal{E}_c \cup \Delta^c(\mathcal{E}_{>c}))$. Thus, by Theorem 3.5.1,
$$|\mathcal{E}_{c-1}| + \mathrm{LL}_c\big(|\mathcal{E}_c| + |\Delta^c(\mathcal{E}_{>c})|\big) \leqslant \binom{n-1}{c-1}. \tag{3.9}$$

We consider two subcases according to the value of $|\mathcal{E}_{>c}|$.

Case 1a. Suppose that $|\mathcal{E}_{>c}| \leqslant \binom{2c+1}{c}$. Then $|\Delta^c(\mathcal{E}_{>c})| \geqslant |\mathcal{E}_{>c}| = |\mathcal{E}| - |\mathcal{E}_c| - |\mathcal{E}_{c-1}|$ by Lemma 3.5.2. So $|\mathcal{E}_c| + |\Delta^c(\mathcal{E}_{>c})| \geqslant \max(p - |\mathcal{E}_{c-1}|, 1)$ because $|\mathcal{E}| \geqslant p$ and $|\mathcal{E}_c| + |\mathcal{E}_{>c}| \geqslant 1$. Thus, using the fact that LL_c is monotonically increasing, (3.9) implies that
$$f(|\mathcal{E}_{c-1}|) \leqslant \binom{n-1}{c-1} \text{ where } f(a) = a + \mathrm{LL}_c\left(\max\left(p-a,1\right)\right).$$

Consider f as a function on the real domain $a_0 \leqslant a \leqslant |\mathcal{E}|$, noting that we have seen $a_0 \leqslant |\mathcal{E}_{c-1}| \leqslant |\mathcal{E}|$. Because $f(|\mathcal{E}_{c-1}|) \leqslant \binom{n-1}{c-1}$, certainly the global minimum of f is at most $\binom{n-1}{c-1}$. Now, f is monotonically increasing for $p-1 < a \leqslant |\mathcal{E}|$ and, because LL_c is concave, f is concave for $a_0 \leqslant a \leqslant p-1$. Thus, f achieves its global minimum either at $a = a_0$ or at $a = p-1$. However, $f(p-1) = p-1+c$ and $p-1 \geqslant \binom{n-1}{c-1}$ by our hypotheses. Thus f achieves its global minimum at $a = a_0$ and we have $f(a_0) \leqslant \binom{n-1}{c-1}$. Now the result follows because

$$f(a_0) = a_0 + \mathrm{LL}_c(p - a_0) = \left\lceil \left(1 - \tfrac{r(c+1)}{n}\right)p \right\rceil + \mathrm{LL}_c\left(\left\lfloor \tfrac{r(c+1)}{n} p \right\rfloor \right).$$

Case 1b. Suppose that $|\mathcal{E}_{>c}| > \binom{2c+1}{c}$. Because $\sum_{E \in \mathcal{E}} \frac{c-|E|}{|E|+1} \geqslant \frac{p(k-r)}{n}$,

$$\tfrac{p(k-r)}{n} \leqslant \tfrac{1}{c}|\mathcal{E}_{c-1}| - \sum_{i=c+1}^{n-1} \tfrac{i-c}{i+1}|\mathcal{E}_i| \leqslant \tfrac{1}{c}|\mathcal{E}_{c-1}| - \tfrac{1}{c+2}|\mathcal{E}_{>c}|,$$

where the last inequality follows because $\frac{i-c}{i+1} \geqslant \frac{1}{c+2}$ for each $i \in \{c+1, \ldots, n-1\}$. Thus $|\mathcal{E}_{>c}| \leqslant \frac{c+2}{c}|\mathcal{E}_{c-1}| - \frac{p(c+2)(k-r)}{n}$. Also, $|\Delta^c(\mathcal{E}_{>c})| > \binom{2c+1}{c}$ by the hypothesis of this subcase and Lemma 3.5.2. Combining these facts and $|\mathcal{E}| \geqslant p$, we have

$$|\mathcal{E}_c| + |\Delta^c(\mathcal{E}_{>c})| = |\mathcal{E}| - |\mathcal{E}_{c-1}| - |\mathcal{E}_{>c}| + |\Delta^c(\mathcal{E}_{>c})| > \max\left(\tfrac{p(c+1)(2k-r)}{n} - \tfrac{2c+2}{c}|\mathcal{E}_{c-1}|, 0\right) + \binom{2c+1}{c}.$$

Thus, (3.9) implies that

$$g(|\mathcal{E}_{c-1}|) < \binom{n-1}{c-1} \text{ where } g(a) = a + \mathrm{LL}_c\left(\max\left(\tfrac{p(c+1)(2k-r)}{n} - \tfrac{2c+2}{c}a, 0\right) + \binom{2c+1}{c}\right).$$

Consider g as function on the real domain $a_0 \leqslant a \leqslant |\mathcal{E}|$ and note that the global minimum of g is less than $\binom{n-1}{c-1}$. Now, g is monotonically increasing for $a_1 < a \leqslant |\mathcal{E}|$ and concave for $a_0 \leqslant a \leqslant a_1$, where $a_1 = \frac{pc(2k-r)}{2n}$. Thus, it achieves its global minimum either at $a = a_0$ or at $a = a_1$. However, $g(a_1) = a_1 + \binom{2c+1}{c-1} \geqslant \binom{n-1}{c-1}$ using the hypothesis that $p \geqslant \frac{2n}{c(2k-r)}\left(\binom{n-1}{c-1} - \binom{2c+1}{c-1}\right)$. Thus we have $g(a_0) < \binom{n-1}{c-1}$. Now, setting $\delta = a_0 - \frac{pc(k-r)}{n}$ and noting that $0 \leqslant \delta < 1$,

$$g(a_0) = a_0 + \mathrm{LL}_c\left(p - a_0 + \binom{2c+1}{c} - \tfrac{c+2}{c}\delta\right) \geqslant a_0 + \mathrm{LL}_c(p - a_0).$$

As in Case 1a, the result follows.

Case 2. Suppose that $|E| \leqslant c-2$ for some $E \in \mathcal{E}$. Using Case 1 as a base case, we may suppose inductively that the minimum cardinality of an edge in $\mathcal{E}$ is $j \leqslant c-2$ and that the lemma holds when the minimum cardinality of an edge in $\mathcal{E}$ is $j+1$. For any family $\mathcal{S}$ of subsets of X, define $d'(\mathcal{S}) = \sum_{S \in \mathcal{S}} \frac{c-|S|}{|S|+1}$. Note we have assumed that $d'(\mathcal{E}) \geqslant \frac{p(k-r)}{n}$.

Let H^* be the hypergraph with vertex set X' and edge set $\mathcal{E}^* = (\mathcal{E} \setminus \mathcal{E}_j) \cup \nabla^{j+1}(\mathcal{E}_j)$. Because H is a clutter, H^* is a clutter and $\nabla^{j+1}(\mathcal{E}_j)$ is disjoint from $\mathcal{E}_{j+1}$. Thus it suffices to show that $d'(\nabla^{j+1}(\mathcal{E}_j)) \geqslant d'(\mathcal{E}_j)$ and $|\nabla^{j+1}(\mathcal{E}_j)| \geqslant |\mathcal{E}_j|$ because then we will be able to apply our inductive hypothesis to H^* to obtain the required result.

Each edge in $\mathcal{E}_j$ is a subset of $n-j-1$ edges in $\nabla^{j+1}(\mathcal{E}_j)$, and each edge in $\nabla^{j+1}(\mathcal{E}_j)$ is a superset of at most $j+1$ edges in $\mathcal{E}_j$. Thus $|\nabla^{j+1}(\mathcal{E}_j)| \geqslant \frac{n-j-1}{j+1}|\mathcal{E}_j|$ and

$$d'(\nabla^{j+1}(\mathcal{E}_j)) = \tfrac{c-j-1}{j+2}|\nabla^{j+1}(\mathcal{E}_j)| \geqslant \tfrac{(c-j-1)(n-j-1)}{(j+1)(j+2)}|\mathcal{E}_j| = \tfrac{(c-j-1)(n-j-1)}{(c-j)(j+2)}d'(\mathcal{E}_j),$$

where the second equality follows because $d'(\mathcal{E}_j) = \frac{c-j}{j+1}|\mathcal{E}_j|$. Thus $d'(\nabla^{j+1}(\mathcal{E}_j)) \geqslant d'(\mathcal{E}_j)$ and $|\mathcal{E}^*| \geqslant |\mathcal{E}|$ as required because, using $j \in \{0,\dots,c-2\}$ and $k \geqslant 3$, we have $c-j-1 \geqslant \frac{1}{2}(c-j)$ and $n-j-1 \geqslant 2(j+2)$. □

Proof of Theorem 3.1.2. Let $p_0 = \mathrm{SP}(n,k)$, let X be a set with $|X| = n$, and let $\mathcal{P}$ be an (n,k)-Sperner partition system on ground set X with p_0 partitions. We may assume $r \neq 0$ because, when $r = 0$, $\mathrm{SP}(n,k) = \binom{n-1}{c-1}$ and (3.1) clearly holds with $p = \binom{n-1}{c-1}$. So, in addition to $n \geqslant 2k+2$, we have $n \geqslant 4c+1$. Let $p_1 = \max(\frac{2n}{c(2k-r)}(\binom{n-1}{c-1} - \binom{2c+1}{c-1}), \binom{n-1}{c-1}+1)$.

Case 1. Suppose that $p_0 \geqslant p_1$. We will find a clutter satisfying the conditions of Lemma 3.5.3 and so complete the proof. For each $x \in X$, let $\mathcal{P}(x)$ be the set of all partition classes of $\mathcal{P}$ that contain x. For a subset S of X we define $d(S) = c+1-|S|$, and for a family $\mathcal{S}$ of subsets of X we define $d(\mathcal{S}) = \sum_{S \in \mathcal{S}} d(S)$. Note that, for each partition π in $\mathcal{P}$, we have $d(\pi) = k-r$ because π has exactly k classes and the sum of the cardinalities of the classes is equal to $n = ck+r$. For a vertex $x \in X$, we further define $d(x) = \sum_{S \in \mathcal{P}(x)} \frac{d(S)}{|S|}$. Thus we have that $\sum_{x \in X} d(x) = \sum_{\pi \in \mathcal{P}} d(\pi) = p_0(k-r)$. Let z be an element of X such that $d(z) \geqslant d(x)$ for each $x \in X$ and observe that $d(z) \geqslant \frac{p_0(k-r)}{n}$. Let H be the hypergraph with vertex set $X' = X \setminus \{z\}$ and edge set $\mathcal{E} = \{S \setminus \{z\} : S \in \mathcal{P}(z)\}$. Note that H is a clutter and $|\mathcal{E}| = p_0$ because $\mathcal{P}$ is a Sperner partition system with p_0 partitions. Thus, because $d(z) \geqslant \frac{p_0(k-r)}{n}$ and $p_0 \geqslant p_1$, H satisfies the conditions of Lemma 3.5.3 and we can apply it to produce the required result.

Case 2. Suppose that $p_0 < p_1$. In this case we show directly that (3.1) holds for some real number $p \geqslant p_1$ and hence that it holds for $p = p_0$ (recall that the left hand side of (3.1) is nondecreasing in p).

Case 2a. Suppose that $c = 2$. Then, when $r = 2$, we have $p_1 = 2k+2$ and (3.1) holds because $\mathrm{LL}_2(6) \leqslant 5$. When $r \geqslant 3$, $p_1 = \frac{2k+r}{2k-r}(2k+r-6)$ and it can be seen that (3.1) holds if and only if $\mathrm{LL}_2(\lfloor\frac{3r(2k+r-6)}{2k-r}\rfloor) \leqslant \lfloor r+5+\frac{2r(r-3)}{2k-r}\rfloor$. This holds for each integer $r \geqslant 4$ because then $\frac{3r(2k+r-6)}{2k-r} < 9r \leqslant \binom{r+5}{2}$. It also holds for $r = 3$ because $\mathrm{LL}_2(9) \leqslant 8$.

Case 2b. Suppose that $c \geqslant 3$. Let $p_2 = \frac{2n}{c(2k-r)}(\binom{n-1}{c-1} - 1)$ and note that $p_2 \geqslant p_1$. Noting that $1 - \frac{r(c+1)}{n} = \frac{c(k-r)}{n}$, it can be seen that (3.1) will hold with $p = p_2$ provided that

$$\mathrm{LL}_c\left(\tfrac{2r(c+1)}{c(2k-r)}\left(\tbinom{n-1}{c-1} - 1\right)\right) \leqslant \left\lfloor \tfrac{r}{2k-r}\tbinom{n-1}{c-1} + \tfrac{2k-2r}{2k-r}\right\rfloor. \tag{3.10}$$

Let $z = \frac{2r(c+1)}{c(2k-r)}(\binom{n-1}{c-1} - 1)$ be the argument of LL_c in (3.10) and note that if $z \geqslant \binom{3c+1}{c}$, then it follows from the definition of LL_c that $\mathrm{LL}_c(z) \leqslant \frac{c}{2c+2}z$ and thus that (3.10) holds. Because $r \geqslant 1$, we have $z \geqslant \frac{2c+2}{2n-c-2}(\binom{n-1}{c-1} - 1)$. This latter expression is an increasing function of n for $n \geqslant 4c+1$. Thus, for $c \geqslant 9$ we have $z \geqslant \binom{3c+1}{c}$ because $z \geqslant \frac{2c+2}{2n-c-2}(\binom{n-1}{c-1} - 1) \geqslant \frac{2c+2}{7c}(\binom{4c}{c-1} - 1)$ and

$$\tfrac{2c+2}{7c}\left(\binom{4c}{c-1} - 1\right)/\binom{3c+1}{c} = \tfrac{2c+2}{21c+7}\binom{4c}{c-1}/\binom{3c}{c-1} - \tfrac{2c+2}{7c}/\binom{3c+1}{c} \geqslant \tfrac{2c+2}{21c+7}(\tfrac{4}{3})^{c-1} - 10^{-7} \geqslant 1.$$

Furthermore, by explicit calculation, we have $z \geqslant \frac{2c+2}{7c}(\binom{4c}{c-1} - 1) \geqslant \binom{3c+1}{c}$ for $c = 8$. We also have $z \geqslant \frac{2c+2}{2n-c-2}(\binom{n-1}{c-1} - 1) \geqslant \binom{3c+1}{c}$ for $c \in \{4,5,6,7\}$ and $n \geqslant 31$ and for $c = 3$ and $n \geqslant 61$. This leaves only a limited number of pairs (n,k) to be checked. Using a computer, it is routine to compute p_1 for each pair and verify that (3.1) holds for $p = p_1$. □

We conclude this section by showing that a slightly weaker version of the upper bound implied by Theorem 3.1.2 can be written in a form that is very reminiscent of the expression for $\mathrm{MMS}(n,k)$, and that this implies that our upper bound is always at least as good as $\mathrm{MMS}(n,k)$.

Corollary 3.5.4. *If n and k are integers such that $n \geqslant 2k+2$, $k \geqslant 4$ and $r \neq 0$,*

$$\mathrm{SP}(n,k) \leqslant \frac{\binom{n}{c}}{(k-r) + \frac{r(c+1)}{q-c+1}} \tag{3.11}$$

where q is the real number such that $q \geqslant c$ and $\binom{q}{c} = \frac{r(c+1)}{n} \cdot \mathrm{SP}(n,k)$. Furthermore, the bound implied by Theorem 3.1.2 is less than $\mathrm{MMS}(n,k)$.

Proof. Observe that $\mathrm{SP}(n,k) \geqslant \mathrm{NLB}(n,k) = \frac{1}{k}\binom{ck}{c}$ implies $\frac{r(c+1)}{n} \cdot \mathrm{SP}(n,k) > \frac{r}{k^2}\binom{ck}{c}$. Further, it is routine to verify $\frac{r}{k^2}\binom{ck}{c} \geqslant \frac{r}{16}\binom{4c}{c} > \binom{2c-1}{c}$ since $r \geqslant 1$ when $c \geqslant 3$ and $r \geqslant 2$ when $c = 2$. Thus we have $\frac{r(c+1)}{n} \cdot \mathrm{SP}(n,k) > \binom{2c-1}{c}$. It follows that q is well defined. Further, because $\mathrm{LL}_c(1) = c$, $\mathrm{LL}_c(\binom{2c-1}{c}) = \binom{2c-1}{c}$ and LL_c is concave, the derivative of $\mathrm{LL}_c(x)$ is less than 1 for all $x \geqslant \binom{2c-1}{c}$ and hence, for any real $\epsilon > 0$,

$$\mathrm{LL}_c\left(\left\lfloor \tfrac{r(c+1)}{n} \cdot \mathrm{SP}(n,k)\right\rfloor + \epsilon\right) < \mathrm{LL}_c\left(\left\lfloor \tfrac{r(c+1)}{n} \cdot \mathrm{SP}(n,k)\right\rfloor\right) + \epsilon.$$

Thus we can deduce from Theorem 3.1.2 the slightly weaker conclusion that

$$\left(1 - \tfrac{r(c+1)}{n}\right) \cdot \mathrm{SP}(n,k) + \mathrm{LL}_c\left(\tfrac{r(c+1)}{n} \cdot \mathrm{SP}(n,k)\right) \leqslant \binom{n-1}{c-1}. \tag{3.12}$$

By applying $\mathrm{LL}_c(x) = \frac{c}{q-c+1}x$ in (3.12) and solving for $\mathrm{SP}(n,k)$ we obtain (3.11).

Now, using $\mathrm{SP}(n,k) \leqslant \mathrm{MMS}(n,k)$, we have

$$\tfrac{r(c+1)}{n} \cdot \mathrm{SP}(n,k) \leqslant \frac{\binom{n-1}{c}}{1+\frac{(n-c)(k-r)}{r(c+1)}} \leqslant \frac{1}{2}\binom{n-1}{c}.$$

Thus, $q < n-1$ and so the bound implied by this corollary, and hence the bound implied by Theorem 3.1.2, is less than $\mathrm{MMS}(n,k)$. □

3.6 Bounds for small $\boldsymbol{n}$ and $\boldsymbol{k}$

We conclude this chapter by displaying the values of the upper and lower bounds we have obtained for some small parameters (n,k).

In Table 3.1 we list, for $4 \leqslant k \leqslant 7$ and $2k+2 \leqslant n \leqslant 33$ a lower bound and an upper bound on $\mathrm{SP}(n,k)$ in the top and bottom rows respectively of the appropriate cell. The upper bound is the bound implied by Theorem 3.1.2 and is followed by the improvement over $\mathrm{MMS}(n,k)$ in brackets. The lower bound is the best one attainable via our results and those of [28, 33] and is followed by the source of the bound according to the following key. "M" refers to a bound obtained through the monotonicity of $\mathrm{SP}(n,k)$ in n; "[28]" refers to one of the bounds given in [28] (and stated in our introduction); "L3.2.5" refers to Lemma 3.2.5 and is followed by the values of m and n used; and finally "L3.3.3" and "L3.4.1" refer to Lemmas 3.3.3 and 3.4.1 and are followed by the value of u used. The exception to the above is when k divides n, in which case the known exact value of $\mathrm{SP}(n,k)$ is placed by itself in the cell. Lemma 3.3.3 and Lemma 3.4.1 give only weak results for the case where $k=3$ and this is why $k=3$ cases were not included in Table 3.6.

Figures 3.1 and 3.2 visualise bounds on $\mathrm{SP}(n,k)$ for the example values $k=5$ and $k=10$ respectively. Values of n between $2k+2$ and 100 appear on the horizontal axis, and above each are a grey and a black line segment. The grey segment gives the interval between $\mathrm{NLB}(n,k)$ and $\mathrm{MMS}(n,k)$, whereas the black segment gives the interval between the best known lower and upper bounds on $\mathrm{SP}(n,k)$ according to the results in this chapter and in [28, 33]. Note that the vertical axis is log scaled.

n	$k=4$	$k=5$	$k=6$	$k=7$
10	10 L3.3.3 (1) 11 (5)			
11	11 [28] 19 (8)			
12	55	12 L3.3.3 (1) 13 (5)		
13	55 M 72 (12)	12 M 19 (8)		
14	55 M 110 (23)	17 L3.3.3 (1) 33 (12)	13 [28] 15 (5)	
15	55 M 190 (37)	91	13 M 20 (8)	
16	455	91 M 114 (16)	28 L3.3.3 (1) 29 (13)	15 [28] 17 (5)
17	455 M 636 (67)	91 M 162 (28)	28 M 51 (17)	15 M 21 (8)
18	648 L3.3.3 (2) 994 (133)	91 M 243 (48)	136	27 L3.3.3 (1) 30 (10)
19	648 M 1 719 (219)	91 M 410 (74)	136 M 167 (17)	27 M 42 (17)
20	3 876	969	210 L3.3.3 (1) 221 (34)	40 L3.3.3 (1) 70 (25)
21	3 876 M 5 601 (428)	969 M 1 290 (103)	210 M 308 (54)	190
22	5 544 L3.3.3 (2) 8 844 (888)	1 008 L3.3.3 (2) 1 849 (208)	210 M 454 (87)	190 M 227 (20)
23	5 544 M 15 355 (1 469)	1 008 M 2 808 (366)	210 M 751 (134)	190 M 291 (36)
24	33 649	3 366 L3.3.3 (2) 4 734 (579)	1 771	190 M 384 (58)
25	33 649 M 49 605 (2 971)	10 626	1 771 M 2 271 (144)	190 M 525 (92)
26	40 898 L3.3.3 (3) 78 927 (6 343)	10 626 M 14 514 (834)	1 771 M 3 071 (285)	286 L3.4.1 (2) 762 (144)
27	40 898 M 137 410 (10 595)	10 626 M 21 020 (1 750)	1 771 M 4 311 (494)	286 M 1 242 (220)
28	296 010	16 016 L3.4.1 (3) 32 169 (3 150)	4 140 L3.3.3 (2) 6 408 (818)	2 925
29	296 010 M 442 270 (21 745)	16 830 L3.2.5 (5, 24) 54 342 (5 035)	4 140 M 10 606 (1 269)	2 925 M 3 643 (187)
30	621 075 L3.3.3 (3) 707 796 (47 420)	118 755	23 751	3 003 L3.3.3 (1) 4 723 (366)
31	621 075 M 1 234 969 (79 818)	118 755 M 164 701 (7 327)	23 751 M 31 093 (1 389)	3 003 M 6 291 (615)
32	2 629 575	139 568 L3.3.3 (2) 240 248 (15 849)	33 600 L3.3.3 (2) 42 433 (2 876)	4 800 L3.3.3 (2) 8 682 (999)
33	2 629 575 M 3 966 925 (165 264)	139 568 M 369 680 (29 044)	33 600 M 60 038 (5 113)	4 800 M 12 696 (1 601)

Table 3.1: Lower and upper bounds on $\mathrm{SP}(n,k)$

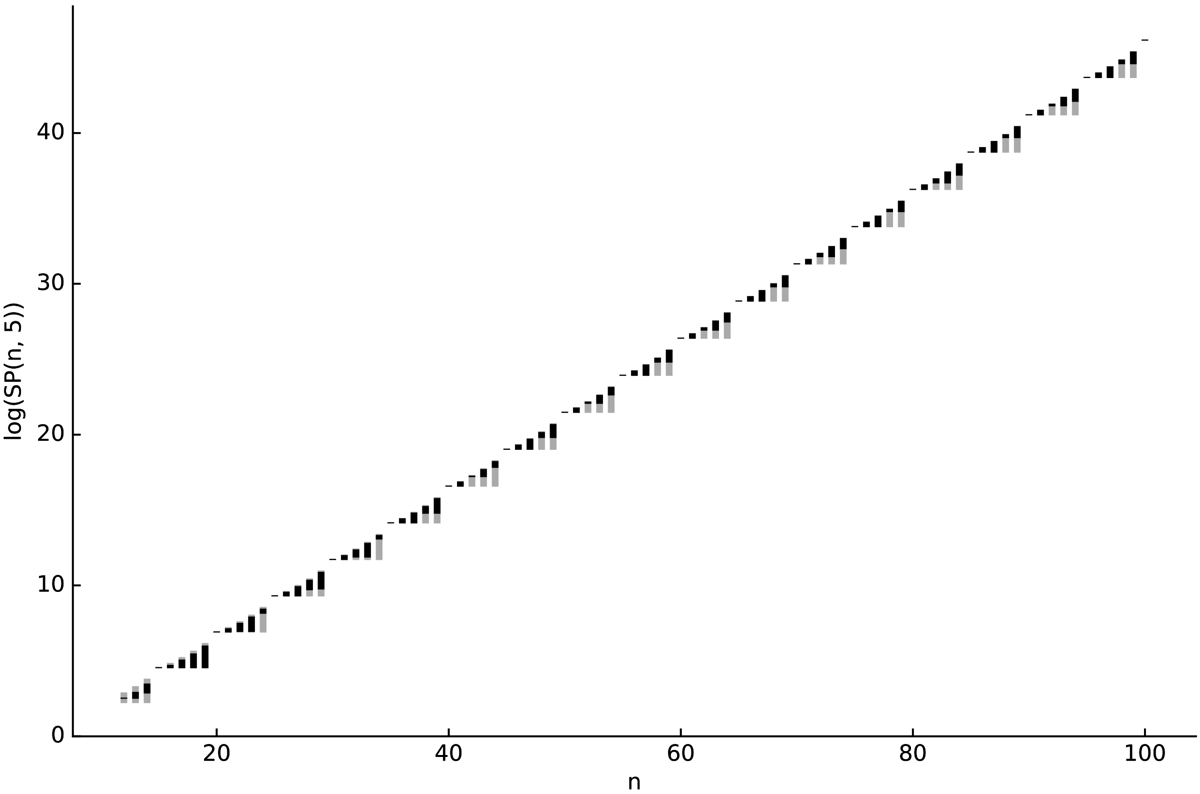

Figure 3.1: Best known bounds on $\mathrm{SP}(n, 5)$ compared to $\mathrm{NLB}(n, 5)$ and $\mathrm{MMS}(n, 5)$

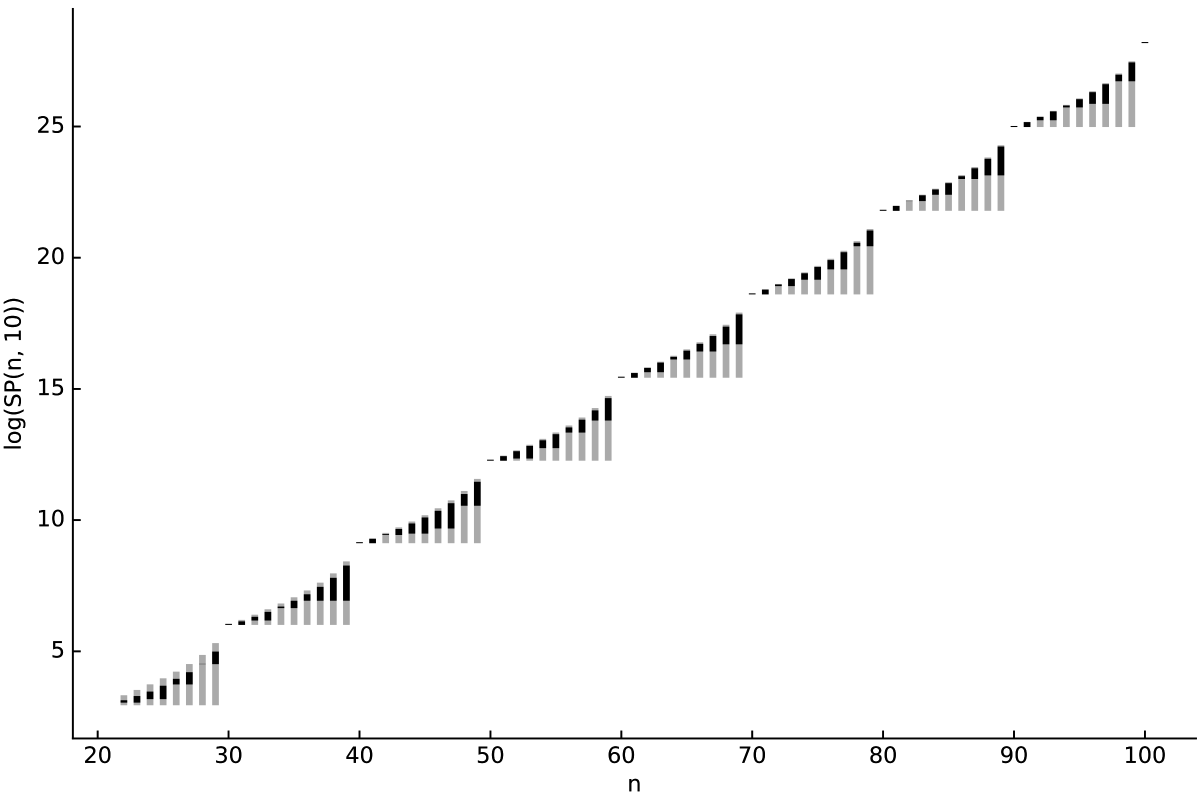

Figure 3.2: Best known bounds on $\mathrm{SP}(n, 10)$ compared to $\mathrm{NLB}(n, 10)$ and $\mathrm{MMS}(n, 10)$

Chapter 4

More constructions for Sperner partition systems

4.1 Introduction

Continuing on from our work in Chapter 3, this chapter looks at answering some of the remaining questions related to Sperner partition systems. Our first main focus is on giving an asymptotic determination of $\mathrm{SP}(n,k)$ when c is bounded, except in cases where r is very close to k. We then present a number of new bounds on $\mathrm{SP}(n,k)$ for the special case of $c=2$. Finally, we go on to show that $\mathrm{SP}(n,k)$ is indeed asymptotic to $\mathrm{MMS}(n,k)$ when n is even and either $r=1$ and k is bounded or $r=k-1$.

Recall that for a pair of integers (n,k) with $n \geqslant k \geqslant 1$, we define $c=c(n,k)$ and $r=r(n,k)$ as the unique integers such that $n=ck+r$ and $r \in \{0,\dots,k-1\}$. These definitions for $c(n,k)$ and $r(n,k)$ are used throughout this chapter and are abbreviated to simply c and r where there is no danger of confusion. As part of Chapter 3, in many cases where c grows along with n, we presented a construction for Sperner partition systems with number of partitions asymptotic to $\mathrm{MMS}(n,k)$.

Theorem 3.1.1. *Let n and k be integers with $n \to \infty$, $k=o(n)$ and $k \geqslant 3$, and let c and r be the integers such that $n=ck+r$ and $r \in \{0,\dots,k-1\}$. Then $\mathrm{SP}(n,k) \sim \mathrm{MMS}(n,k)$ if*

- *n is even and $r \notin \{1,k-1\}$; or*
- *$k-r \to \infty$.*

The condition $k=o(n)$ in Theorem 3.1.1 is equivalent to saying $c \to \infty$, so Theorem 3.1.1 does not cover the case where c is bounded as n grows. Certain very specific cases in the regime where c is bounded have been investigated. When $c=1$, it is not hard to see that $\mathrm{SP}(n,k)=1$. For $c=2$, Li and Meagher [28] found bounds on $\mathrm{SP}(2k+1,k)$, $\mathrm{SP}(2k+2,k)$ and $\mathrm{SP}(3k-1,k)$. As mentioned above, our first main focus in this chapter is on giving an asymptotic determination of $\mathrm{SP}(n,k)$ when c is bounded, except in cases where r is very close to k.

Theorem 4.1.1. *Let n and k be integers with $n\to\infty$, $k\leqslant\frac{n}{2}$ and $k-r=\Theta(n)$ where c and r are the integers such that $n=ck+r$ and $r\in\{0,\dots,k-1\}$. Then $\mathrm{SP}(n,k)\sim\mathrm{MMS}(n,k)$.*

Note that the condition $k-r=\Theta(n)$ implies $k=\Theta(n)$ and hence that $c=O(1)$. Theorem 4.1.1 is proved by introducing a new construction for Sperner partition systems which is based on a division of the ground set into many equal-sized parts (see Lemma 4.3.2). In the special case $c=2$ we are able to say more (see Section 4.4), including determining $\mathrm{SP}(n,k)$ exactly for a number of small parameter sets (n,k) and narrowing it down to either an exact value, or one of two different values for two different infinite families.

Lemma 4.1.2. *Let $k\geqslant 11$ be an integer such that $k\not\equiv 4 \pmod 6$ and let $n=3k-6$. Then $\mathrm{SP}(n,k)=\lfloor\frac{1}{2}(k-2)^2\rfloor$.*

Theorem 4.1.3. *Let $k\geqslant 4$ be an even integer and let $n=3k-2$. Then $\mathrm{SP}(n,k)\in\{\binom{n/2}{2},\binom{n/2}{2}+1\}$.*

Theorem 3.1.1 also does not cover the cases where $r=1$ and k is bounded or where $r=k-1$. Here we show that in most of these cases, if n is even, $\mathrm{SP}(n,k)$ is indeed asymptotic to $\mathrm{MMS}(n,k)$.

Theorem 4.1.4.

(a) *Let n and k be integers such that $n\to\infty$ with $n\equiv k+1 \pmod{2k}$, $k=o(n)$, and $k\geqslant 3$ is odd. Then $\mathrm{SP}(n,k)\sim\mathrm{MMS}(n,k)$.*

(b) *Let n and k be integers such that $n\to\infty$ with $n\equiv k-1 \pmod{2k}$, $k=o(n)$, and $k\geqslant 5$ is odd. Then $\mathrm{SP}(n,k)\sim\mathrm{MMS}(n,k)$.*

We prove Theorem 4.1.4 by extending a construction method used in Chapter 3 and analysing its behaviour. This extended construction method incorporates a solution to a particular integer program, where the objective value of the program gives the size of the Sperner partition system produced. With some effort, we are able to show that in most cases the optimal value of this integer program is asymptotic to $\mathrm{MMS}(n,k)$ and so prove Theorem 4.1.4. In the case $n\equiv k-1 \pmod{2k}$ and $k=3$ we do not prove this, but we present strong numerical evidence that the optimal value of the integer program is such that the result still holds.

This Chapter is organised as follows. Section 4.2 introduces some notation we will require as well as a key result, a consequence of a result of [6], that underlies our constructions. In Section 4.3 we introduce our new construction for Sperner partition systems, based on a division of the ground set into many equal-sized parts, and use this to prove Theorem 4.1.1. In Section 4.4 we then examine the special case where $c = 2$, in the process proving Theorem 4.1.3 and exhibiting many small parameter sets for which the construction from the previous section produces Sperner partition systems of maximum size. Sections 4.5 and 4.6 are then devoted to proving Theorem 4.1.4(a) and (b) respectively, using an extension of the construction for Sperner partition systems given in Section 3.3 and Section 3.4. In the Conclusion (Chapter 7), we provide some numerical evidence that Theorem 4.1.4(b) also holds for $k = 3$ and discuss possible further work.

4.2 Preliminaries

Here we restate a number of definitions and results that are important to the work in this chapter.

An (n, k)-Sperner partition system is called *almost uniform* if each class of each partition in the system is of size $\lfloor \frac{n}{k} \rfloor$ or $\lceil \frac{n}{k} \rceil$. Note that this means that there must be $k - r$ classes of size c and r classes of size $c + 1$ in each partition. It is conjectured in [33] that for all n and k with $n \geqslant k \geqslant 1$ there is an almost uniform Sperner partition system with $\mathrm{SP}(n, k)$ partitions.

In [28], the authors observe that taking an (n, k)-Sperner partition system and adding a new element to an arbitrary class of each partition results in an $(n+1, k)$-Sperner partition system of the same size, as shown in the proof of Lemma 2.2.4. Thus we have

$$\mathrm{SP}(n + 1, k) \geqslant \mathrm{SP}(n, k) \quad \text{for all integers } n \geqslant k \geqslant 1, \tag{4.1}$$

a fact that we will use frequently. If the original Sperner partition system is almost uniform and the new element is added to a class of minimum size in each partition, then the resulting $(n + 1, k)$-Sperner partition system is also almost uniform. Although we do not state it explicitly each time, all the constructions in this chapter produce almost uniform systems. For a set S and a nonnegative integer i, we denote the set of all i-subsets of S by $\binom{S}{i}$. Note that $\left|\binom{S}{i}\right| = \binom{|S|}{i}$.

A *hypergraph* H consists of a vertex set $V(H)$ together with a set $\mathcal{E}(H)$ of edges, each of which is a nonempty subset of $V(H)$. We do not allow multiple edges. A *clutter* is a hypergraph for which no edge is a subset of another. As such, a clutter is exactly a Sperner set system, but we use the term clutter when we wish to consider the object through a hypergraph-theoretic lens.

A *partial edge colouring* of a hypergraph is simply an assignment of colours to some or all of its edges with no further conditions imposed. Let γ be a partial edge colouring of a hypergraph H with colour set C. For each $z \in C$, the set $\gamma^{-1}(z)$ of edges of H assigned colour z is called a *colour class* of γ. For each $z \in C$ and $x \in V(H)$, let the number of edges of H that are assigned the colour z by γ and contain the vertex x be denoted $\deg_z^\gamma(x)$.

Throughout this chapter, we will again make extensive use of Lemma 3.2.2 (which we restate here for convenience), which is a consequence of a more general and powerful result of Bryant [6]. It allows the construction of a Sperner partition system to be reduced to finding a partial edge colouring of a hypergraph with appropriate properties, which can greatly simplify the task.

Lemma 3.2.2. *Let n and k be integers with $n \geqslant k \geqslant 1$, let H be a clutter with $|V(H)| = n$, and let $\{X_1, \ldots, X_t\}$ be a partition of $V(H)$ such that any permutation of X_w is an automorphism of H for each $w \in \{1, \ldots, t\}$. Suppose there is a partial edge colouring γ of H with colour set C such that, for each $z \in C$, $|\gamma^{-1}(z)| = k$ and $\sum_{x \in X_w} \deg_z^\gamma(x) = |X_w|$ for each $w \in \{1, \ldots, t\}$. Then there is an (n, k)-Sperner partition system with $|C|$ partitions such that the classes of the partitions form a subset of $\mathcal{E}(H)$.*

4.3 Proof of Theorem 4.1.1

Our goal in this section is to prove Theorem 4.1.1. We achieve this by first introducing a new construction for Sperner partition systems and then showing that the construction can produce systems with size asymptotic to $\mathrm{MMS}(n, k)$ in the regime where c is bounded and r is not too close to k.

We now introduce a simple lemma which will be useful in detailing our construction. It will eventually allow us to distribute the edges of a hypergraph evenly between colour classes when attempting to define a colouring satisfying the hypotheses of Lemma 3.2.2. The sum of all entries in a row or column of a matrix is referred to as a *row sum* or *column sum* respectively.

Lemma 4.3.1. *Let s_1 and s_2 be positive integers and x and $b \leqslant s_2$ be nonnegative integers. There exists an $s_1 \times s_2$ matrix T such that each row of T has b occurrences of $x + 1$ and $s_2 - b$ occurrences of x, and any two column sums in T differ by at most 1.*

Proof. We proceed by induction on s_1. The result is clearly true when $s_1 = 1$, so let $s_1' \in \{1, \ldots, s_1 - 1\}$ and suppose there exists an $s_1' \times s_2$ matrix T' with the required properties. Let Y be the set of columns of T' whose sum is the minimum column sum in T'. Add to T' a new row with b occurrences of $x+1$ and s_2-b occurrences of x, placed so that each column in Y contains an occurrence of $x+1$ if $b \geqslant |Y|$ and so that each occurrence of $x+1$ is in a column in Y if $b < |Y|$. It can be checked that the resulting matrix has the required properties. $\square$

We now introduce the construction that will be used to prove Theorem 4.1.1. An example of this construction is illustrated in Figure 4.1.

Lemma 4.3.2. *Let n, c, k and r be integers such that $n = ck + r$, $c \geqslant 2$, $k \geqslant 3$ and $r \in \{1, 2, \dots, k-1\}$. Suppose that $n = hm$ for positive integers m and h such that $m \equiv 0 \pmod{c}$ and let*

$$p_1 = \left\lfloor \frac{m(h^c - c - 1)}{c(k-r)} \right\rfloor, \quad p_2 = \left\lfloor \frac{m}{r} \left\lfloor \frac{\binom{h}{c+1}}{\binom{m-1}{c-1}} \right\rfloor \right\rfloor, \quad p_1' = \left\lfloor \frac{mh^c}{c(k-r)} \right\rfloor, \quad p_2' = \left\lfloor \frac{m\binom{h}{c+1}}{r\binom{m-1}{c-1}} \right\rfloor.$$

There exists an almost uniform (n,k)-Sperner partition system with $p\binom{m-1}{c-1}$ partitions if

(a) $p = \min\{p_1, p_2\}$*; or*

(b) $p = \min\{p_1', p_2'\}$ *and* $pr \equiv 0 \pmod{m}$.

Proof. Suppose the hypotheses of (a) hold or that those of (b) do. First note that $r \equiv 0 \pmod{c}$ since $n \equiv 0 \pmod{c}$ and $ck \equiv 0 \pmod{c}$. We will construct our Sperner partition system on a ground set $X = X_1 \cup \dots \cup X_m$, where $X_1, \dots, X_m$ are pairwise disjoint sets such that $|X_1| = \dots = |X_m| = h$. Let $\mathcal{M} = \binom{\{1,\dots,m\}}{c}$ and let $J = \{1, \dots, \binom{m-1}{c-1}\} \times \{1, \dots, \frac{m}{c}\}$. By Baranyai's theorem [2], we can index the sets in $\mathcal{M}$ so that $\mathcal{M} = \{S_{\ell,i} : (\ell, i) \in J\}$ and $\{S_{\ell,i} : i \in \{1, \dots, \frac{m}{c}\}\}$ is a partition of $\{1, \dots, m\}$ for each $\ell \in \{1, \dots, \binom{m-1}{c-1}\}$. Let H be a hypergraph with vertex set X and edge set $\mathcal{A} \cup \mathcal{B}$ where

$$\mathcal{A} = \bigcup_{(\ell,i)\in J} \mathcal{A}_{\ell,i} \qquad \text{for} \qquad \mathcal{A}_{\ell,i} = \left\{E \in \tbinom{X}{c} : |E \cap X_w| = 1 \text{ for each } w \in S_{\ell,i}\right\},$$

$$\mathcal{B} = \bigcup_{w=1}^{m} \mathcal{B}_w \qquad \text{for} \qquad \mathcal{B}_w = \tbinom{X_w}{c+1}.$$

The indexing of the sets in $\mathcal{M}$ will act as a guide for a partial edge colouring of H. Let C' be a set with $|C'| = p$ and let $C = C' \times \{1, 2, \dots, \binom{m-1}{c-1}\}$ be a set of colours. It is clear that any permutation of X_w is an automorphism of H for each $w \in \{1, \dots, m\}$. Thus, by Lemma 3.2.2, to find an (n,k)-Sperner partition system with $p\binom{m-1}{c-1}$ partitions, it suffices to find a partial edge colouring γ of H with colour set C such that for each $z \in C$, $|\gamma^{-1}(z)| = k$ and $\sum_{x \in X_w} \deg_z^{\gamma}(x) = h$ for each $w \in \{1, \dots, m\}$. We proceed to show that such a partial edge colouring γ exists.

Let $x = \lfloor \frac{r}{m} \rfloor$ and $b = \frac{1}{c}(r - mx)$, noting that b will be a non-negative integer and let $T = (t_{z,i})$ be a $p \times \frac{m}{c}$ matrix with b occurrences of $x+1$ and $\frac{m}{c} - b$ occurrences of x in each row such that any two column sums differ by at most 1. Such a matrix exists by Lemma 4.3.1. We consider the rows of T to be indexed by the elements of C'. It follows from our definition of b that each row sum in T is $\frac{r}{c}$. Thus the sum of all the entries in T is $\frac{pr}{c}$ and hence, because T has $\frac{m}{c}$ columns and any two column sums differ by at most 1, each column sum in T is in $\{\lfloor \frac{pr}{m} \rfloor, \lceil \frac{pr}{m} \rceil\}$. So we have

$$\sum_{i=1}^{m/c} t_{z,i} = \tfrac{r}{c} \text{ for each } z \in C', \quad \text{and} \quad \sum_{z \in C'} t_{z,i} \in \left\{ \left\lfloor \tfrac{pr}{m} \right\rfloor, \left\lceil \tfrac{pr}{m} \right\rceil \right\} \text{ for each } i \in \{1, \dots, \tfrac{m}{c}\}. \tag{4.2}$$

We create our partial edge colouring γ of H by, for all $(z, \ell) \in C$ and $i \in \{1, \dots, \frac{m}{c}\}$, one at a time in arbitrary order, performing the following process.

- Assign the colour (z, ℓ) to $h - (c+1)t_{z,i}$ previously uncoloured edges in $\mathcal{A}_{\ell,i}$.
- For each $w \in S_{\ell,i}$, assign the colour (z, ℓ) to $t_{z,i}$ previously uncoloured edges in $\mathcal{B}_w$.

After performing this process for all $(z, \ell) \in C$ and $i \in \{1, \dots, \frac{m}{c}\}$, we call the resulting colouring γ. We will show that there are always uncoloured edges available throughout this process and that γ satisfies the conditions we require of it.

(i) Let $(\ell, i) \in J$. We show that the number of edges in $\mathcal{A}_{\ell,i}$ assigned colours is at most $|\mathcal{A}_{\ell,i}| = h^c$. These edges only receive colours in $C' \times \{\ell\}$ and, for each $z \in C'$, the number that receive colour (z, ℓ) is $h - (c+1)t_{z,i}$. So the total number that are assigned a colour is $\sum_{z \in C'}(h - (c+1)t_{z,j})$, which is at most $ph - (c+1)\lfloor \frac{pr}{m} \rfloor$ by (4.2). If the hypotheses of (a) hold then

$$ph - (c+1)\left\lfloor \frac{pr}{m} \right\rfloor < ph - (c+1)\left(\frac{pr}{m} - 1\right) = \frac{pc(k-r)}{m} + c + 1 \leqslant h^c$$

where the equality follows by substituting $h = \frac{1}{m}(ck + r)$ and the last inequality is obtained using $p \leqslant p_1$ and the definition of p_1. If the hypotheses of (b) hold then similarly we have

$$ph - (c+1)\left\lfloor \frac{pr}{m} \right\rfloor = ph - (c+1)\frac{pr}{m} = \frac{pc(k-r)}{m} \leqslant h^c$$

where the last inequality is obtained using $p \leqslant p'_1$ and the definition of p'_1.

(ii) Let $w \in \{1, \dots, m\}$. We show that we do not run out of uncoloured edges in $\mathcal{B}_w$ by showing that, for each $\ell \in \{1, \dots, \binom{m-1}{c-1}\}$, the number of edges in $\mathcal{B}_w$ assigned a colour in $C' \times \{\ell\}$ is at most $|\mathcal{B}_w| / \binom{m-1}{c-1} = \binom{h}{c+1} / \binom{m-1}{c-1}$. Let $\ell \in \{1, \dots, \binom{m-1}{c-1}\}$ and let i be the unique element of $\{1,\ldots,\frac{m}{c}\}$ such that $w \in S_{\ell,i}$. Then the number of edges in $\mathcal{B}_w$ assigned a colour in $C' \times \{\ell\}$ is $\sum_{z\in C'} t_{z,i}$, and this is at most $\lceil\frac{pr}{m}\rceil$ by (4.2). If the hypotheses of (a) hold then, $\lceil\frac{pr}{m}\rceil \leqslant \lceil\frac{p_2 r}{m}\rceil$ and we obtain the required bound using the definition of p_2. If the hypotheses of (b) hold then, $\lceil\frac{pr}{m}\rceil = \frac{pr}{m} \leqslant \frac{p_2' r}{m}$ and we can obtain the required bound using the definition of p_2'.

(iii) Let (z,ℓ) be a colour in C. We show that $|\gamma^{-1}((z,\ell))| = k$. For each $i \in \{1,\ldots,\frac{m}{c}\}$, we assign (z,ℓ) to $h-(c+1)t_{z,i}$ edges in $\mathcal{A}$ and, because $|S_{\ell,i}| = c$, to $ct_{z,i}$ edges in $\mathcal{B}$. So

$$\left|\gamma^{-1}\big((z,\ell)\big)\right| = \sum_{i=1}^{m/c}(h-(c+1)t_{z,i}+ct_{z,i}) = \frac{hm}{c} - \sum_{i=1}^{m/c} t_{z,i} = \frac{hm}{c} - \frac{r}{c} = k$$

where the third equality follows by (4.2) and the last because $hm = ck + r$.

(iv) Let $w \in \{1,\ldots,m\}$ and $(z,\ell) \in C$. We show that $\sum_{x\in X_w} \deg^{\gamma}_{(z,\ell)}(x) = h$. Let i be the unique element of $\{1,\ldots,\frac{m}{c}\}$ such that $w \in S_{\ell,i}$. Then (z,ℓ) is assigned to $h-(c+1)t_{z,i}$ edges in $\mathcal{A}_{\ell,i}$, each of which contains one vertex in X_w, and to $t_{z,i}$ edges in $\mathcal{B}_w$, each of which contains $c+1$ vertices in X_w. Any other edges assigned (z,ℓ) are disjoint from X_w. Thus

$$\sum_{x\in X_w} \deg^{\gamma}_{(z,\ell)}(x) = (h-(c+1)t_{z,j}) + (c+1)t_{z,j} = h.$$

So by (i) and (ii) we can indeed obtain the partial edge colouring γ as we claimed and by (iii) and (iv) γ has the required properties. So we can apply Lemma 3.2.2 to obtain an almost uniform (n,k)-Sperner partition system with $p\binom{m-1}{c-1}$ partitions as discussed. □

Note that we could potentially include all c-subsets of X that are not subsets of an X_i as edges of our clutter above. However, attempting to use all of these would make finding a suitable colouring γ more difficult. Moreover, as m becomes large, the number of c-sets we do not use is an asymptotically insignificant fraction of the number of those that we do. With this new construction for Sperner partition systems in hand, we are now able to prove Theorem 4.1.1.

Proof of Theorem 4.1.1. Observe that we have $n = \Theta(k) = \Theta(k-r)$ and hence $c = O(1)$. We consider two cases according to the value of r.

Case 1. Suppose that $r \leqslant k^{(2c-1)/2c}$. So $r = o(k)$ and $n \sim ck$. Clearly $\mathrm{SP}(ck,k) \leqslant \mathrm{SP}(n,k) \leqslant \mathrm{MMS}(n,k)$ using (4.1), so to complete the proof it suffices to show that

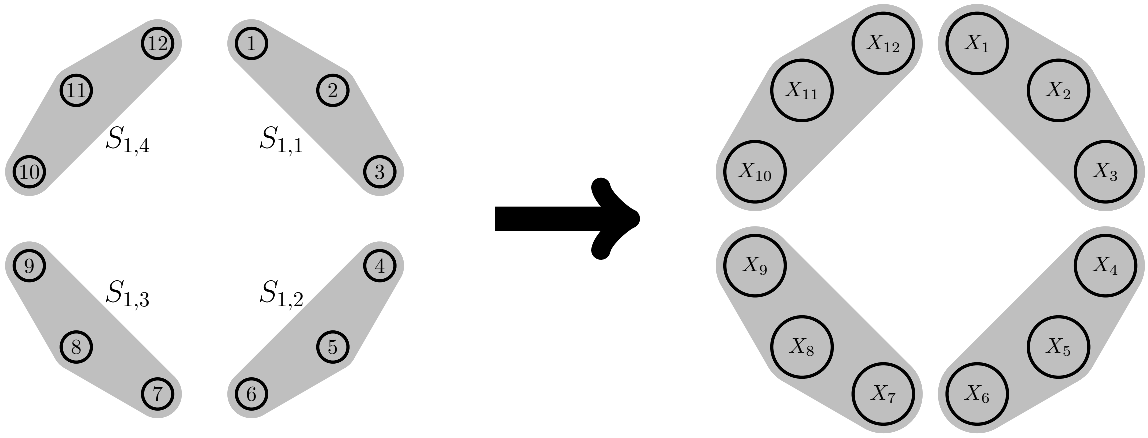

Figure 4.1: An illustration of the construction of Lemma 4.3.2 with $c = 3$ and $m = 12$. The left hand side depicts a partition $\{S_{1,1}, S_{1,2}, S_{1,3}, S_{1,4}\}$ of $\{1, \dots, 12\}$. On the right side this has been 'blown up' so that each set X_w contains h vertices. Each $\mathcal{B}_w$ is composed of all the 4-subsets of X_w. Each of $\mathcal{A}_{1,1}$, $\mathcal{A}_{1,2}$, $\mathcal{A}_{1,3}$ and $\mathcal{A}_{1,4}$ is composed of all the 3-subsets containing exactly one vertex from each X_w in the appropriate grey shaded region. The partition $\{S_{1,1}, S_{1,2}, S_{1,3}, S_{1,4}\}$ gives rise to the p colour classes $\{(z, 1) : z \in C'\}$. Each colour class $(z, 1)$ contains $t_{z,i}$ edges in $\mathcal{B}_w$ and $h - 4t_{z,i}$ edges in $\mathcal{A}_{1,i}$ for all $i \in \{1, 2, 3, 4\}$ and $w \in S_{1,i}$. In total these colour classes may use up to all of the edges in $\mathcal{A}_{1,1}, \mathcal{A}_{1,2}, \mathcal{A}_{1,3}, \mathcal{A}_{1,4}$ and at most $|\mathcal{B}_w|/\binom{m-1}{c-1} = \frac{1}{55}|\mathcal{B}_w|$ edges in each $\mathcal{B}_w$. Each of the other partitions $\{S_{\ell,1}, S_{\ell,2}, S_{\ell,3}, S_{\ell,4}\}$ for $\ell \in \{2, \dots, 55\}$ provided by Baranyai's theorem will similarly give rise to the p colour classes $\{(z, \ell) : z \in C'\}$.

$\mathrm{MMS}(n, k) \sim \mathrm{SP}(ck, k)$. Note that

$$\mathrm{MMS}(n,k) = \frac{\binom{n}{c}}{k - r + \frac{r(c+1)}{n-c}} \sim \frac{1}{k}\binom{n}{c} \sim \frac{n^c}{c!\,k} \sim \frac{n^{c-1}}{(c-1)!} \sim \frac{(ck)^{c-1}}{(c-1)!} \sim \binom{ck-1}{c-1} = \mathrm{SP}(ck, k)$$

where the first $\sim$ follows because $r = o(k)$, we use $n \sim ck$ frequently throughout and the final equality comes from [33, Theorem 1] as discussed in the introduction.

Case 2. Suppose that $r > k^{(2c-1)/2c}$. Let

$$h = \left\lceil \left(\frac{(c+1)rn^{c-1}}{k-r} \right)^{1/c} \right\rceil \qquad \text{and} \qquad m = \left\lfloor \frac{n}{h} \right\rfloor - \delta,$$

where $\delta \in \{0, 1, \dots, c-1\}$ is chosen such that $m \equiv 0 \pmod{c}$. Since $k^{(2c-1)/2c} < r < k$ and $k - r = \Theta(k) = \Theta(n)$, we have $h = O(n^{(c-1)/c})$ but $h = \Omega(n^{(2c^2-2c-1)/2c^2})$. So $mh \leqslant n$ and $mh = n - O(n^{(c-1)/c}) = ck + r - o(r)$. Let $q = mh - ck$ and note that $q \leqslant r$ and $q = r - o(r)$.

Using $mh \leqslant n$ and (4.1), we have $\mathrm{SP}(mh, k) \leqslant \mathrm{SP}(n, k) \leqslant \mathrm{MMS}(n, k)$. We will complete the proof by showing that $\mathrm{SP}(mh, k) \geqslant \mathrm{MMS}(n, k)(1 + o(1))$. We will use Lemma 4.3.2(a) to obtain this lower bound on $\mathrm{SP}(mh, k)$. Let p_1 and p_2 be as defined in the Lemma 4.3.2

statement, except with q in place of r (noting that $mh = ck + q$). Now,

$$p_2 \sim \frac{m\binom{h}{c+1}}{q\binom{m-1}{c-1}} \sim \frac{mh^{c+1}}{qc(c+1)m^{c-1}} \sim \frac{mh^{2c}}{qc(c+1)n^{c-1}} \sim \frac{mh^c}{c(k-r)} \sim \frac{mh^c}{c(k-q)} \sim p_1 \tag{4.3}$$

where the first $\sim$ holds because $\binom{m-1}{c-1} = o(\binom{h}{c+1})$ since $h = \Omega(n^{(2c^2-2c-1)/2c^2})$, the third holds because $m \sim \frac{n}{h}$, the fourth holds by applying the definition of h and then using $q \sim r$, and the fifth holds using $k - r \sim k - q$. By Lemma 4.3.2(a) and (4.3), we have

$$\mathrm{SP}(mh, k) \geqslant \binom{m-1}{c-1} \min\{p_1, p_2\} \sim \binom{m-1}{c-1} \frac{mh^c}{c(k-r)} \sim \frac{m^c h^c}{c!(k-r)} \sim \frac{n^c}{c!(k-r)} \sim \mathrm{MMS}(n, k)$$

where the first $\sim$ uses (4.3), the third uses $mh \sim n$ and the last uses the definition of $\mathrm{MMS}(n, k)$ together with $k - r = \Theta(k)$ and $\frac{r(c+1)}{n-c} = O(1)$. □

4.4 The case $c = 2$

Recall from the introduction that $\mathrm{SP}(n, k) = 1$ when $c = 1$. Thus, of the cases where c is constant, the first nontrivial case of $c = 2$ is of particular interest. Here we first observe two consequences of the upper bound on $\mathrm{SP}(n, k)$ given in Section 3.5 which slightly improves on $\mathrm{MMS}(n, k)$, which we will restate here for convenience. We again extend the usual binomial coefficient notation by defining $\binom{q}{t} = \frac{1}{t!}\prod_{i=0}^{t-1}(q - i)$, for any real number q and integer t with $q \geqslant t \geqslant 0$.

Theorem 3.1.2. *If n and k are integers such that $n \geqslant 2k + 2$ and $k \geqslant 4$, then*

$$\left\lceil \left(1 - \tfrac{r(c+1)}{n}\right) \cdot \mathrm{SP}(n, k) \right\rceil + \mathrm{LL}_c\left(\left\lfloor \tfrac{r(c+1)}{n} \cdot \mathrm{SP}(n, k) \right\rfloor \right) \leqslant \binom{n-1}{c-1}, \tag{4.4}$$

where c and r are the integers such that $n = ck+r$ and $r \in \{0, \dots, k-1\}$ and $\mathrm{LL}_c(x) = \binom{q}{c-1}$ with q being the unique nonnegative real number for which $q \geqslant c$ and $x = \binom{q}{c}$.

Recall that this acts as an upper bound due to the fact that for fixed nonnegative integers n and k, the left hand side of (4.4) is nondecreasing in $\mathrm{SP}(n, k)$. Computation reveals that, in the case $c = 2$, there are numerous small parameter sets (n, k) for which we can exactly determine $\mathrm{SP}(n, k)$ because the upper bound given by Theorem 3.1.2 equals the lower bound implied by Lemma 4.3.2(b) for some choice of m and h with $mh = n$. In Table 4.1 (see page 53), we list all the parameter sets with $n \leqslant 1000$ for which this occurs, together with the associated values of m, h and $\mathrm{SP}(n, k)$. We have not found any such parameter sets for $c \geqslant 3$. This is perhaps not surprising because, as discussed immediately after Lemma 4.3.2, the construction "wastes" some c-sets when $c \geqslant 3$.

n	k	m	h	$\mathrm{SP}(n,k)$
36	15	4	9	54
44	18	4	11	72
56	22	4	14	117
88	33	4	22	264
128	54	8	16	210
138	54	6	23	330
144	56	6	24	360
144	60	8	18	252
150	58	6	25	390
150	65	10	15	225
160	66	8	20	294
168	77	14	12	208
230	95	10	23	432
252	111	14	18	364
288	105	6	48	1280
288	128	16	18	405
300	120	10	30	675
306	111	6	51	1445
318	115	6	53	1560
324	117	6	54	1620
330	119	6	55	1680
336	144	14	24	546
336	160	28	12	378
342	123	6	57	1805
360	129	6	60	2000
360	135	8	45	1260
368	138	8	46	1288
378	135	6	63	2205
420	175	14	30	780
480	176	8	60	2100
528	192	8	66	2541
528	220	16	33	990
546	221	14	39	1183

n	k	m	h	$\mathrm{SP}(n,k)$
560	203	8	70	2800
560	232	16	35	1080
564	220	12	47	1518
576	224	12	48	1584
588	228	12	49	1650
600	224	10	60	2250
600	260	20	30	950
624	304	52	12	663
640	230	8	80	3584
672	266	14	48	1664
672	273	16	42	1440
680	323	40	17	780
700	275	14	50	1820
720	290	16	45	1620
720	330	30	24	928
750	275	10	75	3375
756	360	42	18	861
768	352	32	24	992
770	282	10	77	3510
800	335	20	40	1482
812	315	14	58	2301
816	289	8	102	5712
840	315	12	70	3080
840	350	20	42	1596
840	378	30	28	1160
852	319	12	71	3168
864	342	16	54	2160
880	365	20	44	1710
936	348	12	78	3718
938	358	14	67	3003
944	332	8	118	7497
960	448	40	24	1170
994	378	14	71	3276

Table 4.1: Parameter sets (n, k) for which $\mathrm{SP}(n, k)$ is exactly determined by Theorem 3.1.2 and Lemma 4.3.2(b), and the associated values of m, h and $\mathrm{SP}(n, k)$.

In the special case where $c = 2$ and r is small compared to n, we are able to give a more explicit form of the bound implied by Theorem 3.1.2.

Lemma 4.4.1. *Let $k \geqslant 4$ and r be integers such that $2 \leqslant r \leqslant \frac{1}{3}\sqrt{2k}$ and let $t = \lceil \mathrm{LL}_2(3r) \rceil$. Then $\mathrm{SP}(2k + r, k) \leqslant 2k + 4r - t - 1$.*

Proof. Suppose for a contradiction that $\mathrm{SP}(2k + r, k) \geqslant 2k + 4r - t$. Then, because the left side of (4.4) is monotonically increasing in $\mathrm{SP}(n, k)$, Theorem 3.1.2 implies that

$$\left\lceil \left(1 - \tfrac{3r}{2k+r}\right)(2k + 4r - t) \right\rceil + \mathrm{LL}_2\left(\left\lfloor \tfrac{3r}{2k+r}(2k + 4r - t) \right\rfloor\right) \leqslant 2k + r - 1. \tag{4.5}$$

Observe that $\lfloor \frac{3r}{2k+r}(2k + 4r - t) \rfloor = \lfloor 3r + \frac{3r(3r-t)}{2k+r} \rfloor = 3r$ because $9r^2 \leqslant 2k$. Using this fact, we have $\lceil(1-\frac{3r}{2k+r})(2k+4r-t)\rceil = 2k+4r-t-\lfloor\frac{3r}{2k+r}(2k+4r-t)\rfloor = 2k+r-t$. So (4.5) is equivalent to $2k+r-t+\mathrm{LL}_2(3r) \leqslant 2k+r-1$ which is impossible, because $t-\mathrm{LL}_2(3r) < 1$ by the definition of t. □

More routine calculations establish that Theorem 3.1.2 does not rule out the possibility that $\mathrm{SP}(2k+r,k) = 2k+4r-1-t$.

In [28], it is shown that $2k-1 \leqslant \mathrm{SP}(2k+1,k) \leqslant 2k$ and $2k+1 \leqslant \mathrm{SP}(2k+2,k) \leqslant 2k+3$. As a consequence of (4.1), the latter means that $2k+1$ is a lower bound for $\mathrm{SP}(2k+r,k)$ for all $r \geqslant 3$. For small values of $r \geqslant 3$ we give the upper bound provided by Lemma 4.4.1 on $\mathrm{SP}(2k+r,k)$, together with the range of k values it applies for in Table 4.2 below. Lemma 4.4.1 guarantees that this bound will hold for $k \geqslant \frac{9}{2}r^2$, but in Table 4.2 we give a more precise lower bound on k, obtained computationally by searching for which values of $k < \frac{9}{2}r^2$ Theorem 3.1.2 guarantees the desired bound on $\mathrm{SP}(2k+r,k)$.

r	3	4	5	6	7	8	9	10
for $k \geqslant$	17	35	32	97	71	189	253	311
$\mathrm{SP}(2k+r,k) \leqslant$	$2k+6$	$2k+9$	$2k+13$	$2k+16$	$2k+20$	$2k+23$	$2k+27$	$2k+30$

Table 4.2: Upper bounds on $\mathrm{SP}(2k+r,k)$ and the values of k for which they hold.

In this section we exhibit a new infinite family of parameter sets (n,k) for which we can precisely determine $\mathrm{SP}(n,k)$. For this family, the value of $\mathrm{SP}(n,k)$ matches the upper bound given by Theorem 3.1.2, and hence it supplies examples both of the theorem's usefulness and of situations in which its bound is tight.

Lemma 4.4.2. *Let $k \geqslant 11$ be an integer such that $k \not\equiv 4 \pmod 6$ and let $n = 3k-6$. Then $\mathrm{SP}(n,k) = \lfloor\frac{1}{2}(k-2)^2\rfloor$.*

Proof. First, suppose for a contradiction that $\mathrm{SP}(n,k) \geqslant \lfloor\frac{1}{2}(k-2)^2\rfloor + 1$. Note that $\lfloor\frac{1}{2}(k-2)^2\rfloor + 1 = \frac{1}{2}(k-2)^2 + \delta$ where $\delta = \frac{1}{2}$ if k is odd and $\delta = 1$ if k is even. Then Theorem 3.1.2 implies that (4.4) holds with $n = 3k-6$ and $p = \frac{1}{2}(k-2)^2 + \delta$ and hence, via routine calculation,

$$2k-3+\mathrm{LL}_2(\lfloor\tfrac{1}{2}(k^2-8k+12)\rfloor) \leqslant 3k-7.$$

However, because $\binom{k-4}{2} = \frac{1}{2}(k^2-9k+20) < \lfloor\frac{1}{2}(k^2-8k+12)\rfloor$ for $k \geqslant 11$, we have that $\mathrm{LL}_2(\lfloor\frac{1}{2}(k^2-8k+12)\rfloor) > k-4$ and hence a contradiction.

Now we construct an (n,k)-Sperner partition system with $\lfloor\frac{1}{2}(k-2)^2\rfloor$ partitions and so complete the proof. Let $p = \lfloor\frac{1}{2}(k-2)^2\rfloor$, let X_1, X_2 and X_3 be disjoint sets such that

$|X_1| = |X_2| = |X_3| = k-2$, and let $X = X_1 \cup X_2 \cup X_3$. For each $i \in \{1,2,3\}$, let

$$\mathcal{A}_i = \{A \subseteq X : |A| = 2 \text{ and } |A \cap X_j| = 1 \text{ for each } j \in \{1,2,3\} \setminus \{i\}\}$$
$$\mathcal{B}_i = \{B \subseteq X : |B| = 3 \text{ and } B \subseteq X_i\}.$$

Let $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3$ and $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3$, and let H be the hypergraph with vertex set X and edge set $\mathcal{A} \cup \mathcal{B}$. Note that no set in $\mathcal{A}$ is a subset of a set in $\mathcal{B}$ and thus H is a clutter. Observe that, for each $i \in \{1,2,3\}$, any permutation of X_i is an automorphism of H. Let C be a set of p colours other than black. By Lemma 3.2.2, it suffices to find an edge colouring γ of H with colour set $C \cup \{\text{black}\}$ such that, for each $c \in C$, colour c is assigned to 6 edges in $\mathcal{A}$ and $k-6$ edges in $\mathcal{B}$ and $\sum_{x \in X_i} \deg_c^\gamma(x) = k-2$ for each $i \in \{1,2,3\}$.

We now describe how to find an edge colouring that satisfies the conditions we have specified. If $k \equiv 1$ or $2 \pmod 6$, then $p \equiv 0 \pmod 3$ and we let $\{C_1, C_2, C_3\}$ be a partition of C such that $|C_1| = |C_2| = |C_3| = \frac{p}{3}$. If $k \equiv 5 \pmod 6$, then $p \equiv 1 \pmod 3$ and we let $\{C_1, C_2, C_3\}$ be a partition of C such that $|C_1| = \frac{p+2}{3}$ and $|C_2| = |C_3| = \frac{p-1}{3}$. We describe how to choose the edges from $\mathcal{A}$ in each non-black colour class of γ; the remaining edges in each non-black class can be chosen from $\mathcal{B}$ arbitrarily subject to our specified conditions, and then any remaining edges are coloured black.

- If $k \equiv 0 \pmod 3$ then, for each $c \in C$, assign colour c to two edges in $\mathcal{A}_i$ for each $i \in \{1,2,3\}$;
- If $k \equiv 1 \pmod 6$ then, for each $j \in \{1,2,3\}$ and $c \in C_j$, assign colour c to four edges in $\mathcal{A}_j$ and one edge in $\mathcal{A}_i$ for each $i \in \{1,2,3\} \setminus \{j\}$;
- If $k \equiv 2 \pmod 3$ then, for each $j \in \{1,2,3\}$ and $c \in C_j$, assign colour c to three edges in $\mathcal{A}_i$ for each $i \in \{1,2,3\} \setminus \{j\}$.

It only remains to check that there are sufficiently many edges in $\mathcal{A}_i$ and $\mathcal{B}_i$ for each $i \in \{1,2,3\}$ that we can choose an edge colouring in this manner. Using the fact that $|\mathcal{A}_i| = (k-2)^2$ and $|\mathcal{B}_i| = \binom{k-2}{3}$ for each $i \in \{1,2,3\}$, it is routine to check this by considering cases according to the congruence class of k modulo 6. □

Again, the Sperner partition systems constructed to prove Lemma 4.4.2 are almost uniform.

Finally in this section we prove Theorem 4.1.3 by showing that, when $c = 2$ and $r = k-2$, Lemma 4.3.2(b) and Theorem 3.1.2 allow us to narrow $\mathrm{SP}(n,k)$ down to one of two possible values.

Proof of Theorem 4.1.3. Let $\ell \geqslant 2$ be the integer such that $k = 2\ell$. First, suppose for a contradiction that $\mathrm{SP}(n,k) \geqslant \binom{n/2}{2} + 2$. Using this, together with $n = 6\ell - 2$, $c = 2$ and

$r = 2\ell - 2$, Theorem 3.1.2 implies that

$$\left\lceil \left(1 - \tfrac{6\ell-6}{6\ell-2}\right)\left(\tbinom{3\ell-1}{2} + 2\right)\right\rceil + \mathrm{LL}_2\left(\left\lfloor \tfrac{6\ell-6}{6\ell-2}\left(\tbinom{3\ell-1}{2} + 2\right)\right\rfloor\right) \leqslant 6\ell - 3. \tag{4.6}$$

Observe that $\lceil (1 - \frac{6\ell-6}{6\ell-2})(\binom{3\ell-1}{2} + 2)\rceil = \lceil 3\ell - 2 + \frac{4}{3\ell-1}\rceil = 3\ell - 1$ and $\lfloor \frac{6\ell-6}{6\ell-2}(\binom{3\ell-1}{2} + 2)\rfloor = \lfloor \binom{3\ell-2}{2} + \frac{6\ell-6}{3\ell-1}\rfloor = \binom{3\ell-2}{2} + 1$. So (4.6) is equivalent to $3\ell - 1 + \mathrm{LL}_2\left(\binom{3\ell-2}{2} + 1\right) \leqslant 6\ell - 3$. Clearly $\mathrm{LL}_2\left(\binom{3\ell-2}{2} + 1\right) > 3\ell - 2$, and hence we have a contradiction. Thus $\mathrm{SP}(n,k) \leqslant \binom{n/2}{2} + 1$.

Now we proceed to show $\mathrm{SP}(n,k) \geqslant \binom{n/2}{2}$. Observe that $k - 2 \equiv 0 \pmod 2$ since k is even. Thus, by Lemma 4.3.2(b), with $m = 2$ and $h = \frac{n}{2}$, we know there exists an (n,k)-Sperner partition system with p partitions, where

$$p = \min\left\{\left\lfloor \frac{n^2}{8}\right\rfloor, \left\lfloor \frac{2\binom{n/2}{3}}{k-2}\right\rfloor\right\}.$$

Noting that $\frac{2}{k-2}\binom{n/2}{3} = \binom{n/2}{2}$ for $n = 3k - 2$ and $\binom{n/2}{2} < \frac{1}{8}n^2$, it is apparent that $p = \binom{n/2}{2}$ and the result therefore follows. $\square$

In [28, Theorem 4.1], it was shown that $\mathrm{SP}(3k-1,k) \geqslant 3k - 1$ for all integers $k \geqslant 3$. Using (4.1), Theorem 4.1.3 provides a substantial improvement to this bound for even integers $k \geqslant 6$.

4.5 Proof of Theorem 4.1.4(a)

In this section and the next we extend the approach detailed in Section 3.4 and Section 3.3 to prove, respectively, Theorem 4.1.4(a) and Theorem 4.1.4(b). In this section we are interested in parameter sets (n,k) such that k is odd and $n \equiv k+1 \pmod{2k}$, in accordance with the hypotheses of Theorem 4.1.4(a). For a given parameter set (n,k), our overall approach will be as follows. In Definition 4.5.1 we define an integer program $\mathcal{I}_{(n,k)}$ and then, in Lemma 4.5.2, show that we can use an optimal solution of $\mathcal{I}_{(n,k)}$ to construct an (n,k)-Sperner partition system whose size is the optimal value of $\mathcal{I}_{(n,k)}$. Next, in Lemma 4.5.3, we establish that an obvious upper bound on the optimal value of $\mathcal{I}_{(n,k)}$ is asymptotic to $\mathrm{MMS}(n,k)$ and we then finally prove Theorem 4.1.4(a) by showing that $\mathcal{I}_{(n,k)}$ achieves an optimal value asymptotic to this upper bound. For the basic concepts and definitions of integer and linear programming we direct the reader to [35].

We now introduce some definitions and notation that we will use extensively throughout this section. We let d be the integer such that $c = 2d+1$, that is, such that $n = (2d+1)k+1$. We will construct our Sperner partition systems on a set $X = X_1 \cup X_2$ where X_1 and X_2

are disjoint sets such that $|X_1| = |X_2| = \frac{n}{2}$. For each nonnegative integer i, let

$$\mathcal{E}_i = \{E \subseteq X : |E \cap X_1| = i, |E \cap X_2| = 2d + 1 - i\}$$
$$\mathcal{E}_i^* = \{E \subseteq X : |E \cap X_1| = i, |E \cap X_2| = 2d + 2 - i\}.$$

Note that the elements of $\mathcal{E}_i$ are c-sets and the elements of $\mathcal{E}_i^*$ are $(c+1)$-sets. For each $\ell \in \{0, \dots, d\}$ define $\varepsilon_\ell = \binom{n/2}{d-\ell}\binom{n/2}{d+1+\ell}$ so that we have $|\mathcal{E}_{d-\ell}| = |\mathcal{E}_{d+1+\ell}| = \varepsilon_\ell$ and for each $\ell \in \{0, \dots, d+1\}$ define $\varepsilon_\ell^* = \binom{n/2}{d+1-\ell}\binom{n/2}{d+1+\ell}$ so that we have $|\mathcal{E}^*_{d+1-\ell}| = |\mathcal{E}^*_{d+1+\ell}| = \varepsilon^*_\ell$. Of the integers in $\{0, \dots, d\}$, let u be the smallest that satisfies $a(u) \leqslant (k-1)b(u)$ where, for $x \in \{0, \dots, d\}$,

$$a(x) = 2\sum_{\ell=x+1}^{d} \varepsilon_\ell \quad \text{and} \quad b(x) = \varepsilon_0^* + 2\sum_{\ell=1}^{x} \varepsilon_\ell^*.$$

Let Q be the largest even integer that is at most $\frac{1}{k-1}a(u)$ and also let

$$\mathcal{A} = \bigcup_{i \in I} \mathcal{E}_i \qquad \text{where} \qquad I = \{0, \dots, d-u-1\} \cup \{d+u+2, \dots, 2d+1\}$$
$$\mathcal{B} = \bigcup_{i \in I^*} \mathcal{E}_i^* \qquad \text{where} \qquad I^* = \{d+1-u, \dots, d+1+u\}.$$

Let $\mathcal{F} = \mathcal{A} \cup \mathcal{B}$. It is not hard to see that $u \leqslant d-1$ since $a(d-1) = 2\varepsilon_d < 2\varepsilon_0 < 2\varepsilon_0^* \leqslant 2b(d-1)$ and $k \geqslant 3$. Note that $\mathcal{A}$ contains c-sets and $\mathcal{B}$ contains $(c+1)$-sets, and that the sets in $\mathcal{B}$ are more "balanced" between X_1 and X_2 than the sets in $\mathcal{A}$. Obviously, no set in $\mathcal{B}$ can be a subset of a set in $\mathcal{A}$. Furthermore, no set in $\mathcal{A}$ can be a subset of a set in $\mathcal{B}$ because $\max\{|A \cap X_1|, |A \cap X_2|\} \geqslant d+u+2$ for each $A \in \mathcal{A}$ and $\max\{|B \cap X_1|, |B \cap X_2|\} \leqslant d+1+u$ for each $B \in \mathcal{B}$. Thus $(X, \mathcal{F})$ is a clutter. Also observe that $|\mathcal{A}| = a(u)$ and $|\mathcal{B}| = b(u)$. Note that all of the notation we just defined is implicitly dependent on the values of n and k. These values will be clear from context, so this should not cause confusion.

We will construct a Sperner partition system using the sets in $\mathcal{F}$. Note that each partition in such a system will contain $k-1$ sets from $\mathcal{A}$ and one set from $\mathcal{B}$ and hence such a system can have size at most $Q+1$. Our construction depends on finding up to Q disjoint triples of sets from $\mathcal{F}$ such that for each triple $\{E_1, E_2, E_3\}$ we have $E_1 \in \mathcal{B}$, $E_2, E_3 \in \mathcal{A}$ and $\sum_{i=1}^{3} |E_i \cap X_w| = 3d+2$ for each $w \in \{1, 2\}$. We encode this task in the integer program below. We define $\eta_0^*, \dots, \eta_u^*$ to be the unique sequence of integers such that $\lfloor \frac{1}{2}\eta_0^* \rfloor + \sum_{\ell=1}^{u} \eta_\ell^* = \frac{1}{2}Q$ and, for some $x \in \{0, \dots, u\}$, we have $\eta_\ell^* = \varepsilon_\ell^*$ for $\ell \in \{0, \dots, x-1\}$, $0 \leqslant \eta_x^* < \varepsilon_x^*$, $\eta_\ell^* = 0$ for $\ell \in \{x+1, \dots, u\}$ and, if $x = 0$, $\eta_0^* \equiv \varepsilon_0^* \pmod 2$. Such a sequence exists since $\frac{1}{2}Q = \lfloor \frac{1}{2(k-1)}a(u) \rfloor \leqslant \lfloor \frac{1}{2}b(u) \rfloor = \lfloor \frac{1}{2}\varepsilon_0^* \rfloor + \sum_{\ell=1}^{u} \varepsilon_\ell^*$.

Definition 4.5.1. *For integers $k \geqslant 3$ and $n > 2k$ with k odd and $n \equiv k+1 \pmod{2k}$,* *define* $\mathcal{I}_{(n,k)}$ *to be the integer program on nonnegative integer variables* $x_{i,j}$ *for all* $(i,j) \in \Phi$*, where*

$$\Phi = \{(i,j) : u+1 \leqslant i \leqslant j \leqslant d \text{ and } j-i \leqslant u\},$$

that maximises $2\sum_{(i,j)\in\Phi} x_{i,j}$ *subject to*

$$\sum_{(i,i+\ell)\in\Phi} x_{i,i+\ell} \leqslant \eta^*_\ell \quad \text{for all } \ell \in \{1,\dots,u\} \tag{4.7}$$

$$\sum_{(i,i)\in\Phi} x_{i,i} \leqslant \lfloor \tfrac{1}{2}\eta^*_0 \rfloor \tag{4.8}$$

$$\sum_{(\ell,j)\in\Phi} x_{\ell,j} + \sum_{(i,\ell)\in\Phi} x_{i,\ell} \leqslant \varepsilon_\ell \quad \text{for all } \ell \in \{u+1,\dots,d\}. \tag{4.9}$$

Note that taking each variable to be 0 in $\mathcal{I}_{(n,k)}$ satisfies all of the constraints and hence a feasible solution exists. Also, twice the sum of (4.7) for $\ell \in \{1,\dots,u\}$ and (4.8) has the objective function of $\mathcal{I}_{(n,k)}$ as its left hand side and, by our definition of $\eta^*_0,\dots,\eta^*_u$, Q as its right hand side. Hence the optimal value of $\mathcal{I}_{(n,k)}$ is at most Q. Further, each variable must be bounded above, since it appears in the objective function with positive coefficient.

Lemma 4.5.2. *Let* $k \geqslant 3$ *and* $n > 2k$ *be integers with* k *odd and* $n \equiv k+1 \pmod{2k}$*, and let* p *be the optimal value of* $\mathcal{I}_{(n,k)}$*. Then there exists a* (n,k)*-Sperner partition system with* p *partitions.*

Proof. Consider an arbitrary optimal solution $\{x_{i,j} : (i,j) \in \Phi\}$. This solution has objective value p where $p \leqslant Q$. We will use this solution to create a partial edge colouring of the clutter $H = (X,\mathcal{F})$ with p colours and then apply Lemma 3.2.2 to construct a Sperner partition system.

Note that any permutation of X_w is an automorphism of H for each $w \in \{1,2\}$. Define a set of colours $C = \bigcup_{(i,j)\in\Phi}(C_{i,j} \cup C'_{i,j})$, where $|C_{i,j}| = |C'_{i,j}| = x_{i,j}$ for each $(i,j) \in \Phi$ and $|C| = p$. By Lemma 3.2.2 it suffices to find a partial edge colouring γ_1 of H with colour set C such that, for each $z \in C$, $|\gamma_1^{-1}(z)| = k$ and $\sum_{x\in X_w} \deg_z^{\gamma_1}(x) = kd + \frac{k+1}{2}$ for $w \in \{1,2\}$. We first create a partial edge colouring γ_0 of H with three sets in each colour class which we will later extend to the desired colouring γ_1. We create this colouring γ_0 by beginning with all edges of H uncoloured and then choosing certain edges to go in colour classes. We first describe this process and then justify that we can in fact perform it to obtain γ_0.

For each $(i,j) \in \Phi$, one at a time in arbitrary order, we proceed as follows. For each $z \in C_{i,j} \cup C'_{i,j}$ we assign colour z to three previously uncoloured edges:

- one from each of $\mathcal{E}_{d-i}$, $\mathcal{E}_{d+1+j}$ and $\mathcal{E}^*_{d+1+i-j}$ if $z \in C_{i,j}$; and
- one from each of $\mathcal{E}_{d-j}$, $\mathcal{E}_{d+1+i}$ and $\mathcal{E}^*_{d+1+j-i}$ if $z \in C'_{i,j}$.

Because $(i,j) \in \Phi$, it can be checked that all the edges we colour are in $\mathcal{F} = \mathcal{A} \cup \mathcal{B}$. Further, observe that we will have $\sum_{x \in X_w} \deg_z^{\gamma_0}(x) = 3d+2$ for each $w \in \{1,2\}$ and $z \in C$.

After this process is completed for each $(i,j) \in \Phi$, we call the resulting colouring γ_0. We will be able to perform this process provided that we do not attempt to colour more than $|\mathcal{E}_i|$ sets in $\mathcal{E}_i$ for any $i \in I$ or more than $|\mathcal{E}_i^*|$ sets in $\mathcal{E}_i^*$ for any $i \in I^*$.

(i) Let $\ell \in \{1, \ldots, u\}$. Each of the $\sum_{(i,i+\ell)\in\Phi} x_{i,i+\ell}$ colours in $\bigcup_{(i,i+\ell)\in\Phi} C_{i,i+\ell}$ is assigned to exactly one of the edges in $\mathcal{E}_{d+1-\ell}^*$ and no other colours are assigned to these edges. Similarly, each of the $\sum_{(i,i+\ell)\in\Phi} x_{i,i+\ell}$ colours in $\bigcup_{(i,i+\ell)\in\Phi} C'_{i,i+\ell}$ is assigned to exactly one of the edges in $\mathcal{E}_{d+1+\ell}^*$ and no other colours are assigned to these edges. Thus, since $\eta_\ell^* \leqslant \varepsilon_\ell^* = |\mathcal{E}_{d+1-\ell}^*| = |\mathcal{E}_{d+1+\ell}^*|$, we do not run out of sets in $\mathcal{E}_{d+1-\ell}^*$ or $\mathcal{E}_{d+1+\ell}^*$ by (4.7).

(ii) Each of the $2\sum_{(i,i)\in\Phi} x_{i,i}$ colours in $\bigcup_{(i,i)\in\Phi}(C_{i,i} \cup C'_{i,i})$ is assigned to exactly one of the edges in $\mathcal{E}_{d+1}^*$ and no other colours are assigned to these edges. Thus, since $\eta_0^* \leqslant \varepsilon_0^* = |\mathcal{E}_{d+1}^*|$, we do not run out of sets in $\mathcal{E}_{d+1}$ by (4.8).

(iii) Let $\ell \in \{u+1, \ldots, d\}$. Each of the $\sum_{(\ell,j)\in\Phi} x_{\ell,j}$ colours in $\bigcup_{(\ell,j)\in\Phi} C_{\ell,j}$ and each of the $\sum_{(i,\ell)\in\Phi} x_{i,\ell}$ colours in $\bigcup_{(i,\ell)\in\Phi} C'_{i,\ell}$ is assigned to exactly one of the edges in $\mathcal{E}_{d-\ell}$ and no other colours are assigned to these edges. Similarly each of the $\sum_{(i,\ell)\in\Phi} x_{i,\ell}$ colours in $\bigcup_{(i,\ell)\in\Phi} C_{i,\ell}$ and each of the $\sum_{(\ell,j)\in\Phi} x_{\ell,j}$ colours in $\bigcup_{(\ell,j)\in\Phi} C'_{\ell,j}$ is assigned to exactly one of the edges in $\mathcal{E}_{d+1+\ell}$ and no other colours are assigned to these edges. Thus, since $\varepsilon_\ell = |\mathcal{E}_{d-\ell}| = |\mathcal{E}_{d+1+\ell}|$, we do not run out of sets in $\mathcal{E}_{d-\ell}$ or $\mathcal{E}_{d+1+\ell}$ by (4.9).

So the colouring γ_0 does indeed exist. We now extend γ_0 to the desired colouring γ_1. Note that if $k=3$ this process will be trivial and γ_1 will equal γ_0. Let $\mathcal{A}^\dagger$ be the set of all edges in $\mathcal{A}$ that are not coloured by γ_0. Because γ_0 has p colour classes, each containing two edges in $\mathcal{A}$, we have $|\mathcal{A}^\dagger| = |\mathcal{A}| - 2p$. Now $|\mathcal{A}| \geqslant p(k-1)$ since $|\mathcal{A}| = a(u)$ and $p \leqslant Q \leqslant \frac{1}{k-1}a(u)$. Thus $|\mathcal{A}^\dagger| \geqslant p(k-3)$. For each $\ell \in \{u+1, \ldots, d\}$ we have $|\mathcal{A}^\dagger \cap \mathcal{E}_{d-\ell}| = |\mathcal{A}^\dagger \cap \mathcal{E}_{d+1+\ell}|$ by the way we created γ_0. Thus we can create a partition $\mathcal{A}^\ddagger$ of $\mathcal{A}^\dagger$ into pairs such that for each pair $\{E, E'\}$ we have $E \in \mathcal{E}_{d-\ell}$ and $E' \in \mathcal{E}_{d+1+\ell}$ for some $\ell \in \{u+1, \ldots, d\}$. We form γ_1 from γ_0 by adding to each colour class the edges from $\frac{k-3}{2}$ pairs in $\mathcal{A}^\ddagger$ in such a way that no pair is allocated to two different colour classes. This is possible because $|\mathcal{A}^\ddagger| = \frac{1}{2}|\mathcal{A}^\dagger| \geqslant p\frac{k-3}{2}$. We claim that γ_1 has the required properties. To see this, note that $|\gamma_1^{-1}(z)| = k$ for each $z \in C$ because each colour class in γ_0 contained 3 edges and had $k-3$ edges added to it to form γ_1. Further, for each $z \in C$ and $w \in \{1,2\}$, $\sum_{x \in X_w} \deg_z^{\gamma_1}(x) = kd + \frac{k+1}{2}$ because $\sum_{x \in X_w} \deg_z^{\gamma_0}(x) = 3d+2$ and $|E \cap X_w| + |E' \cap X_w| = 2d+1$ for each pair $\{E, E'\}$ in $\mathcal{A}^\ddagger$. $\square$

Next we show that Q is asymptotic to $\mathrm{MMS}(n,k)$. We shall require an easy consequence of Stirling's approximation (see [38] for example), namely that as x and y tend to infinity with $y \leqslant \frac{x}{2}$,

$$\binom{x}{y} \sim A(x,y) \quad \text{where} \quad A(x,y) = \frac{x^{x+1/2}}{(2\pi)^{1/2} y^{y+1/2} (x-y)^{x-y+1/2}}. \tag{4.10}$$

Lemma 4.5.3. *Let n and k be integers such that $n \to \infty$ with $n \equiv k+1 \pmod{2k}$, $k = o(n)$, and $k \geqslant 3$ is odd. Then $Q \sim \mathrm{MMS}(n,k)$.*

Proof. Observe that $d \to \infty$ since $k = o(n)$. Furthermore, as $c = 2d+1$ and $r = 1$ in this case, we have

$$\mathrm{MMS}(n,k) = \frac{1}{k-1+\frac{2d+2}{n-2d-1}} \binom{n}{2d+1}. \tag{4.11}$$

Recall that $(X, \mathcal{F})$ is a clutter with $a(u)$ edges of size $2d+1$ and $b(u)$ edges of size $2d+2$. So, by the LYM inequality (see [15, p. 25]), $a(u)/\binom{n}{2d+1} + b(u)/\binom{n}{2d+2} \leqslant 1$ or, equivalently, $a(u) + \frac{2d+2}{n-2d-1} b(u) \leqslant \binom{n}{2d+1}$. Thus, because $\frac{1}{k-1} a(u) \leqslant b(u)$ by the definition of u,

$$Q \leqslant \frac{a(u)}{k-1} \leqslant \frac{1}{k-1+\frac{2d+2}{n-2d-1}} \binom{n}{2d+1} = \mathrm{MMS}(n,k).$$

So it remains to show that $Q \geqslant \mathrm{MMS}(n,k)(1-o(1))$.

For technical reasons, we extend the definitions of a, b and ε given at the start of this section by defining $a(-1) = \binom{n}{2d+1}$, $b(-1) = 0$ and $\varepsilon_{-1} = 0$. Using the definitions of Q and a,

$$Q > \frac{a(u)}{k-1} - 2 = \frac{a(u-1) - 2\varepsilon_u}{k-1}. \tag{4.12}$$

We will bound Q by bounding $a(u-1)$ below and applying (4.12).

We first show that

$$(2d+2)b(u-1) = (n-2d-1)\left(\binom{n}{2d+1} - a(u-1)\right) - (n-2d-2u)\varepsilon_{u-1}. \tag{4.13}$$

We may assume $u \geqslant 1$ for otherwise $u = 0$, $a(u-1) = \binom{n}{2d+1}$, $b(u-1) = 0$, $\varepsilon_{u-1} = 0$ and (4.13) holds. Define

$$\begin{aligned}
\mathcal{C} &= \bigcup_{i \in I} \mathcal{E}_i, && \text{where} && I = \{0, \dots, d-u\} \cup \{d+u+1, \dots, 2d+1\} \\
\mathcal{D} &= \bigcup_{i \in I^*} \mathcal{E}_i^*, && \text{where} && I^* = \{d-u+2, \dots d+u\} \\
\overline{\mathcal{C}} &= \bigcup_{i \in \overline{I}} \mathcal{E}_i, && \text{where} && \overline{I} = \{d-u+1, \dots d+u\}.
\end{aligned}$$

Thus we have $\overline{\mathcal{C}} = \binom{X}{2d+1} \setminus \mathcal{C}$, $|\mathcal{C}| = a(u-1)$, $|\mathcal{D}| = b(u-1)$ and $|\overline{\mathcal{C}}| = \binom{n}{2d+1} - a(u-1)$. We now count, in two ways, the number of pairs (S, B) such that $S \in \overline{\mathcal{C}}$, $B \in \mathcal{D}$ and $S \subseteq B$.

- Each of the $b(u-1)$ sets in $\mathcal{D}$ has exactly $2d+2$ subsets in $\binom{X}{2d+1}$ and each of these is in $\overline{\mathcal{C}}$, because no set in $\mathcal{C}$ is a subset of a set in $\mathcal{D}$.
- Each of the $\binom{n}{2d+1} - a(u-1)$ sets in $\overline{\mathcal{C}}$ has $n-2d-1$ supersets in $\binom{X}{2d+2}$. For each $S \in \overline{\mathcal{C}} \setminus (\mathcal{E}_{d-u+1} \cup \mathcal{E}_{d+u})$, all of these supersets of S are in $\mathcal{D}$. For each of the $2\varepsilon_{u-1}$ sets $S \in \mathcal{E}_{d-u+1} \cup \mathcal{E}_{d+u}$, exactly $\frac{n}{2} - d - u$ of these supersets of S are not in $\mathcal{D}$.

Equating our two counts, we see that (4.13) does indeed hold.

By definition of u, $\frac{1}{k-1}a(u-1) > b(u-1)$. Substituting this into (4.13) and solving for $\frac{1}{k-1}a(u-1)$ we see

$$\frac{a(u-1)}{k-1} > \frac{\binom{n}{2d+1} - \frac{n-2d-2u}{n-2d-1}\varepsilon_{u-1}}{k-1+\frac{2d+2}{n-2d-1}}. \tag{4.14}$$

Substituting (4.14) into (4.12) and then manipulating, we obtain

$$Q > \frac{\binom{n}{2d+1} - \frac{n-2d-2u}{n-2d-1}\varepsilon_{u-1}}{k-1+\frac{2d+2}{n-2d-1}} - \frac{2\varepsilon_u}{k-1} - 2 = \frac{\binom{n}{2d+1} - \frac{n-2d-2u}{n-2d-1}\varepsilon_{u-1} - 2(1+\frac{2d+2}{(k-1)(n-2d-1)})\varepsilon_u}{k-1+\frac{2d+2}{n-2d-1}} - 2 \tag{4.15}$$

Observing that the coefficients of ε_{u-1} and ε_u in the numerator of the final expression above are clearly $O(1)$ and that $\varepsilon_u \leqslant \varepsilon_{u-1} \leqslant \varepsilon_0$, we see from (4.11) that (4.15) implies that $Q > \mathrm{MMS}(n,k)(1-o(1))$ provided that $\varepsilon_0 = o(\binom{n}{2d+1})$. To see this is the case, observe that

$$\frac{\varepsilon_0}{\binom{n}{2d+1}} \sim \frac{A(\frac{n}{2},d)A(\frac{n}{2},d+1)}{A(n,2d+1)} \leqslant \frac{\left(A(\frac{n}{2},d+\frac{1}{2})\right)^2}{A(n,2d+1)} = \sqrt{\frac{2n}{\pi(2d+1)(n-2d-1)}} \sim \sqrt{\frac{k}{\pi d(k-1)}} = o(1).$$

where the first $\sim$ uses (4.10) and the last was obtained using $n \sim 2dk$. So the proof is complete. □

We define the *slack* in an inequality $f \leqslant g$ to be $g - f$. We will refer to this particularly in the case of the constraints (4.7)–(4.9) and, in the next section, the constraints (4.17)–(4.19). Note a candidate solution to an integer program is feasible if and only if the slack in each of its constraints is nonnegative. Our next result shows that the optimal value of $\mathcal{I}_{(n,k)}$ is close to Q.

Lemma 4.5.4. *Let $k \geqslant 3$ and $n > 2k$ be integers with k odd and $n \equiv k+1 \pmod{2k}$. The optimal value of $\mathcal{I}_{(n,k)}$ is at least $Q - 2\binom{u}{2} - \frac{2(d-u+1)}{k-1}$.*

Proof. For $t \in \{1,\dots,u\}$, let β_t be the slack in the constraint of $\mathcal{I}_{(n,k)}$ given by setting $\ell = t$ in (4.7). Also let β_0 be the slack in the constraint (4.8) of $\mathcal{I}_{(n,k)}$. Similarly, for $t \in \{u+1,\dots,d\}$, let α_t be the slack in the constraint of $\mathcal{I}_{(n,k)}$ given by setting $\ell = t$ in (4.9). We will create a solution to $\mathcal{I}_{(n,k)}$ with objective value at least $Q - 2\binom{u}{2} - \frac{2(d-u+1)}{k-1}$.

We do this by beginning with the solution to $\mathcal{I}_{(n,k)}$ in which all the variables are 0 and iteratively improving the solution. Each step of the iteration proceeds as follows.

(i) Take the existing solution. Let y be the largest element of $\{1,\dots,u\}$ for which $\beta_y \geqslant y$ if such an element exists. If no such element exists, let $y=0$ if $\beta_0 \geqslant \frac{d-u+1}{k-1}$ and otherwise terminate the procedure.

(ii) Let z be the largest element of $\{u+1,\dots,d\}$ for which $\alpha_z \geqslant \delta$, where $\delta = 1$ if $y \geqslant 1$ and $\delta = 2$ if $y = 0$. We claim that z exists, $z \geqslant u+2y$ and, if $y \geqslant 1$, then $\alpha_\ell \geqslant 1$ for each $\ell \in \{z-2y+1,\dots,z\}$.

(iii) If $y \geqslant 1$, increase the value of each of the y variables in $\{x_{z-i,z-y-i} : i \in \{0,\dots,y-1\}\}$ by 1. This results in β_y decreasing by y, α_ℓ decreasing by 1 for each $\ell \in \{z-2y+1,\dots,z\}$, and all other α_ℓ and β_ℓ remaining unchanged.

(iv) If $y=0$, increase the value of the variable $x_{z,z}$ by 1. This results in β_0 decreasing by 1, α_z decreasing by 2, and all other α_ℓ and β_ℓ remaining unchanged.

Provided the claim in (ii) holds, it can be seen that this procedure will terminate with a solution in which $\beta_0 \leqslant \frac{d-u+1}{k-1}$ and $\beta_\ell \leqslant \ell-1$ for each $\ell \in \{1,\dots,u\}$. As noted just below Definition 4.5.1, twice the sum of the constraints (4.7) for $\ell \in \{1,\dots,u\}$ and (4.8) has the objective function of $\mathcal{I}_{(n,k)}$ as its left hand side and Q as its right hand side. Thus this solution will have objective value at least $Q - 2\binom{u}{2} - \frac{2(d-u+1)}{k-1}$ since

$$\sum_{\ell=0}^{u} \beta_\ell \leqslant \frac{d-u+1}{k-1} + \sum_{\ell=1}^{u}(\ell-1) = \binom{u}{2} + \frac{d-u+1}{k-1}.$$

So it suffices to show that the claim in (ii) holds in each step.

Throughout the process none of the α_ℓ and β_ℓ ever increase and thus the values of y and z chosen at each step form nonincreasing sequences. Let $\alpha = \sum_{\ell=u+1}^{d} \alpha_\ell$ and $\beta = \sum_{\ell=0}^{u} \beta_\ell$. At the beginning of the process, $\alpha = \frac{1}{2}a(u)$ and $\beta = \frac{1}{2}Q$ and we have $a(u) \geqslant (k-1)Q$ by the definition of Q. Further, at each step of the process the reduction in α is exactly twice the reduction in β. So, because $k-1 \geqslant 2$, at any step of the process $\alpha \geqslant (k-1)\beta$. Fix a step and the values of y and z at this step. We will show the claim in (ii) holds for this step by considering cases according to whether $y=0$.

Case 1. Suppose that $y \geqslant 1$. Since $\eta^*_y \geqslant \beta_y > 0$, we have $\eta^*_\ell = \varepsilon^*_\ell$ for each $\ell \in \{1,\dots,y-1\}$ by our definition of $\eta^*_0,\dots,\eta^*_u$. Further, for each $\ell \in \{1,\dots,y-1\}$, β_ℓ has not so far been decreased and hence $\beta_\ell = \eta^*_\ell = \varepsilon^*_\ell$. Thus

$$\alpha \geqslant (k-1)\beta \geqslant \tfrac{k-1}{2}\varepsilon^*_0 + (k-1)\sum_{\ell=1}^{y-1}\varepsilon^*_\ell. \tag{4.16}$$

In particular α is positive and hence z exists. By our choice of z, $\alpha_\ell = 0$ for each $\ell \in \{z+1,\dots,d\}$. Thus, if $z < u+2y$ then we would have

$$\alpha \leqslant \sum_{\ell=u+1}^{u+2y-1} \varepsilon_\ell < \sum_{\ell=u+1}^{u+2y-1} \varepsilon_\ell^*$$

where the second inequality follows from the definitions of ε_ℓ and ε_ℓ^* using the fact that $d+1 < \frac{n}{2}$. But this can be seen to contradict (4.16) by first using $k-1 \geqslant 2$ in (4.16) and then applying $2y-1$ times the fact that $\varepsilon_{\ell_1}^* > \varepsilon_{\ell_2}^*$ for any integers ℓ_1 and ℓ_2 with $0 \leqslant \ell_1 < \ell_2 \leqslant d+1$. So $z \geqslant u+2y$. Finally note that, for any $\ell \in \{z-2y+1,\dots,z\}$, any previous step of the process which decreased the slack in α_ℓ also decreased the slack in α_z by an equal amount (namely 1). Thus, because at the start of the process $\alpha_\ell > \alpha_z$, this still holds at the present step and hence $\alpha_\ell \geqslant 1$. So the claim is proved.

Case 2. Suppose that $y = 0$. Then $\beta_0 \geqslant \frac{d-u+1}{k-1}$ by our choice of y, so $\beta \geqslant \frac{d-u+1}{k-1}$ and hence $\alpha \geqslant d-u+1$. So, by the pigeonhole principle, α_ℓ has slack at least 2 for some $\ell \in \{u+1,\dots,d\}$. Thus z exists, and $z \geqslant u+2y = u$ trivially. So again the claim is proved. □

In the above result, note that $2\binom{u}{2} + \frac{2(d-u+1)}{k-1}$ is obviously $O(d^2)$ since $u \leqslant d$.

Proof of Theorem 4.1.4(a). This follows from Lemmas 4.5.2, 4.5.3 and 4.5.4, noting that in the last of these our lower bound on the optimal value of the integer program is $Q - O(d^2)$ and hence is asymptotic to Q because $Q \sim \mathrm{MMS}(n,k)$ and clearly $d^2 = o(\mathrm{MMS}(n,k))$. □

4.6 Proof of Theorem 4.1.4(b)

In this section we are interested in parameter sets (n,k) such that k is odd and $n \equiv k-1 \pmod{2k}$, in accordance with the hypotheses of Theorem 4.1.4(b). For a given parameter set (n,k), our overall approach in this section is similar to that of the last section. In Definition 4.6.1 we define an integer program $\mathcal{I}_{(n,k)}$ and then, in Lemma 4.6.2, show that we can use an optimal solution of $\mathcal{I}_{(n,k)}$ to construct an (n,k)-Sperner partition system whose size is the optimal value of $\mathcal{I}_{(n,k)}$. Next, in Lemma 4.6.3, we establish that an obvious upper bound on the optimal value of $\mathcal{I}_{(n,k)}$ is asymptotic to $\mathrm{MMS}(n,k)$ and we then finally prove Theorem 4.1.4(b) by showing that, for $k \geqslant 5$, $\mathcal{I}_{(n,k)}$ achieves an optimal value asymptotic to this upper bound. Our proof of this last result differs substantially from the proof of the analogous result in Section 4.5, however.

We do not retain any of the notation defined in the last section. Instead, we will redefine most of it to suit our purposes here. Throughout we will let d be the integer such that $c = 2d$, that is, such that $n = (2d+1)k - 1$. Again, we will construct our Sperner partition systems on a set $X = X_1 \cup X_2$ where X_1 and X_2 are disjoint sets such that $|X_1| = |X_2| = \frac{n}{2}$. For each nonnegative integer i, let

$$\mathcal{E}_i = \{E \subseteq X : |E \cap X_1| = i, |E \cap X_2| = 2d - i\}$$
$$\mathcal{E}_i^* = \{E \subseteq X : |E \cap X_1| = i, |E \cap X_2| = 2d + 1 - i\}.$$

Note that the elements of $\mathcal{E}_i$ are c-sets and the elements of $\mathcal{E}_i^*$ are $(c+1)$-sets. In fact, when considered in terms of c, these definitions are the same as those given in the last section although they appear different when phrased in terms of d. For each $\ell \in \{0, \dots, d\}$, define $\varepsilon_\ell = \binom{n/2}{d-\ell}\binom{n/2}{d+\ell}$ and $\varepsilon_\ell^* = \binom{n/2}{d-\ell}\binom{n/2}{d+1+\ell}$ so that we have $|\mathcal{E}_{d-\ell}| = |\mathcal{E}_{d+\ell}| = \varepsilon_\ell$ and $|\mathcal{E}_{d-\ell}^*| = |\mathcal{E}_{d+1+\ell}^*| = \varepsilon_\ell^*$. Of the integers in $\{-1, \dots, d-1\}$, let u be the largest that satisfies $(k-1)a(u) \leqslant b(u)$ where $a(-1) = 0$, $b(-1) = \binom{n}{2d+1}$ and, for $x \in \{0, \dots, d\}$,

$$a(x) = \varepsilon_0 + 2\sum_{\ell=1}^{x} \varepsilon_\ell \quad \text{and} \quad b(x) = 2\sum_{\ell=x+1}^{d} \varepsilon_\ell^*.$$

Let $Q = a(u)$ if $a(u)$ is even and $Q = a(u) - 1$ if $a(u)$ is odd. Also let

$$\mathcal{A} = \bigcup_{i \in I} \mathcal{E}_i, \qquad \text{where } I = \{d-u, \dots d+u\}$$
$$\mathcal{B} = \bigcup_{i \in I^*} \mathcal{E}_i^*, \qquad \text{where } I^* = \{0, \dots, d-u-1\} \cup \{d+u+2, \dots, 2d+1\}.$$

We allow u to equal -1 in our definition above to ensure it is always well-defined. When $u = -1$, we have that $\mathcal{A}$ is empty and Definition 4.6.1 and Lemma 4.6.2 below are trivial. However our main interest in this section is in the regime where n is large compared to k and for these cases Lemma 4.6.3 below implies that $u \geqslant 0$ and the systems we construct are nontrivial. Let $\mathcal{F} = \mathcal{A} \cup \mathcal{B}$. As in the previous section, $\mathcal{A}$ contains c-sets and $\mathcal{B}$ contains $(c+1)$-sets but, unlike in the previous section, here the sets in $\mathcal{A}$ are more "balanced" between X_1 and X_2 than those in $\mathcal{B}$. Obviously, no set in $\mathcal{B}$ can be a subset of a set in $\mathcal{A}$. Furthermore, no set in $\mathcal{A}$ can be a subset of a set in $\mathcal{B}$ because $\min\{|A \cap X_1|, |A \cap X_2|\} \geqslant d-u$ for each $A \in \mathcal{A}$ and $\min\{|B \cap X_1|, |B \cap X_2|\} \leqslant d-u-1$ for each $B \in \mathcal{B}$. Thus $(X, \mathcal{F})$ is a clutter. Also observe that $|\mathcal{A}| = a(u)$ and $|\mathcal{B}| = b(u)$. Again, all of the notation just defined is implicitly dependent on the values of (n, k).

We will construct a Sperner partition system using the sets in $\mathcal{F}$. Note that each partition in such a system will contain one set from $\mathcal{A}$ and $k-1$ sets from $\mathcal{B}$ and hence such a system can have size at most $Q+1$. Similarly to the previous section, our construction here depends on finding up to Q disjoint triples of sets from $\mathcal{F}$. Here, for each triple $\{E_1, E_2, E_3\}$, we will have $E_1 \in \mathcal{A}$, $E_2, E_3 \in \mathcal{B}$ and $\sum_{i=1}^{3} |E_i \cap X_w| = 3d + 1$ for each $w \in \{1, 2\}$. Again, we encode this task in an integer program.

Definition 4.6.1. *For integers $k \geqslant 3$ and $n > 2k$ with k odd and $n \equiv k-1 \pmod{2k}$, define $\mathcal{I}_{(n,k)}$ to be the integer program on nonnegative integer variables $x_{i,j}$ for all $(i,j) \in \Phi$, where*

$$\Phi = \{(i,j) : u+1 \leqslant i \leqslant j \leqslant d \text{ and } j - i \leqslant u\},$$

that maximises $2\sum_{(i,j)\in\Phi} x_{i,j}$ subject to

$$\sum_{(i,i+\ell)\in\Phi} x_{i,i+\ell} \leqslant \varepsilon_\ell \qquad \text{for all } \ell \in \{1,\dots,u\} \tag{4.17}$$

$$\sum_{(i,i)\in\Phi} x_{i,i} \leqslant \lfloor \tfrac{1}{2}\varepsilon_0 \rfloor \tag{4.18}$$

$$\sum_{(\ell,j)\in\Phi} x_{\ell,j} + \sum_{(i,\ell)\in\Phi} x_{i,\ell} \leqslant \varepsilon^*_\ell \qquad \text{for all } \ell \in \{u+1,\dots,d\}. \tag{4.19}$$

When $u = -1$ we have $\Phi = \emptyset$ in the above and we consider $\mathcal{I}_{(n,k)}$ to be a trivial program with optimal value 0. Again, taking each variable to be 0 in $\mathcal{I}_{(n,k)}$ satisfies all of the constraints and hence a feasible solution exists. Also, twice the sum of (4.17) for $\ell \in \{1,\dots,u\}$ and (4.18) has the objective function of $\mathcal{I}_{(n,k)}$ as its left hand side and $Q = 2\lfloor \frac{1}{2}a(u) \rfloor$ as its right hand side. Hence the optimal value of $\mathcal{I}_{(n,k)}$ is at most Q and once again each variable must be bounded above.

Lemma 4.6.2. *Let $k \geqslant 3$ and $n > 2k$ be integers with k odd and $n \equiv k-1 \pmod{2k}$, and let p be the optimal value of $\mathcal{I}_{(n,k)}$. Then there exists a (n,k)-Sperner partition system with p partitions.*

Proof. The result is trivial if $u = -1$, so we may assume that $u \geqslant 0$ and hence $\mathcal{A}$ is nonempty. Consider an arbitrary optimal solution $\{x_{i,j} : (i,j) \in \Phi\}$. This solution has objective value p where $p \leqslant Q$. We will use this solution to create a partial edge colouring of the clutter $H = (X, \mathcal{F})$ with p colours and then apply Lemma 3.2.2 to construct a Sperner partition system.

Note that any permutation of X_w is an automorphism of H for each $w \in \{1,2\}$. Define a set of colours $C = \bigcup_{(i,j)\in\Phi}(C_{i,j} \cup C'_{i,j})$, where $|C_{i,j}| = |C'_{i,j}| = x_{i,j}$ for each $(i,j) \in \Phi$ and $|C| = p$. By Lemma 3.2.2 it suffices to find a partial edge colouring γ_1 of H with colour set C such that, for each $z \in C$, $|\gamma_1^{-1}(z)| = k$ and $\sum_{x\in X_w} \deg_z^{\gamma_1}(x) = kd + \frac{k-1}{2}$ for $w \in \{1,2\}$.

We first create a partial edge colouring γ_0 of H with three sets in each colour class which we will later extend to the desired colouring γ_1. We create this colouring γ_0 by beginning with all edges of H uncoloured and then choosing certain edges to go in colour classes. We first describe this process and then justify that we can in fact perform it to obtain γ_0.

For each $(i,j) \in \Phi$ one at a time in arbitrary order we proceed as follows. For each $z \in C_{i,j} \cup C'_{i,j}$ we assign colour z to three previously uncoloured edges:

- one from each of $\mathcal{E}^*_{d-i}$, $\mathcal{E}^*_{d+1+j}$ and $\mathcal{E}_{d+i-j}$ if $z \in C_{i,j}$; and
- one from each of $\mathcal{E}^*_{d-j}$, $\mathcal{E}^*_{d+1+i}$ and $\mathcal{E}_{d+j-i}$ if $z \in C'_{i,j}$.

Because $(i,j) \in \Phi$, it can be checked that all the edges we colour are in $\mathcal{F} = \mathcal{A} \cup \mathcal{B}$. Further, observe that we will have $\sum_{w \in X_w} \deg_z^{\gamma_0}(x) = 3d+1$ for each $w \in \{1,2\}$ and $z \in C$.

After this process is completed for each $(i,j) \in \Phi$, call the resulting colouring γ_0. We will be able to perform this process provided that we do not attempt to colour more than $|\mathcal{E}_i|$ sets in $\mathcal{E}_i$ for any $i \in I$ or more than $|\mathcal{E}^*_i|$ sets in $\mathcal{E}^*_i$ for any $i \in I^*$.

(i) Let $\ell \in \{1,\ldots,u\}$. Each of the $\sum_{(i,i+\ell)\in\Phi} x_{i,i+\ell}$ colours in $\bigcup_{(i,i+\ell)\in\Phi} C_{i,i+\ell}$ is assigned to exactly one of the edges in $\mathcal{E}_{d-\ell}$ and no other colours are assigned to these edges. Similarly, each of the $\sum_{(i,i+\ell)\in\Phi} x_{i,i+\ell}$ colours in $\bigcup_{(i,i+\ell)\in\Phi} C'_{i,i+\ell}$ is assigned to exactly one of the edges in $\mathcal{E}_{d+\ell}$ and no other colours are assigned to these edges. Thus, since $|\mathcal{E}_{d-\ell}| = |\mathcal{E}_{d+\ell}| = \varepsilon_\ell$, we do not run out of sets in $\mathcal{E}_{d-\ell}$ or $\mathcal{E}_{d+\ell}$ by (4.17).

(ii) Each of the $2\sum_{(i,i)\in\Phi} x_{i,i}$ colours in $\bigcup_{(i,i)\in\Phi}(C_{i,i} \cup C'_{i,i})$ is assigned to exactly one of the edges in $\mathcal{E}_d$ and no other colours are assigned to these edges. Thus, since $|\mathcal{E}_d| = \varepsilon_0$, we do not run out of sets in $\mathcal{E}_d$ by (4.18).

(iii) Let $\ell \in \{u+1,\ldots,d\}$. Each of the $\sum_{(\ell,j)\in\Phi} x_{\ell,j}$ colours in $\bigcup_{(\ell,j)\in\Phi} C_{\ell,j}$ and each of the $\sum_{(i,\ell)\in\Phi} x_{i,\ell}$ colours in $\bigcup_{(i,\ell)\in\Phi} C'_{i,\ell}$ is assigned to exactly one of the edges in $\mathcal{E}^*_{d-\ell}$, and no other colours are assigned to these edges. Similarly, each of the $\sum_{(i,\ell)\in\Phi} x_{i,\ell}$ colours in $\bigcup_{(i,\ell)\in\Phi} C_{i,\ell}$ and each of the $\sum_{(\ell,j)\in\Phi} x_{\ell,j}$ colours in $\bigcup_{(\ell,j)\in\Phi} C'_{\ell,j}$ is assigned to exactly one of the edges in $\mathcal{E}^*_{d+1+\ell}$, and no other colours are assigned to these edges. Thus, since $|\mathcal{E}^*_{d-\ell}| = |\mathcal{E}^*_{d+1+\ell}| = \varepsilon^*_\ell$, we do not run out of sets in $\mathcal{E}^*_{d-\ell}$ or $\mathcal{E}^*_{d+1+\ell}$ by (4.19).

So the colouring γ_0 does indeed exist. We now extend γ_0 to the desired colouring γ_1. Note that if $k=3$ this process will be trivial and γ_1 will equal γ_0. Let $\mathcal{B}^\dagger$ be the set of all edges in $\mathcal{B}$ that are not coloured by γ_0. Because γ_0 has p colour classes, each containing two edges in $\mathcal{B}$, we have $|\mathcal{B}^\dagger| = |\mathcal{B}| - 2p$. Now $|\mathcal{B}| \geqslant (k-1)|\mathcal{A}| \geqslant p(k-1)$ since $|\mathcal{A}| = a(u)$, $|\mathcal{B}| = b(u)$ and $p \leqslant Q \leqslant a(u)$. Thus $|\mathcal{B}^\dagger| \geqslant p(k-3)$. For each $\ell \in \{u+1,\ldots,d\}$ we have $|\mathcal{B}^\dagger \cap \mathcal{E}^*_{d-\ell}| = |\mathcal{B}^\dagger \cap \mathcal{E}^*_{d+1+\ell}|$ by way we defined γ_0. Thus we can create a partition $\mathcal{B}^\ddagger$ of $\mathcal{B}^\dagger$ into pairs such that for each pair $\{E,E'\}$ we have $E \in \mathcal{E}^*_{d-\ell}$ and $E' \in \mathcal{E}^*_{d+1+\ell}$

for some $\ell \in \{u+1,\dots,d\}$. We form γ_1 from γ_0 by adding to each colour class the edges from $\frac{k-3}{2}$ pairs in $\mathcal{B}^\ddagger$ in such a way that no pair is allocated to two different colour classes. This is possible because $|\mathcal{B}^\ddagger| = \frac{1}{2}|\mathcal{B}^\dagger| \geqslant p\frac{k-3}{2}$. We claim that γ_1 has the required properties. To see this, note that $|\gamma_1^{-1}(z)| = k$ for each $z \in C$ because each colour class in γ_0 contained 3 edges and had $k-3$ edges added to it to form γ_1. Further, for each $z \in C$ and $w \in \{1,2\}$, $\sum_{x\in X_w} \deg_z^{\gamma_1}(x) = kd + \frac{k-1}{2}$ because $\sum_{x\in X_w} \deg_z^{\gamma_0}(x) = 3d+1$ and $|E \cap X_w| + |E' \cap X_w| = 2d+1$ for each pair $\{E, E'\}$ in $\mathcal{B}^\ddagger$. □

Next we show that Q is asymptotic to $\mathrm{MMS}(n,k)$. We do this in a different fashion to the proof of Lemma 4.5.3, one that allows us to also obtain an estimate of u and some other technical results that will be required in our proof of Theorem 4.1.4(b). Recall that the error function, denoted erf, is defined for any real number x by $\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x \exp(-t^2)\,dt$.

Lemma 4.6.3. *Let n and k be integers such that $n \to \infty$ with $n \equiv k-1 \pmod{2k}$, $k = o(n)$, and $k \geqslant 3$ is odd. Then*

(a) $\varepsilon_\ell^* \sim (k-1)\varepsilon_\ell$ *for each integer $\ell \geqslant 0$ such that $\ell = O(\sqrt{d})$;*

(b) $\varepsilon_\ell \sim \varepsilon_0 \exp(-\frac{k}{d(k-1)}\ell^2)$ *for each integer $\ell \geqslant 0$ such that $\ell = O(\sqrt{d})$;*

(c) $u \sim \mathrm{erf}^{-1}(\frac{1}{2})\sqrt{d(k-1)/k}$*; and*

(d) $Q \sim \mathrm{MMS}(n,k)$*;*

Proof. Let $n \equiv k-1 \pmod{2k}$ be a positive integer. Observe that $d \to \infty$ since $k = o(n)$. It will often be useful to note that $n - 2d = (2d+1)(k-1)$.

For each integer $\ell \geqslant 0$ such that $\ell = O(\sqrt{d})$,

$$\varepsilon_\ell^* = \frac{n-2d-2\ell}{2d+2\ell+2}\varepsilon_\ell \sim (k-1)\varepsilon_\ell$$

where the equality follows by the definitions of ε_ℓ and ε_ℓ^* and the $\sim$ follows because $n-2d = (2d+1)(k-1)$ and $\ell = O(\sqrt{d})$. So (a) holds.

For each integer $\ell \geqslant 0$ such that $\ell = O(\sqrt{d})$,

$$\begin{aligned}
\frac{\varepsilon_\ell}{\varepsilon_0} &\sim \frac{d^{2d+1}(\frac{n}{2}-d)^{n-2d+1}}{(d-\ell)^{d-\ell+1/2}(d+\ell)^{d+\ell+1/2}(\frac{n}{2}-d-\ell)^{(n+1)/2-d-\ell}(\frac{n}{2}-d+\ell)^{(n+1)/2-d+\ell}} \\
&= \left(\frac{d^2}{d^2-\ell^2}\right)^{d-\ell+1/2}\left(\frac{(\frac{n}{2}-d)^2}{(\frac{n}{2}-d)^2-\ell^2}\right)^{(n+1)/2-d-\ell}\left(\frac{d}{d+\ell}\right)^{2\ell}\left(\frac{\frac{n}{2}-d}{\frac{n}{2}-d+\ell}\right)^{2\ell} \\
&= \left(\frac{d}{d-\frac{1}{d}\ell^2}\right)^{d-\ell+1/2}\left(\frac{\frac{n}{2}-d}{\frac{n}{2}-d-\frac{1}{n/2-d}\ell^2}\right)^{(n+1)/2-d-\ell}\left(\frac{\ell}{\ell+\frac{1}{d}\ell^2}\right)^{2\ell}\left(\frac{\ell}{\ell+\frac{1}{n/2-d}\ell^2}\right)^{2\ell} \\
&\sim \exp\left(\tfrac{1}{d}\ell^2\right)\exp\left(\tfrac{1}{d(k-1)}\ell^2\right)\exp\left(-\tfrac{2}{d}\ell^2\right)\exp\left(-\tfrac{2}{d(k-1)}\ell^2\right) \\
&= \exp\left(-\tfrac{k}{d(k-1)}\ell^2\right)
\end{aligned}$$

where the first line follows by applying (4.10) and simplifying, and the fourth line follows by applying limit identities, remembering that $n-2d=(2d+1)(k-1)$ and $\ell=O(\sqrt{d})$. So we have proved (b). In particular, note that $\varepsilon_\ell=\Theta(\varepsilon_0)$.

For any positive real constant $\kappa>0$, let $u_\kappa=\kappa\sqrt{d(k-1)/k}$ and note that $u_\kappa=\Theta(\sqrt{d})$. Now,

$$\begin{aligned}
a(\lfloor u_\kappa\rfloor) &= \varepsilon_0+2\sum_{\ell=1}^{\lfloor u_\kappa\rfloor}\varepsilon_\ell\\
&\sim \varepsilon_0\left(1+2\sum_{\ell=1}^{\lfloor u_\kappa\rfloor}\exp\left(-\tfrac{k}{d(k-1)}\ell^2\right)\right)\\
&\sim 2\varepsilon_0\left(\sum_{\ell=0}^{\lfloor u_\kappa\rfloor}\exp\left(-\tfrac{k}{d(k-1)}\ell^2\right)\right)\\
&\sim 2\varepsilon_0\int_0^{u_\kappa}\exp\left(-\tfrac{k}{d(k-1)}\ell^2\right)\,d\ell\\
&= \varepsilon_0\sqrt{\tfrac{k-1}{k}d\pi}\,\mathrm{erf}(\kappa)\\
&\sim \binom{n}{2d}\mathrm{erf}(\kappa)
\end{aligned}\tag{4.20}$$

where in the second line we used part (b) of this lemma and the definition of the function a, in the fourth line we approximated the sum with an integral, in the fifth we changed the variable of integration to $t=\sqrt{k/d(k-1)}\ell$ and applied the definition of u_κ, and in the last we applied (4.10) to ε_0 and $\binom{n}{2d}$ and performed a routine calculation recalling $n\sim 2kd$. In the first three lines, recall that $u_\kappa=\Theta(\sqrt{d})$ and the terms of the sum are comparable and hence each term is insignificant compared to the whole. On the other hand,

$$\begin{aligned}
b(\lfloor u_\kappa\rfloor) &= \binom{n}{2d+1}-2\sum_{\ell=0}^{\lfloor u_\kappa\rfloor}\varepsilon^*_\ell\\
&\sim (k-1)\left(\binom{n}{2d}-2\sum_{\ell=0}^{\lfloor u_\kappa\rfloor}\varepsilon_\ell\right)\\
&\sim (k-1)\left(\binom{n}{2d}-a(\lfloor u_\kappa\rfloor)\right)\\
&\sim (k-1)\binom{n}{2d}(1-\mathrm{erf}(\kappa))
\end{aligned}\tag{4.21}$$

where the first line follows from the definition of b, the second using part (a) of this lemma and the fact that $\binom{n}{2d+1}=(k-1)\binom{n}{2d}$ since $n-2d=(2d+1)(k-1)$, the third by the definition of a and because any term of the sum is insignificant compared to the whole, and the last by (4.20). Let $\kappa_0=\mathrm{erf}^{-1}(\frac{1}{2})$. For any $\kappa<\kappa_0$, using (4.20) and (4.21), we have $(k-1)a(\lfloor u_\kappa\rfloor)<b(\lfloor u_\kappa\rfloor)$ and hence $u\geqslant\lfloor u_\kappa\rfloor$ by the definition of u. Similarly, $(k-1)a(\lfloor u_\kappa\rfloor)>b(\lfloor u_\kappa\rfloor)$ and hence $u<\lfloor u_\kappa\rfloor$ for any $\kappa>\kappa_0$. This proves (c). Finally, we have $Q \sim a(u) \sim \frac{1}{2}\binom{n}{2d}$ using (4.20) and the fact that $u \sim u_{\kappa_0}$ by part (c) of this lemma. Since $c = 2d$, $r = k-1$ and $n - 2d = (2d+1)(k-1)$ in this case, $\mathrm{MMS}(n,k) = \frac{1}{2}\binom{n}{2d}$ and the proof of (d) is complete. □

Lemma 4.6.4. *Let n and k be integers such that $n \to \infty$ with $n \equiv k-1 \pmod{2k}$, $k = o(n)$, and $k \geqslant 5$ is odd. Then the optimal value of $\mathcal{I}_{(n,k)}$ is Q.*

Proof. Observe that $d \to \infty$ since $k = o(n)$. For $t \in \{1,\ldots,u\}$, let α_t be the slack in the constraint of $\mathcal{I}_{(n,k)}$ given by setting $\ell = t$ in (4.17), and let α_0 be the slack in constraint (4.18) of $\mathcal{I}_{(n,k)}$. Similarly, for $t \in \{u+1,\ldots,d\}$, let β_t be the slack in the constraint of $\mathcal{I}_{(n,k)}$ given by setting $\ell = t$ in (4.19). Consider the candidate solution for $\mathcal{I}_{(n,k)}$ in which every variable is 0 except

- $x_{u+1+i,2u+1-i} = \varepsilon_{u-2i}$ for each $i \in \{0,\ldots,\lfloor\frac{u-1}{2}\rfloor\}$,
- $x_{u+1+i,2u-i} = \varepsilon_{u-2i-1}$ for each $i \in \{0,\ldots,\lfloor\frac{u-2}{2}\rfloor\}$,
- $x_{\lfloor 3u/2\rfloor+1,\lfloor 3u/2\rfloor+1} = \lfloor\frac{1}{2}\varepsilon_0\rfloor$.

We claim that this is indeed a feasible solution with objective value Q. Note that, for this assignment, $\alpha_\ell = 0$ for each $\ell \in \{0,\ldots,u\}$. Because twice the sum of the constraints (4.17) for $\ell \in \{1,\ldots,u\}$ and (4.18) has the objective function of $\mathcal{I}_{(n,k)}$ as its left hand side and $Q = 2\lfloor\frac{1}{2}a(u)\rfloor$ as its right hand side, this assignment has objective value Q. So it only remains to show that $\beta_\ell \geqslant 0$ for each $\ell \in \{u+1,\ldots,d\}$ and hence that the solution is feasible. For each $\ell \in \{2u+2,\ldots,d\}$, (4.19) is clearly satisfied since its left hand side is 0. Furthermore, for each $\ell \in \{u+1,\ldots,2u+1\}$, it can be seen that at most two of the variables contributing to the left hand side are nonzero. In fact, we have

$$\beta_{2u+1-i} = \varepsilon^*_{2u+1-i} - \varepsilon_{u-2i+1} - \varepsilon_{u-2i} \qquad \text{for each } i \in \{1,\ldots,\lfloor\tfrac{u-1}{2}\rfloor\} \tag{4.22}$$

$$\beta_{u+1+i} = \varepsilon^*_{u+1+i} - \varepsilon_{u-2i} - \varepsilon_{u-2i-1} \qquad \text{for each } i \in \{0,\ldots,\lfloor\tfrac{u-2}{2}\rfloor\} \tag{4.23}$$

$$\beta_{2u+1} = \varepsilon^*_{2u+1} - \varepsilon_u \tag{4.24}$$

$$\beta_{\lfloor 3u/2\rfloor+1} = \varepsilon^*_{\lfloor 3u/2\rfloor+1} - \varepsilon_1 - 2\lfloor\tfrac{1}{2}\varepsilon_0\rfloor. \tag{4.25}$$

We now obtain two expressions which we will use to approximate, respectively, the positive and negative terms in the expressions on the right hand sides of (4.22)–(4.25). For each $\ell \in \{0,\ldots,2u+1\}$, we have from Lemma 4.6.3(a) and (b) that

$$\varepsilon^*_\ell \sim (k-1)\varepsilon_\ell \sim (k-1)\varepsilon_0 \exp\left(-\tfrac{k}{d(k-1)}\ell^2\right). \tag{4.26}$$

Also, for each $\ell \in \{0, \dots, u\}$, using Lemma 4.6.3(b) we obtain

$$\varepsilon_\ell + \varepsilon_{\ell+1} < 2\varepsilon_\ell \sim 2\varepsilon_0 \exp\left(-\tfrac{k}{d(k-1)}\ell^2\right). \tag{4.27}$$

Using (4.26) and (4.27) it is not too difficult to establish that the expressions in (4.22)–(4.25) are nonnegative in each case. We demonstrate how to do this in the case of (4.22), where the expressions come closest to being negative.

Let i be an arbitrary element of $\{1, \dots, \lfloor \frac{u-1}{2} \rfloor\}$. Using (4.22), (4.26) and (4.27) and then factorising, we have

$$\begin{aligned}\beta_{2u+1-i} > \varepsilon^*_{2u+1-i} - 2\varepsilon_{u-2i} &\sim (k-1)\varepsilon_{2u+1-i} - 2\varepsilon_{u-2i} \\ &\sim (k-1)\varepsilon_0 \exp\left(-\tfrac{k(2u+1-i)^2}{d(k-1)}\right) - 2\varepsilon_0 \exp\left(-\tfrac{k(u-2i)^2}{d(k-1)}\right) \\ &= \varepsilon_0 \exp\left(-\tfrac{k(2u+1-i)^2}{d(k-1)}\right)\left(k-1-2\exp\left(\tfrac{k(3u+1-3i)(u+1+i)}{d(k-1)}\right)\right).\end{aligned} \tag{4.28}$$

The first exponential in (4.28) approaches a positive constant since by Lemma 4.6.3(c), we know that $u = \Theta(\sqrt{d})$, so to prove that $\beta_{2u+1-i} \geqslant 0$ it suffices to show that the second exponential approaches a constant strictly less that $\frac{k-1}{2}$. Now

$$\tfrac{k(3u+1-3i)(u+1+i)}{d(k-1)} \leqslant \tfrac{k(3u-2)(u+2)}{d(k-1)} \sim \tfrac{3ku^2}{d(k-1)} \sim 3\big(\mathrm{erf}^{-1}(\tfrac{1}{2})\big)^2$$

where the first inequality follows because $(3u+1-3i)(u+1+i)$ is maximised when $i = 1$ given that $i \geqslant 1$, the first $\sim$ follows because $u = \Theta(\sqrt{d})$ and the second by Lemma 4.6.3(c). Thus, because $\mathrm{erf}^{-1}(\frac{1}{2}) < 0.477$, it is easy to calculate that the second exponential in (4.28) approaches a constant less than 1.98. Since $k \geqslant 5$, this is less than $\frac{k-1}{2}$. Thus $\beta_{2u+1-i} \geqslant 0$. Very similar arguments show that the expressions in (4.23)–(4.25) are nonnegative in each case. □

Proof of Theorem 4.1.4(b). This follows immediately from Lemmas 4.6.2, 4.6.3(d) and 4.6.4. □

It is of interest to note that as result of Theorem 4.1.4(b), the asymptotics of $\mathrm{SP}(n, k)$ when $k = o(n)$ is odd and $n \equiv k - 1 \pmod{2k}$ are only unknown when $k = 3$. We believe the statement is still true in this case.

Conjecture 4.6.5. *Let n be an integer such that $n \to \infty$ with $n \equiv 2 \pmod 6$. Then* $\mathrm{SP}(n, 3) \sim \mathrm{MMS}(n, 3)$.

In an attempt to find evidence supporting this conjecture, we observed that if we relax one of the integer programs $\mathcal{I}_{(n,k)}$ to a linear program $\mathcal{L}_{(n,k)}$, then the optimal value of $\mathcal{L}_{(n,k)}$ exceeds the optimal value of $\mathcal{I}_{(n,k)}$ by at most $2|\Phi| < 2u(d-u)$. This is because a feasible solution for $\mathcal{I}_{(n,k)}$ can be obtained from an optimal solution for $\mathcal{L}_{(n,k)}$ by simply taking the floor of each variable. So clearly the two optimal values are asymptotic to each other as n becomes large. Thus we implemented the linear relaxation $\mathcal{L}_{(n,k)}$ in the linear programming solver Gurobi [19]. We proceeded to solve $\mathcal{L}_{(n,3)}$ for all $n \equiv 2 \pmod 6$ where $26 \leqslant n \leqslant 18000$ (the program is trivial for $n < 26$), and in all cases the optimal solution found gave an objective value that matched Q to at least 11 significant figures. Since the objective value for $\mathcal{L}_{(n,3)}$ is asymptotic to the objective value of $\mathcal{I}_{(n,3)}$, and in view of Lemma 4.6.2 and Lemma 4.6.3(d), these calculations support our belief that the conjecture holds.

Chapter 5

Intersecting cross-Sperner families

5.1 Introduction

As previously mentioned in Section 2.3, Wong and Tay recently considered pairs of families of sets such that each family is individually a Sperner set system and the pair of families is cross-intersecting. They obtained the following result.

Theorem 2.3.6 ([41]). *If $n \geqslant 3$ is an integer and $\mathcal{F}, \mathcal{G} \subseteq 2^{[n]}$ are Sperner set systems such that $|F \cap G| > 0$ for all $F \in \mathcal{F}$ and $G \in \mathcal{G}$, then*

$$|\mathcal{F}| + |\mathcal{G}| \leqslant \begin{cases} 2\binom{n}{\lceil n/2 \rceil} & \text{if } n \text{ is odd} \\ \binom{n}{n/2} + \binom{n}{n/2+1} & \text{if } n \text{ is even.} \end{cases} \tag{2.5}$$

Furthermore there are Sperner set systems $\mathcal{F}, \mathcal{G} \subseteq 2^{[n]}$ such that $|F \cap G| > 0$ for all $F \in \mathcal{F}$ and $G \in \mathcal{G}$ and for which equality holds in (2.5).

In this short chapter we examine what can be thought of as the "dual" problem concerning pairs of families of sets such that each family is individually intersecting and the pair of families is cross-Sperner. A family $\mathcal{F}$ of sets is *intersecting* if $F \cap F' \neq \emptyset$ for all $F, F' \in \mathcal{F}$. Recall that a pair $(\mathcal{F}, \mathcal{G})$ of families of subsets of $[n]$ is *cross-Sperner* if $F \nsubseteq G$ and $G \nsubseteq F$ for all $F \in \mathcal{F}$ and $G \in \mathcal{G}$. Thus an *intersecting cross-Sperner pair on* $[n]$ is a cross-Sperner pair $(\mathcal{F}, \mathcal{G})$ of families of subsets of $[n]$ such that both $\mathcal{F}$ and $\mathcal{G}$ are both intersecting families. Note that it is not necessary for $\mathcal{F}$ or $\mathcal{G}$ to be Sperner families themselves, and in most cases they will not. In this chapter we determine, for each positive integer n, the maximum value of $|\mathcal{F}| + |\mathcal{G}|$ for an intersecting cross-Sperner pair $(\mathcal{F}, \mathcal{G})$ on $[n]$.

Theorem 5.1.1. *If n is a positive integer and $(\mathcal{F}, \mathcal{G})$ is an intersecting cross-Sperner pair on $[n]$, then $|\mathcal{F}| + |\mathcal{G}| \leqslant 2^{n-1}$. Furthermore, for each $m \in \{0, \ldots, 2^{n-1}\}$, there is an* *intersecting cross-Sperner pair $(\mathcal{F},\mathcal{G})$ on $[n]$ such that $|\mathcal{F}|+|\mathcal{G}|=2^{n-1}$ and $|\mathcal{F}|=m$.*

It is of interest to recall that Gerbner et al. had previously considered the cross-Sperner property in its own right in [16] . They obtained the following tight upper bound on the value of $|\mathcal{F}||\mathcal{G}|$ for a cross-Sperner pair on $[n]$, which we restate for convenience.

Lemma 2.3.4 ([16])**.** *Let $n \geqslant 2$ be an integer and let $\mathcal{F},\mathcal{G} \subseteq 2^{[n]}$. If $(\mathcal{F},\mathcal{G})$ is a cross-Sperner pair, then $|\mathcal{F}||\mathcal{G}| \leqslant 2^{2n-4}$.*

As observed in [16], the tightness of the bound above is witnessed for each $n \geqslant 2$ by the cross-Sperner pair $(\mathcal{F},\mathcal{G})$ where $\mathcal{F}=\{F \in 2^{[n]} : 1 \in F, n \notin F\}$ and $\mathcal{G}=\{G \in 2^{[n]} : 1 \notin G, n \in G\}$. Importantly for us, this is in fact an intersecting cross-Sperner pair. These results immediately tell us that, for any $n \geqslant 2$, the maximum value of $|\mathcal{F}||\mathcal{G}|$ over all intersecting cross-Sperner pairs $(\mathcal{F},\mathcal{G})$ on $[n]$ is exactly 2^{2n-4}. So we concentrate our attention on determining the maximum value of $|\mathcal{F}|+|\mathcal{G}|$ over all intersecting cross-Sperner pairs $(\mathcal{F},\mathcal{G})$ on $[n]$. The example above shows that this maximum value is at least 2^{n-1} for all $n \geqslant 2$. By proving Theorem 5.1.1, we will show that this maximum is exactly 2^{n-1} and, further, that it can be achieved with any choice of sizes of $\mathcal{F}$ and $\mathcal{G}$ such that $|\mathcal{F}|+|\mathcal{G}|=2^{n-1}$.

5.2 An upper bound on a wider class of pairs

Let $\mathcal{F}$ be a family of subsets of a ground set X. Let the *up-set* on X of a family of sets $\mathcal{F} \subseteq 2^X$ be the collection of sets $\mathcal{F}^{\uparrow}=\{S \in 2^X : S \supseteq F$ for some $F \in \mathcal{F}\}$, and the *down-set* of a family of sets $\mathcal{F} \subseteq 2^X$ be the collection of sets $\mathcal{F}^{\downarrow}=\{S \in 2^X : S \subseteq F$ for some $F \in \mathcal{F}\}$. We say that $\mathcal{F}$ is an up-set if $\mathcal{F}^{\uparrow}=\mathcal{F}$ and that $\mathcal{F}$ is a down-set if $\mathcal{F}^{\downarrow}=\mathcal{F}$. Note that unlike the down-set, the up-set of a family depends on its ground set and we will clarify what ground set we are considering whenever it may be unclear. Otherwise assume that the up-set is on $[n]$. The down-set and up-set of a family $\mathcal{F}$ of sets are sometimes also referred to as its *ideal* and *filter* respectively.

We first observe that we can use an approach similar to the one used in [16] to establish an upper bound on $|\mathcal{F}|+|\mathcal{G}|$ for a class of pairs $(\mathcal{F},\mathcal{G})$ that is wider than the class of intersecting cross-Sperner pairs. We begin with a simple observation that will be useful throughout this chapter.

Lemma 5.2.1. *If n is a positive integer and $\mathcal{F}$ is an intersecting family of subsets of $[n]$, then $|\mathcal{F}| \leqslant 2^{n-1}$. Furthermore, for each $m \in \{0,\dots,2^{n-1}\}$, there is an intersecting family $\mathcal{F}$ of subsets of $[n]$ such that $|\mathcal{F}|=m$ and $\mathcal{F}$ is an up-set.*

Proof. For any intersecting family $\mathcal{F}$ of subsets of $[n]$ and any $F \in \mathcal{F}$, we have $[n] \setminus F \notin \mathcal{F}$. This shows that $|\mathcal{F}| \leqslant 2^{n-1}$. To prove the second part of the lemma, we proceed by induction on $2^{n-1} - m$. Let

$$\mathcal{F}_0 = \left\{F \in 2^{[n]} : 1 \in F\right\}$$

and observe that clearly $|\mathcal{F}_0| = 2^{n-1}$ and $\mathcal{F}_0$ is an intersecting family. Thus the second part of the lemma holds for $2^{n-1} - m = 0$.

Now suppose that for some $t \in \{0, \dots, 2^{n-1}\}$ there exists a family $\mathcal{F}_t$ of subsets of $[n]$ such that $|\mathcal{F}_t| = 2^{n-1} - t$ and $\mathcal{F}_t$ is an up-set. Let $\mathcal{F}_{t+1} = \mathcal{F}_t \setminus \{F_t\}$ where F_t is a set of minimum size in $\mathcal{F}_t$. The fact that $\mathcal{F}_t$ is an intersecting family obviously implies that $\mathcal{F}_{t+1}$ is an intersecting family. Furthermore, the facts that F_t is a set of minimum size in $\mathcal{F}_t$ and that $\mathcal{F}_t$ is an up-set imply that $\mathcal{F}_{t+1}$ is an up-set. $\square$

As a consequence of this result, we have that $|\mathcal{F}| \leqslant 2^{n-1}$ and $|\mathcal{G}| \leqslant 2^{n-1}$ for any intersecting cross-Sperner pair $(\mathcal{F}, \mathcal{G})$. We will establish an upper bound on $|\mathcal{F}| + |\mathcal{G}|$ for any (not necessarily intersecting) cross-Sperner pair $(\mathcal{F}, \mathcal{G})$ with $|\mathcal{F}| \leqslant 2^{n-1}$ and $|\mathcal{G}| \leqslant 2^{n-1}$.

Our proof of this bound, like the proof of Theorem 2.3.4 in [16], relies on two existing results. The first is a simple observation about the so called meets and joins of cross-Sperner families, while the second is the Ahlswede and Daykin inequality. We restate both here for convenience.

Lemma 2.3.2 ([16]). *Let n be a positive integer and let $\mathcal{F}, \mathcal{G} \subseteq 2^{[n]}$. If $(\mathcal{F}, \mathcal{G})$ is a cross-Sperner pair, then the families $\mathcal{F}$, $\mathcal{G}$, $\mathcal{F} \wedge \mathcal{G}$, and $\mathcal{F} \vee \mathcal{G}$ are pairwise disjoint, where $\mathcal{F} \wedge \mathcal{G} = \{F \cap G : F \in \mathcal{F}$ and $G \in \mathcal{G}\}$ and $\mathcal{F} \vee \mathcal{G} = \{F \cup G : F \in \mathcal{F}$ and $G \in \mathcal{G}\}$.*

Lemma 2.3.3 ([1]). *Let $\mathcal{F}, \mathcal{G} \subseteq 2^{[n]}$ for some integer n. Then*

$$|\mathcal{F}||\mathcal{G}| \leqslant |\mathcal{F} \wedge \mathcal{G}||\mathcal{F} \vee \mathcal{G}|.$$

Lemma 5.2.2. *Let $n > 2$ be an integer and let $\mathcal{F}, \mathcal{G} \subseteq 2^{[n]}$. Assume $|\mathcal{F}| \leqslant 2^{n-1}$ and $|\mathcal{G}| \leqslant 2^{n-1}$ and that $(\mathcal{F}, \mathcal{G})$ is a cross-Sperner pair. Then $|\mathcal{F}| + |\mathcal{G}| \leqslant (4 - 2\sqrt{2})2^{n-1}$.*

Proof. Assume without loss of generality that $|\mathcal{F}| \geqslant |\mathcal{G}|$ and remember that $|\mathcal{F}| \leqslant 2^{n-1}$. If $\mathcal{F}$ or $\mathcal{G}$ is empty, the bound trivially holds, so assume both are nonempty. Let $X_1 = \mathcal{F} \wedge \mathcal{G}$ and $X_2 = \mathcal{F} \vee \mathcal{G}$. Let $t = |\mathcal{F}| + |\mathcal{G}|$ and $x = |X_1| + |X_2|$. As a result of Lemma 2.3.2, we know that $\mathcal{F}$, $\mathcal{G}$, X_1 and X_2 are pairwise disjoint because $\mathcal{F}$ and $\mathcal{G}$ are cross-Sperner, and thus $|X_1| + |X_2| + |\mathcal{F}| + |\mathcal{G}| \leqslant 2^n$ or, equivalently

$$t \leqslant 2^n - x. \tag{5.1}$$

As a result of Lemma 2.3.3, we have that $|\mathcal{F}||\mathcal{G}| \leqslant |X_1||X_2|$. We also know that $|X_1||X_2| \leqslant \frac{1}{4}x^2$ because by the definition of x, we have that $x^2 \geqslant x^2 - (|X_1| - |X_2|)^2 = 4|X_1||X_2|$.

We claim that $|\mathcal{F}||\mathcal{G}| = |\mathcal{F}|(t - |\mathcal{F}|) \geqslant 2^{n-1}(t - 2^{n-1})$. To see this, recall our earlier assumptions that $|\mathcal{F}| \geqslant |\mathcal{G}| = t - |\mathcal{F}|$ and $|\mathcal{F}| \leqslant 2^{n-1}$. The inequality is obviously true for $t \leqslant 2^{n-1}$, so all that is left is to check when $t > 2^{n-1}$. Let $a = |\mathcal{F}|$, $b = |\mathcal{G}|$ and $2^{n-1} = a + c$. So $t = a + b > a + c$. Then

$$2^{n-1}(t - 2^{n-1}) = (a + c)((a + b) - (a + c)) = (a + c)(b - c) = ab - c(a - b) - c^2 \leqslant ab$$

with the final inequality following from the fact that $-c(a - b) \leqslant 0$ for $a > b$ (which we initially assumed), and so our claim holds.

So we have $2^{n-1}(t - 2^{n-1}) \leqslant \frac{1}{4}x^2$ or, equivalently,

$$t \leqslant \frac{x^2}{2^{n+1}} + 2^{n-1}. \tag{5.2}$$

Now if $x \leqslant (\sqrt{2} - 1)2^n$, then $t \leqslant (4 - 2\sqrt{2})2^{n-1}$ by (5.2). Otherwise $x > (\sqrt{2} - 1)2^n$ and $t < (4 - 2\sqrt{2})2^{n-1}$ by (5.1). $\square$

5.3 Proof of Theorem 5.1.1

We now move on to proving our main result for the chapter. To do so, we first prove the upper bound on $|\mathcal{F}| + |\mathcal{G}|$ we require for Theorem 5.1.1. Our proof relies on the following result of Erdős, Herzog and Schönheim on down-sets and their complements, where we denote the complement of a family $\mathcal{F}$ of subsets of $[n]$ by $\overline{\mathcal{F}} = \{[n] \setminus F : F \in \mathcal{F}\}$ (see [11, Theorem 1] for the full result).

Theorem 5.3.1 ([11]). *Let n be a positive integer and let $\mathcal{F} \subseteq 2^{[n]}$ be a down-set. Then there exists a bijection $\varphi : \mathcal{F} \to \overline{\mathcal{F}}$ such that $F \subseteq \varphi(F)$ for all $F \in \mathcal{F}$.*

Lemma 5.3.2. *If $(\mathcal{F}, \mathcal{G})$ is an intersecting cross-Sperner pair on $[n]$, then $|\mathcal{F}| + |\mathcal{G}| \leqslant 2^{n-1}$.*

Proof. Suppose that $(\mathcal{F}, \mathcal{G})$ is an intersecting cross-Sperner pair on $[n]$. Note that $\mathcal{F}^{\uparrow}$ is an intersecting family because $\mathcal{F}$ is an intersecting family. It follows that $\overline{\mathcal{F}^{\uparrow}}$ is disjoint from $\mathcal{F}^{\uparrow}$. Let $\mathcal{H} = 2^{[n]} \setminus (\mathcal{F}^{\uparrow} \cup \overline{\mathcal{F}^{\uparrow}})$, and note that $\{\mathcal{F}^{\uparrow}, \overline{\mathcal{F}^{\uparrow}}, \mathcal{H}\}$ is a partition of $2^{[n]}$. Our proof is based on considering how $\mathcal{F}$ and $\mathcal{G}$ intersect the classes of this partition. Obviously $\mathcal{F} \subseteq \mathcal{F}^{\uparrow}$ and, because $(\mathcal{F}, \mathcal{G})$ is a cross-Sperner pair, we have that $\mathcal{G}$ is disjoint from $\mathcal{F}^{\uparrow}$. Thus $\mathcal{G} \subseteq \overline{\mathcal{F}^{\uparrow}} \cup \mathcal{H}$.

We first observe that for each $H \in \mathcal{H}$ we also have that $\overline{H} \in \mathcal{H}$ and thus, because $\mathcal{G}$ is an intersecting family, at most one of H and $\overline{H}$ is in $\mathcal{G}$. Therefore,

$$|\mathcal{G} \cap \mathcal{H}| \leqslant \tfrac{1}{2}|\mathcal{H}| = \tfrac{1}{2}(2^n - 2|\mathcal{F}^{\uparrow}|) = 2^{n-1} - |\mathcal{F}^{\uparrow}|. \tag{5.3}$$

Now, observe that $\overline{\mathcal{F}^{\uparrow}} = (\overline{\mathcal{F}})^{\downarrow}$ and hence is a down-set. So, by Theorem 5.3.1, there exists a bijection $\varphi : \overline{\mathcal{F}^{\uparrow}} \to \mathcal{F}^{\uparrow}$ such that $S \subseteq \varphi(S)$ for all $S \in \overline{\mathcal{F}^{\uparrow}}$. Thus no set in $\{\varphi^{-1}(F) : F \in \mathcal{F}\}$ can appear in $\mathcal{G}$ because $(\mathcal{F}, \mathcal{G})$ is a cross-Sperner pair. Thus

$$|\mathcal{G} \cap \overline{\mathcal{F}^{\uparrow}}| \leqslant |\overline{\mathcal{F}^{\uparrow}} \setminus \{\phi^{-1}(F) : F \in \mathcal{F}\}| \leqslant |\overline{\mathcal{F}^{\uparrow}}| - |\mathcal{F}|. \tag{5.4}$$

So, by first using the fact that $\overline{\mathcal{F}^{\uparrow}}$ and $\mathcal{H}$ are disjoint and then applying (5.3) and (5.4), we have that

$$|\mathcal{F}| + |\mathcal{G}| = |\mathcal{F}| + |\mathcal{G} \cap \overline{\mathcal{F}^{\uparrow}}| + |\mathcal{G} \cap \mathcal{H}| \leqslant 2^{n-1}. \qquad \square$$

Next, we show that the upper bound obtained in Lemma 5.3.2 is tight in the strong sense indicated by Theorem 5.1.1. To do so we must first present a general construction of intersecting cross-Sperner pairs $(\mathcal{F}, \mathcal{G})$ for which $|\mathcal{F}| + |\mathcal{G}| = 2^{n-1}$.

Lemma 5.3.3. *Let n be a positive integer and $s \in [n]$. Let $\mathcal{R}$ be an intersecting family on $[n-s]$ such that $\mathcal{R}$ is an up-set on $[n-s]$ and $\mathcal{Q}$ be an intersecting family on $\{n-s+1, \ldots, n\}$ such that $|\mathcal{Q}| = 2^{s-1}$ and $\mathcal{Q}$ is an up-set on $\{n-s+1, \ldots, n\}$. Then $(\mathcal{F}, \mathcal{G})$ where*

$$\mathcal{F} = \{Q \cup R' : Q \in \mathcal{Q}, R' \in 2^{[n-s]} \setminus \mathcal{R}\} \quad \textit{and} \quad \mathcal{G} = \{Q' \cup R : Q' \in 2^{\{n-s+1,\ldots,n\}} \setminus \mathcal{Q}, R \in \mathcal{R}\}$$

is an intersecting cross-Sperner pair with $|\mathcal{F}| = 2^{s-1}(2^{n-s} - |\mathcal{R}|)$, $|\mathcal{G}| = 2^{s-1}|\mathcal{R}|$ and $|\mathcal{F}| + |\mathcal{G}| = 2^{n-1}$.

Proof. It is clear from the definitions of $\mathcal{F}$ and $\mathcal{G}$ that $|\mathcal{F}| = 2^{s-1}(2^{n-s} - |\mathcal{R}|)$ and $|\mathcal{G}| = 2^{s-1}|\mathcal{R}|$, and hence that $|\mathcal{F}| + |\mathcal{G}| = 2^{n-1}$. So it remains to show that both $\mathcal{F}$ and $\mathcal{G}$ are intersecting families and that $(\mathcal{F}, \mathcal{G})$ is cross-Sperner. We can see that $\mathcal{F}$ and $\mathcal{G}$ are intersecting families as $\mathcal{Q}$ and $\mathcal{R}$ are both intersecting families, respectively.

Now we proceed to show that the two families are cross-Sperner. If there were sets $F \in \mathcal{F}$ and $G \in \mathcal{G}$ such that $F \subseteq G$, then $F \cap \{n-s+1, \ldots, n\} \subseteq G \cap \{n-s+1, \ldots, n\}$ and, since $F \cap \{n-s+1, \ldots, n\} \in \mathcal{Q}$ and $G \cap \{n-s+1, \ldots, n\} \in 2^{\{n-s+1,\ldots,n\}} \setminus \mathcal{Q}$, this contradicts the fact that $\mathcal{Q}$ is an up-set on $\{n-s+1, \ldots, n\}$. Similarly if there were $F \in \mathcal{F}$ and $G \in \mathcal{G}$ such that $G \subseteq F$, then $G \cap [n-s] \subseteq F \cap [n-s]$ and, since $G \cap [n-s] \in \mathcal{R}$ and $F \cap [n-s] \in 2^{[n-s]} \setminus \mathcal{R}$, this contradicts the fact that $\mathcal{R}$ is an up-set on $[n-s]$. Thus for any $F \in \mathcal{F}$ and $G \in \mathcal{G}$ we have $F \not\subseteq G$ and $G \not\subseteq F$. So $(\mathcal{F}, \mathcal{G})$ is indeed a cross-Sperner pair. $\square$

Observe that applying Lemma 5.3.3 with $s = 1$, $\mathcal{R} = \{R \in 2^{[n-1]} : 1 \in R\}$ and $\mathcal{Q} = \{\{n\}\}$ recovers the construction of Gerbner et al. that shows the tightness of Theorem 2.3.4. We can apply Lemma 5.3.3 to prove the second part of Theorem 5.1.1.

Lemma 5.3.4. *Let n be a positive integer. For each $m \in \{0, \ldots, 2^{n-1}\}$, there exists an intersecting cross-Sperner pair $(\mathcal{F}, \mathcal{G})$ on $[n]$ such that $|\mathcal{F}| + |\mathcal{G}| = 2^{n-1}$ and $|\mathcal{F}| = m$.*

Proof. We may assume $m \geqslant 2^{n-2}$ for otherwise we can simply exchange the roles of $\mathcal{F}$ and $\mathcal{G}$. Applying Lemma 5.3.3 with $s = 1$, $\mathcal{Q} = \{\{n\}\}$, and $\mathcal{R}$ chosen to have cardinality $2^{n-2} - x$ for some $x \in \{0, \ldots, 2^{n-2}\}$, produces an intersecting cross-Sperner pair $(\mathcal{F}, \mathcal{G})$ with $|\mathcal{F}| = 2^{n-2} + x$ and $|\mathcal{F}| + |\mathcal{G}| = 2^{n-1}$. Thus it suffices to show that for each $x \in \{0, \ldots, 2^{n-2}\}$ there is an intersecting family $\mathcal{R}_x \subseteq 2^{[n-1]}$ such that $|\mathcal{R}_x| = 2^{n-2} - x$ and $\mathcal{R}_x$ is an up-set on $[n-1]$. This follows from Lemma 5.2.1.

$\square$

Proof of Theorem 5.1.1. This follows immediately from Lemmas 5.3.2 and 5.3.4. $\square$

Chapter 6

Cross-Sperner families with one family size specified

6.1 Introduction

This chapter deals with the problem of finding the minimum size of $\mathcal{F}^{\uparrow} \cup \mathcal{F}^{\downarrow}$ when the size of $\mathcal{F}$ is specified. For our purposes in this chapter it will be convenient to extend our usual notation by letting $[n] = \emptyset$ when $n = 0$. Let $\mathcal{F}$ be a family of subsets of a ground set X. Recall from the previous chapter that $\mathcal{F}^{\uparrow} = \{S \subseteq X : F \subseteq S \text{ for some } F \in \mathcal{F}\}$ and $\mathcal{F}^{\downarrow} = \{S \subseteq X : S \subseteq F \text{ for some } F \in \mathcal{F}\}$. Here we further define the *closed neighbourhood* $\mathcal{F}^{\updownarrow}$ of $\mathcal{F}$ to be the family $\mathcal{F}^{\uparrow} \cup \mathcal{F}^{\downarrow}$. For nonnegative integers m and n such that $m \leqslant 2^n$, let

$$\Phi(n, m) = \min \left\{ \left|\mathcal{F}^{\updownarrow}\right| : \mathcal{F} \subseteq 2^{[n]} \text{ and } |\mathcal{F}| = m \right\}.$$

For any nonnegative integer n it is clear that $\Phi(n, 0) = 0$ and $\Phi(n, 2^n) = 2^n$. Note that this includes the trivial cases $\Phi(0, 0) = 0$ and $\Phi(0, 1) = 1$ which will be important as base cases for a recursion. Also note that clearly $\Phi(n, m)$ is nondecreasing in m. Our main result in this chapter is to determine $\Phi(n, m)$, for all nonnegative integers m and n such that $m \leqslant 2^n$, by means of a recursive formula.

Theorem 6.1.1. *Let m and n be nonnegative integers such that $0 \leqslant m \leqslant 2^n$. Let $a = \lfloor \log_2 m \rfloor$, and*

$$a' = \begin{cases} a & \text{if } a \equiv n \text{ (mod 2) or } a = 0 \\ a - 1 & \text{otherwise.} \end{cases}$$

Then $\Phi(n,m) = m$ *if* $m \in \{0, 2^n\}$ *and otherwise*

$$\Phi(n,m) = 2^{\lfloor (n+a)/2 \rfloor} + 2^{\lceil (n-a)/2 \rceil}(m - t + \tfrac{1}{2}\delta) - m$$

where $t \in [2^{a'}]$ *is the greatest integer such that* $m - 2^{a'} \geqslant \Phi(a', t) + 2t - 1$*, and*

$$\delta = \begin{cases} 1 & \text{if } \Phi(a',t) = m - 2^{a'} - 2t + 1 \\ 0 & \text{otherwise.} \end{cases}$$

Although our determination of $\Phi(n,m)$ is recursive in general, in some cases we are able to give $\Phi(n,m)$ explicitly.

Corollary 6.1.2. *Let* n *and* m *be nonnegative integers with* $m \in \{1, \ldots, 2^n - 1\}$ *and let* $a = \lfloor \log_2 m \rfloor$*. Then* $\Phi(n,m) = 2^{\lceil (n+a)/2 \rceil} + 2^{\lfloor (n-a)/2 \rfloor} m - m$ *if and only if* $a = 0$ *or*

- $a \geqslant 1$*,* $a \equiv n \pmod 2$ *and* $m \leqslant 2^a + 2^{\lceil a/2 \rceil} + 2^{\lfloor a/2 \rfloor} - 1$*; or*
- $a \geqslant 1$*,* $a \not\equiv n \pmod 2$ *and* $m \geqslant 2^{a+1} - 2^{\lceil (a+1)/2 \rceil} - 2^{\lfloor (a+1)/2 \rfloor} + 2$*.*

One motivation for investigating $\Phi(n,m)$ is the fact that it is closely linked to the problem of finding the maximum size of $\mathcal{G}$ for a cross-Sperner pair $(\mathcal{F}, \mathcal{G})$ where the size of $\mathcal{F}$ is specified. For nonnegative integers m and n such that $m \leqslant 2^n$, let

$$g(n,m) = \max\left\{ |\mathcal{G}| : \mathcal{F}, \mathcal{G} \subseteq 2^{[n]} \text{ are cross-Sperner and } |\mathcal{F}| = m \right\}.$$

The problem of finding values of $g(n,m)$ was mentioned by Gerbner et al. in [16]. The following proposition shows the exact link between $\Phi(n,m)$ and $g(n,m)$.

Proposition 6.1.3. *For nonnegative integers* m *and* n *such that* $m \leqslant 2^n$*, we have* $g(n,m) = 2^n - \Phi(n,m)$*.*

Proof. Suppose $\mathcal{F}$ is a family of subsets of $[n]$ such that $|\mathcal{F}^{\updownarrow}| = \Phi(n,m)$ and let $\mathcal{G} = 2^{[n]} \setminus \mathcal{F}^{\updownarrow}$. Then $(\mathcal{F}, \mathcal{G})$ is a cross-Sperner pair and $|\mathcal{G}| = 2^n - |\mathcal{F}^{\updownarrow}| = 2^n - \Phi(n,m)$. Thus $g(n,m) \geqslant 2^n - \Phi(n,m)$.

Now suppose that $(\mathcal{F}, \mathcal{G})$ is a cross-Sperner pair such that $|\mathcal{F}| = m$ and $|\mathcal{G}| = g(n,m)$. Then $\mathcal{F}^{\updownarrow} \subseteq 2^{[n]} \setminus \mathcal{G}$ and hence $|\mathcal{F}^{\updownarrow}| \leqslant 2^n - g(n,m)$. Thus $\Phi(n,m) \leqslant 2^n - g(n,m)$ or, equivalently, $g(n,m) \leqslant 2^n - \Phi(n,m)$. □

Beyond the question concerning $g(n,m)$, there are at least two further motivations to study the quantity $\Phi(n,m)$. The first relates to the isoperimetric problem which we discussed in Section 2.5. Recall that the *boundary* $\partial_G(U)$ of a set of vertices U in a graph G is defined as $\partial_G(U) = (\bigcup_{x \in U} N_G(x)) \setminus U$ and that the isoperimetric problem on G asks us to determine $\partial_G(m) = \min\{\partial_G(U) : U \subseteq V(G), |U| = m\}$ for a given value of m in $\{0, \ldots, |V(G)|\}$. For a nonnegative integer n, the *Boolean lattice graph* B_n is the graph with vertex set $2^{[n]}$ and edge set $\{XY : X \subsetneq Y \text{ for } X, Y \in 2^{[n]}\}$. Note that the hypercube Q_n is the spanning subgraph of this graph with edge set $\{XY : X \subsetneq Y, |X| = |Y| - 1 \text{ for } X, Y \in 2^{[n]}\}$. It can be seen that, for any family $\mathcal{F} \subseteq 2^{[n]}$, we have that $\mathcal{F}^{\updownarrow}$ is exactly the disjoint union of $\mathcal{F}$ and $\partial_{B_n}(\mathcal{F})$. Thus we can clearly see the following proposition.

Proposition 6.1.4. *or nonnegative integers m and n such that $m \leqslant 2^n$, we have $\partial_{B_n}(m) = \Phi(n, m) - m$.*

So determining the values of $\Phi(n, m)$ also solves the isoperimetric problem for the Boolean lattice graph. The relationship between $g(n, m)$ and the isoperimetric problem on the Boolean lattice graph was noted by Gerbner et al. in [16].

The second motivation for studying the quantity $\Phi(n, m)$ relates to the Kruskal-Katona theorem (see Section 2.4). Recall that the *shadow* $\Delta(\mathcal{F})$ of a family of k-subsets of $[n]$ is defined to be the family of all $(k-1)$-sets that are a subset of at least one set in $\mathcal{F}$. The Kruskal-Katona theorem states that, over all families $\mathcal{F}$ of k-subsets of $[n]$ with some fixed size m, the size of the shadow $\Delta(\mathcal{F})$ of $\mathcal{F}$ is minimised when $\mathcal{F}$ is taken to be the first m k-subsets of $[n]$ in colexicographic order. Further, it is known that this same choice of $\mathcal{F}$ also minimises the size of $\mathcal{F}^{\downarrow}$. It is also known that the similar results are true for $\nabla(\mathcal{F})$ and $\mathcal{F}^{\uparrow}$ when one takes the first m k-subsets of $[n]$ in lexicographic order. Thus determining the values of $\Phi(n, m)$ can be seen as proving a certain "two-sided" variant of the Kruskal-Katona theorem in which $\mathcal{F}$ is allowed to contain sets of different sizes and we are interested in minimising the size of $\mathcal{F}^{\updownarrow}$. However, note that the previous discussion means that it is not immediately apparent what families of a given size might minimise $|\mathcal{F}^{\updownarrow}|$. In [3] and [4], Bashov studied a different two-sided version of the Kruskal-Katona theorem in which $\mathcal{F}$ remained restricted to contain sets of uniform size and the size of the union of the shade and shadow (rather than the union of the up-set and down-set) was to be minimised. His results were discussed in Section 2.5.

6.2 Preliminaries

Throughout this chapter, for nonnegative integers n and m we call a family of subsets $\mathcal{F}$ of $[n]$ with $|\mathcal{F}| = m$ an (n, m)-*family*. It is easy to determine $\Phi(n, m)$ when $m = 1$ and, consequently, when m is close to 2^n. Part (i) of the lemma below was already observed in [16], and the next two parts follow quickly from the first.

Lemma 6.2.1. *Let n be a nonnegative integer. Then*

(i) $\Phi(n, 1) = 2^{\lceil n/2\rceil} + 2^{\lfloor n/2\rfloor} - 1$;

(ii) $\Phi(n, m) = 2^n$ *for each* $m \in \{2^n - 2^{\lceil n/2\rceil} - 2^{\lfloor n/2\rfloor} + 2, \ldots, 2^n\}$;

(iii) $\Phi(n, m) \leqslant 2^n - 1$ *for each* $m \in \{0, \ldots, 2^n - 2^{\lceil n/2\rceil} - 2^{\lfloor n/2\rfloor} + 1\}$.

Proof. We prove each part separately.

(i). Let F be a subset of $[n]$ and let $k = |F|$. Then $\{F\}^{\downarrow} = 2^k$ and $\{F\}^{\uparrow} = 2^{n-k}$, and hence $\{F\}^{\updownarrow} = 2^k + 2^{n-k} - 1$. Clearly the minimum value of $2^k + 2^{n-k}$ over $k \in \{0, \ldots, n\}$ is $2^{\lceil n/2\rceil} + 2^{\lfloor n/2\rfloor}$ and the result follows.

(ii). Suppose for a contradiction that, for some $m \in \{2^n - 2^{\lceil n/2\rceil} - 2^{\lfloor n/2\rfloor} + 2, \ldots, 2^n\}$, there is an (n, m)-family $\mathcal{F}$ such that $|\mathcal{F}^{\updownarrow}| < 2^n$. Let G be a set in $2^{[n]} \setminus \mathcal{F}^{\updownarrow}$, and note that $\{G\}^{\updownarrow} \subseteq 2^{[n]} \setminus \mathcal{F}$. Thus $|\{G\}^{\updownarrow}| \leqslant 2^n - m$, which contradicts (i).

(iii). By (i), there is a subset F of $[n]$ such that $\{F\}^{\updownarrow} = 2^{\lceil n/2\rceil} + 2^{\lfloor n/2\rfloor} - 1$. So, for any $m \in \{0, \ldots, 2^n - 2^{\lceil n/2\rceil} - 2^{\lfloor n/2\rfloor} + 1\}$, we can choose an (n, m) family $\mathcal{G}$ such that $\mathcal{G} \subseteq 2^{[n]} \setminus \{F\}^{\updownarrow}$. Then $F \notin \mathcal{G}^{\updownarrow}$ and the result follows. $\square$

We define the colexicographic order on subsets of $[n]$ as follows. For two subsets F and G of $[n]$, we say $F \prec G$ if $F \cap \{i+1, \ldots, n\} = G \cap \{i+1, \ldots, n\}$ for some $i \in G \setminus F$. Note that this defines a strict total order on the subsets of $[n]$ (of all sizes). We can then extend this definition to families of subsets of $[n]$. For two families $\mathcal{F}$ and $\mathcal{G}$ of subsets of $[n]$, we say $\mathcal{F} \prec \mathcal{G}$ if $\mathcal{F} \cap \{Y \subseteq [n] : X \prec Y\} = \mathcal{G} \cap \{Y \subseteq [n] : X \prec Y\}$ for some $X \in \mathcal{G} \setminus \mathcal{F}$. This defines a strict total order on families of subsets of $[n]$ (of all sizes). We write $\preceq$ when we also wish to allow equality.

For a family $\mathcal{F}$ of sets, we say a set $X \in \mathcal{F}$ is *minimal in* $\mathcal{F}$ if no set in $\mathcal{F}$ is a proper subset of X and is *maximal in* $\mathcal{F}$ if no set in $\mathcal{F}$ is a proper superset of X. We call a family of sets $\mathcal{F}$ *convex* if $\mathcal{F}^{\uparrow} \cap \mathcal{F}^{\downarrow} = \mathcal{F}$. Equivalently, $\mathcal{F}$ is convex if, for any $F_1, F_2 \in \mathcal{F}$ and any set X such that $F_1 \subseteq X \subseteq F_2$, we have $X \in \mathcal{F}$. Our next lemma shows that if we are attempting to find values of $\Phi(n, m)$, then it suffices to consider convex families.

Lemma 6.2.2. *For any (n, m)-family $\mathcal{F}$, there exists a convex (n, m)-family $\mathcal{G}$ such that $\mathcal{G}^{\updownarrow} \subseteq \mathcal{F}^{\updownarrow}$ and $\mathcal{G} \preceq \mathcal{F}$.*

Proof. Assume that $\mathcal{F} = \mathcal{F}^{\downarrow} \cap \mathcal{F}^{\uparrow}$. Then $\mathcal{F}$ is convex and $\mathcal{G} = \mathcal{F}$ is a suitable choice for $\mathcal{G}$. Assume that $\mathcal{F} \neq \mathcal{F}^{\downarrow} \cap \mathcal{F}^{\uparrow}$, then there exists sets $F_1, F_2 \in \mathcal{F}$ and $X \in 2^{[n]} \setminus \mathcal{F}$ such that $F_1 \subseteq X \subseteq F_2$. We may further assume that F_2 is maximal in $\mathcal{F}$. Let $\mathcal{H} = (\mathcal{F} \setminus \{F_2\}) \cup \{X\}$

and note $\mathcal{H} \preceq \mathcal{F}$. Also $\mathcal{H}^\uparrow = \mathcal{F}^\uparrow$ because F_1 is in both $\mathcal{F}$ and $\mathcal{H}$ and $\{F_2\}^\uparrow \subseteq \{X\}^\uparrow \subseteq \{F_1\}^\uparrow$ since $F_1 \subseteq X \subseteq F_2$. Further, $\mathcal{H}^\downarrow \subseteq \mathcal{F}^\downarrow \setminus \{F_2\}$ because $\{X\}^\downarrow \subseteq \{F_2\}^\downarrow$ since $X \subseteq F_2$ and because $F_2 \notin \mathcal{H}^\downarrow$ since no proper superset of F_2 is in $\mathcal{F}$. It follows that $\mathcal{H}^\updownarrow \subseteq \mathcal{F}^\updownarrow$ and further that $\mathcal{H}^\uparrow \cap \mathcal{H}^\downarrow \subseteq (\mathcal{F}^\uparrow \cap \mathcal{F}^\downarrow) \setminus \{F_2\}$. Thus we can iterate this process and it will eventually terminate with a convex family $\mathcal{G} \subseteq 2^{[n]}$ such that $\mathcal{G}^\updownarrow \subseteq \mathcal{F}^\updownarrow$ and $\mathcal{G} \preceq \mathcal{F}$. □

Finally in this section we give a result that determines $|\mathcal{F}^\updownarrow|$ for convex families $\mathcal{F}$ that contain a set which is a superset of every other set in $\mathcal{F}$, also known as a maximum set. This lemma allows us to give an upper bound on $\Phi(n, m)$ that is sometimes tight.

Lemma 6.2.3. *Let $\mathcal{F}$ be a convex (n, m)-family such that, for some $\ell \in [n]$, we have $[\ell] \in \mathcal{F}$ and $\mathcal{F} \subseteq \{[\ell]\}^\downarrow$. Then*

(i) $|\mathcal{F}^\updownarrow| = 2^\ell + m2^{n-\ell} - m$;

(ii) $|\mathcal{F}^\updownarrow| \geqslant 2^{\lceil (n+a)/2 \rceil} + m2^{\lfloor (n-a)/2 \rfloor} - m$ *where* $a = \lfloor \log_2 m \rfloor$.

Proof. We first prove (i). Since $\mathcal{F}$ is convex, $|\mathcal{F}^\updownarrow| = |\mathcal{F}^\uparrow| + |\mathcal{F}^\downarrow| - m$. Since $[\ell] \in \mathcal{F}$ and $\mathcal{F} \subseteq [\ell]^\downarrow$, we have $|\mathcal{F}^\downarrow| = 2^\ell$. So to prove (i) it suffices to show that $|\mathcal{F}^\uparrow| = m2^{n-\ell}$. For each set $F \in \mathcal{F}$ let $\mathcal{U}_F = \{X \subseteq [n] : X \cap [\ell] = F\}$ and note that $|\mathcal{U}_F| = 2^{n-\ell}$. Let $\mathcal{U} = \bigcup_{F \in \mathcal{F}} \mathcal{U}_F$ and note that the union is disjoint and hence that $|\mathcal{U}| = m2^{n-\ell}$. We will complete the proof of (i) by showing that $\mathcal{F}^\uparrow = \mathcal{U}$. Clearly $\mathcal{U} \subseteq \mathcal{F}^\uparrow$. Let $Y \in \mathcal{F}^\uparrow$. Hence $F \subseteq Y$ for some $F \in \mathcal{F}$. Let $F' = Y \cap [\ell]$ and observe that $F' \in \mathcal{F}$ since $F \subseteq F' \subseteq [\ell]$ and $\mathcal{F}$ is convex. Thus $Y \in \mathcal{U}_{F'} \subseteq \mathcal{U}$ and hence $\mathcal{F}^\uparrow \subseteq \mathcal{U}$. So we do indeed have that $\mathcal{F}^\uparrow = \mathcal{U}$ and hence (i) holds.

We now prove (ii). Let ℓ_0 be an element of $[n]$ such that $2^\ell + m2^{n-\ell} - m$ is minimal for $\ell = \ell_0$ and, subject to this, let ℓ_0 be maximal. Then we have $2^{\ell_0} + m2^{n-\ell_0} < 2^{\ell_0+1} + m2^{n-\ell_0-1}$ and $2^{\ell_0} + m2^{n-\ell_0} \leqslant 2^{\ell_0-1} + m2^{n-\ell_0+1}$, and from these it follows that $2^{2\ell_0-n-1} \leqslant m < 2^{2\ell_0-n+1}$. Thus, by the definition of a, we have $a \in \{2\ell_0 - n - 1, 2\ell_0 - n\}$ and hence $\ell_0 = \lceil \frac{1}{2}(n + a) \rceil$ since ℓ_0 is an integer. So the result follows by the definition of ℓ_0 and part (i). □

Note that the set $[\ell]$ was the maximum set in $\mathcal{F}$ in Lemma 6.2.3. For any $m \leqslant 2^\ell$, it is easy to construct an (n, m)-family satisfying the hypotheses of Lemma 6.2.3 by beginning with $[\ell]^\downarrow$ and iteratively removing a minimal set until a family with m sets is reached. In particular, we can construct an (n, m)-family that satisfies the hypotheses of Lemma 6.2.3 for $\ell = \lceil \frac{1}{2}(n + a) \rceil$ where $a = \lfloor \log_2 m \rfloor$. Thus $\Phi(n, m) \leqslant 2^{\lceil (n+a)/2 \rceil} + m2^{\lfloor (n-a)/2 \rfloor} - m$ by part (i) of Lemma 6.2.3, recalling that $\Phi(n, m) = \min\left\{|\mathcal{F}^\updownarrow| : \mathcal{F} \subseteq 2^{[n]} \text{ and } |\mathcal{F}| = m\right\}$ and noting that $|\mathcal{F}^\updownarrow| \leqslant |\mathcal{F}^\uparrow| + |\mathcal{F}^\downarrow|$. As asserted by Corollary 6.1.2, we shall see that this bound is sometimes tight.

Example 6.2.4. *For $\ell = 4$ and $m \in \{0, \dots, 16\}$, we can construct an (n, m)-family satisfying the hypotheses of Lemma 6.2.3 by beginning with $\{1, 2, 3, 4\}^{\downarrow}$, ordered as follows:*

$$\emptyset, 1, 2, 3, 4, 12, 13, 23, 14, 24, 34, 123, 124, 134, 234, 1234$$

and removing sets from the left until a family with m sets is obtained (note that each set will be minimal as it is removed).

6.3 Shifting

We make use of a variant of the classical notion of shifting. Most significantly, our notion differs from the usual one in that it allows sets to be replaced with sets of a different size. For a family $\mathcal{F}$ of subsets of $[n]$ and two nonempty disjoint subsets I and J of $[n]$ we define, for each $F \in \mathcal{F}$,

$$S^{\mathcal{F}}_{I,J}(F) = \begin{cases} (F \setminus J) \cup I & \text{if } J \subseteq F,\ I \cap F = \emptyset \text{ and } (F \setminus J) \cup I \notin \mathcal{F} \\ F & \text{otherwise.} \end{cases}$$

We further define $S_{I,J}(\mathcal{F}) = \{S^{\mathcal{F}}_{I,J}(F) : F \in \mathcal{F}\}$. Observe that $|S_{I,J}(\mathcal{F})| = |\mathcal{F}|$. If $I = \{i\}$ and $J = \{j\}$, this definition agrees with the conventional definition of a shift introduced in Section 2.4.

Let $\mathcal{F}$ be a family of subsets of $[n]$. Recall from the last chapter that $\overline{\mathcal{F}}$ denotes $\{[n] \setminus F : F \in \mathcal{F}\}$, and for any set $F \in \mathcal{F}$, we let $\overline{F}$ denote the set $[n] \setminus F$. Let ρ be the permutation of $[n]$ such that $\rho(i) = n + 1 - i$ for each $i \in [n]$ and for a subset F of $[n]$, let $\rho(F) = \{\rho(x) : x \in F\}$. The *reverse* of $\mathcal{F}$ is the family $\rho(\mathcal{F}) = \{\rho(F) : F \in \mathcal{F}\}$. Obviously $|\rho(\mathcal{F})| = |\mathcal{F}|$. We prove a number of properties of complements, reverses and shifts.

Lemma 6.3.1. *Let $\mathcal{F}$ be a family of subsets of $[n]$ and I and J be disjoint subsets of $[n]$. Then*

(i) $\left(\overline{\mathcal{F}}\right)^{\downarrow} = \overline{\mathcal{F}^{\uparrow}}$, $\quad \left(\overline{\mathcal{F}}\right)^{\uparrow} = \overline{\mathcal{F}^{\downarrow}}$ *and* $\left(\overline{\mathcal{F}}\right)^{\updownarrow} = \overline{\mathcal{F}^{\updownarrow}}$;

(ii) $\overline{\rho(\mathcal{F})} = \rho\left(\overline{\mathcal{F}}\right)$, $\quad \rho(\mathcal{F}^{\downarrow}) = (\rho(\mathcal{F}))^{\downarrow}$, $\quad \rho(\mathcal{F}^{\uparrow}) = (\rho(\mathcal{F}))^{\uparrow}$, *and* $\rho(\mathcal{F}^{\updownarrow}) = (\rho(\mathcal{F}))^{\updownarrow}$;

(iii) $\overline{S_{I,J}(\mathcal{F})} = S_{J,I}(\overline{\mathcal{F}})$.

Proof. It is a simple exercise to prove (i) by noting that $X \subseteq Y$ if and only if $\overline{Y} \subseteq \overline{X}$. The properties in (ii) are immediate from the fact that ρ is a permutation of the ground set $[n]$.

For (iii), we abbreviate $S^{\mathcal{F}}_{I,J}$ to S and $S^{\overline{\mathcal{F}}}_{J,I}$ to S'. Let $F \in \mathcal{F}$. Note that $F \cap I = \emptyset$ if and only if $I \subseteq \overline{F}$, $J \subseteq F$ if and only if $\overline{F} \cap J = \emptyset$, and $(F \setminus J) \cup I \in \mathcal{F}$ if and only if

$(\overline{F} \setminus I) \cup J \in \overline{\mathcal{F}}$, since $\overline{(F \setminus J) \cup I} = (\overline{F} \setminus I) \cup J$. Hence $S(F) = F$ if and only if $S'(\overline{F}) = \overline{F}$ and when both $S(F) \neq F$ and $S'(\overline{F}) \neq \overline{F}$ we have

$$\overline{S(F)} = \overline{(F \setminus J) \cup I} = \big(\overline{F} \setminus I\big) \cup J = S'(F)\,.$$

Thus, $\overline{S(F)} = S'(\overline{F})$ for all $F \in \mathcal{F}$ and we have $\overline{S_{I,J}(\mathcal{F})} = S_{J,I}(\overline{\mathcal{F}})$. $\square$

We now prove a result that shows that, under certain conditions, the size of the closed neighbourhood of a family of sets is not increased when a shift is applied to the family. This result is analogous to results for conventional shifts that are often used to prove the Kruskal-Katona theorem (see, for example, [12]).

Lemma 6.3.2. *Let $\mathcal{F}$ be a family of subsets of $[n]$ and let I and J be nonempty disjoint subsets of $[n]$. If $S_{I',J}(\mathcal{F}) = \mathcal{F}$ for all nonempty proper subsets I' of I and $S_{I,J'}(\mathcal{F}) = \mathcal{F}$ for all nonempty proper subsets J' of J, then*

(i) $(S_{I,J}(\mathcal{F}))^{\downarrow} \subseteq S_{I,J}(\mathcal{F}^{\downarrow})$;

(ii) $(S_{I,J}(\mathcal{F}))^{\uparrow} \subseteq S_{I,J}(\mathcal{F}^{\uparrow})$;

(iii) $(S_{I,J}(\mathcal{F}))^{\updownarrow} \subseteq S_{I,J}(\mathcal{F}^{\updownarrow})$;

(iv) $|(S_{I,J}(\mathcal{F}))^{\updownarrow}| \leqslant |\mathcal{F}^{\updownarrow}|$.

Proof. **(i).** We abbreviate $S^{\mathcal{F}}_{I,J}$ to S and $S^{\mathcal{F}^{\downarrow}}_{I,J}$ to S'. Suppose for a contradiction that there is a set X in $(S_{I,J}(\mathcal{F}))^{\downarrow} \setminus S_{I,J}(\mathcal{F}^{\downarrow})$. Since $X \in (S_{I,J}(\mathcal{F}))^{\downarrow}$, we have that $X \subseteq S(F)$ for some $F \in \mathcal{F}$. We consider two cases according to whether $S(F) = F$.

Suppose first that $S(F) \neq F$. Then $J \subseteq F$, $I \cap F = \emptyset$ and $(F \setminus J) \cup I \notin \mathcal{F}$. Further, $J \cap X = \emptyset$ since $X \subseteq S(F)$. We must have $I \nsubseteq X$ for otherwise the set $Y = (X \setminus I) \cup J$ is a subset of F since $X \subseteq S(F)$, and the fact that $Y \in \mathcal{F}^{\downarrow}$ contradicts our assumption that $X \notin S_{I,J}(\mathcal{F}^{\downarrow})$. We will show that $X \in \mathcal{F}^{\downarrow}$ and hence, since $J \cap X = \emptyset$, that we have $S'(X) = X$ contradicting $X \notin S_{I,J}(\mathcal{F}^{\downarrow})$. If $X \cap I = \emptyset$, then $X \subseteq F$ and so $X \in \mathcal{F}^{\downarrow}$. If $X \cap I \neq \emptyset$, then by our hypotheses and since $I \nsubseteq X$, we have $S_{I',J}(\mathcal{F}) = \mathcal{F}$ where $I' = I \setminus X$. Thus $(F \setminus J) \cup I' \in \mathcal{F}$, which implies that $X \in \mathcal{F}^{\downarrow}$.

Now suppose that $S(F) = F$. Then $X \subseteq F$ and therefore $X \in \mathcal{F}^{\downarrow}$. So, since $X \notin S_{I,J}(\mathcal{F}^{\downarrow})$, we must have $S'(X) \neq X$. So $J \subseteq X$, $I \cap X = \emptyset$ and $S'(X) = (X \setminus J) \cup I \notin \mathcal{F}^{\downarrow}$. The first of these facts implies that $J \subseteq F$ since $X \subseteq F$ and the last of these facts implies that $I \nsubseteq F$ for otherwise $S'(X) = (X \setminus J) \cup I \subseteq F$ contradicting $S'(X) \notin \mathcal{F}^{\downarrow}$.

If $I \cap F \neq \emptyset$, then $\emptyset \subsetneq I \setminus F \subsetneq I$ and hence, by our hypotheses, $S_{I',J}(\mathcal{F}) = \mathcal{F}$ where $I' = I \setminus F$. Hence, we have that $(F \setminus J) \cup I = (F \setminus J) \cup I' \in \mathcal{F}$, and so $S'(X) = (X \setminus J) \cup I \subseteq (F \setminus J) \cup I$ contradicts $S'(X) \notin \mathcal{F}^{\downarrow}$. So it must be that $I \cap F = \emptyset$ and, as shown earlier,

$J \subseteq F$. Thus, since $S(F) = F$, it must be the case that the set $(F \setminus J) \cup I$ is in $\mathcal{F}$, but this contradicts $S'(X) = (X \setminus J) \cup I \notin \mathcal{F}^{\downarrow}$.

(ii). By using (i) and (iii) of Lemma 6.3.1 and applying (i) of this lemma, we have

$$\overline{(S_{I,J}(\mathcal{F}))^{\uparrow}} = \overline{S_{I,J}(\mathcal{F})}^{\downarrow} = \left(S_{J,I}(\overline{\mathcal{F}})\right)^{\downarrow} \subseteq S_{J,I}\left((\overline{F})^{\downarrow}\right) = S_{J,I}\left(\overline{\mathcal{F}^{\uparrow}}\right) = \overline{S_{I,J}(\mathcal{F}^{\uparrow})}$$

and hence $(S_{I,J}(\mathcal{F}))^{\uparrow} \subseteq S_{I,J}(\mathcal{F}^{\uparrow})$. Thus (ii) follows from (i).

(iii). This follows immediately from (i) and (ii).

(iv). This follows from (iii) by noting that $|S_{I,J}(\mathcal{F}^{\updownarrow})| = |\mathcal{F}^{\updownarrow}|$. □

For establishing lower bounds on $\Phi(n,m)$ it will be useful to consider a particular family that we now define. For nonnegative integers m and n with $m \leqslant 2^n$, we define the *canonical minimal* (n,m)*-family* to be the unique (n,m)-family $\mathcal{F}$ such that $|\mathcal{F}^{\updownarrow}| = \Phi(n,m)$ and $\mathcal{F} \preceq \mathcal{G}$ for each (n,m)-family $\mathcal{G}$ such that $|\mathcal{G}^{\updownarrow}| = \Phi(n,m)$. This family is unique because $\preceq$ is a total order on (n,m)-families. We proceed to show that a canonical minimal family must be convex and invariant under certain shifts. For a family $\mathcal{F}$ of subsets of $[n]$, we say $\mathcal{F}$ is *strongly shifted* if $S_{I,J}(\mathcal{F}) = \mathcal{F}$ for all $I, J \subseteq [n]$ such that $\max(I) < \min(J)$.

Lemma 6.3.3. *The canonical minimal* (n,m)*-family is convex and strongly shifted.*

Proof. Let $\mathcal{F}$ be the canonical minimal (n,m)-family. Then $\mathcal{F}$ must be convex, for otherwise by Lemma 6.2.2 there would be an (n,m)-family $\mathcal{G}$ such that $|\mathcal{G}^{\updownarrow}| \leqslant |\mathcal{F}^{\updownarrow}|$ and $\mathcal{G} \prec \mathcal{F}$, and this would contradict the definition of $\mathcal{F}$. Suppose for a contradiction that $\mathcal{F}$ is not strongly shifted. Then there are subsets I and J of $[n]$ such that $\max(I) < \min(J)$ and $S_{I,J}(\mathcal{F}) \neq \mathcal{F}$. We may further suppose that I and J have been chosen so that $S_{I',J}(\mathcal{F}) = \mathcal{F}$ for all nonempty proper subsets I' of I and $S_{I,J'}(\mathcal{F}) = \mathcal{F}$ for all nonempty proper subsets J' of J (note that we necessarily have $\max(I') \leqslant \min(J)$ and $\max(I) \leqslant \min(J')$). Then by Lemma 6.3.2, $S_{I,J}(\mathcal{F})$ is an (n,m)-family such that $|(S_{I,J}(\mathcal{F}))^{\updownarrow}| \leqslant |\mathcal{F}^{\updownarrow}|$. Furthermore, it is easy to see that $S_{I,J}(\mathcal{F}) \prec \mathcal{F}$ since $\max(I) < \min(J)$. So we have a contradiction to the definition of $\mathcal{F}$. □

6.4 Standard form

We will see in the next lemma that any convex strongly shifted family $\mathcal{F}$ must have a number of properties including that $[s-1] \subseteq F \subseteq [\ell+1]$ for all $F \in \mathcal{F}$ where s and ℓ are the smallest and largest sizes, respectively, of a set in $\mathcal{F}$. We introduce some notation to help us describe such families. Let s, ℓ and n be positive integers with $s \leqslant \ell \leqslant n$. We call a subset of $[\ell] \setminus [s]$ an (s, ℓ)-*pattern* and define the (s, ℓ)-pattern of a subset X of $[n]$ to be $X \cap ([\ell] \setminus [s])$. We always consider a family $\mathcal{P}$ of (s, ℓ)-patterns as a family on the ground set $[\ell] \setminus [s]$. So, in particular, $\mathcal{P}^\uparrow$ denotes the set of all subsets of $[\ell] \setminus [s]$ that are supersets of at least one set in $\mathcal{P}$. We define the (s, ℓ)-*type* of a subset X of $[n]$ to be

$$\begin{array}{ll} 10 & \text{if } s \in X \text{ and } \ell+1 \notin X \\ 00 & \text{if } s \notin X \text{ and } \ell+1 \notin X \\ 11 & \text{if } s \in X \text{ and } \ell+1 \in X \\ 01 & \text{if } s \notin X \text{ and } \ell+1 \in X. \end{array}$$

We say a family $\mathcal{F}$ of subsets of $[n]$ has (s, ℓ)-*pattern list* $(\mathcal{P}_{10}, \mathcal{P}_{00}, \mathcal{P}_{11}, \mathcal{P}_{01})$ where $\mathcal{P}_{10}, \mathcal{P}_{00}, \mathcal{P}_{11}, \mathcal{P}_{01}$ are the families of all patterns belonging to the sets in $\mathcal{F}$ with types $01, 00, 11, 01$ respectively (note that $\mathcal{P}_{10}, \mathcal{P}_{00}, \mathcal{P}_{11}, \mathcal{P}_{01}$ will not be disjoint in general). We say that a family $\mathcal{F}$ with (s, ℓ)-pattern list $(\mathcal{P}_{10}, \mathcal{P}_{00}, \mathcal{P}_{11}, \mathcal{P}_{01})$ is in (s, ℓ)-*standard form* if it satisfies the following properties.

(SF1) $[s-1] \subseteq F \subseteq [\ell+1]$ for each $F \in \mathcal{F}$

(SF2) $\mathcal{P}_{10} = 2^{[\ell]\setminus[s]}$

(SF3) $\mathcal{P}_{00} = \mathcal{P}_{00}^\uparrow$

(SF4) $\mathcal{P}_{11} = \mathcal{P}_{11}^\downarrow$

(SF5) $\{P\}^\uparrow \setminus \{P\} \subseteq \mathcal{P}_{00}$ and $\{P\}^\downarrow \setminus \{P\} \subseteq \mathcal{P}_{11}$ for each $P \in \mathcal{P}_{01}$

Note that a family $\mathcal{F}$ that obeys (SF1) is uniquely determined by its (s, ℓ)-pattern list, as each set is then uniquely determined by its intersection with $[\ell] \setminus [s]$ (its pattern) and its intersection with $\{s, l+1\}$ (given by its type). When we are dealing with a family that we have specified to be in (s, ℓ)-standard form for particular values of s and ℓ, the only types and patterns we will be interested in will be (s, ℓ)-types and (s, ℓ)-patterns and so we usually drop the (s, ℓ) prefix.

Lemma 6.4.1. *Let $\mathcal{F}$ be a strongly shifted and convex family of subsets of $[n]$. Then $\mathcal{F}$ is in (s, ℓ)-standard form where s and ℓ are the sizes of the smallest and largest sets in $\mathcal{F}$, respectively.*

Proof. Let $(\mathcal{P}_{10}, \mathcal{P}_{00}, \mathcal{P}_{11}, \mathcal{P}_{01})$ be the pattern list of $\mathcal{F}$. We will prove that $\mathcal{F}$ satisfies each of (SF1)–(SF5) in turn.

(SF1). Suppose that $F \nsubseteq [\ell+1]$ for some $F \in \mathcal{F}$. Because $S_{I,J}(\mathcal{F}) = \mathcal{F}$ where $I = [\ell+1] \setminus F$ and $J = F \setminus [\ell+1]$, we must have that $(F \setminus J) \cup I = [\ell+1]$ is in $\mathcal{F}$, a contradiction. Thus $F \subseteq [\ell+1]$ for each $F \in \mathcal{F}$.

Now suppose that $[s-1] \nsubseteq F$ for some $F \in \mathcal{F}$. Because $S_{I,J}(\mathcal{F}) = \mathcal{F}$ where $I = [s-1] \setminus F$

and $J = F \setminus [s-1]$, we must have that $(F \setminus J) \cup I = [s-1]$ is in $\mathcal{F}$, a contradiction. Thus $[s-1] \subseteq F$ for each $F \in \mathcal{F}$.

(SF2). As $\mathcal{F}$ is strongly shifted and contains sets of sizes s and ℓ, we have that $[s]$ and $[\ell]$ are in $\mathcal{F}$. Therefore, as $\mathcal{F}$ is convex $[s] \cup P \in \mathcal{F}$ for all $P \subseteq [\ell] \setminus [s]$ and so $\mathcal{P}_{10} = 2^{[\ell]\setminus[s]}$.

(SF3). Let P be a pattern in $\mathcal{P}_{00}$ and $F = [s-1] \cup P$ be the corresponding set in $\mathcal{F}$. Then for any set Q such that $P \subseteq Q \subseteq [\ell] \setminus [s]$, the set $G = [s-1] \cup Q$ is in $\mathcal{F}$ since $F \subseteq G \subseteq [\ell]$ and $\mathcal{F}$ is convex. Thus $Q \in \mathcal{P}_{00}$. It follows that $\mathcal{P}_{00} = \mathcal{P}_{00}^{\uparrow}$.

(SF4). Let P be a pattern in $\mathcal{P}_{11}$ and $F = [s] \cup P \cup \{\ell+1\}$ be the corresponding set in $\mathcal{F}$. Then for any set Q such that $\emptyset \subseteq Q \subseteq P$, the set $G = [s] \cup Q \cup \{\ell+1\}$ is in $\mathcal{F}$ since $[s] \subseteq G \subseteq F$ and $\mathcal{F}$ is convex. Thus $Q \in \mathcal{P}_{11}$. It follows that $\mathcal{P}_{11} = \mathcal{P}_{11}^{\downarrow}$.

(SF5). Let $P \in \mathcal{P}_{01}$, $F = [s-1] \cup P \cup \{\ell+1\}$ be the corresponding set in $\mathcal{F}$ and Q be a set such that $P \subsetneq Q \subseteq [\ell] \setminus [s]$. Because $S_{I,J}(\mathcal{F}) = \mathcal{F}$ where $I = Q \setminus P$ and $J = \{\ell+1\}$, the set $(F \setminus J) \cup I = [s-1] \cup Q$ is in $\mathcal{F}$. This implies that $Q \in \mathcal{P}_{00}$. So $\{P\}^{\uparrow} \setminus \{P\} \subseteq \mathcal{P}_{00}$. By an analogous argument $\{P\}^{\downarrow} \setminus \{P\} \subseteq \mathcal{P}_{11}$. $\square$

For an (n, m)-family $\mathcal{F}$ in (s, ℓ)-standard form, we can determine $|\mathcal{F}^{\uparrow}| + |\mathcal{F}^{\downarrow}|$ exactly in terms of the sizes of the families in its pattern list and their intersections.

Lemma 6.4.2. *Let $\mathcal{F}$ be an (n, m)-family in (s, ℓ)-standard form with pattern list $(\mathcal{P}_{10}, \mathcal{P}_{00}, \mathcal{P}_{11}, \mathcal{P}_{01})$. Then $|\mathcal{F}^{\updownarrow}| \leqslant |\mathcal{F}^{\uparrow}| + |\mathcal{F}^{\downarrow}| - m$ and $|\mathcal{F}^{\uparrow}| + |\mathcal{F}^{\downarrow}|$ is given by*

$$2^{n-s} + 2^s m + 2^{n-\ell-1}\left(2|\mathcal{P}_{00}| + |\mathcal{P}_{01}^{10}| + |\mathcal{P}_{01}^{11}|\right) - 2^{s-1}\left(2|\mathcal{P}_{00}| + |\mathcal{P}_{01}^{10}| + |\mathcal{P}_{01}^{00}| + 2|\mathcal{P}_{01}^{11}| + 2|\mathcal{P}_{01}^{*}|\right)$$

where $\mathcal{P}_{01}^{10} = \mathcal{P}_{01} \setminus (\mathcal{P}_{00} \cup \mathcal{P}_{11})$, $\mathcal{P}_{01}^{00} = \mathcal{P}_{01} \cap (\mathcal{P}_{00} \setminus \mathcal{P}_{11})$, $\mathcal{P}_{01}^{11} = \mathcal{P}_{01} \cap (\mathcal{P}_{11} \setminus \mathcal{P}_{00})$ and $\mathcal{P}_{01}^{*} = \mathcal{P}_{01} \cap \mathcal{P}_{00} \cap \mathcal{P}_{11}$.

Proof. We have $|\mathcal{F}^{\updownarrow}| \leqslant |\mathcal{F}^{\uparrow}| + |\mathcal{F}^{\downarrow}| - m$ since $\mathcal{F} \subseteq \mathcal{F}^{\uparrow} \cap \mathcal{F}^{\downarrow}$, so it remains to show that $|\mathcal{F}^{\uparrow}| + |\mathcal{F}^{\downarrow}|$ is as given in the lemma statement. Clearly $\mathcal{P}_{01}$ is the disjoint union of $\mathcal{P}_{01}^{10}, \mathcal{P}_{01}^{00}, \mathcal{P}_{01}^{11}$ and $\mathcal{P}_{01}^{*}$. We first show that

$$|\mathcal{F}^{\uparrow}| = 2^{n-s} + 2^{n-\ell-1}(2|\mathcal{P}_{00}| + |\mathcal{P}_{01}^{10} \cup \mathcal{P}_{01}^{11}|). \tag{6.1}$$

For a pattern P and a subset A of $\{s, \ell+1\}$, we say the pair (P, A) is *up-covered* if $[s-1] \cup P \cup A \in \mathcal{F}^{\uparrow}$. By (SF1), $[s-1] \subseteq F \subseteq [\ell+1]$ for all $F \in \mathcal{F}$, so $\mathcal{F}^{\uparrow}$ contains exactly the sets $[s-1] \cup P \cup A \cup B$ where (P, A) is up-covered and $B \subseteq [n] \setminus [\ell+1]$. Therefore $|\mathcal{F}^{\uparrow}| = 2^{n-\ell-1}u$, where u is the number of up-covered pairs. So to establish (6.1) it suffices to show that

$$u = 2^{\ell-s+1} + 2|\mathcal{P}_{00}| + |\mathcal{P}_{01}^{10} \cup \mathcal{P}_{01}^{11}|. \tag{6.2}$$

As $[s] \cup P \in \mathcal{F}$ for all patterns P by (SF2), the pairs $(P, \{s\})$ and $(P, \{s, \ell+1\})$ are up-covered for each of the $2^{\ell-s}$ patterns P. Also, the pair $(P, \emptyset)$ is up-covered if and only if $P \in \mathcal{P}_{00}^{\uparrow}$ as the sets in $\mathcal{F}$ that contain neither s nor $\ell+1$ are exactly those of type 00. Finally $(P, \{\ell+1\})$ is up-covered if and only if $P \in (\mathcal{P}_{00} \cup \mathcal{P}_{01})^{\uparrow}$ as the sets in $\mathcal{F}$ that do not contain s are exactly those of type 00 or 01. So

$$u = 2^{\ell-s+1} + |\mathcal{P}_{00}^{\uparrow}| + |(\mathcal{P}_{00} \cup \mathcal{P}_{01})^{\uparrow}| = 2^{\ell-s+1} + |\mathcal{P}_{00}| + |(\mathcal{P}_{00} \cup \mathcal{P}_{01})^{\uparrow}|$$

where the second equality follows by (SF3). So it suffices to prove that $(\mathcal{P}_{00} \cup \mathcal{P}_{01})^{\uparrow} = \mathcal{P}_{00} \cup \mathcal{P}_{01}^{10} \cup \mathcal{P}_{01}^{11}$ to establish (6.2) and therefore (6.1). Noting that $\mathcal{P}_{01}^{\uparrow} \setminus \mathcal{P}_{01} \subseteq \mathcal{P}_{00}$ by (SF5) and $\mathcal{P}_{00}^{\uparrow} = \mathcal{P}_{00}$ by (SF3), it follows that $(\mathcal{P}_{00} \cup \mathcal{P}_{01})^{\uparrow} = \mathcal{P}_{00} \cup \mathcal{P}_{01}$ which is exactly $\mathcal{P}_{00} \cup \mathcal{P}_{01}^{10} \cup \mathcal{P}_{01}^{11}$ by the definitions of $\mathcal{P}_{01}^{10}$ and $\mathcal{P}_{01}^{11}$. So (6.1) does indeed hold.

Next we establish that

$$|\mathcal{F}^{\downarrow}| = 2^{\ell} + 2^{s-1}(2|\mathcal{P}_{11}| + |\mathcal{P}_{01}^{10} \cup \mathcal{P}_{01}^{00}|). \tag{6.3}$$

For a pattern P and a subset A of $\{s, \ell+1\}$, we say the pair (P, A) is *down-covered* if $[s-1] \cup P \cup A \in \mathcal{F}^{\downarrow}$. By (SF1), $[s-1] \subseteq F \subseteq [\ell+1]$ for all $F \in \mathcal{F}$, so $\mathcal{F}^{\downarrow}$ contains exactly the sets $([s-1] \setminus B) \cup P \cup A$ where (P, A) is down-covered and $B \subseteq [s-1]$. Hence, $|\mathcal{F}^{\downarrow}| = 2^{s-1}v$, where v is the number of down-covered pairs. By similar arguments to those used for $|\mathcal{F}^{\uparrow}|$, it can be shown that

$$v = 2^{\ell-s+1} + |\mathcal{P}_{11}^{\downarrow}| + |(\mathcal{P}_{11} \cup \mathcal{P}_{01})^{\downarrow}| = 2^{\ell-s+1} + 2|\mathcal{P}_{11}| + |\mathcal{P}_{01}^{10} \cup \mathcal{P}_{01}^{00}|$$

from which (6.3) follows.

That $|\mathcal{F}^{\uparrow}| + |\mathcal{F}^{\downarrow}|$ is as given in the lemma statement now follows using (6.1) and (6.3) and making the substitutions $|\mathcal{P}_{11}| = m - |\mathcal{P}_{10}| - |\mathcal{P}_{00}| - |\mathcal{P}_{01}|$, $|\mathcal{P}_{01}| = |\mathcal{P}_{01}^{10}| + |\mathcal{P}_{01}^{00}| + |\mathcal{P}_{01}^{11}| + |\mathcal{P}_{01}^{*}|$ and $|\mathcal{P}_{10}| = 2^{\ell-s}$. The first two of these substitutions are valid by the definitions of the families involved and the last is valid by (SF2). $\square$

It turns out that there are two critical cases for establishing $\Phi(n, m)$. These involve a convex (n, m)-family $\mathcal{F}$ in (s, ℓ)-standard form with $\ell = n - s$ or $\ell = n - s - 1$. Our final two lemmas in this section use Lemma 6.4.2 to give lower bounds on $|F^{\updownarrow}|$ in these two situations.

Lemma 6.4.3. *Let n be a nonnegative integer and let $\mathcal{F}$ be a convex (n, m)-family that is in (s, ℓ)-standard form with $\ell = n - s$. Let t be the maximum integer in $\{0, \ldots, 2^{\ell-s}\}$ such that $m - 2^{\ell-s} \geqslant \Phi(\ell - s, t) + 2t - 1$ and let $\delta = 1$ if $m - 2^{\ell-s} = \Phi(\ell - s, t) + 2t - 1$ and*

$\delta = 0$ *otherwise. Then*

$$|F^{\updownarrow}| \geqslant 2^{n-s} + 2^s\left(m - t + \tfrac{1}{2}\delta\right) - m.$$

Proof. Let $(\mathcal{P}_{10}, \mathcal{P}_{00}, \mathcal{P}_{11}, \mathcal{P}_{01})$ be the pattern list of $\mathcal{F}$ and let $\mathcal{P}_{01}^{10}, \mathcal{P}_{01}^{00}, \mathcal{P}_{01}^{11}, \mathcal{P}_{01}^{*}$ be as defined in Lemma 6.4.2. Since $\mathcal{F}$ is convex, using Lemma 6.4.2 and $\ell = n - s$ we have

$$|\mathcal{F}^{\updownarrow}| = |\mathcal{F}^{\uparrow}| + |\mathcal{F}^{\downarrow}| - m = 2^{n-s} + (2^s - 1)m - 2^{s-1}\left(|\mathcal{P}_{01}^{00}| + |\mathcal{P}_{01}^{11}| + 2|\mathcal{P}_{01}^{*}|\right).$$

This completes the proof if $|\mathcal{P}_{01}^{00}| + |\mathcal{P}_{01}^{11}| + 2|\mathcal{P}_{01}^{*}| \leqslant 2t - \delta$. So suppose for a contradiction that $|\mathcal{P}_{01}^{00}| + |\mathcal{P}_{01}^{11}| + 2|\mathcal{P}_{01}^{*}| \geqslant 2t - \delta + 1$. As $\mathcal{F}$ is in standard form, $\mathcal{P}_{01}^{\uparrow} \setminus \mathcal{P}_{01} \subseteq \mathcal{P}_{00}$ by (SF5) and by definition $\mathcal{P}_{01}^{00} \cup \mathcal{P}_{01}^{*} \subseteq \mathcal{P}_{00}$. Similarly, $\mathcal{P}_{01}^{\downarrow} \setminus \mathcal{P}_{01} \subseteq \mathcal{P}_{11}$ and $\mathcal{P}_{01}^{11} \cup \mathcal{P}_{01}^{*} \subseteq \mathcal{P}_{11}$. Therefore

$$|\mathcal{P}_{00}| + |\mathcal{P}_{11}| \geqslant |\mathcal{P}_{01}^{\uparrow} \setminus \mathcal{P}_{01}| + |\mathcal{P}_{01}^{\downarrow} \setminus \mathcal{P}_{01}| + |\mathcal{P}_{01}^{00}| + |\mathcal{P}_{01}^{11}| + 2|\mathcal{P}_{01}^{*}| \geqslant |\mathcal{P}_{01}^{\updownarrow}| - |\mathcal{P}_{01}| + 2t - \delta + 1$$

where the last inequality follows using our assumption and since the convexity of $\mathcal{F}$ implies the convexity of $\mathcal{P}_{01}$. Thus,

$$m - 2^{\ell - s} = |\mathcal{P}_{00}| + |\mathcal{P}_{11}| + |\mathcal{P}_{01}| \geqslant |\mathcal{P}_{01}^{\updownarrow}| + 2t - \delta + 1 \geqslant \Phi(\ell - s, t - \delta + 1) + 2t - \delta + 1$$

where the last inequality follows from the definition of Φ, since $2|\mathcal{P}_{01}| \geqslant |\mathcal{P}_{01}^{00}| + |\mathcal{P}_{01}^{11}| + 2|\mathcal{P}_{01}^{*}| \geqslant 2t - \delta + 1$ by our assumption and hence $|\mathcal{P}_{01}| \geqslant t - \delta + 1$. This contradicts the definition of t when $\delta = 0$ and contradicts the definition of δ when $\delta = 1$. □

Lemma 6.4.4. *Let n be a nonnegative integer and let $\mathcal{F}$ be a convex (n, m)-family that is in (s, ℓ)-standard form with $\ell = n - s - 1$. Then*

$$|F^{\updownarrow}| \geqslant 2^{n-s} + 2^s m - m\,.$$

Proof. Let $(\mathcal{P}_{10}, \mathcal{P}_{00}, \mathcal{P}_{11}, \mathcal{P}_{01})$ be the pattern list of $\mathcal{F}$ and let $\mathcal{P}_{01}^{10}, \mathcal{P}_{01}^{00}, \mathcal{P}_{01}^{11}, \mathcal{P}_{01}^{*}$ be as defined in Lemma 6.4.2. Since $\mathcal{F}$ is convex, using Lemma 6.4.2 and $\ell = n - s - 1$ we have

$$|\mathcal{F}^{\updownarrow}| = |\mathcal{F}^{\uparrow}| + |\mathcal{F}^{\downarrow}| - m = 2^{n-s} + (2^s - 1)m + 2^{s-1}\left(2|\mathcal{P}_{00}| + |\mathcal{P}_{01}^{10}| - |\mathcal{P}_{01}^{00}| - 2|\mathcal{P}_{01}^{*}|\right).$$

Thus we have the desired result because $|\mathcal{P}_{00}| \geqslant |\mathcal{P}_{01}^{00}| + |\mathcal{P}_{01}^{*}|$ by the definitions of $\mathcal{P}_{01}^{00}$ and $\mathcal{P}_{01}^{*}$. □

6.5 Proof of main result

Throughout this section assume that $n, m \in \mathbb{Z}^+$ with $m \in \{1, \ldots, 2^{n-1} - 1\}$. Motivated by Lemmas 6.4.3 and 6.4.4, define

$$
\begin{aligned}
A(n,m) &= 2^{\lceil (n+a)/2 \rceil} + 2^{\lfloor (n-a)/2 \rfloor} m - m \\
B(n,m) &= 2^{\lfloor (n+a)/2 \rfloor} + 2^{\lceil (n-a)/2 \rceil} (m - t + \tfrac{1}{2}\delta) - m = A(n,m) - 2^{\lceil (n-a)/2 \rceil (t - \delta/2)}
\end{aligned}
$$

where $a = \lfloor \log_2 m \rfloor$,

$$
a' = \begin{cases} a & \text{if } a \equiv n \pmod 2 \text{ or } a = 0 \\ a - 1 & \text{otherwise,} \end{cases}
$$

$t \in [2^{a'}]$ is the greatest integer such that $m - 2^{a'} \geqslant \Phi(a', t) + 2t - 1$, and

$$
\delta = \begin{cases} 1 & \text{if } \Phi(a', t) = m - 2^{a'} - 2t + 1 \\ 0 & \text{otherwise.} \end{cases}
$$

The aim is to prove that Theorem 6.1.1, which asserts that $\Phi(n,m) = B(n,m)$ for $m \in \{1, \ldots, 2^n - 1\}$. Note that Lemma 6.2.3(ii) asserts that $|\mathcal{F}^{\updownarrow}| \geqslant A(n,m)$ under the conditions it specifies. Note that $B(n,m) = A(n,m) - 2^{\lceil (n-a)/2 \rceil}(t - \delta/2)$, so $B(n,m) \leqslant A(n,m)$ whenever $t \geqslant \delta/2$. We now show that $B(n,m) \leqslant A(n,m)$ in all cases, and characterise when $A(n,m) = B(n,m)$.

Lemma 6.5.1. *Let n and m be nonnegative integers with $m \in \{1, \ldots, 2^n - 1\}$ and let $a = \lfloor \log_2 m \rfloor$. Then $B(n,m) \leqslant A(n,m)$ with equality if and only if $a = 0$ or*

- *$a \geqslant 1$, $a \equiv n \pmod 2$ and $m \leqslant 2^a + 2^{\lceil a/2 \rceil} + 2^{\lfloor a/2 \rfloor} - 1$; or*
- *$a \geqslant 1$, $a \not\equiv n \pmod 2$ and $m \geqslant 2^{a+1} - 2^{\lceil (a+1)/2 \rceil} - 2^{\lfloor (a+1)/2 \rfloor} + 2$.*

Proof. Let a', t and δ be as in the definition of $B(n,m)$. When $a = 0$, then $m = 1$ and clearly $t = 0$ and $\delta = 0$, so $A(n,m) = B(n,m)$. So we may assume $1 \leqslant a < n$.

Case 1. Suppose that $a \equiv n \pmod 2$. Then

$$
A(n,m) - B(n,m) = 2^{(n-a)/2} \left(t - \tfrac{1}{2}\delta \right).
$$

So, since t cannot be 0 when $\delta = 1$, we have $B(n,m) \leqslant A(m,n)$ with equality if and only if $t = \delta = 0$. By the definition of t, we have $t = 0$ if and only if $m - 2^a \leqslant \Phi(a, 1)$. By Lemma 6.2.1(i), this is equivalent to $m \leqslant 2^a + 2^{\lceil a/2 \rceil} + 2^{\lfloor a/2 \rfloor} - 1$.

Case 2. Suppose that $a \not\equiv n \pmod 2$. Define b to be the element of $\{0,\ldots,2^a-1\}$ such that $m = 2^a + b$. Then

$$A(n,m) - B(n,m) = 2^{(n+a-1)/2} - 2^{(n-a-1)/2}m + 2^{(n-a+1)/2}\left(t - \tfrac{1}{2}\delta\right) = 2^{(n-a-1)/2}\left(2t - b - \delta\right).$$

We first prove that $2t \geqslant b+\delta$ and hence that $A(n,m) \leqslant B(n,m)$ with equality if and only if $2t = b+\delta$. If $t = 2^{a-1}$, then $2t \geqslant b+\delta$, since $b \leqslant 2^a - 1$ and $\delta \leqslant 1$. Otherwise, by the definition of t,

$$\Phi(a-1,t+1) + 2t \geqslant m - 2^{a-1} = 2^{a-1} + b\,. \tag{6.4}$$

Since $\Phi(a-1,t+1) \leqslant 2^{a-1}$ by definition, (6.4) implies that $2t \geqslant b$. Further, if $2t = b$ then

$$\Phi(a-1,t) + 2t - 1 \leqslant 2^{a-1} + 2t - 1 = 2^{a-1} + b - 1 = m - 2^{a-1} - 1$$

and hence $\delta = 0$. Thus $2t \geqslant b+\delta$ and $A(n,m) \leqslant B(n,m)$ with equality if and only if $2t = b+\delta$.

Let $m_0 = 2^{a+1} - 2^{\lceil (a+1)/2\rceil} - 2^{\lfloor (a+1)/2\rfloor} + 2$ and $t_0 = 2^{a-1} - 2^{\lceil (a-1)/2\rceil} - 2^{\lfloor (a-1)/2\rfloor} + 1$. We will complete the proof by showing that $2t = b+\delta$ if and only if $m \geqslant m_0$. By Lemma 6.2.1(ii) and (iii), $\Phi(a-1,t) = 2^{a-1}$ if $t \geqslant t_0+1$ and $\Phi(a-1,t) \leqslant 2^{a-1}-1$ if $t \leqslant t_0$. Using these facts and the definition of t_0, we have

$$\Phi(a-1,t_0+1) + 2(t_0+1) - 1 = 3\cdot 2^{a-1} - 2^{\lceil (a+1)/2\rceil} - 2^{\lfloor (a+1)/2\rfloor} + 3 \tag{6.5}$$

$$\Phi(a-1,t_0) + 2t_0 - 1 \leqslant 3\cdot 2^{a-1} - 2^{\lceil (a+1)/2\rceil} - 2^{\lfloor (a+1)/2\rfloor}. \tag{6.6}$$

Case 2a. Suppose that $m \geqslant m_0+1$. We show that $2t = b+\delta$. By the hypothesis of this subcase, $m - 2^{a-1}$ is greater than or equal to the right hand side of (6.5) and hence $t \geqslant t_0+1$. So we have that $\Phi(a-1,t) = 2^{a-1}$ and hence by definition t is the greatest integer such that $2t \leqslant m - 2^a + 1 = b+1$. So we have $2t = b$ and $\delta = 0$ if b is even and $2t = b+1$ and $\delta = 1$ if b is odd. Hence $2t = b+\delta$.

Case 2b. Suppose that $m \leqslant m_0$. We show that $2t = b+\delta$ if and only if $m = m_0$. By the hypothesis of this subcase, $m - 2^{a-1}$ is less than the right hand side of (6.5) and so $t \leqslant t_0$. Further, if $m \in \{m_0, m_0-1\}$, then $m - 2^{a-1}$ is greater than the right hand side of (6.6) and so $t = t_0$ and $\delta = 0$. We separately consider the situations when $t = t_0$ and when $t \leqslant t_0 - 1$.

If $t = t_0$ then using $b = m - 2^a$ and the definition of t_0 shows that $b = 2t_0$ if $m = m_0$, $b = 2t_0 - 1$ if $m = m_0 - 1$, and $b \leqslant 2t_0 - 2$ otherwise. So, since we have seen that $\delta = 0$ when $m \in \{m_0, m_0-1\}$, we have that $2t = b+\delta$ if and only if $m = m_0$.

If $t \leqslant t_0 - 1$ then, by the definition of t, we have $m - 2^{a-1} \leqslant \Phi(a-1,t+1) + 2t$. Since

$\Phi(a-1,t+1) \leqslant \Phi(a-1,t_0) \leqslant 2^{a-1}-1$, this implies $2t \geqslant b+1$. Further, if $\delta = 1$, then by the definition of δ we have $m - 2^{a-1} = \Phi(a-1,t) + 2t - 1$. Since $\Phi(a-1,t) \leqslant 2^{a-1}-1$, this implies $2t \geqslant b+2$. Thus $2t > b+\delta$. □

For $m \in \{1,\dots,2^n-1\}$ we can now prove Theorem 6.1.1 by first showing that $\Phi(n,m) \geqslant B(n,m)$ and then showing that $\Phi(n,m) \leqslant B(n,m)$.

Lemma 6.5.2. *For nonnegative integers n and m with $m \in \{1,\dots,2^n-1\}$, $\Phi(n,m) \geqslant B(n,m)$.*

Proof. Let $\mathcal{F}$ be the (n,m)-canonical minimal family, let s and ℓ be the least and greatest sizes of a set in $\mathcal{F}$ respectively. It suffices to show that $|\mathcal{F}^{\updownarrow}| \geqslant B(n,m)$. By Lemmas 6.3.3 and 6.4.1 $\mathcal{F}$ is convex and in (s,ℓ)-standard form. Let a, a', t and δ be as in the definition of $B(n,m)$. We have $2^{\ell-s} \leqslant m \leqslant 2^{\ell-s+2}-2$, where the lower bound on m follows by (SF2) and the upper bound follows by (SF1) and the fact that $[s-1]$ and $[\ell+1]$ cannot be sets in $\mathcal{F}$ by the definitions of s and ℓ. We consider two cases according to whether $\mathcal{F} \subseteq \{[\ell]\}^{\downarrow}$.
Case 1. Suppose that $\mathcal{F} \subseteq \{[\ell]\}^{\downarrow}$. Then, since $[\ell]$ is the only set of size ℓ in $\{[\ell]\}^{\downarrow}\}$, we have that $[\ell] \in \mathcal{F}$, and so by Lemma 6.2.3(ii) and Lemma 6.5.1 we have

$$|\mathcal{F}^{\updownarrow}| \geqslant A(n,m) \geqslant B(n,m)\,.$$

Case 2. Suppose that $\mathcal{F} \nsubseteq \{[\ell]\}^{\downarrow}$. We first show that $s \in \{n-\ell-1, n-\ell\}$.

Suppose for a contradiction that $s \geqslant n-\ell+1$. Recall that ρ is the permutation of $[n]$ such that $\rho(i) = n+1-i$ for each $i \in [n]$ and for a subset F of $[n]$, $\rho(F) = \{\rho(x) : x \in F\}$. By (SF1), $\mathcal{F} \subseteq \{[s-1]\}^{\uparrow}$ and hence we have that $\rho(\overline{\mathcal{F}}) \subseteq \{[n-s+1]\}^{\downarrow}$. So $\rho(\overline{\mathcal{F}}) \subseteq \{[\ell]\}^{\downarrow}$ since $s \geqslant n-\ell+1$ and hence $\rho(\overline{\mathcal{F}}) \prec \mathcal{F}$ since $\mathcal{F} \nsubseteq \{[\ell]\}^{\downarrow}$. This contradicts the definition of $\mathcal{F}$ since $|\mathcal{F}^{\updownarrow}| = |(\rho(\overline{\mathcal{F}}))^{\updownarrow}|$ by Lemma 6.3.1 (i) and (ii).

Now suppose for a contradiction that $s \leqslant n-\ell-2$. Then no set in $\mathcal{F}$ contains n by (SF1) because $s \leqslant n-\ell-2$ implies $\ell \leqslant n-2$. Let $\mathcal{G} = \{F \cup \{n\} : F \in \mathcal{F}\}$ and note that $|\mathcal{G}^{\uparrow}| = \frac{1}{2}|\mathcal{F}^{\uparrow}|$, that $|\mathcal{G}^{\downarrow}| = 2|\mathcal{F}^{\downarrow}|$, and that $\mathcal{G}$ is convex because $\mathcal{F}$ is. So

$$|\mathcal{F}^{\updownarrow}| - |\mathcal{G}^{\updownarrow}| = |\mathcal{F}^{\uparrow}| + |\mathcal{F}^{\downarrow}| - |\mathcal{G}^{\uparrow}| - |\mathcal{G}^{\downarrow}| = \tfrac{1}{2}|\mathcal{F}^{\uparrow}| - |\mathcal{F}^{\downarrow}| > 2^{n-s-1} - 2^{\ell+1} \geqslant 0$$

where the second last inequality follows because $\{[s]\}^{\uparrow} \subseteq \mathcal{F}^{\uparrow}$ by (SF2) and $\mathcal{F}^{\downarrow} \subsetneq \{[\ell+1]\}^{\downarrow}$ by (SF1) and the definition of ℓ, and the last follows because $s \leqslant n-\ell-2$. So $|\mathcal{G}^{\updownarrow}| < |\mathcal{F}^{\updownarrow}|$ which contradicts the definition of $\mathcal{F}$. So we do indeed have that $s \in \{n-\ell-1, n-\ell\}$. We divide the proof into two subcases accordingly. In each case, note that $a \in \{l-s, l-s+1\}$ because we saw above that $2^{l-s} \leqslant m < 2^{l-s+1}$.

Case 2a. Suppose that $s = n - \ell - 1$. If $a \equiv n \pmod 2$, then $a = \ell - s + 1$ and hence $s = \frac{n-a}{2}$. If $a \not\equiv n \pmod 2$, then $a = \ell - s$ and hence $s = \frac{n-a-1}{2}$. So $s = \lfloor \frac{n-a}{2} \rfloor$ and, by Lemmas 6.4.4 and 6.5.1, we have

$$|\mathcal{F}^{\updownarrow}| \geqslant 2^{n-s} + 2^s m - m \geqslant 2^{\lceil (n+a)/2 \rceil} + m 2^{\lfloor (n-a)/2 \rfloor} - m = A(n,m) \geqslant B(n,m)\,.$$

Case 2b. Suppose that $s = n - \ell$. If $a \equiv n \pmod 2$, then $a = \ell - s$ and hence $s = \frac{n-a}{2}$. If $a \not\equiv n \pmod 2$, then $a = \ell - s + 1$ and hence $s = \frac{n-a+1}{2}$. So we have that $s = \lceil \frac{n-a}{2} \rceil$. Thus by Lemma 6.4.3,

$$|\mathcal{F}^{\updownarrow}| \geqslant 2^{\lfloor (n+a)/2 \rfloor} + 2^{\lceil (n-a)/2 \rceil}\left(m - t + \tfrac{1}{2}\delta\right) - m = B(n,m)\,. \qquad \square$$

Lemma 6.5.3. *For nonnegative integers n and m with $m \in \{1, \ldots, 2^n - 1\}$, $\Phi(n,m) \leqslant B(n,m)$.*

Proof. Let a, a', t and δ be as in the definition of $B(n,m)$. Let $s = \lceil \frac{n-a}{2} \rceil$ and $\ell = \lfloor \frac{n+a}{2} \rfloor$, and note that $s = n - \ell$ and $\ell - s = a'$. We say a family of subsets of $[n]$ is *suitable* if

- it is in (s,ℓ)-standard form; and
- $|\mathcal{P}_{01}| = t$, $\mathcal{P}_{01} \subseteq \mathcal{P}_{00}$ and $|\mathcal{P}_{01} \setminus \mathcal{P}_{11}| = \delta$, where $(\mathcal{P}_{10}, \mathcal{P}_{00}, \mathcal{P}_{11}, \mathcal{P}_{01})$ is its pattern list.

We will construct a suitable family $\mathcal{F}$ such that $|\mathcal{F}| = m$. This will suffice to complete the proof because then, if $(\mathcal{P}_{10}, \mathcal{P}_{00}, \mathcal{P}_{11}, \mathcal{P}_{01})$ is the pattern list of $\mathcal{F}$ and $\mathcal{P}_{01}^{10}, \mathcal{P}_{01}^{00}, \mathcal{P}_{01}^{11}, \mathcal{P}_{01}^{*}$ are as defined in Lemma 6.4.2, we have

$$\begin{aligned}
|\mathcal{F}^{\updownarrow}| &\leqslant 2^{n-s} + (2^s - 1)m - 2^{s-1}\left(|\mathcal{P}_{01}^{00}| + 2|\mathcal{P}_{01}^{*}|\right) \\
&= 2^{\ell} + 2^s\left(m - t + \tfrac{1}{2}\delta\right) - m \\
&= B(n,m)
\end{aligned}$$

where the first line follows from Lemma 6.4.2 because $\ell = n - s$ and $\mathcal{P}_{01} \subseteq \mathcal{P}_{00}$ implies $|\mathcal{P}_{01}^{10}| = |\mathcal{P}_{01}^{11}| = 0$, the second line follows since $|\mathcal{P}_{01}| = t$, $\mathcal{P}_{10} \subseteq \mathcal{P}_{00}$ and $|\mathcal{P}_{01} \setminus \mathcal{P}_{11}| = \delta$ imply $|\mathcal{P}_{01}^{*}| = t - \delta$ and $|\mathcal{P}_{01}^{-}| = \delta$, and the final line follows from the definitions of ℓ and s. Hence $\Phi(n,m)$ will be at most $B(n,m)$ as required. So it suffices to show there is a suitable family $\mathcal{F}$ with $|\mathcal{F}| = m$.

Let $m_0 = 2^{\ell - s} + \Phi(\ell - s, t) + 2t$. Note that we have

$$m_0 - 1 \leqslant m < 3 \cdot 2^{\ell - s} + t \tag{6.7}$$

and that $\delta = 1$ if and only if $m = m_0 - 1$. The lower bound and the statement concerning

δ follow by the definition of t. The upper bound follows by the definition of $a' = \ell - s$ if $t = 2^{\ell-s}$ and follows because $m \leqslant 2^{\ell-s} + \Phi(\ell - s, t+1) + 2t$ and $\Phi(\ell - s, t+1) \leqslant 2^{\ell-s}$ if $t < 2^{\ell-s}$. Let $\mathcal{Q}$ be a convex family of subsets of $[\ell] \setminus [s]$ such that $|\mathcal{Q}| = t$ and $|\mathcal{Q}^{\updownarrow}| = \Phi(\ell - s, t)$. Such a family exists by the definition of $\Phi(\ell - s, t)$ and Lemma 6.2.2. Noting the bounds on m given by (6.7), we divide the proof into two cases.

Case 1. Suppose that $m = m_0 - 1$. In this case $\delta = 1$ and hence $t \geqslant 1$. Take $\mathcal{F}$ to be the unique family satisfying (SF1) with pattern list $(\mathcal{P}_{10}, \mathcal{P}_{00}, \mathcal{P}_{11}, \mathcal{P}_{01})$ where $\mathcal{P}_{10} = [\ell] \setminus [s]$, $\mathcal{P}_{01} = \mathcal{Q}$, $\mathcal{P}_{00} = \mathcal{Q}^{\uparrow}$, $\mathcal{P}_{11} = \mathcal{Q}^{\downarrow} \setminus \{A\}$ and A is a maximal set in $\mathcal{Q}$. It can be confirmed that $\mathcal{F}$ is in (s, ℓ)-standard form using the fact that A is maximal in $\mathcal{Q}$. By definition, $|\mathcal{P}_{01}| = |\mathcal{Q}| = t$, $\mathcal{P}_{01} \subseteq \mathcal{P}_{00}$ and $|\mathcal{P}_{01} \setminus \mathcal{P}_{11}| = |\{A\}| = 1 = \delta$. So $\mathcal{F}$ is suitable.

Case 2. Suppose that $m \geqslant m_0$. In this case $\delta = 0$, so the condition $|\mathcal{P}_{01} \setminus \mathcal{P}_{11}| = \delta$ in the definition of suitable reduces to $\mathcal{P}_{01} \subseteq \mathcal{P}_{11}$. We complete the proof by showing via induction on k that there is a suitable family of size k for each $k \in \{m_0, \ldots, m\}$.

For $k = m_0$, we take $\mathcal{F}$ to be the unique family satisfying (SF1) with pattern list $(\mathcal{P}_{10}, \mathcal{P}_{00}, \mathcal{P}_{11}, \mathcal{P}_{01})$ where $\mathcal{P}_{10} = 2^{[\ell]\setminus[s]}$, $\mathcal{P}_{01} = \mathcal{Q}$, $\mathcal{P}_{00} = \mathcal{Q}^{\uparrow}$, $\mathcal{P}_{11} = \mathcal{Q}^{\downarrow}$. As in the case above it can be seen that $\mathcal{F}$ is suitable.

Now suppose that $k \in \{m_0 + 1, \ldots, m\}$ and that there is a suitable family $\mathcal{F}$ with $|\mathcal{F}| = k - 1$. Let $(\mathcal{P}_{10}, \mathcal{P}_{00}, \mathcal{P}_{11}, \mathcal{P}_{01})$ be the pattern list of $\mathcal{F}$. We will add a set to $\mathcal{F}$ to obtain a suitable family $\mathcal{F}'$ with $|\mathcal{F}'| = k$ in one of two slightly different ways depending on whether $\mathcal{P}_{00} = 2^{[\ell]\setminus[s]}$.

If $\mathcal{P}_{00} \neq 2^{[\ell]\setminus[s]}$, then take $\mathcal{F}' = \mathcal{F} \cup \{[s-1] \cup X\}$ where X is a maximal set in $2^{[\ell]\setminus[s]} \setminus \mathcal{P}_{00}$. So $\mathcal{F}'$ has pattern list $(\mathcal{P}_{10}, \mathcal{P}_{00} \cup \{X\}, \mathcal{P}_{11}, \mathcal{P}_{01})$. Obviously $|\mathcal{F}'| = k$, and it can be verified that $\mathcal{F}'$ is in (s, ℓ)-standard form since $\mathcal{F}$ is and since X is maximal in $2^{[\ell]\setminus[s]} \setminus \mathcal{P}_{00}$. Then it is easy to see that $\mathcal{F}'$ is suitable, noting that $\mathcal{P}_{01} \subseteq \mathcal{P}_{11}$.

If $\mathcal{P}_{00} = 2^{[\ell]\setminus[s]}$, then note that $\mathcal{P}_{11} \neq 2^{[\ell]\setminus[s]}$ since

$$|\mathcal{P}_{11}| + 2^{\ell-s+1} + t = |\mathcal{F}| = k - 1 < 3 \cdot 2^{\ell-s} + t$$

where the first equality holds because $|\mathcal{P}_{00}| = |\mathcal{P}_{10}| = 2^{\ell-s}$ and $|\mathcal{P}_{01}| = t$ and the inequality holds by (6.7). Take $\mathcal{F}' = \mathcal{F} \cup \{[s-1] \cup \{s, \ell+1\} \cup Y\}$ where Y is a minimal set in $2^{[\ell]\setminus[s]} \setminus \mathcal{P}_{11}$. So $\mathcal{F}'$ has pattern list $(\mathcal{P}_{10}, \mathcal{P}_{00}, \mathcal{P}_{11} \cup \{Y\}, \mathcal{P}_{01})$. Obviously $|\mathcal{F}'| = k$, and it can be verified that $\mathcal{F}'$ is in (s, ℓ)-standard form since $\mathcal{F}$ is and since Y is minimal in $2^{[\ell]\setminus[s]} \setminus \mathcal{P}_{11}$. Then it is easy to see that $\mathcal{F}'$ is suitable, noting that $\mathcal{P}_{01} \subseteq \mathcal{P}_{11}$. This completes the induction and hence the proof of the lemma. $\square$

Proof of Theorem 6.1.1. If $m \in \{0, 2^n\}$ then it is clear that $\Phi(n, m) = m$. Otherwise $m \in \{1, \ldots, 2^n - 1\}$ and $\Phi(n, m) = B(n, m)$ by Lemmas 6.5.2 and 6.5.3. $\square$

Proof of Corollary 6.1.2. By Theorem 6.1.1 we have that $\Phi(n,m) = B(n,m)$ and by Lemma 6.5.1 we have that $B(n,m) = A(n,m)$ exactly in the cases given in the corollary statement. $\square$

In Table 6.1 we use Theorem 6.1.1 to compute the values of $\Phi(n,m)$ for $n \in \{2,3,4,5,6\}$. In the vast majority of these cases we have $\Phi(n,m) = A(n,m)$, with the only exceptions as follows. For $n = 4$, we have $\Phi(n,m) = A(n,m) - 1$ when $m \in \{8,9\}$. For $n = 5$, we have $\Phi(n,m) = A(n,m) - 2$ when $m \in \{15,16,17,18\}$ and $\Phi(n,m) = A(n,m) - 1$ when $m \in \{14,19,20,21\}$. For $n = 6$, we have $\Phi(n,m) = A(n,m) - 5$ when $m \in \{32,33\}$, $\Phi(n,m) = A(n,m) - 4$ when $m \in \{30,31,34,35,36\}$, $\Phi(n,m) = A(n,m) - 3$ when $m \in \{29,37,38,39\}$, $\Phi(n,m) = A(n,m) - 2$ when $m \in \{8,9,25,26,27,28,40,41,42\}$ and $\Phi(n,m) = A(n,m) - 1$ when $m \in \{24,43,44,45,46,47,48,49\}$.

In Table 6.2we also present, for $n = 4$ and each $m \in \{0,...,16\}$, a family $\mathcal{F}$ of subsets of $\{1,2,3,4\}$ such that $|\mathcal{F}| = m$ and $|\mathcal{F}^{\updownarrow}| = \Phi(n,m)$. We also include $\mathcal{F}^{\updownarrow}$ and the family $\mathcal{G} = 2^{\{1,2,3,4\}} \setminus \mathcal{F}^{\updownarrow}$ that would maximise $|\mathcal{F}| + |\mathcal{G}|$ under the condition that $(\mathcal{F},\mathcal{G})$ is a cross-Sperner pair.

m	$\Phi(2,m)$	$\Phi(3,m)$	$\Phi(4,m)$	$\Phi(5,m)$	$\Phi(6,m)$
0	0	0	0	0	0
1	3	5	7	11	15
2	4	6	10	14	22
3	4	7	11	17	25
4	4	8	12	20	28
5		8	13	21	31
6		8	14	22	34
7		8	15	23	37
8		8	15	24	38
9			15	25	39
10			16	26	42
11			16	27	43
12			16	28	44
13			16	29	45
14			16	29	46
15			16	29	47
16			16	30	48
17				30	49
18				30	50
19				31	51
20				31	52
21				31	53
22				32	54
23				32	55
24				32	55
25				32	55
26				32	56
27				32	57
28				32	58
29				32	58
30				32	58
31				32	59
32				32	59
33					59
34					60
35					60
36					60
37					61
38					61
39					61
40					62
41					62
42					62
43					63
44					63
45					63
46					63
47					63
48					63
49					63
$50-64$					64

Table 6.1: Values of $\Phi(n,m)$ for $n \in \{2,3,4,5,6\}$.

m	$\mathcal{F}$	$\mathcal{F}^{\updownarrow}$	$2^{\{1,2,3,4\}} \setminus \mathcal{F}^{\updownarrow}$
0			$2^{\{1,2,3,4\}}$
1	12	$\emptyset, 1, 2, 12, 123, 124, 1234$	$3, 4, 13, 14, 23, 24, 34, 134, 234$
2	$12, 123$	$\emptyset, 1, 2, 3, 12, 13, 23, 123, 124, 1234$	$4, 14, 24, 34, 134, 234$
3	$12, 13, 123$	$\emptyset, 1, 2, 3, 12, 13, 23, 123, 124, 134, 1234$	$4, 14, 24, 34, 234$
4	$12, 13, 23, 123$	$\emptyset, 1, 2, 3, 12, 13, 23, 123, 124, 134, 234, 1234$	$4, 14, 24, 34$
5	$1, 12, 13, 23, 123$	$\emptyset, 1, 2, 3, 12, 13, 14, 23, 123, 124, 134, 234, 1234$	$4, 24, 34$
6	$1, 2, 12, 13, 23, 123$	$\emptyset, 1, 2, 3, 12, 13, 14, 23, 24, 123, 124, 134, 234, 1234$	$4, 34$
7	$1, 2, 3, 12, 13, 23, 123$	$\emptyset, 1, 2, 3, 12, 13, 14, 23, 24, 34, 123, 124, 134, 234, 1234$	4
8	$4, 13, 14, 23, 24, 34, 134, 234$	$\emptyset, 1, 2, 3, 4, 13, 14, 23, 24, 34, 123, 124, 134, 234, 1234$	12
9	$3, 4, 13, 14, 23, 24, 34, 134, 234$	$\emptyset, 1, 2, 3, 4, 13, 14, 23, 24, 34, 123, 124, 134, 234, 1234$	12
10	$3, 4, 12, 13, 14, 23, 24, 34, 134, 234$	$2^{\{1,2,3,4\}}$	
11	$3, 4, 12, 13, 14, 23, 24, 34, 123, 134, 234$	$2^{\{1,2,3,4\}}$	
12	$2, 3, 4, 12, 13, 14, 23, 24, 34, 123, 134, 234$	$2^{\{1,2,3,4\}}$	
13	$1, 2, 3, 4, 12, 13, 14, 23, 24, 34, 123, 134, 234$	$2^{\{1,2,3,4\}}$	
14	$1, 2, 3, 4, 12, 13, 14, 23, 24, 34, 123, 124, 134, 234$	$2^{\{1,2,3,4\}}$	
15	$\emptyset, 1, 2, 3, 4, 12, 13, 14, 23, 24, 34, 123, 124, 134, 234$	$2^{\{1,2,3,4\}}$	
16	$2^{\{1,2,3,4\}}$	$2^{\{1,2,3,4\}}$	

Table 6.2: Examples of families $\mathcal{F}$ of subsets of $\{1, 2, 3, 4\}$ such that $|\mathcal{F}| = m$ and $|\mathcal{F}^{\updownarrow}| = \Phi(n, m)$.

Chapter 7

Conclusion

There are still many questions left open related to our work with different "Sperner-like" families.

The simplest question comes as a result of Theorem 4.1.4(b), where the asymptotics of $\mathrm{SP}(n,k)$ when $k = o(n)$ is odd and $n \equiv k-1 \pmod{2k}$ are only unknown when $k=3$. We believe that in this case, the theorem statement is still true, giving us the following conjecture which was discussed at the end of Section 4.6.

Conjecture 7.0.1. *Let n be an integer such that $n \to \infty$ with $n \equiv 2 \pmod 6$. Then $\mathrm{SP}(n,3) \sim \mathrm{MMS}(n,3)$.*

As per the discussion there, we have computational evidence which supports our belief that this conjecture holds.

In Chapters 3 and 4, we established that $\mathrm{SP}(n,k)$ is asymptotic to $\mathrm{MMS}(n,k)$ in a very wide variety of cases, with the main unresolved situation being when n is odd, $k = o(n)$ and $k-r$ is bounded. It seems likely that addressing these unresolved cases will require developing new constructions for Sperner partition systems. Throughout Chapter 3 and Chapter 4, we provided explicit constructions for (n,k)-Sperner partition systems in cases where n satisfied a particular divisibility condition (n needed to be even in Chapter 3 and in Sections 4.5 and 4.6, and needed to be divisible by some fixed h in Section 4.3) and then used Lemma 2.2.4 to construct all of the non divisible cases. We then proceeded to show that the resulting systems had size asymptotic to $\mathrm{MMS}(n,k)$. When $k-r$ is bounded, we no longer have that $\mathrm{MMS}(n,k) \sim \mathrm{MMS}(n+1,k)$ as

$$\frac{\mathrm{MMS}(n,k)}{\mathrm{MMS}(n+1,k)} = \frac{k-r-1+\frac{(r+1)(c+1)}{n-c+1}}{k-r+\frac{r(c+1)}{n-c}} \cdot \frac{n-c+1}{n+1} \leqslant \frac{k-r-1}{k-r},$$

which is bounded away from 1 since $k-r$ is bounded. Thus systems constructed using

Lemma 2.2.4 will not have a number of partitions asymptotic to $\mathrm{MMS}(n+1,k)$. In spite of this difficulty, we suspect the following conjecture to be true.

Conjecture 7.0.2. *Let n and k be integers such that $n \to \infty$ and $k > 1$. Then $\mathrm{SP}(n,k) \sim \mathrm{MMS}(n,k)$.*

The work in Section 4.4, as well as the work of Li and Meagher in [28], leads to the obvious problem of determining $\mathrm{SP}(n,k)$ exactly for various (n,k) with $n > k > 1$. All of the existing results are for cases where $c = 2$, so it would be of great interest to find cases where we can find an exact value for $\mathrm{SP}(n,k)$ when $c \neq 2$. Even beginning to tackle this problem has proven difficult. When $c = 2$, a simple counting argument gives us that any partition in a nontrivial $(2k+r,k)$-Sperner partition system must have at least $k-r$ 2-sets, and no sets of size 1. As soon as we fix $c \neq 2$, even in the simplest case of $c = 3$, we no longer have a lower bound on the number of c-sets in a partition, nor can we easily bound the number of sets of size less than or equal to c, making any attempt at proving that a given construction is optimal all the more difficult.

Another major open problem related to Sperner partition systems concerns the structure of extremal examples. Li and Meagher conjectured in [28] that for all integers $n > k > 1$, there exists an almost uniform (n,k)-Sperner partition system with $\mathrm{SP}(n,k)$ partitions. Recall that an almost uniform (n,k)-Sperner partition system is one in which all classes contain either $\lfloor n/k \rfloor$ or $\lceil n/k \rceil$ elements. We also believe this conjecture to be true, as supported by the fact that every construction for Sperner partition systems we present in this thesis produces almost uniform Sperner partition systems. We attempted to prove this conjecture in the special case of $c = 2$, as we felt that knowing that there would always be a minimum number of 2-sets in the system would be helpful, but even then we were unable to make any progress.

As part of our work in Chapter 5, we presented a construction for a wide array of pairs of intersecting cross-Sperner families that meet the bound given in Theorem 5.1.1. It would be interesting to find other constructions for families meeting this bound or, on the other hand, show that our construction gives every extremal family. It would also be of interest to extend our result in Theorem 5.1.1 in a fashion similar to that of Tay and Wong, who not only classified all possible Sperner cross-intersecting pairs that met their proven upper bound, but also classified all pairs with one fewer set than this. Unfortunately, we suspect that achieving a similar stability result for cross-Sperner pairs is unlikely due to the wide range of extremal examples produced by Lemma 5.3.3, which are not necessarily exhaustive.

Theorem 6.1.1 gives exact values for $\Phi(n,m)$, which we recall is the minimum size of $|\mathcal{F}^{\updownarrow}|$ over all families $\mathcal{F} \subseteq 2^{[n]}$ with $|\mathcal{F}| = m$. Although the values are recursively determined in general, Corollary 6.1.2 then goes on to identify a number of cases where $\Phi(n, m)$ can be explicitly stated. We are greatly interested in the possibility of extending this explicit determination to other cases, perhaps even to all n and m. We have made some initial progress in finding explicit lower and upper bound for $\Phi(n, m)$, but there is still much work to be done here.

We are also interested in determining if there exists a total order on subsets of $[n]$ such that, for each $m \in \{0, \ldots 2^n\}$, we have that $|\mathcal{F}^{\updownarrow}| = \Phi(n, m)$ where $\mathcal{F}$ is the family containing the first m subsets of $[n]$ under the total order. Bashov proved that no such order exists for the double sided shadow ([4]). We plan for this to be the subject of future investigation.

Another point of interest in our work in Chapter 6, is the introduction of a new extension to the classical technique of shifting that allowed us to prove our main result. It is natural to investigate if this technique would be of use to other problems. Such problems would have to involve families of sets that are allowed to be different sizes.

Chapter 8

References

[1] Rudolf Ahlswede and David Daykin. An Inequality for the Weights of Two Families of Sets, their Unions and Intersections. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, 43(3):183–185, 1978. ISSN 0044-3719.

[2] Zsolt Baranyai. On the Factorization of the Complete Uniform Hypergraph. Infinite and Finite sets, vol. 1. In *Proceedings of a Colloquium held at Keszthely, June*, pages 91–108, 1975.

[3] Maksim Bashov. On Minimisation of the Double-Sided Shadow in the Unit Cube. *Discrete Mathematics and Applications*, 21, 01 2011. doi: 10.1515/dma.2011.033.

[4] Maksim Bashov. Nonexistence of a Kruskal-Katona Type Theorem for Double-sided Shadow Minimization in the Boolean Cube Layer. *Acta Universitatis Sapientiae. Informatica*, 5, 07 2013. doi: 10.2478/ausi-2014-0004.

[5] Béla Bollobás. On Generalized Graphs. *Acta Mathematica Academiae Scientiarum Hungarica*, 16:447–452, 1965. doi: 10.1007/BF01904851.

[6] Darryn Bryant. On Almost-Regular Edge Colourings of Hypergraphs. *The Electronic Journal of Combinatorics*, 23(#P4.7), 2016. doi: 10.37236/5826.

[7] Boris Bukh. Multidimensional Kruskal–Katona Theorem. *SIAM Journal on Discrete Mathematics*, 26(2):548–554, 2012. doi: 10.1137/100808630.

[8] Yanxun Chang, Charles J. Colbourn, Adam Gowty, Daniel Horsley, and Junling Zhou. New Bounds on the Maximum Size of Sperner Partition Systems. *European Journal of Combinatorics*, 90, 2020. doi: 10.1016/j.ejc.2020.103165.

[9] Shagnik Das, Wenying Gan, and Benny Sudakov. Sperner's Theorem and a Problem of Erdős, Katona and Kleitman. *Combinatorics, Probability and Computing*, 24:585–608, 07 2015. doi: 10.1017/S0963548314000273.

[10] Paul Erdős, Chao Ko, and Richard Rado. Intersection Theorems for Systems of Finite Sets. *The Quarterly Journal of Mathematics*, 12(1):313–320, 01 1961. ISSN 0033-5606. doi: 10.1093/qmath/12.1.313.

[11] Paul Erdös, Marcel Herzog, and Jochanan Schönheim. An Extremal Problem on the Set of Noncoprime Divisors of a Number. *Israel Journal of Mathematics*, 8(4):408–412, 1970.

[12] Peter Frankl. A New Short Proof for the Kruskal-Katona Theorem. *Discrete Mathematics*, 48(2):327–329, 1984. ISSN 0012-365X. doi: 10.1016/0012-365X(84)90193-6.

[13] Peter Frankl and Andrey Kupavskii. A size-sensitive inequality for cross-intersecting families. *European Journal of Combinatorics*, 62:263–271, 2017. doi: 10.1016/j.ejc.2017.01.004.

[14] Peter Frankl and Andrey Kupavskii. Uniform s-Cross-Intersecting Families. *Combinatorics, Probability and Computing*, 26(4):517–524, 2017. doi: 10.1017/S0963548317000062.

[15] Peter Frankl and Norihide Tokushige. *Extremal Problems for Finite Sets*, volume 86 of *Student Mathematical Library*. American Mathematical Society, Providence, RI, 2018. ISBN 978-1-4704-4039-8. doi: 10.1090/stml/086.

[16] Dániel Gerbner, Nathan Lemons, Cory Palmer, Balázs Patkós, and Vajk Szécsi. Cross-Sperner Families. *Studia Scientiarum Mathematicarum Hungarica*, 49(1):44–51, 2012. ISSN 0081-6906.

[17] Adam Gowty and Daniel Horsley. More Constructions for Sperner Partition Systems. *Journal of Combinatorial Designs*, 29(9):579–606, 2021. doi: 10.1002/jcd.21780.

[18] Curtis Greene and Daniel J. Kleitman. Strong Versions of Sperner's Theorem. *Journal of Combinatorial Theory, Series A*, 20(1):80–88, 1976. doi: 10.1016/0097-3165(76)90079-0.

[19] Gurobi Optimization, LLC. Gurobi optimizer reference manual, 2020. URL http://www.gurobi.com.

[20] Lawrence H. Harper. Optimal Numberings and Isoperimetric Problems on Graphs. *Journal of Combinatorial Theory*, 1(3):385–393, 1966. ISSN 0021-9800. doi: 10.1016/S0021-9800(66)80059-5.

[21] Anthony J. W. Hilton and Eric C. Milner. Some Intersection Theorems for Systems of Finite Sets. *The Quarterly Journal of Mathematics*, 18(1):369–384, 1967.

[22] Sanpei Kageyama. A Survey of Resolvable Solutions of Balanced Incomplete Block Designs. *International Statistical Review*, pages 269–273, 1972.

[23] Ravindran Kannan, László Lovász, and Miklós Simonovits. Isoperimetric Problems for Convex Bodies and a Localization Lemma. *Discrete & Computational Geometry*, 13 (3-4):541–559, 1995. ISSN 0179-5376. doi: 10.1007/BF02574061.

[24] Gyula Katona. A Theorem of Finite Sets. In *Theory of Graphs*, pages 187–207. Academic Press, New York, 1968.

[25] Daniel A. Klain and Gian-Carlo Rota. A Continuous Analogue of Sperner's Theorem. *Communications on Pure and Applied Mathematics*, 50(3):205–223, 1997. doi: 10.1002/(SICI)1097-0312(199703)50:3⟨205::AID-CPA1⟩3.0.CO;2-F.

[26] Daniel Kleitman. On a Conjecture of Milner on k-Graphs with Non-Disjoint Edges. *Journal of combinatorial theory*, 5(2):153–156, 1968.

[27] Joseph B. Kruskal. The Number of Simplices in a Complex. 1963.

[28] Pak C. Li and Karen Meagher. Sperner Partition Systems. *Journal of Combinatorial Designs*, 21(7):267–279, 2013.

[29] Làszlò Lovàsz. *Combinatorial Problems and Exercises*. North-Holland, Amsterdam, 2nd edition, 1993. doi: 10.1016/B978-0-444-81504-0.50013-8.

[30] David Lubell. A Short Proof of Sperner's Lemma. *Journal of Combinatorial Theory*, 1(2):299, 1966. doi: 10.1016/S0021-9800(66)80035-2.

[31] Erwin Lutwak, Deane Yang, and Gaoyong Zhang. L_p Affine Isoperimetric Inequalities. *J. Differential Geom.*, 56(1):111–132, 2000. ISSN 0022-040X.

[32] Frédéric Maire. On the Flat Antichain Conjecture. *Australasian Journal of Combinatorics*, 15:241–246, 1997.

[33] Karen Meagher, Lucia Moura, and Brett Stevens. A Sperner-Type Theorem for Set-Partition Systems. *The Electronic Journal of Combinatorics*, 12(#N20), 2005.

[34] Lev Meshalkin. Generalization of Sperner's Theorem on the Number of Subsets of a Finite Set. *Theory of Probability & Its Applications*, 8(2):203–204, 1963. doi: 10.1137/1108023.

[35] Manfred Padberg. *Linear Optimization and Extensions*, volume 12 of *Algorithms and Combinatorics*. Springer-Verlag, Berlin, expanded edition, 1999. doi: 10.1007/978-3-662-12273-0.

[36] George Pólya and Gábor Szegö. *Isoperimetric Inequalities in Mathematical Physics*. Annals of Mathematics Studies, No. 27. Princeton University Press, Princeton, N. J., 1951.

[37] Làszlò Pyber. A New Generalization of the Erdös-Ko-Rado Theorem. *Journal of combinatorial theory. Series A*, 43(1):85–90, 1986.

[38] Dan Romik. Stirling's Approximation for $n!$: the Ultimate Short Proof? *American Mathematical Monthly*, 107(6):556–557, 2000. doi: 10.2307/2589351.

[39] Eero Räty. Uniqueness in Harper's Vertex-Isoperimetric Theorem. *Discrete Mathematics*, 343(4):111696, 2020. ISSN 0012-365X. doi: 10.1016/j.disc.2019.111696.

[40] Emanuel Sperner. Ein Satz über Untermengen einer endlichen Menge. *Mathematische Zeitschrift*, 27(1):544–548, 1928.

[41] Willie H. W. Wong and Eng G. Tay. Kruskal-Katona Function and Variants of Cross-Intersecting Antichains. *Discrete Mathematics*, 345(3):112709, 2022. ISSN 0012-365X. doi: 10.1016/j.disc.2021.112709.

[42] Koichi Yamamoto. Logarithmic Order of Free Distributive Lattice. *Journal of the Mathematical Society of Japan*, 6(3-4):343–353, 12 1954. doi: 10.2969/jmsj/00630343.