Kauffman's clock theorem and its generalisation to more complicated structures and surfaces

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Abstract

The thesis aims to generalise Kauffman's clock theorem as much as possible. The starting point is Kauffman's clock theorem applied to universes (or equivalently, strings). A *universe* is a plane graph where every vertex is 4-valent and where the unbounded face and a face adjacent to it are marked with a star. Associated with a universe is its set of states. A *state* is created by assigning to each vertex of a universe a maker lying in one of the four faces adjacent to it such that each unstarred face (face without a star) has exactly one marker. A *transposition* is a simultaneous 90º clockwise or counterclockwise rotation of two markers which turns a state into another state. If a state *A* is turned into a state *B* by a clockwise transposition, then *A* is said to *cover B* and this relation is a covering relation. Kauffman's clock theorem claims that the set of states of a universe equipped with this covering relation is a lattice. We then generalise a universe to an *n*-string multiverse and prove that the theorem still holds for a multiverse embedded on a surface with more than one boundary component and non-zero genus.

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Supervised by Dr Daniel Mathews

Introduction

The thesis aims to generalise Kauffman's clock theorem as much as possible. The starting point is Kauffman's clock theorem applied to universes (or equivalently, strings) as discussed in [13]. A *universe* is a plane graph where every vertex is 4-valent and where the unbounded face and a face adjacent to it are marked with a star. Associated with a universe is its set of states. A *state* is created by assigning to each vertex of a universe a maker lying in one of the four faces adjacent to it such that each unstarred face (face without a star) has exactly one marker. Figure 1 illustrates an example of a universe and one of its states. A *transposition* is a simultaneous 90° clockwise or counterclockwise rotation of two markers which turns a state into another state. If a state A is turned into a state B by a clockwise transposition, then A is said to *cover* B and this relation is a covering relation. Kauffman's clock theorem claims that the set of states of a universe equipped with this covering relation is a lattice where a lattice is defined as a partially ordered set such that every finite non-empty subset has a least upper bound and a greatest lower bound. Kauffman provided a proof of the clock theorem in [13] but it lacks many details and some notions are vaguely defined. Chapter 1 of the thesis presents a more rigorous and detailed version of his proof including missing details and cases. Some important notions skimmed over by Kaufmann are clearly defined here.

Cutting the edge that separates the two starred faces of a universe (faces marked with a star) and extending the resulting parts to infinity gives a *string* or a 1-string whose clock lattice is isomorphic to the clock lattice of its original universe. Figure 1.37 shows the lattice of a string. This inspires the next level of generalisation as follows. Would the clock theorem still hold for a 4-valent graph with an arbitrary number of free edges (edges that are extended to infinity) and the correct number of starred faces (which equals the difference between the number of faces and the number of vertices)? This object is called an *n-string multiverse* where n is half the number of free edges (it turns out that the number of free edges can only be even). The answer is yes and the proof is provided by Zibrowius in [23]. The lattice of a 3-string multiverse is shown in Figure 2.

The main results of this thesis are shown in Chapter 2, where we proceed to the highest level of generalisation which forfeits the condition that a multiverse is embedded

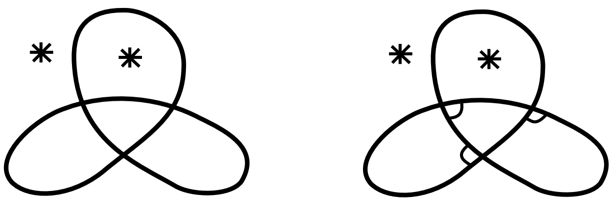


Figure 1. Example of a universe and one of its states.

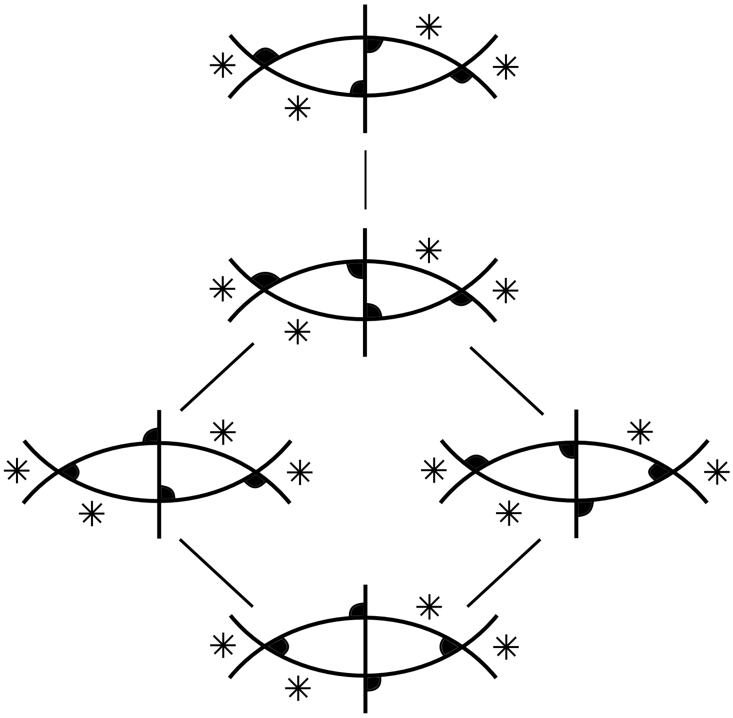


Figure 2. The lattice of a 3-string multiverse.

on a plane. Instead, we allow the underlying surface to have more than one boundary component but no genus. If the same definition of transpositions is kept, the theorem does not hold. A more general definition is formulated and the resulting clock theorem is proved in Section 1 of Chapter 2 with the help of one of Propp's theorems published in [21]. A natural subsequent generalisation is to allow the surface on which a multiverse is embedded to have non-zero genus. For the clock theorem to be true in this case, we restrict a multiverse to be almost 2-cell embedded, a wider version of a 2-cell embedding. At the same time, an even more general definition of transpositions was formulated. Section 2 of Chapter 2 provides the proof of the corresponding clock theorem, which involves the application of another theorem by Propp given in [21]. Note that Chapter 2 presents two new clock theorems. The first applies to surfaces of genus 0 whereas the second applies to more general surfaces. Although both apply to a surface of genus 0, they give different lattices from the same multiverse. In other words, the second theorem is not a generalisation of the first. This shows that there are two similar, reasonable ways to define a clock lattice, but that they are not the same. The third section describes briefly the connection between the clock theorem and contact geometry, namely the similarity between transpositions and bypass additions.

Acknowledgements

My utmost gratitude is reserved for my supervisors Dr Daniel Mathews and Dr Norman Do. Without their patience and sympathy, the composition of this thesis would never see its finishing line. Their directions, hints, comments and suggestions constitute an invaluable boost to the progress of my research and carry me through countless roadblocks. The psychology of a doctoral candidate usually resembles a roller-coaster, riddled with impostor syndrome, self-doubt and disappointment. All these challenges would never be overcome without their optimistic encouragement and compassionate understanding. I consider myself exceptionally lucky for having them as my supervisors.

Contents

Chapter 1

Kauffman's clock theorem on a plane

This chapter provides a detailed and rigorous exposition of Kauffman's proof of his clock theorem which was published in [13]. Despite having no new result, some new definitions are presented along with lemmas related to them which serve to fill the gaps in Kauffman's arguments.

1.1 All concepts that lead to the notion of lattice

Definition 1.1.1 (Partial order)**.** A non-strict partial order is a binary relation over a set X which satisfies the following properties. Let $a, b, c \in X$.

1. Reflexivity: $a \leqslant a$.
2. Antisymmetry: if $a \leqslant b$ and $b \leqslant a$, then $a = b$.
3. Transitivity: if $a \leqslant b$ and $b \leqslant c$, then $a \leqslant c$.

A strict partial order over X is a binary relation over X with the following properties.

1. Irreflexivity: $a < a$ is false.
2. Transitivity: if $a < b$ and $b < c$, then $a < c$.
3. Asymmetry: if $a < b$, then $b < a$ is false. Note that asymmetry is implied by irreflexivity and transitivity.

A set equipped with a partial order is called a *partially ordered set*.

It can be seen that the set of non-strict partial orders over a set X corresponds bijectively to the set of strict partial orders over X. Namely, to convert a non-strict partial order to a strict one, we add the condition $a < b$ if $a \leqslant b$ and $a \neq b$. Conversely, to convert a strict partial order to a non-strict one, we add the condition $a \leqslant b$ if $a < b$ or $a = b$. See [22].

Definition 1.1.2 (Covering relation)**.** Let X be a set equipped with a strict partial order and let $x, y \in X$. Then, y is said to *cover* x, denoted $x \lessdot y$ if $x < y$ and there is no $z \in X$ such that $x < z < y$.

If we have the covering relation derived from a partial order over a set, we can represent graphically the latter by the former using a Hasse diagram.

Definition 1.1.3 (Hasse diagram)**.** Let X be a finite partially ordered set and let $\lessdot$ be the covering relation derived from the partial order over X. Construct a directed graph H as follows. The vertices of H are elements of X. Draw a directed edge upwards from x to y if and only if $x \lessdot y$. The graph H is called a *Hasse diagram* of X.

Definition 1.1.4 (Greatest, least, maximal and minimal elements)**.** Let X be a set equipped with a partial order $\leqslant$.

1. An element $x \in X$ is called the *greatest* element if $a \leqslant x$ for all $a \in X$. An element $y \in X$ is called the *least* element if $y \leqslant a$ for all $a \in X$. It can be seen that the greatest and least elements of X are unique if they exist.

2. An element $x \in X$ is called a *maximal* element if $x \leqslant a$ implies $x = a$ for all $a \in X$. An element $y \in X$ is called a *minimal* element if $a \leqslant y$ implies $y = a$ for all $a \in X$. It can be seen that there can be more than one maximal and minimal elements in X.

3. Let $Q \subseteq X$. An element $x \in X$ is called an *upper bound* of Q if $q \leqslant x$ for all $q \in Q$. An element $y \in X$ is called a *lower bound* of Q if $y \leqslant q$ for all $q \in Q$. If $x \leqslant x'$ for all upper bounds x' of Q, then x is called the *least upper bound* or *supremum* or *join* of Q. If $y' \leqslant y$ for all lower bounds y' of Q, then y is called the *greatest lower bound* or *infimum* or *meet* of Q. It can be seen that the least upper bound and the greatest lower bound of Q are unique if they exist.

The following are standard definitions of lattices and distributive lattices commonly found in textbooks. See [4].

Definition 1.1.5 (Lattice)**.** A partially ordered set X is called a *lattice* if every two-element subset of X has a join and a meet.

Since every two-element subset of a lattice has a join and a meet, we can consider join and meet as binary operations on the lattice. This leads to the following type of lattices.

Definition 1.1.6 (Distributive lattice)**.** Let X be a lattice. Denote the join of $\{a, b\} \subseteq X$ by $a \vee b$ and its meet by $a \wedge b$. If $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ for all $x, y, z \in X$, then X is said to be *distributive*.

It can be shown that the distributive relation $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ in Definition (1.1.6) implies that $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ holds for all $x, y, z \in X$, which can be thought of as its dual.

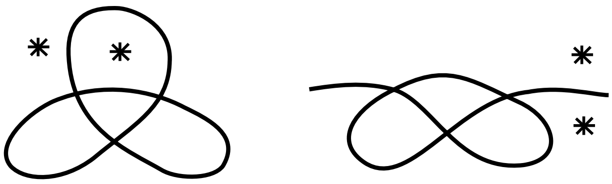


Figure 1.1

1.2 Basic definitions and results

Definition 1.2.1 (Universe)**.** Let A be a connected finite plane graph where every vertex is 4-valent. Mark the unbounded face and a face adjacent to it by a star and call them *starred faces*. The resulting object is called a *universe*.

Definition 1.2.2 (String)**.** Let U be a universe. Select the edge in U that separates the two starred faces and break it. Then, extend the resulting parts infinitely to the left and right. What is obtained is called a *universe in string form*, or a *string* for short. The infinite edges on the left and right are called the *input* and *output edges*, respectively. The vertices to which these edges are incident are called the *input* and *output vertices*, correspondingly. In a string, the starred faces are assumed to be the two unbounded faces.

Usually, plane graphs will be drawn so that edges are smoothly embedded in the plane and so that at each 4-valent vertex, opposite edges join smoothly. We will also allow edges to be drawn with corners in their interior, which we will call *cusps*.

An example of a string is shown in the right diagram of Figure 1.1. The left diagram is the universe version of it. For the rest of this chapter, every strings are considered to be drawn in the canonical way, that is, the input and output edges are drawn horizontally with the former to the left and the latter to the right and the starred faces are above and below them.

Definition 1.2.3 (Corner)**.** Let v be a vertex of degree l of a graph X embedded on a surface Σ, draw a circle on Σ centred at v. If the circle is small enough, it will cut out from the k faces adjacent to v l small faces called the *corners* of v. Note that $k \leqslant l$.

Definition 1.2.4 (State)**.** Let U be a universe or a string. Assign to each vertex of U a marker lying in one corner of the vertex such that the starred faces do not have any marker and each of the remaining faces has exactly one marker. The object obtained is called a *state* of U. The set of all states of a universe (or string) is denoted $\mathscr{S}$.

From the above definition, if two corners at a vertex belong to the same face, then putting the state marker in these two corners yields two distinct states. Kauffman did not introduce the notion of corner but it helps to distinguish between states.

Kauffman's clock theorem asserts that the set of states of a universe forms a lattice, with respect to a certain partial order. This partial order is defined by using an operation on states called transpositions. Because the proof is long, we give a rough outline of the scheme of the proof.

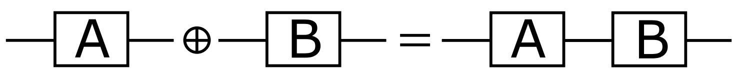
In the remainder of this section, we discuss how a string may be decomposed into "atomic" parts. We introduce a specific type of universe called a "shell composition", and an operation on universes called the "boundary". We show that from a state of a universe, we may "split" the vertices of the universe to obtain a single path called a "Jordan trail", or just "trail". The states of a universe are bijective with its trails.

In Section 1.3, we define some more complicated constructions, such as the "derivatives" of a universe and show that iterating derivatives eventually leads to a shell composition with an extra structure called "sites", which is called a "dissection". In fact, we show that a dissection is equivalent to an "elaboration" on a shell composition, consisting of certain "interactions" following certain rules. We also introduce a "clockwise" and "counterclockwise" splitting of a universe, which will yield the maximal and minimal elements of the lattice of states.

In Section 1.4, we define "transpositions", which move between states and prove some of their properties. We introduce the important notions of "reassembly" and "exchange", which are operations on trails, and prove some of their properties. We show that any two trails are connected by a sequence of exchanges and that each exchange in the sequence factors into a composition of transpositions. By using the bijection between states and trails, we show that any two states are connected by a sequence of transpositions and that they form a lattice, proving the clock theorem.

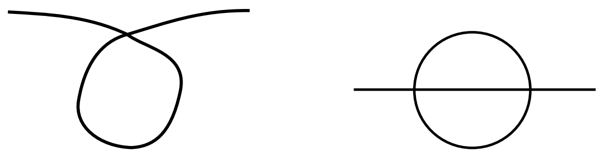
Next, we present some methods of constructing a string from other strings.

Definition 1.2.5 (Irreducibility)**.** Let A and B be strings. The string formed by joining the output edge of A to the input edge of B is denoted by $A \oplus B$, as illustrated below. A string C is said to be *irreducible* if it cannot be written as $A \oplus B$ where neither A nor B are the trivial string.



Remark. By definition, the trivial string has a unique state.

Definition 1.2.6 (Interior, connecting and boundary edges)**.** In a string A, an *interior edge* is an edge that separates two bounded faces in A. A *connecting edge* is an edge that separates two unbounded faces (this includes the input and output edges). A *boundary edge* is an edge that separates an unbounded face and a bounded face.

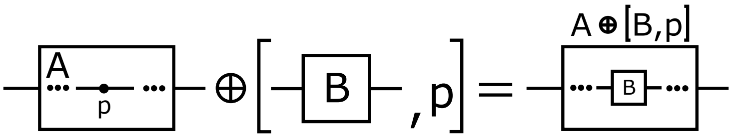


(a) A curl (b) A shell

Figure 1.2

Each edge of a string A can be seen as a trivial string if the rest of A is ignored. If this trivial string is replaced by a non-trivial string, then a new string is created having a string-within-string structure. This leads to another construction defined as follows.

Definition 1.2.7 (Atomicity)**.** Let A and B be strings where B is not trivial and let p be a point on an edge of A that is not a connecting edge. The string obtained by replacing the trivial string at p by B such that B does not intersect the rest of A is denoted by $A \oplus [B, p]$, as illustrated below. In $A \oplus [B, p]$, B is said to *ride* A at p. A string X is said to *have (or carry) rider* R if there exists a non-trivial string Y such that $X = Y \oplus [R, p]$ where p belongs to Y. In this case, R is said to be a *rider* of X. A string is said to be *atomic* if it is irreducible and has no riders. By convention, a rider is always non-trivial and a string is not a rider of itself. Sometimes, the point p is omitted if no confusion arises.



Remark. By $A \oplus [A_1, p_1] \oplus [A_2, p_2] \oplus \cdots \oplus [A_n, p_n]$ for some $n \in \mathbb{N}$, we mean that each A_i $(1 \leqslant i \leqslant n)$ rides A, that is, each p_i belongs to A. To denote a rider riding a rider, we use nested brackets. For example, $A \oplus [B \oplus [C, q], p] \oplus [D, r]$ means that B rides A at p belonging to A, C rides B at q belonging to B and D rides A at r belonging to A.

Despite its definition, an atomic string can be very complicated. However, there are two special atomic strings called a *curl* and a *shell* as depicted in Figure 1.2. The following is an important type of strings having riders.

Definition 1.2.8 (Shell composition)**.** A string is called a *shell composition* if it belongs to a sequence of strings described as follows. Let A_0 be a shell and A_{k+1} a string constructed from the string A_k, where $k \in \mathbb{N}$, by riding a shell on any edge of A_k. This procedure for constructing a sequence of strings is called a *composition procedure*.

Lemma 1.2.9. *Riding a shell composition on a shell composition produces a shell composition.*

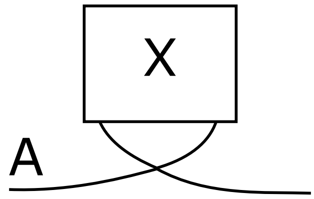


Figure 1.3

Proof. Let A, B be shell compositions and p a point on an edge of A. This implies that both A and B are some terms in two sequences of strings constructed by two composition procedures. Suppose $A = \mathcal{A}_k$ for some sequence $\mathcal{A}_0, \mathcal{A}_1, \ldots, \mathcal{A}_k = A$ and $B = \mathcal{B}_l$ for some sequence $\mathcal{B}_0, \mathcal{B}_1, \ldots, \mathcal{B}_l = B$. Let $\mathcal{A}_{k+i} = \mathcal{A}_k \oplus [\mathcal{B}_i, p]$ for some $0 \leqslant i \leqslant l$. Since $\mathcal{B}_l$ can be constructed from $\mathcal{B}_0$ by a composition procedure, $\mathcal{A}_{k+l} = \mathcal{A}_k \oplus [\mathcal{B}_l, p] = A \oplus [B, p]$ can be constructed from $\mathcal{A}_k = \mathcal{A}_k \oplus [\mathcal{B}_0, p] = A$ by exactly the same composition procedure. Hence, $A \oplus [B, p]$ is a shell composition, as required. □

Lemma 1.2.10. *If A is a string whose input and output vertices coincide, then A is a curl or $A = C \oplus [X, p]$ where C is a curl and X is some string.*

Proof. Let A be a string whose input and output vertices coincide. Since every vertex in a string is 4-valent, there are exactly two edges emanating from the input (output) vertex other than the input and output edges. Thus, A can be represented by the diagram in Figure 1.3 where X is some string. Therefore, we have $A = C \oplus [X, p]$ where C is a curl. If X happens to be trivial, then A becomes a curl. □

Lemma 1.2.11 (Decomposition of a string, definition of core)**.** *Let A be an arbitrary string. We have*

1. *If A is reducible (that is, not irreducible), then A can be written uniquely as $A_1 \oplus A_2 \oplus \cdots \oplus A_n$ where $n \in \mathbb{N}$, $n \geqslant 2$ and each A_i $(1 \leqslant i \leqslant n)$ is irreducible.*

2. *If A is irreducible but not atomic (that is, A has rider(s)), then A can be written uniquely as $X \oplus [A_1, p_1] \oplus [A_2, p_2] \oplus \cdots \oplus [A_n, p_n]$, up to permutations of the A_i's $(1 \leqslant i \leqslant n)$, where X is atomic and non-trivial, $n \in \mathbb{N}$, each A_i is irreducible and each p_i belongs to an edge of X. The string X is called the* core *of A. If A is atomic, then the* core *of A is defined to be A itself.*

Kauffman did not prove this lemma nor did he introduce the notion of a core but apparently, he assumed it to be true while proving other results. The concept of a core is useful in proving some important lemmas.

Proof. To prove part 1, let A be a reducible string which is drawn in such a way that its input edge is extended to the left, its output edge is extended to the right and the two starred faces are the top and bottom faces. The edges adjacent to both starred faces are the connecting edges of A and the input and output edges. They can be ordered from left to right. Therefore, all irreducible strings whose input and output edges are among them can be ordered uniquely from left to right. By labeling them $A_1, A_2, \ldots, A_n$ in the same order, we have $A = A_1 \oplus A_2 \oplus \cdots \oplus A_n$ and this expression is unique.

Consider a rooted tree. In graph theory, a *tree* is a connected simple graph whose any two vertices are connected by a unique path. A *rooted tree* is a tree whose one vertex is designated the root. This creates a natural orientation: *towards* or *away from* the root. A *descendant* of a vertex a is a vertex b such that, if each edge of the rooted tree is given the direction away from the root, then there is a directed path from a to b. Define a *branch* of A to be a subtree of A (i.e. a subgraph of A which is also a tree) that consists of a vertex and all its descendants (and all the edges connecting them). It can be seen that the whole tree corresponds to A and each branch represents an irreducible rider of A. Moreover, one branch B_1, corresponding to an irreducible rider R_1, contains another branch B_2, corresponding to an irreducible rider R_2, precisely when R_1 contains R_2 (i.e. R_2 is a rider of R_1). Cutting from A a branch X, corresponding to an irreducible rider B, yields a smaller rooted tree, and this smaller rooted tree corresponds to the simpler universe C such that $A = C \oplus [B, p]$ for some p.

Pick a branch B_1 and cut it from A, we obtain the branch B_2. Thus, $A = B_2 \oplus [B_1]$. A branch is atomic (corresponding to an atomic string) if it is just a single vertex, that is, it does not have any descendant (in the language of graph theory, this vertex is called a *leaf*). If B_2 is atomic, we obtain something that looks like Figure 1.4. In this case, B_2 is also the root of A. If B_2 is not atomic, then either there is some vertex between B_1 and the root (the diagram on the left of Figure 1.5) or B_1 is adjacent to the root but there are also other vertices adjacent to the root (the diagram on the right of Figure 1.5). Analyse B_4 and B_6 in the same way as we analyse B_2. Eventually, we will reach a situation where the root is the only thing that remains and therefore atomic. This is when we obtain the desired decomposition. Regarding the uniqueness of such an expression, it can be seen that the process above consists in cutting off branches of A until all branches adjacent to the root are cut off. Since all branches adjacent to the root are unique and the root itself is unique, the expression obtained is unique and the root is none other than the core of the string A. The whole rooted tree shows a full decomposition of the string A into atomic riders. The root corresponds to the core; each branch corresponds to an irreducible rider; each leaf corresponds to an atomic rider and each branch adjacent to the root is one of the A_i's in claim 2 of Lemma (1.2.11). The key fact is that any two distinct irreducible riders of A are either disjoint or one carries the other.

Note that the correspondence between A and a rooted tree relies on a subtle assump-

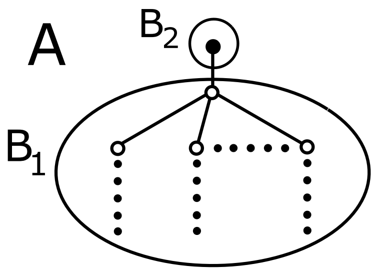


Figure 1.4

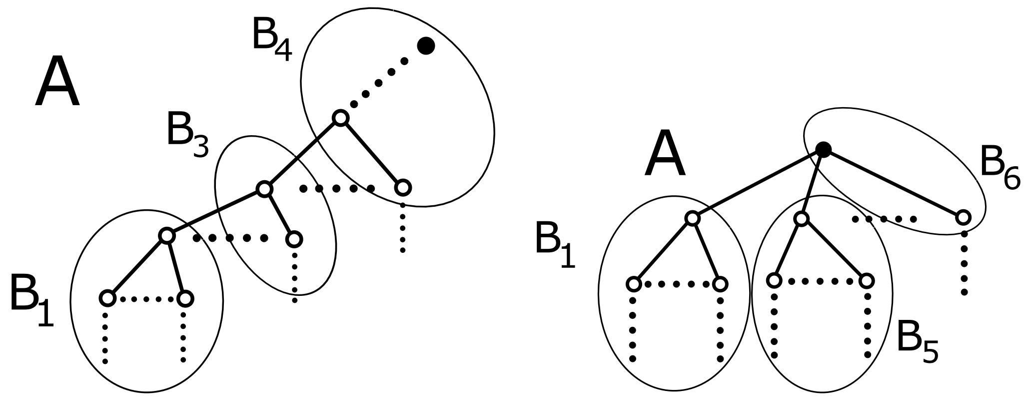


Figure 1.5

tion that the string A can be represented by a unique rooted tree. One may think that we can choose any vertex in a tree to be its root. Different root may produce different rooted trees. However, if we defined the root to be the atomic string containing the input and output vertices of A, which is unique, then the root is unique and so is the rooted tree. □

Kauffman did not use tree graphs to visualise the structure of the riders in a string. All methods of constructing a new string from other strings described so far produce more complicated strings. Here is one which creates a simpler string from a given string.

Definition 1.2.12 (Boundary string)**.** Let A be an atomic string. The *boundary string* of A or the *boundary* of A for short, denoted ∂A, is the string (possibly with cusps) created by deleting all interior edges of A and joining the input and output vertices by a new interior edge. In the case where these two vertices are the same, no such edge needs to be added.

The boundary of an arbitrary string is defined inductively as follows (B and C are any strings):

1. $\partial(B \oplus C) = \partial B \oplus \partial C$.

2. $\partial(B \oplus [C, p]) = \partial B \oplus [\partial C, p]$ if p belongs to a boundary edge of ∂B.

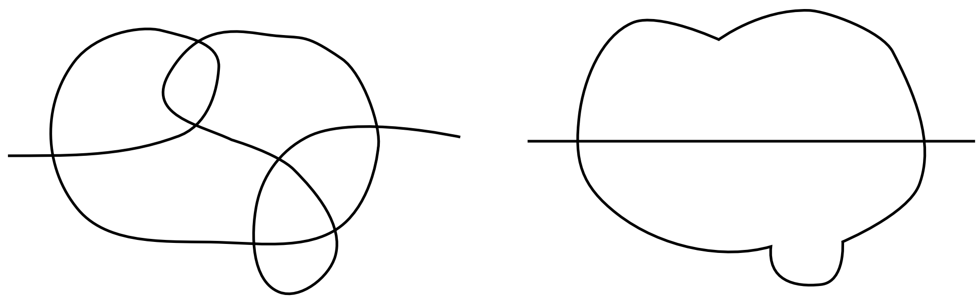


Figure 1.6

3. $\partial(B \oplus [C,p]) = \partial B$ if p does not belong to a boundary edge of ∂B.

4. The boundary of the trivial string is the trivial string.

Example. The right diagram in Figure 1.6 shows the boundary string of the atomic string on the left. Note that cusps arise naturally when certain interior edges are deleted.

Remark. It immediately follows from Definition (1.2.12) that, for any string A, the input and output vertices of A coincide if and only if the input and output vertices of the boundary of A coincide. Moreover, if a vertex of A has at least one edge incident to it deleted when ∂A is constructed, then it is not a vertex of ∂A.

It can be seen that when A is atomic, ∂A is well-defined and unique. When A is irreducible, it is obtained from an atomic string by adding riders and the same result is obtained for ∂A regardless of the order in which riders are added. A general A is of the form $A_1 \oplus \cdots \oplus A_n$ where each A_i is irreducible and in this case $\partial A = \partial A_1 \oplus \cdots \oplus \partial A_n$. Thus, for a general string A, its boundary is well-defined and unique. Also, if a string A is atomic, then ∂A looks like a curl or a shell, except with some cusps. This observation is made precise in the following lemma.

Lemma 1.2.13. *The boundary ∂A of an irreducible string A is either a shell or a curl (with possibly some cusps) if and only if A is atomic or A carries riders but all riders riding the core of A ride it at points on its interior edges.*

This lemma was not proved by Kauffman but it is useful in visualising the concept of a derivative presented in Definition (1.3.1).

Proof. Let A be a string such that ∂A is either a shell or a curl (possibly with cusps) and let R be a rider riding the core B of A. By Lemma (1.2.11), A can be written uniquely as $B \oplus [R,p] \oplus \cdots$ where the ellipsis stands for other riders riding B. Proceeding by contradiction. Suppose R rides B on one of its non-interior edges. A non-interior edge can only be a boundary edge or an input edge or an output edge. We consider these cases separately.

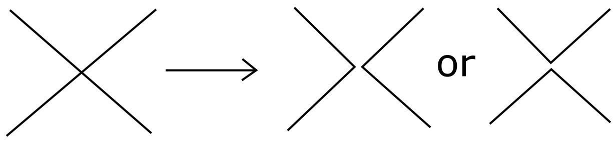
Case 1: R rides B on an input or output edge of B. Thus, $A = C \oplus R$ where C is some string. Hence, $\partial A = \partial B \oplus \partial R$, by Definition (1.2.12). This implies that ∂A is not atomic, which leads to a contradiction because, by assumption, ∂A is a shell or a curl and therefore atomic.

Case 2: R rides B on a boundary edge of B. Since a boundary edge of B is also a boundary edge of ∂B, we have $\partial A = \partial B \oplus [\partial R, p] \oplus \cdots$, by Definition (1.2.12). Again, this implies that ∂A is not atomic, which leads to a contradiction because, by assumption, ∂A is a shell or a curl and therefore atomic.

From the two cases above, we have shown that, if ∂A is either a shell or a curl, then A is atomic or A carries riders but all riders riding the core of A ride it on its interior edges.

The converse follows directly from Definition (1.2.12). □

Definition 1.2.14 (String with sites)**.** Let A be a string. Split some vertices of A in the following way



such that the resulting object A' is also a string. Where each vertex is split, the string A' obtains a pair of cusps, called a *site*. The resulting string A' is called a *string with sites*. Note that at each cusp of A', two edges of A are joined into a single edge with a cusp. The cusps in a diagram may be partitioned into pairs and the pairs correspond precisely to sites.

Strictly speaking, a site should include the data of an arc drawn between the two cusps of the site. However, we will follow Kauffman in not drawing such arcs since the arc joining the two cusps is unique up to homotopy.

Definition 1.2.15 (Jordan trail)**.** Let A be a string. When all vertices of A are split, the edges of A then form a collection of paths. When the splitting is such that there is a single connected path, we call the result path a *Jordan trail* of A or *trail* for short. Note that a Jordan trail is a string with sites.

The set of all strings with sites is denoted by $\hat{\mathscr{U}}$, which contains the set of all strings $\mathscr{U}$ and the set of all trails $\mathscr{T}$ since a trail is just a string whose vertices are all split according to one of its states. It can be seen that a string with sites does not contain any more information than a state or a trail (see Lemma (1.4.6)). It is just a convenience for other notions to be defined on it.

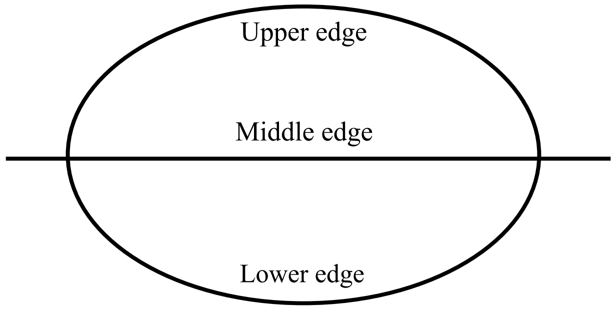
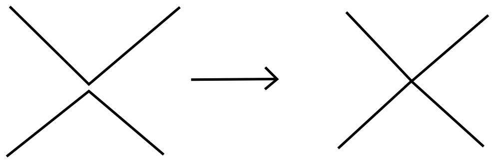


Figure 1.7

Definition 1.2.16 (Closing all sites)**.** Define $J : \hat{\mathscr{U}} \to \mathscr{U}$ to be the map that closes all sites of a string with sites, as illustrated below.



The following lemma is a special case of Lemma (1.2.10) but we state it explicitly here for use in subsequent proofs.

Lemma 1.2.17. *If A is atomic and its input and output vertices coincide, then A is a curl.*

Before going on to more elaborate concepts, we introduce the following notations to simplify future explanations. In a shell A, denote by $p_{\in A}$ a point on the middle edge of A and by $p^{\in A}$ a point on either of the two boundary edges of A. Figure 1.7 shows how the edges of a shell are named. The following lemma was not proved by Kauffman but it shows the connection between the boundary string and the inner string and how even an atomic string can be "decomposed" into simpler strings.

Lemma 1.2.18. *For any atomic string A which is not a curl or a shell, there exist a point $p_{\in \partial A}$ and a unique string A' (possibly with cusps) such that $A = J(\partial A \oplus [A', p_{\in \partial A}])$. We call A' the* inner string *of A.*

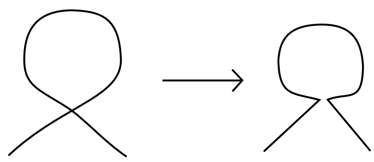
Remark. If A is atomic and not a curl, then by Lemma (1.2.17), A has distinct input and output vertices. Thus, by Definition (1.2.12), ∂A has distinct input and output vertices; and by Lemma (1.2.13), ∂A is a shell or a curl. But as ∂A has distinct input and output vertices, it cannot be a curl, hence it is a shell. Therefore, the notation $p_{\in \partial A}$ is valid.

Proof of Lemma (1.2.18). Let A be an atomic string which is not a curl or a shell. From the remark above, ∂A is a shell. Split all the vertices of A that belong to ∂A (except the input and output vertices) in such a way that ∂A remains intact. We will obtain a string with sites $\tilde{A}$. Delete the upper and lower edges of ∂A from $\tilde{A}$. We will obtain a non-trivial string A' whose cusps coincide with the split vertices. By riding A' on the middle edge of ∂A in such a way that each cusp of A' coincides with the cusp of ∂A that belongs to the same split vertex with it, we reproduce the string with sites $\tilde{A}$. Closing all the site of $\tilde{A}$ gives back the original string A. Therefore, there exists a non-trivial string A' such that $A = J(\partial A \oplus [A', p_{\in \partial A}])$ where $p_{\in \partial A}$ is any point on the middle edge of ∂A. It can be seen that the process described above creates a unique string A'. □

1.3 More complicated constructions

Definition 1.3.1 (First derivative)**.** Let A be an atomic string. The *first derivative* of A is defined as follows

1. If A is a curl, then DA is the curl with site as shown below



2. If A is a shell, then $DA = A$.

3. If A is neither a curl nor a shell, then $DA = \partial A \oplus [A_1, p_{\in \partial A}]$ where A_1 is the inner string of A guaranteed by Lemma (1.2.18).

The first derivative of an arbitrary string is a map $D : \hat{\mathscr{U}} \to \hat{\mathscr{U}}$ defined inductively as follows (B and C are any strings):

1. $D(B \oplus C) = DB \oplus DC$.

2. $D(B \oplus [C, p]) = DB \oplus [DC, p]$.

3. The derivative of the trivial string is the trivial string.

Remark. The equation $DA = \partial A \oplus [A_1, p_{\in \partial A}]$ in the above definition applies to a shell too if we allow A_1 to be trivial.

Definition 1.3.2 (Higher derivative)**.** Let A be a string. The *derivative of order n of* A, where $n \geqslant 1$, is a map $D^n : \hat{\mathscr{U}} \to \hat{\mathscr{U}}$ defined inductively as $D^n A = D(D^{n-1} A)$. By convention, we let $D^0 A = A$.

Let A be an atomic string. To find the first derivative of A, we find ∂A and the inner string A_1 of A, then let A_1 ride A at some point $p_{\in \partial A}$ such that $A = J(\partial A \oplus [A_1, p_{\in \partial A}])$. Thus, we have $DA = \partial A \oplus [A_1, p_{\in \partial A}]$. For the second derivative, we have

$$\begin{aligned} D^2 A = D(DA) &= D(\partial A \oplus [A_1, p_{\in \partial A}]) \\ &= D(\partial A) \oplus [DA_1, p_{\in \partial A}] && \text{Using the second rule in Definition (1.3.1)} \\ &= \partial A \oplus [\partial A_1 \oplus [A_2, p_{\in \partial A_1}], p_{\in \partial A}] && \text{Assuming } A_1 \text{ is atomic.} \end{aligned}$$

Lemma 1.3.3. *For any string A, $DA = A$ if and only if A is a shell composition.*

Kauffman did not prove this lemma but we believe a proof will demonstrate what is special about a shell composition.

Proof. The gist of this proof is based on the fact that each vertex in a shell composition is either the input or output vertex of some rider and the fact that, for a string to equal its derivative, the derivative has to be site-free.

Let A be a string such that $DA = A$. Since $DA = A$, A cannot be a curl or a curl with riders. As a result, the input and output vertices of A are distinct. We consider the following cases.

Case 1: Suppose that A is atomic. If A is trivial, then the lemma holds trivially because the trivial string is also a shell composition. So, we exclude the fact that A is trivial. Since A is atomic and not a curl, it has distinct input and output vertices, by Lemma (1.2.17). Hence, by Definition (1.2.12), ∂A has distinct input and output vertices; and by Lemma (1.2.13), ∂A is a shell or a curl. But as ∂A has distinct input and output vertices, it cannot be a curl, hence it is a shell. Moreover, if A has nontrivial inner string A', then $DA = \partial A \oplus [A', p_{\in \partial A}]$. But then, since $DA = A$, we have $\partial A \oplus [A', p_{\in \partial A}] = A$, where the left-hand side is a string with some rider while the right-hand side is an atomic string. This is a contradiction. Thus, A has a trivial inner string. Hence, $\partial A = A$, which implies that A is a shell, as required.

Case 2: Suppose that A is irreducible but has rider(s), so it can be written uniquely as $A = B \oplus [A_1, p_1] \oplus \cdots \oplus [A_n, p_n]$ for some nontrivial atomic string B and nontrivial irreducible strings $A_n, \ldots, A_n$. Proceed by induction on n. The base case $n = 0$ is Case 1.

Consider a point M moving to the right along the input edge of A. Since the input vertex is 4-valent, it has three edges emanating from it (excluding the input edge). Thus, after passing through the input edge, M can travel along an upper, middle or lower edge.

Suppose M proceeds along the upper edge. If it meets a vertex, say V_1, then V_1 is either split into a site in DA or remains intact. But since $DA = A$, DA has no site and hence V_1 remains intact. Therefore, V_1 is the input vertex of some rider of A, say A_1.

Thus, $A = C \oplus [A_1, p_1]$ for some irreducible string C and $DA = DC \oplus [DA_1, p_1]$. Since $DA = A$, we have $DC \oplus [DA_1, p_1] = C \oplus [A_1, p_1]$. As the derivative can only decrease the number of vertices, we can obtain equality when no vertices are split. Hence, we have $DC = C$ and $DA_1 = A_1$. Both C and A_1 are simpler than A, so by induction, C and A_1 are both shell compositions. As a result, $A = C \oplus [A_1, p_1]$ is a shell composition.

A similar argument applies if M proceeds along the lower edge and meets a vertex there. Thus, we may assume there are no vertices along the upper and lower edges between the input and output vertices. Now, consider M proceeding along the middle edge. If M meets a vertex, then $A = C \oplus [A_1, p_1]$ for some irreducible string C and a point p_1 on the middle edge. Again, we have $DC = C$ and $DA_1 = A_1$ and again, by induction, obtain a shell composition.

Case 3: Suppose that A is reducible. By Lemma (1.2.11), A can be written as $A_1 \oplus A_2 \oplus \cdots \oplus A_n$ where each A_i $(1 \leqslant i \leqslant n)$ is irreducible. Using the same arguments in case 2 for each A_i, it can be seen that A is still a shell composition.

The three cases above cover all possible cases of which a string A can be. As a result, we have shown that, if $DA = A$, then A is either a shell composition or trivial.

The converse is obvious. □

Remark. Lemma (1.3.3) still holds for a string with sites, that is, for any string with sites A, $DA = A$ if and only if A is a shell composition with sites. The proof of this generalised lemma is exactly the same as that of Lemma (1.3.3).

Lemma 1.3.4. *For every string A, there exists $n \in \mathbb{N} \cup \{0\}$ such that $D^{n+1}A = D^n A$. The string with site $D^n A$ is denoted by $\hat{D}A$ and is called the* dissection *of A. The smallest n satisfying $D^{n+1}A = D^n A$ is called the* order *of $\hat{D}A$.*

Proof. From Definition (1.3.1), it can be seen that, if $DA \neq A$, then some vertices of A are split in DA. More generally, if $D^{k+1}A \neq D^k A$ for some $k \in \mathbb{N} \cup \{0\}$, then some vertices of $D^k A$ are split in $D^{k+1}A$. Since the number of vertices of A is finite, the process of iterating derivatives of A has to stop at some $n \in \mathbb{N} \cup \{0\}$, which implies that $D^{n+1} = D^n A$. □

Here is one of the main results that we aim to prove.

Lemma 1.3.5. *The dissection of any string is a shell composition with sites.*

Proof. Let A be a string. Since $D(\hat{D}A) = \hat{D}A$, $\hat{D}A$ is a shell composition, by Lemma (1.3.3). □

Definition 1.3.6 (Room)**.** A *room* of a string with sites A is a subset of the plane corresponding to a face of $J(A)$.

Lemma 1.3.7. *The core of an irreducible and non-trivial shell composition is a shell.*

Proof. Let A be an irreducible and non-trivial shell composition. If A is atomic, then it is obvious from Definition (1.2.8) that A is a shell. Since the core of an atomic string is defined to be itself (see Lemma (1.2.11) part 3), the core of A is a shell. If A is not atomic, then, by Lemma (1.2.11) part 2, we have $A = X \oplus [A_1, p_1] \oplus [A_2, p_2] \oplus \cdots \oplus [A_n, p_n]$ where the core X is atomic, non-trivial and is therefore a shell. □

Let A be a shell. The three edges incident to the input (output) vertex of A apart from the input (output) edge can be unambiguously labeled *upper*, *middle* and *lower edges* according to their position relative to the input (output) edge. For a general shell composition, we have the following definition.

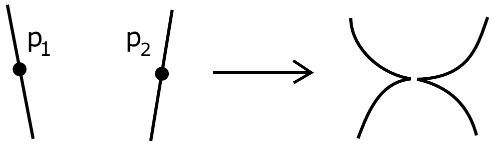
Lemma 1.3.8. *An irreducible rider B of a shell composition A is a shell composition.*

Proof. By definition, A is obtained from a shell by successive additions of shell riders. Some of these shells lie in B and some do not. The composition of those shells lying in B expresses B as a shell with shell riders. Hence, B is a shell composition. □

Definition 1.3.9 (Upper, lower and middle components)**.** Let B be an irreducible rider of a shell composition A. It can be seen that B itself is an irreducible and non-trivial shell composition (a rider is always non-trivial, by Definition (1.2.7)). By Lemma (1.3.7), the core of B is a shell. The string consisting of the upper edge of the core of B and all riders riding it (and all riders riding them and so on) is called the *upper component* of B and denoted by B_+. The *middle* and *lower components* of B are defined similarly and denoted by B_0 and B_-, respectively. Note that B_-, B_0, B_+ can be considered as strings in their own right.

Lemma (1.3.5) is the most important lemma so far since it reveals a common pattern shared by every string: any string can be turned to a shell composition with sites by splitting its vertices according to the rules of derivative. This common pattern leads directly to the construction of the extremal states in the clock lattice. However, we need some more definitions. Lemma (1.3.5) says that dissecting a string produces a shell composition with sites. From this comes the question: from a shell composition, can we pull close together pairs of points to create sites such that the resulting shell composition with sites is identical to the dissection of some string? To answer this question, we introduce the following rules of interaction.

Definition 1.3.10 (Interaction of two points)**.** Let A be a shell composition with sites. Let p_1 and p_2 be two distinct points on some edge(s) of the same room of A. The process of creating a site from p_1 and p_2 described as follows



is called the *interaction* of p_1 and p_2. In the case where the two rooms adjacent to p_1 are the same as the two rooms adjacent to p_2, one of the two rooms adjacent to these two points must be specified for the interaction to be defined. (Strictly speaking, with sites defined to include the data of an arc, the points p_1, p_2 would be joined along an arc by which they would form a site.)

Definition 1.3.11 (Smallest shell composition)**.** Let A be a shell composition with sites. Let p_1 and p_2 be two distinct points on some edge(s) of A. A rider of A is said to *contain* p_1 and p_2 or equivalently, these points are said to *belong* to a rider of A if they lie on some edge(s) of the rider which is (are) not its input or output edges. The irreducible rider of A which contains both p_1 and p_2 and which does not have any irreducible rider containing both p_1 and p_2 is called the *smallest shell composition* containing p_1 and p_2, denoted B. If A is irreducible and has no irreducible rider containing p_1 and p_2, then the smallest shell composition containing p_2 and p_2 is defined to be A itself (note that, according to Definition (1.2.7), a string is not a rider of itself). If there exists an interaction of p_1 and p_2 which creates a site p, then B is said to be the *smallest shell composition* containing p and p is said to belong to B.

It seems that we can construct any string from an appropriate shell composition by interacting pairs of points on it. In order to prove this, we need to formalise the types of interactions that serve this purpose.

Definition 1.3.12 (Allowed and forbidden interactions)**.** Let A be a shell composition with sites. Let p_1 and p_2 be two distinct points on some edge(s) of the same room of A. Let B be the smallest shell composition containing p_1 and p_2. The interaction of p_1 and p_2 is said to be *allowed* if either of the following conditions is satisfied.

1. Both p_1 and p_2 belong to some connecting edge(s) (see Definition (1.2.6)) of the same component B_0 or B_+ or B_- of B.

2. If p_1 and p_2 belong to different components of B, then one of them must belong to a connecting edge of the component B_+ or B_- of B.

3. If there is no smallest shell composition containing p_1 and p_2, that is, if B does not exist, then both p_1 and p_2 belong to some connecting edge(s) of A.

If none of the conditions above is satisfied, then the interaction of p_1 and p_2 is said to be *forbidden*.

It can be seen that, the three rules above are mutually exclusive, that is, any pair of points that can be interacted satisfies at most one of them.

Definition 1.3.13 (Elaboration)**.** An *elaboration* of a shell composition $A \in \mathscr{U}$ is a shell composition with sites obtained from successive applications of allowed interactions on A. The set of all elaborations of all shell compositions is denoted by SH.

Remark. By convention, a shell composition is an elaboration (without any interaction). The order in which the interactions are applied is important. If an elaboration of a shell composition A is obtained from A by more than 1 interaction, an allowed interaction may no longer be allowed (or even defined in some cases) if it is preceded by another allowed interaction.

It is obvious that the codomain of the dissection map $\hat{D}$ is $\hat{\mathscr{U}}$ (the set of all strings with sites). The next lemma restricts this codomain to a smaller set. Kauffman gave a very short proof of this lemma. Here we provide the full details of the proof.

Lemma 1.3.14. *The dissection of a string is an elaboration, that is, $\hat{D}$ maps $\mathscr{U}$ into SH.*

Proof. Let A be a string. Let V_i be the set of all vertices and sites of D^iA for all $0 \leqslant i \leqslant n$ where $D^0A = A$ by convention and n is the order of $\hat{D}A$ (the smallest number satisfying $D^{n+1}A = D^nA$). The derivative of any order of A is a string with sites obtained from A by splitting some vertices of A in a certain way. Therefore, there is a natural bijection f_{ij} mapping a vertex or site in D^iA to the corresponding vertex (if this vertex is not split) or site (if this vertex is split) in D^jA. We consider the following cases.

Case 1: Suppose A is atomic. Again, we consider the following sub-cases.

Case 1.1: Suppose that the input and output vertices of A coincide. Since A is atomic, A is a curl, by Lemma (1.2.10). Hence, $\hat{D}A$ is the trivial string and therefore a shell composition. As remarked above, this implies that $\hat{D}A$ is an elaboration.

Case 1.2: Suppose that the input and output vertices of A are distinct. Since A is atomic and not a curl, $DA = \partial A \oplus [A_1, p_{\in A}]$ where A_1 is the inner string of A, by Definition (1.3.1). Although DA is not necessarily a shell composition, it can be seen that DA is a shell (∂A) whose middle edge is ridden by A_1. Therefore, all sites created by the process of taking the derivative of A lie in the region between A_1 and the upper edge of ∂A and the region between A_1 and the lower edge of ∂A, which are the shaded regions in Figure 1.8. Here come some technical difficulties, since DA is not necessarily a shell composition and the notions of interactions, upper, lower and middle components are all defined on a shell composition with sites, they cannot be used here. However, we can make use of the fact that $\hat{D}A$ is a shell composition with sites (Lemma (1.3.5)) and the

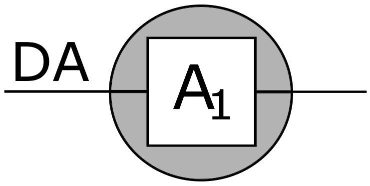


Figure 1.8. Case 1.2

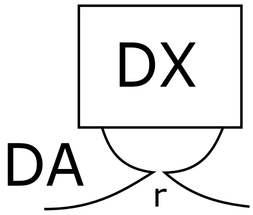


Figure 1.9. Case 2.1

bijections f_{ij}'s defined above to relate a site (or vertex) in any D^iA to a site (or vertex) in $\hat{D}A$ where these notions are well defined. It can be seen that the upper and lower edges of ∂A become the (trivial) upper and lower components of $\hat{D}A$ and the inner string A_1 become the middle component of $\hat{D}A$. Also, from the fact that $DA = \partial A \oplus [A_1, p_{\in A}]$, it can be seen that every site s in DA consists of a cusp coming from the upper edge or the lower edge of ∂A. Therefore, the interaction creating $f_{1n}(s)$ in $\hat{D}A$ is allowed by rule 2. We have shown that every site of A that is not a site of A_1 comes from an allowed interaction. In a sense, we have reduced A to a smaller string A_1. This is the inductive step in a proof by induction on the number of sites (or vertices) of the original string A. The base case is where A is a shell composition without any site. Hence, by the remark after Definition (1.3.13), it is an elaboration. Note that A_1 is not necessarily atomic, if it is, repeat the arguments in case 1 to arrive at a smaller string. If it is not, apply the arguments in the remaining cases to obtain a smaller string.

Case 2: Suppose A is irreducible but has rider(s). We have the following sub-cases.

Case 2.1: Suppose that the input and output vertices of A coincide. Since A has rider(s), by Lemma (1.2.10), we have $A = C \oplus [X, p]$ where C is a curl and X is a non-trivial string. Thus, we have $DA = DC \oplus [DX, p]$, which looks like Figure 1.9. The input (and also output) vertex of C is split into the site r. It can be seen that the interaction creating $f_{1n}(r)$ in $\hat{D}A$ is allowed by rule 3. As in case 1, we have reduced A to a smaller string X such that each site of A which is not a site of X (namely r) comes from an allowed interaction. If X is atomic, apply case 1 to X, if it is irreducible but has rider(s) and its input and output vertices coincide, apply this case to X. Otherwise, apply the remaining cases to obtain a smaller string.

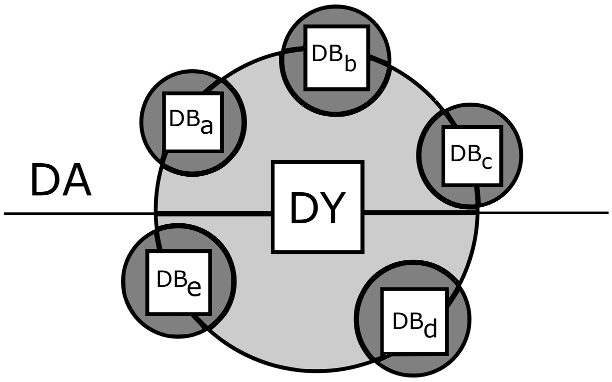


Figure 1.10. Case 2.2

Case 2.2: Suppose that the input and output vertices of A are distinct. We have

$$A = Y \oplus [B_1, p_1] \oplus [B_2, p_2] \oplus \cdots \oplus [B_m, p_m] \qquad \text{by Lemma (1.2.11)} \quad (1.3.1)$$
$$\Rightarrow DA = DY \oplus [DB_1, p_1] \oplus [DB_2, p_2] \oplus \cdots \oplus [DB_m, p_m] \quad (1.3.2)$$

where Y is atomic and non-trivial. Thus, DY looks like Figure 1.8. Since each B_i ($1 \leqslant i \leqslant m$) is irreducible, we can again express it and its derivative in a way similar to Equations (1.3.1) and (1.3.2). It can be seen that each DB_i can only ride DY on the circle or inside the square of Figure 1.8. Thus, we have the diagram in Figure 1.10 illustrating DA. Note that some circles with DB_i's may lie inside DY but they are omitted from the diagram. Let t be a site in the lightly shaded region of Figure 1.10. It can be seen that t consists of a cusp coming from a connecting edge of the big circle (∂DY). Hence, $f_{1n}(t)$ is created by an interaction satisfying rule 2. Let u be a site in one of the deeply shaded regions of Figure 1.10. Again, it can be seen that u consists of a cusp from a connecting edge of the corresponding small circle (∂DB_i for some i). Thus, $f_{1n}(u)$ is created by an interaction satisfying rule 2. So far, we have not reduced A to a smaller string but we have shown that all sites created by taking the first derivative of A come from allowed interactions. Since Y is atomic and each B_i is irreducible, we can apply the arguments in this and previous cases to arrive at a smaller string.

Case 3: Suppose A is reducible. By Lemma (1.2.11), we have $A = E_1 \oplus E_2 \oplus \cdots \oplus E_l$. This implies $DA = DE_1 \oplus DE_2 \oplus \cdots \oplus DE_l$ where each E_i and DE_i ($1 \leqslant i \leqslant l$) is irreducible. Thus, we have the diagram in Figure 1.11 illustrating DA where each DE_i looks like Figure 1.9 or 1.10 (possibly Figure 1.8 if DE_i is atomic but Figure 1.8 is just a special case of Figure 1.10 where each DB_i on the big circle is trivial). Apply the arguments in the previous cases to each DE_i in order to arrive at a smaller string.

The cases above cover all possible cases of which a string A can be. Using these cases, we can analyse each rider of DA, then each rider of D^2A and so on until we reach $\hat{D}A$. Each site created in D^iA after taking the derivative of $D^{i-1}A$ is mapped to its

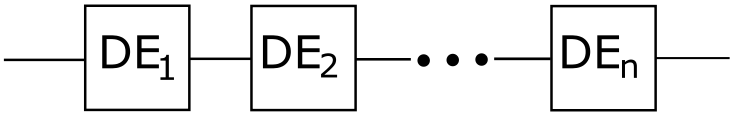


Figure 1.11. Case 3

corresponding site in $\hat{D}A$ which, according to the argument in each case, comes from an allowed interaction. As a result, $\hat{D}A$ is an elaboration, that is, $\hat{D}A \in SH$ and $\hat{D}$ maps $\mathscr{U}$ into SH. This completes the proof.

It can be seen that rule 1 of an allowed interaction is never mentioned in the proof. In fact rule 1 and rule 3 are essentially the same, except that when two points p_1 and p_2 are on some connecting edge of the *whole* shell composition with sites, there is no smallest shell composition containing them. Rule 3 is imposed to cover this case. If case 2.1 is used to analyse a rider of A instead of A itself, then rule 3 will be replaced by rule 1 to justify why $f_{in}(r)$ comes from an allowed interaction, where i is the order of the higher derivative where r first appears. □

Lemma 1.3.15. *The map $J : SH \to \mathscr{U}$ is a bijection and the map $\hat{D} : \mathscr{U} \to SH$ is its inverse, that is, $\hat{D} \circ J = 1_{SH}$ and $J \circ \hat{D} = 1_{\mathscr{U}}$ where 1_{SH} and $1_{\mathscr{U}}$ are the identity maps on SH and $\mathscr{U}$, respectively.*

Kauffman almost left the proof of this lemma to the reader. Here we detail the full proof.

Proof. Let X be a string. The map $\hat{D}$ splits some vertices of X into sites. The map J closes these sites, turning them back to vertices. This process does not change anything about X. Therefore, $J \circ \hat{D}$ is the identity on $\mathscr{U}$.

Now, we prove that $\hat{D} \circ J = 1_{SH}$. The gist of this proof is to prove that any vertex created from a site being closed by J will be split back into the same site by $\hat{D}$. Let E be an elaboration of a shell composition A. Let $\{x_1, y_1\}, \{x_2, y_2\}, \ldots, \{x_n, y_n\}$ be pairs of points on A such that their interactions are allowed when applied in this order and that these interactions result in E. Proceeding by induction, for the basis of induction, we prove that the elaboration E_1 obtained from the interaction of $\{x_1, y_1\}$ satisfies $\hat{D} \circ J(E_1) = E_1$.

Suppose that the pair of points $\{x_1, y_1\}$ satisfies rule 1 of Definition (1.3.12). Let X be the smallest shell composition containing $\{x_1, y_1\}$. Suppose in addition that $\{x_1, y_1\}$ belongs to X_+. Since $\{x_1, y_1\}$ either belongs to the same connecting edge or 2 different ones, the two cases shown in Figure 1.12 cases are the only possible cases. The box represents the rider of X_+ between the 2 interacting points. The ellipses stand for other riders of X_+ or the trivial string. The map J will close the cusps in the two diagrams of Figure 1.12 to make a curl or a curl ridden by a rider and therefore create a new rider of

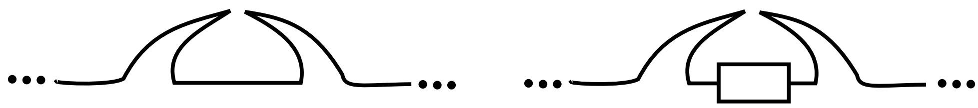


Figure 1.12

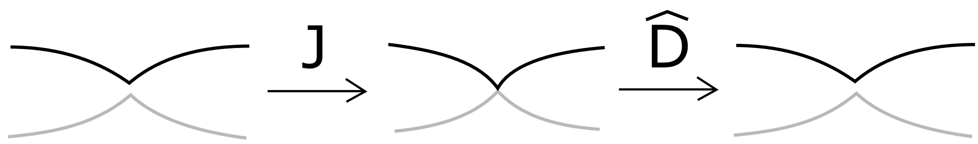


Figure 1.13

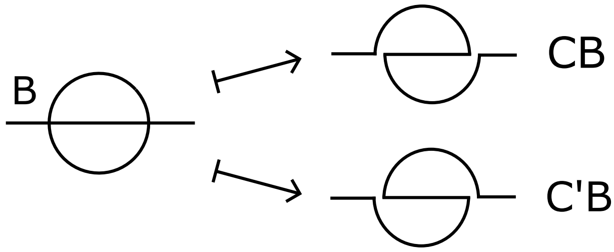
X_+. By the rules of derivatives, $\hat{D}$ will split the input (which is also the output) vertex of the curl (or the curl ridden by a rider) back to the site created by the interaction of $\{x_1, y_1\}$. As a result, $\hat{D} \circ J(E_1) = E_1$. Exactly the same arguments apply in the cases where $\{x_1, y_1\}$ belongs to X_0 or X_- and also in the case where $\{x_1, y_1\}$ satisfies rule 3.

Next, suppose that $\{x_1, y_1\}$ satisfies rule 2. Thus, one among the pair, say x_1, belongs to a connecting edge of X_+ or X_-, say X_+. Hence, y_1 belongs to X_0. Consider the vertex v created by closing the site obtained from interacting $\{x_1, y_1\}$. Since x_1 belongs to a connecting edge of X_+, two edges emanating from v are boundary edges of X (coloured black in Figure 1.13). The other two edges are interior edges (coloured grey in Figure 1.13). Therefore, by the rules of derivative, $\hat{D}$ will split v back to the site previously closed by J, as illustrated in Figure 1.13. As a result, $\hat{D} \circ J(E_1) = E_1$. This completes the basis of induction.

For the inductive step, assume that the elaboration E_k $(k < n)$ resulting from the interactions of $\{x_1, y_1\}, \ldots, \{x_k, y_k\}$ satisfies $\hat{D} \circ J(E_k) = E_k$. Let E_{k+1} be the elaboration obtained from the interactions of $\{x_1, y_1\}, \ldots, \{x_{k+1}, y_{k+1}\}$. Thus, E_{k+1} is obtained from E_k by interacting $\{x_{k+1}, y_{k+1}\}$ on E_k. Since the proof of the fact that $\hat{D} \circ J(E_1) = E_1$ does not refer to the fact that E_1 is obtained from a shell composition without site A instead of an elaboration, the same reasoning can be used to prove $\hat{D} \circ J(E_{k+1}) = E_{k+1}$. Thus, $\hat{D} \circ J(E) = E$ for any elaboration E of A. As a result, J is a bijection form SH to $\mathscr{U}$ and $\hat{D} : \mathscr{U} \to SH$ is its inverse. □

Next, we introduce the ultimate maps that turn an elaboration to trails having the special property of being the trails of the clocked and counterclocked states.

Definition 1.3.16 (Clockwise and counterclockwise splitting)**.** Let B be an irreducible rider of an elaboration A (if B does not exist, then A is considered instead). Split the input and output vertices of B according to either of the two following methods.



The maps $C, C' : SH \to \mathscr{T}$ defined by splitting all riders of A according to the upper and lower methods in the figure above, respectively, are called the *clockwise splitting* and *counterclockwise splitting* maps, respectively. The images CA and $C'A$ of A via these maps are called the *clocked* and *counterclocked trails* of A.

Lemma (1.3.15) has established a one-to-one correspondence between the set of all strings and the set of all elaborated shell compositions. This means that, instead of placing markers on the vertices of a string, we can place markers on the sites of an elaborated shell composition without introducing any ambiguity. Such a method of placing markers is described in the next definition. Recall from the proof of Lemma (1.2.11) that a tree is a connected simple graph whose any two vertices are connected by a unique path. A rooted tree is a tree with a unique vertex designated as the root which creates a natural orientation: towards or away from the root. In the following definition and afterwards, a site is said to be adjacent to a room if, when it is closed, two of the four newly formed edges bound the room.

Definition 1.3.17 (Tree-growing procedure)**.** Let X be a trail of a string A. Grow a rooted tree as follows. Let the star in one of the two starred (unbounded) rooms of X be the root. Search for site(s) adjacent to the same starred room which allow an edge from the root to pass through without being intersected by X. Place a vertex at these sites (call it a_i with $i \leqslant k$ where k is the number of sites found) and join it to the root. Each of these vertices separates two rooms. One is the starred room just mentioned. In the other room, search again for sites which allow an edge from the corresponding vertex a_i to pass through without being intersected by X. Place a vertex at these sites (called it b_j with $j \leqslant l$ where l is the number of sites found) and join it to a_i. Repeat this process for each b_j, a_i and the other star until every site of X has a vertex. Thus, we have obtained two rooted trees for X. Place a marker at each vertex of each tree (except the roots) such that the room where it lies is furthest away from the root of the corresponding tree. Finally, closing all sites gives a state of A. Let $\mathscr{T}_A$ be the set of trails of A and $\mathscr{S}_A$ the set of its states. The map $b : \mathscr{T}_A \to \mathscr{S}_A$ which maps a trail $X \in \mathscr{T}_A$ to the state $S \in \mathscr{S}_A$ obtained by growing trees on X is called the *tree-growing map.*

Example. The right diagram of Figure 1.14 shows a trail of a string, the two trees obtained from the tree-growing procedure (the green graphs) and the makers placed at

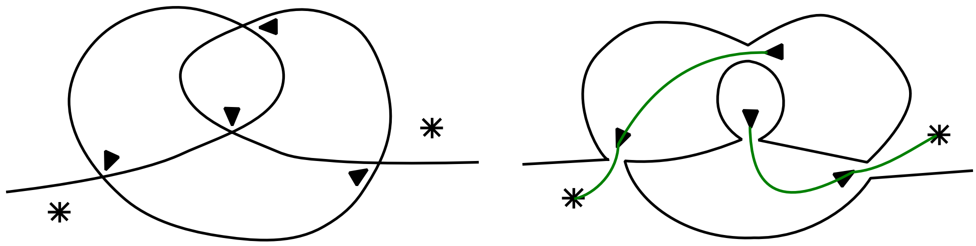


Figure 1.14

each vertex of each tree (except the roots). The left diagram show the state resulting from this procedure.

Lemma 1.3.18. *The tree-growing procedure described in Definition* (1.3.17) *actually produces two trees for each trail of a string U and the tree-growing map b actually maps a trail to a state, that is, the codomain of b is really $\mathscr{S}_U$, which is the set of states of U.*

Proof. Let T be a trail of a string U. Proceeding by contradiction, suppose that, among the graphs obtained from the tree-growing procedure, one of them, say G, is not a tree, that is, there is a cycle within G. From Definition (1.3.17), this implies that there is a cycle within T, as illustrated in Figure 1.15. This leads to a contradiction because a trail cannot have any cycle. Therefore, any graph that can be constructed by the tree-growing procedure is a tree. Moreover, two trees constructed from two roots cannot be connected since the two roots belong to two faces of T separated by a path. As a result, exactly two rooted trees can be grown on T.

There are only two ways $b(T)$ can fail to be a state. The first one is the situation where not every vertex of $b(T)$ is assigned a marker. This is impossible because every site of T separates two rooms of T which belong to the same face of T. This face must have a star. Thus, a tree growing from this star will eventually reach the site and a marker will therefore be placed at that site. The second situation where $b(T)$ fails to be a state is the one where some face of $b(T)$ contains more than one marker. A necessary condition for this to happen is that the tree which places markers in the room of T corresponding to this face has a cycle (see Figure 1.15). Since it is a tree, it cannot have a cycle, which makes this situation impossible. Hence, $b(T)$ is a state of U and $\mathscr{S}_U$ is the codomain of b. □

Lemma 1.3.19. *Let E be an elaboration and CE ($C'E$) its clocked trail (counterclocked trail) with markers placed according to the tree-growing procedure. Let v be a site created from the interaction of $\{x, y\}$. Let R be the smallest shell composition containing $\{x, y\}$ and CR ($C'R$) its clocked trail (counterclocked trail) with markers inherited from CE ($C'E$). Suppose the interaction of $\{x, y\}$ satisfies rule 2 of Definition* (1.3.12). *If x*

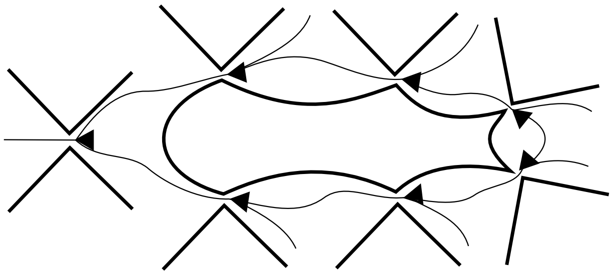
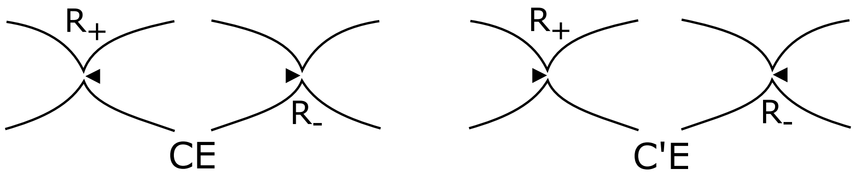


Figure 1.15

belongs to R_+*, then the marker at* v *will be on its right (left). If* x *belongs to* R_-*, then the marker at* v *will be on its left (right), as illustrated below.*



Before beginning the proof, we make clear what "left" and "right" mean. If a site is formed by an interaction of two points satisfying rule 2, then one of the points comes from the upper or lower component of the smallest shell composition containing the two points. The other comes from the middle component. These points create a "vertical" direction with respect to which the left and right of the site are understood.

Proof. Suppose x belongs to R_+. Since a tree can only grow from the left or the right of v in order not to be intersected, the marker at v has to be on its left or right. Consider the room on the left of v. Exactly one answer "Yes" or "No" can be given to the following question: Is the input vertex of R adjacent to this room? We consider each of the answers.

Suppose the answer is "Yes". Since CR is clocked, the input vertex of R is split in CR such that a rooted tree can grow through it from left to right. Therefore, the room furthest from the root will be on the right of v, which is where the marker at v lies. See the left diagram of Figure 1.16.

Suppose the answer is "No". This implies that this room is separated from the room to which the input vertex is adjacent by one or more rooms. Each of these rooms is separated from the ones on its left and its right by sites created from interactions satisfying rule 2. Since CR is clocked, a rooted tree can only grow through these rooms by passing through the input site first (formed by splitting the input vertex) to enter the room on the right of the leftmost room through the site separating them. It continues to grow until it passes through v. As a result, the room furthest from the root is again on the right of v, which is where the marker at v lies. See the right diagram of Figure 1.16. In this figure, the boxes stand for trivial strings or riders with no interaction between them and the ellipses

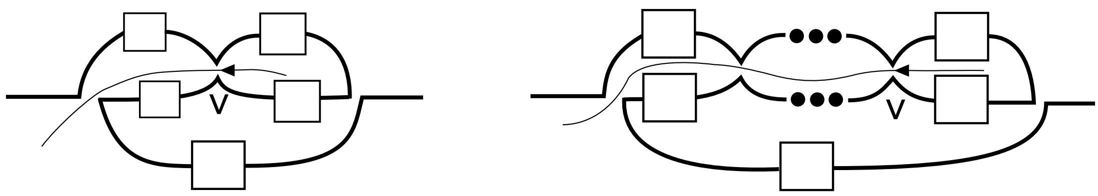


Figure 1.16

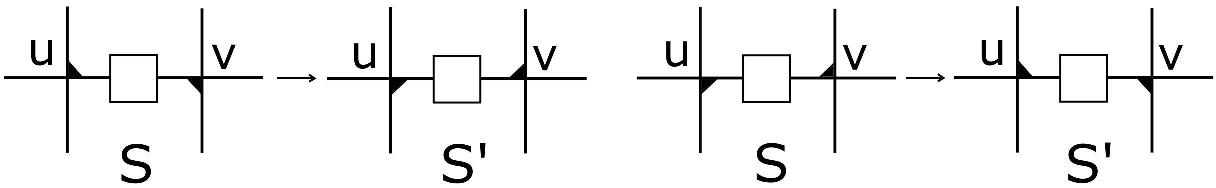
stand for trivial strings or riders with possibly some interactions between them.

Exactly the same arguments are used to prove that, if x belongs to R_{-}, then the marker at v will be on its left. □

1.4 Moving between states and trails of the same string

Now we present one of the crucial concepts in the clock theorem.

Definition 1.4.1 (Transposition)**.** Let S be a state of a string A. Let u and v be two vertices on the same edge or on the input and output edges of a rider of A. Suppose the markers at u and v are placed relative to each other as shown in the figure below. A *transposition* on S is a swap of the faces occupied by these two markers such that another state S' is produced as shown in the figure below where the boxes represent a rider or the trivial string. The transpositions on the left and the right are said to be *clockwise* and *counterclockwise*, respectively.



Definition 1.4.2 (Clocked and counterclocked states)**.** A state of a string A where no counterclockwise transposition can be performed is called a *clocked state*. Similarly, a state where no clockwise transposition can be performed is called a *counterclocked state.*

Coming next is one of the most important lemmas constituting the proof of the clock theorem.

Lemma 1.4.3. *The clocked and counterclocked trails of the dissection of a string correspond to a clocked and a counterclocked states respectively by the tree-growing procedure.*

Remark. In the following proof, we call a marker by the same name as the vertex or site at which it is placed, if this causes no confusion. Also, a site is sometimes treated as a

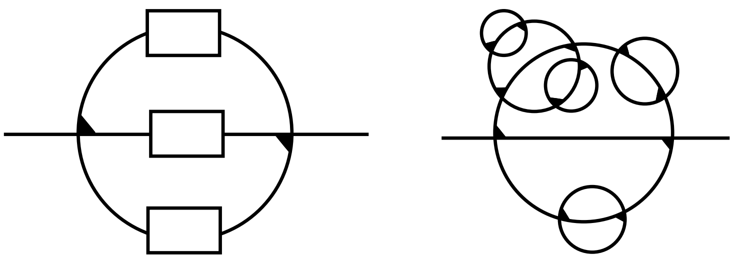


Figure 1.17

vertex as if it has been closed as long as no confusion arises. Also, the *outside* of a shell composition, whether it is a stand-alone string or a rider, refers to its two unbounded regions as if it is a stand-alone string.

Proof of Lemma (1.4.3). Let $K, K' : \mathscr{U} \to \mathscr{T}$ be maps defined by $K = C \circ \hat{D}$ and $K' = C' \circ \hat{D}$. The lemma above is equivalent to the fact that, for any string A, KA and $K'A$ correspond to a clocked and a counterclocked state of A, respectively. We first prove this fact for a shell composition A (without sites). Since A is a shell composition, $\hat{D}A = A$, by Lemma (1.3.3). Thus, $KA = CA$. The tree-growing procedure implies that the markers at the input and output vertices of any irreducible rider of A are placed as shown on the left of Figure 1.17, where the boxes represent riders or trivial strings. An example of the state of a shell composition produced by K is shown on the right of Figure 1.17. Since every vertex in a shell composition is the input or output vertex of some rider, Figure 1.17 depicts how the state of A corresponding to KA looks. It can be seen that every pair of vertices which is on the same edge or on the input and output edges of a rider of A and which can be transposed have markers in positions identical to those of the vertices u and v in state S on the left of the figure in Definition (1.4.1) (the leftmost state). This implies that a clockwise transposition is the only transposition possible for the state corresponding to KA. Hence, KA corresponds to a clocked state of A. The same arguments are used to prove that $K'A$ corresponds to a counterclocked state of A.

By Lemma (1.3.15), any string corresponds to an elaboration of a shell composition. Thus, if we can prove that the clocked and counterclocked trails of an elaboration correspond to a clocked and a counterclocked states of the string obtained by closing all its sites, then we are done. To avoid cumbersome repetition in the future, from now on the fact stated in the previous sentence is referred to as *closure invariance* of an elaboration. Proving closure invariance of an elaboration amounts to proving that an allowed interaction of two points in an elaboration having closure invariance produces an elaboration having closure invariance. Let E be an elaboration having closure invariance of a shell composition A. Let a and b be points in E such that their interaction is allowed. Suppose their interaction satisfies rule 2. Let R be the smallest shell composition containing a

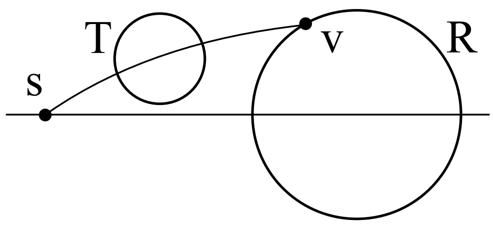


Figure 1.18

and b. Suppose in addition that a belongs to R_+. Hence, in CR, the marker at the site created from the interaction of $\{a, b\}$ (denoted v) is on the right of v, by Lemma (1.3.19). Any counterclockwise transposition involving this marker will rotate it to a room outside R which is either occupied by a star (in the case where R is the rider riding the trivial string to make the whole E, that is, R is the "biggest" rider of E) or by a marker that cannot be coupled with v in any transposition and therefore cannot rotate together with v to avoid occupying the same face with v. Here, we explain why this is true. It can be seen that Lemma (1.3.19) prevents the marker at any vertex inside R to form a counterclockwise transposition with v because they are not in the correct positions. Regarding a vertex outside R, it is obvious that a vertex outside a rider cannot belong to the same edge as a vertex inside the rider if the latter is not its input or output vertex (which is clearly the case of v). Thus, because of Definition (1.4.1), we only have to show that v and some vertex outside R, say s, cannot be on the input and output edges of some rider, say T. Suppose that they are. It can be seen that any path from a vertex inside a rider to a vertex outside it must go through the input or the output vertex of the rider, that is, any such path must contain the input or the output vertex. Hence, any path l from v to s must go through the input or the output vertex of R. However, since v and s are also on the input and output edges of T, there is another path l' from v to s such that l' goes through the input and the output vertices of T. Since s is outside R, T is outside R. Therefore, l' does not go through the input or the output vertex of R, which leads to a contradiction. Figure 1.18 helps visualise these arguments. As a result, no counterclockwise transposition involving v is possible. The same arguments apply when a belongs to R_- and when $C'R$ is considered instead of CR. As a result, the elaboration E' obtained from E by the interaction of $\{a, b\}$ has closure invariance.

Now suppose that the interaction of a and b satisfies rule 1. Let R be the smallest shell composition containing a and b. Suppose in addition that a and b belong to R_0. From the tree-growing procedure, in CR, the marker at the site created by the interaction of $\{a, b\}$ (denoted u) lies inside the room created by this interaction. Closing u creates a curl or a curl ridden by a rider. Call this curl (or curl ridden by a rider) M. Any counterclockwise transposition involving the marker at u will rotate it to a room outside

M. Thus, a marker at some vertex, say w, in M (except u) has to move in to occupy the room left empty by the marker at u. Let us have a closer look at this marker. If M is just a curl, then w obviously does not exist. Hence, M is ridden by some rider, say N. Since E is an elaboration, N is either a curl or a curl ridden by a rider or a shell composition with sites. From the tree-growing procedure, the marker at any vertex of N lies inside N, which makes it unable to be coupled with u in any transposition and therefore unable to occupy the room left empty by the marker at u. As a result, the marker at u cannot move out, making a counterclockwise transposition involving it impossible. Since we did not use any intrinsic property of R_0 in our arguments, they apply to the cases where a and b belong to R_+ or R_-. They also apply if $C'R$ is considered instead of CR. Hence, the elaboration E' obtained from E by the interaction of $\{a, b\}$ has closure invariance.

Finally, suppose that the interaction of a and b satisfies rule 3. From the tree-growing procedure, in CE the marker at the site created by the interaction of $\{a, b\}$ (denoted w) lies inside the room created by this interaction. Closing w creates a curl or a curl ridden by a rider (denoted O). Any counterclockwise transposition involving the marker at w will rotate it to a room outside O which is occupied by a star because a and b belong to some connecting edges of E. Thus, the marker at w cannot move out, making a counterclockwise transposition involving it impossible. The same reasoning applies if $C'E$ is considered instead of CE. Hence, the elaboration of E' obtained from E by the interaction of $\{a, b\}$ has closure invariance, completing the proof. □

We now enter the next big part of the clock theorem, which basically declares that any two states of a string can be connected by a sequences of transpositions. Note the important fact that there is a one-to-one correspondence between the set of all states of a string and the set of all its trails. To find the corresponding trail of a state, we split the vertices of the state according to their markers. To find the corresponding state of a trail, we use the tree-growing procedure to put a marker at each site of the trail and close all its sites. This one-to-one correspondence is proved in the next lemmas.

Lemma 1.4.4. *Let S be a state of a string U. Split some vertices of S in a certain order to create a sequence of states of strings with sites. For every state in this sequence, there is a natural bijection from the set of vertices to the set of unstarred faces, that is, faces that do not have a star. This bijection assigns to each vertex the face containing its marker.*

Proof. Let S_0 be a state of a string U. Label k vertices of S_0 by $v_1, v_2, \ldots, v_k$ and the states of the strings with sites obtained from splitting them one by one in the same order by $S_1, S_2, \ldots, S_k$. Note that the S_i's are connected. Define f to be the map from the set V_{S_0} of vertices of S_0 to the set R_{S_0} of its unstarred faces which assigns to each element

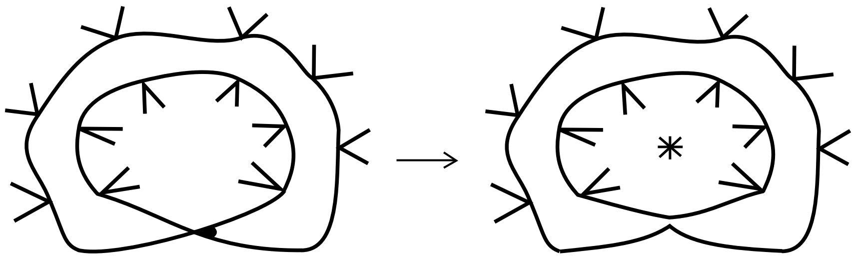


Figure 1.19

$v \in V_{S_0}$ the element of R_{S_0} that contains the marker at v (S_0 is a state so all its vertices have markers). By Definition (1.2.4), f is well defined and surjective. It is easy to see that V_{S_0} and R_{S_0} have the same cardinality. Thus, f is bijective. Our goal is to prove that f is still bijective when it maps from V_{S_i} to R_{S_i} where V_{S_i} and R_{S_i} are the set of vertices and the set of unstarred faces of S_i, respectively ($1 \leqslant i \leqslant k$). In other words, we prove by induction on i that f yields a bijective map from V_{S_i} to R_{S_i}. Since splitting a vertex v_i decreases the number of vertices of S_{i-1} by 1, that is, $|V_{S_i}| = |V_{S_{i-1}}| - 1$, to achieve this goal, it is sufficient to prove that whenever a vertex v_i is split to turn S_{i-1} into S_i, the number of unstarred faces decreases by 1, that is, $|R_{S_i}| = |R_{S_{i-1}}| - 1$. We divide the proof into 2 parts. Part 1 consists in proving that splitting a vertex v_i decreases the number of faces of S_{i-1} by 1. In part 2, we prove that it is the number of unstarred faces of S_{i-1} that decreases by 1 after splitting v_i, not the number of starred faces. In the rest of this proof, when we split a vertex, we do not delete its marker but leave it there (what we obtain is a "site with marker").

We prove part 1 by contradiction. Suppose that, there exists a vertex v_i such that, after v_i is split, the number of faces of S_{i-1} does not decrease and that v_i is the first vertex among the v_j's ($1 \leqslant j \leqslant k$) for which this event occurs. This implies that the face containing the marker at v_i is the same as the face opposite to it. Call this face M. There is then an arc in M from one corner of v_i to the opposite corner of v_i. After splitting v_i, this arc becomes a loop enclosing a nonempty plane graph. See Figure 1.19. Thus, splitting v_i disconnect S_{i-1}. This is a contradiction as the string S_i is assumed to be connected. As a result, every time a vertex v_i is split, the number of faces of S_{i-1} decreases by 1, completing part 1.

Proceeding by contradiction again to prove part 2. Suppose that there exists a vertex v_i such that, after splitting it, the number of unstarred faces of S_{i-1} does not decrease, that is, $|R_{S_{i-1}}| = |R_{S_i}|$. Hence, the number of starred faces must decrease by 1. This implies that two starred faces A and B of S_{i-1} are joined together in S_i after v_i is split. In S_{i-1}, since A is a starred face, the markers at the vertices and sites adjacent to A must point away from A as illustrated in Figure 1.20. Similarly, the markers at vertices and sites adjacent to each face and room adjacent to A must again point away from A.

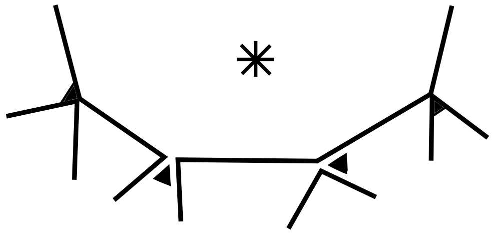


Figure 1.20

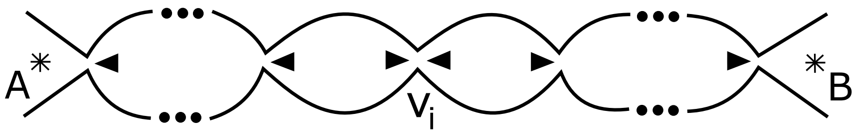


Figure 1.21

The same argument applies to B. In S_{i-1}, let A' and B' be the two rooms adjacent to v_i which belong to A and B, respectively. Since v_i is adjacent to the starred face A, by the argument above, the marker at v_i lies in B'. However, since v_i is also adjacent to the starred face B, the marker at v_i lies in A'. This leads to a contradiction because a vertex has only one marker which lies in only one face. Therefore, every time a vertex in a state of a string with sites is split, the number of unstarred faces decreases by 1, completing part 2. Figure 1.21 illustrates this proof. As a result, by induction, f is bijective when it maps from V_{S_i} to R_{S_i} for all $1 \leqslant i \leqslant k$. □

Lemma 1.4.5. *Let $\mathscr{S}_U$ be the set of states of a string U and $\mathscr{T}_U$ the set of trails of U. Splitting all the vertices of a state S of U gives a trail of U, that is, the map a on $\mathscr{S}_U$ defined by splitting all vertices of a state $S \in \mathscr{S}_U$ has codomain $\mathscr{T}_U$. We call $a : \mathscr{S}_U \to \mathscr{T}_U$ the* resolving map*.*

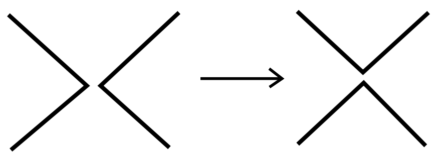
Proof. Let S be a state of a string U. By Lemma (1.4.4), splitting the vertices of S one by one only decreases the number of unstarred faces, not the number of starred faces. Therefore, since there are two starred faces in S, there are two starred faces in $a(S)$. Moreover, by the same lemma, there is a bijection from the set of vertices of $a(S)$ to the set of its unstarred faces. Since there is no vertex in $a(S)$ (all vertices in S have been split), there is no unstarred face in $a(S)$. Hence, there are only starred faces in $a(S)$ and more exactly two of them. Since all the starred faces in S are unbounded, all the (starred) faces in $a(S)$ are unbounded. Hence, there is no cycle in $a(S)$. This implies that the two (starred) faces in $a(S)$ are separated by one non-crossing paths (there is no crossing because all vertices have been split). As a result, $a(S)$ has two faces separated by one non-crossing paths and each of them has a star. This implies that $a(S)$ is a trail. □

Lemma 1.4.6. *For every string U, the resolving map $a : \mathscr{S}_U \to \mathscr{T}_U$ is a bijection and* *the tree-growing map $b : \mathscr{T}_U \to \mathscr{S}_U$ is its inverse, that is, $a \circ b = 1_{\mathscr{T}_U}$ and $b \circ a = 1_{\mathscr{S}_U}$ where $1_{\mathscr{T}_U}$ and $1_{\mathscr{S}_U}$ are the identity maps on $\mathscr{T}_U$ and $\mathscr{S}_U$, respectively.*

Proof. We remark that the vertices of a state do not have to be split in any specific order to obtain its corresponding trail. Let T be a trail of a string U and v be a site of T. The map b places a marker at v and closes it to create a vertex. The map a split this vertex back into a site according to the marker placed there by b. This recreates v. This reasoning applies to every site of T. Hence, $a \circ b = 1_{\mathscr{T}_U}$.

Let S be a state of U. Consider a starred face R of S. We repeat the same argument regarding a starred face as in part 2 of the proof of Lemma (1.4.4). The markers at the vertices adjacent to R must point away from R as illustrated in Figure 1.20 (however, in this case there is no site in S). Similarly, the markers at the vertices adjacent to each face adjacent to R must again point away from R. This pattern of how the markers are placed corresponds exactly to the way markers are placed at each vertex of a tree after a tree is grown according to the procedure described in Definition (1.3.17). Therefore, after a vertex u of S is split into a site by the map a according to its marker, the map b assigns a new marker to this site at the same position as the previous maker and closes it, recreating the vertex u with its marker appearing the way it was in S. This reasoning applies to every vertex of S. As a result, $b \circ a = 1_{\mathscr{S}_U}$. □

Definition 1.4.7 (Reassembly and exchange)**.** Let s be a site of a trail T. The *reassembly* of s is the process described as follows.



The reassemblies of two distinct sites of T are called an *exchange* on T if they turn the trail T into another trail.

It can be seen that reassembling only one site of a trail T breaks it into two disjoint parts. However, reassembling two arbitrary sites of T does not necessarily result in another trail. It may even break T into three parts. The reassemblies of two sites are an exchange only when they turn a trail to another one. There is a lemma telling us what an exchange must look like.

Definition 1.4.8 (Path)**.** Let a and b be two cusps belonging to some site(s) of a trail T. A point which starts from one cusp, moves along T towards the other cusp and stops at the other cusp traces a *path* connecting a and b. A path is said to connect two sites u and v if it connects two cusps u' and v' with u' belonging to u and v' belonging to v

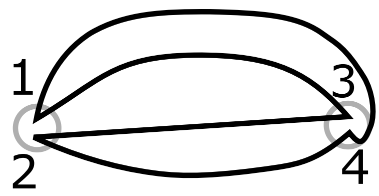


Figure 1.22

without passing through any other cusp of u or v. Note that u and v are not necessarily distinct (a path can connect a site to itself). Note that joining two paths connecting cusps, which intersect only at one cusp (i.e. have one common endpoint), yields another path connecting cusps. However, with this definition, joining two paths connecting sites does not necessarily yield a path connecting sites.

Lemma 1.4.9. *There exists a unique path connecting two distinct cusps and there exist at most three different paths connecting two distinct sites.*

Proof. Let T be a trail and let a and b be two distinct cusps on T. Since T is a trivial string with sites, it is homeomorphic to a straight line. Let f be a homeomorphism mapping T to a straight line ℓ. Thus, $f(a)$ and $f(b)$ are two distinct points on ℓ. It is obvious that there is a unique line segment connecting them. Therefore, the preimage of this line segment via f is the unique path connecting a and b.

We now prove the second claim. Let u and v be distinct sites of T. It can be seen that there are four pairs of cusps from u and v in which each cusp belongs to a different site. Since each pair has a unique path connecting them (by the first claim), there cannot be more that 4 different paths connecting u and v. Proceeding by contradiction, suppose there are 4 different paths connecting these sites. Label the cusps of u 1 and 2 and the cusps of v 3 and 4. Since there are 4 different paths connecting u and v, each of the pairs $\{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}$ has its own path which we label by the label of the pair. It can be seen that the pair $\{1,3\}$ is connected by the path (between cusps) $\{1,3\}$ and also by the path (between cusps) formed by joining the paths $\{1,4\}, \{4,2\}, \{2,3\}$. If these two paths are distinct, then this contradicts the first claim. If they aren't, then they form a cycle in T, contradicting the fact that T is a trail. Figure 1.22 illustrates this proof, where the gray circles mark the sites being considered. □

Since there is a unique path connecting two distinct cusps, from now on we label a path by the pair of cusps it connects.

Lemma 1.4.10. *The reassemblies of two sites are an exchange if and only if there are exactly three different paths connecting them.*

Proof. Let u and v be distinct sites of a trail T. Label the cusps of u 1 and 2 and the cusps of v 3 and 4. We first prove that, if the reassemblies of u and v are an exchange,

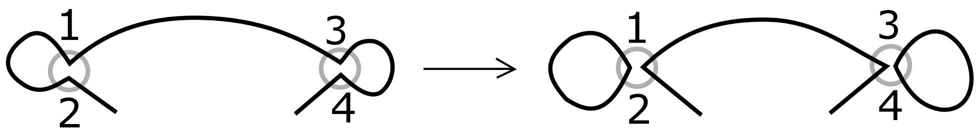


Figure 1.23

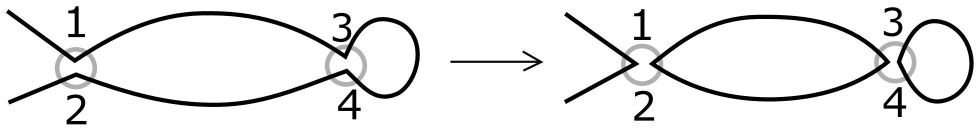


Figure 1.24

then there are exactly three different paths connecting them. We prove this by proving its contrapositive. Suppose that there are either one or two paths connecting u and v (there cannot be more than 3 paths connecting them by Lemma (1.4.9)). We consider these cases separately.

Case 1: Suppose that there is only one path connecting u and v, say $\{1, 3\}$. This implies that the paths $\{1, 4\}, \{2, 3\}, \{2, 4\}$ cannot exist. Therefore, there must be paths $\{1, 2\}, \{3, 4\}$ to guarantee that there is a unique path connecting each pair of cusps. It can be seen that the reassemblies of u and v break T into 3 disjoint parts, as illustrated in Figure 1.23. Therefore, they are not an exchange.

Case 2: Suppose that there are two distinct paths connecting u and v. Suppose further that each cusp appears in a path exactly once, say $\{1, 3\}$ and $\{2, 4\}$. This implies that the paths $\{2, 3\}$ and $\{1, 4\}$ cannot exist. Therefore, precisely one among the paths $\{1, 2\}$ and $\{3, 4\}$ must exist to guarantee that there is a unique path connecting each pair of cusps (after relabeling, these possibilities are really the same). It can be seen that the reassemblies of u and v break T into 3 disjoint parts, as illustrated in Figure 1.24. Therefore, they are not an exchange. Now, suppose that a cusp appears in both paths, say 1. Thus, the two paths connecting u and v are $\{1, 3\}$ and $\{1, 4\}$. This implies that the path $\{2, 3\}$ and $\{2, 4\}$ cannot exist. As a result, the cusp 2 is not connected to any of the remaining three cusps, which contradicts the fact that T is a trail.

The converse is almost obvious. Suppose that there are three different paths connecting u and v, say $\{1, 3\}, \{2, 4\}, \{2, 3\}$. This implies that the path $\{1, 4\}$ cannot exist. It can be seen that the reassemblies of u and v turn T to another trail T', as illustrated in Figure 1.25. Therefore, they are an exchange. □

It can be seen that an exchange looks very similar to the clocked and counterclocked trail of a shell. In fact, it is actually more complicated. Figure 1.25 gives us the impression that the two reassembled sites are in horizontal positions relative to each other. However, there is no reason why they should be so. In other words, an exchange can involve two

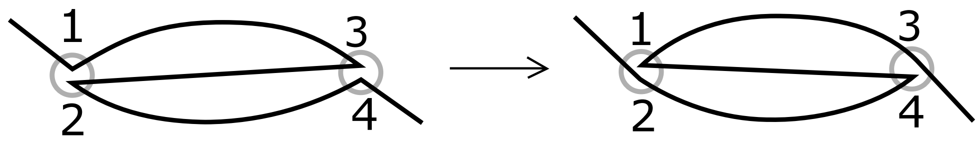


Figure 1.25

arbitrary sites, not necessarily the input and output sites of a shell (with sites) which we can view as horizontal. Moreover, we are considering an arbitrary trail. Inside it, there is no guarantee that we can find a trail of a shell composition. As a result, it is hard to label the three paths connecting the reassembled sites in an exchange in the same way as we label the three non-connecting edges of a shell. Fortunately, we can label one of them, which turns out to be the most important path among the three.

Definition 1.4.11 (Middle path, side path and middle string)**.** Let u and v be the reassembled sites in an exchange E on a trail T. A point tracing T from its input edge to its output edge goes through the three paths connecting u and v successively in a unique order. The second path in this order is called the *middle path* of E. The two other paths are called the *side paths* of E. The string obtained by closing all sites whose both cusps belong to the middle path is called the *middle string* of E.

Lemma 1.4.12. *If T and T' are two different trails of a string, then there is a sequence of exchanges turning T into T' such that each reassembled site in this sequence belongs to a unique exchange in the sequence.*

Proof. Since T and T' are different trails of the same string, there is a natural one-to-one correspondence between the sites in T and the sites in T'. Because of this bijection, when we refer to a site, we do not have to specify which trail it belongs to. The only way a site in T' can differ from its corresponding site in T is that the former results from the reassembly of the latter. Thus, T' is obtained from T by reassembling all the sites at which they differ. Moreover, the number of sites at which they differ is even because reassembling an odd number of sites in a trail will break it into disjoint parts. Therefore, we can pair up these sites in such a way that each pair forms an exchange upon being reassembled. Here is the reason why this pairing is possible. Let x be a site at which T and T' differ. We claim that, in T, there is at least one site at which T and T' differ and which can make an exchange with x. The proof of this claim is as follows. In T, it can be seen that x is represented (without loss of generality) by the left diagram of Figure 1.26. In fact, any site in any trail can be represented by this diagram. Let α be the path connecting the cusps of x. Suppose there is no site y distinct from x such that T and T' differ at y and that y has at least one cusp belong to α. Hence, reassembling x will break the trail into two disjoint parts and there is no way to glue these two parts together by

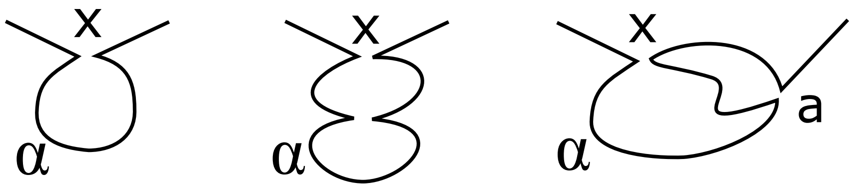


Figure 1.26

reassembling another site at which T and T' differ. This leads to a contradiction because reassembling all sites at which T and T' differ produces T' from T (we allow objects that are not trails to be produced while reassembling these sites). Therefore, y exists. Suppose that all sites having the same properties as y have both cusps belong to α (middle diagram of Figure 1.26). Thus, reassembling x and any of these sites will still break the trail and there is no way to glue all the disjoint parts together by reassembling other sites at which T and T' differ. Again, this leads to a contradiction with the same reason as the previous contradiction. As a result, there is a site a whose one cusps belongs to α and the other does not and at which T and T' differ (right diagram of Figure 1.26). This implies that there are three paths in T connecting x and a. Thus, by Lemma (1.4.10), the reassemblies of x and a are an exchange, completing the proof of the claim. Let T_1 be the trail obtained from T by the exchange of x and a. We are now faced with two trails T_1 and T' which differ at a number of sites fewer than the number of sites at which T and T' differ by 2. In other words, we have "pushed" T two sites closer to T' by exchanging x and a. By repeating the same arguments as above, we can "push" T_1 closer to T' by finding two sites at which they differ and whose reassemblies are an exchange and exchanging them. When we finally reach T', we have created a sequence of exchanges turning T into T' by pairing up the sites at which they differ. Each of these sites belongs to a unique exchange in the sequence. □

Definition 1.4.13 (Clockwise and counterclockwise exchanges)**.** Let E be an exchange turning a trail T into a trail T'. Let u and v be the reassembled sites in E. By the tree-growing procedure, the markers at u and v are placed inside the two unions of rooms bounded by the middle path and the two side paths connecting them. In T, rotating these markers 90° either clockwise or counterclockwise will give the positions of the corresponding markers in T'. If the rotation is clockwise, E is called a *clockwise exchange*. If the rotation is counterclockwise, E is called a *counterclockwise exchange*.

The definition above provides a way to classify exchanges. Strictly speaking, we need to prove that an exchange is either clockwise or counterclockwise but not both. However, the proof is so straightforward that we can omit it. In addition, it is easy to see that, if a trail T is turned into a trail T' by a clockwise (counterclockwise) exchange, then T' can be turned back to T by the corresponding counterclockwise (clockwise) exchange. In other words, each clockwise exchange has a counterclockwise exchange which undoes what the former does and vice versa.

Lemma 1.4.14. *If a clockwise (counterclockwise) exchange E turns a trail T into a trail T', then there is a sequence of clockwise (counterclockwise) transpositions turning the state S corresponding to T into the state S' corresponding to T'. In other words, E* factors *into a sequence of clockwise (counterclockwise) transpositions.*

The following proof proves stronger results than what is stated in the lemma. It shows precisely the markers at which sites rotate in the sequence of transpositions turning S to S' and how much in total these markers rotate through the sequence. Kauffman also gave a proof of this lemma but many details were glossed over and many cases were omitted.

Proof. In this proof, sites and vertices are used interchangeably because we will only deal with trails, not strings with sites. Moreover, "a site of a rider/string/path" means a site whose both cusps belong to the rider/string/path. Likewise, "a vertex of a rider/string/path" means a vertex obtained from closing a site whose both cusps belong to the rider/string/path. Since riders are defined using strings, not trails, a rider of a path means a rider of the string formed by closing all sites whose both cusps belong to that path. A site whose one cusp belongs to the middle path of an exchange (see Definition (1.4.11)) and the other belongs to a side path of the exchange is called *an interaction between the middle path and a side path* (see Definition (1.3.10)). A site whose both cusps belong to the middle path of an exchange is called *a self-interaction of the middle path.* A site whose both cusps belong to a side path is called *a self-interaction of a side path.* In other words, we essentially consider the portion of the trail T involved in an exchange as a string with sites, by closing all sites in T both of whose cusps lie in this portion. This string then has its own riders and interactions.

Note that removing a site from a trail gives a unique trail although the latter is not a trail of the same string as the former. Therefore, we can prove this lemma by strong induction on the number of sites in a trail (the number of vertices in a string). The induction hypothesis is as follows. Kauffman had a similar list but concepts such as involved and uninvolved riders were not carefully defined, which led to some loopholes and some cases not being covered.

Induction hypothesis: The clockwise exchange E factors into a sequence of clockwise transpositions having the following properties.

1. The markers at the two exchange sites (the sites reassembled in E) rotate clockwise by 90°.

2. Markers at interactions between the middle path and a side path rotate 180° clockwise.

3. If A is an atomic rider of the middle string (see Definition (1.4.11)) and there is a site whose one cusp is on A and the other is on a side path, then the marker at this site rotates 180° whereas the markers at sites whose both cusps are on A rotate 360°. An atomic rider is said to be involved if the marker at every site (vertex) of the rider rotates in the factorisation of E. It is said to be *uninvolved* if none of the markers at its sites (vertex) and cusp rotates.

4. If A is a rider of the middle string (not necessarily atomic) and there is no site whose one cusp is on A and the other is on a side path, the markers at all sites of A do not rotate.

5. If A is an involved atomic rider of B, where B is an irreducible rider of the middle string, then the markers at all sites of the core of B rotate 360°. An irreducible rider is said to be *involved* if the marker at every site (vertex) of the core of the rider rotates in the factorisation of E. (The *core* of an irreducible string was defined in Lemma (1.2.11) and the same definition applies to riders when regarded as strings.) An irreducible rider is said to be *uninvolved* if none of the markers at the sites (vertices) of its core rotates. Note that this generalises the notion of an atomic rider being involved or uninvolved from property 3 above, since the core of an atomic rider is itself.

6. Markers at self-interactions of a side path do not rotate.

We observe that these six properties predict precisely which sites and riders of the middle string will be involved in the factorisation. Indeed, consider the tree T of riders of the middle string M, as discussed in Lemma (1.2.11). The vertices of T correspond bijectively to the irreducible riders of M. We can associate to a vertex of T all the sites in the core of the corresponding irreducible rider of M. Each site of M is then associated with a unique vertex of T. Each interaction of the midline string with a side path lies in a smallest irreducible rider R of M, corresponding to a vertex v of T. Property 3 then says that R is involved, and property 5 says that the irreducible riders corresponding to the ancestors of v in T are involved. Property 4 says that any vertex not of this type is uninvolved. The properties taken together thus say that every irreducible rider is involved or uninvolved. Precisely, an irreducible rider is involved if and only if it contains an interaction with a side path.

For the base case, it can be seen that the smallest trail where an exchange can be performed is the trail of a shell. The only clockwise exchange on a trail of a shell is the one turning a shell's clocked trail into its counterclocked trail because these two trails are the only trails of a shell. The reassembled sites in this exchange are the input and output sites. It is easy to see that the markers at these sites can be coupled in a single transposition turning the clocked trail into the counterclocked trail. Hence, the base case holds trivially. If we then add self-interactions of side paths, the two markers at the reassembled sites of the exchange can still be coupled in a single transposition. The markers at the sites of the self-interactions do not move, in accordance with property 6.

An exchange is said to be *simple* if there are neither interactions between the middle path and a side path nor self-interactions of the middle path (self-interactions of a side path may exist). In other words, a simple exchange on a trail is the equivalent of a transposition on a state where there is no rider between the vertices whose markers rotate in the transposition. Thus, any exchange can be obtained from a simple exchange by first adding interactions between the middle path and a side path; then adding self-interactions of the middle path.

Now we come to the main part of the proof. Throughout the remainder of this proof, to avoid using the word "marker" repeatedly, we will call a marker by the same name as the site at which it is placed if this causes no confusion. In addition, two markers are said to be *coupled* if a transposition can be performed on them. Let E_1 be an exchange on a trail T_1 and let T_2 be the trail obtained by adding a site s to T_1. Let E_2 be the exchange on T_2 with the same exchange sites as E_1. Consider the following cases.

Case 1: Suppose s is an interaction between the middle path and a side path of E_2. Suppose in addition that there is no self-interaction of the middle path of E_1, that is, every cusp on the middle path comes from a site where the other cusp is on a side path. Label the two sites adjacent to s on the middle path by a and b. Since the induction hypothesis holds for T_1, a and b have property 2, which means they rotate 180°. Since a and b are joined by one single edge in T_1, they make one unique transposition in the sequence of transpositions which E_1 factors into. Thus, when s is inserted, the transposition of a and b in T_1 induces two transpositions in T_2. One involves s and b. The other involves a and s. These two transpositions rotate s 180° clockwise. Since the transposition of a and b in T_1 is unique, no more rotation can be done on s. Hence, s satisfies property 2. The sequence of transpositions which E_2 factors into consists of all transpositions in the factorisation of E_1 except the transposition of a and b which is replaced by the transposition of s and b and that of a and s. As a result, the induction hypothesis holds for T_2. This case is illustrated in Figure 1.27. The exchange sites are enclosed in grey circles. The $+$ and $-$ symbols are used to label the side paths (arbitrarily). The ellipses stand for other cusps which may be present on the three paths, including self-interactions of side paths. The numbered arrows represent rotations. The order in which the rotations occur is indicated by the number on each arrow. Two arrows having the same number represent a transposition. When a number appears only once, this means the marker whose rotation is represented

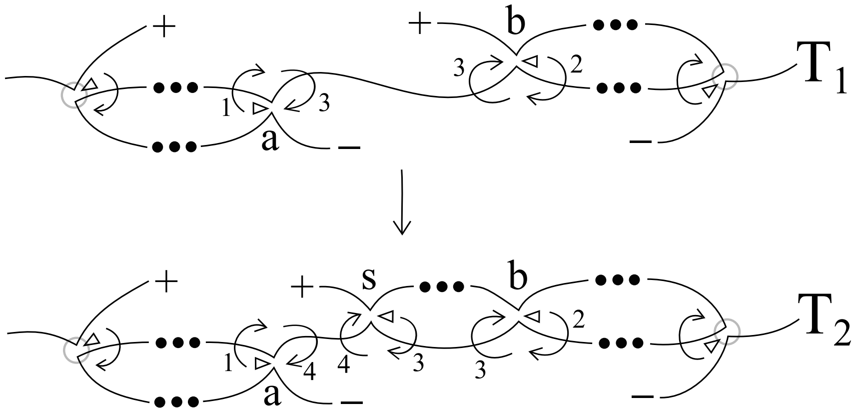


Figure 1.27

by that number is coupled with some unknown marker. In Figure 1.27, each of a and b rotates 180° (property 2). In T_1, they make transposition 3 which induces transpositions 3 and 4 in T_2.

We have proved in Case 1 that the lemma holds for every exchange in which each interaction is a self-interaction of a side path, or an interaction between the middle path and a side path. Therefore, to prove the whole lemma, it suffices to only consider adding self-interactions of the middle path to the exchanges discussed in Case 1. It can be observed in the subsequent cases that as self-interactions of the middle path are added, any marker at a self-interaction of a side path never moves, preserving property 6.

Case 2: Suppose s is a self-interaction of the middle path of E_2. Consider the following sub-cases.

Case 2.1: Suppose that, before s is inserted, all sites adjacent to s rotate in the factorisation of E_1. There are at least two and at most four sites adjacent to s. We consider each number of sites adjacent to s in the following sub-cases.

Case 2.1.1: Suppose there are two sites adjacent to s called a and b. Then, one of the following two situations happens.

- The first one is shown in Figure 1.28 where the addition of s creates a curl on the edge joining a and b in T_1. In T_1, while rotating, a and b make a unique transposition. In T_2, they still make the same transposition because the curl created by s is just a rider on the edge joining them. Hence, s does not rotate. This agrees with property 4 of the induction hypothesis which therefore holds for T_2.

- The second situation is shown in Figure 1.29. Since b is a self-interaction of the middle path, it rotates 360° (property 3). The site a can be a self-interaction of the middle path or an interaction between the middle path and a side path. Thus, it rotates 360° or 180° (property 3). Regardless of how much a rotates, a and b make precisely two transpositions, namely 1 and 4 in T_1. Transposition 1 in T_1 induces

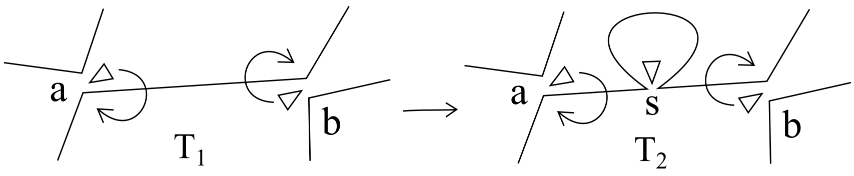


Figure 1.28

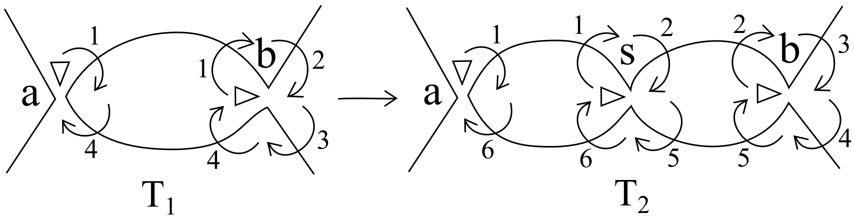


Figure 1.29

transpositions 1 and 2 in T_2, which rotate s 180°. Transposition 4 in T_1 induces transpositions 5 and 6 in T_2, which rotate s another 180°. Therefore, s rotates 360° in total, which agrees with property 3. As a result, the induction hypothesis holds for T_2.

Case 2.1.2: Suppose there are three sites adjacent to s called a, b and c as shown in Figure 1.30. The unique transposition of a and b in T_1, namely 1, induces transpositions 1 and 2 in T_2, which rotate s 180°. The unique transposition of a and c in T_1, namely 2, induces transposition 3 and 4 in T_2, which rotate s another 180°. Thus, s rotates 360° in total, which agrees with property 3. Therefore, the induction hypothesis holds for T_2.

Case 2.1.3: Suppose there are four sites adjacent to s called a, b, c and d as shown in Figure 1.31. Assume further that, in T_1 where s is absent, a is adjacent to b and c is adjacent to d. The unique transposition of a and b in T_1, namely 9, induces transpositions 7 and 10 in T_2, which rotate s 180°. The unique transposition of c and d in T_1, namely 5, induces transpositions 5 and 6 in T_2, which rotate s another 180°. Hence, s rotates 360° in total, which agrees with property 3. This implies the induction hypothesis holds for T_2. Note that the positions of the markers and the order of their transpositions

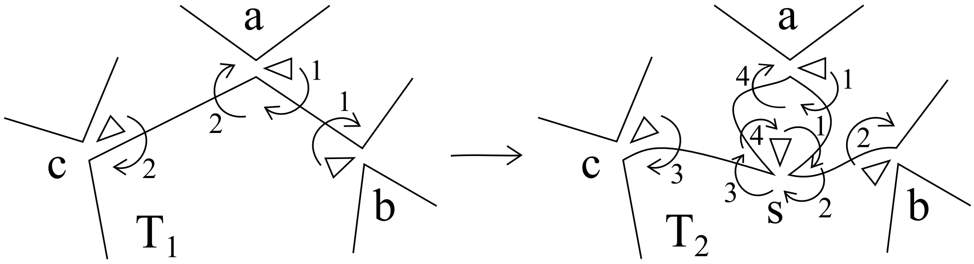


Figure 1.30

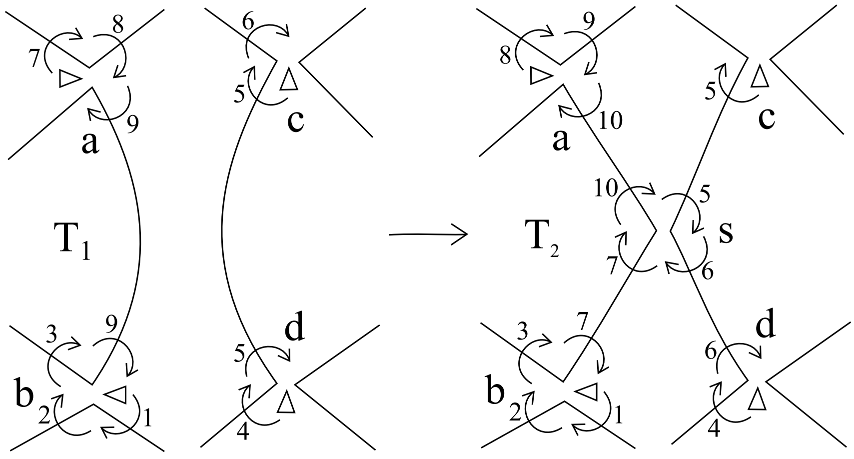


Figure 1.31

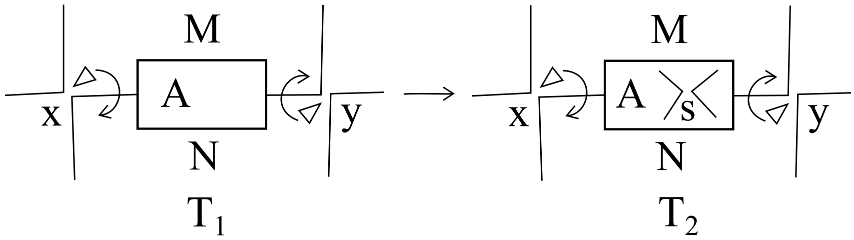


Figure 1.32

in Figure 1.31 are for illustrative purposes only. Not every instance of this subcase have the exact same positions of the markers and the exact same order of their transpositions. What is always true is that the transposition of a and b in T_1 and the transposition of c and d in T_1 cannot happen consecutively because, before s is inserted, a, b, c and d are adjacent to the same room which can be occupied by only 1 marker. Both a and b must move out first before either c or d moves in.

Case 2.2: Suppose that, before s is inserted, at least one site adjacent to s does not rotate.

Case 2.2.1: Suppose that, before s is inserted, both cusps of s are on one single uninvolved atomic rider, denoted A. A necessary condition for the transposition of two sites x and y to happen is that both of them are adjacent to two rooms, say M and N. See Figure 1.32. Since A is uninvolved before the insertion of s, it rides on the path connecting x and y. Since both cusps of s are on A, both x and y are still adjacent to both M and N after s is inserted. Hence, A remains uninvolved after the insertion of s. Therefore, s does not rotate in the factorisation of E_2, which agrees with property 4. As a result, the induction hypothesis holds for T_2. Note that, in Figure 1.32, there can be other riders riding on the path connecting x and y and A may be a rider of a bigger rider riding on this path.

For the remaining cases 2.2.2–2.2.4, we will consider the riders involved when we insert the self-interaction s. Let the cusps of s be s_1, s_2. These cusps can be regarded as separate cusps on E_1. On E_2, the two cusps s_1, s_2 are joined together into the site s. On E_1, the cusps s_1, s_2 respectively lie on smallest irreducible riders A_1, A_2. The riders A_1, A_2 may be the same or different. By induction, we have a factorisation of E_1 into transposition in which each of A_1 and A_2 is either involved or uninvolved, obeying properties 1-6. On E_2, the two cusps are joined into s, and s lies in a smallest irreducible rider B. This rider B will contain all the sites of A_1 and A_2 (and possibly more). Although we do not yet have a factorisation of E_2 into transpositions (indeed, this is what we must prove), properties 1-6 predict which irreducible riders of E_2 are involved or uninvolved: as we discussed earlier, the irreducible riders of the midline trail should be precisely those containing an interaction with a side path. Thus we can say that B "should" be involved or not. However we must prove that there is a factorisation of E_2 into transpositions obeying properties 1-6, under which B is accordingly actually involved or not.

Case 2.2.2: Suppose that, using the terminology of the previous paragraph, the irreducible riders A_1, A_2 containing the cusps of s before s is inserted are both uninvolved. (Note the riders A_1, A_2 may be the same or different.) Suppose also that, after s is inserted, the irreducible rider B containing s should also be uninvolved. (Note this case overlaps with case 2.2.1 but neither contains the other.) Then as none of A_1, A_2 or B contain an interaction with a side path, the factorisation of E_1 only involves sites outside A_1 and A_2 (and any other rider that becomes part of B). This provides a factorisation of E_2 in which no site of B rotates, including s. If A_1 and A_2 are the same, then B contains $A_1 = A_2$ and the new site s inserted. Thus, B satisfies property 4. If A_1 and A_2 are different, then A_1 and A_2 are merged into B due to the insertion of s. Hence, B again satisfies property 4. As a result, the induction hypothesis holds for T_2.

Case 2.2.3: Suppose that, using the same terminology again, the irreducible riders A_1 and A_2 containing the cusps s_1, s_2 of s before s is inserted are distinct and uninvolved. Suppose also that, after s is inserted, the irreducible rider B containing s should become involved (so B contains an interaction with a side path). See Figures 1.33 and 1.34. Since A_1 is uninvolved in the absence of s, there exist two site x and y such that A_1 rides on the path connecting them and that a transposition can be performed on them in T_a where T_a is a trail of the same string as T_1 (remember that T_1 and T_2 are trails of different strings due to the insertion of s). Note that T_1 and T_a are not necessarily the same because we do not know if the transposition of x and y can be done immediately in T_1; it is possible that other transpositions must be done first so that the markers at x and y are in the correct positions for a transposition to be done on them. This implies that an exchange F_1 can also be done on x and y in T_a. Thus, by Lemma (1.4.10), there are three paths connecting x and y in T_a and therefore in T_2. Hence, a corresponding exchange F_2

can also be done on x and y in T_2. Since A_1 is uninvolved in T_a but involved in T_2, it rides the middle path of F_2 and s is an interaction between this middle path and a side path of F_2. This is the situation described in Case 1. Therefore, by the result of Case 1, the exchange F_2 factors into a sequence of clockwise transpositions, in which s rotates 180° and every site of A_1 rotates 360° (if it is in the core of A_1, or in the core of an irreducible rider of A_1 containing s_1) or 0° (if not). Similarly, since A_2 is uninvolved in the absence of s, there exist two sites p and q such that A_2 rides on the path connecting them and that a transposition can be performed on them in T_b where T_b is a trail of the same string as T_1. This implies that an exchange G_1 can also be done on p and q in T_b. Thus, there are three paths connecting p and q in T_b and therefore in T_2. Hence, a corresponding exchange G_2 can also be done on p and q in T_2. Since A_2 is uninvolved in T_b but involved in T_2, it rides the middle path of G_2 and s is an interaction between this middle path and a side path of G_2. Again, this is the situation described in Case 1. Therefore, by the result of Case 1, the exchange G_2 factors into a sequence of clockwise transpositions, in which s rotates another 180° and every site of A_2 rotates 360° (if it is in the core of A_2, or in the core of an irreducible rider of A_2 containing s_2) or 0° (if not). The two 180° rotations due to F_2 and G_2 combine to give s a 360° rotation, which agrees with property 3. As a result, the exchange E_2 factors into a sequence of transpositions, which can be obtained from the sequence of transpositions factorising E_1 by replacing the exchange F_1 (which was a transposition in the factorisation of E_1) with the sequence of transpositions factorising F_2, and replacing the exchange G_1 (which was also a transposition in the factorisation of E_1) with the sequence of transpositions factorising G_2. In this sequence of transpositions, s rotates 360°, each site of A_1 and A_2 rotates 360° or 0° (accordingly as it lies in the core of A_1 or A_2 or an irreducible rider thereof containing s, or not), and each other site rotates as it did in the factorisation of E_1. Now B contains an interaction with a side path, and any irreducible rider of B either lies inside A_1, or inside A_2, or outside both A_1 and A_2. Thus, the irreducible riders of B not containing an interaction with a side path are precisely those which had that property in E_1 and remain unchanged, together with those irreducible riders of A_1 or A_2 not containing s_1 or s_2. It follows that B is involved, as it should be, and the rotation of markers in the factorisation of E_2 follows properties 1-6. Thus, the induction hypothesis holds for T_2.

Case 2.2.4: Using the same terminology, suppose that one of the irreducible riders A_1 (the smallest irreducible rider of E_1 containing the cusp s_1 of s, before s is inserted) is uninvolved, while the other irreducible rider A_2 (the smallest irreducible rider of E_1 containing the cusp s_2 of s, before s is inserted) is involved. Since A_2 is involved, A_2 contains an interaction with a side path, and after inserting s, the merged rider B (the smallest irreducible rider of E_2 containing s) also contains an interaction with a

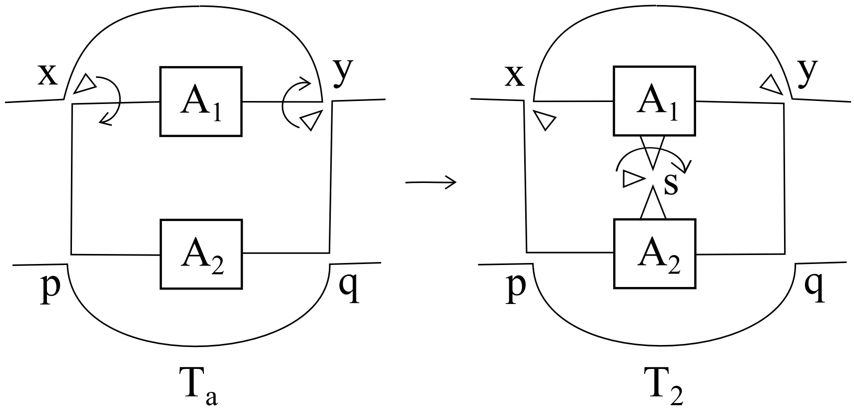


Figure 1.33

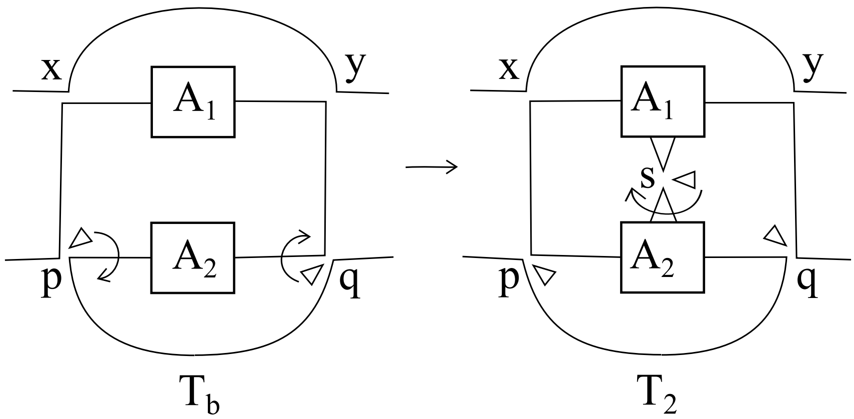


Figure 1.34

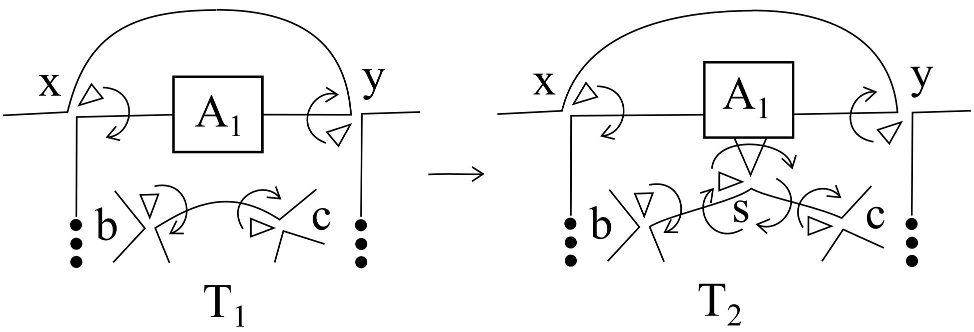


Figure 1.35

side path. Thus B should be involved. As to A_1, we have the same situation as the uninvolved atomic rider A_1 described in Case 2.2.3. Thus, by a similar argument, we arrive at the conclusion that the factorisation of E_1 involves a transposition at two sites x and y, such that A_1 rides on the path connecting x and y, and that in E_2 this becomes a clockwise exchange which factorises into a sequence of clockwise transpositions, under which s is rotated 180°, and the sites of the core of A_1 rotate 360°, as do sites in the core of irreducible riders of A_1 containing s. See Figure 1.35. Observe that each cusp of s is adjacent to at most two sites. Suppose that the cusp s_2 of s on A_2 is adjacent to only one site of A_2, denoted a. Then, in T_1 where s is absent, there is an edge connecting a to itself, that is, there is a loop at a. Hence, the marker at a always occupies the room enclosed by the loop. This contradicts the fact that the marker at a rotates 360° because A_2 is involved. Therefore, there are exactly two sites of A_2 adjacent to s_2, denoted b and c. Since A_2 is involved, the markers at b and c rotate 180° (if they are interactions with a side path) or 360° (if they are self-interactions of the middle path). Whatever the case, they make a unique transposition across the edge containing s_2. This transposition induce a unique transposition involving b and s and a unique transposition involving s and c. These two transpositions rotate s 180°. Combining this rotation with the 180° rotation caused by the exchange of x and y gives s a 360° rotation, which agrees with property 3. As a result, the exchange E_2 factors into a sequence of transpositions, which can be obtained from the sequence of transpositions factorising E_1 by replacing the transposition involving (b, c) with transpositions involving (b, s) and (s, c), and replacing the exchange around A_1 with the sequence of transpositions factorising that exchange in E_2. As in Case 2.2.3, it follows that B is involved, as it should be, and the rotation of markers follows properties 1-6. Thus, the induction hypothesis holds for T_2. □

We finally arrive at the last part in the proof of the clock theorem, which is about the uniqueness of the clocked and counterclocked states. Before proving it, the following lemma is needed.

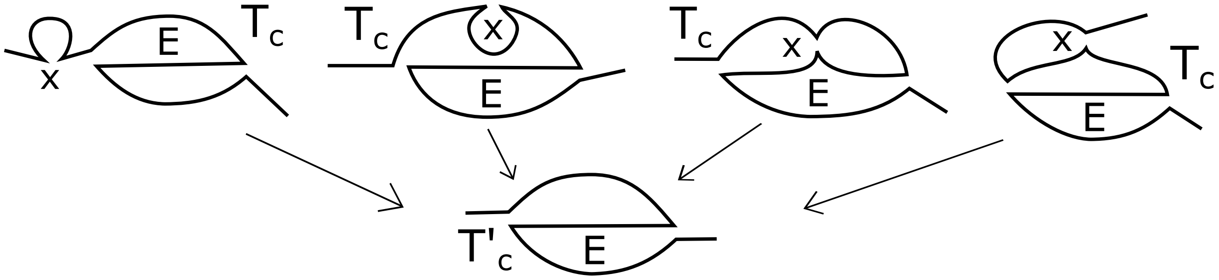


Figure 1.36

Lemma 1.4.15. *Removing a site of a clocked (counterclocked) trail gives a clocked (counterclocked) trail.*

Proof. Let T_c be a clocked trail of a string A, let x be a site of T_c and let T'_c be the trail obtained after removing x from T_c (thus, T'_c is a trail of the string obtained from A by removing the vertex corresponding to x). If x is the only site of T_c, that is, A is a curl, then removing x produces the trail of a trivial string, which makes the lemma hold trivially. If T_c has only 2 sites including x, then removing x gives the trail of a curl, which is obviously clocked (and counterclocked). Thus, we can assume that T_c has more than 2 sites. If no exchange can be performed on T_c, then T_c is the unique trail of A. Hence, removing x gives the unique trail of some string, which is both clocked and counterclocked, making the lemma true. Therefore, we can assume that at least one exchange which does not involve x can be carried out on T_c. Let E be such an exchange. Since T_c is clocked, E is clockwise. If both cusps of x do not belong to either the middle path or any of the side paths of E (leftmost diagram in Figure 1.36), then it is easy to see that removing x does not turn E into a counterclockwise exchange on T'_c. If both cusps of x belong to the middle path or one of the side paths of E (second diagram from the left in Figure 1.36), then E still remains clockwise in T'_c after x is removed. If one cusp of x belongs to the middle path and the other belongs to a side path of E (second diagram from the right in Figure 1.36), then, again, E is still clockwise on T'_c. If a cusp of x belongs to a side path and the other does not belong to any path of E (rightmost diagram in Figure 1.36), then removing x still keeps E clockwise in T'_c. Since we have covered all possible positions x can have relative to E, an arbitrary exchange on T_c, we have shown that removing x does not affect the nature (clockwise or counterclockwise) of any possible exchange on T_c whose both reassembled sites are not x. As a result, since T_c is clocked, T'_c is also clocked. The same arguments apply in the case where T_c is a counterclocked trail. □

Lemma 1.4.16. *The clocked and counterclocked states of a string exist and are unique.*

Proof. We prove this lemma by induction on the number of vertices n of the string. By Lemma (1.4.3) a clocked and a counterclocked states exist. They are given by the clocked and counterclocked trails of the dissection of a string. Therefore, it only remains to prove uniqueness. The base case ($n = 0$) is obvious. The trivial string has only one state which is both clocked and counterclocked. Assume the result holds when $n = k$. We prove the case $n = k + 1$ by contradiction. Let A be a string having $k + 1$ vertices. Suppose that A has at least two clocked states S_1 and S_2. Let T_1 and T_2 be the two clocked trails corresponding to S_1 and S_2, respectively. From the proof of Lemma (1.4.12), T_1 and T_2 differ at an even number of sites. Let x be a site at which T_1 and T_2 do not differ. Note that x exists because of the following reason. If it does not, then T_1 and T_2 differ at every site. By the argument of Lemma (1.4.12), we can find two reassemblies of sites of T_1 which form an exchange E. This exchange E on T_1 turns its sites into their corresponding sites in T_2. Then, we can perform an exchange E' involving these sites on T_2 to turn them back to how they appeared in T_1. Since T_1 is clocked, E is clockwise. Therefore, E' is counterclockwise, which is a contradiction because T_2 is clocked. As a result, x exists. Since T_1 is clocked, removing x from T_1 gives a clocked trail T_1' of the string A' obtained by removing the vertex corresponding to x from A (by Lemma (1.4.15)). Similarly, removing x from T_2 gives a clocked trail T_2' of the same string A'. Since T_1 and T_2 differ at an even number of sites and x is a site at which they do not differ, T_1' and T_2' are different clocked trails of A' which is a string having k vertices. This contradicts the inductive assumption that the result holds when $n = k$. Therefore, A has a unique clocked state. Repeat the same arguments to show that A has a unique counterclocked state. As a result, the claim holds when $n = k + 1$, completing the inductive step.

There is a subtlety worth noticing about the fact that x is a site at which T_1 and T_2 do not differ. If we remove from them a site at which they differ, then the two trails obtained will not be the trails of the same string. Therefore, a contradiction cannot be drawn. □

Lemma (1.4.16) is practically the end of all we need to prove the clock theorem. We restate it here.

Theorem 1.4.17 (The clock theorem on a plane). *Let S be a string and let $\mathscr{S}$ be the set of all states of S. Define a partial order on $\mathscr{S}$ as follows. For all $A, B \in \mathscr{S}$, $A \leqslant B$ if there exists a sequence of clockwise transpositions turning B to A. Then, $\mathscr{S}$ equipped with this partial order is a lattice, called the* clock lattice *of S.*

Proof. Lemma (1.4.12) shows that any two trails are connected by a sequence of exchanges, and Lemma (1.4.14) shows that any exchange factors into a sequence of transpositions. Thus any two trails can be connected by a sequence of transpositions. Moreover there is a greatest element given by the clocked state (due to its uniqueness, see Lemma (1.4.16)), and a least element given by the counterclocked state. Thus any two elements

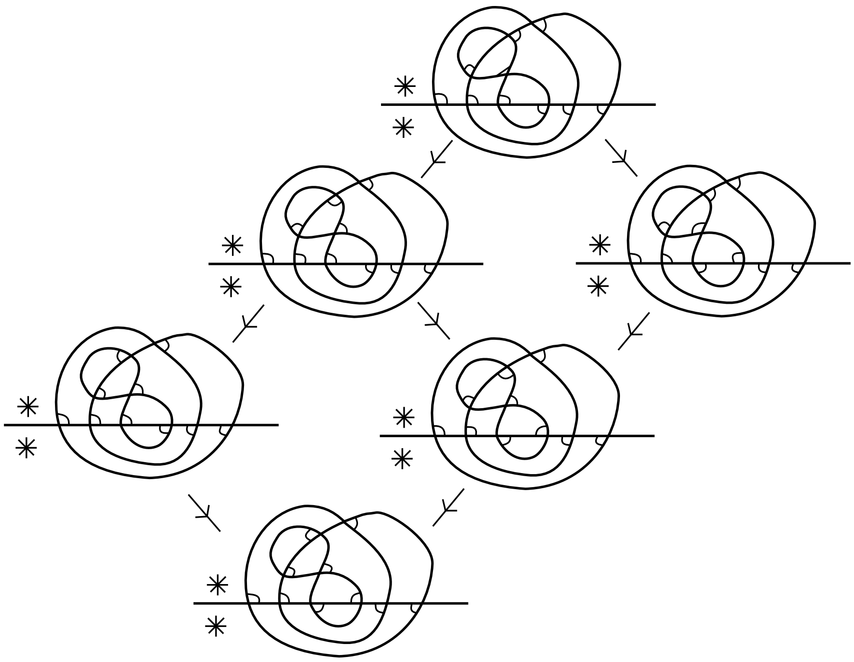


Figure 1.37

have an upper bound and lower bound. As there are only finitely many trails on a string, least upper bounds (joins) and greatest lower bounds (meets) exist. This makes $\mathscr{S}$ a lattice. $\square$

Example. An example of the clock lattice of a string is shown in Figure 1.37.

In [7], Gilmer and Litherland provided another proof of the clock theorem. They constructed a parallel interpretation in terms of spanning trees, which correspond to states in Kauffman's proof. A *spanning tree* of a graph is a 1-connected subcomplex which includes each vertex. Now suppose that a connected graph is embedded in the plane, with one of its exterior vertices (i.e. a vertex in the closure of the unbounded component of its complement) distinguished as a root vertex. Given a spanning tree of a graph, assign to all the edges of the tree arrows that point away from the base (the base is an exterior vertex of the graph). Across each edge of the graph that does not belong to the tree, draw an arrow pointing into the bounded region of the complement of the graph. This region is obtained by adjoining this edge to the tree. Suppose an edge in the tree and an edge not in the tree share a common vertex and a common face such that the edge in the tree points toward this vertex and the edge not in the tree has its arrow point toward this face. Obtain a new tree by deleting the edge in the tree and adding the edge not in the tree. If the edge not in the tree swings in a (counter) clockwise direction across the common face, we call this a (counter) clockwise move. This move corresponds to a transposition in Kauffman's proof. Let T and T' be trees. If there is a sequence of clockwise moves turning T into T', then $T \geqslant T'$, which is a partial order. Gilmer and

Litherland's version of the clock theorem says that the collection of spanning trees of a connected plane graph based at an exterior vertex becomes a graded distributive lattice under this partial order.

Given a universe U in string form, one can form a connected plane graph G as follows. The complementary regions of U can alternately be shaded and unshaded, so that each edge of U has an unshaded region on one side and a shaded region on the other. Of the two stars of U, one then lies in an unshaded region and one lies in a shaded region. Form a plane graph G, with one vertex in each shaded region, and an edge between two shaded regions for each crossing of U shared by those two shaded regions. Choose the root vertex of G to be the vertex in the starred shaded region of U. Then the Kauffman states of U naturally correspond to the Gilmer-Litherland states of G, and the Kauffman clock lattice of U is isomorphic to the Gilmer-Litherland clock lattice of G. Their version is equivalent to Theorem (1.4.17): from each universe U one may construct a connected plane graph G with an isomorphic clock lattice as above; and conversely, Gilmer-Litherland show that, from a connected plane graph G, one can recover the universe U. It seems that the proof presented in this chapter is more cumbersome but the concept of a transposition is related to that of a bypass addition in contact geometry, which will be described at the end of this thesis. Propp also proved in [21] a theorem that says the set of spanning trees of a finte connected plane graph form a distributive lattice under a covering relation called "swinging down". That theorem of Propp is a corollary of a more general theorem proved by Propp in the same paper, which provides a lattice structure on the set of orientations of a graph with a fixed circulation. This more general theorem is the basis of our main result in Section 2.2, where it is stated as Theorem (2.2.28).

Chapter 2

Multiverses on a surface of zero and positive genus

2.1 Multiverses on a surface of genus 0

We now define the notion of an embedded graph X on a surface Σ. Note that our notion of an embedded graph is slightly unusual: we allow some edges of the graph X to run to the boundary $\partial\Sigma$, and we do not count the endpoints on $\partial\Sigma$ as vertices. We call such edges *free edges*. This unorthodox definition is made in order to agree with the notion of a string on a plane, introduced previously in Definition (1.2.2), which also has such "free" edges, namely the input and output edges.

Definition 2.1.1 (Graph embedding)**.** A graph X is said to be *embedded* on a compact orientable surface Σ if the vertices of X are distinct points of Σ and each edge of X is a simple arc connecting in Σ the two vertices it joins in X such that its interior is disjoint from other edges and vertices. If an edge of X is free, that is, it is incident to a vertex at one end and intersects the boundary of Σ at the other, then it is a simple arc connecting the vertex and the point of intersection such that its interior is disjoint from other edges and vertices. An *embedding* of a graph X on Σ is an isomorphism of X with a graph X' embedded on Σ. Let Y be the union of the points and arcs associated with the vertices and edges of X. Each connected component of $\Sigma \setminus Y$ is a called a *face* of X. If each face of X is a disk, then the embedding of X on Σ is called a 2-*cell embedding*.

We note that this definition is quite non-standard. But it follows Kauffman's convention with a string universe not to have vertices at infinity or the boundary of the surface.

Definition 2.1.2 (n-string multiverse on a surface of genus 0)**.** Let Π be a compact orientable surface of genus 0 with non-empty boundary. Let U be a (possibly disconnected)

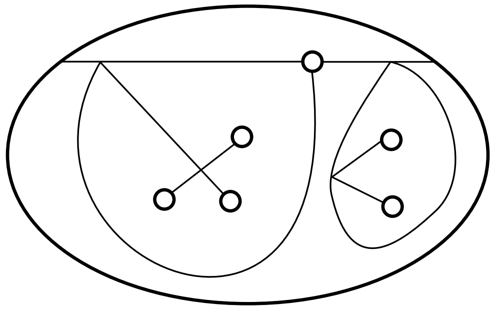


Figure 2.1

graph embedded on Π (loops and multiple edges are allowed) such that every vertex of U is 4-valent and the free edges make $2n$ intersections with the boundary of Π. Pick a boundary component C of Π and embed Π on a plane with C being the outermost boundary component. This enables us to identify the interior of any simple closed curve on Π. We impose the condition that U is a 2-cell embedding on Π. Mark some faces of U which are adjacent to C by a star and call them *starred faces*. If U has $F - V$ starred faces such that $F - V \geqslant 0$ where F and V are the number of faces and vertices of U, respectively, then U is called an *n-string multiverse*.

Example. An example of a multiverse as defined in Definition (2.1.2) is shown in Figure 2.1. In this example, the number of vertices equals the number of faces. Hence, there is no starred face. The little circles are boundary components of the surface.

For the rest of this section, when we speak of a multiverse, we assume that it is embedded on a compact orientable surface of genus 0 with non-empty boundary unless stated otherwise.

Definition 2.1.3 (Spine)**.** Let U be a multiverse embedded on a surface Π. The *spine* of U is a graph G embedded on Π constructed as follows. Place a vertex of G at each vertex of U and call it a *white vertex*. Place a vertex of G at each face of U that is not starred and call it a *black vertex*. The vertex-set of G consists of white and black vertices. A black vertex b and a white vertex w are said to be *adjacent* to each other if the face of U where b is placed is adjacent to the vertex of U where w is placed. Let b and w be a pair of adjacent black and white vertices, respectively and let F be the face of U where b is placed. Let $\alpha_1, \ldots, \alpha_k$ ($k \leqslant 4$) be the corners of w that belong to F. For each α_i, join b and w by exactly one edge incident to w through α_i. Repeat the same process to every pair of adjacent black and white vertices. The edge-set of G consists of all these edges.

Example. Figure 2.2 gives an example of a spine of a multiverse, which is the green graph. This will be the common colour for all spines depicted in this chapter.

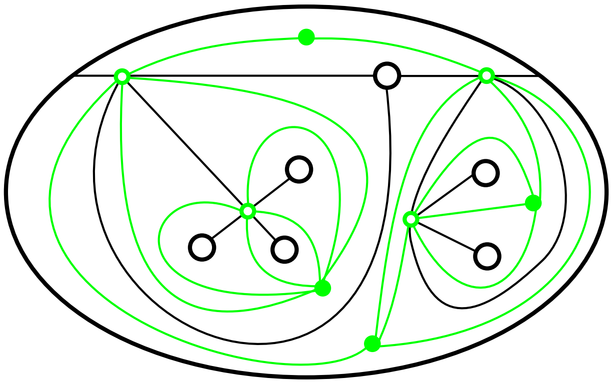


Figure 2.2

It can be seen from the above definition that the spine G of a multiverse U has some uniqueness associated with it. Firstly, the white vertices of G are uniquely determined by U. Secondly, the black vertices are unique up to homotopy within a face of U. Thirdly, each edge of G lies in the closure of a face F of U; once the vertices are placed, each edge is unique up to homotopy relative to endpoints in F. In short, the spine is unique as a graph rather than as an embedded graph on a surface. The following definition is a generalisation of Definition (1.2.4) of a state from the context of a string to the context of an n-string multiverse on a surface of genus 0.

Definition 2.1.4 (State)**.** Let U be a multiverse. Assign to each vertex of U a marker lying in one of its four corners such that the starred faces do not have any marker and each of the remaining faces has exactly one marker. The object obtained is called a *state* of U. The set of all states of a multiverse U is denoted $\mathscr{S}_U$.

One can show that the set of states is nonempty using a generalisation of the state-trail correspondence (Lemma (1.4.6)). It is not too hard to prove by induction on the number of vertices that every multiverse has at least one trail.

The clock theorem we will prove in this section again asserts that the set of states of a multiverse forms a lattice. This lattice is defined with respect to a notion of "transposition", generalised from Kauffman's notion of transposition in Chapter 1. The proof uses a theorem of Propp [21]. We give a brief outline of the scheme of the proof here.

First, we consider matchings on the spine. We show there is a bijection between states of U and matchings of its spine. We show that these are also the matchings of a smaller graph called the "reduced spine".

Second, we define an "n-transposition" for any $n \geqslant 1$. The case $n = 2$ corresponds to Kauffman's notion of transposition. We introduce a notion of "twisting" on matchings and show that a twisting done on a matching corresponds to a transposition done on a state. Propp's theorem says that the set matchings forms a lattice and by applying it to our situation appropriately, we can prove the clock theorem.

Definition 2.1.5 (Matching)**.** Let X be a graph. A *matching* M of X is a set of edges of X such that no two edges of M are adjacent and each vertex of X whose degree is at least 1 belongs to an edge of M. The set of all matchings of the spine of a multiverse U is denoted $\mathscr{M}_U$.

When a graph G is bipartite (say with black and white vertices), each edge in a matching of G has one black and one white endpoint, so the matching provides a bijection between black and white vertices. It can therefore be regarded as "matching up" the black and white vertices via this bijection.

Lemma 2.1.6. *Let U be a multiverse embedded on a surface Π. There is a bijection between $\mathscr{S}_U$ and $\mathscr{M}_U$.*

Proof. We first define a map $f : \mathscr{S}_U \to \mathscr{M}_U$. Let S be a state of U. Construct a matching M of the spine G of U from S as follows. Since every white vertex coincides with a vertex of U, every white vertex is assigned a marker by S. For convenience, we call it the marker at the white vertex. Let e be an edge of G; let b and w be the black and white vertices joined by e and let α be the corner where the marker at w lies (it is one of the four corners determined by the four edges of U incident to w). The edge e belongs to M if and only if the following two conditions are satisfied. Firstly, the marker at w occupies the face of U where b is placed. Secondly, e is incident to w through α. Now, we prove that M is really a matching of G. Since a marker can only occupy an unstarred face (a face which is not starred), this face has a black vertex in it. Therefore, every white vertex is joined to a black vertex by an edge that is in M. Since every unstarred face is occupied by a marker, every black vertex is joined to a white vertex by an edge that is in M. Thus every vertex of G belongs to an edge of M. Suppose that there are two edges of M which share a common black vertex but are incident to different white vertices. This implies that there is a face of U occupied by two markers, which contradicts the fact that S is a state. Suppose that there are two edges of M which share a common white vertex but are incident to different black vertices. Thus, there are two distinct markers at that vertex, which also contradicts the fact that S is a state. Now, assume that there are two distinct edges of M, denoted l and l', which share a common black vertex and a common white vertex v. If l and l' are incident to v through different corners, say β and γ, then there are two different markers at v, one in β and one in γ. This again contradicts the fact that S is a state. If l and l' are incident to v through the same corner, then they are the same edge by Definition (2.1.3). This contradicts the fact that they are distinct. Therefore, the edges of M are pairwise non-adjacent, completing the proof that M is a matching. It is clear from the construction of M that M is well-defined. The matching M is the image of S via f, that is, $M = f(S)$.

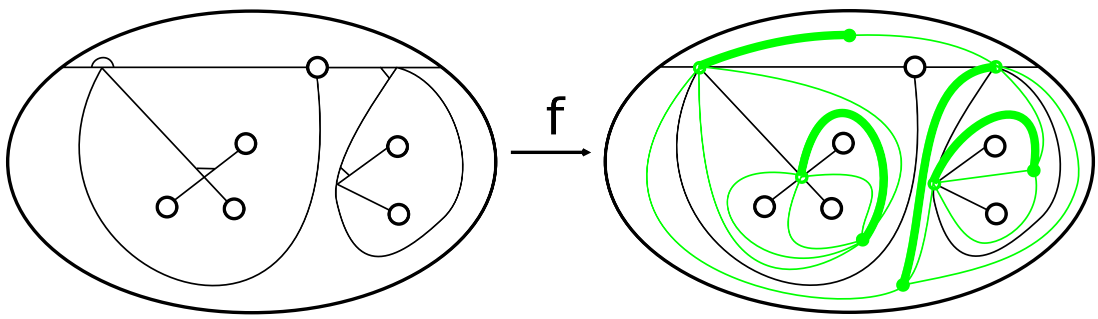


Figure 2.3

Now we define a map $g : \mathscr{M}_U \to \mathscr{S}_U$. Let M be a matching of G. Construct a state S of U from M as follows. Let w be a vertex of U. Since w coincides with a white vertex of G (also denoted w), it belongs to an edge e of M. Place a marker at w in the corner α through which e is incident to w (it is one of the four corners determined by the four edges of U incident to w). Repeat the same operation to the remaining vertices of U to obtain S. To prove that S is really a state, it is sufficient to prove that there is a bijection between the set of white vertices W and the set of black vertices B because this implies that there is a bijection between the set of vertices of U and the set of unstarred faces, which is equivalent to Definition (2.1.4). By Definition (2.1.2), the number of vertices of U equals the number of unstarred faces. Hence, $|W| = |B|$. Define a map $h : W \to B$ by letting the image of a white vertex w via h be the black vertex b such that the edge joining w and b belongs to M. Since no two edges of M are adjacent, h is well-defined and injective. This, along with the fact that $|W| = |B|$, implies that h is bijective. It is clear from the construction of S that S is well-defined. The state S is the image of M via g, that is $S = g(M)$.

It is clear from the definitions of f and g that they are inverses of each other. As a result, they are bijections. □

Example. An example of a state and its corresponding matching is shown in Figure 2.3. The edges of the spine that belong to the matching are thickened. This will be how matchings are illustrated throughout the chapter.

Definition 2.1.7 (Reduced spine)**.** Let G be the spine of a multiverse U. Remove all edges of G that do not belong to any matching. The graph obtained is called the *reduced spine* of U.

Example. Figure 2.4 shows an example where the reduced spine of a multiverse U is different from its spine. Let F be the indicated face of U. Let u and v be the indicated white vertices. The face F is always occupied by the marker at v. Thus, the marker at u lies outside F. This implies that the marker at U never lies in the two corners belonging

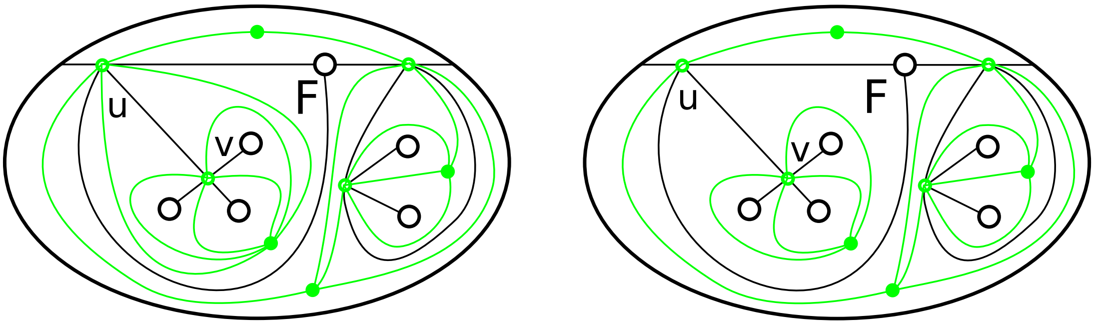


Figure 2.4

to F. As a result, the edges of the spine joining u to the black vertex in F never belong to any matching, which makes them absent in the reduced spine.

The reason why we need to construct the reduced spine is because Propp's theorem [21] about twisting (Theorem (2.1.11)) requires every edge of the graph G to belong to some matching. Since the spine G of a multiverse U is unique and each edge of G belongs to either no matching or some matchings, the reduced spine of U is also unique. The following definition is a generalisation of Definition (1.4.1) of a transposition from the context of a string to the context of an n-string multiverse on a surface of genus 0.

Definition 2.1.8 (n-transposition)**.** Let S be a state of a multiverse U; let $v_0, \ldots, v_{n-1}$ be n ($n \geqslant 1$) vertices of U and let R_i be the face occupied by the marker at v_i. Suppose that the following conditions are satisfied.

1. Firstly, for each $0 \leqslant i \leqslant n-1$, there is a nonzero rotation of the marker at v_i from R_i to $R_{i-1 \bmod n}$, which turns S into another state S'. All these rotations must be in the same direction (clockwise or counterclockwise), but different markers may rotate by different angles (90° or 180° or 270°).

2. Secondly, let γ_i be a curve in R_i joining v_i to $v_{i+1 \bmod n}$, which only intersects U at its endpoints, which is incident to v_i through the corner where the marker at v_i lies in S, and which is incident to $v_{i+1 \bmod n}$ through the corner where the marker at $v_{i+1 \bmod n}$ lies in S'. (Such a curve exists since R_i is connected; it is unique up to homotopy in R_i relative to endpoints because U is 2-cell embedded, so each R_i is topologically a disc.) It can be seen that the union of all these γ_i's is a simple closed curve γ. The second condition requires that, for any v_i which has one or more corners lying entirely in the interior of γ, no marker lies in any of these corners in any state of U.

Then, the n simultaneous rotations of the markers at the v_i's are called an *n-transposition* on S. If all these markers rotate clockwise (counterclockwise) from their positions in S to their positions in S' through the interior of γ, the n-transposition is said to be *clockwise (counterclockwise)*. The transposition is said to be done on the curve γ. Note that n is unrelated to the number of strings of U.

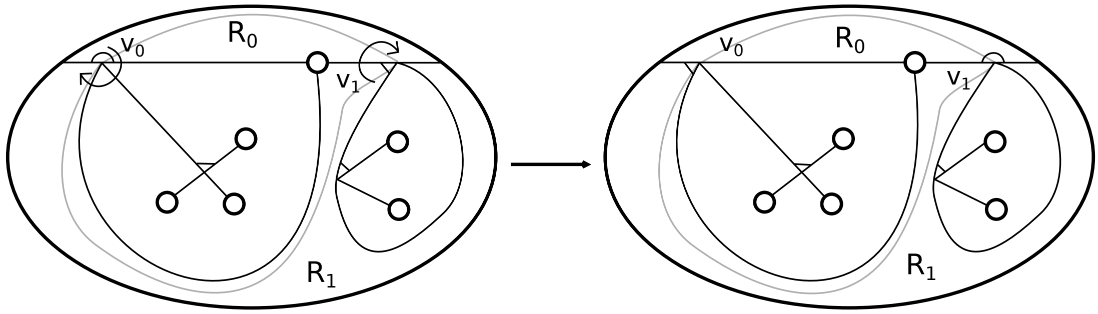


Figure 2.5

Example. Figure 2.5 provides an example of a clockwise 2-transposition involving the vertices v_0 and v_1 whose markers occupy the faces R_0 and R_1 before the transposition, respectively. After the transposition, the markers at v_0 and v_1 occupy R_1 and R_0, respectively. The grey curve is the curve γ. The clockwise direction is determined by the direction in which the markers rotate through the interior of γ to go from their initial positions to their final positions. Since γ is a closed curve on a surface embedded on a plane, its interior is well-defined.

The motivation behind the conditions in Definition (2.1.8) is to establish a correspondence between the set of n-transpositions that can be done on a state of a multiverse and the set of twisting that can be done on the matching corresponding to this state. The details are presented in Definition (2.1.9) and Lemma (2.1.10).

The following definition is taken from Propp's paper [21].

Definition 2.1.9 (Twisting)**.** In a bipartite plane graph G where the vertices in the two independent sets are called black and white vertices, an *elementary cycle* is a simple cycle (in the graph-theoretic sense) that encircles a single face, excluding the faces of any connected component of G that may exist in the interior of the cycle. An elementary cycle is said to be *alternating* relative to a matching M if its edges alternately belong to M and the complement of M in G. Let A be an alternating cycle encircling a face F. Direct the edges of A that belong to M from their black vertices to their white vertices. If these edges encircle F in the counterclockwise direction, then A is said to be *positive relative to* M. If they encircle F in the clockwise direction, then A is said to be *negative relative to* M. Removing from M the edges of A which belong to M and adding to M the edges of A which do not belong to M turns M into another matching M'. If A is positive relative to M and negative relative to M', then this operation is called *twisting down.* On the other hand, if A is negative relative to M and positive relative to M', then this operation is called *twisting up.*

Lemma 2.1.10. *Let S be a state of a multiverse U and let M be its corresponding matching on the* reduced *spine of U as determined by the bijection in Lemma* (2.1.6)*. Let*

Tr_S be the set of all clockwise transpositions that can be done on S and let Tw_M be the set of all twisting down that can be done on M. Define a map $p : \mathrm{Tr}_S \to \mathrm{Tw}_M$ as follows. Let T be a clockwise transposition that turns S into another state S'. The twisting down $O = p(T)$ is defined to be the one that turns M into the matching M' corresponding to S'. The map p is a bijection. Replacing 'clockwise' by 'counterclockwise' and 'down' by 'up' in these statements gives another true statement.

Proof. We proceed to prove that O is indeed a twisting down. Let $v_0, \ldots, v_{n-1}$ be the vertices whose markers rotate in T; let α_i and α'_i be the corners where the marker at v_i lies in S and S', respectively (that is, α_i and α'_i are the positions of the marker at v_i before and after T is performed) and let R_i be the face occupied by the marker at v_i. For each v_i, label the white vertex coinciding with it also by v_i. For each R_i, label the black vertex inside it also by R_i. When confusion may arise, we will clarify what we mean by v_i and R_i. By Definition (2.1.3), every edge of the spine G and the reduced spine G_0 of U is uniquely specified by three objects: the black and white vertices it joins and the corner of the white vertex through which the edge is incident to it. Thus, we represent each edge in G and G_0 by a triple (black vertex, white vertex, corner). Since the marker at v_i is in $R_{i-1 \bmod n}$ after T is performed, the face $R_{i-1 \bmod n}$ is adjacent to the vertex v_i. Hence, the edge $(R_{i-1 \bmod n}, v_i, \alpha'_i)$ exists. Replacing i by $i+1$, we rewrite this edge as $(R_i, v_{i+1 \bmod n}, \alpha'_{i+1 \bmod n})$. It is easy to see that, as i goes from 0 to $n-1$, the edges (R_i, v_i, α_i) and $(R_i, v_{i+1 \bmod n}, \alpha'_{i+1 \bmod n})$ form a simple cycle A.

We need to prove that A is elementary. In order to do so, it is sufficient to prove that all the black vertices R_i's are adjacent to the same face of G_0 in the interior of A, regarded as a simple closed curve. Consider the white vertex v_0. Let B be the face inside the interior of A and adjacent to the edge (R_0, v_0, α_0) (B is unique because there are only two faces adjacent to (R_0, v_0, α_0), one in the interior of A, the other outside it). Suppose that the black vertex R_{n-1} is not adjacent to B. This implies that there is an edge (X, v_0, β) which joins v_0 to the black vertex X of a face X of U in the interior of A and is incident to v_0 through the corner β (Figure 2.6). The existence of (X, v_0, β) in G_0, the reduced spine of U, means that there is a matching K of G_0 which contains (X, v_0, β). Hence, by the state-matching correspondence, the state P corresponding to K has a marker in β, a corner lying entirely in the interior of A. This contradicts the second condition in the definition of a transposition, which says that no marker can lie in β in any state of U (Definition (2.1.8)). Thus, both black vertices R_0 and R_{n-1} are adjacent to B. Repeat this argument to the white vertex v_1, then v_2 and so on up to v_{n-1} to arrive at the conclusion that all the black vertices R_i's are adjacent to B which is in the interior of A. Therefore, A encircles a single face, namely B, which implies that A is elementary.

We proceed to prove that A is alternating relative to both M and M'. Previously, we

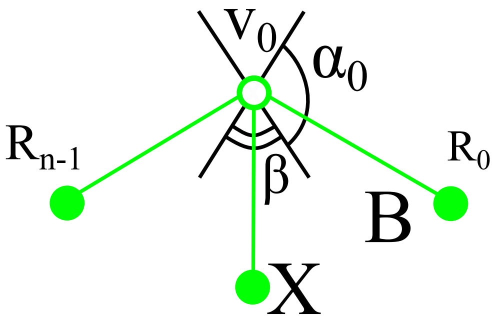


Figure 2.6

note that A, when considered as a set of edges, consists of two families of edges: (R_i, v_i, α_i) and $(R_i, v_{i+1 \bmod n}, \alpha'_{i+1 \bmod n})$. Since, in S, the marker at v_i lies in R_i, the edges (R_i, v_i, α_i) belong to M. Since, in S' the marker at v_i lies in $R_{i-1 \bmod n}$, the edges $(R_{i-1 \bmod n}, v_i, \alpha'_i)$ belong to M'. These edges are exactly the edges $(R_i, v_{i+1 \bmod n}, \alpha'_{i+1 \bmod n})$ after replacing i by $i+1$. Moreover, the edges of A alternately belong to those two families. As a result, A is alternating relative to both M and M'. Thus, the operation O that turns M to M' is a twisting. Since T is clockwise, the marker at each v_i rotates clockwise through the interior of A from its position in S to its position in S'. Thus, in M, the edges (R_i, v_i, α_i), when directed from their black vertices to their white vertices, encircle B in the counterclockwise direction, making A positive. In M', the edges $(R_i, v_{i+1 \bmod n}, \alpha'_{i+1 \bmod n})$, when directed from their black vertices to their white vertices, encircle B in the clockwise direction, making A negative (Figure 2.7). Since A is positive relative to M and negative relative to M', the operation O is a twisting down.

Define a map $q : \mathrm{Tw}_M \to \mathrm{Tr}_S$ as follows. Let O be a twisting down that turns M into another matching M'. The clockwise transposition $T = q(M)$ is defined to be the one that turns S into the state S' corresponding to M'. We now prove that T is indeed a clockwise transposition. Let A be the elementary cycle of G_0 where O is performed and let B be the face A encircles. Thus, A is alternating relative to both M and M'. This implies that A has an even number of vertices. Half of them are white and the other half are black. Let $v_0, \ldots, v_{n-1}$ be the white vertices of A and let R_i be the black vertex of A such that the edge of A joining R_i and v_i belongs to M. This means that the edge of A joining $R_{i-1 \bmod n}$ and v_i belongs to M'. Let α_i be the corner of U at v_i through which the edge of A joining R_i and v_i is incident to v_i; let α'_i be the corner of U at v_i through which the edge of A joining $R_{i-1 \bmod n}$ and v_i is incident to v_i. Hence, using the triple notation to label an edge, the edges (R_i, v_i, α_i) belong to M and the edges $(R_{i-1 \bmod n}, v_i, \alpha'_i)$ belong to M'. This is equivalent to the fact that, in S, the marker at v_i lies in α_i while in S', it lies in α'_i. Since α_i is a corner of the region of U where the black vertex R_i is placed (denote this region also by R_i) and α'_i is a corner of the region of U where the black vertex

$R_{i-1 \bmod n}$ is placed (denote this region also by $R_{i-1 \bmod n}$), the marker at v_i rotates from R_i to $R_{i-1 \bmod n}$ as T turns S into S'. As a result, T satisfies the first condition in Definition (2.1.8).

We prove the second condition by contradiction. Note that the union of all edges of A is a simple closed curve which satisfies the definition of the curve γ in Definition (2.1.8). Therefore, A can be treated as γ. Suppose that there is a vertex v_j which has at least one corner η lying entirely in the interior of A and this corner is occupied by a marker in some state of U. Thus, the edge (Y, v_j, η) exists in G_0 where Y is the black vertex of a face of U in the interior of A. Let D be the subgraph of G_0 which consists of v_j and all vertices and edges in the interior of A connected to v_j. Since A is elementary, no face of G_0 other than B is in the interior of A (except faces that belong to a component of G_0 disconnected from the component that contains A). Hence, D does not have any cycle and is therefore a tree. Let x be a leaf of D other than v_j and let y be the only vertex joined to x by an edge $\{x, y\}$. Suppose that x is a white vertex. Thus, y is black. Since x has degree 1 and $\{x, y\}$ is also an edge of the reduced spine G_0, the face of U where y is placed, also denoted y, is occupied by the marker at x is all states. However, in some state, say S^*, this face is occupied by the marker at another white vertex z joined to y by an edge of D, which is also an edge of G_0. Note that z exists because D would be otherwise disconnected. Hence, in S^*, the face y of U is occupied by two markers; one at x and one at z, which leads to a contradiction. Now, assume that x is a black vertex. Thus, y is white. Since x has degree 1 and $\{x, y\}$ is also an edge of the reduced spine G_0, the face of U where x is placed, also denoted x, is occupied by the marker at y in all states. Let z be another black vertex joined to y by an edge of D (z exists because D would be otherwise disconnected). This implies that, in some state S^*, the marker at y also occupies the face of U where z is placed. Denote this face also by z. Hence, in S^*, the marker at y occupies two different faces of U; one is x and the other is z, which again leads to a contradiction. As a result, no marker lies in η in every state of U, completing the proof that T satisfies the second condition in Definition (2.1.8). Therefore, T is really a transposition.

What remains to prove is that T is clockwise. The fact that O is a twisting down implies that, in M, the edges (R_i, v_i, α_i), when directed from their black vertices to their white vertices, encircle B in the counterclockwise direction, making A positive and that, in M', the edges $(R_{i-1 \bmod n}, v_i, \alpha'_i)$, when directed from their black vertices to their white vertices, encircle B in the clockwise direction, making A negative. Therefore, the marker at v_i rotates clockwise from α_i to α'_i through the interior of A, making T clockwise.

It can be seen from the definitions of p and q that they are inverses of each other. As a result, they are bijections. Moreover, a clockwise n-transposition corresponds to a

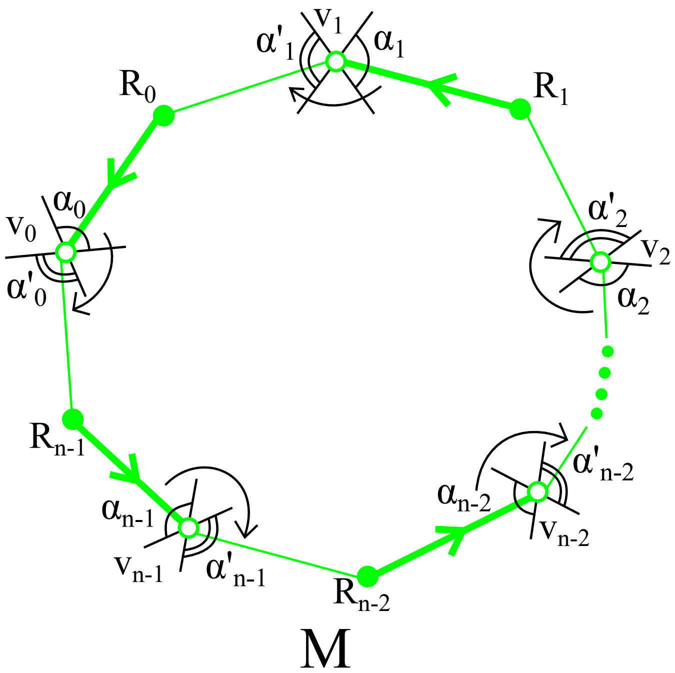


Figure 2.7

twisting down on a cycle of length $2n$ and vice versa. Following a similar proof, we can prove that there is a bijection between the set of all counterclockwise transpositions that can be done on S and the set of all twisting up that can be done on M. $\square$

Remark. What is good about p is that it preserves structure and relates nicely to the bijection f in Lemma (2.1.6) in the following sense. Consider a state $S \in \mathscr{S}_U$ and a clockwise transposition $T \in \mathrm{Tr}_S$ which can be applied to S because T is itself a map from $\mathscr{S}_U$ to $\mathscr{S}_U$. Under the bijection f, the state S corresponds to a matching $f(S) \in \mathscr{M}_U$. Under the bijection p, the transposition T corresponds to a twisting down $p(T)$ which is itself a map from $\mathscr{M}_U$ to $\mathscr{M}_U$. The two bijections relate nicely to each other in that the matching corresponding to the state $T(S)$ obtained by applying T to S is the matching obtained by applying $p(T)$ to $f(S)$. In other words, $f(T(S)) = p(T)(f(S))$. The commutative diagram in Figure 2.8 illustrates this. A fancy way of expressing this would be to say that we have an isomorphism of categories. There is a category with states as objects and a unique morphism $A \mapsto B$ if and only if there is a sequence of clockwise transpositions taking A to B (including the empty sequence, which gives identity morphisms). Composition is defined in the unique possible way. In this sense, we obtain a category generated by clockwise transpositions. Similarly, there is a category with matchings as objects and a unique morphism $A \mapsto B$ if and only if there is a sequence of twisting down taking A to B. The bijections f and p provide a functor realising an isomorphism between these categories.

Example. Figure 2.9 shows a concrete example of the commutative diagram in Figure

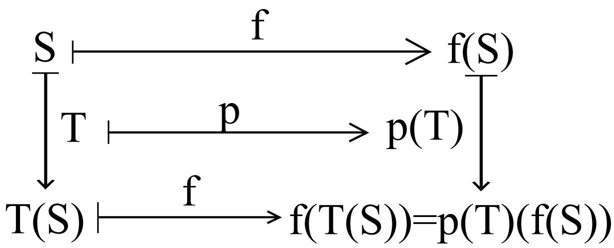


Figure 2.8

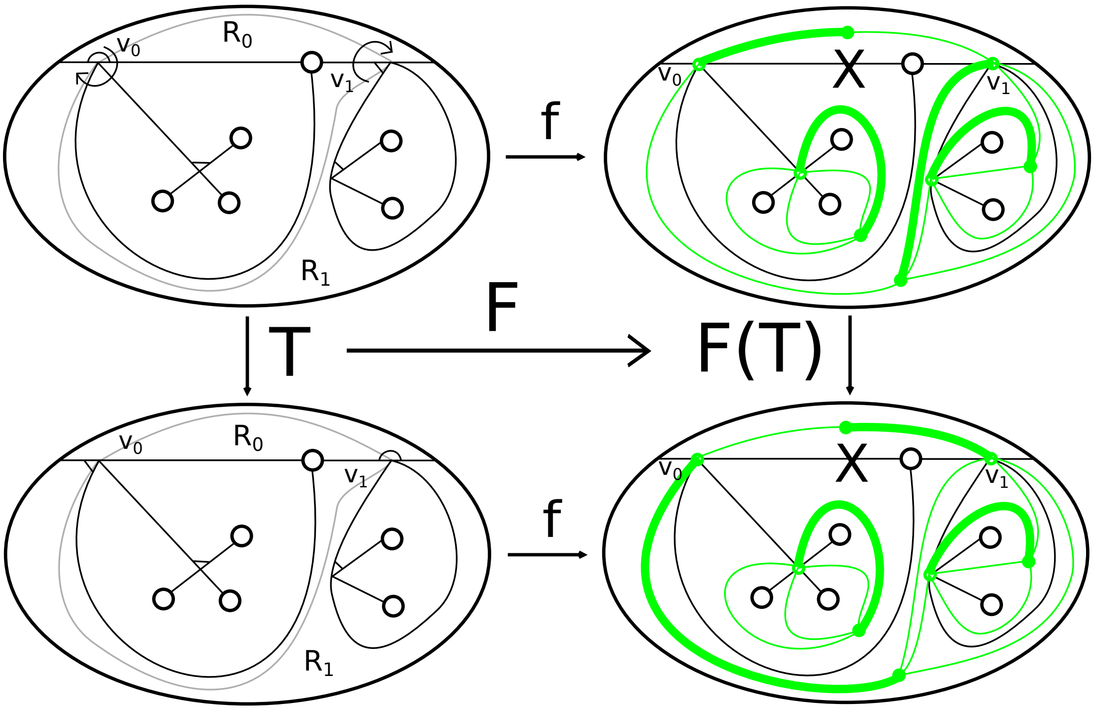


Figure 2.9

2.8 where X is the elementary cycle where the twisting is done.

Now we are almost ready to prove the clock theorem for a multiverse on a surface of genus 0 using the following theorem by Propp [21] where the definitions of a matching and a twisting down are precisely Definitions (2.1.5) and (2.1.9).

Theorem 2.1.11 (Propp's theorem about twisting). *Let $\mathscr{M}$ be the set of matchings of a connected bipartite planar graph G embedded on a sphere and let f^* be a face of G, called the* starred face. *A matching M is said to cover another matching M' if M' is obtained from M by twisting down at a face other than f^*. Assume that every edge of G belongs to at least one matching. Then, the covering relation makes $\mathscr{M}$ into a poset that is a distributive lattice, called the* Propp's lattice *of G with respect to f^*, denoted $\mathcal{P}_G$. The partial order of this poset is naturally generated from the covering relation, namely, $M' < M$ if M' is obtained from M by a sequence of twisting down.*

Remark. Given a planar graph, we can compactify it by adding a point at infinity to the unbounded region to obtain a graph on a sphere. Therefore, Propp's theorem also applies to plane graphs.

Since the graph G in the theorem above is connected while the reduced spine of a multiverse is not necessarily so (see Figure 2.4 for an example), we need the following lemma, preceded by the relevant definitions.

Definition 2.1.12 (Product of finitely many lattices)**.** For each $1 \leqslant i \leqslant n$, let L_i be the distributive lattice generated by a set L_i equipped with a covering relation $\lessdot_i$. The product of the L_i's, denoted $L_1 \times \cdots \times L_n$ is the distributive lattice generated by the set $L_1 \times \cdots \times L_n$ equipped with the covering relation $\lessdot$ defined by $(x_1, \ldots, x_n) \lessdot (y_1, \ldots, y_n)$ if $x_j \lessdot_j y_j$ for a unique j and $x_i = y_i$ for all $i \neq j$, where $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in L_1 \times \cdots \times L_n$.

Definition 2.1.13 (Lattice isomorphism)**.** Two lattices M and N are said to be *isomorphic*, denoted $M \cong N$, if there is a bijection $f : M \to N$ such that $x \lessdot_M y$ implies $f(x) \lessdot_N f(y)$ for all $x, y \in M$, where $\lessdot_M$ and $\lessdot_N$ are the covering relations of M and N, respectively.

Lemma 2.1.14. *Let $G_1, \ldots, G_n$ be n connected bipartite plane graphs and let the starred face f_i^* of G_i be the unbounded face of G_i. Let G be the disjoint union of all the G_i's and let $\mathscr{M}_G$ be the set of matchings of G. A matching M of G is said to cover another matching M' if M' is obtained from M by twisting down at a face of some G_j other than f_j^*. Assume that every edge of G belongs to at least one matching. Then, $\mathscr{M}_G$ equipped with this covering relation is a lattice, called the Propp's lattice of G, denoted $\mathcal{P}_G$, and $\mathcal{P}_G \cong \mathcal{P}_{G_1} \times \cdots \times \mathcal{P}_{G_n}$.*

Proof. Since there is no edge joining a vertex of G_i and a vertex of G_j whenever $i \neq j$, any matching M_G of G is the disjoint union of n matchings $M_{G_1}, \ldots, M_{G_n}$ where M_{G_i} is a matching of G_i. Thus, we can write $M_G = M_{G_1} \cup \cdots \cup M_{G_n}$. Let $\mathscr{M}_G$ and $\mathscr{M}_{G_i}$ be the set of matchings of G and G_i, respectively. Define $f : \mathscr{M}_G \to \mathscr{M}_{G_1} \times \cdots \times \mathscr{M}_{G_n}$ to be the map that assigns $(M_{G_1}, \ldots, M_{G_n})$ to $M_G = M_{G_1} \cup \cdots \cup M_{G_n}$. This map is obviously well-defined and bijective. Let $A, B \in \mathscr{M}_G$ such that $A \lessdot_G B$ where $\lessdot_G$ is the covering relation of $\mathscr{M}_G$ as defined in the statement of the lemma. We have $A = A_1 \cup \cdots \cup A_n$ and $B = B_1 \cup \cdots \cup B_n$ for some $A_i \in \mathscr{M}_{G_i}$ and $B_i \in \mathscr{M}_{G_i}$. Since a twisting down is performed on an elementary cycle and the G_i's are pairwise disjoint, a twisting down can only be performed within a single G_j. Hence, $A \lessdot_G B$ implies that $A_j \lessdot_{G_j} B_j$ for a unique j and $A_i = B_i$ for all $i \neq j$ where $\lessdot_{G_j}$ is the covering relation of $\mathscr{M}_{G_j}$ as defined in Theorem (2.1.11). This, in turn, implies that $(A_1, \ldots, A_n) \lessdot (B_1, \ldots, B_n)$ where $\lessdot$ is the covering relation of $\mathscr{M}_{G_1} \times \cdots \times \mathscr{M}_{G_n}$ as defined in Definition (2.1.12). Since $(A_1, \ldots, A_n) = f(A)$

and $(B_1, \ldots, B_n) = f(B)$, we have $f(A) \lessdot f(B)$. Therefore, $\mathscr{M}_G$ equipped with $\lessdot_G$ is isomorphic to $\mathscr{M}_{G_1} \times \cdots \times \mathscr{M}_{G_n}$ equipped with $\lessdot$, that is, $\mathcal{P}_G \cong \mathcal{P}_{G_1} \times \cdots \times \mathcal{P}_{G_n}$. □

Here is the proof of the clock theorem.

Theorem 2.1.15 (The clock theorem on a surface of genus 0)**.** *Let U be a multiverse as defined in Definition* (2.1.2) *and let $\mathscr{S}$ be the set of states of U. A state S is said to cover another state S' if S' is obtained from S by a clockwise transposition. Then, $\mathscr{S}$ equipped with this covering relation is a lattice, called the* clock lattice *of U. The partial order of this lattice is naturally generated from the covering relation, namely, $S' < S$ if S' is obtained from S by a sequence of clockwise transpositions.*

Proof. Consider the reduced spine G of U. Since every edge of G joins a white vertex and a black vertex, G is bipartite. It is obvious that G is also planar. Suppose G is connected. The starred face f^* of G is defined to be the unbounded one. Since G is connected, bipartite and planar, Propp's theorem applies (Propp's theorem holds on a sphere and hence holds on a plane). Therefore, the Propp's lattice of G exists, denoted $\mathcal{P}_G$. Assume that G is disconnected. Let $G_1, \ldots, G_n$ be the connected components of G. The starred face f_i^* of G_i is defined to be the unbounded one, ignoring the presence of all other connected components. Since each G_i is connected, bipartite and planar, Lemma (2.1.14) applies and therefore the Propp's lattice of G exists, also denoted $\mathcal{P}_G$. Let $\mathscr{M}$ be the set of matchings of G and let $\lessdot_{\mathscr{M}}$ be the covering relation of $\mathscr{M}$ as defined in Theorem (2.1.11) or Lemma (2.1.14) depending on whether G is connected or disconnected. Denote the covering relation of $\mathscr{S}$ defined in the statement of the theorem by $\lessdot_{\mathscr{S}}$. By Lemma (2.1.6), there is a bijection $f : \mathscr{S} \to \mathscr{M}$. By Lemma (2.1.10), we have $S' \lessdot_{\mathscr{S}} S$ implies $f(S') \lessdot_{\mathscr{M}} f(S)$. Therefore, the Hasse diagram of $\mathscr{S}$ equipped with $\lessdot_{\mathscr{S}}$ is isomorphic to $\mathcal{P}_G$, which is a lattice. As a result, $\mathscr{S}$ equipped with $\lessdot_{\mathscr{S}}$ is a lattice. □

Example. Figure 2.10 presents an example of the lattice of a multiverse (black graph) along with its spine (green graph including dotted edges) and its reduced spine (green graph excluding dotted edges). It can be seen that this lattice is the product of two lattices. Each of them corresponds to a component of the reduced spine.

2.2 Multiverses on a surface of positive genus

Definition 2.2.1 (n-string multiverse on a surface of positive genus)**.** In a graph embedded on a surface, an edge is said to be *free* if it intersects the boundary of the surface (the points of intersection are not vertices of the graph). Let Π be a compact orientable surface of positive genus with non-empty boundary. Let U be a (possibly disconnected)

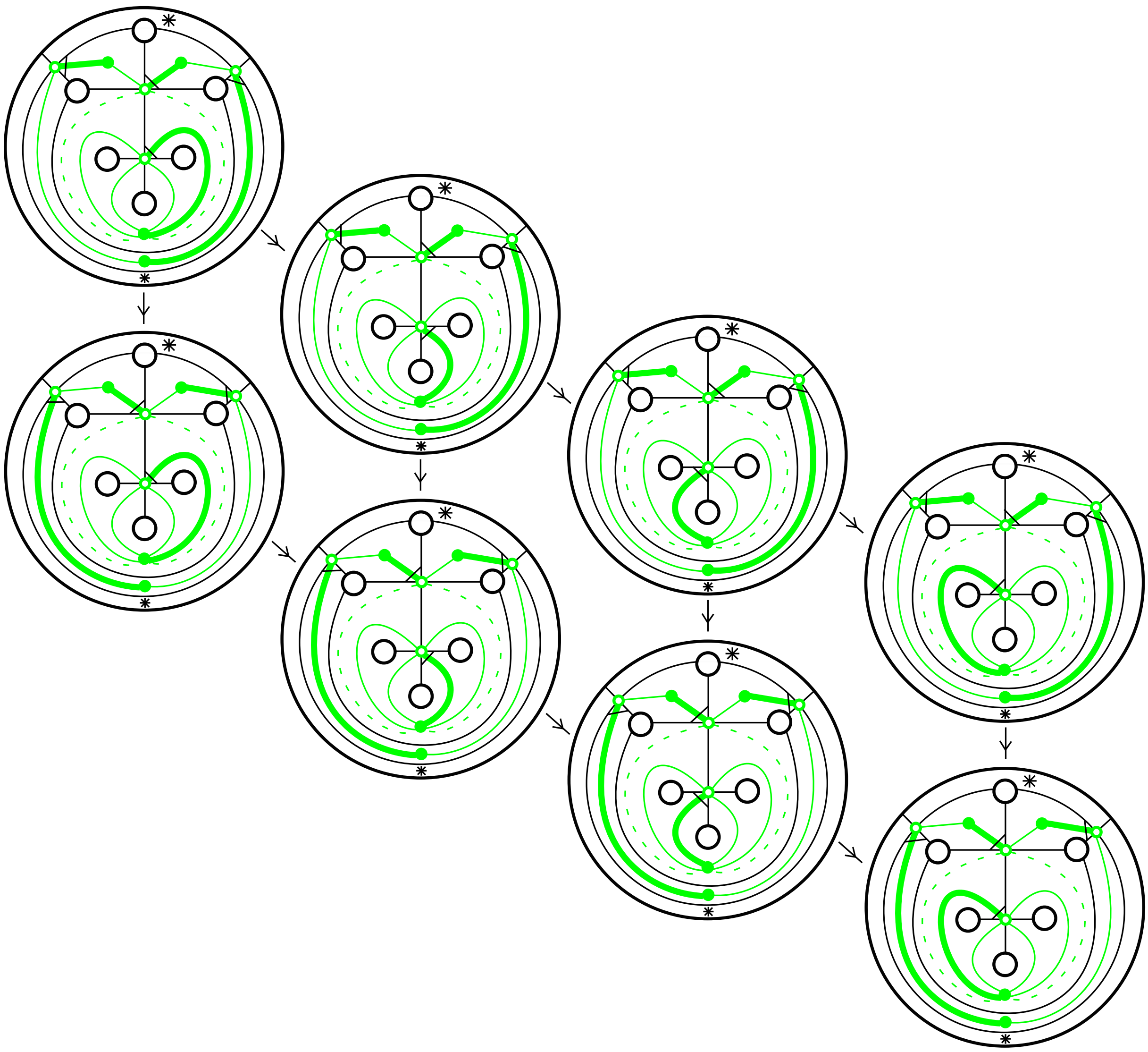


Figure 2.10

graph embedded on Π (loops and multiple edges are allowed) such that every vertex of U is 4-valent and the free edges make $2n$ intersections with the boundary of Π. Pick a boundary component C of Π and call it *the outermost boundary component*. The *interior* of any separating simple closed curve δ on Π is defined to be one of the two subsurfaces of Π separated by δ, say S, such that C does not belong to the boundary of S. We impose the condition that U is a 2-cell embedding on Π. Mark some faces of U which are adjacent to C by a star and call them *starred faces*. If U has $F - V$ starred faces such that $F - V \geqslant 0$ where F and V are the number of faces and vertices of U, respectively, then U is called an *n-string multiverse*.

Definition (2.2.1) is different from Definition (2.1.2) in the following two details. Firstly, Π is a compact orientable surface of positive genus with non-empty boundary. Secondly, when Π has genus 0, it can be flattened, that is, it can be embedded on a plane with one chosen boundary component to be the outermost one. Here, being outermost is intuitive. However, when Π has positive genus, it cannot be flattened. Therefore, a chosen boundary component is labeled "outermost" just for the purpose of identifying the interior of any separating simple closed curve on Π.

The spine of a multiverse embedded on a surface of non-zero genus is defined in the exact same way as Definition (2.1.3). Note that this definition does not depend on the topology of Π.

The clock theorem of this section, again, asserts that the set of states of a multiverse as defined in Section 2.1, forms a lattice. This lattice is again defined with respect to a notion of "transposition", which is defined in a distinct way from Section 2.1. The proof again uses a theorem of Propp [21]. We again give a brief outline of the scheme of the proof here.

First, we consider the spine of the multiverse (as previously defined in Section 2.1) and show it is "almost" 2-cell embedded. We also consider a "dual" of the spine, appropriately defined.

Second, we introduce several notions from Propp's work, involving orientations on the edges of a graph, "accessibility" classes of vertices and the "circulation" of an orientation around a cycle, including "elementary" cycles. We consider various types of orientations on a dual of the spine, including a "standard" orientation and orientations "prescribed" by a matching of the spine. We consider the relationship between orientations and circulations and show there are bijections between (i) the matchings of the spine, (ii) the prescribed orientations of a dual of the spine and (iii) the states of the multiverse.

Third, we introduce a notion of "twisting" a matching of the spine around a cycle and a generalised notion we call "big twisting". We then introduce a notion of "pushing" done on accessibility classes in a graph with an orientation. We also introduce a notion of "n-transposition" done on states. We show that the three notions are equivalent: (i) big twisting done on matchings of the spine, (ii) pushing done on accessibility classes of a prescribed orientation of a dual of spine and (iii) n-transpositions done on states. Propp's theorem shows that a certain set of orientations on a graph forms a lattice with respect to pushing and by applying this theorem, we are able to show we obtain lattices in our cases and prove the clock theorem.

Definition 2.2.2 (Almost 2-cell embedding)**.** An *almost* 2*-cell embedding* of a graph X on a surface Π is an embedding of X on Π where each face that is not adjacent to a boundary component of Π is a disk and each boundary component of Π is adjacent to a unique face and this face is not a disk.

Lemma 2.2.3. *Let U be a multiverse embedded on a surface Π and let G be its spine. If U is* 2*-cell embedded on Π, then G is almost* 2*-cell embedded on Π.*

Proof. Since U is 2-cell embedded on Π, cutting Π along U produces disks, each of which is a polygon. Based on how the spine G is constructed (Definition (2.1.3)), cutting further along G cuts each disk that does not come from a starred face into polygons which are classified into the following types. Type 1 consists of triangles where two sides are edges of G and the remaining one is a non-free edge of U, that is, an edge of U that does not intersect the boundary of Π. Type 1 is further divided into two smaller types. Type 1a includes triangles where the one edge belonging to U is adjacent to a starred face. Type 1b includes triangles where the one edge belonging to U is not adjacent to a starred face. Type 2 consists of pentagons where two consecutive sides are edges of G and the remaining ones form a sequence including a free edge of U, an arc of a boundary component of Π and another free edge of U, all placed in the said order.

Let M be a face of G which is not adjacent to a boundary component of Π. This implies that the boundary of M, denoted ∂M, consists only of edges of G, that is, no part of the boundary of Π is contained in the boundary of M. Let p be an edge of G which belongs to ∂M and let B_p and W_p be the black and white vertices joined by p, respectively. Starting from p, proceed along ∂M through B_p and denote the next edge by q. Let W_q be the white vertex q is incident to. Among all the edges of U joining W_p and W_q, precisely one of them forms a triangle with p and q that is a subset of M. Call this edge a. Since M is not adjacent to a boundary component of Π, both faces of U which a is adjacent to are unstarred. One of them is where B_p is located. Since the other is unstarred, there is a unique black vertex in it, denoted B. This vertex is joined to W_p and W_q by two edges of G, say r and s. The triangle bounded by a, r and s is also a subset of M. Combining the fact that the triangle bounded by p, q, a and the one bounded by a, r, s both belong to M with the fact that p, q, r, s form a cycle, we conclude that M is

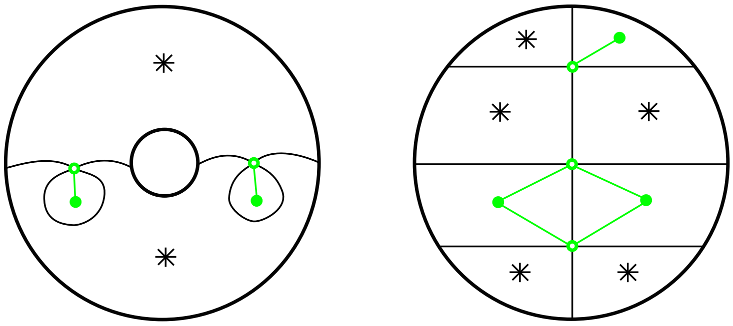


Figure 2.11

the union of those two triangles and ∂M is the union of p, q, r and s. Therefore, M is a disk. The above argument also implies that triangles of type 1b come in pairs. Each pair glues together to give a quadrilateral which is a face of G.

Let C be a boundary component of Π. Since the intersections between the free edges of U and C are not vertices of U, no white vertex of G is placed at these intersections. This implies that no edge of G intersects C. Therefore, C is adjacent to a unique face of G, denoted C_G, where C is one boundary component and at least one other boundary component is made of edges of G. As a result, C_G is not a disk. □

Example. The left diagram of Figure 2.11 shows an example where the spine of the multiverse has only one face and since it is adjacent to a boundary component of the surface on which the multiverse is embedded, it is not a disk (it has four boundary components in this specific example). The right diagram of Figure 2.11 gives an example where the face of the spine adjacent to the only boundary component of the surface has three boundary components whereas the face that is not adjacent to the boundary component is a disk.

Definition 2.2.4 (Dual of spine)**.** Let U be a multiverse embedded on a surface Π and let G be its spine. A *dual of spine* of U is a graph $G^\perp$ embedded on Π constructed as follows. Place a vertex of $G^\perp$ at each face of G. If two faces of G share an edge e, then join the vertices of $G^\perp$ corresponding to them by an edge $e^\perp$ of $G^\perp$ which intersects e exactly once. If the same face appears on both side of an edge e, add a loop $e^\perp$ to the vertex corresponding to the face in such a way that $e^\perp$ intersects e exactly once. Thus, each edge e of G has a corresponding edge in $G^\perp$, called the *dual edge* of e. Let A be a face of G that is adjacent to a boundary component of Π. Thus, by Lemma (2.2.3), A is not a disk. Let e be an edge between A and another face B of G. Let $A^\perp$ and $B^\perp$ be vertices of $G^\perp$ dual to A and B, respectively. Since A is not a disk, the edge $e^\perp$ dual to e can be drawn in at least two ways which are not homotopic to each other relative to endpoints as illustrated in Figure 2.12. Thus, $G^\perp$ is not unique as an embedded graph on Π but it is unique as a graph, that is, the vertex-set, edge-set and the incidence relations

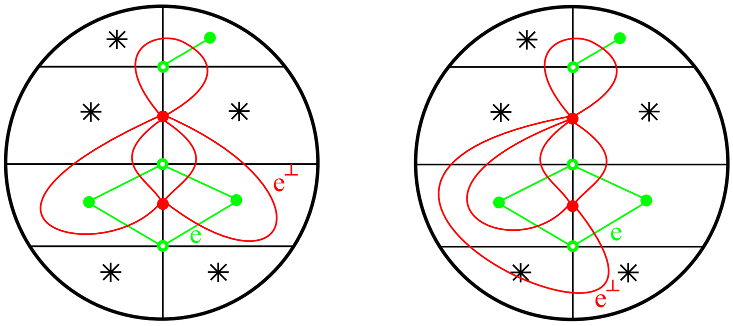


Figure 2.12

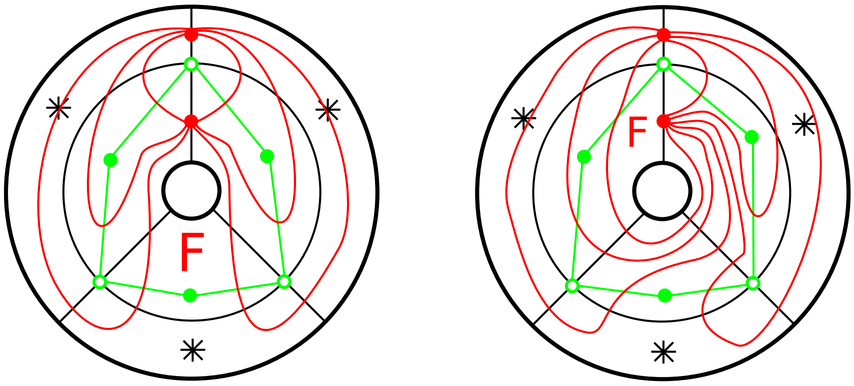


Figure 2.13

between them are uniquely determined. However, this lack of uniqueness does not affect any statement that refers to a dual of spine.

Remark. It can be seen that a dual of spine is somewhat similar to the dual of a plane graph such as the bijective correspondence between the edges of the spine and the edges of a dual of spine. However, there are some notable differences. Each face of the spine has a unique vertex of a dual of spine but each face of a dual of spine may have more than one vertices of the spine, as shown in Figure 2.12. Another difference is the non-uniqueness of a dual of spine as opposed to the uniqueness of the dual of a plane graph.

Example. Figure 2.12 illustrates how the dual of the edge e, denoted $e^{\perp}$, can be drawn in two ways that are not homotopic to each other relative to endpoints within a face of $G^{\perp}$ (the red graph). A face F of a dual of spine that is adjacent to a boundary component C of Π may have more than two boundary components depending on how its edges are drawn around C. Figure 2.13 shows how F can be a pair of pants (left) or an annulus (right).

The next definition is taken from Propp's paper [21].

Definition 2.2.5 (Various definitions from Propp's paper)**.** Let X be a graph. Assign a direction to each edge of X (from one vertex the other). Denote a directed edge e joining vertices a and b with direction from a to b by (e, a, b).

1. The set of all edges of X with their directions is called an *orientation* of X. In other words, the edge-set of a digraph is an orientation of its undirected version.

2. A *directed path* is a sequence of directed edges having the form $(e_1, v_0, v_1), (e_2, v_1, v_2), \ldots, (e_n, v_{n-1}, v_n)$ where v_0 and v_n are the initial and terminal vertices of the path, respectively. A directed path whose initial and terminal vertices coincide is called a *directed cycle*.

3. Let R be an orientation of X. Pick an edge e and assign an arbitrary direction to it. Denote the resulting directed e by $\vec{e}$. If this direction agrees with the direction of e in R, then $\vec{e}$ is called a *forward edge* relative to R. If it does not, $\vec{e}$ is called a *backward edge*. A directed path consisting purely of forward edges is called a *forward path*.

4. Let x and y be vertices of X. The vertex y is said to be *accessible* from x (relative to R) if there is forward path from x to y. The vertices x and y are said to be *mutually accessible* (relative to R) if there is a forward path from x to y and vice versa. It can be seen that mutual accessibility is a equivalence relation. The resulting equivalence classes are called *accessibility classes*.

5. Let C be a directed cycle in X. The *circulation* of R around C is defined to be $|C_R^+| - |C_R^-|$, where C_R^+ is the set of forward edges of C and C_R^- is the set of backward edges of C. The *circulation* of R is overall defined to be the function c which assigns to each directed cycle C its circulation $c(C)$ of R around C.

The notion of an elementary cycle was first introduced in Definition (2.1.9) in the context of a planar graph. We now make it suit the context of a graph embedded on a surface.

Definition 2.2.6 (Elementary cycle)**.** Let Σ be a compact orientable surface with nonempty boundary and a distinguished boundary component C which we call *outermost*. Let X be a graph embedded on Σ and let A be a simple cycle on X such that the union of the edges of A is separating. Thus, A determines two subsurfaces of Σ. The *interior* of A is the subsurface of Σ separated by A whose boundary does not contain C. If A encircles a single face of X in its interior, then A is called an *elementary* cycle.

The following definition is also taken from Propp's paper [21] but modified to suit a dual of spine specifically.

Definition 2.2.7 (Standard and prescribed orientations)**.** Let G be the spine of a multiverse and let $G^\perp$ be its dual. Assign a direction to the edges of $G^\perp$ in such a way that every elementary cycle of $G^\perp$ encircling a black vertex of G has the clockwise direction and every elementary cycle of $G^\perp$ encircling a white vertex of G has the counterclockwise direction. The orientation obtained is called the *standard orientation* of $G^\perp$, denoted R^0.

Let M be a matching of G. Assign to an edge of $G^{\perp}$ its direction in R^0 if its dual edge in G does not belong to M. Assign to an edge of $G^{\perp}$ the direction opposite to its direction in R^0 if its dual edge in G belongs to M. The orientation obtained is called the *prescribed orientation of* $G^{\perp}$ *corresponding to* M.

A more general version of the following lemma is stated in Propp's paper [21] but he did not prove it. We restrict it to suit our context.

Lemma 2.2.8. *Let G be the spine of a multiverse and let $G^{\perp}$ be a dual of G. Let v be a vertex of G with degree d_v and let C be an elementary cycle of $G^{\perp}$ around v. Let R_M be the prescribed orientation of $G^{\perp}$ corresponding to a matching M of G. The circulation of R_M around C is $2 - d_v$ if v is black and $d_v - 2$ if v is white.*

Proof. Since C is an elementary cycle of $G^{\perp}$ around v, the face of $G^{\perp}$ containing v contains no other vertex of G. Since $G^{\perp}$ is the dual of G, the degree of v, denoted d_v equals the length of C. Assign a counterclockwise direction to C. Let M be a matching of G. Only one edge incident to v belongs to M, say e. Let $e^{\perp}$ be the edge of $G^{\perp}$ dual to e. Suppose that v is black. According to the prescribed orientation of $G^{\perp}$ corresponding to M, denoted R_M, the edges of C are directed clockwise in R_M except $e^{\perp}$. Thus, the numbers of forward edges and backward edges in C relative to R_M are 1 and $d_v - 1$, respectively. Hence, the circulation of R_M around C is $1 - (d_v - 1) = 2 - d_v$. Suppose that v is white. According to R_M, the edges of C are directed counterclockwise in R_M, except $e^{\perp}$. Therefore, the numbers of forward edges and backward edges in C relative to R_M are $d_v - 1$ and 1, respectively. As a result, the circulation of R_M around C is $(d_v - 1) - 1 = d_v - 2$. □

Remark. If G is embedded on a surface of genus 0, the circulation of R_M around an arbitrary cycle is completely determined by the circulation around every elementary cycle. Thus, by Lemma (2.2.8), all the prescribed orientations of $G^{\perp}$ corresponding to some matching of G have the same circulation. However, this is no longer true if G is embedded on a surface of positive genus because there may be cycles which form non-separating curves. In fact, different prescribed orientations of $G^{\perp}$ corresponding to different matchings of G may have different circulations. An example illustrating this remark is shown in Figure 2.19 (on page 93) where the multiverse is embedded on a punctured torus obtained by edge identifications on an octagon. All octagons show the same multiverse (black graph) along with its spine (green graph) and a dual of spine (red graph). Different octagons represent different states of the multiverse with its corresponding matching (thickened green edges belong to the matching) and its corresponding orientation (arrows on red edges). These correspondences are established in Lemmas (2.2.10) and (2.2.11). The four orientations which form the vertical column on the left have the same circulation which is different to each of the circulations of the four orientations on the right. The four orientations on the right are pairwise distinct.

Definition 2.2.9 (Set of prescribed orientations)**.** Let G be the spine of a multiverse and let $G^{\perp}$ be a dual of G. The set of all prescribed orientations of $G^{\perp}$ corresponding to some matching of G is henceforth denoted $\mathscr{R}$.

Lemma 2.2.10. *Let G be the spine of a multiverse and let $G^{\perp}$ be a dual of G. Let $\mathscr{M}$ be the set of all matchings of G. There is a bijection between $\mathscr{M}$ and $\mathscr{R}$.*

Proof. Let $k : \mathscr{M} \to \mathscr{R}$ be a map defined in the exact same way as how the prescribed orientation R of $G^{\perp}$ corresponding to a matching $M \in \mathscr{M}$ is constructed in Definition (2.2.7). It can be seen that k is injective, that is, two different matchings give different prescribed orientations. Now we define a map $l : \mathscr{R} \to \mathscr{M}$. Let O be the prescribed orientation of $G^{\perp}$ corresponding to a matching N of G. Then, $l(O)$ is defined to be N. It is clear from the definitions of k and l that they are inverses of each other. As a result, they are bijections. □

Lemma 2.2.11. *Let U be a multiverse; let G be its spine and let $G^{\perp}$ be a dual of G. There is a bijection between the set of all states of U, denoted $\mathscr{S}$, and $\mathscr{R}$.*

Proof. Let $\mathscr{M}$ be the set of all matchings of G. By Lemma (2.1.6), there is a bijection between $\mathscr{S}$ and $\mathscr{M}$. Note that the proof of this lemma is independent of the topology of the surface on which U is embedded. Thus, the lemma can be used here. By Lemma (2.2.10), there is a bijection between $\mathscr{M}$ and $\mathscr{R}$. As a result, there is a bijection between $\mathscr{S}$ and $\mathscr{R}$. □

A more general version of Definition (2.1.9) of twisting is given below.

Definition 2.2.12 (Twisting around a cycle)**.** Let G be the spine of a multiverse. A simple cycle in G is said to be *alternating* relative to a matching M if its edges alternately belong to M and the complement of M in G. Let C be an alternating cycle relative to M. Removing from M the edges of C which belong to M and adding to M the edges of C which do not belong to M turns M into another matching. This operation is called *twisting around C*.

Definition 2.2.13 (Positive and negative subsurfaces)**.** Let G be the spine of a multiverse embedded on a surface Π and let M be a matching of G. Let Δ be a subsurface of Π bounded by edges of G and boundary components of Π such that each boundary component of Δ that is made of edges of G is an alternating cycle relative to M. Recall that Π is orientable. An orientation of Π determines an orientation of Δ which, in turn, determines an orientation for all its boundary components. If, for all boundary components of Δ that are made of edges of G, the edges belonging to M are directed from their black vertices to their white vertices following the orientation inherited from Π, the subsurface Δ is said to be *positive relative to* M. If they are directed from their white vertices to their black vertices, Δ is said to be *negative relative to* M.

Definition 2.2.14 (Big twisting)**.** Let G be the spine of a multiverse embedded on a surface Π and let $G^{\perp}$ be a dual of G. Let $F_1, \ldots, F_m$ be m faces of G such that the closure of their union is a connected subsurface F of Π. Note that each boundary component of F either is made of edges of G or is a boundary component of Π. Suppose that the following conditions are also satisfied.

1. Each boundary component of F that is made of edges of G is an alternating cycle relative to a matching M. Let c be the circulation of the prescribed orientation of $G^{\perp}$ corresponding to M.
2. The subsurface F is either positive or negative relative to M.
3. Every edge of G which is adjacent to a face F_i on each side either belongs to all matchings whose corresponding orientations of $G^{\perp}$ have circulation c or does not belong to any matching whose corresponding orientation has circulation c.
4. No boundary component of F is the outermost boundary component of Π.

For all boundary components of F, removing from M the edges which belong to M and adding to M the edges which do not belong to M turns M into another matching M'. If F is positive relative to M and negative relative to M', then this operation is called a *big twisting down.* On the other hand, if F is negative relative to M and positive relative to M', then this operation is called a *big twisting up.*

Remark. It can be seen that twisting around a single face of the spine G, as defined in Definition (2.1.9) is also a big twisting where that face is the face F in the above definition. However, in the case of a multiverse on a surface of genus 0, transpositions correspond to twisting around a face of the *reduced* spine, not the spine. If we perform big twisting on the spine of a multiverse on a surface of genus 0, we will still obtain a lattice but it will be different from the lattice obtain by performing twisting on the reduced spine of the same multiverse. Similarly, the clock lattice of a multiverse on a surface of genus 0 where the covering relation is a transposition defined by Definition (2.1.8) is different from the clock lattice of the same multiverse where the covering relation is another type of transposition which will be defined by Definition (2.2.25). Figures 2.10 and 2.14 illustrate this difference. The motivation behind the conditions in Definition (2.2.14) is that a correspondence will be established between the set of big twisting that can be done on a matching and the set of pushing that can be done on an orientation. The details are presented in the following definitions and lemmas.

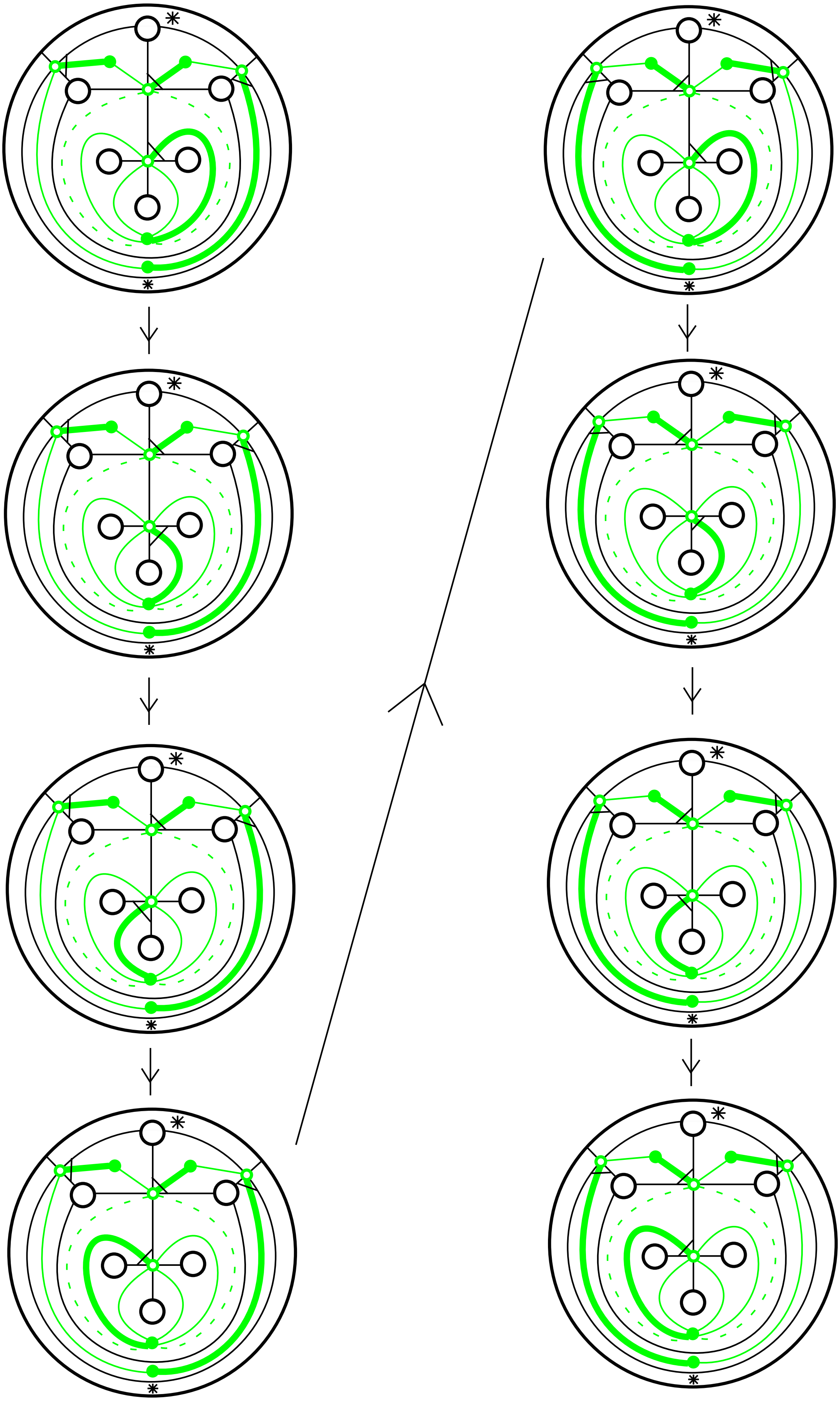


Figure 2.14

The following definitions are taken from Propp's paper [21].

Definition 2.2.15 (Pushing)**.** Let R be an orientation of a graph X. An accessibility class K is said to be *maximal relative to* R if all directed edges in R between K and its complement in the vertex-set of X, denoted K^c, point towards K. It is said to be *minimal relative to* R if these edges point toward K^c. If K is maximal, the operation of reversing the directed edges between K and K^c (which thereby makes K minimal) is called *pushing down.* If K is minimal, the operation of reversing the directed edges between K and K^c (which thereby makes K maximal) is called *pushing up.*

If we pick an arbitrary circulation, it's not necessarily true that at least some orientation will have it as its circulation. This is why Propp calls a circulation c *feasible* if there is an orientation having c as its circulation. However, not every feasible orientation is the prescribed orientation corresponding to some matching. Thus, we introduce the notion of *viable* circulation to further restrict the class of feasible circulations. Note that, in Definition (2.2.5), a circulation is defined based on an orientation but we can understand circulation as a function that maps each directed cycle in a graph to an integer.

Definition 2.2.16 (Feasible and viable circulation)**.** Let X be a graph. A *circulation* on X is a function that maps each directed cycle in X to an integer. A circulation c on X is said to be *feasible* if there is at least an orientation of X that has circulation c. Let G be the spine of a multiverse and $G^{\perp}$ a dual of G. A circulation c on $G^{\perp}$ is said to be *viable* if there exists a matching of G whose prescribed orientation of $G^{\perp}$ has circulation c.

It can be seen that every viable circulation is also feasible. The following proposition is precisely Proposition 1.1 in Propp's paper [21], which was proved by him.

Proposition 2.2.17. *Let c be a feasible circulation on a graph. The following statements are equivalent.*

(i) A directed edge e belongs to either every orientation having circulation c or no orientation having circulation c.

(ii) The endpoints of e belong to the same accessibility class relative to all orientations having circulation c.

Applying Proposition (2.2.17) to the spine and its dual gives the following corollary.

Corollary 2.2.18. *Let G be the spine of a multiverse and let $G^{\perp}$ be a dual of G. Fix a viable circulation c on $G^{\perp}$. Let e be an edge of G and let $e^{\perp}$ be its dual in $G^{\perp}$. The following statements are equivalent.*

(i) Either e belongs to every matching of G whose corresponding orientation of $G^{\perp}$ has circulation c or e belongs to no matching of G whose corresponding orientation of $G^{\perp}$ has circulation c.

(ii) $e^{\perp}$ joins two vertices in the same accessibility class relative to all orientations having circulation c.

Proof. Suppose (i) is true. This implies that $e^{\perp}$ (when given some specific orientation) either belongs to every orientation of $G^{\perp}$ which has circulation c *and* which corresponds to a matching of G or belongs to no orientation of $G^{\perp}$ which has circulation c *and* which corresponds to a matching of G. However, in order to invoke Proposition (2.2.17), we need to prove that $e^{\perp}$ either belongs to every orientation which has circulation c or belongs to no orientation which has circulation c. This amounts to proving that, if there exists an orientation of $G^{\perp}$ which has circulation c and is the prescribed orientation corresponding to some matching of G (the existence is guaranteed since c is viable), then every orientation of $G^{\perp}$ with circulation c is the prescribed orientation corresponding to some matching of G. Here is the proof. Let O be an orientation of $G^{\perp}$ with circulation c. Construct a matching N of G from O as follows. Let b be a black vertex and let C be the elementary cycle of $G^{\perp}$ around b. Assign to C a counterclockwise direction. Since b is black, Lemma (2.2.8) gives $c(C) = 2 - d_b$ where d_b is the degree of b. Thus, by Definition (2.2.5), we have $\left|C_O^+\right| - \left|C_O^-\right| = 2 - d_b$ where C_O^+ and C_O^- are the set of forward edges and the set backward edges of C relative to O, respectively. Moreover, $\left|C_O^+\right| + \left|C_O^-\right| = d_v$. Solving the system of equations for $\left|C_O^+\right|$ and $\left|C_O^-\right|$ gives $\left|C_O^+\right| = 1$ and $\left|C_O^-\right| = d_b$. Hence, there is precisely one edge in C that is forward relative to O. Since C is assigned a counterclockwise direction, there is precisely one edge in C that is directed counterclockwise, say x. The edge of G dual to x is chosen to belong to the matching N.

Similarly, let w be a white vertex and let D be the elementary cycle of $G^{\perp}$ around w. Assign to D a counterclockwise direction. Since w is white, Lemma (2.2.8) gives $c(D) = d_w - 2$ where d_w is the degree of w. Therefore, by Definition (2.2.5), we have $\left|C_D^+\right| - \left|C_D^-\right| = d_w - 2$ where C_D^+ and C_D^- are the set of forward edges and the set backward edges of D relative to O, respectively. Also, $\left|C_D^+\right| + \left|C_D^-\right| = d_w$. Solving the system of equations for $\left|C_D^+\right|$ and $\left|C_D^-\right|$ gives $\left|C_D^+\right| = d_w - 1$ and $\left|C_D^-\right| = 1$. Thus, there is exactly one edge in D that is backward relative to O. Since D is assigned a counterclockwise direction, there is exactly one edge in D that is directed clockwise, say y. The edge of G dual to y is chosen to belong to the matching N. Repeat the same procedure to all vertices of G to finish the construction of N. It can be seen that N is a set of edges of G where each vertex of G is adjacent to exactly one edge of N. This implies that N is a matching of G. Therefore, O is the prescribed orientation of $G^{\perp}$ corresponding to N, completing the proof. Having proved this, we deduce that the set of orientations of $G^{\perp}$ with circulation c is equal to the set of orientations of $G^{\perp}$ with circulation c that are prescribed orientations of matchings on G and, in fact, bijective with the set of matchings of G whose prescribed orientations of $G^{\perp}$ have circulation c. By Proposition (2.2.17), the endpoints of $e^\perp$ belong to the same accessibility class relative to all orientations having circulation c, which is statement (ii).

Conversely, suppose that (ii) is true. By Proposition (2.2.17), either $e^\perp$ belongs to every orientation having circulation c or $e^\perp$ belongs to no orientation having circulation c. From how the prescribed orientation of $G^\perp$ corresponding to a matching of G is constructed (Definition (2.2.7)), (i) holds. □

Lemma 2.2.19. *Let R be an orientation of a graph X and let c be the circulation of R. If C is a directed cycle in R, then the direction of every edge of C is the same in all orientations of X having circulation c.*

Proof. Since C is a directed cycle in R, the circulation of R around C is either n or $-n$ where n is the length of C. Changing the direction of at least one edge of C results in a circulation other than n or $-n$. As a result, the direction of each edge of C is unchanged in all orientations of X having circulation c. □

Lemma 2.2.20 (Big twisting preserves circulation)**.** *Let G be the spine of a multiverse embedded on a surface Π and let $G^\perp$ be a dual of G. Let M be a matching of G and let T be a big twisting that turns M into another matching M'. Let R and R' be the prescribed orientations of $G^\perp$ corresponding to M and M', respectively. Then, R and R' have the same circulation. In other words, a big twisting preserves circulation.*

Proof. Let F be the subsurface of Π where T is performed. Let C be a boundary component of F that is made of edges of G. Since T is a big twisting, C is an alternating cycle in G relative to M. Label the black vertices of C by $b_1, \dots, b_n$ and the white vertices of C by $w_1, \dots, w_n$ in such a way that the edge joining b_i and w_i belongs to M for all $1 \leqslant i \leqslant n$. The cycle C identifies two sides on a small neighbourhood around it, say α and β. We proceed to prove that all edges of $G^\perp$ dual to an edge of C are directed from one side to the other in R and in the opposite direction in R'. In the standard orientation R^0 of $G^\perp$, the edges around the face dual to b_i are oriented clockwise and the edges around the face dual to w_i are oriented counterclockwise. This implies that the edges of $G^\perp$ dual to an edge of C are directed alternately from α to β and vice versa. Denote an edge of C by (black vertex, white vertex). Assume that $(b_i, w_i)^\perp$ is directed from β to α in R^0. Thus, $(b_i, w_{i-1 \bmod n})^\perp$ is directed from α to β in R^0. Since $(b_i, w_i) \in M$, the direction of $(b_i, w_i)^\perp$ is flipped in R. Hence, all edges dual to an edge of C are directed from α to β in R (see Figure 2.15). A similar argument shows that all edges dual to an edge of C are directed from β to α in R'.

Let F be the subsurface of Π where T is performed and let $C_1, \dots, C_m$ be the boundary components of F that are made of edges of G. Fix an orientation for Π. Since T is a big twisting, F is either positive or negative relative to M (depending on whether T is

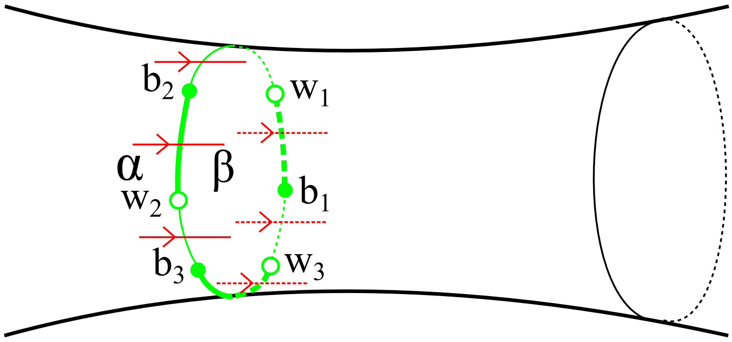


Figure 2.15

down or up). Suppose that F is positive relative to M. Thus, for all C_i, the edges of C_i belonging to M are directed from their black vertices to their white vertices following the fixed orientation of F inherited from Π. Combining this with the fact that all edges dual to an edge of C_i are directed from one side of C_i to the other (proved in the previous paragraph), we come to the conclusion that all edges dual to an edge of C_i are directed towards F in R and away from F in R'. Let D be a directed cycle in $G^\perp$. Suppose that D does not contain any edge dual to an edge of C_i. Therefore, every edge of D has the same direction in R and R', which implies that the circulations of R and R' around D are identical. Now, assume that D has at least one edge dual to an edge of C_i. Since F is a subsurface of Π, this implies that D has an even number of edges dual to an edge of C_i. Combining this with the fact that all edges dual to an edge of C_i are directed towards F in R, we conclude that the number of forward edges of D dual to an edge of C_i equals the number of backward edges of D dual to an edge of C_i. When the directions of all edges of D dual to an edge of C_i are flipped as R is turned into R', the forward edges of D dual to an edge of C_i become backward and vice versa. Hence, the total number of forward edges of D gains and loses by the same amount and so does the total number of backward edges. Therefore, the circulations of R and R' around D are again identical. As a result, R and R' have the same circulation. Apply the same arguments in the case where F is negative relative to M. □

Definition 2.2.21 (Special accessibility class)**.** Let G be the spine of a multiverse embedded on a surface Π and let $G^\perp$ be a dual of G. The accessibility class (relative to an orientation) which contains the vertex dual to the face of G one of whose boundary components is the outermost boundary component of Π is called the *special accessibility class*.

Note that, by Lemma (2.2.19), the special accessibility class is unique and the same in all orientations having the same circulation. In fact, all accessibility classes are the same in all orientations with the same circulation, as formally stated in the following lemma.

Lemma 2.2.22. *Let X be a graph and let c be a feasible circulation on X. All accessibility classes are the same in all orientations with circulation c.*

Proof. This is a consequence of Lemma (2.2.19). □

Lemma 2.2.23. *Let M be a matching of the spine G of a multiverse U embedded on a surface Π and let R be the prescribed orientation of a dual $G^\perp$ of G corresponding to M. There is a bijection between the set BTw_M of all big twisting down (up) that can be done on M and the set Pu_R of all pushing down (up) that can be done on R except any pushing down (up) done on the special accessibility class of R.*

Proof. Define a map $r : \mathrm{BTw}_M \to \mathrm{Pu}_R$ as follows. Let T be a big twisting down that turns M into another matching M'. The pushing down $P = r(T)$ is defined to be the one that turns R into the orientation R' corresponding to M' according to the bijection established in Lemma (2.2.10). We now prove that P is indeed a pushing down. Note that, by Lemma (2.2.20), R and R' have the same circulation, denoted c. Let F be the subsurface of Π where T is performed. Thus, F is the closure of the union of some faces of G, labeled $F_1, \ldots, F_m$. Let V_i be the vertex of $G^\perp$ dual to F_i. Our first step is to prove that all vertices of $\{V_i\}_{i=1}^m$ belong to the same accessibility class which consists of only those in $\{V_i\}_{i=1}^m$. Let V_a and V_b be two vertices in $\{V_i\}_{i=1}^m$ that are joined by an edge $e^\perp$ whose dual in G is denoted e. Since e is adjacent to F_a on one side and F_b on the other, by Definition (2.2.14), e either belongs to all matchings whose corresponding orientations of $G^\perp$ have circulation c, or does not belong to any matching whose corresponding orientation has circulation c. Thus, by Corollary (2.2.18), $e^\perp$ joins two vertices in the same accessibility class relative to all orientations having circulation c. Hence, V_a and V_b belong to the same accessibility class. Repeating the same argument to the remaining vertices in $\{V_i\}_{i=1}^m$ proves that all those in $\{V_i\}_{i=1}^m$ belong to the same accessibility class, say K. Suppose that there is a vertex V which is not dual to any F_i but also belongs to K. Therefore, by the fourth item in Definition (2.2.5), there is a directed cycle D (which is directed by R) containing V and at least one of the vertices in $\{V_i\}_{i=1}^m$. Since V is not dual to any F_i, there is an edge $d^\perp$ of D joining one of the vertices in $\{V_i\}_{i=1}^m$, say V_c, to a vertex outside $\{V_i\}_{i=1}^m$. This implies that the edge d of G dual to $d^\perp$ is adjacent to F_c on one side and to a face that does not belong to $\{F_i\}_{i=1}^m$ on the other side. Thus, d is one of the edges of G that form a boundary component of F. Since T is a big twisting, by Definition (2.2.14), d belongs to some matchings whose corresponding orientations have circulation c but not to some other matchings whose corresponding orientations also have circulation c. Therefore, the direction of $d^\perp$ is not always the same in all orientations with circulation c. However, since $d^\perp$ belongs to the directed cycle D, its direction is unchanged in all orientations having circulation c by Lemma (2.2.19). This leads to a contradiction. As a

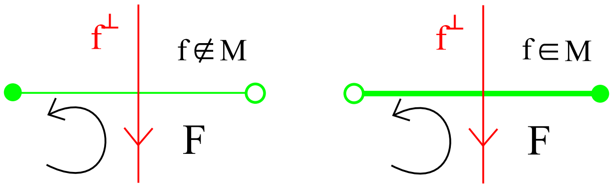


Figure 2.16

result, all of $\{V_i\}_{i=1}^m$ belong to the same accessibility class K which consists of only the vertices in $\{V_i\}_{i=1}^m$.

Next, we prove that all edges between K and its complement in the vertex-set of $G^\perp$, denoted K^c, point towards K in R. Let $f^\perp$ be one of these edges. Since $f^\perp$ joins a vertex in K and a vertex outside K, it crosses a boundary component B of F ($f^\perp$ is similar to $d^\perp$ in the previous paragraph). Hence, the edge f dual to $f^\perp$ is one of the edges of G that form B. Since T is a big twisting, the edges of B alternately belong to M and the complement of M in G. Since T is a big twisting down, F is positive relative to M. Therefore, the edges of B belonging to M are directed from their black vertices to their white vertices following the orientation inherited from F which inherits its orientation from Π. In the following arguments, we look at F from the side to which its normal vector points, that is, its positive side. Suppose that f does not belong to M. Thus, the direction of $f^\perp$ in R is the same as its direction in the standard orientation R^0, that is, clockwise around black vertices. Hence, $f^\perp$ points towards F as shown in the left diagram of Figure 2.16. Similarly, assume that f belongs to M. Therefore, the direction of $f^\perp$ in R is opposite to its direction in R^0. Thus, $f^\perp$ again points towards F as shown in the right diagram of Figure 2.16. As a result, all edges between K and K^c point towards K in R. This implies that K is a maximal accessibility class in R. Since T is a big twisting down, F is negative relative to M'. Repeat the same argument as above to arrive at the conclusion that K is a minimal accessibility class in R'. Since K is maximal relative to R and minimal relative to R', the operation $P = r(T)$ is a pushing down.

Since no boundary component of F is the outermost boundary component C_Π of Π (the fourth condition in Definition (2.2.14)), none of the faces in $\{F_i\}_{i=1}^m$ is adjacent to C_Π. Thus, none of the vertices in $\{V_i\}_{i=1}^m$ is dual to the face of G one of whose boundary components is C_Π. Hence, K is not the special accessibility class of R. Therefore, P really belongs to Pu_R.

Third condition of a big twisting Define a map $s : \mathrm{Pu}_R \to \mathrm{BTw}_M$ as follows. Let P be the pushing down that turns R into another orientation R'. The big twisting down $T = s(P)$ is defined to be the one that turns M into the matching M' corresponding to R'. We now prove that T is indeed a big twisting down. Note that, since R' is obtained from R by a pushing down, they have the same circulation, denoted x. Let

L be the accessibility class to which P is applied. Let $W_i, \ldots, W_n$ be the vertices in L. Let E_i be the face of G dual to W_i and let E be the closure of their union. If E is disconnected, then some pair of vertices of L are not mutually accessible, which contradicts the definition of an accessibility class. Thus, E is connected. Let g be an edge of G which is adjacent to a face E_i on each side. Thus, the edge $g^\perp$ dual to g joins two vertices of L. Since $g^\perp$ joins two vertices of an accessibility class, it belongs to a directed cycle in R. Therefore, by Lemma (2.2.19), the direction of $g^\perp$ is the same in all orientations having circulation x. This implies that g belongs to either every matching of G whose corresponding orientation of $G^\perp$ has circulation x or no matching of G whose corresponding orientation of $G^\perp$ has circulation x. This is precisely the third condition in Definition (2.2.14).

Fourth condition of a big twisting Let H be a boundary component of E that is made of edges of G. Since every edge h of H is adjacent to a face E_i on one side but not to any face of $\{E_i\}_{i=1}^n$ on the other side, its dual $h^\perp$ joins a vertex in the accessibility class L and a vertex outside L. Since P is a pushing down, L is maximal relative to R and minimal relative to R'. Therefore, $h^\perp$ points towards E in R and away from E in R'. Since H is a cycle in the bipartite graph G, its vertices alternate between black and white. Label the black vertices by $b_1, \ldots, b_l$ and the white vertices by $w_1, \ldots, w_l$ in the orientation inherited from the orientation of E so that b_i is placed between w_i and $b_{i-1 \bmod l}$. Label each edge of H by an ordered pair (u, v) where u and v are its endpoints and the edge is directed from u to v following the orientation around H inherited from E. In the following arguments, we look at E from the side to which its normal vector points, that is, its positive side. Since $(b_i, w_i)^\perp$ points towards E in R, this direction is opposite to its direction in the standard orientation R^0 of $G^\perp$ (left diagram of Figure 2.17). Thus, (b_i, w_i) belongs to the matching M corresponding to R. Since $(w_i, b_{i+1 \bmod l})^\perp$ points towards E in R, this direction is the same as its direction in R^0 (right diagram of Figure 2.17). Hence, $(w_i, b_{i+1 \bmod l})$ does not belong to M. As a result, the edges of H alternately belong to M and the complement of M in G. This implies that H is an alternating cycle relative to M, which is precisely the first condition in Definition (2.2.14). Moreover, we have proved that, as we proceed around H following its orientation inherited from E, all edges belonging to M are always directed from their black vertices to their white vertices. This is precisely the second condition in Definition (2.2.14) and also implies that E is positive relative to M. Repeat the same argument as above to arrive at the conclusion that E is negative relative to M'. Since E is positive relative to M and negative relative to M', the operation $T = s(P)$ is a big twisting down. Note that, since P cannot be done on the special

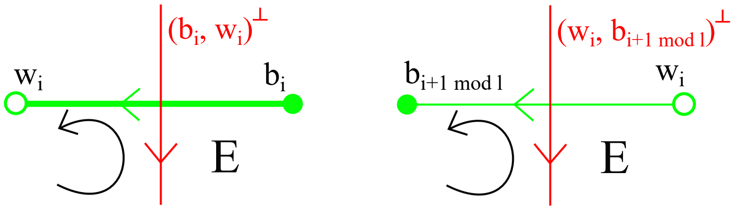


Figure 2.17. The red edge is given its orientation in R. The green edge is oriented according to the boundary orientation of E.

accessibility class of R, L is not the special accessibility class of R. Thus, E has no boundary component that is the outermost boundary component of Π, which is the fourth condition in Definition (2.2.14).

There is a subtle point to be made. The closure of a union of faces of a graph embedded on a surface is not necessarily a surface whereas Definition (2.2.14) requires E to be a connected subsurface of Π. Thus, we need to prove that E is really a connected subsurface of Π. We have already proven that E is connected. The only way in which E fails to be a surface is when two faces of $\{E_i\}_{i=1}^n$ meet only at some vertices but not along any edge. We prove this cannot happen by contradiction. Suppose that it happens. Let a be one of these vertices. Then, we have two alternating cycles X and Y relative to the matching M that intersect each other at a. Let x_1 and x_2 be the two edges of X incident to a. Let y_1 and y_2 be the two edges of Y incident to a. Since X is alternating relative to M, precisely one among x_1 and x_2 belongs to M. Assume it is x_1. Thus, none of x_2, y_1 and y_2 belongs to M, which contradicts the fact that Y is alternating relative to M. Therefore, no two faces in $\{E_i\}_{i=1}^n$ meet only at some vertices but not along any edge. As a result, E is a connected subsurface of Π.

It can be seen from the definitions of r and s that they are inverses of each other. As a result, they are bijections. Following a similar proof, we can prove that there is a bijection between the set of all big twisting up that can be done on M and the set of all pushing up that can be done on R. □

Definition 2.2.24 (Partial n-transposition)**.** Let S be a state of a multiverse U embedded on a surface Π. Let $v_0, \dots, v_{n-1}$ be n vertices of U and let Y_i be the face occupied by the marker at v_i. Suppose that, for each i, rotating the marker at v_i to a different corner moves the marker to $Y_{i-1 \bmod n}$ and turns S into another state S'. The rotations are required to be all clockwise or all counterclockwise with respect to an orientation of Π. Then, the n simultaneous rotations of the markers at the v_i's are called a *partial n-transposition* on S.

Let α_i and α_i' be the corners where the marker at v_i lies in S and S', respectively. Draw a curve γ_i joining v_i and $v_{i+1 \bmod n}$ in such a way that γ_i lies entirely in Y_i, does not intersect U at any point other than v_i and $v_{i+1 \bmod n}$, is incident to v_i through α_i and is incident to $v_{i+1 \bmod n}$ through α'_i. It can be seen that the union of all γ_i's is a simple closed curve γ which is called the *contour curve* of the partial transposition.

The partial n-transposition is purposely defined to be equivalent to twisting around a cycle of length n.

Definition 2.2.25 (Transposition)**.** Let S be a state of a multiverse U embedded on a surface Π and let $G^{\perp}$ be a dual of spine of U. Let $Z_1, \ldots, Z_t$ be t partial transpositions that can be done on S. Let γ_{Z_i} be the contour curve of Z_i. Let c be the circulation of the orientation of $G^{\perp}$ which corresponds to S according to the bijection in Lemma (2.2.11). Let Ψ be a subsurface of Π such that the following conditions are satisfied.

1. Every γ_{Z_i} is a boundary component of Ψ. A boundary component of Ψ is either a contour curve of Z_i or a boundary component of Π.
2. Every corner in the interior of Ψ is either occupied by a marker in every state of U whose corresponding orientation of $G^{\perp}$ has circulation c or unoccupied by a marker in every state of U whose corresponding orientation of $G^{\perp}$ has circulation c.
3. The markers at the vertices in different Z_i's rotate in the same direction through the interior of Ψ when going from their initial positions to their final positions.
4. No boundary component of Ψ is the outermost boundary component of Π.

The operation of performing all Z_i simultaneously is called a transposition on S, denoted Z. If the markers in Z rotate clockwise (counterclockwise), Z is said to be *clockwise* (*counterclockwise*).

Lemma 2.2.26. *Let S be a state of a multiverse U embedded on a surface Π and let M be its corresponding matching of the spine G of U as determined by the bijection in Lemma* (2.1.6)*. There is a bijection between the set Tr_S of all clockwise (counterclockwise) transpositions that can be done on S and the set BTw_M of all big twisting down (up) that can be done on M.*

Proof. In this proof, we will denote an edge of G using the same triple notation (black vertex, white vertex, corner) as in the proof of Lemma (2.1.10). Define a map $\mu : \mathrm{Tr}_S \to \mathrm{BTw}_M$ as follows. Let Z be a clockwise transposition that turns S into another state S'. The big twisting down $T = \mu(Z)$ is defined to be the one that turns M into the matching M' corresponding to S'. We now prove that T is indeed a big twisting down. Let Ψ be the subsurface of Π to which Z is applied. Let γ_1 be a boundary component of Ψ. Thus, γ_1 is the contour curve of a partial transposition Z_1 among the partial transpositions which Z consists of. Let $v_0, \ldots, v_{n-1}$ be the vertices of S whose markers are rotated by Z_1 and let S_1 be the state obtained from performing Z_1 on S (S_1 and S' are not necessarily identical). Let α_i and α'_i be the corners where the marker at v_i lies in S and S', respectively. Let Y_i be the face occupied by the marker at v_i in S. It can be seen that the edges (Y_i, v_i, α_i)

and $(Y_{i-1 \bmod n}, v_i, \alpha'_i)$ form a cycle in G which satisfies Definition (2.2.24) of the contour curve of Z_1. Thus, we can call this cycle γ_1. Since, in S, the marker at v_i lies in Y_i, the edges (Y_i, v_i, α_i) belong to M, the matching corresponding to S (according to the bijection defined in the proof of Lemma (2.1.6)). Moreover, the edges of γ_1 alternately belong to those two collections of edges. Therefore, γ_1 is alternating relative to M. This implies that Ψ satisfies the first condition in Definition (2.2.14).

Let $G^\perp$ be the dual of G. Let R be the orientation of $G^\perp$ which corresponds to M (and therefore corresponds to S according to the bijection in Lemma (2.2.11)). Let c be the circulation of R. Let β be a corner in the interior of Ψ. Thus, β is either occupied by a marker in every state of U whose corresponding orientation of $G^\perp$ has circulation c or unoccupied by a marker in every state of U whose corresponding orientation of $G^\perp$ has circulation c. Let v_β be the vertex of U where β is located and call the white vertex of G at v_β by the same name. Let Y_β be the face of U to which β belongs and call the black vertex of G at Y_β by the same name. Therefore, by the bijection in Lemma (2.1.6), the edge $(Y_\beta, v_\beta, \beta)$ of G either belongs to all matchings whose corresponding orientations of $G^\perp$ have circulation c or does not belong to any matching whose corresponding orientation of $G^\perp$ has circulation c. Since every boundary component of Ψ that is not a boundary component of Π is made of edges of G, Ψ is the closure of the union of some faces of G, denoted $\Psi_1, \ldots, \Psi_m$. Since β is in the interior of Ψ, $(Y_\beta, v_\beta, \beta)$ is adjacent to one of the Ψ_i's on each side. As a result, the third condition of Definition (2.2.14) is satisfied. Since no boundary component of Ψ is the outermost boundary component of Π, the fourth condition of Definition (2.2.14) is satisfied as well.

In the following arguments, we look at Ψ from the side to which its normal vector points, that is, its positive side. Since Z is clockwise, the marker at v_i rotates clockwise through the interior of Ψ to go from α_i to α'_i. Thus, the edges (Y_i, v_i, α_i) are traversed from their black vertices to their white vertices following the positive orientation of γ_1 (Figure 2.18). Since these edges belong to M, the second condition in Definition (2.2.14) is satisfied. This also implies that Ψ is positive relative to M. Let M_1 be the matching corresponding to S_1 (M_1 and M' are not necessarily identical). Since γ_1 is alternating relative to M, it is also alternating relative to M_1. Since the edges (Y_i, v_i, α_i) belong to M, the edges $(Y_{i-1 \bmod n}, v_i, \alpha'_i)$ belong to M_1. Since the edges (Y_i, v_i, α_i) are traversed from their black vertices to their white vertices following the positive orientation of γ_1, the edges $(Y_{i-1 \bmod n}, v_i, \alpha'_i)$ are traversed from their white vertices to their black vertices following the positive orientation of γ_1. Since Z_1 is one among the partial transpositions which Z consists of, the position of the marker at v_i in S_1 is the same as in S', for all i. Therefore, the edges of γ_1 which belong to M_1 also belong to M', namely the edges $(Y_{i-1 \bmod n}, v_i, \alpha'_i)$. Since these edges are traversed from their white vertices to their black

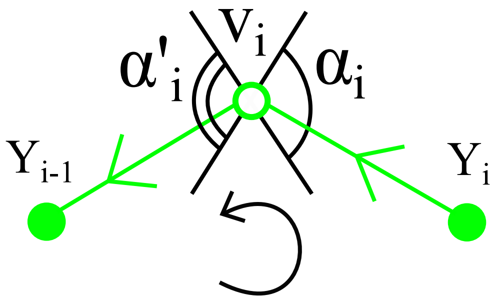


Figure 2.18

vertices following the positive orientation of γ_1, Ψ is negative relative to M'. Since Ψ is positive relative to M and negative relative to M', the operation $T = \mu(Z)$ is a big twisting down.

Define a map $\nu : \mathrm{BTw}_M \to \mathrm{Tr}_S$ as follows. Let T be the big twisting down that turns M into another matching M'. The clockwise transposition $Z = \nu(T)$ is defined to be the one that turns S into the state S' corresponding to M'. We now prove that Z is indeed a clockwise transposition. Let Φ be the subsurface of Π to which T is applied. Thus, Φ is the closure of the union of some faces of G, labeled $\Phi_1, \ldots, \Phi_k$. Let Q be a boundary component of Φ made of edges of G. Thus, Q is an alternating cycle relative to M and M'. Label the white vertices of Q by $w_1, \ldots, w_l$ and the black vertices of Q by $b_1, \ldots, b_l$ in such a way that, for all i, the edge (b_i, w_i, δ_i) belongs to M and the edge $(b_{i-1 \bmod n}, w_i, \delta'_i)$ belongs to M', where δ_i and δ'_i are the corners of w_i through which (b_i, w_i, δ_i) and $(b_{i-1 \bmod n}, w_i, \delta'_i)$ are incident to w_i, respectively. Thus, the marker at w_i occupies δ_i in S and δ'_i in S'. Label each face of U by the same name as the black vertex of G placed in it. Hence, the marker at w_i occupies the face b_i in S and the face $b_{i-1 \bmod n}$ in S'. As a result, there is a partial l-transposition Z_Q rotating the marker at each v_i from its position in S to its position in S'. In addition, Q is the contour curve of Z_Q. This is precisely the first condition in Definition (2.2.25).

Let e be an edge of G which is adjacent to a face Φ_i on each side. Thus, e either belongs to all matchings whose corresponding orientations of $G^\perp$ have circulation c, the circulation of the orientation R corresponding to M or does not belong to any matching whose corresponding orientation has circulation c. Let θ be the corner of a vertex v_θ of U through which e is incident to v_θ. Since e is adjacent to one of the Φ_i's on each side, θ is in the interior of Φ. Therefore, θ is either occupied by a marker in every state of U whose corresponding orientation of $G^\perp$ has circulation c or unoccupied by a marker in every state of U whose corresponding orientation of $G^\perp$ has circulation c. This is precisely the second condition in Definition (2.2.25).

Let Q be a boundary component of Φ made of edges of G. Since T is a big twisting down, Φ is positive relative to M and negative relative to M'. Since the edge (b_i, w_i, δ_i)

belongs to M for all i, it is traversed from b_i to w_i following the positive orientation of Q inherited from Φ. Similarly, since the edge $(b_{i-1 \bmod n}, w_i, \delta'_i)$ belongs to M' for all i, it is traversed from w_i to $b_{i-1 \bmod n}$ following the positive orientation of Q. Therefore, for all i, the marker at v_i rotates clockwise through the interior of Φ from δ_i to δ'_i. This is precisely the third condition in Definition (2.2.25). Since no boundary component of Φ is the outermost boundary component of Π, the fourth condition of Definition (2.2.25) is satisfied as well. As a result, the operation $Z = \nu(T)$ is a clockwise transposition.

It can be seen from the definitions of μ and ν that they are inverses of each other. As a result, they are bijections. Following a similar proof, we can prove that there is a bijection between the set of all counterclockwise transpositions that can be done on S and the set of all big twisting down that can be done on M. □

Lemma 2.2.27. *Let S be a state of a multiverse U and let R be its corresponding orientation of the dual of spine $G^\perp$ of U, as determined by the bijection described in Lemma (2.2.11). There is a bijection between the set Tr_S of all clockwise (counterclockwise) transpositions that can be done on S and the set Pu_R of all pushing down (up) that can be done on R except any pushing down (up) done on the special accessibility class of R.*

Proof. By Lemma (2.1.6), there is a bijection between the set of states of U and the set of matchings of the spine G of U. Note that the proof of this lemma is independent of the topology of the surface on which U is embedded. Therefore, the lemma can be used here. Let M be the matching of G which corresponds to S and let BTw_M be the set of all big twisting down that can be done on M. By Lemma (2.2.26), there is a bijection between Tr_S and BTw_M. By Lemma (2.2.23), there is a bijection between BTw_M and Pu_R. Therefore, there is a bijection between Tr_S and Pu_R. An injective map from Tr_S to Pu_R can be defined as $\kappa = r \circ \mu$ where μ is the map μ from the proof of Lemma (2.2.26) and r is the map r from the proof of Lemma (2.2.23). An injective map from Pu_R to Tr_S can be defined as $\lambda = \nu \circ s$ where s is the map s from the proof of Lemma (2.2.23) and ν is the map ν from the proof of Lemma (2.2.26). It can be seen that κ and λ are inverses of each other and therefore bijections. □

Now we are a few steps away from proving the clock theorem for a multiverse on a surface of positive genus using the following theorem by Propp [21] where the definitions of an orientation and a pushing down are precisely Definitions (2.2.5) and (2.2.15).

Theorem 2.2.28 (Propp's theorem about pushing)**.** *Let $\mathscr{R}$ be the set of orientations having a fixed circulation of a connected graph X and let K be an accessibility class relative to all orientations in $\mathscr{R}$ (if a set of vertices is an accessibility class relative to an orientation, it is an accessibility class relative to all other orientations having the same circulation). An orientation R is said to cover another orientation R' if R' is obtained* *from R by pushing down at a maximal accessibility class other than K. Then, the covering relation makes $\mathscr{R}$ into a distributive lattice, called the* Propp's lattice *of X with respect to K, denoted $\mathcal{P}_X$. The accessibility class K is said to be* unpushable.

By definition, a lattice is a connected graph. However, in the case of a multiverse embedded on a surface with positive genus, it is not guaranteed that a lattice can be constructed from the set of states with the current definition of transpositions. However, we can construct a disconnected lattice which is defined as follows.

Definition 2.2.29 (Disconnected lattice)**.** A disconnected lattice is a disconnected graph where each connected component is a lattice.

Here is the proof of the clock theorem in the case of a surface of positive genus.

Theorem 2.2.30 (The clock theorem on a surface of positive genus)**.** *Let U be a multiverse as defined in Definition* (2.2.1) *and let $\mathscr{S}$ be the set of all states of U. A state S is said to cover another state S' if S' is obtained from S by a clockwise transposition. Then, $\mathscr{S}$ equipped with this covering relation is a possibly disconnected lattice, called the* clock lattice *of U.*

Proof. Let $G^{\perp}$ be the dual of spine of U. Let $\mathscr{R}$ be the set of all orientations of $G^{\perp}$ corresponding to some state of U (by Lemma (2.2.11)). Let $\omega : \mathscr{S} \to \mathscr{R}$ be the bijection mapping a state of U to the orientation of $G^{\perp}$ corresponding to it. Note that the orientations in $\mathscr{R}$ may have different circulations. Partition $\mathscr{R}$ into subsets each of which contains orientations having the same circulation. Thus, $\mathscr{R} = \mathscr{R}_1 \cup \cdots \cup \mathscr{R}_n$ where the orientations in $\mathscr{R}_i$ have circulation c_i. This partition induces a partition of $\mathscr{S}$ through ω. Hence, $\mathscr{S} = \mathscr{S}_i \cup \cdots \cup \mathscr{S}_n$ where $\mathscr{S}_i = \omega(\mathscr{R}_i)$. In other words, $\mathscr{S}_i$ is the set of all states of U whose corresponding orientations of $G^{\perp}$ have circulation c_i. Since all orientations of $\mathscr{R}_i$ have the same circulation c_i, they have the same special accessibility class, denoted K_i. Let $\lessdot_{\mathscr{R}_i}$ be the covering relation of $\mathscr{R}_i$ as defined in Theorem (2.2.28). By Theorem (2.2.28), each $\mathscr{R}_i$ equipped with $\lessdot_{\mathscr{R}_i}$ is a lattice where K_i is the unpushable accessibility class. Let $\lessdot_{\mathscr{S}}$ be the covering relation defined in the statement of this theorem. Assume that S and S' belong to the same $\mathscr{S}_i$. By Lemma (2.2.27), we have $S \lessdot_{\mathscr{S}} S'$ implies $\omega(S) \lessdot_{\mathscr{R}_i} \omega(S')$. Therefore, the Hasse diagram of $\mathscr{S}_i$ equipped with $\lessdot_{\mathscr{S}}$ is isomorphic to the Propp's lattice of $\mathscr{R}_i$ equipped with $\lessdot_{\mathscr{R}_i}$. As a result, $\mathscr{S}$ equipped with $\lessdot_{\mathscr{S}}$ is a possibly disconnected lattice. It is connected when all orientations of $G^{\perp}$ corresponding to some state of U have a unique circulation, that is, when all the c_i's are the same. It is disconnected when the orientations of $G^{\perp}$ corresponding to some state of U have different circulations. In this case, each connected component of the lattice consists of states whose corresponding orientations have the same circulation. In other words, each $\mathscr{S}_i$ is a connected lattice. □

Remark. If we apply Theorem (2.2.30) to a string as defined in Definition (1.2.2), we will obtain the same lattice as the one obtained from Theorem (1.4.17).

2.3 Connection with contact topology

Part of the motivation for this project, considering generalisations of Kauffman's clock lattice, comes from contact topology. As we have seen, a clock lattice is defined from states and transpositions on a multiverse, which is just a surface with a collection of curves satisfying some properties, together with some decorations. As such, clock lattices can be studied in their own right. But similar structures arise in studying 3-dimensional contact geometry and topology, in particular in the study of *contact categories*, as suggested by Honda in [10, 11].

A contact structure on a 3-manifold is a nowhere integrable 2-plane field. We do not study contact structures in detail here, but we simply note some of the possible connections between contact categories and clock lattice. A contact category for a compact oriented surface Σ is an algebraic object which encodes the possible contact structures on $\Sigma \times [0,1]$. Let $\mathscr{C}(\Sigma, F)$ be a contact category where $F \subseteq \partial\Sigma$ is a finite set of alternating signed points on the boundary of Σ. The objects of $\mathscr{C}(\Sigma, F)$ represent contact structures near a surface in $\Sigma \times [0,1]$ while the morphisms represent contact structures in the submanifold of $\Sigma \times [0,1]$ bounded by two such surfaces. Giroux has proved in [8] that the contact structures near a surface in $\Sigma \times [0,1]$, that is, the objects of $\mathscr{C}(\Sigma, F)$, are described by dividing sets on (Σ, F) up to isotopy. A *dividing set* Γ on (Σ, F) is an oriented 1-manifold $\Gamma \subseteq \Sigma$ such that (i) $\partial\Gamma = F$ is an oriented 0-manifold and (ii) $\Gamma \setminus F = R_+ \sqcup R_-$ where R_+, R_- are oriented substructures of Σ and $\partial R_\pm$ contains $\pm\Gamma$. A dividing set is trivial if it contains a contractible closed curve (i.e. a curve which bounds a disc in Σ). Two dividing sets Γ_0 and Γ_1 are isotopic if there is a homotopy (relative to endpoints) Γ_t connecting them. Figure 2.20 gives an example of a dividing set on an annulus.

The morphisms of $\mathscr{C}(\Sigma, F)$ are constructed as follows. Consider $\Sigma \times I$. We put dividing sets Γ_0 and Γ_1 on $\Sigma \times \{0\}$ and $\Sigma \times \{1\}$ respectively. Next, we put dividing curves on $\partial\Sigma \times [0,1]$, which are of the form $\{\text{point}\} \times [0,1]$, such that they are spaced evenly between elements of F. Then, we join the endpoints of these dividing curves that are in $\Sigma \times \{0\}$ with elements of $F \times \{0\}$ and the endpoints that are in $\Sigma \times \{1\}$ with elements of $F \times \{1\}$ in such a way that the resulting curve (or curves) are closed. The morphisms that send Γ_0 to Γ_1 are the isotopy classes of contact structures on $\Sigma \times \{1\}$ with the boundary conditions just described (there may be many such isotopy classes of contact structures and each of them gives a distinct morphism). With the objects and the morphisms between them at hand, a directed graph can be drawn to represent the contact category with objects

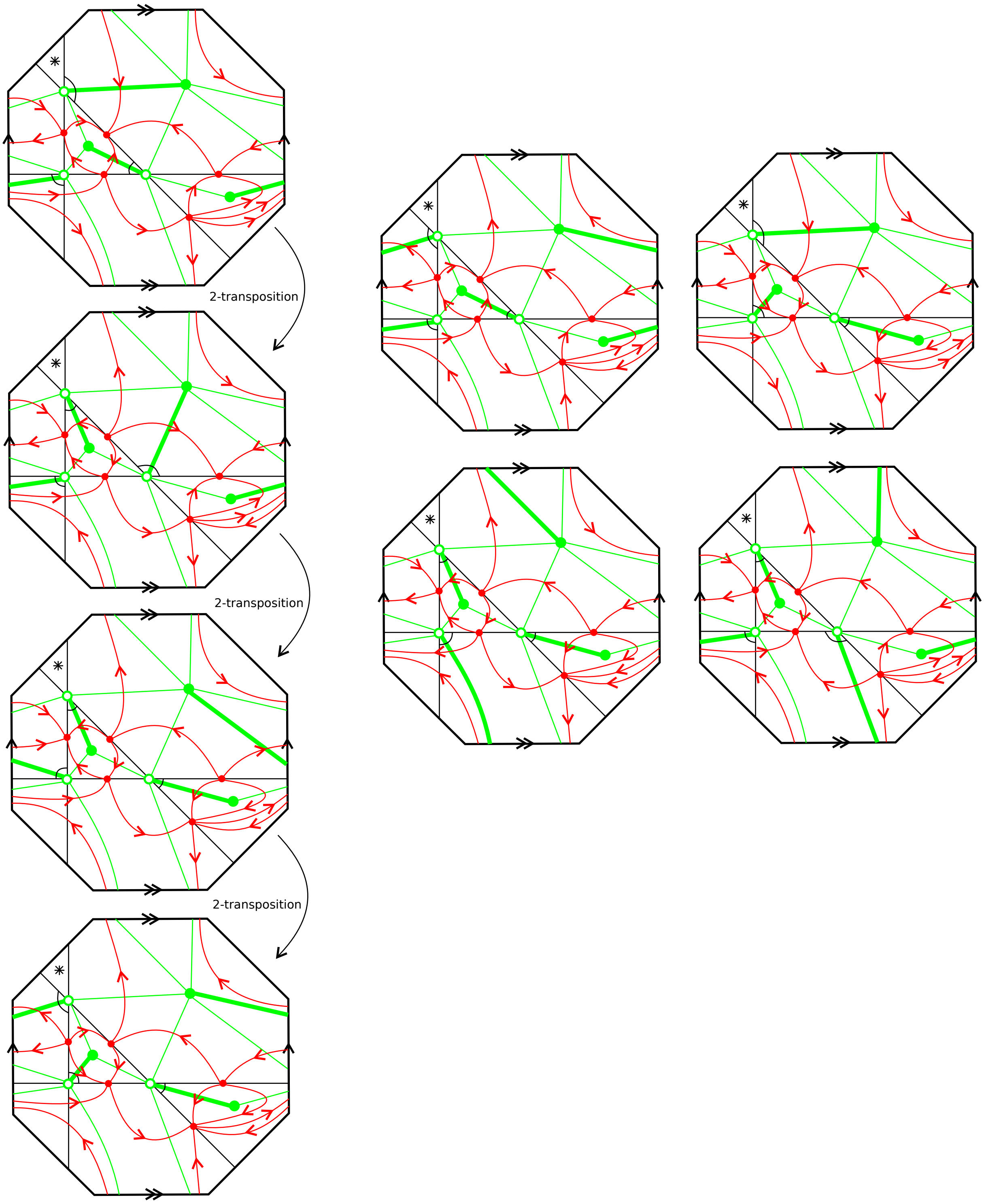


Figure 2.19

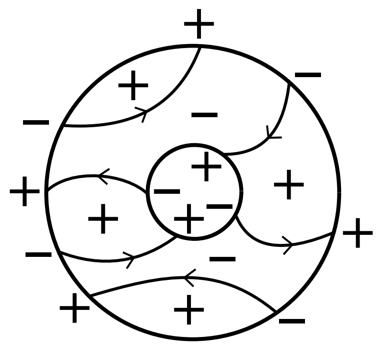


Figure 2.20. A dividing set on an annulus

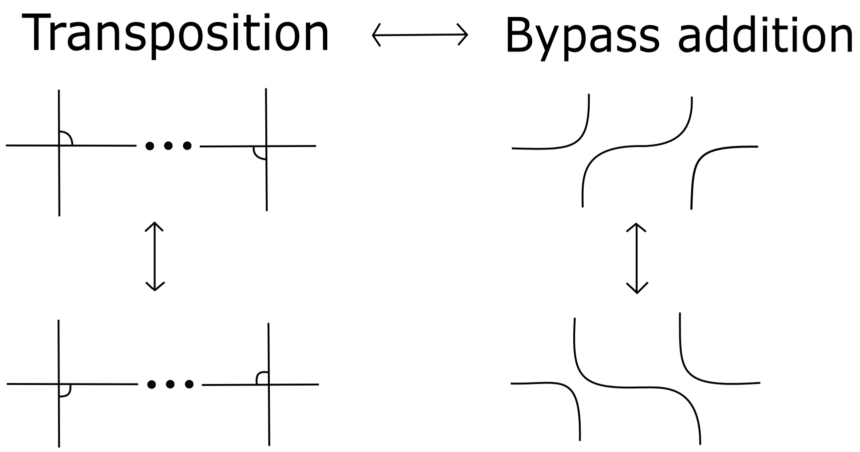


Figure 2.21. Similarity between transposition and bypass addition

as vertices and morphisms as directed edges. However, this graph cannot convey all the structures of the contact category it represents since two morphisms can be composed whereas the graph does not show what is their composition.

It has recently been observed that the two structures described above (contact category and clock lattice) possess several interesting and intriguing similarities described as follows. Firstly, transpositions (moves from one element of a clock lattice to another adjacent to it or, in other words, "elementary morphisms" in the clock lattice) resemble a contact geometry operation called bypass addition (an operation introduced by Honda). More precisely, bypass additions are the "elementary morphisms" in the contact category (every morphism factorises as a composition of bypass additions). Secondly, the condition for a resolution of crossings to give a state is very similar to the condition for a dividing set to describe a tight contact structure (thanks to a result of Eliashberg, [5]). The first observation is illustrated in figure 2.21.

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