Real θ-groups and orbit classifications

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Abstract

We study a class of algebraic groups called θ-groups. These are constructed from graded Lie algebras and possess a linear group action on subspaces of the Lie algebra called the θ-representation. Multiple symmetry groups of interest arise as θ-groups, such as the natural action of general linear groups with small dimension on trivectors. A special feature of such symmetry groups is that their orbits can be classified despite the groups often being infinite. The original construction of θ-groups and classification of the nilpotent orbits, both due to Vinberg, are over the complex numbers. We describe Vinberg's work and provide an overview of the newer area of real θ-groups, which appear in areas such as black hole theory in physics. The main aims of our work are to provide a comprehensive overview of complex and real θ-groups, especially with respect to the classification of nilpotent orbits.

Declaration

This thesis is an original work of my research and contains no material which has been accepted for the award of any other degree or diploma at any university or equivalent institution and that, to the best of my knowledge and belief, this thesis contains no material previously published or written by another person, or any use of generative artificial intelligence technologies, except where due reference is made in the text of the thesis.

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Chapter summaries

- Chapter 1 - Recap of Lie algebras. Basic definitions and concepts like homomorphisms and ideals are defined, before going over the theory of complex semisimple Lie algebras. The Killing form is covered before going over Cartan subalgebras, which lead into the root space decomposition, which leads to root systems. Since semisimple Lie algebras are direct sums of simple Lie algebras as ideals, and simple Lie algebras have connected root systems, the focus turns to classifying connected root systems. In the course of reaching this, we discuss Cartan matrices and Dynkin diagrams and present the classification of connected root systems using these Dynkin diagrams. Lastly, we cover Serre's theorem and explain why, as a result, this classification implies the classification of simple complex Lie algebras.

- Chapter 2 - Real Lie algebras. The basic theory is covered, including the realification, real forms and real structures of complex Lie algebras. The remainder of this chapter concerns the classification of all real simple Lie algebras, which is obtained by classifying the real forms of complex Lie algebras, as realifications are unique for each complex Lie algebra. To this end, we discuss a "canonical" real form, the compact real form, and its relationship with the Cartan decomposition of the Lie algebra. Then we show how the other real forms are constructed from the compact real form and an involutive automorphism of the Lie algebra, and how all the involutive automorphisms have a combinatorial description. This completes the classification, as the complex simple Lie algebras are classified, per Chapter 1. The chapter concludes with an addendum on how the classification can be described in terms of Satake diagrams, analogous to the Dynkin diagrams from the real case.

- Chapter 3 - Complex θ-groups. The content of the chapter may be considered as three parts. The first introduces graded complex Lie algebras and θ-groups, covering their definitions and constructions, as well as some key properties and theorems. These include the Jacobson-Morozov-Vinberg theorem describing $\mathfrak{sl}_2$-triples in the graded Lie algebras, a version of the Jordan decomposition, the classification of $\mathbb{Z}$-graded semisimple Lie algebras via weighted Dynkin and Satake diagrams, and topological characterisations of semisimplicity and nilpotency of elements with respect to orbits under the θ-group. The second part covers Vinberg's description of nilpotent orbits of complex θ-groups in 1987. This was done by defining the support of a nilpotent element, and considering nestled subclasses, culminating in a subset of supports that Vinberg showed is in one-to-one correspondence with the nilpotent orbits. The third part is an overview of Kac's description of $\mathbb{Z}_m$-graded complex semisimple Lie algebras, for $m \in \mathbb{Z}$, via a classification of the possible covering algebras and all their $\mathbb{Z}$-gradings that match the grading on the original Lie algebra. This part also covers Vinberg's utilisation of this work to describe θ-groups, such as the description of the θ-group from the root system of the covering algebra, and a decomposition of the θ-representation into irreducible representations.

- Chapter 4 - Real θ-groups. This chapter covers one of the main objectives of the report, which is to summarise and elaborate on four main approaches of classifying the nilpotent orbits of real θ-groups, since Vinberg's supports do not give a one-to-one correspondence in the real case. The first is via real supports, used to reduce the problem to one in real algebraic geometry. The difficulty in this approach lies in solving this corresponding problem. The second is using carrier algebras, which are another real analogue to supports. This method shows how to construct all the carrier algebras, then prune them until a one-to-one correspondence with the nilpotent orbits is reached, although it relies on heuristics to work. The third is using Galois cohomology, an approach which has led to the classification of orbits of classical modules by reducing the problem to that of θ-groups. While this approach is used effectively in isolated cases, the Galois cohomology groups are non-trivial to calculate, making this difficult to generalise. The last way uses Cayley triples to find a large, but complete, set of orbit representatives, then utilises various invariants to eliminate them until the set is irredundant. An alternative means of classifying nilpotent orbits in the complex case is also given to better illustrate the various invariants and constructions utilised. While the heuristic techniques are significantly less general in scope, they have been effective in areas where θ-groups are applied, particularly in physics papers investigating the structure of black holes.

Chapter 1

Introduction

1.1 First concepts

A Lie algebra $\mathfrak{g}$ is a vector space over a field F equipped with a bilinear map

$$[\cdot,\ \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g},\ (x, y) \mapsto [x, y],$$

such that, for all $x, y, z \in \mathfrak{g}$, the following hold:

(1) $[x, x] = 0$.
(2) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ (Jacobi identity).

Unless otherwise stated, Lie algebras are finite-dimensional. When both real and complex Lie algebras are being discussed, we sometimes use a superscript c to denote a complex Lie algebra (e.g. $\mathfrak{g}^c$) and a letter without superscript to denote a real Lie algebra. The material in this chapter is generally known and may be found in standard books, such as Erdmann and Wildon [12] and Humphreys [17].

If the characteristic of the field is not 2, the first condition is equivalent to $[x, y] = -[y, x]$ for all $x, y \in \mathfrak{g}$. An example of a Lie algebra is the general linear algebra $\mathfrak{gl}(n, F)$ of $n \times n$ matrices with entries in a field F with Lie bracket $[x, y] = xy - yx$. A **homomorphism** of Lie algebras is a linear map $\phi : \mathfrak{g} \to \mathfrak{h}$ between Lie algebras such that $\phi([x, y]) = [\phi(x), \phi(y)]$ for all $x, y \in \mathfrak{g}$. If the map is also bijective, then ϕ is a Lie algebra **isomorphism**; $\mathfrak{g}$ and $\mathfrak{h}$ are then isomorphic, denoted by $\mathfrak{g} \cong \mathfrak{h}$.

Example 1.1. Given a vector space V over a field F, the vector space of endomorphisms $\text{End}(V)$ becomes a Lie algebra, denoted by $\mathfrak{gl}(V)$, when given the Lie bracket $[S, T] = (S \circ T) - (T \circ S)$. If V has dimension n, then $\mathfrak{gl}(V)$ is isomorphic to $\mathfrak{gl}(n, F)$ via the isomorphism between linear transformations and matrices. By Ado's theorem, given in [19, Section IV.2], every finite-dimensional Lie algebra over a field F of characteristic zero is isomorphic to a subalgebra of $\mathfrak{gl}(n, F)$ for some $n \in \mathbb{N}$. Since the field is $\mathbb{R}$ or $\mathbb{C}$ and the Lie algebras are finite-dimensional in this report, every Lie algebra here can be treated as a matrix Lie algebra or a Lie algebra of linear transformations. When this is desirable, we will generally choose the matrix representation. •

If $[x, y] = 0$ for all $x, y \in \mathfrak{g}$, then $\mathfrak{g}$ is **abelian**. Clearly, there is only one abelian Lie algebra for each dimension up to isomorphism, so such cases are generally trivial. Hence, unless otherwise stated, **all given Lie algebras are non-abelian**. A vector subspace $\mathfrak{h} \subseteq \mathfrak{g}$ is a **subalgebra** of $\mathfrak{g}$, denoted $\mathfrak{h} \leq \mathfrak{g}$, if $[x, y] \in \mathfrak{h}$ for all $x, y \in \mathfrak{h}$. For example, the Lie algebra $\mathfrak{b}(n, F)$ of upper triangular matrices is a subalgebra of $\mathfrak{gl}(n, F)$. If $[x, a] \in \mathfrak{h}$ for all $x \in \mathfrak{h}$ and $a \in \mathfrak{g}$, then $\mathfrak{h}$ is an **ideal** of $\mathfrak{g}$, denoted $\mathfrak{h} \trianglelefteq \mathfrak{g}$. Analogously to rings, the kernel of a Lie algebra homomorphism is an ideal, and every ideal induces a canonical homomorphism onto the quotient by the ideal. For example, since the trace function, tr, satisfies $\mathrm{tr}(XY) = \mathrm{tr}(YX)$ for all $n \times n$ matrices X and Y, it follows that the special linear algebra $\mathfrak{sl}(n, F)$ of matrices with trace zero is an ideal of $\mathfrak{gl}(n, F)$.

A case of particular interest is $\mathfrak{sl}(2, F)$, which is 3-dimensional with *canonical basis*

$$E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \ F = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \ H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Observe that $[E, F] = H$, $[H, E] = 2E$ and $[H, F] = -2F$. A triplet $\{f, h, e\} \subset \mathfrak{g}$ which has the same brackets as the canonical basis for $\mathfrak{sl}(2, F)$ is an **$\mathfrak{sl}_2$-triple**, and its span is isomorphic to $\mathfrak{sl}(2, F)$. These appear throughout the report; in the case of $F = \mathbb{C}$, we will later see their significance in the classification of complex semisimple Lie algebras.

As with other algebraic structures, there are multiple ways of constructing new Lie algebras from old ones. If the Lie algebra $\mathfrak{g}$ has ideals $\mathfrak{h}_1$ and $\mathfrak{h}_2$, the following are also ideals of $\mathfrak{g}$, hence Lie algebras in their own right:

- $\mathfrak{h}_1 \cap \mathfrak{h}_2$,
- $\mathfrak{h}_1 + \mathfrak{h}_2 = \{h_1 + h_2 : \ h_1 \in \mathfrak{h}_1, h_2 \in \mathfrak{h}_2\}$,
- $[\mathfrak{h}_1, \mathfrak{h}_2] = \mathrm{Span}(\{[h_1, h_2] : \ h_1 \in \mathfrak{h}_1, h_2 \in \mathfrak{h}_2\})$.

For two Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$, the external direct sum $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ of vector spaces may be given a Lie bracket by letting $[g_1, g_2] = 0$ for all $g_1 \in \mathfrak{g}_1$ and $g_2 \in \mathfrak{g}_2$ and keeping the original brackets on the Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$. The resulting Lie algebra is the **Lie direct sum** of $\mathfrak{g}_1$ and $\mathfrak{g}_2$. While some authors use $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ to refer to the Lie direct sum of $\mathfrak{g}_1$ and $\mathfrak{g}_2$, we only use the symbol $\oplus$ to refer to a direct sum of vector spaces, unless stated otherwise. Also observe that, if $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ as a direct sum of vector spaces, then this is a Lie direct sum if and only if $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are ideals in $\mathfrak{g}$.

A **derivation** on $\mathfrak{g}$ is a linear map $\delta \colon \mathfrak{g} \to \mathfrak{g}$ such that $\delta([x, y]) = [\delta(x), y] + [x, \delta(y)]$ for all $x, y \in \mathfrak{g}$. The space of all derivations of $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{gl}(\mathfrak{g})$, denoted $\mathrm{Der}\,\mathfrak{g}$. Furthermore, the map $\mathrm{ad} \colon \mathfrak{g} \to \mathrm{Der}\,\mathfrak{g}, (\mathrm{ad}\,x)(y) = [x, y]$ is the **adjoint homomorphism** on $\mathfrak{g}$. It is linear by linearity of the first term in the Lie bracket, and the Jacobi identity implies that it is a homomorphism whose image is in $\mathrm{Der}\,\mathfrak{g}$.

For a set $S \subset \mathfrak{g}$, the **centraliser** of S in $\mathfrak{g}$ is $C_{\mathfrak{g}}(S) = \{x \in \mathfrak{g} : [x, y] = 0 \quad \forall y \in S\}$. This is clearly a subalgebra of $\mathfrak{g}$ by the Jacobi identity: if $a, b \in \mathfrak{g}$, then, for all $y \in S$, $[[a, b], y] = -[[y, a], b] - [[b, y], a] = 0$. In particular, $C_{\mathfrak{g}}(\mathfrak{g}) = Z(\mathfrak{g})$ is the **centre** of $\mathfrak{g}$. It is also an ideal of $\mathfrak{g}$, and $\mathfrak{g}$ is abelian if and only if $Z(\mathfrak{g}) = \mathfrak{g}$. For a subspace $K \subseteq \mathfrak{g}$, the **normaliser** of K in $\mathfrak{g}$ is $N_{\mathfrak{g}}(K) = \{x \in \mathfrak{g} : [x, K] \subseteq K\}$. Again, this is a subalgebra of $\mathfrak{g}$ by the Jacobi identity. If K is a subalgebra of $\mathfrak{g}$, then $N_{\mathfrak{g}}(K)$ is the largest subalgebra containing K as an ideal. We illustrate these definitions with two examples.

Example 1.2. Let $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$ (for any $n \in \mathbb{N}$). For a matrix M, we let M_{ij} be the entry of M at row i and column j.

a) Consider $\mathfrak{n}(n, \mathbb{C})$, the subalgebra of $\mathfrak{g}$ of diagonal matrices. We compute the centraliser of $\mathfrak{n}(n, \mathbb{C})$ in $\mathfrak{g}$. If $X, Y \in \mathfrak{n}(n, \mathbb{C})$, we see that $[X, Y]$ is diagonal, and the kth entry of $[X, Y]$ is $X_{kk}Y_{kk} - Y_{kk}X_{kk} = 0$. Therefore, $[X, Y] = 0$, so $\mathfrak{n}(n, \mathbb{C})$ is abelian and $C_{\mathfrak{g}}(\mathfrak{n}(n, \mathbb{C})) \geq \mathfrak{n}(n, \mathbb{C})$. Now, let $X \in \mathfrak{gl}(n, \mathbb{C}) \setminus \mathfrak{n}(n, \mathbb{C})$. Then X has an off-diagonal entry $X_{ij} \neq 0$ for some $i \neq j$. Let $M_i \in \mathfrak{n}(n, \mathbb{C})$ be the diagonal matrix with entry 1 at position (i, i) and 0 elsewhere. We can see that $[X, M_i]$ has (i, j)th entry

$$\begin{aligned}[M_i, X]_{ij} = (M_iX)_{ij} - (XM_i)_{ij} &= \sum_{q=1}^{n} (M_i)_{iq}X_{qj} - \sum_{q=1}^{n} X_{iq}(M_i)_{qj} \\ &= (M_i)_{ii}X_{ij} = X_{ij} \neq 0.\end{aligned}$$

Therefore, $[M_i, X] \neq 0$, so $X \notin C_{\mathfrak{g}}(\mathfrak{n}(n, \mathbb{C}))$. We conclude $C_{\mathfrak{g}}(\mathfrak{n}(n, \mathbb{C})) = \mathfrak{n}(n, \mathbb{C})$.

b) Let $\mathfrak{s}(n, \mathbb{C})$ be the subalgebra of $\mathfrak{g}$ of strictly lower triangular matrices. We compute the normaliser of $\mathfrak{s}(n, \mathbb{C})$ in $\mathfrak{g}$. Let $X \in \mathfrak{g}$ and $Y \in \mathfrak{s}(n, \mathbb{C})$. Then, for all $1 \leq i \leq j \leq n$, the (i, j)th entry of $[X, Y]$ is

$$\sum_{q=1}^{n} X_{iq}Y_{qj} - \sum_{q=1}^{n} Y_{iq}X_{qj} = \sum_{q=j+1}^{n} X_{iq}Y_{qj} - \sum_{q=1}^{i-1} Y_{iq}X_{qj},$$

since Y is strictly lower triangular. If Y has entry 1 at position (m, p) and entry 0 everywhere else for $m > p$, then the (k, p)th entry of $[X, Y]$ for $k \leq p$ is X_{km}, which is always equal to 0 if and only if X is in the normaliser. Since $m > p \geq k$, this occurs if and only if $X_{km} = 0$ for all $k < m$. Thus, the normaliser $N_{\mathfrak{g}}(\mathfrak{s}(n, \mathbb{C}))$ is the subalgebra of lower triangular matrices. •

The **derived subalgebra** of $\mathfrak{g}$ is $[\mathfrak{g}, \mathfrak{g}]$, also denoted $\mathfrak{g}'$. Let $\mathfrak{g}^0 = \mathfrak{g}$, and $\mathfrak{g}^{i+1} = [\mathfrak{g}^i, \mathfrak{g}^i]$ for $i \geq 0$. Then

$$\mathfrak{g} = \mathfrak{g}^0 \trianglerighteq \mathfrak{g}^1 \trianglerighteq \dots$$

is the **derived series** for $\mathfrak{g}$, and $\mathfrak{g}$ is **solvable** if $\mathfrak{g}^k = 0$ for some $k \geq 1$. Similarly, defining $\mathfrak{g}^{(0)} = \mathfrak{g}$ and $\mathfrak{g}^{(i+1)} = [\mathfrak{g}, \mathfrak{g}^{(i)}]$ for $i \geq 0$ gives another descending series of ideals, called the **lower central series** of $\mathfrak{g}$, and $\mathfrak{g}$ is **nilpotent** if $\mathfrak{g}^{(k)} = 0$ for some $k \geq 1$. In particular, since $\mathfrak{g}^i \leq \mathfrak{g}^{(i)}$ for all $i \geq 0$, if $\mathfrak{g}$ is nilpotent, then it is also solvable.

Every Lie algebra has a unique largest solvable ideal, called the **radical** of $\mathfrak{g}$. If $\mathfrak{g}$ has no nontrivial solvable ideal, $\mathfrak{g}$ is **semisimple**. This is equivalent to $\mathfrak{g}$ having trivial radical, and to $\mathfrak{g}$ having no nontrivial abelian ideal. More generally, $\mathfrak{g}$ is **simple** if it is non-abelian and has no proper nontrivial ideal. The first restriction only removes the 1-dimensional abelian Lie algebra, since every vector subspace of an abelian Lie algebra is an ideal. This Lie algebra has different properties to simple Lie algebras, like not being semisimple and having nontrivial centre. The following examples are taken from [12, Exercise 1.13, Exercise 4.5].

Example 1.3.

a) $\mathfrak{sl}(2, \mathbb{C})$ is a simple Lie algebra, so it is also semisimple.

b) The Lie algebra $\mathfrak{b}(n, F)$ of upper triangular matrices is solvable. •

Proof.

a) Let $\{f, h, e\}$ be the canonical basis of $\mathfrak{sl}(2, \mathbb{C})$ and suppose $\mathfrak{h} \neq \{0\}$ is an ideal of $\mathfrak{sl}(2, \mathbb{C})$. Then $\mathfrak{h}$ contains an element of the form $\alpha e + \beta f + \gamma h$ for $\alpha, \beta, \gamma \in \mathbb{C}$ not all equal to 0. We now show that $h \in \mathfrak{h}$. First, suppose $\alpha = \beta = 0$, so $\gamma \neq 0$. Then $\frac{1}{\gamma}(\gamma h) = h \in \mathfrak{h}$. Now suppose $\alpha \neq 0$ or $\beta \neq 0$, and observe that $[\alpha e + \beta f + \gamma h, h] = -2\alpha e + 2\beta f \in \mathfrak{h}$. If $\alpha \neq 0$, then $[-2\alpha e + 2\beta f, -\frac{1}{2\alpha} f] = h \in \mathfrak{h}$. If $\beta \neq 0$, then $[-2\alpha e + 2\beta f, -\frac{1}{2\beta} e] = h \in \mathfrak{h}$. Thus, in any case, we have $h \in \mathfrak{h}$.
As $\mathfrak{h}$ is an ideal, $[h, \frac{1}{2} e] = e$ and $[h, -\frac{1}{2} f] = f$ are also in $\mathfrak{h}$. Thus, $\{e, f, h\} \subseteq \mathfrak{h}$, so $\mathfrak{h} = \mathfrak{sl}(2, \mathbb{C})$. Hence, the only ideals of $\mathfrak{sl}(2, \mathbb{C})$ are $\{0\}$ and $\mathfrak{sl}(2, \mathbb{C})$, so $\mathfrak{sl}(2, \mathbb{C})$ is simple.

b) Here, we abbreviate $\mathfrak{g} = \mathfrak{b}(n, F)$. For every $X \in \mathfrak{g}$, we see that $X_{ij} = 0$ if $i > j$. We define the **depth** $d \in \mathbb{Z}$ of X to be the largest integer such that $X_{ij} = 0$ for all i, j such that $i + d > j$, and let $\mathcal{A}_d \subseteq \mathfrak{b}(n, F)$ be the set of matrices with depth at least d. Observe that the depth of a matrix is zero when X has a nonzero diagonal entry, and n precisely when $X = 0$. We now show that $\mathfrak{g}^{d+1} \subseteq \mathcal{A}_d$ for all $0 \leq d \leq n$, via induction on d. For $d = 0$, as the depth of every matrix in $\mathfrak{g}$ is at least zero, we have that $\mathcal{A}_0 = \mathfrak{g}$, so $\mathfrak{g}^1 = \mathfrak{g} \subseteq \mathcal{A}_0$, as desired. Now we assume that $\mathfrak{g}^{d+1} \subseteq \mathcal{A}_d$ up to some d and show that $\mathfrak{g}^{d+2} \subseteq \mathcal{A}_{d+1}$. Recall that $\mathfrak{g}^{d+2}$ is spanned by $[X, Y]$ for $X, Y \in \mathfrak{g}^{d+1}$, and by the induction hypothesis $X, Y \in \mathcal{A}_d$. Therefore, $X_{ij} = Y_{ij} = 0$ when $i + d > j$. We see that

$$[X, Y]_{ij} = (XY - YX)_{ij} = \sum_{k=1}^{n} X_{ik} Y_{kj} - Y_{ik} X_{kj}.$$

Suppose $d = 0$. Then letting $i = j$, we have that $[X, Y]_{ii} = X_{ii} Y_{ii} - Y_{ii} X_{ii} = 0$, so the depth of $[X, Y]$ is at least 1, so $\mathfrak{g}^2 \subseteq \mathcal{A}_1$. If $d \geq 1$, the terms in the equation for $[X, Y]_{ij}$ can be nonzero only if $i + d \leq k$ and $k + d \leq j$, so $i + d \leq k \leq j - d$. Therefore, $[X, Y]_{ij} = 0$ if $i + d > j - d$, or $i + (d + d) > j$. Since $i + d + d \geq i + d + 1$, this shows that $[X, Y]_{ij} = 0$ when $i + (d + 1) > j$. Thus, $\mathfrak{g}^{d+2} \subseteq \mathcal{A}_{d+1}$ in both cases, so the inductive step is proven. Hence, $\mathfrak{g}^{d+1} \subseteq \mathcal{A}_d$ for all $0 \leq d \leq n$. Since $\mathcal{A}_n = \{0\}$, we have that $\mathfrak{g}^{n+1} = \{0\}$, so the derived series terminates and $\mathfrak{g}$ is indeed solvable. □

1.2 Complex semisimple Lie algebras

We now focus on Lie algebras over the complex field $\mathbb{C}$. By Levi's theorem [19, Ch. 2], the solvable ideal of such a Lie algebra has a vector space complement that is also a semisimple subalgebra, so such Lie algebras decompose into a (vector space) direct sum of a solvable and a semisimple Lie algebra. The classification of solvable Lie algebras is an ongoing project, and is done by progressively classifying those with additional properties like low dimension. For example, de Graaf develops techniques in [13] to classify solvable Lie algebras of dimensions 3 and 4 over any field. In contrast, semisimple Lie algebras have been fully classified by the work of Killing and Cartan. This classification was a significant achievement of 20th century mathematics, utilising both algebra and combinatorics. We summarise this work in the rest of Chapter 1 and, along the way, introduce important notation and concepts that are used in the rest of the report.

1.2.1 The Killing form

An important concept in the classification of semisimple Lie algebras, as well as in the general study of complex Lie algebras, is the **Killing form**. Utilising the fact that, for a complex Lie algebra $\mathfrak{g}$ and every $x \in \mathfrak{g}$, we can treat $\operatorname{ad} x \in \mathfrak{gl}(\mathfrak{g})$ as a matrix to define its trace, the Killing form is defined as

$$\kappa : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}, \quad \kappa(x, y) = \operatorname{tr}(\operatorname{ad} x \circ \operatorname{ad} y).$$

This is well-defined since the trace is independent of the choice of basis. The Killing form is a symmetric bilinear form that is also **associative**; that is, $\kappa([x, y], z) = \kappa(x, [y, z])$, for all $x, y, z \in \mathfrak{g}$. Furthermore, we see from [26, Theorem I.3] that $\kappa(\alpha(x), \alpha(y)) = \kappa(x, y)$ for all $x, y \in \mathfrak{g}$ and $\alpha \in \operatorname{Aut} \mathfrak{g}$. This establishes the Killing form as **invariant**.

Given a symmetric bilinear form β on a vector space V, its **radical** is the space

$$V^{\perp} = \{x \in V : \beta(x, v) = 0 \quad \forall v \in V\}.$$

The bilinear form is **nondegenerate** if its radical is trivial; that is, $V^{\perp} = \{0\}$. The Killing form is used to prove the Cartan criteria for semisimplicity and solvability of a complex Lie algebra, as given in [12, Theorem 9.6 and Theorem 9.9].

Theorem 1.4 (Cartan Criteria)**.**

a) *First Cartan criterion: A complex Lie algebra $\mathfrak{g}$ is solvable if and only if $\kappa(x, y) = 0$ for all $x \in \mathfrak{g}$ and $y \in [\mathfrak{g}, \mathfrak{g}]$.*
b) *Second Cartan criterion: A complex Lie algebra $\mathfrak{g}$ is semisimple if and only if κ is nondegenerate.*

Lastly, we introduce two key theorems that are derived from the Killing form and the Cartan criteria. Henceforth, we suppose $\mathfrak{g}$ is semisimple. A linear map $T \colon V \to V$

on a finite-dimensional complex vector space V is **diagonalisable** if V has a basis consisting of eigenvectors of T. The map T is **nilpotent** if $T^k = 0$ for some $k > 0$. As a consequence of the Cartan criteria, we have the following:

Theorem 1.5 (Abstract Jordan decomposition). *For a complex semisimple Lie algebra $\mathfrak{g}$, every element $x \in \mathfrak{g}$ has a unique **abstract Jordan decomposition**, given as $x = d + n$, for $d, n \in \mathfrak{g}$, such that $\operatorname{ad} d$ is diagonalisable, $\operatorname{ad} n$ is nilpotent and $[d, n] = 0$.*

An element $x \in \mathfrak{g}$ is **nilpotent/semisimple** if $\operatorname{ad} x$ is nilpotent/diagonalisable. There is no ambiguity when $\mathfrak{g}$ is a matrix Lie algebra, as some $x \in \mathfrak{g}$ is diagonalisable or nilpotent as a matrix if and only if $\operatorname{ad} x$ is diagonalisable or nilpotent, hence if and only if x is semisimple or nilpotent as an element of the Lie algebra. The second theorem is what allows the later analysis to focus on simple Lie algebras:

Theorem 1.6. *A complex Lie algebra is semisimple if and only if it can be expressed as a direct sum of simple ideals.*

This is directly related to the later analysis of abstract root systems, since the direct summands of a semisimple Lie algebra correspond precisely to the irreducible components of the corresponding root system.

1.2.2 Toral and Cartan subalgebras, and the root space decomposition

Key to the structure theory of complex semisimple Lie algebras $\mathfrak{g}$ is the concept of a Cartan subalgebra. More generally, a subalgebra $\mathfrak{h} \leq \mathfrak{g}$ is **toral** if every element of $\mathfrak{h}$ is semisimple. In fact, this provides extra structure, as shown below, with the proof adapted from [17, Section 8.1].

Lemma 1.7. *A toral subalgebra is abelian.*

Proof. Let d be the dimension of $\mathfrak{h}$ and consider any $x \in \mathfrak{h}$. Since $\operatorname{ad} x$ is diagonalisable and $\mathfrak{h}$ is invariant under $\operatorname{ad} x$, the restriction of $\operatorname{ad} x$ to $\mathfrak{h}$ is also diagonalisable, so $\mathfrak{h}$ has a basis consisting of eigenvectors of $\operatorname{ad} x$. Then $\mathfrak{h}$ is abelian if and only if zero is the only eigenvalue of any $\operatorname{ad} x$ when restricted to $\mathfrak{h}$. Indeed, suppose this were not the case, so there exists a $y \in \mathfrak{h}$ such that $(\operatorname{ad} x)(y) = [x, y] = cy$ for $c \neq 0$. Then $[y, x] = (\operatorname{ad} y)(x) = -cy$. Since y is an eigenvector of $\operatorname{ad} y$, we can extend it to a basis of eigenvectors of $\operatorname{ad} y$, denoted as $\{y, v_1, v_2, ..., v_{d-1}\}$, with v_i having eigenvalue λ_i. As a result, $x = c_0 y + \sum_{i=1}^{d-1} c_i v_i$ for $c_0, ..., c_{d-1}$, so

$$-cy = (\operatorname{ad} y)(x) = \sum_{i=1}^{r-1} c_i \lambda_i v_i.$$

Since $c \neq 0$, this shows that the chosen basis is linearly dependent, a contradiction. Therefore, $\operatorname{ad} x$ restricted to $\mathfrak{h}$ only has zero eigenvalue for any $x \in \mathfrak{h}$, so $\mathfrak{h}$ is abelian, as desired. □

If there are no semisimple elements in $\mathfrak{g}$, then the abstract Jordan decomposition shows that every element of $\mathfrak{g}$ is nilpotent. By Engel's theorem (see [17, Section 3.2]), if every element of $\mathfrak{g}$ is nilpotent, $\mathfrak{g}$ itself is nilpotent, contradicting the semisimplicity of $\mathfrak{g}$. Thus, $\mathfrak{g}$ must have a nontrivial toral subalgebra: given a semisimple element $s \in \mathfrak{g}$, one case is the one-dimensional space spanned by s, which is trivially abelian.

Example 1.8. Let $\mathfrak{n}(n, \mathbb{C}) \subseteq \mathfrak{gl}(n, \mathbb{C})$ be the abelian subalgebra of diagonal matrices mentioned in Exercise 1.1. Since the intersection of subalgebras is itself a subalgebra, it follows that $\mathfrak{g} \cap \mathfrak{n}(n, \mathbb{C})$ is a toral subalgebra of $\mathfrak{g}$ for any $\mathfrak{g} \leq \mathfrak{gl}(n, \mathbb{C})$. •

We can now describe the main object of interest. A **Cartan subalgebra** (CSA) $\mathfrak{h}$ is a toral subalgebra that is maximal with respect to inclusion among all toral subalgebras. As $\mathfrak{g}$ is finite-dimensional, it has a Cartan subalgebra $\mathfrak{h}$, which is nonzero since $\mathfrak{g}$ is semisimple. Note that some texts, such as [17, 20], define a Cartan subalgebra as a nilpotent Lie algebra that is equal to its normaliser in $\mathfrak{g}$. We do not concern ourselves with this because the definitions are equivalent if $\mathfrak{g}$ is semisimple and defined over a field of characteristic zero (see [17, Section 15.3] for details). When $\mathfrak{g}$ is complex and semisimple, the Cartan subalgebras have a clear classification: they are unique up to conjugacy by the action of the adjoint group, a structure that is defined and considered in more detail in Chapter 3. A consequence of this is that the Cartan subalgebras of a complex semisimple Lie algebra have the same dimension. This is the **rank** of the Lie algebra.

Example 1.9. Let $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{C})$. As we will show in Section 1.2.3, the Lie algebras $\mathfrak{sl}(n, \mathbb{C})$ are part of the classical Lie algebras, which are all simple, hence semisimple.

a) Consider
$$x = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in \mathfrak{g}.$$
Since x has 3 distinct eigenvalues, it is diagonalisable, so its span is a toral subalgebra of $\mathfrak{g}$. In particular, there exists a Cartan subalgebra $\mathfrak{h}_1$ containing x. Since $\mathfrak{h}_1$ is abelian, it must be contained in the centraliser $C_{\mathfrak{g}}(\{x\})$. Note that $y \in C_{\mathfrak{g}}(\{x\})$ if and only if
$$[x, y] = xy - yx = \begin{bmatrix} -2y_{12} & 0 & y_{13} \\ 2y_{11} - 2y_{21} - 2y_{22} & 2y_{12} & 2y_{13} - y_{23} \\ -(y_{31} + 2y_{32}) & y_{32} & 0 \end{bmatrix} = 0.$$
We see that $y_{12} = y_{13} = y_{32} = 0$, so $y_{23} = y_{31} = 0$ as well. Also, $y_{11} = y_{21} + y_{22}$ and, since $y \in \mathfrak{g}$, it must have zero trace, so $-y_{11} = y_{22} + y_{33}$. Now, since x is in its centraliser and the centraliser is a Lie subalgebra, hence a vector subspace, it follows that $\tilde{y} = y - y_{11}x$ is also in the centraliser. Therefore, $\tilde{y}_{12} = \tilde{y}_{13} = \tilde{y}_{32} = \tilde{y}_{23} = \tilde{y}_{31} = 0$, as well as $\tilde{y}_{11} = 0$ since $x_{11} = 1$, so $\tilde{y}_{21} = -\tilde{y}_{22}$ and $\tilde{y}_{33} = -\tilde{y}_{22}$. Having checked all the entries and confirming that $\tilde{y}$ is in $\mathfrak{g}$, we conclude that $\tilde{y} = \gamma z$, where $\gamma = \tilde{y}_{22}$ and
$$z = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

As y is a linear combination of x and $\tilde{y}$, it is also a linear combination of x and z, so $C_{\mathfrak{g}}(\{x\}) = \mathrm{Span}(\{x, z\}) \geq \mathfrak{h}_1$. Since z is also lower triangular with 3 distinct eigenvalues, it is also semisimple, hence also in $\mathfrak{h}_1$. Thus, we have equality, i.e.

$$\mathfrak{h}_1 = \mathrm{Span}\left(\left\{\begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\right\}\right).$$

b) Each classical Lie algebra has a natural choice for a Cartan subalgebra, which is the subalgebra of diagonal matrices. For example, let $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{C})$ and let $\mathfrak{h}_2$ be this CSA in $\mathfrak{g}$, which is spanned by $c = \mathrm{diag}(1, -1, 0)$ and $d = \mathrm{diag}(0, 1, -1)$. As previously mentioned, Cartan subalgebras are unique up to conjugacy by the adjoint group. In the case of classical Lie algebras, this action can be expressed relatively simply as matrix conjugation by elements of its corresponding Lie group G. The Lie group of $\mathfrak{g}$ is $\mathrm{SL}(3, \mathbb{C})$, the 3×3 complex matrices with determinant 1, so there exists an element of $\mathrm{SL}(3, \mathbb{C})$ which acts on $\mathfrak{h}_2$ via matrix conjugation to give $\mathfrak{h}_1$. Indeed, for

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

we can see that $AcA^{-1} = x$ and $AdA^{-1} = z$, so $\mathfrak{h}_2^A = A\mathfrak{h}_2 A^{-1} = \mathfrak{h}_1$, as desired.

•

Let $\mathfrak{g}$ be a complex semisimple Lie algebra with Cartan subalgebra $\mathfrak{h}$. For any two elements $x, y \in \mathfrak{h}$, $\mathrm{ad}\ x$ and $\mathrm{ad}\ y$ are diagonalisable by definition, and they commute since $\mathfrak{h}$ is abelian and the adjoint map is a homomorphism, so $0 = \mathrm{ad}[x, y] = (\mathrm{ad}\, x \circ \mathrm{ad}\, y) - (\mathrm{ad}\, y \circ \mathrm{ad}\, x)$. By a well-known theorem from linear algebra, see [22, Section III.8], commuting diagonalisable maps are simultaneously diagonalisable. The notion of an eigenspace decomposition can then be generalised as follows. For a vector space V over a field F, let $V^\star$ denote the dual vector space of linear functionals mapping from V to F. For the Lie algebra $\mathfrak{g}$ previously described, a functional $\alpha \in \mathfrak{h}^\star$ is a **root** if $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} : [h, x] = \alpha(h)x,\ \forall h \in \mathfrak{h}\} \neq 0$. In this case, the space $\mathfrak{g}_\alpha$ is the **root space** of α. As the elements of $\mathrm{ad}\,\mathfrak{h}$ are simultaneously diagonalisable, the root spaces lead to the **root space decomposition**

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha,$$

where $\Phi = \{\alpha \in \mathfrak{h}^\star : \mathfrak{g}_\alpha \neq \{0\}\}$ is the set of **roots** of $\mathfrak{h}$.

Remark 1.10. As this construction only depends on $\mathfrak{h}$ being abelian and consisting of semisimple elements, the analogous decomposition can be defined if one replaces $\mathfrak{h}$ with any toral subalgebra. That decomposition is a **weight space decomposition**, which is less frequently used as it lacks many of the properties of the root space decomposition. The weight space decomposition still surfaces, and its similarity with the root space decomposition is discussed, when reviewing the decomposition of graded Lie algebras in Chapter 3. •

The root spaces satisfy multiple properties that aid in the analysis of semisimple complex Lie algebras. Every $\mathfrak{g}_\alpha$ is 1-dimensional and the only scalar multiples of α in Φ are $\pm\alpha$. Furthermore, $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \leq \mathfrak{g}_{\alpha+\beta}$ for all $\alpha, \beta \in \Phi$ and, for every $\alpha \in \Phi$, there exist $e_\alpha \in \mathfrak{g}_\alpha, f_\alpha \in \mathfrak{g}_{-\alpha}$ and $h_\alpha \in \mathfrak{h}$ such that $\mathrm{Span}(\{e_\alpha, f_\alpha, h_\alpha\}) = \mathfrak{sl}_2(\alpha) \cong \mathfrak{sl}(2, \mathbb{C})$. The set $\{e_\alpha, f_\alpha, h_\alpha\}$ is an $\mathfrak{sl}_2$-triple, so one can consider the adjoint representation of $\mathfrak{sl}_2(\alpha)$ on $\mathfrak{g}$ and utilise the developed body of theory on $\mathfrak{sl}(2, \mathbb{C})$-representations. This shows that, for example, every eigenvalue of $\operatorname{ad} h_\alpha$ is an integer. Then, for $\alpha, \beta \in \Phi$, it is seen that $(\operatorname{ad} h_\alpha)(e_\beta) = [h_\alpha, e_\beta] = \beta(h_\alpha)e_\beta$, so the previous statement shows that $\beta(h_\alpha) \in \mathbb{Z}$.

Example 1.11.

- From Example 1.8, for every $n \geq 2$, the Lie algebra $\mathfrak{sl}(n, \mathbb{C})$ has an n-dimensional toral subalgebra $\mathfrak{h}$ spanned by its diagonal matrices. We review the proof given in [12, Lemma 12.2] to show that $\mathfrak{h}$ is a CSA. The elements of $\mathfrak{h}$ are trivially diagonalisable and diagonal matrices commute, so what remains is to show that $\mathfrak{h}$ is maximal with these properties. From the lemma, $\mathfrak{h}$ is maximal if, for every nonzero $h \in \mathfrak{h}$, there exists some $\alpha \in \Phi$ such that $\alpha(h) \neq 0$. Let e_{ij} be the matrix with entry 1 at position (i, j) and 0 elsewhere. Then, for any $1 \leq i \leq n-1$, observe that the map $\alpha_i(h) = h_{i,i} - h_{i+1,i+1}$ is a root, since $[h, e_{i,i+1}] = \alpha_i(h)e_{i,i+1}$ for any $h \in \mathfrak{h}$. If $\alpha_i(h) = 0$ for all $1 \leq i \leq n-1$, then $h_{ii} = h_{(i+1),(i+1)}$ for all i, so the diagonal entries of h are all equal. Since $h \in \mathfrak{sl}(n, \mathbb{C})$, this shows that the entries must all be zero, so $h = 0$, showing that $\mathfrak{h}$ is maximal by the contrapositive. As $\mathfrak{h}$ is maximal, it induces a root space decomposition with set of roots Φ. The resultant roots are linear combinations of the n roots $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $1 \leq i \leq n-1$, where the $\epsilon_i \in \mathfrak{h}^\star$ are defined by $\epsilon_i(h) = (h)_{ii}$, matching the rank of $\mathfrak{sl}(n, \mathbb{C})$.

  This shows that $\mathfrak{sl}(n, \mathbb{C})$ has a CSA consisting of its diagonal matrices, justifying the second part of Example 1.9. As is later explored, the same reasoning can be used with little modification for all the classical Lie algebras, giving a natural choice of CSAs for them.

- Given $S \in \mathfrak{gl}(n, \mathbb{C})$, the subset

  $$\mathfrak{gl}_S(n, \mathbb{C}) = \{A \in \mathfrak{gl}(n, \mathbb{C}) : A^\intercal S = -SA\}$$

  is a Lie subalgebra of $\mathfrak{gl}(n, \mathbb{C})$. When $n = 5$ and

  $$S = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix},$$

  this gives a simple Lie algebra, called the **orthogonal Lie algebra** of dimension 5, denoted $\mathfrak{so}(5, \mathbb{C})$, with elements of the form

  $$\begin{bmatrix} 0 & C & -B \\ B^\intercal & M & P \\ -C^\intercal & Q & -M^\intercal \end{bmatrix},$$

where $B = [b_1, b_2]$ and $C = [c_1, c_2]$ are 1×2 matrices and M, P, Q are 2×2 matrices with $P = -P^\intercal$ and $Q = -Q^\intercal$. Like the other classical Lie algebras, this has a Cartan subalgebra comprising its diagonal matrices. As with the special linear algebra, its roots are linear combinations of ϵ_i, and they are generated by $\{\epsilon_2 - \epsilon_4, \epsilon_3 - \epsilon_5\}$. •

The roots of Φ in fact span $\mathfrak{h}^\star$, so Φ contains a basis B of that space. Furthermore, Φ is contained in the real span E of B, so Φ can be treated as a subset of a real vector space. The last two statements imply that $\dim_{\mathbb{R}}(E) = |B| = \dim_{\mathbb{C}}(\mathfrak{h}^\star) = \dim_{\mathbb{C}}(\mathfrak{h})$, so the dimension of E is equal to the rank of $\mathfrak{g}$. Lastly, the Killing form gives an inner product $(\cdot, \cdot)$ on the roots, which extends to E. Importantly, this inner product satisfies the relation $2(\alpha, \beta)/(\beta, \beta) = \alpha(h_\beta)$, where h_β is in the $\mathfrak{sl}_2$-triple for the root β. Turning E into an inner product space gives enough structure to analyse Φ independent of its Lie algebra. We refer to [17, Section II.8] for details and proofs.

1.2.3 Structure theory and classification

The structure theory now turns to the roots of the root space decomposition of a semisimple Lie algebra. We will later recall that these are sufficient to fully classify the underlying Lie algebra. Start by considering a general inner product space $(E, (\cdot, \cdot))$. For an element $\alpha \in E$, we consider a **reflection** with respect to α to be the unique linear endomorphism on E that maps α to $-\alpha$ and fixes the vectors in the hyperplane perpendicular to α. This map has the explicit form: $s_\alpha : E \to E, s_\alpha(x) = x - \frac{2(x,\alpha)}{(\alpha,\alpha)}\alpha$. It is convenient to abbreviate

$$\langle \alpha, \beta \rangle = \frac{2(\alpha, \beta)}{(\beta, \beta)},$$

so $s_\alpha(x) = x - \langle x, \alpha \rangle \alpha$. Note this is linear in x but not in α. A subset $R \subseteq E$ is an **abstract root system** if it satisfies the following conditions:

(1) R is finite, spans E and does not contain 0.
(2) For $\alpha \in R$, the only scalar multiples of α in R are $\pm\alpha$.
(3) For $\alpha \in R$, the map s_α permutes the elements of R.
(4) $\langle \alpha, \beta \rangle \in \mathbb{Z}$ for all $\alpha, \beta \in R$.

Example 1.12. Let $\mathfrak{g}$ be a semisimple complex Lie algebra with a CSA $\mathfrak{h}$. Let Φ be the set of roots contained in the real inner product space E constructed in the previous section. Since $\mathfrak{g}$ is finite-dimensional, Φ is clearly finite. From the previous section, Φ spans E, and Φ does not contain 0 since it is known that $C_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$, so Φ satisfies Condition (1). It is already mentioned in the previous section that Φ satisfies Condition (2). For $\alpha, \beta \in \Phi$, recall that $\langle \alpha, \beta \rangle = \frac{2(\alpha,\beta)}{(\beta,\beta)} = \alpha(h_\beta)$. Since this is always an integer, Φ satisfies Condition (4). Lastly, $s_\alpha(\beta) = \beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha = \beta - \beta(h_\alpha)\alpha$. From [17, Section 8.5], this is also an element of Φ. As it is clear from direct computation that $s_\alpha^{-1} = s_\alpha$, since s_α maps R into itself, it must be a bijection, hence a permutation, on Φ, establishing Condition (3). Thus, $\Phi \subset E$ is an example of an abstract root system. •

Let R be an abstract root system in a real inner product space E. Then $\dim(E)$ is the **rank** of the root system and R is **irreducible** if there is no proper disjoint union $R = R_1 \cup R_2$ such that $(\alpha, \beta) = 0$ for all $\alpha \in R_1$ and $\beta \in R_2$. By the following theorem, given in [12, Lemma 11.8], it is sufficient to study the irreducible root systems:

Theorem 1.13. *Every abstract root system R in E is a disjoint union*

$$R = R_1 \cup R_2 \cup ... \cup R_k,$$

such that each R_i is an irreducible root system in the space E_i spanned by R_i. Moreover, the space E spanned by R is a direct and orthogonal sum of the spaces $E_1, ..., E_k$.

Remark 1.14. In the next parts of this section, we focus on the irreducible root systems and the simple Lie algebras. From Theorems 1.6 and 1.13, all abstract root systems and semisimple Lie algebras build up from these. To complete the correspondence, [17, Corollary, Section 14.1] establishes that the simple ideals of a semisimple Lie algebra precisely corresponds to the irreducible components of its root system. We summarise the justification for this here.

Proposition 1.15. *Let $\mathfrak{g}$ be a complex semisimple Lie algebra with Cartan subalgebra $\mathfrak{h}$ and root system Φ. Applying Theorems 1.6 and 1.13, let $\mathfrak{g} = \mathfrak{g}_1 \oplus ... \oplus \mathfrak{g}_k$ be the decomposition of $\mathfrak{g}$ into simple ideals, and let $\Phi = \Phi_1 \cup ... \cup \Phi_m$ be the decomposition of Φ into irreducible root systems. Then each Φ_i corresponds to the irreducible root system of one of the $\mathfrak{g}_i$.*

Proof. First suppose the root system is not irreducible, so there is a proper disjoint union $\Phi = \Psi_1 \cup \Psi_2$ such that $(\alpha, \beta) = 0$ for all $\alpha \in \Psi_1$ and $\beta \in \Psi_2$. Since $\mathfrak{g}$ is semisimple, we know that $Z(\mathfrak{g})$ is trivial. Recall from the root space decomposition that $\mathfrak{g}$ is spanned by $\mathfrak{h}$ and the eigenvectors e_η for $\eta \in \Phi$. Let $\mathfrak{k} = \mathrm{Span}(\{e_\alpha : \alpha \in \Psi_1\})$, which is a vector subspace of $\mathfrak{g}$ by construction. For $h \in \mathfrak{h}$ and $\alpha \in \Psi_1$, we have $[h, e_\alpha] = \alpha(h)e_\alpha \in \mathfrak{k}$. We now show that $[e_\eta, e_\alpha] \in \mathfrak{k}$ for all $\eta \in \Phi$ and $\alpha \in \Psi_1$. We recall that the inner product is positive definite, that is, $(\eta, \eta) > 0$ for all nonzero $\eta \in \Phi$. For all $\alpha \in \Psi_1$ and $\beta \in \Psi_2$, if $\alpha + \beta \in \Phi$, then $(\alpha + \beta, \alpha) = (\alpha, \alpha) \neq 0$, so $\alpha + \beta \notin \Psi_2$, and $(\alpha + \beta, \beta) = (\beta, \beta) \neq 0$, so $\alpha + \beta \notin \Psi_1$. This is impossible as Φ is equal to the union of Ψ_1 and Ψ_2, so by contradiction we have that $\alpha + \beta \notin \Phi$, so $[e_\beta, e_\alpha] = 0 \in \mathfrak{k}$. Similarly, for all $\alpha, \tilde{\alpha} \in \Psi_1$, if $\alpha + \tilde{\alpha} \notin \Phi$, then $[e_\alpha, e_{\tilde{\alpha}}] = 0 \in \mathfrak{k}$. If $\alpha + \tilde{\alpha} \in \Phi$, then it is nonzero, so observe that

$$(\alpha, \alpha + \tilde{\alpha}) + (\tilde{\alpha}, \alpha + \tilde{\alpha}) = (\alpha + \tilde{\alpha}, \alpha + \tilde{\alpha}) > 0,$$

so $(\alpha, \alpha + \tilde{\alpha}), (\tilde{\alpha}, \alpha + \tilde{\alpha})$ cannot both be zero. Supposing without loss of generality that $(\alpha, \alpha + \tilde{\alpha}) \neq 0$, it follows that $\alpha + \tilde{\alpha} \notin \Psi_2$. Therefore, we have $\alpha + \tilde{\alpha} \in \Psi_1$ and so $[e_\alpha, e_{\tilde{\alpha}}] \in \mathrm{Span}(e_{\alpha+\tilde{\alpha}}) \subseteq \mathfrak{k}$. Thus, $[\mathfrak{g}, \mathfrak{k}] \subseteq \mathfrak{k}$, so $\mathfrak{k}$ is an ideal of $\mathfrak{g}$. Observe that

$\mathfrak{k}$ is nontrivial by construction. Recall that $[e_\beta, e_\alpha] = 0$ for $\alpha \in \Psi_1$ and $\beta \in \Psi_2$, so $[e_\beta, \mathfrak{k}] = 0$. This shows that $e_\beta \in C_{\mathfrak{g}}(\mathfrak{k})$; since $Z(\mathfrak{g})$ is trivial, this shows that $\mathfrak{k} < \mathfrak{g}$, hence $\mathfrak{k}$ is a proper nontrivial ideal of $\mathfrak{g}$, which shows that $\mathfrak{g}$ is not simple. By the contrapositive, it follows that, if $\mathfrak{g}$ is simple, then its root system is irreducible.

Now let $\mathfrak{h}_i = \mathfrak{h} \cap \mathfrak{g}_i$ for all $i = 1, ..., k$. For $h_i \in \mathfrak{h}_i$, we have that $[h_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_j$ for all $j = 1, ..., k$, as the $\mathfrak{g}_j$ are ideals in $\mathfrak{g}$, so $(\operatorname{ad} h_i)|_{\mathfrak{g}_i}$ is diagonalisable since $\operatorname{ad} h$ is diagonalisable in $\mathfrak{g}$. Hence, the $\mathfrak{h}_i$ are toral subalgebras in $\mathfrak{g}_i$. As the $\mathfrak{g}_i$ are ideals in $\mathfrak{g}$, we have that $[\mathfrak{g}_i, \mathfrak{g}_j] = 0$ whenever $i \neq j$, so if $H_i > \mathfrak{h}_i$ is also toral in $\mathfrak{g}_i$, then $H_i \oplus_{j \neq i} \mathfrak{h}_j$ would be a toral subalgebra of $\mathfrak{g}$ greater than $\mathfrak{h}$, contradicting the maximality of $\mathfrak{h}$. Therefore, $\mathfrak{h}_i$ is a maximal toral subalgebra of $\mathfrak{g}_i$, hence a CSA of $\mathfrak{g}_i$ with a set Θ_i of roots. We then see that $[\mathfrak{h}_i, \mathfrak{g}_j] = 0$ for all $j \neq i$, so a root θ_i in Θ_i precisely corresponds to a root α in Φ where $\alpha(h) = \theta_i(h)$ for $h \in \mathfrak{h}_i$ and $\alpha(h) = 0$ otherwise. This gives a one-to-one correspondence from the disjoint union of Θ_i onto Φ.

Let this correspondence map Θ_i onto $\tilde{\Theta}_i$. Clearly each Θ_i is nonempty since $\mathfrak{g}_i$ is simple, hence nonabelian, so $\tilde{\Theta}_i$ is also nonempty. The previous paragraph shows that Θ_i is irreducible, so clearly $\tilde{\Theta}_i$ is also irreducible, which shows that $\tilde{\Theta}_i$ is contained in some Φ_k. We also see that $e_{\theta_i} \in \mathfrak{g}_i$ and $h_{\theta_i} \in \mathfrak{h}_i$ for all $\theta_i \in \Theta_i$. Therefore, for all $\alpha_i \in \tilde{\Theta}_i$ and $\alpha_j \in \tilde{\Theta}_j$ where $i \neq j$, we have that $[h_{\alpha_i}, e_{\alpha_j}] = \alpha_j(h_{\alpha_i}) e_{\alpha_j} \in [\mathfrak{h}_i, \mathfrak{g}_j] = \{0\}$, so $\alpha_j(h_{\alpha_i}) = 2(\alpha_j, \alpha_i)/(\alpha_i, \alpha_i) = 0$, showing that $(\alpha_j, \alpha_i) = 0$. Hence, $\tilde{\Theta}_i$ and $\tilde{\Theta}_j$ are disjoint for $i \neq j$, implying that all Φ_k contain precisely one $\tilde{\Theta}_i$. As the original correspondence is one-to-one, this implies that $\tilde{\Theta}_i = \Phi_k$, as desired. Thus, the decomposition of Φ into irreducible components has a natural correspondence with the decomposition of $\mathfrak{g}$ into a direct sum of simple ideals. This clarifies the remark from the end of Section 1.2.1, and justifies our later focus on simple complex Lie algebras. □

A **base** B of a root system R in the inner product space E is a vector space basis of E such that each root $\alpha \in R$ has a basis decomposition $\alpha = \sum_{\beta \in B} k_\beta \beta$ where the coefficients are integers and either all nonnegative or all nonpositive. Note that $\langle \alpha, \beta \rangle \leq 0$ for all $\alpha, \beta \in B$, namely the angle between base elements is always obtuse, since otherwise $s_\alpha(\beta) = \beta - \langle \beta, \alpha \rangle \alpha$ would have coefficients of different signs. Every root system has a base, as shown in [17, Section 10.1], so we now focus on the base since it generates, and hence defines, the root system.

Example 1.16 (Small root systems)**.** There is a unique root system of rank 1, denoted A_1. For a Lie algebra of this rank, the only roots are α and $-\alpha$. As the rank matches the dimension of $\mathfrak{h}$, we have that $\mathfrak{h}$ is 1-dimensional, so from the root space decomposition, the Lie algebra is spanned by a single $\mathfrak{sl}_2$-triple, so it is isomorphic to $\mathfrak{sl}(2, \mathbb{C})$. We later cover how $\mathfrak{sl}(l+1, \mathbb{C})$ has a root system of rank l for every $l \geq 1$, called A_l.

There is a unique reducible root system of rank 2, occurring when the base $\{\alpha, \beta\}$ satisfies $(\alpha, \beta) = 0$. This is the root system of type $A_1 \times A_1$. Following Remark 1.14, a Lie algebra with this root system is isomorphic to a direct sum of ideals with root system A_1. Illustrating Remark 1.14, a Lie algebra with root system $A_1 \times A_1$ is then semisimple but not simple, and isomorphic to the direct sum of ideals $\mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$. •

Once B is fixed, the elements of B are the **simple** roots of R. The elements of R with nonnegative coefficients with respect to the base are the **positive** roots, and the elements with nonpositive coefficients are the **negative** roots. For $\alpha \in R$, consider the reflection s_α. Since reflections permute R, the restriction of s_α to R is an element of $\mathrm{Sym}(R)$. The reflections then generate a subgroup $W \leq \mathrm{Sym}(R)$ called the **Weyl group** of R. The Weyl group is also generated by just the reflections s_β for simple roots $\beta \in B$.

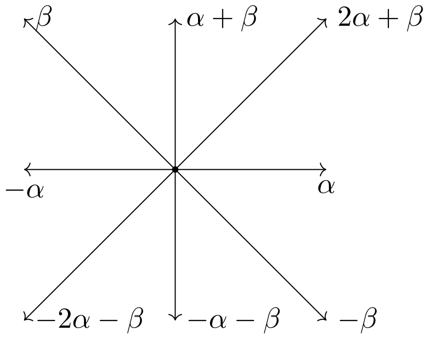
For every root $\alpha \in \Phi$, let H_α be the hyperplane in E perpendicular to α. Since finite hyperplanes cannot cover the entire space, they instead divide the space $E \setminus \bigcup_{\alpha\in\Phi} H_\alpha$ into disjoint regions, called **Weyl chambers**. In particular, for any base B, there is precisely one Weyl chamber on the positive side of the hyperplane H_β for all $\beta \in B$, called the **fundamental Weyl chamber** with respect to B. Just as the Weyl group permutes bases, it permutes Weyl chambers as well; in particular [17, Lemma B, Section 10.3], shows that the closure of the fundamental Weyl chamber is a **fundamental domain**, that is, it contains precisely one element of every W-orbit in E.

For a root system R with base $B = \{\alpha_1, ..., \alpha_n\}$, the **Cartan matrix** of the root system is the $n \times n$ matrix C with entries $C_{ij} = \langle \alpha_i, \alpha_j \rangle$. The Finiteness Lemma, given in [12, Lemma 11.4], states that $\langle \alpha, \beta\rangle\langle \beta, \alpha\rangle \in \{0, 1, 2, 3\}$ for all $\alpha, \beta \in R$ such that $\alpha \neq \pm\beta$. Geometrically, this constrains the angle between two different roots to be multiples of $\pi/6$ or $\pi/4$ strictly between 0 and π. Two root systems R and R' in inner product spaces E and E' are **isomorphic** if there exists a vector space isomorphism $\psi : E \to E'$ such that $\psi(R) = R'$, and $\langle \psi(\alpha), \psi(\beta)\rangle_{E'} = \langle \alpha, \beta\rangle_E$ for all $\alpha, \beta \in R$. Up to such an isomorphism, the Cartan matrix is sufficient to characterise the root system, by the following theorem given in [17, Section 11.1].

Theorem 1.17. *If there are two root systems with bases $B = \{\alpha_1, ..., \alpha_n\}$ and $B' = \{\alpha'_1, ..., \alpha'_n\}$ such that $\langle \alpha_i, \alpha_j\rangle_E = \langle \alpha'_i, \alpha'_j\rangle_{E'}$ for $1 \leq i, j \leq n$, the map $\alpha_i \mapsto \alpha'_i$ induces an isomorphism of root systems.*

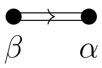
There is another way to represent the information in the Cartan matrix: via graphs. Let D be a directed multigraph with vertices the base of the root system. For vertices α_i, α_j, let there be $\langle \alpha_i, \alpha_j\rangle\langle \alpha_j, \alpha_i\rangle$ edges between them. By the Finiteness Lemma, the number of edges between any two vertices lies in $\{0, 1, 2, 3\}$. If there is more than one edge between vertices, the roots must have different lengths, so direct the edges from long root to short root. The resulting graph is the **Dynkin diagram** of the root system (defined in [17, Section 11.2]). The Dynkin diagram characterises the root system in the same way as the Cartan matrix, since [12, Proposition 11.21] shows that isomorphic Dynkin diagrams also induce an isomorphism of root systems and vice versa.

Example 1.18. Here, we consider the abstract root system of type B_2, with base $\{\alpha, \beta\}$. As it has rank 2, we can present it visually as vectors:

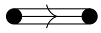


With respect to this base, the Cartan matrix is given by

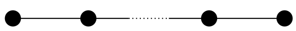
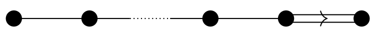
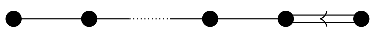
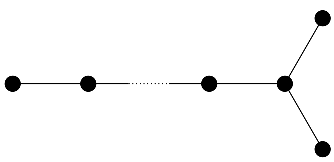
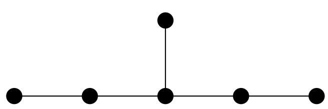
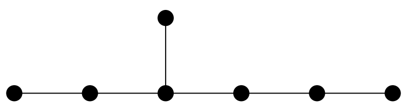
$$\begin{bmatrix} 2 & -1 \\ -2 & 2 \end{bmatrix},$$

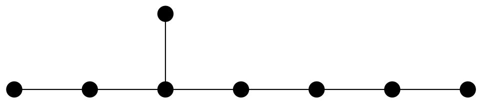
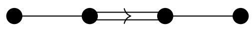
so the Dynkin diagram for this root system is 

. •

To summarise, a complex semisimple Lie algebra has a root system and a Dynkin diagram. The advantage of using Dynkin diagrams is that there are theorems on the valid abstract root systems that can be expressed in graph-theoretic terms. Two particular theorems are that the resulting Dynkin diagram cannot have any cycles, and that every vertex is connected to at most 3 edges. This already shows that the only Dynkin diagram of an irreducible root system with a triple edge is . Continuing this analysis, the Dynkin diagrams of irreducible root systems have been classified, as given in [17, Section 11.4]:

Theorem 1.19 (Classification of irreducible root systems)**.** *The (unlabelled) Dynkin diagram for an irreducible root system R has the form of one of*

- $A_n, n \geq 1$*:* 
- $B_n, n \geq 2$*:* 
- $C_n, n \geq 3$*:* 
- $D_n, n \geq 4$*:* 
- E_6 : 
- E_7 : 

- E_8 : 
- F_4 : 
- G_2 : 

where the subscript represents the number of vertices.

Conversely, each of the Dynkin diagrams given in Theorem 1.19 is associated with a unique Cartan matrix, which, in turn, is associated with a unique simple complex Lie algebra up to isomorphism. This follows naturally from the following result, shown in [12, Theorem 14.6].

Theorem 1.20 (Serre's theorem)**.** *Let C be an $n \times n$ Cartan matrix for one of the Dynkin diagrams in Theorem 1.19. Then there exists a complex finite-dimensional simple Lie algebra $\mathfrak{g}$ generated by taking linear combinations and Lie brackets from elements $\{e_1, f_1, h_1, ..., e_n, f_n, h_n\}$ satisfying the following relations:*

- $[h_i, h_j] = 0$ *for all* $1 \leq i, j \leq n$,
- $[h_i, e_j] = C_{ji} e_j$ *and* $[h_i, f_j] = -C_{ji} f_j$ *for all* $1 \leq i, j \leq n$,
- $[e_i, f_i] = h_i$ *for all* $1 \leq i \leq n$ *and* $[e_i, f_j] = 0$ *if* $i \neq j$,
- $(\operatorname{ad} e_i)^{1-C_{ji}}(e_j) = 0$ *and* $(\operatorname{ad} f_i)^{1-C_{ji}}(f_j) = 0$ *if* $i \neq j$.

Note that $C_{ji} \leq 0$ since the angle between elements of the base is always obtuse. Furthermore, $\mathfrak{g}$ has a Cartan subalgebra spanned by $\{h_i : 1 \leq i \leq n\}$, and its root system has Cartan matrix C up to some permutation of the base.

Every complex simple Lie algebra $\mathfrak{g}$ satisfies the relations given in Serre's theorem by [17, Section 18.1], so the Dynkin diagrams do fully classify the simple Lie algebras up to isomorphism. By Remark 1.14, every complex semisimple Lie algebra is a direct sum of ideals, each being a simple Lie algebra, so this classifies the complex semisimple Lie algebras.

Example 1.21 (Classical Lie algebras)**.** Recalling Example 1.11, the special linear algebra $\mathfrak{sl}(n+1, \mathbb{C})$ has a Dynkin diagram of type A_n, and the second example can be generalised to $\mathfrak{so}(2l+1, \mathbb{C})$ for $l \geq 2$, which has a Dynkin diagram of type B_l. The two remaining families correspond to families of Lie algebras constructed in the same manner as $\mathfrak{so}(2l+1, \mathbb{C})$. The family C_l corresponds to the Lie subalgebra $\mathfrak{sp}(2l, \mathbb{C}) \leq \mathfrak{gl}(2l, \mathbb{C})$ of matrices A that satisfy the relation $A^\intercal S = -SA$ for $S = \begin{bmatrix} 0_l & I_l \\ -I_l & 0_l \end{bmatrix}$, where 0_l and I_l are the $l \times l$ zero and identity matrices, respectively. These are the **symplectic algebras**. Lastly, the family D_l corresponds to the even case of the orthogonal algebras, $\mathfrak{so}(2l, \mathbb{C})$, comprising matrices A that satisfy $A^\intercal S = -SA$ for $S = \begin{bmatrix} 0_l & I_l \\ I_l & 0_l \end{bmatrix}$.

From the classification in Theorem 1.19, it follows that $\mathfrak{sl}(n, \mathbb{C})$ and $\mathfrak{so}(2n+1, \mathbb{C})$ are simple for all $n \geq 2$ (from the Dynkin diagrams, we note that $\mathfrak{so}(3, \mathbb{C}) \cong \mathfrak{sl}(2, \mathbb{C})$). Furthermore, $\mathfrak{sp}(2l, \mathbb{C})$ is simple for all $l \geq 3$, with $\mathfrak{sp}(2, \mathbb{C}) \cong \mathfrak{sl}(2, \mathbb{C})$ and $\mathfrak{sp}(4, \mathbb{C}) \cong \mathfrak{so}(5, \mathbb{C})$ from the Dynkin diagrams. Lastly, $\mathfrak{so}(2l, \mathbb{C})$ is simple for $l \geq 4$. Neither $\mathfrak{so}(2, \mathbb{C})$ nor $\mathfrak{so}(4, \mathbb{C})$ are simple, so they do not have Dynkin diagrams as indicated in Theorem 1.19. The former is actually the 1-dimensional abelian Lie algebra, which is also not semisimple, and the latter has the reducible root system $A_1 \times A_1$. For $l = 3$, from the Dynkin diagrams it follows that $\mathfrak{so}(6, \mathbb{C}) \cong \mathfrak{sl}(4, \mathbb{C})$. •

We conclude by noting that for a Lie algebra $\mathfrak{g}$, it is often beneficial, both in computational and theoretical contexts, to have a basis with integer structure constants. This is a **Chevalley basis** of $\mathfrak{g}$. When $\mathfrak{g}$ is complex and semisimple, we can see after letting $e_{-k} = f_k$, that $\{h_\alpha, e_\alpha : \alpha \in \Phi\}$ is almost such a basis, except that $[e_\alpha, e_\beta] = N(\alpha, \beta) e_{\alpha+\beta}$ and $N(\alpha, \beta)$ might not be an integer. However, in this case, [17, Section 25.2] shows that, by replacing some of the e_α with a scalar multiple, one can construct a Chevalley basis for $\mathfrak{g}$. Such a basis is generally used in the study of Lie algebras, since integer structure constants make it easier to, for example, classify subclasses of Lie algebras, and form constructions from Lie algebras, like groups of Lie type as presented in [2].

Chapter 2

Simple Lie algebras over $\mathbb{R}$

2.1 Introduction to real Lie algebras

In this chapter, we discuss Lie algebras over the real numbers. We illustrate basic terminology and their relationship with the complex Lie algebras. Focusing on the simple case, we recall how real Lie algebras can be classified in terms of the complex simple Lie algebras and certain automorphisms on those complex Lie algebras. For the rest of this report, we also assume Lie algebras to be non-abelian unless specified otherwise. Most of the content in this chapter can be found in [26].

For a real Lie algebra $\mathfrak{g}$, its **complexification** is defined to be $\mathfrak{g}(\mathbb{C}) = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{g} \oplus \imath\mathfrak{g}$, as given in [7, Section 1.2]. This is the Lie algebra $\mathfrak{g}$ with the underlying field extended to $\mathbb{C}$; in particular it has the same basis as $\mathfrak{g}$, so it can be defined by the same structure constants as $\mathfrak{g}$. Given a homomorphism $f : \mathfrak{g} \to \mathfrak{h}$ of real Lie algebras, it also has a complexification $f(\mathbb{C}) : \mathfrak{g}(\mathbb{C}) \to \mathfrak{h}(\mathbb{C})$, defined by $f(\mathbb{C})(x + \imath y) = f(x) + \imath f(y)$ for $x, y \in \mathfrak{g}$. Some objects defined over the complex Lie algebras need to be modified in the real case. For example, since $\mathbb{R}$ is not algebraically closed, the definition of semisimplicity of an element $x \in \mathfrak{g}$ is relaxed; following [17, Section 4.2], for any field F, an endomorphism $x \in \operatorname{End}(V)$ over an F-vector space V is **semisimple** if the roots of the minimal polynomial of x are distinct in V. This definition coincides with that given in the previous section (that x is diagonalisable) when F is algebraically closed.

For a complex Lie algebra $\mathfrak{g}^c$, a **real form** of $\mathfrak{g}^c$ is a real Lie subalgebra $\mathfrak{g} \leq \mathfrak{g}^c$ such that $\mathfrak{g}^c = \mathfrak{g}(\mathbb{C})$. Observe that $\dim_{\mathbb{R}}(\mathfrak{g}^c) = 2\dim_{\mathbb{R}}(\mathfrak{g})$, since if $\mathfrak{g}$ has basis $\{x_j\}_{1\leq j\leq n}$, then clearly the set $\{x_j, \imath x_j\}_{1\leq j\leq n}$ forms an $\mathbb{R}$-basis for $\mathfrak{g}^c$. The **realification** $\mathfrak{g}^c(\mathbb{R})$ of a complex Lie algebra $\mathfrak{g}^c$ is the real Lie algebra with the same elements as $\mathfrak{g}^c$, so $\dim_{\mathbb{R}}(\mathfrak{g}^c(\mathbb{R})) = 2\dim_{\mathbb{C}}(\mathfrak{g}^c)$. We present how the real forms are related to automorphisms on Lie algebras. A **real structure** on a complex Lie algebra $\mathfrak{g}^c$ is an automorphism $\alpha : \mathfrak{g}^c \to \mathfrak{g}^c$ that is both involutive (that is, $\alpha^2 = \mathrm{id}$) and **anti-linear**: $\alpha(x + cy) = \alpha(x) + \bar{c}\alpha(y)$ for $x, y \in \mathfrak{g}^c$ and $c \in \mathbb{C}$. Any real form $\mathfrak{g}$ of $\mathfrak{g}^c$ induces a real structure via the map $\sigma : \mathfrak{g}^c \to \mathfrak{g}^c$, $\sigma(a+\imath b) = a-\imath b$, where $a, b \in \mathfrak{g}$. Clearly σ is an involution. Observe that it is anti-linear since, for $a_1, a_2, b_1, b_2 \in \mathfrak{g}$ and $c_1, c_2 \in \mathbb{R}$, we have

$$\sigma((a_1 + \imath a_2) + (b_1 + \imath b_2)) = \sigma(a_1 + b_1 + \imath(a_2 + b_2)) = a_1 + b_1 - \imath(a_2 + b_2)$$
$$= (a_1 - \imath a_2) + (b_1 - \imath b_2) = \sigma(a_1 + \imath a_2) + \sigma(b_1 + \imath b_2).$$

and

$$\begin{aligned}\sigma((c_1 + c_2\imath)(a_1 + \imath a_2)) &= \sigma(a_1c_1 - a_2c_2 + \imath(a_1c_2 + a_2c_1)) \\ &= a_1c_1 - a_2c_2 - \imath(a_1c_2 + a_2c_1) \\ &= (c_1 - \imath c_2)\sigma(a_1 + \imath a_2).\end{aligned}$$

Conversely, a real structure on $\mathfrak{g}^c$ induces a real form from its set of fixed points, so there is a one-to-one correspondence between real forms and real structures of $\mathfrak{g}^c$. It is natural to ask whether this can be extended into a characterisation of real forms up to isomorphism. In fact, there is also a one-to-one correspondence between the isomorphism classes of real forms, and conjugacy classes of real structures. This follows from the previous correspondence via the following theorem, given in [26, Proposition 2.1].

Theorem 2.1. *For a complex Lie algebra $\mathfrak{g}^c$, let $\mathfrak{g}_1, \mathfrak{g}_2$ be real forms with corresponding real structures σ_1, σ_2. Then $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are isomorphic Lie algebras, written $\mathfrak{g}_1 \cong \mathfrak{g}_2$, if and only if $\sigma_2 = \alpha\sigma_1\alpha^{-1} = \sigma_1^\alpha$ for some $\alpha \in \operatorname{Aut}\mathfrak{g}^c$.*

Proof. First suppose $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are isomorphic, with isomorphism $h : \mathfrak{g}_1 \to \mathfrak{g}_2$. Then the complexification of h gives an automorphism $\alpha = h(\mathbb{C}) \in \operatorname{Aut}\mathfrak{g}^c$. For $x \in \mathfrak{g}_1$, we have $\alpha(x) = h(x) \in \mathfrak{g}_2$, so $(\sigma_2\alpha)(x) = \sigma_2(\alpha(x)) = h(x)$. By definition, $\sigma_1(x) = x$, so $(\alpha\sigma_1)(x) = \alpha(x) = h(x)$ as well. This shows $\sigma_2\alpha = \alpha\sigma_1 = h$ on $\mathfrak{g}_1$. Furthermore, as the products of an anti-involution and an automorphism, both $\sigma_2\alpha$ and $\alpha\sigma_1$ are anti-automorphisms, so their values on $\mathfrak{g}_1$ extend anti-linearly on the complexification $\mathfrak{g}^c$, so $\sigma_2\alpha = \alpha\sigma_1$, whereupon $\sigma_2 = \alpha\sigma_1\alpha^{-1}$.

Conversely, let $\sigma_2 = \alpha\sigma_1\alpha^{-1}$ for some $\alpha \in \operatorname{Aut}\mathfrak{g}^c$. Since $\mathfrak{g}_2$ is the set of fixed points of σ_2, it is also the set of fixed points of $\alpha\sigma_1\alpha^{-1}$. For any point x in this set, we have $\sigma_1(\alpha^{-1}(x)) = \alpha^{-1}(x)$. Since $\mathfrak{g}_1$ is the set of fixed points of σ_1, it follows that $\mathfrak{g}_1 = \alpha^{-1}(\mathfrak{g}_2)$, so $\mathfrak{g}_2 = \alpha(\mathfrak{g}_1)$. As α is an automorphism, $\alpha|_{\mathfrak{g}_1}$ is an isomorphism between $\mathfrak{g}_1$ and $\mathfrak{g}_2$, as desired. □

In what follows, we require the following lemma, presented in [26, Section 2].

Lemma 2.2. *For a real Lie algebra $\mathfrak{g}$ and a vector subspace $\mathfrak{b} \subseteq \mathfrak{g}(\mathbb{C})$, let $\bar{\mathfrak{b}} = \sigma(\mathfrak{b})$. Then $\mathfrak{b} = \mathfrak{a}(\mathbb{C})$ for a subspace $\mathfrak{a} \subseteq \mathfrak{g}$ if and only if $\mathfrak{b} = \bar{\mathfrak{b}}$. If the latter condition holds, $\mathfrak{a} = \mathfrak{b} \cap \mathfrak{g}$ is an explicit construction.*

Proof. If $\mathfrak{b} = \mathfrak{a}(\mathbb{C})$, for $b \in \mathfrak{b}$, we have $b = a_1 + a_2\imath$ for $a_1, a_2 \in \mathfrak{a}$. Also, $\mathfrak{a} \leq \mathfrak{g}$, so $\bar{b} = a_1 - a_2\imath = a_1 + (-a_2)\imath \in \mathfrak{b}$, so $\bar{\mathfrak{b}} \subseteq \mathfrak{b}$. Since $\bar{\bar{x}} = x$, this also shows that $\mathfrak{b} \subseteq \bar{\mathfrak{b}}$, so $\bar{\mathfrak{b}} = \mathfrak{b}$.

Conversely, if $\bar{\mathfrak{b}} = \mathfrak{b}$, let $b = b_1 + \imath b_2 \in \mathfrak{b}$ for $b_1, b_2 \in \mathfrak{g}$. Then $\bar{b} = b_1 - \imath b_2 \in \mathfrak{b}$. Since $\mathfrak{b}$ is a complex vector space, $\frac{1}{2}(b + \bar{b}) = b_1 \in \mathfrak{b}$ and $-\imath(b - b_1) = b_2 \in \mathfrak{b}$, so we see $\mathfrak{b} = \mathfrak{a} \oplus \imath\mathfrak{a}$ for $\mathfrak{a} = \mathfrak{b} \cap \mathfrak{g}$. Clearly, $\mathfrak{a}$ is a real vector space, so $\mathfrak{a}$ is a vector subspace of $\mathfrak{g}$, as desired. □

Example 2.3. Since the realification $\mathfrak{g}^c(\mathbb{R})$ of a complex semisimple Lie algebra $\mathfrak{g}^c$ is a real semisimple Lie algebra, we can consider its complexification, $\mathfrak{g}^\dagger = [\mathfrak{g}^c(\mathbb{R})](\mathbb{C})$. This can be considered in a simpler way as follows, following [26, Proposition 2.3]. For all complex Lie algebras $\mathfrak{a}$, one can consider a Lie algebra with the same elements and Lie bracket, but with scalar multiplication given by $c \cdot x = \bar{c}x$ for all $c \in \mathbb{C}$ and $x \in \mathfrak{a}$, where the second scalar multiplication is computed in $\mathfrak{a}$. This Lie algebra is denoted as $\bar{\mathfrak{a}}$. Now, consider the Lie direct sum $\mathfrak{g}^\diamond = \mathfrak{g}^c \oplus \bar{\mathfrak{g}}^c$. The map σ on $\mathfrak{g}^\diamond$ given by $\sigma(x, y) = (y, x)$ is clearly an involution. It is also an anti-automorphism, since

$$\sigma(c(x, y)) = \sigma(cx, \bar{c}y) = (\bar{c}y, cx) = \bar{c}(y, x) = \bar{c}\sigma(x, y).$$

Hence, the fixed points $\mathfrak{d} = \{(x, x) : \ x \in \mathfrak{g}^c\}$ of σ give a real form in $\mathfrak{g}^\diamond$. Clearly, the map $\psi : \mathfrak{g}^c(\mathbb{R}) \to \mathfrak{d}$ defined by $\psi(x) = (x, x)$ is an isomorphism of Lie algebras, so their complexifications must be isomorphic too. Therefore, $\mathfrak{g}^\dagger \cong \mathfrak{g}^c \oplus \bar{\mathfrak{g}}^c$, with the real form $\mathfrak{g}^c(\mathbb{R})$ associated with the diagonal subalgebra $\mathfrak{d}$. Since $\bar{\mathfrak{g}}^c$ only differs from $\mathfrak{g}^c$ with respect to the scalar multiplication, elements of $\mathfrak{g}^\dagger$ are often treated as being in $\mathfrak{g}^c \oplus \mathfrak{g}^c$. •

Example 2.4. Despite the previous example, it is possible to perform computations directly on $\mathfrak{g}^\dagger$, as long as the scalar multiplication is distinguished from that of $\mathfrak{g}^c$. Letting $\mathfrak{g}^c = \mathfrak{sl}(n, \mathbb{C})$, we saw in the previous chapter how $\mathfrak{g}^c$ has a CSA comprising its diagonal matrices, a resulting set of roots Φ and a root space decomposition on $\mathfrak{g}^c$. Here, we show how to construct these for $\mathfrak{g}^\dagger$.

Let $\mathfrak{h}^c$ be the CSA in $\mathfrak{g}^c$ and consider a maximal subalgebra $\mathfrak{m}$ consisting of semisimple elements. Having σ be the real structure of $\mathfrak{g}^c(\mathbb{R})$ in $\mathfrak{g}^\dagger$, we must have $\mathfrak{m} = \sigma(\mathfrak{m})$, since otherwise $\mathfrak{m} + \sigma(\mathfrak{m})$ would be a larger subalgebra consisting of semisimple elements (real structures preserve semisimplicity as they are anti-linear involutions). From Lemma 2.2, this shows that $\mathfrak{m}$ is the complexification of a subalgebra of $\mathfrak{g}^c$ consisting of semisimple elements. Since $\mathfrak{h}^c$ is maximal in $\mathfrak{g}^c$ with semisimple elements, $\mathfrak{h}^\dagger = \mathfrak{m}$ must be a CSA of $\mathfrak{g}^\dagger$.

Now, for roots $\alpha \in \Phi$, we have the $\mathfrak{sl}_2$-triple $\{f_\alpha, h_\alpha, e_\alpha\}$ in $\mathfrak{g}^c$. Since the h_α's span $\mathfrak{h}^c$ in $\mathfrak{g}^c$, the space $\mathfrak{h}^\dagger$ is spanned by the set of h_α and $\imath h_\alpha$. We use $\imath$ to represent the imaginary unit for $\mathfrak{g}^c$ and $\jmath$ for the imaginary unit in the field of scalars for $\mathfrak{g}^\dagger$. For $\beta \in \Phi$, since $[\imath h_\beta, e_\alpha] = \alpha(h_\beta)\imath e_\alpha$ and the elements e_α and $\imath e_\alpha$ are linearly independent in $\mathfrak{g}^\dagger$, we see that the root α must be modified to get a root for $\mathfrak{g}^\dagger$. This is done by observing that

$$[\imath h_\beta, e_\alpha \pm \jmath(\imath e_\alpha)] = \alpha(h_\beta)\imath e_\alpha \mp \jmath\alpha(h_\beta)e_\alpha = \mp\jmath\alpha(h_\beta)(e_\alpha \pm \jmath(\imath e_\alpha)).$$

Since it is clear from linearity that $[h_\beta, e_\alpha \pm \jmath(\imath e_\alpha)] = \alpha(h_\beta)(e_\alpha \pm \jmath(\imath e_\alpha))$, this shows us that $e_\alpha \pm \jmath(\imath e_\alpha)$ is an eigenvector, and that α induces two roots $\alpha^\pm$ for the roots in $\mathfrak{g}^\dagger$, which map h_β to $\alpha(h_\beta)$ and map $\imath h_\beta$ to $\pm\jmath\alpha(h_\beta)$. The set $\tilde{\Phi}$ of all $\alpha^\pm$ is then the set of roots in $\mathfrak{g}^\dagger$. The resulting $\mathfrak{sl}_2$-triple for the root is

$$\{f_{\alpha^\pm}, h_{\alpha^\pm}, e_{\alpha^\pm}\} = \left\{\frac{1}{2}(f_\alpha \pm \jmath(\imath f_\alpha)), h_\alpha \pm \jmath(\imath h_\alpha), \frac{1}{2}((e_\alpha \pm \jmath(\imath e_\alpha))\right\}.$$

This then induces the desired root space decomposition

$$\mathfrak{g}^\dagger = \mathfrak{h}^\dagger \oplus \bigoplus_{\alpha^\pm \in \tilde{\Phi}} \langle e_{\alpha^\pm} \rangle.$$

There is another use for real structures. Given a real structure σ with real form $\mathfrak{g}$, consider the function $h_\sigma : \mathfrak{g}^c \times \mathfrak{g}^c \to \mathbb{C}$, defined by $h_\sigma(x, y) = -\kappa(x, \sigma(y))$, where κ is the Killing form on $\mathfrak{g}$. As the Killing form is bilinear, and real structures are anti-linear, this function is linear in the first argument and anti-linear in the second. For a general F-vector space V where $F = \mathbb{R}$ or $\mathbb{C}$, a **Hermitian form** is a map $f : V \times V \to F$ that is linear in the first argument and satisfies $f(x, y) = \overline{f(y, x)}$ for all $x, y \in V$. Note that f is symmetric when $F = \mathbb{R}$. It follows that Hermitian forms are also anti-linear in the second argument as well. With this, we establish some key properties of the form h_σ, following [26, Proposition 3.2(i - iii)] and shown below.

Theorem 2.5. *Let $\mathfrak{g}^c$ be a complex semisimple Lie algebra with Killing form κ. Furthermore, let $\mathfrak{g}$ be a real form with real structure σ on $\mathfrak{g}^c$, and let h_σ be the corresponding Hermitian form. Then the following hold:*

a) *h_σ is a Hermitian form, and equal to $-\kappa$ on $\mathfrak{g}$.*
b) *For $x, y \in \mathfrak{g}^c$, we have $h_\sigma(\gamma(x), \gamma(y)) = h_\sigma(x, y)$ for all $\gamma \in \operatorname{Aut}\mathfrak{g}^c$ that commute with σ.*
c) *For $x \in \mathfrak{g}$ and $u, v \in \mathfrak{g}^c$, we have $h_\sigma([x, u], v) = -h_\sigma(u, [x, v])$.*

Proof. a) For $x, y \in \mathfrak{g}^c$, we have $h_\sigma(x, y) = -\kappa(x, \sigma(y)) = -\kappa(\sigma(y), x)$. From [26, Proposition 2.2(iii)], this is equal to $\overline{-\kappa(y, \sigma(x))} = \overline{h_\sigma(y, x)}$, which shows that h_σ is Hermitian. Furthermore, if $x, y \in \mathfrak{g}$, then $\sigma(y) = y$ so $h_\sigma(x, y) = -\kappa(x, y)$ on $\mathfrak{g}$.

b) For $x, y \in \mathfrak{g}^c$, observe $h_\sigma(x, y) = -\kappa(x, \sigma(y))$. Since the Killing form is invariant, this is equal to $-\kappa(\gamma(x), \gamma(\sigma(y)))$. Since $\gamma\sigma = \sigma\gamma$, we then have $h_\sigma(x, y) = -\kappa(\gamma(x), \sigma(\gamma(y))) = -h_\sigma(\gamma(x), \gamma(y))$, as desired.

c) As the Killing form is associative, we have that $h_\sigma([x, u], v) = \kappa([x, u], \sigma(v)) = -\kappa(u, [x, \sigma(v)])$. As $x \in \mathfrak{g}$, it is fixed by σ, so this is equal to $-\kappa(u, [\sigma(x), \sigma(v)])$. As σ is a real structure, it is an anti-automorphism, so in particular the above is equal to $-\kappa(u, \sigma([x, v])) = h_\sigma(u, [x, v])$, as desired. □

For a real Lie algebra $\mathfrak{g}$, a **complex structure** on $\mathfrak{g}$ is an automorphism J on $\mathfrak{g}$ such that $J^2 = -\mathrm{id}$ and $J([x, y]) = [x, J(y)]$ for $x, y \in \mathfrak{g}$. A complex Lie algebra $\mathfrak{g}^c$ induces a complex structure on its realification by $J(x) = \imath x$. Conversely a complex structure on a real Lie algebra $\mathfrak{g}$ induces a complex Lie algebra with realification $\mathfrak{g}$, with complex multiplication defined by $(a + \imath b)x = ax + bJ(x)$.

For a complex simple Lie algebra $\mathfrak{g}^c$, a real form $\mathfrak{g}$ of $\mathfrak{g}^c$ and the realification $\mathfrak{g}^c(\mathbb{R})$ are simple, as shown in [6, Section 2], since vector subspaces are preserved under extension of scalars, and the condition for an ideal is independent of the underlying field. The following theorem, from [26, Theorem 1, Chapter 2], presents the converse and justifies the subsequent shift in focus to the real simple Lie algebras constructed from the complex case.

Theorem 2.6. *Every real simple Lie algebra is the real form or the realification of a complex simple Lie algebra.*

Proof. Let $\mathfrak{g}$ be a real simple Lie algebra. If $\mathfrak{g}(\mathbb{C})$ is simple, then $\mathfrak{g}$ is the real form of the complex simple Lie algebra $\mathfrak{g}(\mathbb{C})$. Now suppose $\mathfrak{g}(\mathbb{C})$ is not simple, with ideal $0 < \mathfrak{a} < \mathfrak{g}(\mathbb{C})$, so all $a \in \mathfrak{a}$ and $x \in \mathfrak{g}(\mathbb{C})$ satisfy the equation $[a, x] \in \mathfrak{a}$. As the complex conjugate σ is an anti-involution, $[\sigma(a), x] = \sigma([a, \sigma(x)]) \in \sigma(\mathfrak{a})$ holds for all $\sigma(a) \in \sigma(\mathfrak{a})$ and $x \in \mathfrak{g}(\mathbb{C})$, so $\bar{\mathfrak{a}} = \sigma(\mathfrak{a})$ is also an ideal of $\mathfrak{g}(\mathbb{C})$. Then $\mathfrak{b} = \mathfrak{a} + \bar{\mathfrak{a}}$ and $\mathfrak{c} = \mathfrak{a} \cap \bar{\mathfrak{a}}$ are also ideals. It is clear that $\bar{\mathfrak{b}} = \mathfrak{b}$ and $\bar{\mathfrak{c}} = \mathfrak{c}$ by construction, so the prior lemma shows that $\mathfrak{b} = (\mathfrak{b} \cap \mathfrak{g})(\mathbb{C})$ and $\mathfrak{c} = (\mathfrak{c} \cap \mathfrak{g})(\mathbb{C})$. This implies that $\mathfrak{b} \cap \mathfrak{g}$ and $\mathfrak{c} \cap \mathfrak{g}$ are ideals of $\mathfrak{g}$. Since $\mathfrak{g}$ is simple by assumption, these ideals must be $\mathfrak{g}$ or $\{0\}$. As $\mathfrak{b} \geq \mathfrak{a} > 0$ and $\mathfrak{c} \leq \mathfrak{a}$, we must have $\mathfrak{b} \cap \mathfrak{g} = \mathfrak{g}$ and $\mathfrak{c} \cap \mathfrak{g} = 0$.

This shows that $\mathfrak{g}(\mathbb{C}) = \mathfrak{b} = \mathfrak{a} \oplus \bar{\mathfrak{a}}$. We can now establish a complex structure on $\mathfrak{g}$, given by $J(a + \bar{b}) = \imath a - \imath \bar{b}$ for $a, b \in \mathfrak{a}$. As $J^2(a + \bar{b}) = J(\imath a - \imath \bar{b}) = -a - \bar{b}$, we have $J^2 = -\mathrm{id}$. Since $\mathfrak{a}$ is an ideal of $\mathfrak{g}$ and $\mathfrak{g}$ is a Lie direct sum of $\mathfrak{a}$ and $\bar{\mathfrak{a}}$, we have

$$\begin{aligned} J([a_1 + \bar{b}_1, a_2 + \bar{b}_2]) &= J([a_1, a_2] + [\bar{b}_1, \bar{b}_2]) \\ &= \imath[a_1, a_2] - \imath[\bar{b}_1, \bar{b}_2] = [a_1, \imath a_2] + [\bar{b}_1, -\imath \bar{b}_2] \\ &= [a_1 + \bar{b}_1, \imath a_2 - \imath \bar{b}_2] = [a_1 + \bar{b}_1, J(a_2 + \bar{b}_2)], \end{aligned}$$

so J is indeed a complex structure. Then J induces a complex Lie algebra with realification $\mathfrak{g}$, which is known to be simple since $\mathfrak{g}$ is simple.

Hence, $\mathfrak{g}$ is the real form or the realification of a complex simple Lie algebra, as desired. ☐

Example 2.7. Consider the complex simple Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ of 2×2 complex matrices with the canonical basis

$$e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Taking the real span of this gives the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$, which is then a real form of $\mathfrak{sl}(2, \mathbb{C})$. Recall that the structure constants of $\mathfrak{sl}(2, \mathbb{C})$ are defined by $[e, f] = h$, $[e, h] = -2e$ and $[f, h] = 2f$. Then $[\imath e, \imath f] = -h$, $[\imath f, h] = 2\imath f$ and $[\imath e, h] = -2\imath e$, so the Lie brackets of the basis $\{\imath e, \imath f, h\}$ are contained in the real span. Thus, $\mathrm{Span}_{\mathbb{R}}(\{\imath e, \imath f, h\})$ is also a real form of $\mathfrak{sl}(2, \mathbb{C})$. As we later see, these are in fact the only real forms of $\mathfrak{sl}(2, \mathbb{C})$ up to isomorphism.

2.2 Compact real forms and the Cartan decomposition

From [26, Proposition I.12], a (general) real Lie algebra $\mathfrak{g}$ is **compact** if $\kappa(x, x) \leq 0$ for all $x \in \mathfrak{g}$, and it is both compact and semisimple if and only if the Killing form is negative definite. For this section, $\mathfrak{g}^c$ and $\mathfrak{g}$ are assumed to be semisimple unless stated otherwise. There are multiple reasons why compactness is of interest, especially when studying compact real forms of a complex semisimple Lie algebra $\mathfrak{g}^c$. By [26, Corollary, pp. 25], there is a unique compact real form of $\mathfrak{g}^c$ up to conjugation by elements of the adjoint group. Furthermore, as it is later shown, they are central to the study of the remaining real forms of $\mathfrak{g}^c$, and the construction of Cartan decompositions. The compactness of a real form can also be determined from the form of the induced real structure, as shown below [26, Proposition 3.2(iv)]

Proposition 2.8. *Let $\mathfrak{g}^c$ be a complex semisimple Lie algebra and $\mathfrak{u}$ be a real form of $\mathfrak{g}^c$ with real structure τ. Then $\mathfrak{u}$ is compact if and only if the form h_τ is positive definite.*

Proof. Suppose h_τ is positive definite, so $h_\tau(x,x) > 0$ for all nonzero $x \in \mathfrak{g}^c$. Since, in particular, this holds for all $x \in \mathfrak{u}$, it follows from the first part of Theorem 2.5 that $-\kappa(x,\tau(x)) = -\kappa(x,x) > 0$, so $\kappa_{\mathfrak{g}^c}|_{\mathfrak{u}}$ is negative definite. Now, $\kappa_{\mathfrak{g}^c}|_{\mathfrak{u}} = \kappa_{\mathfrak{u}}$ by [26, Proposition 2.2] since, given a basis for $\mathfrak{g}$, the complexification $\mathfrak{g}^c$ has the same basis and, for any $x \in \mathfrak{g}$, the map $\operatorname{ad} x$ is an endomorphism on $\mathfrak{g}$, so it has the same matrix form for both $\mathfrak{g}$ and $\mathfrak{g}^c$. Thus, $\kappa_{\mathfrak{u}}$ is negative definite so $\mathfrak{u}$ is compact.

Conversely, if $\mathfrak{u}$ is compact, then by definition $-\kappa(x,x) = -\kappa(x,\tau(x)) = h_\tau(x,x) > 0$ for all $x \in \mathfrak{u}$, so h_τ is positive definite on $\mathfrak{u}$. Since $\mathfrak{u}$ is a real form of $\mathfrak{g}^c$, any nonzero element of $\mathfrak{g}^c$ can be expressed in the form $a + \imath b$ for $a, b \in \mathfrak{u}$ where one of them is nonzero, whereupon

$$\begin{aligned} h_\tau(a+\imath b, a+\imath b) &= h_\tau(a,a) + h_\tau(a,\imath b) + h_\tau(\imath b, a) + h_\tau(\imath b, \imath b) \\ &= h_\tau(a,a) + h_\tau(b,b) - \imath h_\tau(a,b) + \imath h_\tau(b,a) \\ &= h_\tau(a,a) + h_\tau(b,b) + \imath(\kappa(a,\tau(b)) - \kappa(b,\tau(a))) \\ &= h_\tau(a,a) + h_\tau(b,b), \end{aligned}$$

where the last equality follows from $a, b \in \mathfrak{u}$ and symmetry of the Killing form. Since h_τ is positive definite on $\mathfrak{u}$, it follows that the above expression is strictly positive, as desired. □

As Proposition 2.8 shows, if τ is a compact real form of the complex semisimple Lie algebra, $\mathfrak{g}^c$, then h_τ is a positive definite Hermitian form. Using this, we can apply functional analysis techniques as shown in [26, Section 3] to analyse maps on real or complex Lie algebras, culminating in a construction of the exponential map that does not rely on series or Lie groups. We first introduce some concepts from functional analysis, also shown in [22, Section 13.9 - 13.10]. Let E be a real or complex vector space. If E is real, then a **real inner product** is a symmetric, positive definite bilinear form on E, as seen in Section 1.2.3. Similarly, if E is complex, then a **complex inner product** is a positive definite Hermitian form on E. The space together with the form is a (real or complex) inner product space. For a real or complex vector space with inner product and an endomorphism α on E, the **adjoint operator** α^* is uniquely defined by the condition $(\alpha(x), y) = (x, \alpha^*(y))$ for all $x, y \in E$. The endomorphism α is **self-adjoint** if $\alpha^* = \alpha$. If the underlying field is $\mathbb{R}$, it is more common to call these operators **symmetric**. Let $S(E) \subseteq \mathfrak{gl}(E)$ be the subset of self-adjoint operators. Importantly, we have an equivalent condition from [22, Section 7.9], given in the following proposition.

Proposition 2.9. *For a real or complex inner product space $(E, (-,-))$, an endomorphism $\alpha \in \mathfrak{gl}(E)$ is self-adjoint if and only if E has an orthonormal basis of eigenvectors of α with real eigenvalues.*

An endomorphism $\alpha \in \mathfrak{gl}(E)$ is **positive-definite** if $(\alpha(x), x) > 0$ for all nonzero $x \in E$. This is equivalent to E having an orthonormal basis of eigenvectors of α with strictly positive eigenvalues; from Proposition 2.9, α must also be self-adjoint. Let $P(E) \subseteq \mathfrak{gl}(E)$ be the subset of positive-definite operators; the previous sentence implies that $P(E) \subseteq S(E)$. Since none of the eigenvalues of a positive-definite operator are zero, the operator is invertible, so $P(E) \subseteq \mathrm{GL}(E)$ as well. Now, for some $\alpha \in S(E)$, Proposition 2.9 shows that E is a direct sum of eigenspaces E_{λ_i} for real eigenvalues λ_i. There then exists an operator with the same eigenspaces E_{λ_i} but with corresponding eigenvalue $\exp \lambda_i$, denoted $\exp \alpha$. Since the exponential map is positive, it follows that $\exp \alpha$ is positive definite, giving a map $\exp : S(E) \to P(E)$.

This map is also bijective since, given any $\beta \in P(E)$, there is a basis of eigenvectors with positive eigenvalues, so there is an operator $\log \beta \in S(E)$ with the same eigenspaces E_{λ_i} having eigenvalues $\log \lambda_i \in \mathbb{R}$. This map has the same form as the exponential map given in Section 1.3; since any $\alpha \in S(E) \subseteq \mathfrak{gl}(E)$ has a basis of eigenvectors, the matrix A corresponding to α with respect to this basis is diagonal, so the image under the first exponential map has, as entries, the power series acting on A_i, which is known to be $\exp A_i$.

For a complex semisimple Lie algebra $\mathfrak{g}^c$ with real form $\mathfrak{g}$, a **Cartan decomposition** of $\mathfrak{g}$ is a decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ of subspaces, such that $[\mathfrak{k}, \mathfrak{k}] \leq \mathfrak{k}$, $[\mathfrak{p}, \mathfrak{p}] \leq \mathfrak{k}$, $[\mathfrak{p}, \mathfrak{k}] \leq \mathfrak{p}$, and the Killing form of $\mathfrak{g}$ satisfies $\kappa(k, k) < 0$ for $k \in \mathfrak{k}$ and $\kappa(p, p) > 0$ for $p \in \mathfrak{p}$. The associated **Cartan involution** is defined by $\theta : \mathfrak{g} \to \mathfrak{g}$ with $\theta(k + p) = k - p$ for $k \in \mathfrak{k}$ and $p \in \mathfrak{p}$.

Theorem 2.10. *For a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, the Cartan involution θ induces a positive definite form $\kappa_\theta(x, y) = -\kappa(x, \theta(y))$ on $\mathfrak{g}$, where κ is the Killing form on $\mathfrak{g}^c$.*

Proof. Letting $x = k_1 + p_1$ and $y = k_2 + p_2$, we have

$$\begin{aligned} \kappa_\theta(x, y) &= -\kappa(x, \theta(y)) = -\kappa(k_1 + p_1, k_2 - p_2) \\ &= -\kappa(k_1, k_2) + \kappa(k_1, p_2) - \kappa(p_1, k_2) + \kappa(p_1, p_2), \end{aligned}$$

since the Killing form is bilinear. By the first part of the definition of the Cartan decomposition, it follows that $\operatorname{ad} k_1$ and $\operatorname{ad} k_2$ have block matrix form $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$, and $\operatorname{ad} p_1$ and $\operatorname{ad} p_2$ have block matrix form $\begin{bmatrix} 0 & C \\ D & 0 \end{bmatrix}$. Then $(\operatorname{ad} k_1) \circ (\operatorname{ad} p_2)$ has block matrix form $\begin{bmatrix} 0 & AC \\ BD & 0 \end{bmatrix}$ and $(\operatorname{ad} p_1) \circ (\operatorname{ad} k_2)$ has block matrix form $\begin{bmatrix} 0 & CB \\ DA & 0 \end{bmatrix}$, which both have zero trace. Thus, $\kappa(k_1, p_2) = \kappa(p_1, k_2) = 0$.

Then $\kappa_\theta(x, y) = -\kappa(k_1, k_2) + \kappa(p_1, p_2)$. From the second part of the definition of a Cartan decomposition, $\kappa(k_1, k_2) < 0$ and $\kappa(p_1, p_2) > 0$, so $\kappa_\theta(x, y) > 0$ and κ_θ is positive definite. □

Recall that Theorem 2.1 establishes a bijection between the conjugacy classes of real structures under the group of automorphisms of $\mathfrak{g}^c$ and isomorphism classes of real forms. Using compact real forms, this can be used to establish a correspondence between isomorphism classes of real forms and conjugacy classes of involutive automorphisms under the adjoint group. Given a compact real form τ, let σ be a real structure on the semisimple complex Lie algebra $\mathfrak{g}^c$. Then [26, Proposition 3.5] shows that there is an inner automorphism α such that $\sigma' = \alpha\sigma\alpha^{-1}$ and σ' and τ commute. Then let $\theta = \sigma'\tau$. As a product of two anti-automorphisms, θ is an automorphism and, since $\theta^2 = (\sigma'\tau)(\sigma'\tau) = \sigma'\sigma'\tau\tau = \mathrm{id}$, it is involutive. We then have the following theorem from [26, Theorem 3.2]. Note that the adjoint group, a Lie group acting on $\mathfrak{g}^c$ is discussed in more detail in Chapter 3.

Theorem 2.11. *For a complex semisimple Lie algebra $\mathfrak{g}^c$ with compact real form τ, the map $\sigma \mapsto (\alpha\sigma\alpha^{-1})\tau$ shown above extends to a bijection between the conjugacy classes of real structures of $\mathfrak{g}^c$, and the conjugacy classes of involutive automorphisms of $\mathfrak{g}^c$ under the adjoint group. Furthermore, the bijection is independent of the choice of compact real form τ.*

This theorem is of interest as, combined with Theorem 2.1, it shows that there is a bijection between the real forms of a complex semisimple Lie algebra up to isomorphism, and the conjugacy classes of involutive automorphisms under the adjoint group. For a real structure σ and a compact structure τ on the complex semisimple Lie algebra $\mathfrak{g}^c$, Theorem 2.11 gives a class of involutive automorphisms on $\mathfrak{g}^c$. By [26, Proposition 7], there is an element θ of this class which commutes with τ. By construction $\theta\tau = \sigma$ is a real structure with real form $\mathfrak{g}$. As shown in [26, Section 5], $\theta(\mathfrak{g}) \subseteq \mathfrak{g}$, and $\theta|_{\mathfrak{g}}$ is a Cartan involution on $\mathfrak{g}$.

2.3 Classification of real forms

For the remainder of the chapter, let $\mathfrak{g}^c$ be a complex simple Lie algebra. Let $\mathfrak{g}^c$ have Cartan subalgebra $\mathfrak{h}^c$ and a corresponding set of roots Φ. Let $\Delta = \{\alpha_1, ..., \alpha_n\}$ be a base for the root system, so $\mathfrak{g}^c$ has rank n, and $\mathfrak{g}^c$ has the Chevalley basis $\mathcal{B} = \{h_k, e_\alpha : \ 1 \le k \le n, \alpha \in \Phi\}$, where $h_k = h_{\alpha_k}$. Recall $h_\alpha = [e_\alpha, e_{-\alpha}]$ for each $\alpha \in \Phi$. Set $H_\alpha = \imath h_\alpha$, $E_\alpha = e_\alpha - e_{-\alpha}$ and $F_\alpha = \imath(e_\alpha + e_{-\alpha})$. Then $B = \{H_\alpha, E_\beta, F_\beta : \ \alpha \in \Delta, \beta \in \Phi^+\}$ is linearly independent, where Φ^+ is the set of positive roots defined by Δ. Furthermore, the real span of B, denoted $\mathfrak{u} = \mathrm{Span}_{\mathbb{R}}(B)$, is a compact real form of $\mathfrak{g}^c$, see [6, Section 2.2]. Let τ be the corresponding real structure for $\mathfrak{u}$. From [26, Corollary, pp. 25], the compact real form is unique up to conjugacy by the adjoint group.

Example 2.12. For $\mathfrak{g}^c = \mathfrak{sl}(2, \mathbb{C})$, the previously shown basis $\{e, f, h\}$ is a Chevalley basis, whereupon $E_\alpha = e - f$, $F_\alpha = \imath(e + f)$ and $H_\alpha = \imath h$. Then $\mathfrak{u} = \mathrm{Span}_{\mathbb{R}}(\{e - f, \imath(e + f), \imath h\})$.

To find the non-compact real forms of $\mathfrak{g}^c$, let θ be an involutive automorphism of $\mathfrak{g}^c$ (so $\theta^2 = 1$), which commutes with τ. We now cover how a real form $\mathfrak{g}(\mathfrak{u}, \theta)$ is constructed from θ and $\mathfrak{u}$. It follows from $\theta\tau = \tau\theta$ that $\tau(\theta(\mathfrak{u})) = \theta(\tau(\mathfrak{u})) = \theta(\mathfrak{u})$. As τ is the real structure of $\mathfrak{u}$, we know $\mathfrak{u} = \{x \in \mathfrak{g}^c : \tau(x) = x\}$, so $\theta(\mathfrak{u}) \subseteq \mathfrak{u}$. Thus, $\mathfrak{u}$ is invariant under θ. Now, θ is linear and $\theta^2 - 1 = 0$, so the minimal polynomial of θ divides $x^2 - 1 = (x-1)(x+1)$. As the minimal polynomial of θ is a product of linear factors, θ is diagonalisable with eigenvalues ± 1. Since $\mathfrak{u}$ is invariant under θ, the restriction $\theta|_{\mathfrak{u}}$ is also diagonalisable with eigenvalues ± 1 . That is, $\mathfrak{u}$ has eigenspace decomposition $\mathfrak{u}_1 \oplus \mathfrak{u}_{-1}$.

Let $\mathfrak{u}_+ = \mathfrak{u}_1$ and $\mathfrak{u}_- = \mathfrak{u}_{-1}$. For $u, v \in \mathfrak{u}_+$, observe that $[u, v] = [\theta(u), \theta(v)] = \theta([u, v])$, so $[u, v] \in \mathfrak{u}_+$. As u and v were arbitrary, $[\mathfrak{u}_+, \mathfrak{u}_+] \subseteq \mathfrak{u}_+$. The same computation shows that $[\mathfrak{u}_-, \mathfrak{u}_-] \subseteq \mathfrak{u}_+$ and $[\mathfrak{u}_+, \mathfrak{u}_-] \subseteq \mathfrak{u}_-$. Since $\mathfrak{u}$ is compact, [26, Chapter 5, pp. 36] shows that $\mathfrak{u} = \mathfrak{u}_+ \oplus \imath\mathfrak{u}_-$ is a Cartan decomposition, so $\mathfrak{g}(\mathfrak{u}, \theta) = \mathfrak{u}_+ \oplus \imath\mathfrak{u}_-$ is a real Lie algebra, hence a real form for $\mathfrak{g}^c$. Since $\mathfrak{g}^c = (\mathfrak{u}_+ \oplus \imath\mathfrak{u}_-) \oplus (\imath\mathfrak{u}_+ \oplus \mathfrak{u}_-)$, the real structure of $\mathfrak{g}(\mathfrak{u}, \theta)$ is the composition of θ (which negates $\mathfrak{u}_-$) and τ (which negates $\imath\mathfrak{u}$), $\sigma = \tau\theta$. From the previous reasoning, it is also clear that θ restricted to $\mathfrak{g}(\mathfrak{u}, \theta)$ is the corresponding Cartan involution.

Every real form of $\mathfrak{g}^c$ is isomorphic to $\mathfrak{g}(\mathfrak{u}, \theta)$ for some involutive automorphism θ commuting with τ, and $\mathfrak{g}(\mathfrak{u}, \theta)$ and $\mathfrak{g}(\mathfrak{u}, \theta')$ are isomorphic if and only if θ and θ' are conjugate with respect to an element of the adjoint group, by [6, Section 2.3].

Example 2.13. Following Example 2.12 with $\mathfrak{g}^c = \mathfrak{sl}(2, \mathbb{C})$, let θ be defined as the linear map that fixes $e - f$ and negates both $\imath(e + f)$ and $\imath h$. Then $\mathfrak{u}_+ = \mathrm{Span}_{\mathbb{R}}(\{e - f\})$ and $\mathfrak{u}_- = \mathrm{Span}_{\mathbb{R}}(\{\imath(e + f), \imath h\}$, so $\mathfrak{g}(\mathfrak{u}, \theta) = \mathrm{Span}_{\mathbb{R}}(\{e - f, -(e + f), -h\}) = \mathrm{Span}_{\mathbb{R}}(\{e, f, h\}) = \mathfrak{sl}(2, \mathbb{R})$.

Thus, finding all involutive automorphisms of $\mathfrak{g}^c$ commuting with τ is sufficient to classify the real forms of $\mathfrak{g}^c$.

2.3.1 Classification of automorphisms

The previous section shows the construction of a compact real form $\mathfrak{u}$, with real structure τ, and the construction of the other real forms by an involutive automorphism θ commuting with τ. In this section we finish summarising the classification of real forms of a complex Lie algebra $\mathfrak{g}^c$, by classifying those automorphisms. Since the Cartan subalgebra of $\mathfrak{g}^c$ is unique up to conjugacy, so we may assume it to be fixed. It is also trivial that an automorphism of $\mathfrak{g}^c$ fixes the structure constants, so, in particular, it maps the chosen Chevalley basis $\mathcal{B}$ to another Chevalley basis. As a linear map is determined by its image on a basis, it follows that the automorphisms may be classified by the Chevalley basis that is the image of $\mathcal{B}$ under such an automorphism.

Again let $\mathfrak{g}^c$ be a complex semisimple Lie algebra with a CSA $\mathfrak{h}^c$, a set of roots Φ, and a base of simple roots Δ. The involutive automorphisms commuting with τ are characterised by two pieces of data. One item is an involutive permutation $\pi \in \mathrm{Sym}(\{1, ..., n\})$ such that $\langle \alpha_{\pi(i)}, \alpha_{\pi(j)} \rangle = \langle \alpha_i, \alpha_j \rangle$ for $1 \le i, j \le n$, where $\{\alpha_1, ..., \alpha_n\} = \Delta$. As Δ is a basis for the vector space containing Φ, π extends linearly to an automorphism on Φ, as shown in [6, Section 2.3]. The second item is an n-tuple $\mathcal{E} = (\epsilon_1, ..., \epsilon_n)$ such that $\epsilon_i \in \{1, -1\}$ and $\epsilon_i = \epsilon_{\pi(i)}$ for all $1 \leq i \leq n$.

Then one can define a map θ between Chevalley bases by $\theta(e_{\alpha_i}) = \epsilon_i e_{\alpha_{\pi(i)}}$ for $\alpha_i \in \Delta$ and $\theta(h_i) = h_{\pi(i)}$. As mentioned in Chapter 1, the CSA is unique up to conjugacy by an element of the adjoint group, which is an automorphism on $\mathfrak{g}$. As θ has values for each element in the Chevalley basis, θ extends linearly to an endomorphism on $\mathfrak{g}^c$. Furthermore θ is an involutive automorphism of $\mathfrak{g}^c$ commuting with τ, and every such automorphism is conjugate to an automorphism constructed from some choice of $\mathcal{E}$ and π, by [6, Section 2.3].

Example 2.14. Returning to the example of $\mathfrak{g}^c = \mathfrak{sl}(2, \mathbb{C})$ with CSA $\mathfrak{h}^c$ spanned by h and with the compact real form $\mathfrak{u} = \mathrm{Span}_{\mathbb{R}}(\{e - f, \imath(e + f), \imath h\})$. The root space is $\Phi = \{\alpha, -\alpha\}$, with $\alpha \in \mathfrak{h}^\star$ defined by $\alpha(h) = h_{11} - h_{22}$. Here, π can only be the identity permutation on the singleton. Similarly, $\mathcal{E}$ can only be (1) or (-1), showing there are two real forms. One is the compact real form $\mathfrak{u}$. Considering $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$ as a real form of $\mathfrak{g}^c$, observe that $(\mathrm{ad}\, h \circ \mathrm{ad}\, h)(e) = [h, 2e] = 4e$, $(\mathrm{ad}\, h \circ \mathrm{ad}\, h)(f) = [h, -2f] = 4f$ and $(\mathrm{ad}\, h \circ \mathrm{ad}\, h)(h) = 0$, so $\kappa_{\mathfrak{g}^c}(h, h) = 4 + 4 - 0 = 8 \geq 0$. Since $\kappa_{\mathfrak{g}^c}$ is not negative definite on $\mathfrak{g}$, it follows that $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$ is not a compact real form. Thus, we conclude the two non-conjugate real forms of $\mathfrak{sl}(2, \mathbb{C})$ are $\mathfrak{u}$ and $\mathfrak{sl}(2, \mathbb{R})$. In particular, this shows that $\mathfrak{sl}(2, \mathbb{R})$ and the other real form, $\mathfrak{g}_2 = \mathrm{Span}_{\mathbb{R}}(\{\imath e, \imath f, h\})$, are conjugate by an element of the adjoint group, since $\mathrm{Span}_{\mathbb{R}}(\{\imath e, \imath f, -h\})$ is not compact $\kappa_{\mathfrak{g}^c}(-h, -h) = \kappa_{\mathfrak{g}^c}(h, h) = 8 \geq 0$. It is already clear that they are isomorphic, as the map $\tilde{\psi} : \{e, f, h\} \to \mathfrak{g}_2$ defined by $\tilde{\psi}(e) = \imath e$, $\tilde{\psi}(f) = -\imath f$ and $\tilde{\psi}(h) = h$ extends linearly to an isomorphism $\psi : \mathfrak{sl}(2, \mathbb{R}) \to \mathfrak{g}_2$.

2.4 Satake diagrams

The classification of real semisimple Lie algebras can also be expressed via graphs, in a manner analogous to Dynkin diagrams. The contents of this subsection are taken from [26, Chapter 9]. Let $\mathfrak{g}$ be a real semisimple Lie algebra with $\mathfrak{g}^c$ as its complexification and σ as the real structure. Then $\mathfrak{g}^c$ is a complex semisimple Lie algebra, so it has a corresponding Dynkin diagram; the **Satake diagram** of $\mathfrak{g}$ has the same edges and vertices as this Dynkin diagram, but with additional information given below. Now recall that $\mathfrak{g}^c$ has a compact form $\mathfrak{u}$ with real structure τ, and the real structure σ induces an involutive automorphism θ commuting with τ such that $\sigma = \theta\tau$. Here we know θ is a Cartan involution on $\mathfrak{g}$, so it induces a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.

Since $\mathfrak{p}$ is a subspace of $\mathfrak{g}$, it contains the trivial Lie algebra $\{0\}$. As $\mathfrak{g}$ is finite-dimensional, $\mathfrak{p}$ must contain a maximal subalgebra $\mathfrak{a}$ of $\mathfrak{g}$ in $\mathfrak{p}$. By definition of the Cartan decomposition, $[\mathfrak{a}, \mathfrak{a}] \leq \mathfrak{k}$. Since $\mathfrak{a}$ is a subalgebra, we have $[\mathfrak{a}, \mathfrak{a}] \leq \mathfrak{k} \cap \mathfrak{p} = \{0\}$, so $\mathfrak{a}$ is abelian. We further have the following lemma, inspired by [25, Problem 2, Section 5.4].

Lemma 2.15. *Let $\mathfrak{t}$ be a maximal abelian subalgebra of $\mathfrak{g}$, containing a maximal subalgebra $\mathfrak{a}$ in $\mathfrak{p}$. Then $\mathfrak{t} = \mathfrak{t}^+ \oplus \mathfrak{a}$ for some subalgebra $\mathfrak{t}^+ \leq \mathfrak{k}$.*

Proof. We first show that the centraliser of $\mathfrak{a}$ in $\mathfrak{g}$ has the form $\mathfrak{m}\oplus\mathfrak{a}$ where $\mathfrak{m} \leq \mathfrak{k}$. If $x \in C_{\mathfrak{g}}(\mathfrak{a})$ and $a \in \mathfrak{a}$, then $[a, x] = 0$, so $0 = \theta([a, x]) = [\theta(a), \theta(x)] = -[a, \theta(x)]$ since $a \in \mathfrak{p}$, so $\theta(x) \in C_{\mathfrak{g}}(\mathfrak{a})$. Hence, the space $C_{\mathfrak{g}}(\mathfrak{a})$ is invariant under θ, so the Cartan decomposition induces the decomposition $C_{\mathfrak{g}}(\mathfrak{a}) = \mathfrak{m} \oplus \mathfrak{r}$, for subspaces $\mathfrak{m} \subseteq \mathfrak{k}$ and $\mathfrak{r} \subseteq \mathfrak{p}$. Since $\mathfrak{m} = C_{\mathfrak{g}}(\mathfrak{a}) \cap \mathfrak{k}$, it is a subalgebra of $\mathfrak{k}$. Also, if $\mathfrak{r} > \mathfrak{a}$, then there exists $r \in \mathfrak{r} \setminus \mathfrak{a}$, so let $\tilde{\mathfrak{a}} = \mathfrak{a} \oplus \langle r\rangle \subseteq \mathfrak{p}$. We see that $[\tilde{\mathfrak{a}}, \mathfrak{a}] = \{0\}$ since $\tilde{\mathfrak{a}} \subseteq C_{\mathfrak{g}}(\mathfrak{a})$, and $[\langle r\rangle, \langle r\rangle] = \{0\}$ since $[r, r] = 0$, so $\tilde{\mathfrak{a}}$ is a (abelian) subalgebra contained in $\mathfrak{p}$ and properly containing $\mathfrak{a}$. As this contradicts the maximality of $\mathfrak{a}$, we must have $\mathfrak{r} = \mathfrak{a}$.

By assumption, we have $\mathfrak{a} \leq \mathfrak{t} \leq C_{\mathfrak{g}}(\mathfrak{a}) = \mathfrak{m}\oplus\mathfrak{a}$. Then for $x \in \mathfrak{t}$, we have $x = x_m+x_a$, where $x_m \in \mathfrak{m}$ and $x_a \in \mathfrak{a}$. Since $x_a \in \mathfrak{a} \leq \mathfrak{t}$, we have $x_m = x - x_a \in \mathfrak{t}$. This shows that $\mathfrak{t} = (\mathfrak{m} \cap \mathfrak{t}) \oplus \mathfrak{a}$. Hence, letting $\mathfrak{t}^+ = \mathfrak{t} \cap \mathfrak{m}$, we see that $\mathfrak{t}^+ \leq \mathfrak{t}$ and $\mathfrak{t} = \mathfrak{t}^+ \oplus \mathfrak{a}$, as desired. □

Now, $\mathfrak{t}(\mathbb{C})$ is clearly a maximal toral subalgebra of $\mathfrak{g}^c$, hence a CSA of $\mathfrak{g}^c$. Let Φ be the set of roots for $\mathfrak{t}(\mathbb{C})$. Since $\theta \in \operatorname{Aut} \mathfrak{g}$ and $\theta(\mathfrak{t}) = \mathfrak{t}$, we have from [26, Theorem II.12] that the transpose map $\theta^\intercal$ acting on $\mathfrak{t}^\star$ satisfies $\theta^\intercal(\Phi) = \Phi$. Here, $\theta^\intercal$ is the induced transpose map that acts on $\mathfrak{g}^\star$ by $\theta^\intercal(f) = f \circ \theta$ for all $f \in \mathfrak{g}^\star$, following [22, Section 6.7]. There is then a subset of Φ defined by

$$\Phi_c = \{\alpha \in \Phi : \ \theta^\intercal(\alpha) = \alpha\} = \{\alpha \in \Phi : \ \alpha|_{\mathfrak{a}} = 0\},$$

where the last equality is true since elements of Φ_c are fixed by $\theta^\intercal$, so they must act trivially on the -1 eigenspace of $\mathfrak{g}$ which is known to be $\mathfrak{a}$. The elements of Φ_c are the **compact roots**, and its complement in Φ is the set Φ_{nc} of **non-compact roots**. The set of positive roots Φ^+ of Φ may be chosen such that $\theta^\intercal(\alpha) \in -\Phi^+$ for all non-compact $\alpha \in \Phi^+$, as shown in [26, Chapter 9]. From this, one can choose a base $\Delta \subset \Phi^+$. Letting $\Delta_c = \Delta \cap \Phi_c$ and $\Delta_{nc} = \Delta \cap \Phi_{nc}$, it is known from [26, Lemma 1, Chapter 9] that there exists an involution $\chi : \Delta_{nc} \to \Delta_{nc}$ such that

$$\theta^\intercal(\alpha) = -\chi(\alpha) - \sum_{\gamma\in\Delta_c} c_{\alpha,\gamma}\gamma,$$

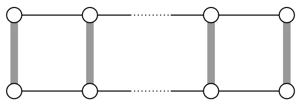
for $\alpha \in \Delta_{nc}$ and $c_{\alpha,\gamma} \in \mathbb{Z}^+$. The Satake diagram for $\mathfrak{g}$ is then defined as the Dynkin diagram for Δ, with compact roots labelled by black circles, non-compact roots labelled by white circles, and any pair $\alpha, \chi(\alpha) \in \Delta_{nc}$ with $\alpha \neq \chi(\alpha)$ linked by a two-headed arrow or grey bar. This is sufficient, as the following theorem shows:

Theorem 2.16. *The Satake diagram defines its Lie algebra uniquely up to isomorphism, and is connected if and only if the corresponding Lie algebra is simple.*

A list of the Satake diagrams for simple Lie algebras may be found in [26, Table 5].

Remark 2.17. For a complex semisimple Lie algebra $\mathfrak{g}^c$, consider the realification $\mathfrak{g}^c(\mathbb{R})$. Here, we show the Satake diagram of this Lie algebra and apply it to the case of $\mathfrak{g}^c = \mathfrak{sl}(n + 1, \mathbb{C})$.

From Example 2.3, the complexification of $\mathfrak{g}^\dagger$ can be identified with the Lie direct sum $\mathfrak{g}^c \oplus \overline{\mathfrak{g}^c}$. Therefore, the overlying Dynkin diagram is two disjoint copies of the Dynkin diagram of $\mathfrak{g}^c$. Following [25, Example 2, Section 5.4] each component is precisely the Satake diagram of $\mathfrak{g}$, with arrows connecting the matching vertices that are labelled white. In particular, the Dynkin diagram of $\mathfrak{sl}(n+1,\mathbb{C})$ is of type A_n and the Satake diagram of $\mathfrak{g} = \mathfrak{sl}(n+1,\mathbb{R})$ has each vertex labelled white with no arrows, so the Satake diagram of $\mathfrak{g}^{\mathrm{c}}(\mathbb{R}) = (\mathfrak{sl}(n+1,\mathbb{C}))(\mathbb{R})$ is



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Chapter 3

Complex θ-groups

3.1 Introduction to complex θ-groups

For a semisimple real Lie algebra $\mathfrak{g}$, a **grading** of $\mathfrak{g}$ with respect to an additively written abelian group A is a vector space decomposition

$$\mathfrak{g} = \bigoplus_{k \in A} \mathfrak{g}_k,$$

such that $[\mathfrak{g}_j, \mathfrak{g}_k] \subseteq \mathfrak{g}_{j+k}$ for all $j, k \in A$, as defined in [7, Section 6]. For example, the Cartan decomposition of a real semisimple Lie algebra $\mathfrak{g}$ is a $\mathbb{Z}_2$-grading of $\mathfrak{g}$. The elements of each summand $\mathfrak{g}_k$ are the **homogeneous** elements of $\mathfrak{g}$. Note that $\mathfrak{g}_0$ is a subalgebra of $\mathfrak{g}$. Since tensor products distribute over direct sums, the complexification of $\mathfrak{g}$, denoted $\mathfrak{g}^{\mathrm{c}}$, has the decomposition $\mathfrak{g}^{\mathrm{c}} = \bigoplus_{k \in A} \mathfrak{g}_k^{\mathrm{c}}$. For the rest of this section, both real and complex Lie algebras will be denoted by $\mathfrak{g}$. These concepts are illustrated in the following example, from [14, Example 1].

Example 3.1. Let $\mathfrak{g} = \mathfrak{sl}(4, \mathbb{C})$ with a basis

$$\{e_{i,j}, h_k :\ 1 \leq i, j \leq 4, 1 \leq k \leq 3, i \neq j\},$$

where $e_{i,j}$ and h_k are elements of $\mathfrak{sl}(4, \mathbb{C})$ such that $(e_{i,j})_{pq} = \delta_p^i \delta_q^j$ and $h_k = e_{k,k} - e_{k+1,k+1}$ (The **Kronecker delta** δ_j^i is equal to 1 if $i = j$ and 0 otherwise). Then $\mathfrak{g}$ has a $\mathbb{Z}_3$-grading given by $\mathfrak{g}_0 = \mathrm{Span}(\{h_1, h_2, h_3, e_{3,4}, e_{4,3}\})$, $\mathfrak{g}_1 = \mathrm{Span}(\{e_{1,2}, e_{2,3}, e_{2,4}, e_{3,1}, e_{4,1}\})$, and $\mathfrak{g}_2 = \mathrm{Span}(\{e_{1,3}, e_{1,4}, e_{2,1}, e_{3,2}, e_{4,2}\})$. This can also be expressed by

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2,$$

where $\mathfrak{g}_0$, $\mathfrak{g}_1$, $\mathfrak{g}_2$ comprise matrices of the forms

$$\begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & d \\ 0 & 0 & e & (-a-b-c) \end{bmatrix}, \begin{bmatrix} 0 & a & 0 & 0 \\ 0 & 0 & b & c \\ d & 0 & 0 & 0 \\ e & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & a & b \\ c & 0 & 0 & 0 \\ 0 & d & 0 & 0 \\ 0 & e & 0 & 0 \end{bmatrix},$$

respectively, where $a, b, c, d, e \in \mathbb{C}$. Note that

$$\begin{aligned}\mathfrak{g}_0 &= \left\{ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & c & d \\ 0 & 0 & e & -c \end{bmatrix} : \ c, d, e \in \mathbb{C} \right\} \oplus \left\{ \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (-a-b) \end{bmatrix} : \ a, b \in \mathbb{C} \right\} \\ &\cong \mathfrak{sl}(2, \mathbb{C}) \oplus T_2,\end{aligned}$$

as a direct sum of ideals, where T_2 is the abelian Lie algebra of dimension 2. •

Here, we introduce the background theory and definition of θ-groups. We refer to Appendix A.1 for the background theory on differential geometry and Lie groups. Every (real or complex) Lie algebra $\mathfrak{g}$ is isomorphic to the tangent space of a Lie group G. The **adjoint action** Ad of G on $\mathfrak{g}$ is then defined by $\mathrm{Ad}(g) = d_e\alpha_g$, with α_g being the conjugation map $h \mapsto ghg^{-1}$ and d_e being the differential map evaluated at the identity. This shows, for example, that $\mathrm{Ad}(g) \in \mathrm{Aut}\,\mathfrak{g}$, being the differential of a group isomorphism. Now, the $\mathrm{Int}(\mathfrak{g})$ is the subgroup of $\mathrm{Aut}\,\mathfrak{g}$ generated by automorphisms of the form $\exp \mathrm{ad}\, X$ for $X \in \mathfrak{g}$. Elements of $\mathrm{Int}(\mathfrak{g})$ are the **inner automorphisms** of $\mathfrak{g}$. By [20, Proposition 1.91], for any $X \in \mathfrak{g}$ we have $\mathrm{Ad}(\exp X) = \exp(\mathrm{ad}\, X)$, so the group $\mathrm{Int}(\mathfrak{g})$ of inner automorphisms is a subgroup of $\mathrm{Ad}(G)$. In fact, [20, Section I.11] shows that $\mathrm{Int}(\mathfrak{g})$ is the connected component of the identity in $\mathrm{Ad}(G)$, and it is the subgroup of $\mathrm{Ad}(G)$ with Lie algebra $\mathrm{ad}\,\mathfrak{g}$.

Now let $\mathfrak{g}$ also be semisimple and possess an A-grading. Since $\mathfrak{g}_0$ is a Lie subalgebra of $\mathfrak{g}$, from the correspondence between Lie subalgebras of $\mathfrak{g}$ and subgroups of G given in [5, Theorem 5.2.5], $\mathfrak{g}_0$ is the Lie algebra of a connected Lie subgroup $G_0 \leq G$. Since G_0 is connected, it is shown in [16, Chapter II, pp. 127] that $\mathrm{Ad}(G_0)$ is generated by $\exp(\mathrm{ad}\, X)$ for $X \in \mathfrak{g}$. Because we have $[\mathfrak{g}_0, \mathfrak{g}_k] \subseteq \mathfrak{g}_k$ for all $k \in \mathbb{Z}_m$, the action of G_0 on each $\mathfrak{g}_k$ is a linear endomorphism for all $k \in \mathbb{Z}_m$. This leads to the central object of study for the remainder of the thesis.

Definition 3.2 (θ-groups). *Let $\mathfrak{g}$ be a real or complex semisimple A-graded Lie algebra, and let G be a connected Lie group that has $\mathfrak{g}$ as its Lie algebra. If G_0 is the connected Lie subgroup of G that has $\mathfrak{g}_0$ as its Lie algebra, the adjoint action of G_0 restricted to $\mathfrak{g}_1$ yields a representation $\rho : G_0 \to \mathrm{GL}(\mathfrak{g}_1)$. Then G_0 is called a **θ-group** and ρ is a **θ-representation**.*

To the best of our knowledge, the term 'θ-group' was coined by Vinberg in the abstract to [30]. While the θ-representation can be computed as the subgroup of $\mathrm{Aut}\,\mathfrak{g}$ generated by $\exp \mathrm{ad}\, X$ for $X \in \mathfrak{g}_0$ acting on $\mathfrak{g}_1$, this computation can be simplified if Lie group is a closed matrix group; that is, a subgroup of $\mathrm{GL}(n, K)$, where $K = \mathbb{R}$ or $\mathbb{C}$, that is closed under the Zariski topology. Following [20, Equation 1.88], the adjoint action of G on $\mathfrak{g}$ in this case is just the conjugation action $h \mapsto ghg^{-1}$ for $h \in \mathfrak{g}$ and $g \in G$. Furthermore, from [20, Corollary 0.20], the connected component of the Lie group with Lie algebra $\mathfrak{g}_0$ containing the identity is generated by the exponentials $\exp A$ for $A \in \mathfrak{g}_0$. Hence, the θ-representation is equivalent to the conjugation action of the group generated by $\{\exp A : \ A \in \mathfrak{g}_0\}$ acting on $\mathfrak{g}_1$.

Remark 3.3 (Matrix exponentials)**.** Since both means of finding the θ-representation involve the computation of the exponential of a matrix or the exponential of the linear map $\operatorname{ad} X$ for $X \in \mathfrak{g}$, here we present how to find the exponential of a complex matrix M, following [12, Section 16.6]. A matrix is in **Jordan normal form** if it has the form

$$\begin{bmatrix} J_1 & 0 & \cdots & 0 \\ 0 & J_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_k \end{bmatrix},$$

where each J_i is a Jordan block; that is, a matrix of the form

$$J_i = \begin{bmatrix} \lambda_i & 1 & 0 & \cdots & 0 \\ 0 & \lambda_i & 1 & \cdots & 0 \\ 0 & 0 & \lambda_i & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_i \end{bmatrix},$$

with $\lambda_i \in \mathbb{C}$. Every matrix over $\mathbb{C}$ is similar to a matrix in Jordan normal form. The Jordan normal form J of M can then be expressed as $J = D + N$, where the nonzero entries of D are the diagonal entries of J, and the nonzero entries of N are the off-diagonal entries of J. Since J is similar to M, we have $M = SJS^{-1}$, giving the Jordan decomposition $M = SDS^{-1} + SNS^{-1} = \tilde{D} + \tilde{N}$, where $\tilde{D}$ is diagonalisable, $\tilde{N}$ is nilpotent, and $\tilde{D}$ and $\tilde{N}$ commute. Since $\tilde{N}$ is nilpotent, $\tilde{N}^k = 0$ for some $k > 0$. Following [17, Section 2.3], the exponential of $\tilde{N}$ is

$$\exp \tilde{N} = \sum_{i=1}^{k} \frac{\tilde{N}^i}{i!}.$$

From [26, Chapter 3], the exponential of a diagonalisable matrix with eigenvectors v_i for its eigenvalues λ_i is a matrix with eigenvectors v_i for the eigenvalues $\exp \lambda_i$. Since $\tilde{D}$ and $\tilde{N}$ commute, $\exp M = \exp(\tilde{D}) \exp(\tilde{N})$, giving the desired exponential of M. •

Example 3.4. Let $\mathfrak{g}$ be defined as in Example 3.1. Since $\mathfrak{g}$ is a classical Lie algebra, we can apply the second method of the prior remark. It is known that a closed matrix group with Lie algebra $\mathfrak{g}$ is $\mathrm{SL}(4, \mathbb{C})$. Then the subgroup with Lie algebra $\mathfrak{g}_0$ is generated by $\exp \mathfrak{g}_0 \leq \mathrm{SL}(4, \mathbb{C})$. •

Here we recall some terminology for actions. Let G be a group acting on a set X. The **orbit** of some $x \in X$ over G is the set $G \cdot x = \{g \cdot x : g \in G\}$, and the orbits partition X. Two elements $x_1, x_2 \in X$ are **conjugate** if they are in the same orbit. Returning to the case of θ-groups, for a connected subgroup H of the adjoint group G with Lie algebra $\mathfrak{h} \leq \mathfrak{g}$, the **centraliser** of $S \subseteq \mathfrak{g}_1$ in H is the Lie subgroup $\mathcal{C}_H(S) = \{g \in H : g \cdot x = x \quad \forall x \in S\}$. By the Lie correspondence given in [5, Theorem 5.2.5], the Lie algebra of this subgroup is a subalgebra of $\mathfrak{h}$, and in fact it is the centraliser $C_{\mathfrak{h}}(S)$ of S as shown in [4, (1.2.14)]. Similarly, for a vector subspace $X \subseteq \mathfrak{g}_1$, the **normaliser** of X in $\mathfrak{g}_1$ over H is the Lie subgroup $\mathcal{N}_H(X) = \{g \in H : g \cdot X = X\}$, and its Lie algebra is the normaliser $N_{\mathfrak{h}}(X)$ of X.

A key objective of this report is to review the analysis of nilpotent homogeneous elements in the graded Lie algebra $\mathfrak{g}$. This is done primarily by studying the orbits under the adjoint action, as shown in [4, Section 1.2]. Recall that the action is via Lie automorphisms on $\mathfrak{g}$, so let $\psi \in \operatorname{Aut}\mathfrak{g}$. Then, for x, y in $\mathfrak{g}$, we have $(\psi \circ (\operatorname{ad} x))(y) = [\psi(x), \psi(y)] = (\operatorname{ad}\psi(x))(\psi(y))$, so $\psi \circ (\operatorname{ad} x) \circ \psi^{-1} = \operatorname{ad}\psi(x)$. As nilpotency and semisimplicity are preserved under automorphisms, if $\operatorname{ad} x$ is nilpotent or semisimple, then so is $\operatorname{ad}\psi(x)$. Hence, the action preserves semisimplicity and nilpotency, and it is well-defined for us to consider nilpotent orbits or semisimple orbits. As is later given, these orbits have a structure which can be effectively analysed.

The main focus for Lie algebras in this report is on the semisimple case. However, although θ-groups are constructed from semisimple Lie algebras, this class is sometimes too restrictive for our purposes. For example, subalgebras of semisimple Lie algebras are not necessarily semisimple. Thus, it will be useful to introduce a broader class of Lie algebras. A Lie algebra is **reductive** if its radical is equal to its centre. By Levi's theorem, $\mathfrak{g}$ has the form $\mathfrak{g} = \mathfrak{s} \oplus Z(\mathfrak{g})$, where $\mathfrak{s}$ is semisimple. This is a Lie direct sum since $Z(\mathfrak{g})$ trivially centralises $\mathfrak{s}$. Thus, $[\mathfrak{g}, \mathfrak{g}] = [\mathfrak{s}, \mathfrak{s}] = \mathfrak{s}$ since $\mathfrak{s}$ is semisimple. This is indeed a generalisation, as every semisimple Lie algebra has trivial centre, so they are also reductive.

From [29, Section 1.2], for a subgroup $B \leq A$, the sum $\bigoplus_{b \in B} \mathfrak{g}_b$ is also a reductive Lie algebra. In particular, the subalgebra $\mathfrak{g}_0$ is reductive. Furthermore, when considering the θ-representation (where we are only interested in the action of G_0 on $\mathfrak{g}_1$), we may assume the group indexing the grading is cyclic. That is, we let $A = \mathbb{Z}_m$ for $m \in \mathbb{N} \cup \{\infty\}$, where $\mathbb{Z}_\infty = \mathbb{Z}$. In particular, when the Lie algebra $\mathfrak{g}^{\mathrm{c}}$ is complex, a $\mathbb{Z}_m$-grading of $\mathfrak{g}^{\mathrm{c}}$ induces a Lie automorphism $\psi : \mathfrak{g}^{\mathrm{c}} \to \mathfrak{g}^{\mathrm{c}}$ given by $\psi(x) = \omega^i x$ for $x \in \mathfrak{g}_i$, where ω is a primitive mth root of unity (if $m = \infty$, then ω is a complex number with unit length and infinite order), such that the grading of $\mathfrak{g}^{\mathrm{c}}$ is the eigenspace decomposition of $\mathfrak{g}^{\mathrm{c}}$ with respect to ψ. In particular, there is a one-to-one correspondence between the $\mathbb{Z}_m$-gradings of $\mathfrak{g}^{\mathrm{c}}$ with $m < \infty$ and the finite-order automorphisms of $\mathfrak{g}^{\mathrm{c}}$.

The following lemma, given in [30, Section 2.1], shows that, although representations of G_0 can be chosen for any $\mathfrak{g}_k$, their study is equivalent to the study of the θ-representation.

Lemma 3.5. *For a $\mathbb{Z}_m$-graded reductive Lie algebra $\mathfrak{g}$ and $k \in \mathbb{Z}_m \setminus \{0\}$, there exists a $\mathbb{Z}_{\tilde{m}}$-graded reductive Lie algebra $\tilde{\mathfrak{g}}$, for some $\tilde{m} \in \mathbb{N} \cup \{\infty\}$, such that $\tilde{\mathfrak{g}}_0 = \mathfrak{g}_0$ and $\tilde{\mathfrak{g}}_1 = \mathfrak{g}_k$.*

Proof. One can construct a $\mathbb{Z}_{\tilde{m}}$-graded Lie algebra $\tilde{\mathfrak{g}}$, where $\tilde{m} = m/\gcd(m, k)$ (or $\tilde{m} = \infty$ if $m = \infty$), by letting $\tilde{\mathfrak{g}} = \bigoplus_{p \in \mathbb{Z}_{\tilde{m}}} \tilde{\mathfrak{g}}_p$, where $\tilde{\mathfrak{g}}_p = \mathfrak{g}_{pk}$ for $p \in \mathbb{Z}_{\tilde{m}}$. Since $k\mathbb{Z}_m \leq \mathbb{Z}_m$, [29, Section 1.2] shows that $\tilde{\mathfrak{g}}$ is indeed a reductive Lie algebra graded by $\mathbb{Z}_{\tilde{m}}$. Furthermore, $\tilde{\mathfrak{g}}$ has $\tilde{\mathfrak{g}}_0 = \mathfrak{g}_0$ and $\tilde{\mathfrak{g}}_1 = \mathfrak{g}_k$, as desired. □

For the rest of this section, every (real or complex) Lie algebra is denoted by $\mathfrak{g}$.

We now summarise some key properties of graded complex reductive Lie algebras required to construct the supports, and link them to the nilpotent orbits in $\mathfrak{g}_1$. First, it will be useful to consider a more general version of the Killing form. By [7, Section 1.2], for a finite-dimensional representation $\phi : \mathfrak{g} \to \mathfrak{gl}(V)$, a **trace form** is defined by $(x, y) = \operatorname{tr}(\phi(x) \circ \phi(y))$ for $x, y \in \mathfrak{g}$. Note that letting $\mathfrak{g}$ be semisimple, $V = \mathfrak{g}$ and $\phi = \operatorname{ad}$ gives the standard Killing form κ.

Example 3.6. Recall that every finite-dimensional Lie algebra over a field of characteristic 0 has a faithful representation by [19, Section IV.2], so every such Lie algebra is isomorphic to a matrix Lie algebra. Assuming $\mathfrak{g}$ to be a matrix Lie algebra, the so-called second trace form is constructed by setting $(x, y) = \operatorname{tr}(xy)$ for $x, y \in \mathfrak{g}$. This form is used extensively by Vinberg [29, 30] and in this report as, if $\mathfrak{g}$ is complex and semisimple, then the adjoint representation $\operatorname{ad} : \mathfrak{g} \to \operatorname{End}(\mathfrak{g})$ is faithful. For example, Vinberg uses the second trace form in [29, Section 1.1] to show that a matrix Lie algebra is reductive if and only if the second trace form is nondegenerate. •

3.1.1 Background for semisimple graded Lie algebras

We provide some key theorems and concepts for graded Lie algebras that are relevant to investigating and understanding the aims of the report. In the special case where the Lie algebra is graded over $\mathbb{Z}$, the grading may be characterised by a **defining element**. This is shown below and given in [29, Section 1.4], with proof from [10, Introduction].

Theorem 3.7. *For a $\mathbb{Z}$-graded complex semisimple Lie algebra $\mathfrak{g}$, there exists a unique element $h_0 \in \mathfrak{g}_0$ such that, for every $k \in \mathbb{Z}$,*

$$\mathfrak{g}_k = \{x \in \mathfrak{g} : [h_0, x] = kx\}.$$

*Conversely, every semisimple element $h_0 \in \mathfrak{g}_0$ such that $\operatorname{ad} h_0$ has only integer eigenvalues induces a $\mathbb{Z}$-grading on $\mathfrak{g}$. This element h_0 is the **defining element** of the grading on $\mathfrak{g}$.*

Proof. First suppose $\mathfrak{g}$ has a $\mathbb{Z}$-grading and consider the linear map $d : \mathfrak{g} \to \mathfrak{g}$ defined by $d(x) = ix$ for $x \in \mathfrak{g}_i$. It is then clear from the vector space decomposition that, for $i \in \mathbb{Z}$ and $x \in \mathfrak{g}$, we have $d(x) = ix$ if and only if $x \in \mathfrak{g}_i$. From the definition of the grading, it follows that, for $x \in \mathfrak{g}_i$ and $y \in \mathfrak{g}_j$, we have

$$d([x, y]) = (i + j)[x, y] = i[x, y] + j[x, y] = [ix, y] + [x, jy] = [d(x), y] + [x, d(y)],$$

so by linearity d is a derivation on $\mathfrak{g}$. Since $\mathfrak{g}$ is semisimple, a consequence of Cartan's second criterion given in [17, Theorem 1, Section 5.3] gives that the map $\operatorname{ad} : \mathfrak{g} \to \operatorname{Der} \mathfrak{g}$ is an isomorphism, so there exists a unique element $h_0 \in \mathfrak{g}$ such that $\operatorname{ad} h_0 = d$. Observe that $d(h_0) = [h_0, h_0] = 0$, so $h_0 \in \mathfrak{g}_0$.

Conversely, let $h_0 \in \mathfrak{g}_0$ be semisimple and such that $\operatorname{ad} h_0$ has integer eigenvalues and define $\mathfrak{g}_i = \{x \in \mathfrak{g} : [h_0, x] = ix]\}$. Clearly these are vector subspaces, and they span $\mathfrak{g}$ since h_0 is semisimple, hence $\operatorname{ad} h_0$ is diagonalisable and has a full set of eigenvectors. Furthermore, for $x \in \mathfrak{g}_i$ and $y \in \mathfrak{g}_j$, we have

$$[h_0, [x, y]] = -[y, [h_0, x]] - [x, [y, h_0]] = -[y, ix] - [x, -jy] = (i + j)[x, y],$$

so $[x, y] \in \mathfrak{g}_{i+j}$, confirming that this is a $\mathbb{Z}$-grading on $\mathfrak{g}$. □

The defining element of a grading provides information on $\mathfrak{g}$, as shown in [29, Section 1, Part 4]:

Corollary 3.8. *Let $\mathfrak{g}$ be a complex reductive $\mathbb{Z}$-graded Lie algebra. Then $\mathfrak{g}_0$ contains a Cartan subalgebra of $\mathfrak{g}$, so the rank of $\mathfrak{g}_0$ is equal to the rank of $\mathfrak{g}$.*

Proof. The defining element h_0 of the grading gives a weight space decomposition of $\mathfrak{g}$ with respect to $\operatorname{ad} h_0$, so $\operatorname{ad} h_0$ is diagonalisable and h_0 is semisimple. Hence, there is a Cartan subalgebra $\mathfrak{t} \leq \mathfrak{g}$ containing h_0. Since $\mathfrak{t}$ is abelian, for any $y \in \mathfrak{t}$ we have $[h_0, y] = 0$, so $\mathfrak{t}$ is contained in $\mathfrak{g}_0$. As the elements of $\mathfrak{t}$ are semisimple in $\mathfrak{g}$, they are also semisimple in $\mathfrak{g}_0$, so it follows that $\mathfrak{t}$ is also a Cartan subalgebra of $\mathfrak{g}_0$. Since the rank of a complex semisimple Lie algebra is equal to the dimension of the Cartan subalgebra, rank $\mathfrak{g}$ = rank $\mathfrak{g}_0$. □

We now summarise concepts from general Lie theory that carry over to complex graded Lie algebras. The Jordan decomposition is one of these, as shown in [21, Chapter 2].

Lemma 3.9 (Jordan decomposition for complex graded Lie algebras)**.** *Let $\mathfrak{g}$ be a complex graded Lie algebra with grading induced from an automorphism θ. For a homogeneous element $x \in \mathfrak{g}_i$, the Jordan decomposition of x is $x = d+n$ for $d, n \in \mathfrak{g}_i$, where d is semisimple, n is nilpotent and $[d, n] = 0$.*

Proof. For $x \in \mathfrak{g}$, recall that x is semisimple if $\operatorname{ad} x$ is diagonalisable. Let B_x be a basis of eigenvectors of $\operatorname{ad} x$. Since θ is a Lie automorphism, for $b \in B_x$ with eigenvalue λ, we have $(\operatorname{ad} \theta(x))(\theta(b)) = [\theta(x), \theta(b)] = \theta([x, b]) = \theta(\lambda b) = \lambda\theta(b)$, so $\theta(b)$ is an eigenvector of $\operatorname{ad} x$. Hence, the $\theta(B_x)$'s are a basis of eigenvectors of $\operatorname{ad} \theta(x)$, so $\theta(x)$ is also semisimple.

If x is nilpotent, $(\operatorname{ad} x)^n = 0$ for some $n \geq 1$. For $y \in \mathfrak{g}$, let $z = \theta^{-1}(y)$. Then $(\operatorname{ad} \theta(x))^n(y) = (\operatorname{ad} \theta(x))^n(\theta(z)) = \theta((\operatorname{ad} x)^n(y)) = 0$, so $(\operatorname{ad} \theta(x))^n = 0$ and $\operatorname{ad} \theta(x)$ is nilpotent. These show that the set of nilpotent elements and the set of semisimple elements of $\mathfrak{g}$ are invariant under automorphisms. From Theorem 1.5, an element $x \in \mathfrak{g}_i$ has an abstract Jordan decomposition $x = d + n$ where $[d, n] = 0$. Since x is in an eigenspace of θ, it follows that $\theta(x) = \theta(d + n) = \theta(d) + \theta(n) = \lambda(d + n)$, so $d, n \in \mathfrak{g}_i$. Hence, we conclude that the abstract Jordan decomposition of $x \in \mathfrak{g}_i$ is $x = d + n$ with both $d, n \in \mathfrak{g}_i$. □

For a $\mathbb{Z}_m$-graded complex semisimple Lie algebra $\mathfrak{g}$, there is an analogue to the existence of $\mathfrak{sl}_2$-triples from the general structure theory of complex semisimple Lie algebras, from [29, Theorem 1, Section 2].

Theorem 3.10 (Jacobson-Morozov-Vinberg (JMV) theorem)**.** *Let $\mathfrak{g}$ be a $\mathbb{Z}_m$-graded semisimple complex Lie algebra and let G_0 be the connected Lie group that has $\mathfrak{g}_0$ as its Lie algebra. If $e \in \mathfrak{g}_a \setminus \{0\}$ is nilpotent for some $a \in \mathbb{Z}_m \setminus \{0\}$, then the following hold.*

i) There is a nilpotent $f \in \mathfrak{g}_{-a}$ and a semisimple $h \in \mathfrak{g}_0$ such that $[e, f] = h$, $[h, e] = 2e$ and $[h, f] = -2f$.

ii) h is unique up to conjugation by an element of the centraliser of e in G_0.

iii) Fixing e and h, f is uniquely determined.

iv) For a given h, the element e is unique up to conjugation by an element of the centraliser of h in G_0.

The set $\{f, h, e\}$ given from part (i) of Theorem 3.10 is the **homogeneous $\mathfrak{sl}_2$-triple** of e; in particular, h is the **characteristic** of e. Recall that one of the primary aims for θ-groups is to classify their nilpotent orbits. Theorem 3.10 will be greatly used for this, as it gives a unique nilpotent element e (up to conjugacy) in the $\mathfrak{sl}_2$-triple for a given characteristic h, and vice versa.

3.1.2 Topological notions and properties of orbits

Let $\mathfrak{g}$ be a reductive complex $\mathbb{Z}_m$-graded Lie algebra. Following [30], the topology on $\mathfrak{g}$ is generally the Zariski topology. A summary of key properties of the Zariski topology is given in Appendix A. In particular, another aspect of θ-groups is the relationship between the nilpotency or semisimplicity of an element of $\mathfrak{g}_1$, and the properties of its orbit under the action of the θ-group. Following Vinberg in [30, Proposition 1-3], two key properties of nilpotent orbits are shown in the following two theorems.

Theorem 3.11. *There are a finite number of nilpotent orbits in $\mathfrak{g}_1$ with respect to G_0.*

Theorem 3.12. *An element $x \in \mathfrak{g}_1$ is nilpotent if and only if the closure of its orbit under the action of G_0 contains 0.*

Proof. For $x \in \mathfrak{g}_1$ nilpotent, by Theorem 3.10 there is a semisimple $h \in \mathfrak{g}_0$ such that $[h, x] = 2x$. For any $t \in \mathbb{R}$, we see that the one-parameter group $A_t = \exp t(\operatorname{ad} h_0)$ is in the orbit. Letting $\tilde{t} = \exp t \in \mathbb{R}^+$, the one-parameter group can be re-parameterised as $A_{\tilde{t}} = \exp[(\ln \tilde{t})(\operatorname{ad} h)]$. Since $[h, x] = 2x$, we have $A_{\tilde{t}}(x) = \tilde{t}^2 x$. Since $\lim_{\tilde{t}\to 0} A_{\tilde{t}}(x) = 0$ and the map A is continuous, it follows that 0 is a limit point of the orbit of x, so it is contained in the closure of the orbit of x.

Conversely, suppose 0 is in the closure of the orbit of x under G_0. As previously stated, the action by an element of G is a Lie automorphism, so it follows from the Jacobi identity that $\operatorname{ad}(g \cdot x) = g \cdot (\operatorname{ad} x) \cdot g^{-1}$. Considering g and $\operatorname{ad} x$ as linear maps acting on $\mathfrak{g}_1$, it follows that $\operatorname{ad} x$ and $\operatorname{ad}(g \cdot x)$ are similar, so they have the same eigenvalues. Since 0 is in the closure, it follows that every point in the orbit has the same eigenvalues as 0, so every point has zero as its only eigenvalue. This implies that $\operatorname{ad} x$ is nilpotent, so x is nilpotent, as desired. □

The following also gives a simple characterisation of semisimple elements of $\mathfrak{g}_1$. The proof, in [31, Proposition 1.3.5.5], goes beyond the scope of this report.

Theorem 3.13. *An element $x \in \mathfrak{g}_1$ is semisimple if and only if its orbit under G_0 is closed.*

From Theorem 3.10 and the Jordan decomposition, we see that the semisimple and nilpotent elements of $\mathfrak{g}_1$ can be considered the building blocks of a Lie algebra. In particular, Theorem 3.11 shows that the number of nilpotent orbits is finite, which is one of our justifications for investigating the multiple attempts to classify them.

A complex graded Lie algebra $\mathfrak{g}$ with grading induced from an automorphism θ is of **zero rank** if every element of $\mathfrak{g}_1$ is nilpotent. A key reason why graded Lie algebras of zero rank are of interest to us is the following theorem, shown in [30, Section 2.6].

Proposition 3.14. *Every $\mathbb{Z}$-graded complex Lie algebra $\mathfrak{g}$ has zero rank.*

Proof. The automorphism that induces the grading for $\mathfrak{g}$ forms a one-parameter group ψ of automorphisms on $\mathfrak{g}$, by $\psi_t(x) = t^k x$ for $x \in \mathfrak{g}_k$ and $t \in \mathbb{C}^*$. Clearly, on $\mathfrak{g}_1$ this automorphism acts as multiplication by $t^1 = t$. Therefore, $x \in \mathfrak{g}_1$ is conjugate to tx under the action ψ_t for any $t \in \mathbb{C}^*$; this implies that there is an inner automorphism of $\mathfrak{g}$ mapping x to tx, by [30, Section 2.1]. Hence, the set $S_x = \{tx : t \in \mathbb{C}^*\}$ is contained in the orbit of x under G_0. Since 0 is clearly a limit point of S_x, the closure of the orbit of x contains 0, so by Theorem 3.12, x is nilpotent. As this holds for every $x \in \mathfrak{g}_1$, the proposition follows. □

Many of our Lie algebra constructions in the study of nilpotent orbits are of zero rank. One reason for this is the following theorem, given in [29, Section 1.3]:

Proposition 3.15. *Let $\mathfrak{g}$ be a complex reductive $\mathbb{Z}_m$-graded Lie algebra with θ-group G_0. If $\mathfrak{g}_1$ contains a finite number of orbits (not necessarily nilpotent) under the action of G_0, then it has a unique open orbit.*

Proof. Supposing that $\mathfrak{g}_1$ contains n orbits, we can label the orbits as $\mathcal{O}_i$ for $i \in \{1, ..., n\}$. From [4, Section 1.4], the orbits have a smooth manifold structure, hence a dimension associated to each. If the dimension of every orbit is less than the dimension of $\mathfrak{g}_1$, since the orbits are disjoint, the union must also have dimension less than the dimension of $\mathfrak{g}_1$. But the union of the orbits is the space $\mathfrak{g}_1$, giving a contradiction. Hence, at least one of the closures has the same dimension as $\mathfrak{g}_1$. Let $\mathcal{O}_k$ be the orbit with $\dim(\mathcal{O}_k) = \dim(\mathfrak{g}_1)$. As algebraic sets, $\mathfrak{g}_1$ is an affine variety and $\bar{\mathcal{O}}_k$ is a closed variety, with the dimensions still equal. But it is well known, for example in [1, Proposition 4.1], that this is only possible if the spaces are equal. Therefore, we have that $\bar{\mathcal{O}}_k = \mathfrak{g}_1$, Since the orbit is open in its closure by [15, Lemma 3.12.1], $\mathcal{O}_k$ is open in $\mathfrak{g}_1$.

Now suppose there are two open orbits $\mathcal{O}_1$ and $\mathcal{O}_2$ in $\mathfrak{g}_1$. If they were disjoint, then the complement of $\mathcal{O}_1$ would be a closed proper subset of $\mathfrak{g}_1$ containing $\mathcal{O}_2$, which is impossible since open sets are dense in the Zariski topology. Hence, $\mathcal{O}_1$ and $\mathcal{O}_2$ have nontrivial intersection. As they are orbits, this implies that $\mathcal{O}_1$ and $\mathcal{O}_2$ are the same orbit. Thus, we conclude that the open orbit in $\mathfrak{g}_1$ is unique. □

In general, we note that computations involving the orbit are hampered by the lack of an immediate, closed-form expression for the θ-group. However, the following proposition, given in [15, Lemma 8.2.1], allows us to find the tangent space of the orbits of the action in a concrete manner with only $\mathfrak{g}_0$ (recall that the orbit has the structure of a smooth manifold). This is highly useful to us as our main interest for θ-groups will be in their orbits.

Proposition 3.16. *For a (complex or real) graded Lie algebra $\mathfrak{g}$ with θ-group G_0, the tangent space of the orbit of some $x \in \mathfrak{g}_1$ is precisely $[\mathfrak{g}_0, x]$.*

Proof. Under the action of the θ-group G_0 on $\mathfrak{g}_1$, the orbit of a point $x \in \mathfrak{g}_1$ is $\mathrm{Ad}(G_0)(x)$, so the tangent space to the orbit of x is the image of the differential map, which is contained in $[\mathfrak{g}_0, x]$. Define the map $\phi_x : G_0 \to \mathfrak{g}_1$ by $\phi_x(g) = (\rho(g))(x)$, so the orbit of x is also the image of ϕ_x. By [15, Lemma 8.2.1], the image of $d_e\phi_x$ is surjective, hence precisely the tangent space of the orbit, giving us the desired result. □

For a complex reductive $\mathbb{Z}$-graded Lie algebra, which must then be of zero rank, an element e of the open orbit of $\mathfrak{g}_1$ is a **point in general position**. Crucially, there is a test to determine points in general position that does not depend on the θ-group itself, hence being much easier to compute, given in [7, Section 9]

Theorem 3.17. *Let $\mathfrak{g}$ be a complex reductive $\mathbb{Z}$-graded Lie algebra. Then an element $e \in \mathfrak{g}_1$ is a point in general position if and only if $[\mathfrak{g}_0, e] = \mathfrak{g}_1$.*

Proof. Let $\mathcal{O}_e$ be the orbit of such a point under the θ-representation. Such an open set is dense, which occurs if and only if the closure of the orbit is equal to $\mathfrak{g}_1$. In this case, the tangent space of the closure of $\mathcal{O}_e$ must then also be $\mathfrak{g}_1$, since the tangent space of the vector space $\mathfrak{g}_1$ is well-known to also be $\mathfrak{g}_1$. From Proposition 3.16, the tangent space of $\bar{\mathcal{O}}_e$ is $[\mathfrak{g}_0, e]$, so a point e is in general position if and only if $[\mathfrak{g}_0, e] = \mathfrak{g}_1$. □

There is another characterisation by Vinberg of points in general position. First, we require the following lemma, from [30, Proposition 5].

Lemma 3.18. *Taking $\mathfrak{g}$ to be a matrix Lie algebra, the radical of the tangent space of the orbit of a point $x \in \mathfrak{g}_1$ is the centraliser of x in the dual space $\mathfrak{g}_{-1}$, where the radical is taken with respect to the bilinear form $(x, y) = \mathrm{tr}(xy)$.*

Proof. First let $y \in \mathfrak{g}_{-1}$ be in the radical of the tangent space of x. By Proposition 3.16, the tangent space to the orbit of x is the image of the differential map, which is $[\mathfrak{g}_0, x]$. Thus, y is orthogonal to $[\mathfrak{g}_0, x]$, so $([a, x], y) = 0$ for all $a \in \mathfrak{g}_0$. By invariance of the form, $(a, [x, y]) = 0$ for all $a \in \mathfrak{g}_0$. It has already been noted that, since $\mathfrak{g}_0$ is reductive, this form is nondegenerate on $\mathfrak{g}_0$, so $[x, y] = 0$ and y is in the centraliser of x in $\mathfrak{g}_1$. Conversely, if y is in the centraliser of x in $\mathfrak{g}_1$, then $[x, y] = 0$ and the same reasoning shows that $([\mathfrak{g}_0, x], y) = 0$, so y is orthogonal to $[\mathfrak{g}_0, x]$, which is the tangent space of the orbit of x. □

Then the other characterisation of points in general position is as follows, from [30, Section 2.6].

Proposition 3.19. *A point $x \in \mathfrak{g}_1$ is in general position if and only if the centraliser of x in $\mathfrak{g}_{-1}$ is trivial.*

Proof. Invoking Ado's theorem, suppose $\mathfrak{g}$ is a matrix Lie algebra. By the previous lemma, the tangent space to the orbit of x is the radical of the centraliser of x in $\mathfrak{g}_{-1}$ with respect to the second trace form. As stated in the previous lemma, the tangent space to the orbit of x is $[\mathfrak{g}_0, x]$. If the centraliser is trivial, its orthogonal complement is $\mathfrak{g}_1$, so the tangent space to the orbit of x is $\mathfrak{g}_1$. Therefore, $[\mathfrak{g}_0, x] = \mathfrak{g}_1$, so by the previous characterisation, x is a point in general position. For the converse, if $[\mathfrak{g}_0, x] = \mathfrak{g}_1$ then the orthogonal complement is trivial, so by the previous lemma the centraliser of x in $\mathfrak{g}_1$ is trivial. □

3.1.3 $\mathbb{Z}$-graded semisimple Lie algebras

Let $\mathfrak{g}^{\mathrm{c}}$ be a complex $\mathbb{Z}$-graded semisimple Lie algebra. The classification of complex semisimple Lie algebras was reviewed in the first chapter. From Theorem 3.7, the grading is induced by a defining element $h \in \mathfrak{g}_0^{\mathrm{c}}$, which is unique since $\mathfrak{g}^{\mathrm{c}}$ is semisimple and so has trivial centre. Thus, one can uniquely represent the complex semisimple Lie algebra $\mathfrak{g}^{\mathrm{c}}$ with $\mathbb{Z}$-grading induced from h via the pair $(\mathfrak{g}^{\mathrm{c}}, h)$. Two such pairs $(\mathfrak{g}^{\mathrm{c}}, h_1)$ and $(\mathfrak{h}^{\mathrm{c}}, h_2)$ are **isomorphic** if there exists a Lie algebra isomorphism $\psi : \mathfrak{g}^{\mathrm{c}} \to \mathfrak{h}^{\mathrm{c}}$ such that $\psi(h_1) = h_2$. Since $x \in \mathfrak{g}_k^{\mathrm{c}}$ if and only if $[h_1, x] = kx$, applying the isomorphism gives $\psi([h_1, x]) = [h_2, \psi(x)] = k\psi(x)$, so $\psi(x) \in \mathfrak{h}_k^{\mathrm{c}}$ as well. Hence, the condition that $\psi(h_1) = h_2$ is equivalent to ψ preserving the grading.

From [10, Definition 1], a **weighted Dynkin diagram** is a Dynkin diagram in which each vertex is given any nonnegative integer, called its **weight**. Two weighted Dynkin diagrams are **isomorphic** if there is a graph isomorphism between them which preserves the weight of each vertex. As will be detailed further, the complex $\mathbb{Z}$-graded semisimple Lie algebras are classified by the weighted Dynkin diagrams. Since $\operatorname{ad} h$ defines a weight space decomposition on $\mathfrak{g}^{\mathrm{c}}$, it is diagonalisable, so h is semisimple and there exists a Cartan subalgebra $\mathfrak{h}$ containing h. Let Φ be the root system arising from the root space decomposition, let E be the overlying real inner product space for the abstract root system Φ, and let $\Phi(+) = \{\alpha \in \Phi : \alpha(h) \geq 0\}$. For $\alpha \in \Phi$, it is known that $-\alpha \in \Phi$ as well. Since Φ spans E and a vector space basis is still a basis if elements are multiplied by -1, this implies that $\Phi(+)$ also contains a vector space basis for E. Also, for $\alpha, \beta \in \Phi(+)$, we have $(\alpha+\beta)(h) = \alpha(h)+\beta(h) \geq 0$, so the elements of $\Phi(+)$ are contained in a half-plane of E, so the proof in [12, Theorem 11.10] shows that $\Phi(+)$ contains a base for Φ. From this base one can construct the Dynkin diagram, which is weighted by giving each vertex α the weight $\alpha(h)$, giving a weighted Dynkin diagram denoted by $\Delta(\mathfrak{g}^c, h)$. From [10, Theorem 1], we then have the following:

Theorem 3.20. *There is a one-to-one correspondence between the isomorphism classes of complex $\mathbb{Z}$-graded semisimple Lie algebras $(\mathfrak{g}^c, h)$ and the isomorphism classes of weighted Dynkin diagrams $\Delta(\mathfrak{g}^c, h)$.*

Example 3.21. As shown in Example 1.11,the CSA of $\mathfrak{g}^c = \mathfrak{sl}(2, \mathbb{C})$ is spanned by h and the root system has rank 1 with base $\{\epsilon_1 - \epsilon_2\}$, where $\epsilon_i(h) = (h)_{ii}$. The Dynkin diagram of $\mathfrak{g}^c$ is then A_1 and the weighted Dynkin diagrams are constructed by giving the vertex a non-negative integer.

Let $\alpha = \epsilon_1 - \epsilon_2$. If the label is k, then the defining element $\tilde{h}$ satisfies $\alpha(\tilde{h}) = k$, so $(\tilde{h})_{11} = -(\tilde{h})_{22} = k/2$ and $\tilde{h} = \frac{k}{2}h$. We see $[\tilde{h}, e] = \frac{k}{2}(2e) = ke$, so $\mathfrak{g}^c_k$ is spanned by e. By the same reasoning, $\mathfrak{g}^c_{-k}$ is spanned by f, so all other summands of the grading are trivial except for $k = 0$. Since the grading with $e \in \mathfrak{g}^c_{-k}$, $f \in \mathfrak{g}^c_k$ and $h \in \mathfrak{g}^c_0$ is isomorphic to the reversed grading via the isomorphism defined by $\psi(e) = f$, $\psi(f) = e$ and $\psi(h) = -h$, it is clear that the weighted Dynkin diagrams classify all $\mathbb{Z}$-gradings of $\mathfrak{sl}(2, \mathbb{C})$ up to isomorphism. •

The real case can also be classified via the classification of the complex case from Theorem 3.20. The grading of a real semisimple $\mathbb{Z}$-graded Lie algebra $\mathfrak{g}$ is again uniquely defined by a defining element h by the same reasoning as shown in Theorem 3.7, so we again denote the graded Lie algebra by $(\mathfrak{g}, h)$. By Theorem 2.16, the real semisimple Lie algebras are classified up to isomorphism by the Satake diagrams. A **weighted Satake diagram** is a Satake diagram to which each vertex is given a weight, such that all black vertices have zero weight and all white vertices connected by two-headed arrows have the same weight. Analogously to the complex case, a real semisimple $\mathbb{Z}$-graded Lie algebra $(\mathfrak{g}, h)$ induces a weighted Satake diagram, denoted $\Delta(\mathfrak{g}, h)$. This classifies the $\mathbb{Z}$-gradings, as stated in [10, Theorem 3]:

Theorem 3.22. *There is a one-to-one correspondence between the isomorphism classes of real $\mathbb{Z}$-graded semisimple Lie algebras and the isomorphism classes of weighted Satake diagrams up to isomorphism, defined by associating $(\mathfrak{g}, h)$ with $\Delta(\mathfrak{g}, h)$ as defined above.*

The forward direction of the proof is given by the construction preceding [10, Theorem 2], with uniqueness given by [10, Theorem 2]. The reverse is given by [10, Theorem 3], which shows that every weighted Satake diagram is isomorphic to $\Delta(\mathfrak{g}, h)$ for some real semisimple Lie algebra $\mathfrak{g}$ with grading defined by some $h \in \mathfrak{g}$. Lastly, a real semisimple $\mathbb{Z}$-graded Lie algebra $(\mathfrak{g}, h)$ has complexification $(\mathfrak{g}^c, h)$ since complexification preserves the grading and so the defining element. Then [10, Theorem 4] shows that the weighted Dynkin diagram corresponding to $(\mathfrak{g}^c, h)$ is just the underlying diagram for the Satake diagram corresponding to $(\mathfrak{g}, h)$. This classification is useful both theoretically, as it allows for the classification of all real forms of a $\mathbb{Z}$-graded complex semisimple Lie algebra, and computationally, as the problem of finding these real forms is reduced to checking the Satake diagrams overlying the known Dynkin diagram and checking if the weights induced from the weighted Dynkin diagram satisfy the criteria for a weighted Satake diagram.

3.1.4 Supports of a $\mathbb{Z}_m$-graded Lie algebra

In the next section, we cover the classification of nilpotent orbits of a θ-group G_0 in a complex semisimple $\mathbb{Z}_m$-graded Lie algebra where $m \in \mathbb{N} \cup \{\infty\}$. To do this, we introduce a key object, the **support** of a nilpotent element, first described by Vinberg in [29, Section 4] to achieve this classification. Recall that a Lie algebra is reductive if it is the direct sum of the semisimple derived subalgebra and the centre. For a $\mathbb{Z}_m$-graded complex reductive Lie algebra $\mathfrak{g}$ and some fixed $a \in \mathbb{Z}_m$, a **$\mathbb{Z}$-graded subalgebra** $\mathfrak{a}$ of $\mathfrak{g}$ is a subalgebra with a $\mathbb{Z}$-grading such that $\mathfrak{a}_k \subseteq \mathfrak{g}_{k\cdot a}$ for all $k \in \mathbb{Z}$.

Remark 3.23 (Characters)**.** For any finitely generated abelian group A, a **character** χ of A in $\mathbb{C}$ is a group homomorphism of A into $\mathbb{C}^*$, the multiplicative group of nonzero elements of $\mathbb{C}$. The set of characters forms a group $X(A)$ with binary operation $(\chi_1\chi_2)(x) = \chi_1(x)\chi_2(x)$. For an A-graded complex reductive Lie algebra $\mathfrak{g}$ and a character $\chi \in X(A)$, one can define the map $\theta_\chi : \mathfrak{g} \to \mathfrak{g}$ by letting $\theta_\chi(x) = \chi(a)x$ for each $x \in \mathfrak{g}_a$ and $a \in A$, and extending this linearly to all $\mathfrak{g}$. For $x \in \mathfrak{g}_a$ and $y \in \mathfrak{g}_b$, we have $[\theta_\chi(x), \theta_\chi(y)] = [\chi(a)x, \chi(b)y] = \chi(a)\chi(b)[x, y]$, since the Lie bracket is bilinear. Furthermore, since $[x, y] \in \mathfrak{g}_{a+b}$ by definition of the grading, $\theta_\chi([x, y]) = \chi(a + b)[x, y] = \chi(a)\chi(b)[x, y]$ since χ is a homomorphism, so $\theta_\chi \in \operatorname{Aut}\mathfrak{g}$. The map $\chi \mapsto \theta_\chi$ is therefore also a homomorphism from $X(A)$ into $\operatorname{Aut}\mathfrak{g}$. Conversely, for any homomorphism $\theta : X(A) \to \operatorname{Aut}\mathfrak{g}$, we have $\theta(\chi) = \theta_\chi$, so there is a unique A-grading of $\mathfrak{g}$ such that $\theta_\chi(x) = \chi(a)x$ for all $x \in \mathfrak{g}_a$, by [29, Section 1, pp. 18].

For a reductive complex $\mathbb{Z}_m$-graded Lie algebra $\mathfrak{g}$ and a nonzero nilpotent element $e \in \mathfrak{g}_1$, let $N_0(e)$ be the normaliser of the span of $\{e\}$ in $\mathfrak{g}_0$. Then $N_0(e)$ contains a maximal toral subalgebra $\mathfrak{h}$, called the **accompanying torus** of e. Since the characteristic h of e satisfies $[h, e] = 2e$ and $h \in \mathfrak{g}_0$, we have $h \in N_0(e)$; since h is semisimple, we can assume $h \in \mathfrak{h}$. As $\mathfrak{h}$ is contained in the normaliser of the span of e, it follows that $[u, e]$ is a scalar multiple of e, so a character ϕ can be defined on $\mathfrak{h}$ by $\phi(u)e = [u, e]$ for $u \in \mathfrak{h}$.

Definition 3.24 (Support)**.** *We can define a $\mathbb{Z}$-graded subalgebra of $\mathfrak{g}$ by letting $\mathfrak{g}(\mathfrak{h}, \phi) = \oplus_{k\in\mathbb{Z}}\mathfrak{g}_k(\mathfrak{h}, \phi)$, where*

$$\mathfrak{g}_k(\mathfrak{h}, \phi) = \{x \in \mathfrak{g}_{k\cdot 1} : [u, x] = k\phi(u)x \quad \forall u \in \mathfrak{h}\}.$$

*The **support** $\mathfrak{s}$ of e is the derived subalgebra $\mathfrak{g}'(\mathfrak{h}, \phi)$. Then $\mathfrak{s}$ is also a $\mathbb{Z}$-graded subalgebra of $\mathfrak{g}$. It is clear from the definition that*

$$\mathfrak{g}_0(\mathfrak{h}, \phi) = \{x \in \mathfrak{g}_0 : [u, x] = 0 \quad \forall u \in \mathfrak{h}\} = C_0(\mathfrak{h}),$$

where $C_0(\mathfrak{h}) = C_{\mathfrak{g}}(\mathfrak{h}) \cap \mathfrak{g}_0$*, and*

$$e \in \mathfrak{g}_1(\mathfrak{h}, \phi) = \{x \in \mathfrak{g}_1 : [u, x] = \phi(u)x \quad \forall u \in \mathfrak{h}\}.$$

From [29, Section 1, Lemma 2], $\mathfrak{g}(\mathfrak{h}, \phi)$ is reductive, so the support is a semisimple $\mathbb{Z}$-graded subalgebra of $\mathfrak{g}$. The construction of $\mathfrak{g}(\mathfrak{h}, \phi)$ depends on the choice of a maximal toral subalgebra $\mathfrak{h}$ and all supports are conjugate by some element of $\mathcal{C}_0(e) = \mathcal{C}(e) \cap G_0$, where $\mathcal{C}(e)$ is the centraliser of e in the Lie group of $\mathfrak{g}$.

Example 3.25. Let $\mathfrak{g}$ be the Lie algebra given in Example 3.1 and consider $e_{1,2} \in \mathfrak{g}_1$, the matrix with entry 1 at (1, 2) and 0 elsewhere. Since e is strictly upper triangular, it is a nilpotent element of $\mathfrak{g}_1$. Following Theorem 3.10, a homogeneous $\mathfrak{sl}_2$-triple of e is $\{e_{2,1}, h_1, e_{1,2}\}$, so a characteristic of $e_{1,2}$ is h_1. To calculate $N_0(e_{1,2})$, recall that, for $x \in \mathfrak{g}_0$,

$$x = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & d \\ 0 & 0 & e & (-a-b-c) \end{bmatrix}$$

for $a, b, c, d, e \in \mathbb{C}$. We calculate that

$$[x, e_{1,2}] = \begin{bmatrix} 0 & a-b & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = (a-b)e_{1,2}.$$

Since $[x, e_{1,2}] \in \text{Span}(\{e_{1,2}\})$ for all $x \in \mathfrak{g}_0$, it follows that $N_0(e_{1,2}) = \mathfrak{g}_0$. This also shows that the character ϕ induced from $e_{1,2}$ maps x to $a - b$. From the initial construction, let $u_1 = h_1 + h_2 + h_3$, $u_2 = h_2 + h_3$, $u_3 = h_3$, $u_4 = e_{3,4}$ and $u_5 = e_{4,3}$, so $\{u_1, ..., u_5\}$ is a basis for $\mathfrak{g}_0$. Clearly, the characteristic of $e_{1,2}$ is contained in the subalgebra of diagonal matrices of $\mathfrak{g}_0$ spanned by $\{u_1, u_2, u_3\}$. Also note that diagonal matrices commute and are semisimple, since the adjoint map for a diagonal matrix is diagonalisable by [12, Exercise 1.17]. Furthermore, $[u_1, u_4] = u_4 \neq 0$ and $[u_1, u_5] = -u_5 \neq 0$, so the span of $\{u_1, u_2, u_3\}$ is a Cartan subalgebra of $N_0(e_{1,2})$, so $\text{Span}(\{u_1, u_2, u_3\}) = \mathfrak{h}$ is the accompanying torus to $e_{1,2}$.

Since diagonal matrices commute, we see that $\mathfrak{h} \leq \mathfrak{g}_0(\mathfrak{h}, \phi)$. Also, $\phi(u_1) = 1$ and, as stated before, $[u_1, u_4] = u_4$ and $[u_1, u_5] = -u_5$ are neither zero nor linearly dependent, so $\mathfrak{g}_0(\mathfrak{h}, \phi) = \mathfrak{h}$. From taking brackets of the basis of $\mathfrak{h}$, it is clear that $\mathfrak{g}_{3k}(\mathfrak{h}, \phi) = \{0\}$ for $k \neq 0$. Similarly to before, one can verify (e.g. by using the computer algebra system GAP), that $\mathfrak{g}_1(\mathfrak{h}, \phi) = \text{Span}(\{e_{1,2}\})$ and $\mathfrak{g}_{3k+1}(\mathfrak{h}, \phi) = \{0\}$ for $k \neq 0$, and $\mathfrak{g}_{-1}(\mathfrak{h}, \phi) = \text{Span}(\{e_{2,1}\})$ and $\mathfrak{g}_{3k+2}(\mathfrak{h}, \phi) = \{0\}$ for $k \neq -1$. Hence,

$$\begin{aligned} \mathfrak{g}(\mathfrak{h}, \phi) &= \mathfrak{g}_{-1}(\mathfrak{h}, \phi) \oplus \mathfrak{g}_0(\mathfrak{h}, \phi) \oplus \mathfrak{g}_1(\mathfrak{h}, \phi) \\ &= \{ae_{2,1} : a \in \mathbb{C}\} \oplus \mathfrak{h} \oplus \{be_{1,2} : b \in \mathbb{C}\}. \end{aligned}$$

As $[e_{1,2}, e_{2,1}] = h_1$ and diagonal matrices commute, we have that

$$\mathfrak{s} = [\mathfrak{g}(\mathfrak{h}, \phi), \mathfrak{g}(\mathfrak{h}, \phi)] = \mathfrak{s}_{-1} \oplus \mathfrak{s}_0 \oplus \mathfrak{s}_1,$$

where

$$\mathfrak{s}_{-1} = \left\{ \begin{bmatrix} 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} : a \in \mathbb{C} \right\},$$

$$\mathfrak{s}_0 = \left\{ \begin{bmatrix} b & 0 & 0 & 0 \\ 0 & -b & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} : b \in \mathbb{C} \right\},$$

$$\mathfrak{s}_1 = \left\{ \begin{bmatrix} 0 & d & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} : d \in \mathbb{C} \right\}.$$

Therefore, $\mathfrak{s}$ is the support of the nilpotent element $e_{1,2}$. Note that $\mathfrak{s} \cong \mathfrak{sl}(2, \mathbb{C})$, which can be seen from the upper-left blocks of $\mathfrak{s}$. •

The interest in these constructions is from the following theorem and corollary, given in [29, Section 4, pp. 28], which shows the association between supports and nilpotent elements of $\mathfrak{g}_1$.

Theorem 3.26. *Let $\mathfrak{g}$ be a reductive $\mathbb{Z}_m$-graded Lie algebra, with a nilpotent element e in $\mathfrak{g}_1$. Furthermore, let $\mathfrak{s}$ be the support of e. Then e is a point in general position in $\mathfrak{s}_1$ and the centraliser of e in $\mathfrak{s}_0$ is trivial.*

Proof. As the support is a complex $\mathbb{Z}_m$-graded reductive Lie algebra, it contains some point $x \in \mathfrak{s}_0$ in general position, so Theorem 3.17 shows that $[\mathfrak{s}_0, x] = \mathfrak{s}_1$. Observe that both $X = (\operatorname{ad} x)|_{\mathfrak{s}_0}$ and $E = (\operatorname{ad} e)|_{\mathfrak{s}_0}$ are linear maps with domain $\mathfrak{s}_0$ and codomain $\mathfrak{s}_1$. The centraliser of e is $\mathfrak{s}_0$ is precisely the kernel of E, so if the former is trivial, then E has trivial kernel and so the dimension of the image of E is greater than or equal to the dimension of X. But X is already surjective, so E must be surjective as well, showing that $[\mathfrak{s}_0, e] = \mathfrak{s}_1$, so e is a point in general position. Hence, the first statement follows from the second.

From Theorem 3.10 applied to $\mathfrak{s}$, the given e has a homogeneous $\mathfrak{sl}_2$-triple generated by e, some $f \in \mathfrak{s}_{-1}$ and some $h \in \mathfrak{s}_0$. Since h is semisimple, there is a maximal toral subalgebra in $N_0(e)$ containing h, denoted $\mathfrak{h}$. Let $\mathfrak{sl}(h)$ be the subalgebra $\langle e, f, h \rangle$. We will now show that $[\mathfrak{h}, \mathfrak{sl}(h)] \leq \mathfrak{sl}(h)$. Since $\mathfrak{h} \leq N_0(e)$, by definition of the normaliser, we have $[\mathfrak{h}, e] \subseteq \operatorname{Span}(\{e\}) \subseteq \mathfrak{sl}(h)$. Since $\mathfrak{h}$ is a maximal toral subalgebra, it is abelian, so $[\mathfrak{h}, h] = 0$. For $x \in \mathfrak{h}$, suppose $[x, e] = \alpha e$ for some $\alpha \in \mathbb{C}$. If $\alpha = 0$, then $\phi(x) = 0$. Then for all $y \in \mathfrak{g}_k(\mathfrak{h}, \phi)$, we have $[x, y] = 0$ by definition, so $[x, z] = 0$ for all $z \in \mathfrak{g}(\mathfrak{h}, \phi)$ and so x is in the centre of $\mathfrak{g}(\mathfrak{h}, \phi)$. In particular, note that e and h are in $\mathfrak{g}(\mathfrak{h}, \phi)$, so since $\mathfrak{g}(\mathfrak{h}, \phi)$ is reductive, we can apply Theorem 3.10(3) to deduce that $f \in \mathfrak{g}(\mathfrak{h}, \phi)$. Thus, $[x, f] = 0 \in \mathfrak{sl}(h)$. Now, for nonzero $\alpha \in \mathbb{C}$, by the Jacobi identity

$$\begin{aligned} [e, \frac{1}{\alpha}[f, x]] &= -\frac{1}{\alpha}([f, [x, e]] + [x, [e, f]]) = -\frac{1}{\alpha}([f, \alpha e] + [x, h]) \\ &= \frac{1}{\alpha}(-\alpha[f, e]) = h, \end{aligned}$$

and

$$[h, \frac{1}{\alpha}[f, x]] = -\frac{1}{\alpha}([f, [x, h]] + [x, [h, f]]) = 0 - \frac{1}{\alpha}[x, [h, f]]$$
$$= -\frac{1}{\alpha}[x, -2f] = -2(\frac{1}{\alpha}[f, x]),$$

so $\{e, [\frac{1}{\alpha}f, x], h\}$ form an $\mathfrak{sl}_2$-triple. Since $[\frac{1}{\alpha}f, x]$ is nilpotent in the subalgebra spanned by this triple, and the abstract Jordan decomposition coincides with the Jordan decomposition per the statement following Theorem 1.5, $\frac{1}{\alpha}[f, x]$ is also nilpotent. But, by condition (3) of Theorem 3.10, f is uniquely determined by e and h, so we conclude that $[f, x]$ lies in $\mathrm{Span}(\{f\}) \subseteq \mathfrak{sl}(h)$. Hence, we conclude $[\mathfrak{h}, \mathfrak{sl}(h)] \leq \mathfrak{sl}(h)$.

It follows that $\mathfrak{t} = \mathfrak{h} + \mathfrak{sl}(h)$ is a subalgebra of $\mathfrak{g}$. Following [28, Theorem 3.16.3], we have the Levi decomposition $\mathfrak{t} = \mathfrak{r} \oplus \mathfrak{u}$, where $\mathfrak{r}$ is the radical of $\mathfrak{t}$ and $\mathfrak{u}$ is a semisimple subalgebra. From the Jacobi identity, for any $x \in \mathfrak{r}$ and $y, z \in \mathfrak{t}$ we have $[[x, y], z] = -[[z, x], y] - [[y, z], x]$. By anti-symmetry of the Lie bracket we know $[[y, z], x] \in [\mathfrak{r}, \mathfrak{t}]$. Since $\mathfrak{r}$ is an ideal, $[z, x] \in \mathfrak{r}$, so $[[z, x], y] \in [\mathfrak{r}, \mathfrak{t}]$. Hence, we conclude that $[[x, y], z] \in [\mathfrak{r}, \mathfrak{t}]$. Since x, y and z were arbitrarily chosen, it follows that $[\mathfrak{r}, \mathfrak{t}]$ is an ideal. But $[\mathfrak{r}, \mathfrak{t}] \leq [\mathfrak{t}, \mathfrak{t}]$ and the previous working also shows that $[\mathfrak{t}, \mathfrak{t}] = \mathfrak{sl}(h)$ since $\mathfrak{sl}(h)$ is semisimple, so $[\mathfrak{t}, \mathfrak{t}]$ is semisimple. By definition of semisimplicity, it follows that $[\mathfrak{r}, \mathfrak{t}] = 0$ so $\mathfrak{r} \subseteq Z(\mathfrak{t})$. Since $\mathfrak{u}$ is semisimple, it has trivial centre, so $Z(\mathfrak{t}) \cap \mathfrak{u}$ is also trivial. Thus, $Z(\mathfrak{t}) = \mathfrak{r}$ which implies that $\mathfrak{t}$ is reductive.

Since the adjoint action is an automorphism, it maps $\mathfrak{sl}_2$-triples to $\mathfrak{sl}_2$-triples. Therefore, if an element in $\mathfrak{g}^c$ centralises e and h, then its exponential maps both e and h to themselves. Applying part (iii) of Theorem 3.10, this shows that f is also mapped to itself. Finally, we have that G_0 is connected and [20, Section I.10, pp. 49] shows that the exponential map is locally invertible about the identity, so the element must also centralise f. Hence $C(\mathfrak{sl}(h)) = C(e) \cap C(h)$. Since $C(\mathfrak{h})$ is contained in $C(h)$, we see that $C(\mathfrak{t}) = C(\mathfrak{h}) \cap C(e)$, and $C_0(\mathfrak{t}) = C_0(\mathfrak{h}) \cap C(e) = \mathfrak{g}_0(\mathfrak{h}, \phi) \cap C(e)$. This is the centraliser of e in $\mathfrak{g}_0(\mathfrak{h}, \phi)$, which is reductive since it is the zero-graded subalgebra of the centraliser of the reductive Lie algebra $\mathfrak{t}$, and both steps preserve the reductive property by [29, Sections 1.1 and 1.2]. Now, since $C_0(\mathfrak{t}) = C(\mathfrak{h}) \cap C_0(e)$, it is also the centraliser of $\mathfrak{h}$ in $C_0(e)$. It follows from maximality of $\mathfrak{h}$ in $N_0(e)$ that $\mathfrak{h} \cap C_0(e)$ is a maximal toral subalgebra of $N_0(e) \cap C_0(e) = C_0(e)$. We also see

$$\mathfrak{g}_0(\mathfrak{h}, \phi) \cap C(e) = \mathfrak{g}(\mathfrak{h}, \phi) \cap \mathfrak{g}_0 \cap C(e) = \mathfrak{g}(\mathfrak{h}, \phi) \cap C_0(e)$$

is a maximal toral subalgebra of $C_0(e)$: in fact, since $\mathfrak{h}$ is maximal, we have $\mathfrak{g}_0(\mathfrak{h}, \phi) \cap C(e) = C_0(e) \cap C(\mathfrak{h}) = C_0(e) \cap \mathfrak{h}$. The elements in the centre of $\mathfrak{g}(\mathfrak{h}, \phi)$ are precisely those that centralise both e and $\mathfrak{h}$, so $C(\mathfrak{g}(\mathfrak{h}, \phi)) = C_0(e) \cap \mathfrak{h}$. Therefore,

$$0 = (C_0(e) \cap \mathfrak{h})' = (\mathfrak{g}_0(\mathfrak{h}, \phi) \cap C(e))' = \mathfrak{g}_0(\mathfrak{h}, \phi)' \cap C(e) = \mathfrak{s}_0 \cap C(e).$$

This shows that the centraliser of e in $\mathfrak{s}_0$ is trivial, as desired. □

Finally, we can show that the supports are sufficient to classify the nilpotent orbits, by showing uniqueness:

Corollary 3.27. *For a complex reductive $\mathbb{Z}_m$-graded Lie algebra $\mathfrak{g}$, a nilpotent element $e \in \mathfrak{g}_1$ is uniquely defined by its support up to conjugacy by G_0.*

Proof. Suppose there are two nilpotent elements $e_1, e_2 \in \mathfrak{g}_1$ with the same support $\mathfrak{s}$. Since $\mathfrak{s}$ is a $\mathbb{Z}$-graded subalgebra of $\mathfrak{g}$, it has zero rank; by the previous theorem, e_1 and e_2 are both points in general position in $\mathfrak{s}_1$, so they must be in the same (open) orbit. This means they are conjugate under the action of S_0, the connected Lie group which has $\mathfrak{s}_0$ as its Lie algebra. As $\mathfrak{s}_0 \leq \mathfrak{g}_0$, it follows that $S_0 \leq G_0$, so e_1 and e_2 are in the same (nilpotent) orbit. □

The construction of supports also shows a partial converse to the previous statement: since the supports of a nilpotent element $e \in \mathfrak{g}_1$ are conjugate by $\mathcal{C}_0(e) \leq G_0$, where $\mathcal{C}_0(e)$ is the centraliser of e in G_0, any two supports of a nilpotent element are also conjugate under the action of G_0.

3.1.5 Classification of nilpotent orbits in a $\mathbb{Z}_m$-graded Lie algebra

Having shown that the supports are sufficient to classify the nilpotent orbits of a reductive $\mathbb{Z}_m$-graded Lie algebra $\mathfrak{g}$ under the action of the θ-group G_0, the culmination of the problem then lies in the effective classification of the supports. This was achieved by Vinberg in the rest of [29, Chapter 4], and reviewed in this section. Recall that, for a chosen $a \in \mathbb{Z}_m$, by Lemma 3.1 a new Lie algebra $\tilde{\mathfrak{g}}$ may be constructed such that $\tilde{\mathfrak{g}}_1 = \mathfrak{g}_a$, so we can assume $a = 1$ without loss of generality. The defining element of the support $\mathfrak{s}$ with characteristic h is $\frac{h}{2}$ by [29, Section 4.2] and, given a nilpotent $e \in \mathfrak{g}_1$, the constructed support is unique up to conjugation by $N_{G_0}(e)$ as previously stated. The usefulness of supports comes from their ability to be further characterised. To do this, we first need to introduce more subclasses of complex Lie algebras. A reductive $\mathbb{Z}$-graded Lie algebra $\mathfrak{g}$ is **locally flat** if $\dim(\mathfrak{g}_0) = \dim(\mathfrak{g}_1)$. From [29, Section 4.2], we have the following:

Theorem 3.28. *Let $\mathfrak{g}$ be a complex reductive $\mathbb{Z}_m$-graded Lie algebra, G_0 the connected subgroup of the Lie group with Lie algebra $\mathfrak{g}_0$ and acting on $\mathfrak{g}_1$ under the θ-representation, and $e \in \mathfrak{g}_1$ a point in general position. The following conditions are then equivalent:*

(1) $\mathcal{C}_0(e) = \mathcal{C}(e) \cap G_0$ *is finite.*
(2) $C_0(e) = C(e) \cap \mathfrak{g}_0 = 0$*.*
(3) $\dim(\mathfrak{g}_0) = \dim(\mathfrak{g}_1)$*.*

Proof. First, since the dimension of the tangent space is equal to the dimension of its manifold, $\dim(\mathfrak{g}_0) = \dim(G_0)$. The θ-representation ρ gives a group action for G_0 on $\mathfrak{g}_1$. Since e is in general position, it is contained in the open orbit of G_0, so $\rho(G_0)(e)$ is open. Using a corollary of Brouwer's theorem on invariance of domain in

[5, Corollary 1.1.10], this has the same dimension as $\mathfrak{g}_1$, which shows that $\dim(G_0) = \dim(\mathfrak{g}_0) \geq \dim(\mathfrak{g}_1)$.

Now assume condition (2). This shows that the restricted adjoint homomorphism $\operatorname{ad} e : \mathfrak{g}_0 \to \mathfrak{g}_1$ has trivial kernel. As this is a linear map over vector spaces, it follows that $\dim \mathfrak{g}_1 \geq \dim \mathfrak{g}_0$. Combined with the previous paragraph, this implies condition (3). Now assume condition (3). Since G_0 is connected, by [26, Theorem I.2, Section 1] we have that the kernel of the adjoint map on the open orbit containing e is $\mathcal{C}_0(e)$. Since $\dim\,(G_0) = \dim\,(\mathfrak{g}_1)$, the kernel of the adjoint map on the open orbit is 0-dimensional. That is, it comprises discrete points. Since it is closed, under the Zariski topology it is an algebraic set, which is a finite union of irreducible, hence connected, components. Hence the number of points must be finite. Now assume condition (1). It is then clear that $\mathcal{C}_0(e)$ is zero-dimensional. Since the Lie algebra of $\mathcal{C}_0(e)$ is $C_0(e)$, it follows that $C_0(e)$ is also zero-dimensional. But $C_0(e)$ is a vector space over $\mathbb{C}$, so the only possibility is that $C_0(e) = 0$. □

For a locally flat, complex reductive Lie algebra $\mathfrak{g}$, since the centre of $\mathfrak{g}$ is in $\mathfrak{g}_0$ and $C(e) \cap \mathfrak{g}_0 = 0$, no element centralises e, so $\mathfrak{g}$ has trivial centre. Since $\mathfrak{g}$ is already reductive, it follows that $\mathfrak{g}$ is semisimple. Vinberg also shows that the local flatness imposes key restrictions on the roots of $\mathfrak{g}$: the simple roots can only have degree 0 or 1 and, letting r_0 and r_1 be the number of roots with degree 0 or 1, respectively, the roots must satisfy the formula

$$\operatorname{rank}(\mathfrak{g}) + r_0 = r_1.$$

In particular, if $\mathfrak{g}$ is of type A_n, then Vinberg showed that $r_0 = 0$. In this case, all the simple roots have degree 1, and the locally flat Lie algebra is **principal**.

For a complex reductive $\mathbb{Z}_m$-graded Lie algebra $\mathfrak{g}^c$, a $\mathbb{Z}$-graded subalgebra $\mathfrak{c}$ is **regular** if it is normalised by a maximal toral subalgebra $\mathfrak{h}$ of $\mathfrak{g}_0$; that is, $\mathfrak{h} \subseteq N_{\mathfrak{g}^c}(\mathfrak{c})$. Furthermore, a regular reductive subalgebra is **complete** if it is not a proper $\mathbb{Z}$-graded subalgebra of another regular reductive $\mathbb{Z}$-graded subalgebra with the same rank. That is, a complete $\mathbb{Z}$-graded subalgebra is maximal while having some fixed rank. These can be obtained systematically from $\mathfrak{g}^c$, as follows.

First, fixing a maximal toral subalgebra $\mathfrak{h}$ of $\mathfrak{g}_0$, a regular $\mathbb{Z}$-graded subalgebra $\mathfrak{c}$ is **standard** if $\mathfrak{c}$ is normalised by $\mathfrak{h}$ and $[\mathfrak{h}, \mathfrak{c}_k] \subseteq \mathfrak{c}_k$ for all $k \in \mathbb{Z}$; that is, the condition for regularity is restricted to the chosen maximal toral subalgebra $\mathfrak{h}$ and $\mathfrak{h}$ normalises the grading subspaces of $\mathfrak{c}$. By [29, Proposition 4(1)], every regular subalgebra is conjugate to a standard subalgebra, via the element of G_0 that maps the torus that normalises the regular subalgebra to $\mathfrak{h}$. Hence, up to conjugacy by G_0, we focus on finding the standard semisimple $\mathbb{Z}$-graded subalgebras. To do this, we follow [29, Section 3].

Here we suppose that $\mathfrak{g}$ is $\mathbb{Z}$-graded. We consider the case where $\mathfrak{g}$ is $\mathbb{Z}_m$-graded for $m \in \mathbb{N}$ after presenting the classification in the $\mathbb{Z}$-graded case. With respect to the given maximal toral subalgebra $\mathfrak{h}$, let Φ be the set of roots of $\mathfrak{g}$, which, from Section 1.3.2, forms a root space contained in an inner product space with inner product $\langle -, - \rangle$. For each $\alpha \in \Phi$, observe that $e_\alpha \in \mathfrak{g}_{q_\alpha}$ for some $q_\alpha \in \mathbb{Z}$, so there is a degree function deg : $\alpha \mapsto q_\alpha$. From properties of the root space decomposition, we have $\deg(\alpha+\beta) = \deg(\alpha) + \deg(\beta)$ if $\alpha, \beta, \alpha+\beta \in \Phi$ and since, for $e_\alpha \in \mathfrak{g}_\alpha$, there is a $f_\alpha \in \mathfrak{g}_{-\alpha}$ such that $[e_\alpha, f_\alpha] = h_\alpha \in \mathfrak{h} \setminus \{0\} \subseteq \mathfrak{g}_0$, we also have $\deg(-\alpha) = -\deg(\alpha)$. Thus, the degree function is defined by its values on a base of the roots. Since the base is a vector space basis, the number of simple roots is the rank of the root system, so there exists a hyperplane with all roots of positive degree on one side, by [12, Section 11.3]. Thus, there exists a base $\mathcal{B}$ such that all simple roots have nonnegative degree.

Given a standard semisimple $\mathbb{Z}$-graded subalgebra $\mathfrak{s}$ of $\mathfrak{g}$, recall from Corollary 3.8 that $\mathfrak{s}_0$ has the same rank as $\mathfrak{s}$, so $\mathfrak{h} \cap \mathfrak{s}_0$ is a maximal toral subalgebra of $\mathfrak{s}$ and we c an consider a base Φ_s and degree function on $\mathfrak{s}$ in the same manner as $\mathfrak{g}^c$. Furthermore, Vinberg shows that, for $\alpha \in \Phi_s$, a root vector e_α for $\mathfrak{s}$ is also a root vector for $\mathfrak{g}^c$ with root β, where $\beta(x) = \alpha(x)$ for $x \in \mathfrak{h} \cap \mathfrak{s}_0$ and $\beta(x) = 0$ for $x \in \mathfrak{h} \cap C_0(\mathfrak{s})$, which is the complement of $\mathfrak{h} \cap \mathfrak{s}_0$ by [29, Section 3.3]. Hence, we can construct the map $\tilde{f} : \Phi_s \to \Phi$ by $\tilde{f}(\alpha) = \beta$. The map $\tilde{f}$ satisfies the same linearity properties as the degree function, so it is also defined by its restriction f to the base $\mathcal{B}_s$ of Φ_s.

As $[\tilde{f}(\alpha)]|_{\mathfrak{h}\cap\mathfrak{s}_0} = \alpha$ and $[\tilde{f}(\alpha)]|_{\mathfrak{h}\cap C_0(\mathfrak{s})} = 0$, it is clear that $\tilde{f}$ is injective, so the restriction f must also be injective. From [29, Section 3.4], we have the following proposition.

Proposition 3.29. *Let $\mathfrak{g}$ be a complex reductive $\mathbb{Z}_m$-graded Lie algebra, with a $\mathbb{Z}$-graded semisimple subalgebra $\mathfrak{s}$ and the map $f : \mathcal{B}_s \to \Phi$ just constructed. Then $\mathfrak{s}$ is standard if*

- $\langle f(\alpha), f(\beta)\rangle = \langle \alpha, \beta \rangle$ *for all* $\alpha, \beta \in \mathcal{B}_s$.
- $f(\alpha) - f(\beta) \notin \Phi$ *for all* $\alpha, \beta \in \mathcal{B}_s$.
- $\deg(f(\alpha)) = \deg(\alpha)$ *for all* $\alpha \in \mathcal{B}_s$.

The main value of this proposition is that the converse also holds: for any semisimple $\mathbb{Z}$-graded Lie algebra $\mathfrak{s}$, a map $f : \mathcal{B}_s \to \Phi$ which has the above properties induces a standard semisimple $\mathbb{Z}$-graded subalgebra $\mathfrak{s}(f)$ isomorphic to $\mathfrak{s}$. This gives a means of classifying the standard subalgebras of $\mathfrak{g}^c$ which, by the above discussion, also classifies the regular subalgebra. Furthermore, the additional property of completeness is quickly deduced: $\mathfrak{s}(f)$ is complete if and only if there exists an element w of the Weyl group of $\mathfrak{g}^c$ such that the image of $w \circ f$ is contained in the simple roots of $\mathfrak{g}^c$.

When the Lie algebra has a $\mathbb{Z}_m$-grading for $m \in \mathbb{N}$, a method by Kac, reiterated in [30, Section 8.1], associates each $\mathbb{Z}_m$-graded Lie algebra to its **covering algebra** $\mathfrak{G} = \oplus_{k\in\mathbb{Z}} \mathfrak{G}_k$, contained in $\mathbb{C}[t, t^{-1}] \otimes \mathfrak{g}$, where each

$$\mathfrak{G}_k = t^k \otimes \mathfrak{g}_{k \bmod m}.$$

This is a $\mathbb{Z}$-graded Lie algebra with each component being the tensor product of the component of the original Lie algebra with the element of a ring used to track the $\mathbb{Z}$-grading. As Vinberg showed in [29, Section 3.5], one can find the regular and complete subalgebras associated to the covering algebra using the techniques in the previous section (with slight adaptation since the covering algebra is infinite-dimensional), then descend back to the original Lie algebra to get its standard and complete subalgebras.

Now, [29, Proposition 1, Section 3] shows that a support in $\mathfrak{g}$ is a complete, regular semisimple $\mathbb{Z}$-graded subalgebra $\mathfrak{g}$. It turns out that additionally imposing that the subalgebra is locally flat is sufficient to get a converse. That is, we have the following theorem, from [29, Theorem 4, Section 4]:

Theorem 3.30. *For a reductive $\mathbb{Z}_m$-graded Lie algebra $\mathfrak{g}$, the supports of the nilpotent elements of $\mathfrak{g}_1$ are, up to conjugacy by G_0, precisely the complete, regular, locally flat, semisimple $\mathbb{Z}$-graded subalgebras of $\mathfrak{g}$.*

Proposition 3.31 (Finding supports from roots)**.** *Suppose one has a semisimple complex $\mathbb{Z}_m$-graded Lie algebra $\mathfrak{g}^c$, a set of roots Φ with simple roots Π and a Dynkin diagram $\mathcal{D}$. The process of finding the supports can be summarised as follows.*

- *If $m \in \mathbb{N}$, replace $\mathfrak{g}^c$ with its covering algebra.*
- *From the first condition of Proposition 3.29, the subgraphs in $\mathcal{D}$ form the candidates for the Dynkin diagram of a support. Each subgraph ∂ defines the support up to isomorphism, so let $\mathfrak{s}$ be a complex semisimple Lie algebra that has Dynkin diagram ∂. Let $\Pi(\mathfrak{s})$ be the set of simple roots of $\mathfrak{s}$, and let these simple roots be labelled α_i.*
- *Search for a map $f : \Pi(\mathfrak{s}) \to \Phi$ satisfying the three conditions in Proposition 3.29. Note that the codomain of f is not just the set to simple roots of Φ. To satisfy local flatness, the degrees of the roots for both $\Pi(\mathfrak{s})$ and $f(\Pi(\mathfrak{s}))$ must be 0 or 1, with the difference in the number of all roots of $\mathfrak{s}$ with degree 1 and the number of all roots of $\mathfrak{s}$ with degree 0 being the rank of $\mathfrak{s}$.*
- *If $\tilde{\Phi}$ is the set of roots in the image of f, the set $\{f_\alpha, h_\alpha, e_\alpha : \alpha \in \tilde{\Phi}\}$ generates (by taking linear combinations and Lie brackets) a $\mathbb{Z}$-graded subalgebra $\mathfrak{s}(f)$ in $\mathfrak{g}^c$ isomorphic to $\mathfrak{s}$.*
- *If there exists an element of the Weyl group of Φ that sends the image of f into Π, the set of simple roots of $\mathfrak{g}^c$, then $\mathfrak{s}$ is also complete. If no such element exists, there must be a complete, locally flat subalgebra containing $\mathfrak{s}(f)$ with the same rank.*

Having found a support $\mathfrak{s}$, Theorem 3.26 shows that the element of the associated nilpotent orbit is an element in general position in the support, which can be found as a linear algebra problem by solving $[\mathfrak{s}_0, e] = \mathfrak{s}_1$.

Example 3.32. Let us apply the proposition to investigate the supports for our example of $\mathfrak{g}^c$ with the given $\mathbb{Z}_3$-grading. The Dynkin diagram of $\mathfrak{g}^c$ is A_3, so the supports can be of type A_1, A_2, $A_1 + A_1$ or A_3. The roots of $\mathfrak{g}^c$ are

$$\Phi = \pm\{\alpha_{12}, \alpha_{23}, \alpha_{34}, \alpha_{13}, \alpha_{24}, \alpha_{14}\},$$

with the first three being the simple roots. Recall that the root vector of α_{ij} is e_{ij}, the matrix with entry 1 at (i, j) and 0 elsewhere. Taking the covering algebra to get a $\mathbb{Z}$-grading, by looking at the grading, we see that $e_{12}, e_{23} \in \mathfrak{g}^c_1$ and $e_{34} \in \mathfrak{g}^c_0$, so $\deg(\alpha_{12}) = \deg(\alpha_{23}) = 1$ and $\deg(\alpha_{34}) = 0$, and we extend this to the other roots so that the degree function deg: $\Phi \to \mathbb{Z}$ is a homomorphism. Note that this is done over the covering algebra, not $\mathfrak{g}^c$. In particular, while e_{31} and e_{41} are elements of $\mathfrak{g}^c_1$, the degrees of their respective roots are $\deg(-\alpha_{12} - \alpha_{23}) = -\deg(\alpha_{12}) - \deg(\alpha_{23}) = -2$ and $\deg(-\alpha_{12}-\alpha_{23}-\alpha_{34}) = -\deg(\alpha_{12})-\deg(\alpha_{23})-\deg(\alpha_{34}) = -2$. Now we consider each type. Since all of the types are sums of A_n, the support must be principal, so the degree of all simple roots in $\mathfrak{s}$ must be 1.

- A_1: A standard locally flat subalgebra is found by finding a map f that sends the simple root α_1 of $\mathfrak{sl}(2, \mathbb{C})$ to any root with degree 1. The image of f can be α_{12}, α_{23} or α_{24}. We see that α_{12} and α_{23} are already simple, and $s_{\alpha_{23}}(\alpha_{24}) = \alpha_{34}$, which is simple. Therefore, for each choice of f, the resulting subalgebra $\mathfrak{s}(f)$ is complete, so $\mathfrak{s}(f)$ is indeed a support.

- $A_1 + A_1$: Since the root system is reducible, the roots of $\mathfrak{s}$ satisfy the equation $(\alpha_1, \alpha_2) = 0$, so if α_{ab}, α_{cd} are the images of f, then $(\alpha_{ab}, \alpha_{cd}) = 0$. Then $\langle \alpha_{ab}, \alpha_{cd} \rangle = \alpha_{ab}(h_{\alpha_{cd}}) = 0$. Recall, for $c < d$, that $e_{\alpha_{cd}} = e_{cd}$, and $h_{\alpha_{cd}} = e_{cc}-e_{dd}$, so $\alpha_{ab}(h_{\alpha_{cd}}) = \alpha_{ab}(e_{cc})-\alpha_{ab}(e_{dd}) = 0$, which implies that $\alpha_{ab}(h_{\alpha_{cd}}) = 0$ if and only if $a \neq c$, $a \neq d$, $b \neq c$ and $b \neq d$. Hence, choosing $f(\alpha_1) = \alpha_{ab}$ gives one other candidate $f(\alpha_2) = \alpha_{cd}$, so all that remains is to check if the degrees of these roots are 1. But the only roots with degree 1 are α_{12}, α_{23} and α_{24}, which cannot be orthogonal, so no such f is possible. Hence, there is no standard locally flat $\mathbb{Z}$-graded subalgebra of $\mathfrak{g}^c$ with type $A_1 + A_1$.

- A_2: Since the simple roots of the simple Lie algebra of type A_2 are $\alpha_1 = \alpha_{12}$ and $\alpha_2 = \alpha_{23}$, it is clear that the map defined by $f(\alpha_1) = \alpha_{12}$ and $f(\alpha_2) = \alpha_{23}$ gives a support with this type, since the images of f are simple roots with degree 1. What remains is to consider if one of the images of f can be α_{24} while still giving a support. Computing the value of $\langle \cdot, \cdot \rangle$ is simpler here since we know $s_\alpha(\beta) = \beta - \langle \beta, \alpha \rangle \alpha$ is another root. Recall that replacing $\langle \beta, \alpha \rangle$ with an integer produces an element of the root string of α passing through β. If α and β are orthogonal (which we already considered), then $\langle \beta, \alpha \rangle = 0$. Otherwise, there is only one value for $\langle \beta, \alpha \rangle$ that will make $s_\alpha(\beta)$ into a root. To begin, $s_{\alpha_{23}}(\alpha_{12}) = \alpha_{12} - \langle \alpha_{12}, \alpha_{23} \rangle \alpha_{23}$ can only equal $\alpha_{13} = \alpha_{12} + \alpha_{23}$, so $\langle \alpha_{12}, \alpha_{23} \rangle = -1 = \langle f(\alpha_1), f(\alpha_2) \rangle$. Checking the map f that has $\alpha_1 \mapsto \alpha_{12}$ and $\alpha_2 \mapsto \alpha_{24}$, we see that $\alpha_{12} - \langle \alpha_{12}, \alpha_{24} \rangle \alpha_{24}$ must equal $\alpha_{14} = \alpha_{12} + \alpha_{24}$, so $\langle \alpha_{12}, \alpha_{24} \rangle = -1$ and the choice of f satisfies the first condition of Proposition 3.29. Clearly $\alpha_{12} - \alpha_{24} = \alpha_{12} - \alpha_{23} - \alpha_{34}$ is not a root, since the summands are simple roots, so the coefficients would need to have the same sign, so the second condition is satisfied. Now, if the standard subalgebra $\mathfrak{s}(f)$ induced by f were not complete, it would be contained in a complete subalgebra with the same rank. But we see from the rank candidates that a subalgebra of type A_2 could only be contained in one of type A_3, which has greater rank. Thus, the choice of f gives a support. By the same reasoning, for the map f induced by $\alpha_1 \mapsto \alpha_{24}$ and $\alpha_2 \mapsto \alpha_{23}$, the reflection $s_{\alpha_{23}}(\alpha_{24})$ can only equal $\alpha_{34} = \alpha_{24} - \alpha_{23}$, so $\langle \alpha_{24}, \alpha_{23} \rangle = 1$, so this choice of f does not give a support.

- A_3: A candidate for f would require elements α_1, α_2 and α_3 in Φ such that $(\alpha_1, \alpha_3) = 0$, $\langle \alpha_1, \alpha_2 \rangle = \langle \alpha_2, \alpha_3 \rangle = -1$. Testing these across the three elements of Φ with degree 1, no such arrangement is possible, so there are no supports of type A_3.

Summarising, $\mathfrak{g}^c$ has three supports of type A_1 and two supports of type A_2.

3.2 Complex graded Lie algebras and the covering algebra

In this section, we investigate results for finite gradings, so let $\mathfrak{g}$ be a complex, finite-dimensional, $\mathbb{Z}_m$-graded Lie algebra for some $m \geq 1$. We will cover Kac's classification of finitely graded automorphisms on complex simple graded Lie algebras, which is equivalent to classifying the finitely graded Lie algebras as stated at the start of Section 3.1. As is later shown, these are applied by Vinberg in [30, Chapter 8] to describe the associated θ-group, and used by Vinberg in [29, Section 5, Chapter 3] to classify the supports of complex θ-groups when the Lie algebra is finitely graded.

Following the work of Kac, as described by Helgason in [16, Chapter X], one can associate the given $\mathfrak{g}$ with its **covering algebra**. As mentioned in Section 3.1.5, this is an infinite-dimensional $\mathbb{Z}$-graded Lie algebra $\mathfrak{G} = \oplus_{k \in \mathbb{Z}} \mathfrak{G}_k$ contained in $\mathbb{C}[t, t^{-1}] \otimes \mathfrak{g}$, where each

$$\mathfrak{G}_k = t^k \otimes \mathfrak{g}_{k \bmod m}.$$

While the underlying vector space is infinite-dimensional, this is a finite-dimensional $\mathbb{C}[x, x^{-1}]$-algebra if one defines $x(t^n \otimes a) = t^{m+n} \otimes a$ for all $a \in \mathfrak{g}$ on $\mathfrak{G}_n$, and extends this linearly to all of $\mathfrak{G}$. Since $x\mathfrak{G}_n = \mathfrak{G}_{n+m}$, from the covering algebra we can recover the original Lie algebra as $\mathfrak{g} \cong \mathfrak{G}/(x-1)\mathfrak{G}$, since by the commutativity of $\mathbb{C}[x, x^{-1}]$, we see that $(x-1)\mathfrak{G}$ is an ideal of $\mathfrak{G}$ and $(x-1)(t^i \otimes a) = (t^{m+i} - t^i) \otimes a$, so two elements of $\mathfrak{G}$ are in the same equivalence class under the quotient if and only if their second components are equal. Hence, this quotient map is equal to the covering epimorphism $\pi : \mathfrak{G} \to \mathfrak{g}$ defined by $(t^k \otimes x) = x$ for $k \in \mathbb{Z}$ and $x \in \mathfrak{g}_{k \bmod m}$. In particular, each $\mathfrak{G}_i$ (especially for $i = 0$) is mapped bijectively onto $\mathfrak{g}_i$, and so the θ-representation for $\mathfrak{G}$ is the same as for $\mathfrak{g}$. Disregarding the grading, in this thesis we define a **Kac algebra** to be a Lie algebra $\mathfrak{G}$ that can be constructed as the covering algebra of a complex semisimple (finite-dimensional) Lie algebra. However, note that other authors use this term to refer to other objects.

Example 3.33. Let $\mathfrak{g}$ be the $\mathbb{Z}_m$-graded Lie algebra given in Example 3.1. If the grading of $\mathfrak{g}$ is labelled as $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$, then the covering algebra of $\mathfrak{g}$ is given as

$$\mathfrak{G} = ... \oplus (t^{-2} \otimes \mathfrak{g}_1) \oplus (t^{-1} \otimes \mathfrak{g}_2) \oplus (1 \otimes \mathfrak{g}_0) \oplus (t \otimes \mathfrak{g}_1) \oplus (t^2 \otimes \mathfrak{g}_2) \oplus ...$$

Kac algebras have multiple properties in common with the Lie algebras from which they are constructed, including the means of their classification. Let $\mathfrak{g}$ be a finite-dimensional complex semisimple Lie algebra. Furthermore, let $\mathfrak{g}$ have a $\mathbb{Z}_m$-grading (with $m \in \mathbb{N}$) induced from a finite-order automorphism σ. From [16, Lemma 5.3, Chapter X], there is a CSA of $\mathfrak{g}$ constructed as the centraliser of a maximal torus $\mathfrak{h}$ of $\mathfrak{g}_0$, denoted by $\bar{\mathfrak{h}} = C_{\mathfrak{g}}(\mathfrak{h})$. Analogously to ungraded complex semisimple Lie algebras, for $\alpha \in \mathfrak{h}^\star$ and $i \in \mathbb{Z}_m$, if the generalised eigenspace

$$\mathfrak{g}_{(\alpha,i)} = \{x \in \mathfrak{g}_i : \ [h,x] = \alpha(h)x, \ \forall h \in \mathfrak{h}\}$$

is nonzero, then the pair (α, i) is called a **root** and the eigenspace is a root space. Let Φ be the set of all these roots of the graded Lie algebra $\mathfrak{g}$, excluding the zero root $(\mathbf{0}, 0)$. Since $\mathfrak{h} \leq \mathfrak{g}_0$ is maximal abelian by construction, it is clear that $\mathfrak{h} = \mathfrak{g}_{(\mathbf{0},0)}$, so we have a decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\tilde{\alpha} \in \Phi} \mathfrak{g}_{\tilde{\alpha}}.$$

While we have already excluded the zero root from Φ, like in the ungraded case, it will be useful to be more specific, justified by the following lemma:

Lemma 3.34. *For a finite-dimensional complex semisimple $\mathbb{Z}_m$-graded Lie algebra $\mathfrak{g}$ with maximal torus $\mathfrak{h} \leq \mathfrak{g}_0$, the Cartan subalgebra $\bar{\mathfrak{h}} = C_{\mathfrak{g}}(\mathfrak{h})$ is the sum of $\mathfrak{h}$ with root spaces of the form $(\mathbf{0}, i)$ for $i \in \mathbb{Z}_m^+$.*

Proof. Let $z \in \bar{\mathfrak{h}}$; this implies that $[h, z] = 0$ for all $h \in \mathfrak{h}$. From the grading, z is uniquely expressed as $z = z_0 + ... + z_{m-1}$ with each $z_i \in \mathfrak{g}_i$. Applying $\operatorname{ad} h$ to the sum gives $0 = [h, z_0] + ... + [h, z_{m-1}]$. As $h \in \mathfrak{h} \leq \mathfrak{g}_0$, see that each $[h, z_i] \in \mathfrak{g}_i$. By linear independence of the eigenspace decomposition, it follows that $[h, z_i] = 0$, for all $h \in \mathfrak{h}$ and $0 \leq i \leq m-1$. For $i \geq 1$, we have $z_i \in \mathfrak{g}_{(\mathbf{0},i)}$ by definition of the root space. For $i = 0$, every element of $\mathfrak{h}$ commutes with z_0, so $z_0 \in \mathfrak{h}$ since $\mathfrak{h}$ was defined to be maximal abelian. Hence, $z_0 \in \mathfrak{h}$ and the remaining z_i are root vectors for roots of the form $(\mathbf{0}, i)$, as desired. □

With this in mind, let $\Phi^0 \subseteq \Phi$ be the subset of roots of the form $(\mathbf{0}, i)$ for some $i \in \mathbb{Z}$ and $\bar{\Phi} = \Phi \setminus \Phi^0$. Then the previous lemma gives the root space decomposition:

$$\mathfrak{g} = \bar{\mathfrak{h}} \oplus \bigoplus_{\tilde{\alpha} \in \bar{\Phi}} \mathfrak{g}_{\tilde{\alpha}}.$$

As Vinberg notes in [30, Chapter 9], the roots in $\bar{\Phi}$ are a close analog to the roots for ungraded semisimple Lie algebras. One reason is that, for any $\tilde{\alpha} \in \bar{\Phi}$, the dimension of $\mathfrak{g}_{\tilde{\alpha}}$ is always 1 and the only scalar multiples of $\tilde{\alpha}$ in $\bar{\Phi}$ are $\pm\tilde{\alpha}$, like with roots of ungraded semisimple Lie algebras.

As each grading subspace is isomorphic to the grading subspace of its covering algebra, the inverse relation of the covering epimorphism induces an analogous decomposition on the covering algebra. That is, let $\mathfrak{G}$ be the covering algebra of $\mathfrak{g}$, and define roots of $\mathfrak{G}$ with respect to $\mathfrak{h}$ in the same way as with $\mathfrak{g}$, but with the grading index $i \in \mathbb{Z}$. Let Δ be the set of roots of $\mathfrak{G}$, with the sets of zero roots $(\mathbf{0}, i)$, $i \in \mathbb{Z}$, and nonzero roots being denoted Δ^0 and $\bar{\Delta}$, analogously to Φ^0 and $\bar{\Phi}$. Then

$$\mathfrak{G} = \bar{\mathfrak{h}} \oplus \bigoplus_{\tilde{\alpha} \in \bar{\Delta}} \mathfrak{G}_{\tilde{\alpha}}.$$

Example 3.35. Let $\mathfrak{g}$ be the Lie algebra of Example 3.1, with $\mathfrak{G}$ as its covering algebra. From [12, Chapter 12], a natural CSA for $\mathfrak{g}$ is the subalgebra $\mathfrak{n}(4,\mathbb{C})\cap\mathfrak{g}$ of diagonal matrices in $\mathfrak{g}$. Since this is contained in $\mathfrak{g}_0$, it is clear that $\mathfrak{n}(4,\mathbb{C})\cap\mathfrak{g}$ is a CSA of $\mathfrak{g}_0$. From Example 1.1, recall that $\mathfrak{n}(4,\mathbb{C})$ is its own centraliser in $\mathfrak{gl}(4,\mathbb{C})$, so $\mathfrak{h}=\bar{\mathfrak{h}}=\mathfrak{n}(4,\mathbb{C})$ in $\mathfrak{g}$.

From [12, Section 12.2], the roots of the ungraded form of $\mathfrak{g}$ are of the form $\epsilon_{ij}=\epsilon_i-\epsilon_j$, where $1\leq i,j\leq 4$ and $\epsilon_i\in\mathfrak{h}^\star$ is defined by $\epsilon_i(h)=h_{ii}$. Furthermore, the root space of ϵ_{ij} is spanned by $e_{i,j}$, as defined in Example 3.1. Hence, the graded roots of $\mathfrak{g}$ are $\{(\epsilon_{ij},a_{ij}):\ 1\leq i,j\leq 4\}$, where each a_{ij} is chosen such that by $e_{i,j}\in\mathfrak{g}_{a_{ij}}$. Similarly, the graded roots of the covering algebra $\mathfrak{G}$ are

$$\{(\epsilon_{ij},a_{ij}+3k):\ k\in\mathbb{Z};1\leq i,j\leq 4\},$$

where the a_{ij} is defined in the same manner as before. •

As is later shown, the roots define the covering algebra in a manner analogous to Serre's theorem in the finite-dimensional case, so our attention shifts to classifying the roots of the covering algebra, via an approach analogous to the finite-dimensional case. From [16, Lemma 5.7a, Chapter X], one can generalise the notions of a basis and positive roots. For $(\alpha,i),(\beta,j)\in\Delta$, define addition by $(\alpha,i)+(\beta,j)=(\alpha+\beta,i+j)$, if $(\alpha+\beta,i+j)\in\Delta$. Then the positive roots of Δ can be described as follows. First, all roots of the form (α,i) for $i>0$ are positive, and all roots of the form (β,j) for $j<0$ are negative. As stated in Section 3.1.4, $\mathfrak{g}_0$ and $\mathfrak{G}_0$ are reductive subalgebras of $\mathfrak{g}$ and $\mathfrak{G}$, respectively, so $\mathfrak{G}_0=[\mathfrak{G}_0,\mathfrak{G}_0]\oplus Z(\mathfrak{G}_0)$, with $[\mathfrak{G}_0,\mathfrak{G}_0]$ being semisimple. As a result, $[\mathfrak{G}_0,\mathfrak{G}_0]$ is a finite-dimensional complex semisimple Lie algebra, and it is clear that $\mathfrak{h}\cap[\mathfrak{G}_0,\mathfrak{G}_0]$ is a Cartan subalgebra since semisimplicity of the elements is given from this being a subset of $\mathfrak{h}$ and the preservation of the Killing form by [17, Lemma, Section 5.1], and maximality follows from maximality of $\mathfrak{h}$: if there exists an abelian subalgebra $\mathfrak{t}$ such that $\mathfrak{h}\cap[\mathfrak{G}_0,\mathfrak{G}_0]<\mathfrak{t}<[\mathfrak{G}_0,\mathfrak{G}_0]$, then $\mathfrak{h}<\mathfrak{t}\oplus Z(\mathfrak{G}_0)<\mathfrak{G}_0$, where it is clear that $\mathfrak{t}\oplus Z(\mathfrak{G}_0)$ is a Cartan subalgebra larger than $\mathfrak{h}$, a contradiction.

Hence, from the finite-dimensional theory one can consider the root space of $[\mathfrak{G}_0,\mathfrak{G}_0]$ over $\mathfrak{h}\cap[\mathfrak{G}_0,\mathfrak{G}_0]$. Since a root α of $\mathfrak{G}_0$ over $\mathfrak{h}$ has a nontrivial root space, let x be a nonzero element of the root space. The choice of x gives that, for $z\in\mathfrak{h}\cap Z(\mathfrak{G}_0)$, we have $[z,x]=\alpha(z)x$. Since z is in the centre of $\mathfrak{G}_0$ and x is nonzero, this gives $\alpha(z)=0$. Therefore, every root of $\mathfrak{G}_0$ over $\mathfrak{h}$ can be found from a root in the dual space of $\mathfrak{h}\cap[\mathfrak{G}_0,\mathfrak{G}_0]$ by letting $\alpha(z)=0$ for $z\in\mathfrak{h}\cap Z(\mathfrak{G}_0)$. This gives a means of constructing the roots of $\mathfrak{G}_0$. As $\mathfrak{G}_0$ is finite-dimensional and semisimple, it is already known that we can find a basis and a set of positive roots. Combining the positive roots of $\mathfrak{G}_0$ and the roots of positive index gives the set Δ^+ of positive roots of $\mathfrak{G}$.

Example 3.36. With $\mathfrak{g}$ and $\mathfrak{G}$ from Example 3.1, recall that $\mathfrak{G}_0=\mathfrak{sl}(2,\mathbb{C})\oplus T_2$. Since this is a direct sum of ideals, it is clear that

$$[\mathfrak{G}_0,\mathfrak{G}_0]=\left\{\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&c&d\\0&0&e&-c\end{bmatrix}:\ c,d,e\in\mathbb{C}\right\}\cong\mathfrak{sl}(2,\mathbb{C}).$$

Recall that $\mathfrak{sl}(2,\mathbb{C})$ has rank 1 so, associating $\mathfrak{sl}_2(\mathbb{C})$ with $[\mathfrak{G}_0,\mathfrak{G}_0]$, it has one positive root $\epsilon_{34} \in \mathfrak{h}^\star$, where $\epsilon_{34}(h) = h_{33} - h_{44} = 2h_{33}$, for all $h \in \mathfrak{h}$. Using the previous example, the positive roots of $\mathfrak{G}$ are the roots $(\epsilon_{ij}, a_{ij}+3k)$ for $i,j \in \{1,2,3,4\}$ and $k \in \mathbb{Z}$ with $a_{ij}+3k > 0$, along with $(\epsilon_{34}, 0)$. •

Now, [16, Lemma 5.7, Chapter X] gives that the elements of Δ^+ that are not the sum of two other elements of Δ^+ form a generating set for Δ. Let $\tilde{\Pi} = \{(\alpha_0, i_0), ..., (\alpha_n, i_n)\}$ be this set, with $\Pi = \{\alpha_0, ..., \alpha_n\}$. By the same lemma, Π comprises distinct elements and spans $\mathfrak{h}^\star$ like in the ungraded case, so it is indeed a finite set. Furthermore, every element $\beta \in \Delta$ can be expressed as $\beta = \sum_{j=0}^{n} k_j(\alpha_j, i_j)$, where the k_j are either all positive or all negative. Thus, $\tilde{\Pi}$ closely resembles the basis of a root system. The one (and key) difference is that Π is not linearly independent: as $\mathfrak{G}_m \equiv \mathfrak{G}_0$, it follows that $\mathfrak{G}_m$ contains an abelian subspace isomorphic to $\mathfrak{h}$, so $(0,m) \in \Delta$. As $m > 0$, it follows that $(0,m) \in \Delta^+$, so there exists a $\mathbb{Z}$-linear combination of elements in $\tilde{\Pi}$ equal to $(0,m)$. Since $(0,m) \in \Delta^0$ as well, by definition of addition in Δ, there exists a $\mathbb{Z}$-linear combination of elements in Π equal to 0, so in fact Π is always a linearly dependent set. Letting $A = \langle \tilde{\Pi} \rangle$ be the abelian group generated by $\tilde{\Pi}$, and defining $\mathfrak{G}_{(\mathbf{0},0)} = \mathfrak{h}$ and $\mathfrak{G}_{\tilde{\beta}} = 0$ if $\tilde{\beta} \notin \Delta$, we see that the root space decomposition can be collected into an A-grading on $\mathfrak{G}$:

$$\mathfrak{G} = \bigoplus_{\tilde{\alpha} \in A} \mathfrak{G}_{\tilde{\alpha}}.$$

In this report, we refer to this grading as a **root grading** of the Kac algebra, to distinguish from the $\mathbb{Z}$-gradings on the ungraded Kac algebra constructed later.

Example 3.37. Let $\mathfrak{g}$ and $\mathfrak{G}$ be as in the example above. From [12, Exercise 11.4], a base for $\mathfrak{sl}(4,\mathbb{C})$ is $\{\epsilon_{12}, \epsilon_{23}, \epsilon_{34}\}$. From Example 3.1, the indices of these roots are 1, 1 and 0, respectively. Combined with the previous example, we see that $(\epsilon_{34}, 0)$, $(\epsilon_{12}, 1)$ and $(\epsilon_{23}, 1)$ can be made elements of a base of the root system of $\mathfrak{G}$. From [16, Theorem 3.28, Chapter X], we have that $-\epsilon_{12} - \epsilon_{23} - \epsilon_{34}$ is an element of the root system that is not a sum of two other roots. Since $-\epsilon_{12} - \epsilon_{23} - \epsilon_{34} = \epsilon_{41}$ and $e_{41} \in \mathfrak{g}_1$, it is clear that $(-\epsilon_{12} - \epsilon_{23} - \epsilon_{34}, 1)$ is a root that is not a sum of the other three given elements of the base. As the rank of $\mathfrak{sl}(4,\mathbb{C})$ is 3, by [16, Lemma 5.9(i), Chapter X], it follows that these 4 elements span the root system, so a base of the given root system of $\mathfrak{G}$ is

$$\tilde{\Pi} = \{(\epsilon_{12}, 1), (\epsilon_{23}, 1), (\epsilon_{34}, 0), (\epsilon_{41}, 1)\}.$$ •

As in the finite-dimensional case, from [16, Lemma 5.4, Chapter X], the Killing form restricted to $\mathfrak{h}$ is nondegenerate, so for any $\alpha \in \mathfrak{h}^\star$ there exists a unique $H_\alpha \in \mathfrak{h}$ such that $\kappa(h, H_\alpha) = \alpha(h)$ for all $h \in \mathfrak{h}$, and one can define an inner product on the roots by $\langle \alpha, \beta \rangle = \kappa(h_\alpha, h_\beta)$. From [16, Lemma 5.8, Chapter X], this gives an analogue to Serre's Theorem: letting $h_{\alpha_i} = 2\frac{H_{\alpha_i}}{\langle \alpha_i, \alpha_i \rangle}$, there exist $e_{\alpha_i} \in \mathfrak{G}_{\alpha_i}$ and $f_{\alpha_i} \in \mathfrak{G}_{\alpha_{-i}}$ such that the set $\{e_{\alpha_i}, f_{\alpha_i}, h_{\alpha_i} : 1 \le i \le n\}$ generates $\mathfrak{G}$, with relations given by:

- $[h_{\alpha_i}, h_{\alpha_j}] = 0$ for all $1 \le i, j \le n$,

- $[h_{\alpha_i}, e_{\alpha_j}] = C_{ji}e_{\alpha_j}$ and $[h_{\alpha_i}, f_{\alpha_j}] = -C_{ji}f_{\alpha_j}$ for all $1 \leq i, j \leq n$,
- $[e_{\alpha_i}, f_{\alpha_i}] = h_{\alpha_i}$ for all $1 \leq i \leq n$ and $[e_{\alpha_i}, f_{\alpha_j}] = 0$ if $i \neq j$,

where the C_{ji} are the entries of the Cartan matrix of the root system. Note the only difference between this and the relations of Serre's theorem is the lack of the fourth condition (that $(\operatorname{ad} e_{\alpha_i})^{1-C_{ji}}(e_{\alpha_j}) = 0$ and $(\operatorname{ad} f_{\alpha_i})^{1-C_{ji}}(f_{\alpha_j}) = 0$, where $1 \leq i, j \leq n$ and C_{ji} is the ijth entry of the Cartan matrix), since $\mathfrak{G}$ is infinite-dimensional. Finally, we stipulate that the set Φ is irreducible like in the finite-dimensional case, which by [16, Lemma 5.8(iii), Chapter X] occurs if $\mathfrak{g}$ does not decompose into a nontrivial direct sum of ideals fixed by σ. Such a σ is said to be **indecomposable**. Note that, since a simple Lie algebra has no nontrivial ideals, every automorphism on a simple Lie algebra is trivially indecomposable. Mirroring Theorem 1.13, if this is not the case then $\mathfrak{g}$ could be expressed as a direct sum of ideals such that σ restricted to each ideal is indecomposable, so the theory would apply to the summands. Analogously to Serre's theorem, [16, Lemma 5.10, Chapter X] shows that a covering algebra is uniquely determined by the Cartan matrix, allowing us to take the same route towards classification as in the ungraded case.

3.2.1 Kac Diagrams and the classification of covering algebras

Let the notation be the same as in the previous section. One can use the data from the Cartan matrix to construct a corresponding directed multigraph, analogous to the Dynkin diagram. This graph has n vertices, with $C_{ij}C_{ji}$ edges connecting the ith and jth vertices, and if $|C_{ij}| < |C_{ji}|$, the edge is directed towards the ith vertex, which is the shorter root. This graph is the **Kac diagram** of the Cartan matrix, denoted by $S(C)$. From the proof of [16, Lemma 5.11, Chapter X], a Kac diagram is not a Dynkin diagram since the number of vertices is one greater than the rank of the root system, but it is connected and every proper subgraph is the disconnected union of Dynkin diagrams, as given in Theorem 1.19. In particular, this shows that the Kac diagrams can be found by augmenting a connected Dynkin diagram by a single point.

The remainder of [16, Lemma 5.11, Chapter X] places conditions on the Kac diagram that allow for the full classification of them. Each vertex is labelled with a positive integer, which represents the coefficients of their linear dependence. That is, the labels a_i attached to the ith vertex, which corresponds to the ungraded root $\alpha_i \in \Pi$, are the smallest positive integers such that

$$\sum_{i=0}^{n} a_i\alpha_i = 0.$$

Observe that, as nonzero graded roots are linear combinations of simple roots with the same sign for coefficients, none of the a_i are negative. Furthermore, from [16, Lemma 5.9(i), Chapter X], every proper subset of Π is linearly independent, so none of the a_i can be zero, as expected. The list of Kac diagrams is given in, for example, [16, pp. 503]. From [30, Section 8.1], the covering algebra of $\mathfrak{g}$ is fixed under inner automorphisms. Hence, the Kac diagrams can be categorised by their **index**, which is defined to be the order of the grading automorphism modulo the subgroup of inner automorphisms, and is equal to the order of the induced graph automorphism on the Dynkin diagram of $\mathfrak{g}$. By [16, Lemma 3.29, Chapter X], this is equal to 1, 2 or 3, which can be seen from the symmetries of the Dynkin diagrams in Theorem 1.19.

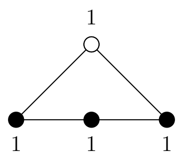
Remark 3.38. The Kac diagrams of index 1 correspond to the covering algebras with trivial grading. That is, the resulting Kac algebras are simply described as the direct sum of countably infinite copies of a simple complex Lie algebra $\mathfrak{g}$. If we consider these cases as trivial, then the only simple Lie algebras which have nontrivial indecomposable grading automorphisms are of index 2 or 3, which by their symmetries must be of type A_n, D_n and E_6. The Kac diagrams of these Kac algebras are shown in [30, Section 8.1].

From the construction of the covering algebras, it is clear that they are precisely the Kac algebras, represented by $\mathfrak{G}$, imbued with a $\mathbb{Z}$-grading and $\mathbb{C}[x, x^{-1}]$-structure such that $x\mathfrak{G}_p = \mathfrak{G}_{m+p}$ for all $p \in \mathbb{Z}$ and some $m \in \mathbb{N}$. These are described as follows, from [16, Example 2, pp. 505]. For a simple complex Lie algebra $\mathfrak{g}$, from Serre's Theorem one gets a Dynkin diagram for $\mathfrak{g}$ and a set of generators $\{e_i, f_i, h_i : 1 \leq i \leq n\}$, where n is the rank of $\mathfrak{g}$. Then an automorphism $\bar{\nu}$ of the Dynkin diagram induces an automorphism ν of $\mathfrak{g}$ by letting $\nu(e_i) = e_{\bar{\nu}(i)}$, $\nu(f_i) = f_{\bar{\nu}(i)}$ and $\nu(h_i) = h_{\bar{\nu}(i)}$. By [16, Lemma 5.12, Chapter X], every Kac diagram corresponds to a covering algebra of this form. The order of ν is the index k, equal to 1, 2 or 3.

Next, let $[s_0, ..., s_n]$ be a list of nonnegative integers which are not all zero. Recall that $\Delta^+ = \{\tilde{\alpha}_0, ..., \tilde{\alpha}_n\}$ is the set of positive graded roots. By [16, Lemma 5.9(ii), Chapter X], this set is independent over $\mathbb{Z}$, so any root is a unique linear combination of the form $\tilde{\alpha} = \sum_{i=0}^{n} k_i \tilde{\alpha}_i$ for $k_i \in \mathbb{Z}$. The **degree** of a graded root with respect to the array is then $\deg \tilde{\alpha} = \sum_{i=0}^{n} s_i k_i$. It is clear that $\deg \tilde{\alpha}_i = s_i$, so letting $\mathfrak{G}_j = \bigoplus_{\deg \tilde{\alpha}=j} \mathfrak{G}_{\tilde{\alpha}}$ for $j \in \mathbb{Z}$ gives a $\mathbb{Z}$-grading for $\mathfrak{G}$. Such a grading is of **type** $(s_0, ..., s_n)$. As [16, Lemma 5.14, Chapter X] shows, any $\mathbb{Z}$-grading of a covering algebra for a simple Lie algebra with grading induced from an indecomposable automorphism is isomorphic to a $\mathbb{Z}$-grading of type $(s_0, ..., s_n)$ on a covering algebra with grading induced from an automorphism induced by an automorphism of the Dynkin diagram. All that remains is to ensure such a construction preserves the $\mathbb{C}[x, x^{-1}]$-algebra structure. This occurs if $x\mathfrak{G}_p \subseteq \mathfrak{G}_{m+p}$ for all $p \in \{0, 1, ..., m-1\}$. From the definition of the degree, this occurs precisely when $m = k \sum_{i=0}^{n} a_i s_i$. The classification is summarised in the following theorem, from [16, Theorem 5.15, Chapter X].

Theorem 3.39. *Let $\mathfrak{g}$ be a complex simple Lie algebra. The covering algebras for $\mathfrak{g}$ with a grading induced from some indecomposable automorphism can be classified up to isomorphism by the Kac algebras of $\mathfrak{g}$ with a grading induced by an automorphism ν induced from an automorphism of the Dynkin diagram. The Kac diagram then has a $\mathbb{Z}$-grading of type $(s_0, ..., s_n)$ with the s_i chosen such that $m = k \sum_{i=0}^{n} a_i s_i$, where m and n are natural numbers. Furthermore, the Kac algebras are further classified by their Kac diagrams, which are enumerated in [16, Page 503]. Thus, every such Lie algebra is classified by a Kac diagram. Furthermore, two such Lie algebras are* *isomorphic only if their Kac diagrams are isomorphic (preserving the label of the indices).*

Example 3.40. For $\mathfrak{g}$ and $\mathfrak{G}$ given in the previous examples, the subset $\{\epsilon_{12}, \epsilon_{23}, \epsilon_{34}\}$ has the same diagram as the Dynkin diagram for $\mathfrak{sl}(4, \mathbb{C})$. Then $\langle \epsilon_{41}, \epsilon_{12} \rangle = \epsilon_{41}(h_{\epsilon_{12}}) = \epsilon_{41}(e_{11} - e_{22}) = [(e_{11})_4 - (e_{11})_1] - [(e_{22})_4 - (e_{22})_1] = -1$. Similarly, $\langle \epsilon_{41}, \epsilon_{23} \rangle = 0$ and $\langle \epsilon_{41}, \epsilon_{34} \rangle = -1$. Thus, there is also one line connecting the vertex for ϵ_{41} to ϵ_{12} and one line connecting ϵ_{41} to ϵ_{34}. This is the diagram $A_4^{(1)}$, shown below.



The graded algebra $\mathfrak{G}$ is isomorphic to the covering algebra of $\mathfrak{g}$ with the standard grading, via the isomorphism that maps the subspace

$$(\mathfrak{g}_{3k} \otimes 3k) \oplus (\mathfrak{g}_{3k+1} \otimes (3k+1)) \oplus (\mathfrak{g}_{3k+2} \otimes (3k+2))$$

isomorphically onto $(t^k, \mathfrak{g})$ for all $k \in \mathbb{Z}$. •

3.2.2 Vinberg's application to θ-groups

Here, we relate the previous work to the area of θ-groups by covering Vinberg's use of Kac diagrams to describe the associated θ-representation. Let $\mathfrak{G}$ be a Kac algebra. While $\mathfrak{G}$ is the covering algebra of a simple, finite-dimensional complex $\mathbb{Z}_m$-graded Lie algebra, denoted by $\mathfrak{g}$, for some $m \in \mathbb{N}$, we cover how aspects of the θ-representation can be described from the graded simple roots Π of $\mathfrak{G}$ and the associated root vectors. In what follows, we require the description of subspaces of $\mathfrak{G}$ associated with subsets of Π, as given below.

Definition 3.41 (Standard subalgebras)**.** *For a Kac algebra $\mathfrak{G}$ with a set Π of simple roots, a **standard subalgebra** of $\mathfrak{G}$ is the Lie subalgebra of $\mathfrak{G}$ generated by $\{e_\alpha, f_\alpha, h_\alpha : \alpha \in \Pi_0\}$ for a proper subset $\Pi_0 \subset \Pi$.*

A standard subalgebra is denoted by $\mathfrak{g}(\Pi_0)$ and satisfies the following properties:

1. As every proper subgraph of a Kac diagram is a Dynkin diagram, $\mathfrak{g}(\Pi_0)$ is a finite-dimensional, semisimple Lie algebra.

2. From 1) and Serre's theorem, it follows that the subset $\{h_\alpha : \alpha \in \Pi_0\}$ generates a Cartan subalgebra of $\mathfrak{g}(\Pi_0)$, denoted by $\mathfrak{t}(\Pi_0)$.

3. Parts 1) and 2) show that the roots of $\mathfrak{g}(\Pi_0)$ with respect to $\mathfrak{t}(\Pi_0)$ are associated with the roots of $\mathfrak{G}$ that are linear combinations of elements in Π_0.

Now, let $\mathfrak{G}$ be a Kac algebra with $\mathbb{Z}$-grading of type $(s_0, ..., s_n)$ as defined in Theorem 3.39. Then Π can be partitioned into subsets Π_0, Π_1 and Π_2, where

$$\Pi_0 = \{\alpha \in \Pi : \deg \alpha = 0\},$$
$$\Pi_1 = \{\alpha \in \Pi : \deg \alpha = 1\},$$
$$\Pi_2 = \{\alpha \in \Pi : \deg \alpha \geq 2\}.$$

As stated in Section 3.1, for a finite-dimensional complex semisimple $\mathbb{Z}_m$-graded Lie algebra $\mathfrak{g}$, the subalgebra $\mathfrak{g}_0 \cong \mathfrak{G}_0$ is reductive, so $\mathfrak{G}_0 = [\mathfrak{G}_0, \mathfrak{G}_0] \oplus Z(\mathfrak{G}_0)$. For the semisimple part, recall from the construction of positive roots of $\mathfrak{G}$ that every root of $\mathfrak{G}_0$ over $\mathfrak{h}$ is an extension of a root of $[\mathfrak{G}_0, \mathfrak{G}_0]$ over $\mathfrak{h} \cap [\mathfrak{G}_0, \mathfrak{G}_0]$. From the definition of the grading, $\mathfrak{G}_0 = \bigoplus_{\deg \alpha = 0} \mathfrak{G}_\alpha$. Thus, as $[\mathfrak{G}_0, \mathfrak{G}_0]$ is semisimple, this space is the standard subalgebra generated by roots with degree 0, so $[\mathfrak{G}_0, \mathfrak{G}_0] = \mathfrak{g}(\Pi_0)$. For the solvable part, from the root space decomposition we have that $Z(\mathfrak{G}_0) \subseteq \mathfrak{G}_{(\mathbf{0},0)} = \mathfrak{h}$. As $\mathfrak{h}$ is already abelian, the elements of $Z(\mathfrak{G}_0)$ are precisely those that commute with the nonzero root spaces in $\mathfrak{G}_0$. But then, for $z \in Z(\mathfrak{G}_0) \leq \mathfrak{h}$ and $e_\alpha \in \mathfrak{G}_\alpha$, we must have that $[z, e_\alpha] = \alpha(z) e_\alpha = 0$. As $\alpha \in \Pi_0$, this shows that $Z(\mathfrak{G}_0) = \{h \in \mathfrak{h} : \alpha(h) = 0 \ \alpha \in \Pi_0\} = C_{\mathfrak{h}}(\Pi_0)$. This is summarised in the following theorem.

Theorem 3.42. *Let $\mathfrak{G}$ be a Kac algebra with a grading of type $(s_0, ..., s_n)$, constructed from a Lie algebra $\mathfrak{g}$ with maximal torus $\mathfrak{h}$. Then $\mathfrak{G}_0 = \mathfrak{g}(\Pi_0) \oplus C_{\mathfrak{h}}(\Pi_0)$, where $\mathfrak{g}(\Pi_0)$ is the commutator of $\mathfrak{G}_0$ and $C_{\mathfrak{h}}(\Pi_0)$ is the centre of $\mathfrak{G}_0$.*

We now cover Vinberg's application to θ-groups, provided in [30, Section 3, Chapter 8]. Consider the complex, simple, $\mathbb{Z}_m$-graded Lie algebra $\mathfrak{g}$ for $m \in \mathbb{N}$. As noted at the start of the section, the θ-representation of $\mathfrak{g}$ is the same as that of its covering algebra $\mathfrak{G}$, justifying a shift in focus to the Kac algebra $\mathfrak{G}$, with a root grading indexed by the group A. Let $A_0 \leq A$ be generated by the subset $\Pi_0 \subset \Pi$. As A_0 is a subgroup of A, its cosets form a partition, so the subspace $\mathfrak{G}_1$ can be decomposed into $\mathfrak{G}_1 = \bigoplus_{\alpha \in \Pi_1} \mathfrak{G}_1(\alpha)$, where

$$\mathfrak{G}_1(\alpha) = \bigoplus_{\beta \in \alpha + A_0} \mathfrak{G}_\beta.$$

As proven in [30, Section 3, Chapter 8], the summands $\mathfrak{G}_1(\alpha)$ are precisely the irreducible components of $\mathfrak{G}_1$ with respect to the action by $\mathfrak{G}_0$ and the elements of Π_1 are the lowest-weight vectors of the representation. Furthermore, both the centraliser and commutator subalgebras of $\mathfrak{G}_0$ can be described in terms of the Kac diagram of $\mathfrak{G}$ due to Theorem 3.42. For the commutator, recall that it is equal to $\mathfrak{g}(\Pi_0)$. Therefore, from the prior discussion of standard subalgebras, it is defined by the Dynkin diagram formed by the subgraph corresponding to elements of Π_0. The centre can be found algebraically as it is $C_{\mathfrak{h}}(\Pi_0) = \{h \in \mathfrak{h} : \alpha(h) = 0 \ \forall \alpha \in \Pi_0\}$.

Example 3.43. Let $\mathfrak{g}$ and $\mathfrak{G}$ be as in the previous examples, and recall that the graded roots of $\mathfrak{G}$ has the previously found base

$$\tilde{\Pi} = \{(\epsilon_{12}, 1), (\epsilon_{23}, 1), (\epsilon_{34}, 0), (\epsilon_{41}, 1)\}.$$

We recall the decomposition $\mathfrak{G}_0 \cong \mathfrak{sl}(2,\mathbb{C}) \oplus T_2$ from Example 3.28, where T_2 is the abelian Lie algebra of dimension 2. We show how to find these summands by the analysis indicated above. The semisimple part is the standard subalgebra generated by all the roots in Π with zero index. We see that there is only one such root, so $\Pi_0 = \{\epsilon_{34}\}$, so the semisimple part is the Lie algebra which has a single point as its Dynkin diagram, which we recall is $\mathfrak{sl}(2,\mathbb{C})$. For the centre, we solve for $C_{\mathfrak{h}}(\Pi_0)$ where $\Pi_0 = \{\epsilon_{34}\}$ and $\mathfrak{h} = \mathfrak{n}(4,\mathbb{C}) \cap \mathfrak{g}$ is the subalgebra of diagonal matrices with zero trace. For any $h \in C_{\mathfrak{h}}(\Pi_0)$, we then have that $\epsilon_{34}(h) = h_{33} - h_{44} = 0$, so $h_{33} = h_{44}$. As $\mathfrak{h} \leq \mathfrak{g} = \mathfrak{sl}(4,\mathbb{C})$, we can express $C_{\mathfrak{h}}(\Pi_0)$ as

$$\left\{ \begin{bmatrix} -b-2c & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & c \end{bmatrix} : b, c \in \mathbb{C} \right\}.$$

This is a 2-dimensional abelian Lie algebra, which is then isomorphic to T_2, as desired. •

Chapter 4

Real θ-groups

4.1 Classification by real supports

Henceforth, we return to the convention that complex Lie algebras are denoted with superscript (e.g. $\mathfrak{g}^c$), and real Lie algebras are denoted without them. Let $\mathfrak{g}$ be a real semisimple $\mathbb{Z}_m$-graded Lie algebra with $m \in \mathbb{N}$. As stated in [21], the abstract Jordan decomposition of an element $x \in \mathfrak{g}_i$ is also contained in $\mathfrak{g}_i$. The JMV theorem (Theorem 3.10) also generalises to the real case, except for part (iv); that is, given a characteristic $h \in \mathfrak{g}_0$, there is not necessarily a unique nilpotent $e \in \mathfrak{g}_1$ up to conjugation by $N_{G_0}(h)$ forming an $\mathfrak{sl}_2$-triple in the real case. This lack of uniqueness complicates the classification of nilpotent elements compared to the complex case, as stated in [21, Remark 2.2].

As Van Le shows in [21, Lemma 2.5], the characterisation of nilpotent and semisimple elements in $\mathfrak{g}_1$ is the same as in the complex case. That is, $x \in \mathfrak{g}_1$ is nilpotent if and only if the closure of its orbit under the θ-representation contains 0, and it is semisimple if and only if the orbit is closed.

We now follow Van Le through [21] in the problem of classifying the nilpotent orbits of real $\mathbb{Z}_m$-graded semisimple Lie algebra, analogous to the problem in the previous section solved by Vinberg. First, we introduce notation that generalises the complex case. From [6, Section 4.1], for an ungraded real semisimple Lie algebra $\mathfrak{g}$ with Cartan involution θ and Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, a Cartan subspace is a maximal abelian subspace of $\mathfrak{p}$. Since the Cartan decomposition is a $\mathbb{Z}_2$-grading, Van Le uses the following natural generalisation, also used by Vinberg in [30, Section 3.1]: A **Cartan subspace** of a $\mathbb{Z}_m$-graded real semisimple Lie algebra $\mathfrak{g}$ is a maximal abelian subspace of $\mathfrak{g}_1$ consisting of semisimple elements. An element of a matrix Lie algebra is **$\mathbb{R}$-diagonalisable** if it is similar to a diagonal matrix via a real matrix, and a Lie algebra $\mathfrak{g}$ is $\mathbb{R}$-diagonalisable if $\operatorname{ad} x \in \mathfrak{gl}(\mathfrak{g})$ is $\mathbb{R}$-diagonalisable for all $x \in \mathfrak{g}$. Note that, with the ungraded definition, every Cartan subspace $C \subseteq \mathfrak{p}$ is $\mathbb{R}$-diagonalisable, since it is an abelian subspace, invariant under θ, and with semisimple elements, so [6, Section 2.3 and Chapter 5] show that C is contained in a CSA of $\mathfrak{g}$ that is invariant under θ, so $\operatorname{ad} x$ has real eigenvalues for all $x \in C$.

However, this is not necessarily the case with our graded definition of a Cartan subspace.

For a real $\mathbb{Z}_m$-graded semisimple Lie algebra and a nilpotent element $e \in \mathfrak{g}_1$, let $\mathfrak{h}$ be a maximal $\mathbb{R}$-diagonalisable Cartan subspace in $N_0(e)$. Such a choice is unique up to conjugacy by $\mathcal{N}_{G_0}(e)$, so by (ii) of the real JMV theorem, we can assert that $h \in \mathfrak{h}$. Observe that the definition of the accompanying torus $\mathfrak{h}$ of e for a complex Lie algebra (the maximal toral subalgebra in $N_0(e)$ containing h) is precisely the same as $\mathfrak{h}$ being a Cartan subspace in $N_{\mathfrak{g}_0^c}(e)$ containing h. Hence, the construction of the real support of e is analogous to that of Vinberg, and the complexification of this support is a complex support of e in $\mathfrak{g}^c$. Hence, given e and $\mathfrak{h}$, the character ϕ is defined by $[u, e] = \phi(u)e$ for all $u \in \mathfrak{h}$, and we define the sum $\mathfrak{g}(\mathfrak{h}, \phi) = \bigoplus_{k\in\mathbb{Z}} \mathfrak{g}_k(\mathfrak{h}, \phi)$, with

$$\mathfrak{g}_k(\mathfrak{h}, \phi) = \{x \in \mathfrak{g}_{k \bmod m} : \; [u, x] = k\phi(u)x \quad \forall u \in \mathfrak{h}\},$$

as in the complex case. The support is then the derived subalgebra of $\mathfrak{g}(\mathfrak{h}, \phi)$. Since $\mathfrak{g}(\mathfrak{h}, \phi)$ is again reductive, the support is a semisimple $\mathbb{Z}$-graded subalgebra of $\mathfrak{g}$. Analogously to the complex supports, [21, Lemma 4.1] shows that there is an injective map from the Ad_{G_0}-orbits of characteristics in $\mathfrak{g}$ to the $\mathrm{Ad}_{G_0^c}$-orbits of characteristics in $\mathfrak{g}^c$.

Applying [21, Remark 4.2] to this shows that the conjugacy classes of characteristics in $\mathfrak{g}$ acted on by G_0 can be characterised as follows. From the previous section, one can find all the (complex) supports $\mathfrak{s}^c$ in $\mathfrak{g}^c$. The real forms of all these supports may be found from the known classification of real forms of complex $\mathbb{Z}$-graded semisimple Lie algebras given by Djokovic in [10]. Lastly, it needs to be checked which of those real forms are embedded in $\mathfrak{g}$ in such a manner that the complexification is $\mathfrak{s}^c$. These are the real supports of $\mathfrak{g}$. As with the complex case, if the defining element of this support is $h_{\mathfrak{s}}$, the characteristic of the associated nilpotent element is $2h_{\mathfrak{s}}$.

The previous paragraph shows how to find the conjugacy classes of characteristics in $\mathfrak{g}$ under the action of G_0. Let $e \in \mathfrak{g}_1$ have $h \in \mathfrak{g}_0$ as its characteristic, so $\tilde{h} = \frac{h}{2}$ is the defining element of the support of e. Now let

$$\mathfrak{g}(\tilde{h}) = \bigoplus_{k\in\mathbb{Z}} \mathfrak{g}_k(\tilde{h}), \quad \mathfrak{g}_k(\tilde{h}) = \{x \in \mathfrak{g}_{k \bmod m} : \; [\tilde{h}, x] = kx\}.$$

It can be seen that the nilpotent orbits in $\mathfrak{g}_1$ can be considered as the Ad_{G_0}-orbits fully contained in $\mathrm{Ad}_{G_0^c}(e) \cap \mathfrak{g}_1$. For an $\mathfrak{sl}_2$-triple (f, h, e) in $\mathfrak{g}$, from [21, Theorem 4.3] it is then known that the open $\mathrm{Ad}_{\mathcal{C}_{G_0}(h)}$-orbits in $\mathfrak{g}_1(\frac{h}{2})$ are in one-to-one correspondence with the nilpotent orbits in $\mathfrak{g}_1$, via the bijection induced from the inclusion $\mathfrak{g}_1(\frac{h}{2}) \hookrightarrow \mathfrak{g}_1$. Hence, a characterisation of the open $\mathrm{Ad}_{\mathcal{C}_{G_0}(h)}$-orbits in $\mathfrak{g}_1(\frac{h}{2})$ classifies the nilpotent orbits in $\mathfrak{g}_1$. However, it is here that this approach becomes problematic, since the characterisation requires a highly technical algorithm from real algebraic geometry which seems to be implemented.

4.2 Classification by carrier algebras

This section represents the second method of attempting to classify the nilpotent orbits of a real θ-group. Let $\mathfrak{g}$ be a real semisimple $\mathbb{Z}_m$-graded Lie algebra with θ-group G_0. Recall that this grading extended to $\mathfrak{g}\otimes_{\mathbb{R}}\mathbb{C}$ is induced by an automorphism ψ. A Cartan involution τ on $\mathfrak{g}$ is **R-compatible** if $\tau \circ \psi = \psi \circ \tau$, so the compact real form $\mathfrak{u}$ of τ is invariant under τ. Equivalently τ is R-compatible if it reverses the grading, so $\tau(\mathfrak{g}_k) = \mathfrak{g}_{-k}$ for all $k \in \mathbb{Z}_m$. By [21, Theorem 3.3], an R-compatible Cartan involution always exists. Denote the corresponding Cartan decomposition as $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Also let $\mathfrak{h}$ be a CSA of $\mathfrak{g}$ that is invariant under τ. By [7, Section 5], the set $\mathfrak{h} \cap \mathfrak{k}$ is equal to the set of elements of $\mathfrak{h}$ such that $\mathrm{ad}_g(h)$ has only purely imaginary eigenvalues, and $\mathfrak{h} \cap \mathfrak{p}$ is equal to the set of elements of $\mathfrak{h}$ such that $\mathrm{ad}_g(h)$ has only real eigenvalues. Then the **noncompact dimension** of $\mathfrak{h}$ is the dimension of $\mathfrak{h} \cap \mathfrak{p}$. More generally, if $\mathfrak{a} \leq \mathfrak{g}$ is reductive and $\mathfrak{h}$ is a CSA of $\mathfrak{a}$, then the noncompact dimension of $\mathfrak{h}$ is the dimension of the subspace of $\mathfrak{h}$ such that $\mathrm{ad}_g(h)$ has only real eigenvalues. Furthermore, such an $\mathfrak{h}$ is **maximally noncompact** if its noncompact dimension is maximal, and this dimension is the **real rank** of $\mathfrak{a}$.

Similar to the approach in the previous section, the work in [7] creates a real analogue of the support of a nilpotent element. A **complex carrier algebra** of $\mathfrak{g}^c$ is a complete regular, locally flat, semisimple $\mathbb{Z}$-graded subalgebra of $\mathfrak{g}^c$, and a $\mathbb{Z}$-graded subalgebra of $\mathfrak{g}$ is a **real carrier algebra** if its complexification is a complex carrier algebra in $\mathfrak{g} \otimes \mathbb{C} = \mathfrak{g}^c$. Henceforth, we follow [7] in calling both of these structures "carrier algebras" unless there is some ambiguity. Applying Theorem 3.30 gives the following :

Theorem 4.1. *For a complex $\mathbb{Z}_m$-graded semisimple Lie algebra $\mathfrak{g}^c$, the definitions of a complex carrier algebra and a support of $\mathfrak{g}^c$ coincide. Thus, the real carrier algebras of a real $\mathbb{Z}_m$-graded semisimple Lie algebra $\mathfrak{g}$ can be found as the real forms of supports of $\mathfrak{g}^c$.*

Hence , the complex carrier algebras of a complex semisimple $\mathbb{Z}_m$-graded Lie algebra $\mathfrak{g}^c$ are precisely the supports of $\mathfrak{g}^c$, which can be classified via the methods of Chapter 3. These classify the real carrier algebras of the real analogue $\mathfrak{g}$, since $\mathfrak{s}^c$ is a complex carrier algebra of $\mathfrak{g} \otimes \mathbb{C}$ if and only if $\mathfrak{s} = \mathfrak{s}^c \cap \mathfrak{g}$ is a real carrier algebra of $\mathfrak{g}$. Furthermore, [7, Proposition 27] gives the conjugacy classes of real carrier algebras under G_0.

Similar to Vinberg's work in [29], there is another construction of carrier algebras that relates to nilpotent elements. Following [7, Definition 31], these are constructed as follows. As in the complex case, for a nilpotent $e \in \mathfrak{g}_1$, apply Theorem 3.10 to obtain an $\mathfrak{sl}_2$-triple (e, f, h), which spans a subalgebra $\mathfrak{a}$ of $\mathfrak{g}$. Let $\mathfrak{h}_z$ be a maximally noncompact CSA of $C_{\mathfrak{g}_0}(\mathfrak{a})$, which exists and is unique up to conjugacy by the adjoint group of $C_{\mathfrak{g}_0}(\mathfrak{a})$. Now consider the space $\mathfrak{t} = \mathrm{Span}_{\mathbb{R}}(h) \oplus \mathfrak{h}_z$. Since $\mathfrak{h}_z$ is a CSA, all the elements in $\mathfrak{h}_z$ commute, and since it is contained in the centraliser of $\mathfrak{h}$, they commute with h, so $\mathfrak{t}$ is abelian. Since $\mathfrak{h}_z$ is a CSA, its elements are semisimple, and, by Theorem 3.10, h is semisimple, so every element of $\mathfrak{t}$ is semisimple, so $\mathfrak{t}$ is toral. By [7, Lemma 30], $\mathfrak{t}$ is maximal in $N_0(\mathrm{Span}_{\mathbb{R}}(e))$. As with the previous method, define the character $\lambda : \mathfrak{t} \to \mathbb{R}$ such that $\lambda(t)e = [t, e]$ for all $t \in \mathfrak{t}$, and let $\mathfrak{g}(\mathfrak{t}, \lambda) = \bigoplus_{\mathbb{Z}} \mathfrak{g}_k(\mathfrak{t}, \lambda)$, where

$$\mathfrak{g}_k(\mathfrak{t}, \lambda) = \{x \in \mathfrak{g}_{k \bmod m} : \; [t, x] = k\lambda(t)x \quad \forall t \in \mathfrak{t}\}.$$

The derived subalgebra of $\mathfrak{g}(\mathfrak{t}, \lambda)$ is then a carrier algebra by [7, Proposition 32]; it is the carrier algebra of e and denoted by $\mathfrak{c} = \mathfrak{c}(e, h, \mathfrak{h}_z)$ since the given parameters are necessary and sufficient to define the carrier algebra. By Theorem 3.10, the element f and so the subalgebra $\mathfrak{a}$ is uniquely defined by the choice of e and h. By [29], the defining element of $\mathfrak{c}^c$, and hence $\mathfrak{c}$, is $h/2$, and $e \in \mathfrak{g}_1$ is in general position with respect to $\mathfrak{c}$; that is, $[\mathfrak{c}_0, e] = \mathfrak{c}_1$. Furthermore, [7, Proposition 34] shows that, given two nilpotent elements $e, e' \in \mathfrak{g}_1$ and carrier algebras $\mathfrak{c}(e, h, \mathfrak{h}_z)$ and $\mathfrak{c}(e', h', \mathfrak{h}'_z)$, if e and e' are in the same orbit under G_0, then $\mathfrak{c}(e, h, \mathfrak{h}_z)$ and $\mathfrak{c}(e', h', \mathfrak{h}'_z)$ are in the same conjugacy class under G_0. Hence, there exists a well-defined map ψ from the nilpotent orbits in $\mathfrak{g}_1$ to the conjugacy classes of carrier algebras.

Recall that a similar connection was established in Chapter 3. In Corollary 3.27, it was stated that such a map is bijective, as every nilpotent orbit corresponds to a unique support up to conjugacy. As in the previous section, this no longer holds in the real case; a conjugacy class of carrier algebras may correspond to multiple nilpotent orbits, one nilpotent orbit or even none, in which case the carrier algebra in the class cannot be constructed by the method above. Thus, given a carrier algebra $\mathfrak{c}$, the problem turns to finding which nilpotent elements in $\mathfrak{c}_1$ produce $\mathfrak{c}$ as its carrier algebra, and finding necessary and sufficient conditions for two nilpotent elements to be in the same orbit. Completing these amounts to finding unique representatives for each nilpotent orbit that intersect $\mathfrak{c}$.

4.2.1 Limitations and heuristics approaches

For a real semisimple $\mathbb{Z}_m$-graded Lie algebra, $\mathfrak{g}$, let $\mathfrak{c}$ be a carrier algebra in $\mathfrak{g}$. Let $e \in \mathfrak{c}_1$ be a point in general position with respect to $\mathfrak{c}$. From the previous section, we can apply Theorem 3.10 to $\mathfrak{c}$ to get an $\mathfrak{sl}_2$-triple that spans a space $\mathfrak{a}$ isomorphic to $\mathfrak{sl}(2, \mathbb{R})$. Since $\mathfrak{a}$ has a $\mathbb{Z}$-grading induced from $\mathfrak{c}$ and is semisimple, [7, Lemma 23] shows that $C_{\mathfrak{g}_0}(\mathfrak{a})$ is reductive, so let $C_{\mathfrak{g}_0}(\mathfrak{a}) = D \oplus T$ be the decomposition into its derived subalgebra and centre. Hence, from [7, Proposition 35(c) - (d)], we have the following theorem for carrier algebras:

Theorem 4.2. *For a real $\mathbb{Z}_m$-graded semisimple Lie algebra $\mathfrak{g}$:*

1. *Letting $\mathfrak{h}$ be a maximally noncompact CSA of $C_{\mathfrak{g}_0}(\mathfrak{c})$, the carrier algebra $\mathfrak{c}$ is a carrier algebra of a nonzero, nilpotent element $e \in \mathfrak{g}_1$ if and only if $\mathfrak{h} \cap D$ is a maximally noncompact CSA of D.*
2. *For nonzero nilpotent elements $e, e' \in \mathfrak{c}_1$ in general position, if e and e' are conjugate under G_0, then $\mathfrak{c}$ is a carrier algebra of e if and only if it is a carrier algebra of e'.*

Hence, the theorem solves the first goal of associating each carrier algebra with a point in general position in $\mathfrak{c}$, which is a nilpotent element in $\mathfrak{c}_1$. Furthermore, it advances the second goal if, given a carrier algebra $\mathfrak{c}$ in $\mathfrak{g}$, one can identify all the elements in $\mathfrak{c}_1$ in general position.

This has not yet been solved in general. However, [7, Section 10.1] notes that, since an element is in general position if and only if $[\mathfrak{c}_0, e] = \mathfrak{c}_1$, the map $\mathrm{ad}_{\mathfrak{g}}(e)|_{\mathfrak{c}_0}$, is a bijection from $\mathfrak{c}_0$ to $\mathfrak{c}_1$. Recall that $\dim \mathfrak{c}_0 = \dim \mathfrak{c}_1 = r$ for some $r \in \mathbb{N}$ since $\mathfrak{c}$ is locally flat. Since $\mathrm{ad}_{\mathfrak{g}}(e)$ is linear, choosing bases $\{v_1, ..., v_r\}$ and $\{w_1, ..., w_r\}$ for $\mathfrak{c}_0$ and $\mathfrak{c}_1$ respectively, the invertible map $\mathrm{ad}_{\mathfrak{g}}(e)|_{\mathfrak{c}_0}$ corresponds to an invertible matrix M. Furthermore, letting $e = \sum_{j=1}^{r} c_j w_j$, the linearity of the adjoint homomorphism shows that $\mathrm{ad}_{\mathfrak{g}}(e)|_{\mathfrak{c}_0} = \sum_{j=1}^{r} c_j \,\mathrm{ad}_{\mathfrak{g}}(w_j)|_{\mathfrak{c}_0}$ so, in particular, the matrix M can be expressed as a linear function $M(c_1, ..., c_r)$. Thus, the invertibility of M is equivalent to the condition that $\det(M(c_1, ...c_r)) = f(c_1, ..., c_r) \neq 0$, where f is a polynomial function. While such a problem does not have a general solution, situations where some roots or factorisations are known are sufficient to give the general solution.

Given the set of elements in general position with respect to $\mathfrak{c}$, the resolution of the third task has two components: reducing this set to one that is amenable to computation, and checking whether elements are in the same orbit or in different orbits, ideally until one has unique representatives for each orbit. For the first part, one can represent elements of $\mathfrak{c}_1$ as linear combinations, so $z = \sum_{j=1}^{r} c_j w_j$. Taking elements from the image of the θ-representation $\rho(G_0)$ and having them act on elements of $\mathfrak{c}_1$ can give restrictions on some c_j. For example, if $x \in \mathfrak{c}_0$ is nilpotent, then $\exp(\mathrm{ad}_{\mathfrak{g}}(x)) \in G_0$. Using [20, Theorem 0.23], we have $\rho(\exp(\mathrm{ad}_{\mathfrak{g}}(x))) = \exp(d\rho(x))$. Again using [20, Theorem 8.2], $d\rho(x)$ is equal to the restriction of $\mathrm{ad}_{\mathfrak{g}}(x)$ to $\mathfrak{g}_1$. Thus, $\rho(\exp(\mathrm{ad}_{\mathfrak{g}}(x)))$ fixes $\mathfrak{c}_1$ and its restriction to $\mathfrak{c}_1$ is equal to $\exp(y)$, where y is the restriction of $\mathrm{ad}_{\mathfrak{g}} x$ to $\mathfrak{c}_1$. Since applying this to a point in general position gives another point in the same orbit, this can be used to eliminate or fix some values of c_j, as stated in [7, Section 10.2.1]. Note that resolving this problem also progresses in finding if points in general position are in the same orbit. Thus, the last task is to find if points are in different orbits. Again, this has not been solved in general. The approach given in [7, Section 10.3] is a heuristic one based on the use of G_0-invariants.

For example, if $e_1, e_2 \in \mathfrak{g}_1$ are in the same orbit, they are conjugate under G_0, so their centralisers under $\mathfrak{g}$ or $\mathfrak{g}_0$ are conjugate as well. Thus, if the centralisers contain different conjugacy classes, under either G_0 or G, then e_1 and e_2 cannot be G_0-conjugate. Letting $\mathfrak{a}_i$ be the subalgebra generated by the $\mathfrak{sl}_2$-triple $\{e_i, f_i, h_i\}$, [7, Lemma 29] gives that, since e_1 and e_2 are nilpotent, they are G_0-conjugate if and only if $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are conjugate. Using the same reasoning as before, these are conjugate if and only their centralisers in $\mathfrak{g}$ are conjugate, giving another test to distinguish the orbits.

4.3 Classification using Galois cohomology

4.3.1 Describing real orbits from complex ones

From Chapter 2, every real semisimple $\mathbb{Z}_m$-graded Lie algebra $\mathfrak{g}$ is the real form of a complex semisimple graded Lie algebra, such as the complexification $\mathfrak{g}^{\mathrm{c}} = \mathfrak{g} \otimes \mathbb{C}$. Furthermore, from Chapter 3, the nilpotent orbits of a θ-group G_0^{c} for $\mathfrak{g}_1^{\mathrm{c}}$ have been classified and, as $\mathfrak{g}$ is a real Lie subalgebra of $\mathfrak{g}^{\mathrm{c}}$, the θ-group G_0 of $\mathfrak{g}$ is a subgroup of G_0^{c}. Hence, it is natural to ask if one can find the nilpotent orbits in the real case by analysing and refining the complex case. In this section, we cover a situation from [11] when this can be done, using Galois cohomology. In doing so, we also illustrate one of the key applications of θ-groups, which is that some key group actions are described by them; in this case, the action of $\mathrm{GL}_8(\mathbb{R})$ on the algebra of trivectors $\bigwedge^3(\mathbb{R}^8)$.

Following Djokovic in [11], for there to be a θ-representation of the Lie group $\mathrm{GL}(8,\mathbb{R})$ acting on $\mathfrak{g}_1$, we need a semisimple $\mathbb{Z}_m$-graded Lie algebra with $\mathfrak{g}_0 \cong \mathfrak{gl}(8,\mathbb{R})$ and vector space $\mathfrak{g}_1 \cong \bigwedge^3(\mathbb{R}^8)$. In [11, Chapter 2 - 4], Djokovic gives the construction of the desired Lie algebra via a $\mathbb{Z}$-grading on the simple Lie algebra E_8. Furthermore, [11, Proposition 3.2] shows that there exists a homomorphism $\phi : \mathrm{GL}_8(\mathbb{R}) \to H_0$, where H_0 is the group of automorphisms on $\mathfrak{g}$ that preserve the grading. Since $G_0 = \mathrm{GL}_8(\mathbb{R})$, we see that ϕ turns $\mathrm{GL}(8,\mathbb{R})$ into a θ-group acting on $\mathfrak{g}_1 = \bigwedge^3(\mathbb{R}^8)$, as desired.

Recall from Chapter 3 that an element $h \in \mathfrak{g}_0$ is a characteristic (of a nilpotent $e \in \mathfrak{g}_1$) if $\{f, h, e\}$ forms an $\mathfrak{sl}_2$-triple for some $f \in \mathfrak{g}_{-1}$. From the theory of $\mathfrak{sl}(2,F)$-representations shown in [17, Section 7.2], for a characteristic $h \in \mathfrak{g}_0$, the eigenvalues of $\operatorname{ad} h$ are integers. From [11, Chapter 6], this can be generalised to any underlying field of characteristic 0. Hence, for a characteristic $h \in \mathfrak{g}_0$ and $j \in \mathbb{Z}$, let $\mathfrak{g}(j;h)$ be the eigenspace of $\operatorname{ad}_{\mathfrak{g}} h$ with eigenvalue j, so

$$\mathfrak{g} = \bigoplus_{j\in\mathbb{Z}} \mathfrak{g}(j;h), \quad \mathfrak{g}(j;h) = \{x \in \mathfrak{g} : [h,x] = jx\}.$$

Furthermore, let $\mathfrak{g}_i(j;h) = \mathfrak{g}_i \cap \mathfrak{g}(j;h)$. Observe that, since $[h,e] = 2e$, it follows that $e \in \mathfrak{g}_1(2;h)$; more specifically, an element $x \in \mathfrak{g}_1(2;h)$ is **generic** if $\operatorname{ad} x$ maps the subspace $\mathfrak{g}_0(0;h)$ onto $\mathfrak{g}_1(2;h)$. Lastly, let $\mathfrak{h} \leq \mathfrak{g}_0$ be the Lie subalgebra of diagonal matrices; that is, elements of $\mathfrak{gl}(\mathbb{R}^8)$ that have the natural basis as eigenvectors. This is a Cartan subalgebra of $\mathfrak{g}$, hence also a Cartan subalgebra of $\mathfrak{g}_0$. By the following theorem, given in [11, Lemma 6.5], we can get a one-to-one correspondence between the real and complex case.

Theorem 4.3. *Let $\mathfrak{g}$ be the given real Lie algebra, with the $\mathrm{GL}_8(\mathbb{R})$-module structure. Then the canonical embedding map of $\mathfrak{g}$ into $\mathfrak{g} \otimes \mathbb{C} = \mathfrak{g}^{\mathrm{c}}$ induces a bijection between the $\mathrm{GL}_8(\mathbb{R})$-orbits of characteristics in $\mathfrak{g}$ and the $\mathrm{GL}_8(\mathbb{C})$-orbits of characteristics in $\mathfrak{g}^{\mathrm{c}}$.*

Proof. For this proof, a superscript 'c' represents the image of a real vector space under complexification. From [11, Section 5], $\mathfrak{h}^{\mathrm{c}}$ is a CSA of $\mathfrak{g}^{\mathrm{c}}$. Let $h \in \mathfrak{g}_0^{\mathrm{c}}$ be a characteristic in $\mathfrak{g}^c$. Since in the complex case, CSAs are unique up to conjugacy, there exists a $u \in \mathrm{GL}_8(\mathbb{C})$ such that $u \cdot h \in \mathfrak{h}^c$. For all roots μ of $\mathfrak{h}^c$, observe that $\mu(u \cdot h) \in \mathbb{Z}$ by $\mathfrak{sl}_2$-theory, so $u \cdot h \in \mathfrak{h}$. Hence, $u \cdot h \in \mathfrak{g}_0$, so it is a characteristic in $\mathfrak{g}$ as well, by [11, Lemma 6.4]. This shows that the $\mathrm{GL}_8(\mathbb{C})$-orbit of a characteristic in $\mathfrak{g}^c$ contains a characteristic in $\mathfrak{g}$.

Now let h_1, h_2 be characteristics of $\mathfrak{g}$ in the same complex orbit i.e. there exists some $u \in \mathrm{GL}_8(\mathbb{C})$ such that $u \cdot h_1 = h_2$. After acting with other elements of $\mathrm{GL}_8(\mathbb{C})$, we can assume without loss of generality that $h_1, h_2 \in \mathfrak{h}$. Recall that $\mathrm{GL}_8(\mathbb{C})$ has $\mathfrak{gl}(8, \mathbb{C}) = \mathfrak{g}_0^c$ as its Lie algebra. Then the action of $u \cdot h_1 = h_2$ can be lifted to a relation $w \cdot h_1 = h_2$ for an element w of the Weyl group of the root system for $\mathfrak{h}^c$ in $\mathfrak{g}_0^c$. Since the root system for a complex Lie algebra inhabits a real inner-product space, and the Weyl group is associated with permutations of the roots, this Weyl group can be embedded in $\mathrm{GL}_8(\mathbb{R})$. Hence, there exists $\tilde{w} \in \mathrm{GL}_8(\mathbb{R})$ such that $\tilde{w} \cdot h_1 = h_2$, as desired. □

As in the beginning of the section, let $\mathfrak{g}$ be the real semisimple Lie algebra that is acted on by $\mathrm{GL}(8, \mathbb{R})$. Then $\mathfrak{g}$ has complexification $\mathfrak{g}^c$, which is acted upon by $\mathrm{GL}(8, \mathbb{C})$, and we see that $\mathrm{GL}(8, \mathbb{R}) \leq \mathrm{GL}(8, \mathbb{C})$ as groups. If we consider $x \in \mathfrak{g}$ to be embedded in $\mathfrak{g}^c$ and with a known $\mathrm{GL}(8, \mathbb{C})$-orbit, it is natural to ask how this informs one to the $\mathrm{GL}(8, \mathbb{R})$-orbits in $\mathfrak{g}$, since we have seen complex orbits to be generally easier to classify than real ones. Clearly, a real orbit is always contained in a complex one, although it is possible for a complex orbit to not contain a real orbit. Since $\mathfrak{g}_1 < \mathfrak{g}_1^c$, we further impose that the real orbits must be contained in $\mathfrak{g}_1$ to study θ-groups. Let $\mathcal{C}_{G_0}(h)$ be the centraliser of h in $\mathrm{GL}_8(\mathbb{R})$ for a characteristic $h \in \mathfrak{g}_0$ of some $x \in \mathfrak{g}_1$. Similar to the orbit-stabiliser theorem and from the previous theorem, [11, Theorem 6.6] shows that such a classification is possible.

Theorem 4.4. *Let $\mathfrak{g}$ be the given real Lie algebra, with the $GL_8(\mathbb{R})$-module structure. Then the inclusion map $\mathfrak{g}_1(2; h) \to \mathfrak{g}_1$ induces a bijection between the generic orbits in $\mathfrak{g}_1(2; h)$ under $\mathcal{C}_{G_0}(h)$ and the $GL_8(\mathbb{R})$-orbits in $(GL_8(\mathbb{C}) \cdot x) \cap \mathfrak{g}_1$.*

4.3.2 Characterisation using Galois cohomology

Following [11, Chapter 8], the extension $\mathbb{C}/\mathbb{R}$ is a finite Galois extension with Galois group $\Gamma = \{1, \gamma\}$, where γ is complex conjugation. The construction of the Galois cohomology groups is taken from [27, Section 4]. Let Γ act on matrix groups via their entries and, for a matrix space A, let A^Γ be the subset of fixed points under Γ. Now, recall that a nilpotent $e \in \mathfrak{g}_1$ is contained in a homogeneous $\mathfrak{sl}_2$-triple $\{f, h, e\}$ which spans a subspace $\mathfrak{p}$. Letting $B = \mathrm{GL}(8, \mathbb{C})$ and $C = \{g(\mathfrak{p}) : g \in B\}$, we see that the action of B on C corresponds to the previous discussion on the action of the group on characteristics. Reducing to the real case to find the real orbits within the complex orbit amounts to finding B^Γ orbits in C^Γ. This leads to the following definitions, from [27, Definition 4.1].

Definition 4.5 (Galois cohomology)**.** *Let M be some complex matrix group acted upon by Γ on its entries. A **1-cocycle** is a map $\alpha : \Gamma \to M$ such that, for all*

$x, y \in \Gamma$, we have

$$\alpha(xy) = \alpha(x)[x \cdot \alpha(y)],$$

so if $\alpha(\gamma) = A$, then $A\bar{A} = \alpha(1) = 1$, and any such A induces a 1-cocycle α_A. The set of all 1-cocycles is denoted $Z^1(\Gamma, M)$. Furthermore, two 1-cocycles α, β are ***equivalent*** *(or* ***cohomologous****), if $\alpha = \beta^U$ for some $U \in M$, where*

$$\beta^U(w) = U^{-1}\beta(w)[w \cdot U].$$

This induces the set $H^1(\Gamma, M) = Z^1(\Gamma, M)/\sim$, where $\alpha \sim \beta$ if they are cohomologous.

Relating to the previous theorem, let $A(\mathbb{R}) = \mathcal{C}(h, \mathbb{R})$ be the centraliser of h in $\mathrm{GL}_8(\mathbb{R})$ and $X = \mathfrak{g}_1(2; h)$. The Galois group Γ can act on $A(\mathbb{R})$ and X since they are defined over $\mathbb{R}$. Continuing from the previous theorem, for some $x \in X$, let $E_x(\mathbb{C}/\mathbb{R})$ be the set of $A(\mathbb{R})$-orbits in $(A(\mathbb{C}) \cdot x) \cap X$. Fixing some $y \in (A(\mathbb{C}) \cdot x) \cap X$ and an $a \in A(\mathbb{C})$ such that $a \cdot x = y$, consider the map $p : \Gamma \to A(\mathbb{C}), s \mapsto a^{-1}s(a)$. As $y \in X$, which is a real vector space, it is fixed by s, so $s(y) = y = s(a \cdot x) = s(a) \cdot x$. Therefore, $p_s(x) = a^{-1}y = x$, so p_s is a stabiliser of x in $A(\mathbb{C})$ and so p can be refined to the map $p : \Gamma \to (A(\mathbb{C}))_x$.

As [11, Chapter 8] shows, the map p is a 1-cocycle, and different choices of $a \in A(\mathbb{C})$ only produce cohomologous elements, so the map $\theta : y \mapsto [p]$, where $[p]$ is the cohomology class of p in $H^1(\Gamma, A(\mathbb{C})_x)$, is well-defined. Furthermore, replacing y with an element of its $A(\mathbb{R})$-orbit also produces a cohomologous 1-cocycle. Hence, the map factors to a well-defined map $\theta : E_x(\mathbb{C}/\mathbb{R}) \to H^1(\Gamma, A(\mathbb{C})_x)$. From [11, Chapter 7], for each of the 23 complex orbits, $\mathcal{C}(h, \mathbb{C}) = A(\mathbb{C})$ is a direct product of general linear groups. A classical result of homology theory shows that the first cohomology group $H^1(\Gamma, A(\mathbb{C}))$ is trivial, so [11, Proposition 8.1] shows that θ is a bijection. This along with the previous theorem shows that the real orbits contained in a complex orbit are in one-to-one correspondence with the first cohomology group that appears as codomain of θ, so computing the latter allows one to classify the former. However, finding cohomology groups is a non-trivial task, which is the issue with this approach when trying to apply to other Lie algebras with such a module structure. Another possible issue is to find all complex orbits which have a real representative.

4.4 Invariant classification for more general representations

Recall that a θ-group is the Lie group G_0 of the subalgebra $\mathfrak{g}_0$ in $\mathfrak{g}$, where $\mathfrak{g}$ is a (real or complex) $\mathbb{Z}_m$-graded semisimple finite-dimensional Lie algebra. In particular, we consider the adjoint action of this group on the subspace $\mathfrak{g}_i$ for $i \neq 0$. From Lemma 3.5, we may let the action be on $\mathfrak{g}_1$. While the previous methods attempt to give general solutions to the classification of nilpotent orbits, in specific cases one can attempt a rougher approach that uses invariants to filter the elements of $\mathfrak{g}_1$. If the invariants are sufficiently strong, the necessary separation of elements into different orbits will become sufficient and the orbits will have then be classified. This is particularly useful in applications, such as in the theory of black holes, shown in [8] and [3].

In this section, we summarise an alternative means of classifying the nilpotent orbits in the complex case, described in [14, Section 3], then the invariants that can be employed in the simpler cases, which uses the analysis for finding orbits in the ungraded case shown in [4]. We then cover how (non-unique) orbit representatives are found in the real case, and how conjugate representatives are eliminated, including via the use of so-called α-, β- and γ-labels, along with tensor classifiers introduced in [8], to complete their classification in specific cases.

4.4.1 The ungraded case

Here, we review the nilpotent orbits of a semisimple Lie algebra under the action of its adjoint group. Following Collingwood and McGovern in [4, Chapter 3, 9], we may start with the complex case, then move to the real case. Hence, let $\mathfrak{g}^c$ be a complex semisimple Lie algebra with adjoint group G^c. Ultimately, the nilpotent orbits are placed in one-to-one correspondence with certain weighted Dynkin diagrams, from [4, Section 5.3]. These labels for the vertices are the **α-labels** that define the orbit.

Let $\mathcal{N}$ be the set of (nonzero) nilpotent orbits in $\mathfrak{g}^c$, and let $e \in \mathfrak{g}^c$ be nilpotent and defining its orbit $\mathcal{O}_e$. Applying the JMV theorem, e is contained in an $\mathfrak{sl}_2$-triple $\{f, h, e\}$. For $\rho \in G^c$, it is clear that $\{\rho(f), \rho(h), \rho(e)\}$ is also an $\mathfrak{sl}_2$-triple, so we may define this natural action of G^c on the set of $\mathfrak{sl}_2$-triples. Letting $\mathcal{A}_{\text{triple}}$ be the set of orbits of $\mathfrak{sl}_2$-triples under the given action of G^c, we have the following lemma, from [4, Theorem 3.2.10].

Lemma 4.6. *The map $\Omega : \mathcal{A}_{triple} \to \mathcal{N}$ defined by $\Omega(\{f, h, e\}) = \mathcal{O}_e$ is a bijection, giving a one-to-one correspondence between the nilpotent orbits and orbits of $\mathfrak{sl}_2$-triples.*

Proof. Clearly, if $\{f, h, e\}$ and $\{\tilde{f}, \tilde{h}, \tilde{e}\}$ are in the same orbit, then $\tilde{e} = \rho(e)$ for some $\rho \in G^c$, so e and $\tilde{e}$ are in the same orbit and Ω is well-defined. Surjectivity of Ω is clear from the Jacobson-Morozov theorem, so all that is left is injectivity. We can see this as a specific application of parts b) and c) of Theorem 3.10, or note that this follows from a theorem of Konstant shown in [4, Theorem 3.4.10]. □

The next step is to associate each orbit of an $\mathfrak{sl}_2$-triple with certain semisimple orbits via the map $\mathcal{Y}(\{f, h, e\}) = \mathcal{O}_h$. The image of $\mathcal{Y}$ is the set $\mathcal{S}$ of **distinguished** semisimple orbits. As before, the map $\mathcal{Y}$ is well-defined, and it maps onto $\mathcal{S}$ by definition. Finally, a well-known theorem of Mal'cev (or part (d) and (c) of Theorem 3.10) gives that two $\mathfrak{sl}_2$-triples with the same semisimple element h are conjugate by an element of the adjoint group, as desired.

Finally, we may classify the distinguished semisimple orbits. Let $\mathcal{O}_h$ be such an orbit. Since the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}^c$ is unique up to conjugacy by the adjoint group, we assume without loss of generality that $h \in \mathfrak{h}$. Let B be a base for the root system. As discussed in Section 1.2.2, the root system spans $\mathfrak{h}^\star$, so the element h is defined by the values $\alpha(h)$ for $\alpha \in B$. From the representation theory of $\mathfrak{sl}(2, \mathbb{C})$-modules mentioned in Section 1.2.2, it is known that $\alpha(h) \in \mathbb{Z}$. As mentioned after [4, Remark 2.2.4], one can conjugate h with an element of the adjoint group such that h is in the fundamental Weyl chamber, so that $\alpha(h)$ is non-negative. Finally, the reasoning of [4, Section 3.5] shows that $\alpha(h) - 2 \leq 0$, so we conclude that $\alpha(h) \in \{0, 1, 2\}$. In particular, this shows that the set of nilpotent orbits is finite. However, not all Dynkin diagrams weighted in this manner correspond to a distinguished semisimple orbit. Collingwood and McGovern approach the case of each simple complex Lie algebra systematically, since these have all been classified (as given in Theorem 1.19) and the semisimple Lie algebras are Lie direct sums of simple ones. The exceptional Lie algebras require the direct classification of weighted Dynkin diagrams, but the orbits of the classical ones can be described using partitions. Here, we cover the simplest case.

Example 4.7. Let $\mathfrak{g}^c = \mathfrak{sl}(n, \mathbb{C})$. Recall that this is a complex simple Lie algebra with a standard Cartan subalgebra consisting of its diagonal matrices, whereupon the roots take the from $\epsilon_i - \epsilon_j$ for $i \neq j$, where $\epsilon_i \in \mathfrak{h}^\star$ acts by $\epsilon_i(h) = h_{ii}$. The natural base for this root system is then $\epsilon_i - \epsilon_{i+1}$ for $1 \leq i \leq n-1$, and the resulting Dynkin diagram is of type A_{n-1}.

For an $n \in \mathbb{N}$, a **partition** of n is a list of natural numbers $[a_1, a_2, ..., a_k]$ such that $a_i \geq a_{i+1}$ for $1 \leq i < k$, and $\sum_{i=1}^k a_i = n$. We now construct representatives of the orbits using matrices in Jordan normal form defined in Section 3.1. It is clear that a nilpotent element of $\mathfrak{g}^c$ in Jordan normal form is strictly upper triangular, so the Jordan blocks will simply be matrices with 1 on the superdiagonal and 0 elsewhere, called an **elementary** Jordan block. Now, for any partition $P = [a_1, a_2, ..., a_k]$, let J_i be an $a_i \times a_i$ elementary Jordan block. Clearly, the resultant matrix is $n \times n$ and nilpotent, denoted by $M_{[a_1, a_2, ..., a_k]}$. It is known that the Jordan normal form is unique up to reordering of the Jordan blocks. Since the partition fixes a descending order on the size of the blocks, and elementary Jordan blocks of the same size are identical, the similarity class of a matrix contains a unique matrix in Jordan normal form, so the two orbits $\mathcal{O}_{P_1}$ and $\mathcal{O}_{P_2}$ are disjoint. Lastly, let $X \in \mathfrak{g}^c$ have Jordan normal form $J = A^{-1}XA$ for some $A \in \mathrm{GL}(n, \mathbb{C})$. Letting $\tilde{A} = A \cdot \mathrm{sgn}(\det(A))/\sqrt[n]{|\det(A)|})^{-1}$, where scalar multiplication is computed entrywise, it is clear from direct computation that $J = \tilde{A}^{-1}X\tilde{A}$ as well. From elementary linear algebra, for an $n \times n$ matrix and any $x \in \mathbb{C}$, we have $\det(x \cdot A) = x^n \det(A)$, so $\det(\tilde{A}) = \det(A)^n / \det(A)^n = 1$. Therefore, every element of $\mathfrak{g}^c$ is conjugate to a matrix in Jordan normal form via an element of $\mathrm{SL}(n, \mathbb{C})$. Since $\mathrm{SL}(n, \mathbb{C})$ is a Lie group with Lie algebra $\mathfrak{g}^c$, this shows that every nilpotent orbit in $\mathfrak{g}^c$ under the adjoint action contains a matrix of the form M_P for a partition P of n. Hence, the partitions of n are in one-to-one correspondence with the nilpotent orbits $\mathfrak{g}^c$ under the map sending $[a_1, a_2, ..., a_k]$ to $M_{[a_1, a_2, ..., a_k]}$. Note that a partition of all 1's maps to the zero matrix, so it corresponds to the trivial orbit. •

A similar process occurs for the real case, which may be considered by means of the Konstant-Sekiguchi theorem, shown in [4, Theorem 9.5.1]. Recall that a real simple Lie algebra $\mathfrak{g}$ has a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and Cartan involution τ. In particular, [4, Theorem 9.4.1] shows that every $\mathfrak{sl}_2$-triple is conjugate to an $\mathfrak{sl}_2$-triple of the form $\{f,h,e\}$ such that $\tau(f) = -e$, $\tau(e) = -f$ and $\tau(h) = -h$. Such a triple is called a **Cayley triple**. Now, associated with such a Cayley triple is an auxiliary $\mathfrak{sl}_2$-triple in $\mathfrak{g}\otimes\mathbb{C}$ that consists of eigenvectors for (the $\mathbb{C}$-linear extension of) τ. This is the **Cayley transform**; given a Cayley triple $\{f,h,e\}$, its Cayley transform is $\{\tilde{f},\tilde{h},\tilde{e}\} = \{\frac{1}{2}(e+f-\imath h), \imath(e-f), \frac{1}{2}(e+f+\imath h)\}$. More specifically, we see that $\tilde{h}$ is fixed by τ, whereas $\tilde{e}$ and $\tilde{f}$ are inverted, so $\tilde{h}\in\mathfrak{k}^{\mathrm{c}}$ and $\tilde{e},\tilde{f}\in\mathfrak{p}^{\mathrm{c}}$. Now, since $\mathfrak{k}\leq\mathfrak{g}$, we clearly have that $\mathfrak{k}\otimes_{\mathbb{R}}\mathbb{C}$ is a subalgebra of the complexification $\mathfrak{g}^{\mathrm{c}}$, so there is a subgroup $K^{\mathrm{c}} < G^{\mathrm{c}}$, such that K^{c} has Lie algebra $\mathfrak{k}^{\mathrm{c}}$ and G^{c} is the adjoint group of $\mathfrak{g}^{\mathrm{c}}$. Letting G be the adjoint group of $\mathfrak{g}$, the subgroup K^{c} is precisely the right size to make the previous correspondence bijective, yielding the following theorem.

Theorem 4.8. *For a real semisimple Lie algebra $\mathfrak{g}$ with Lie group G, there is a one-to-one correspondence between the nilpotent G-orbits in $\mathfrak{g}$, and the nilpotent K^{c}-orbits in $\mathfrak{p}^{\mathrm{c}}$. For a G-orbit with representative lying in a Cayley triple $\{f,h,e\}$, the corresponding representative for the K^{c}-orbit is $\frac{1}{2}(e+f+\imath h)$, which lies in the Cayley transform of the Cayley triple.*

Like with the complex case, this may also be represented in terms of a weighted Dynkin diagram. Since $\mathfrak{k}^{\mathrm{c}}$ is reductive, it has a Cartan subalgebra and hence a root system with simple roots β_i and a Weyl group W. For a Cayley transform $\{\tilde{f},\tilde{h},\tilde{e}\}$, the characteristic $\tilde{h}$ is W-conjugate to an element in a fixed Weyl chamber, like in the complex case. With the new characteristic, the vertex of the Dynkin diagram corresponding to β_i may be weighted with $\beta_i(\tilde{h})$. Like with the complex case, the labels are natural numbers, although they can be larger than 2. Lastly, as noted in [4, Section 9.5], it is possible for $\mathfrak{k}^{\mathrm{c}}$ to have a 1-dimensional centre, requiring the Dynkin diagram to be extended by another label.

4.4.2 Alternative classification for the complex case

We proceed to the $\mathbb{Z}_m$-grading on a semisimple Lie algebra. First, we review the complex case, following [8, Section 4.1] and [14, Section 3.3], giving representatives for the complex nilpotent orbits, then cover the invariants applied in [8, Section 5.1] to find the real nilpotent orbits for the θ-representation.

Let $\mathfrak{g}^{\mathrm{c}}$ be a complex semisimple $\mathbb{Z}_m$-graded Lie algebra, and let $G_0^{\mathrm{c}}\leq G^{\mathrm{c}}$ be the subgroup of the adjoint group of $\mathfrak{g}^{\mathrm{c}}$ with Lie algebra $\mathfrak{g}_0^{\mathrm{c}}$. Since $\mathfrak{g}$ and $\mathfrak{g}^{\mathrm{c}}$ are semisimple, let $\mathfrak{h}$ and $\mathfrak{h}_0$ be Cartan subalgebras of $\mathfrak{g}$ and $\mathfrak{g}_0$, respectively, and let their respective Weyl groups be W and W_0. From the previous subsection, one can find the nilpotent G^{c}-orbits in $\mathfrak{g}^{\mathrm{c}}$. This yields representatives for the orbits of $\mathfrak{sl}_2$-triples, given as $\{f_i,h_i,e_i\}$ where $1\leq i\leq k$ indexes the representatives. Since $G_0^{\mathrm{c}}\leq G^{\mathrm{c}}$ and $\mathfrak{g}_1^{\mathrm{c}}\subset\mathfrak{g}^{\mathrm{c}}$, the nilpotent orbits for the θ-representation are contained in the orbits already found. For each G^{c}-orbit, there are two items to consider: finding the points for orbits that lie specifically in $\mathfrak{g}_1$, and separating the orbits that are disjoint under G_0^{c}. Recall that the Weyl group has a canonical action on the CSA such that $w(h_\alpha) = h_{w(\alpha)}$ for all roots α, where h_α arises from the $\mathfrak{sl}_2$-triple containing the root vector e_α as mentioned in Section 1.2.2. Two elements in the CSA are then conjugate if and only if they are conjugate under the action of the Weyl group. As de Graaf shows in [14, Section 1-3], while an h_i might not lie in $\mathfrak{h}_0^c$, it lies in some Cartan subalgebra of $\mathfrak{g}_0^c$; since all Cartan subalgebras are conjugate under the adjoint group, the G_0^c-orbit of h_i contains an element $\tilde{h}_i \in \mathfrak{h}_0^c$. Since the representative h_i for the larger orbit was chosen to lie in the fundamental Weyl chamber, the set of all semisimple h in $\mathcal{O}_{h_i}$ that form part of an $\mathfrak{sl}_2$-triple is then $W \cdot h_i$. Finally, as conjugacy under G_0^c corresponds to conjugacy under the smaller Weyl group W_0, representatives for the G_0^c-orbits are then precisely the elements of $W \cdot h_i$ contained in the fundamental Weyl chamber for W_0. Hence, we arrive at the following, from [14, Proposition 13].

Proposition 4.9. *Let $\mathfrak{g}^c$ be a $\mathbb{Z}_m$-graded complex semisimple Lie algebra, let G^c be the Lie group of $\mathfrak{g}^c$ and let G_0^c be the subgroup that has $\mathfrak{g}_0^c$ as its Lie algebra. Let $\mathfrak{h}_0^c$ be a CSA for $\mathfrak{g}_0$ and $\mathfrak{h}^c \geq \mathfrak{h}_0^c$ be a CSA for $\mathfrak{g}$, with Weyl groups W_0 and W, respectively. Finally, let the G^c-orbits of $\mathfrak{sl}_2$-triples in $\mathfrak{g}^c$ have representatives $\{f_i, h_i, e_i\}$, with all h_i lying in the fundamental Weyl chamber D for W. Letting D_0 be the fundamental Weyl chamber for W_0, the set of representatives for the G_0^c-orbit of elements in $\mathfrak{h}_0^c$ that are part of an $\mathfrak{sl}_2$-triple is precisely*

$$\mathcal{H} = \bigcup_{i=1}^{k} (W \cdot h_i) \cap D_0.$$

All that remains is to check for which representatives leads to a homogeneous $\mathfrak{sl}_2$-triple $\{\bar{f}, \bar{h}, \bar{e}\}$, as only then will $\bar{e} \in \mathfrak{g}_1^c$ be a representative for a nilpotent orbit under the θ-representation. This is done heuristically in [14, Algorithm 1], as (recalling the notation from the previous section) there is a dense subset $S \subset \mathfrak{g}_1^c(2; h)$ such that, for any $x \in S$, either $\{f, h, x\}$ defines an $\mathfrak{sl}_2$-triple for some $f \in \mathfrak{g}_{-1}^c$, or there does not exist a homogeneous $\mathfrak{sl}_2$-triple for the given h.

4.4.3 Progression to the real case

Now, let $\mathfrak{g}$ be a real $\mathbb{Z}_m$-graded semisimple Lie algebra. The previous subsection gives representatives for the orbits of homogeneous $\mathfrak{sl}_2$-triples for the θ-representation on $\mathfrak{g}^c$. Let these representatives be $\{f_i, h_i, e_i\}$ for $1 \leq i \leq k$. Following [8], we restrict ourselves to a $\mathbb{Z}_2$-grading, and assume $\mathfrak{g}$ is the split real form of a complex semisimple Lie algebra $\mathfrak{g}^c$ (recall that, if $\{\hat{e}_i, \hat{f}_i, \hat{h}_i\}$ are the canonical generators of $\mathfrak{g}^c$, the real structure σ corresponding to $\mathfrak{g}$ fixes each of the generators). From [7, Chapter 6], the Cartan involution θ on $\mathfrak{g}$ is defined by $\theta(\hat{e}_i) = -\hat{f}_i$, $\theta(\hat{f}_i) = -\hat{e}_i$, and $\theta(\hat{h}_i) = -\hat{h}_i$.

Remark 4.10 (Non-unique representatives of orbits)**.** Following [8, Section 4.2], we proceed to find a characterisation for representatives of every nilpotent orbit contained in the complex orbit, albeit with representatives not being unique to each orbit. Similarly to the previous section, a **real Cayley triple** is a homogeneous $\mathfrak{sl}_2$-triple $\{f, h, e\}$ such that $\theta(f) = -e$, $\theta(e) = -f$ and $\theta(h) = -h$. From [8, Proposition 6], every homogeneous $\mathfrak{sl}_2$-triple is G_0-conjugate to a real Cayley triple $\{f, h, e\}$ such that $h = h_i$ is from one of the representatives of the complex orbit.

The case of all homogeneous $\mathfrak{sl}_2$-triples is restricted to just the real Cayley triples as follows.

Fixing a characteristic $h = h_i$, for $e \in \mathfrak{g}_1$ to be part of a homogeneous $\mathfrak{sl}_2$-triple with h, we must have $[h, e] = 2e$, so the representatives of the nilpotent orbit in $\mathfrak{g}_1$ must all be in $\mathfrak{g}_1(2; h)$. Since we need only consider real Cayley triples, we can assume the homogeneous $\mathfrak{sl}_2$-triple corresponding to the representative e is $\{-\theta(e), h, e\}$. For this to be a real Cayley triple, we require $[e, -\theta(e)] = h$, $[h, e] = 2e$ and $[h, -\theta(e)] = 2\theta(e)$. Fixing a basis $\{v_1, ..., v_d\}$ of $\mathfrak{g}_1(2; h)$ means that the unknown e decomposes as $e = \sum_{i=1}^{d} c_i v_i$. Since $\mathfrak{g}$ is real, this reduces the equations to a list of polynomials equations $p_1(c_1, ..., c_d) = p_2(c_1, ..., c_d) = ... = p_m(c_1, ..., c_d) = 0$. The points $(c_1, ..., c_d) \in \mathbb{R}^d$ that satisfy these equations form an algebraic set in $\mathfrak{g}_1(2; h)$.

Generally, the representatives found above are excessive to the number of nilpotent orbits, being infinite even when the number of nilpotent orbits is finite. Following [8, Section 4.2, Chapter 6], the representatives are filtered in the following manner:

- Centralisers: Similarly to the second method taken from [7], we may directly reduce the number of representatives by considering their orbits with respect to elements of G_0. In particular, consider the centraliser $C_{\mathfrak{g}_0}(h)$. For $z \in C_{\mathfrak{g}_0}(h)$, we have $\exp(\operatorname{ad}_{\mathfrak{g}} z) \in G_0$; since the Lie group of the centraliser fixes h, $\exp(\operatorname{ad}_{\mathfrak{g}} z)$ fixes h. By also restricting z to be in $\mathfrak{k}$, $\exp(\operatorname{ad}_{\mathfrak{g}} z)$ then also maps real Cayley triples to other real Cayley triples. Up to G_0-conjugacy, then one can pick a real Cayley triple and eliminate every other real Cayley triple in the image under $\exp(\operatorname{ad}_{\mathfrak{g}} z)$.

- α-labels: Clearly, if two representatives e_1, e_2 lie in separate G^c-orbits, they cannot be in the same G_0-orbit. Recall from the ungraded case that the G^c-orbit of nilpotent elements is determined by their weighted Dynkin diagrams. After specifying the simple roots $\alpha_1, ..., \alpha_l$ of the root system, acting on h with an element of the Weyl group gives an $\tilde{h} \in \mathfrak{h}^c$ in the fundamental Weyl chamber, so that $\alpha_i(\tilde{h}) \in \{0, 1, 2\}$. These are the **$\alpha$-labels** of the orbit, and if they differ for e_1 and e_2, then they must not be G_0-conjugate.

- β-labels: Similarly to the previous case, since $G_0 \leq G$, if two representatives lie in different G-orbits, they cannot be G_0-conjugate. Since $\mathfrak{k}^c$ is reductive, it has a CSA $\mathfrak{h}$, giving a root system, simple roots $\{\beta_1, ..., \beta_k\}$ and a Weyl group. As with the α-labels, after acting on h to send it to an element $\tilde{h}$ in a fixed Weyl chamber, the weighted Dynkin diagram has labels $\beta_i(\tilde{h}) \geq 0$. These are the **$\beta$-labels** of the orbit, so the representatives cannot be G-conjugate, hence not G_0-conjugate, if they differ.

- γ-labels: The last label follows the same pattern as the prior two, but is to distinguish orbits that are in different G_0^c-orbits. Since $\mathfrak{g}_0^c$ is reductive, it has a Cartan subalgebra $\mathfrak{h}_0^c$, yielding a root system, Weyl group and simple roots $\{\gamma_1, ..., \gamma_j\}$. As the algebra is complex, we can again act on h by an element of the Weyl group sending it to an element $\tilde{h}$ in the fundamental Weyl chamber, so that $\gamma_i(\tilde{h}) \geq 0$. Now, two nilpotent elements $e_1, e_2 \in \mathfrak{g}_1$ that are in the same G_0-orbit are also in the same G_0^c-orbit. Since the adjoint action preserves $\mathfrak{sl}_2$-triples, part (iv) of Theorem 3.10 shows that their corresponding characteristics are also G_0^c-conjugate. From [4, Theorem 2.2.4], the semisimple orbits are in one-to-one correspondence with the orbits of $\mathfrak{h}^c$ under the action of the Weyl group, so the two characteristics will associate with the same element $\tilde{h}$ in the fundamental Weyl chamber for fixed $\mathfrak{h}^c$. Therefore, the γ-labels are invariant under the action of G_0^c, so two representatives cannot be G_0^c-conjugate, hence not G_0-conjugate, if their γ-labels differ.

The last invariant we consider are the tensor classifiers. Recall that a θ-representation is a map $\rho : G_0 \to \mathrm{GL}(\mathfrak{g}_1)$. Following the material in Appendix B3, consider the $(p, 0)$-representation $\rho^{(p,0)} = \rho \otimes ... \otimes \rho$ which acts on $\mathfrak{g}_1^{(p,0)}$. As it is still a representation, it can be decomposed into irreducible representations $\rho^{(p,0)} = \oplus_{i=1}^{l} \rho_{(i)}$, where l is the number of components and each $\rho_{(i)}$ acts on a subspace $S_i \leq \mathfrak{g}_1^{(p,0)}$. In particular, any $v \in \mathfrak{g}_1$ induces a tensor $v \otimes ... \otimes v \in \mathfrak{g}_1^{(p,0)}$ that decomposes with respect to these subspaces, so $v \otimes ... \otimes v = \sum_{i=1}^{l} s_i$.

It is then possible for one of these restricted representations to be a symmetric $(2, 0)$- or $(0, 2)$-tensor, $T^{ij}(v)$ or $T_{ij}(v)$, called a **tensor classifier**. Now, applying the formula in Appendix B3 and using the symmetry of the tensor, a tensor classifier transforms under the action of $g \in G_0$ by $\tilde{T} = \rho(g)T\rho(g)^\intercal$ in the contravariant case, or by $\tilde{T} = \rho^\star(g)T\rho^\star(g)^\intercal$ in the covariant case. In either case, we see that the real symmetric matrix transforms by a congruence transformation; by Sylvester's Law, its **signature** (n_+, n_-), which is the number of positive and negative eigenvalues of the tensor classifier, is invariant with respect to G_0 (but not necessarily G_0^c), giving a means of separating points in different orbits, as two points v and w in $\mathfrak{g}_1$ must be in different orbits if their tensor classifiers, say $T^{ij}(v)$ and $T^{ij}(w)$, have different signatures.

While these methods are not universal, they have been used to fully classify nilpotent orbits in several applied cases, especially regarding the physics of black holes, such as [8] and [3].

4.5 The Karplevich-Mostow theorem for graded Lie algebras

From the Karplevich-Mostow theorem [26, Corollary 1, Chapter 6], it is known that, for a homomorphism $f : \mathfrak{g}_1 \to \mathfrak{g}_2$ of real semisimple Lie algebras, if $\mathfrak{g}_1 = \mathfrak{k}_1 \oplus \mathfrak{p}_1$ is a Cartan decomposition, there is a Cartan decomposition $\mathfrak{g}_2 = \mathfrak{k}_2 \oplus \mathfrak{p}_2$ such that $f(\mathfrak{k}_1) \subseteq \mathfrak{k}_2$ and $f(\mathfrak{p}_1) \subseteq \mathfrak{p}_2$. Now, for a real semisimple Lie algebra $\mathfrak{g}$ with semisimple subalgebra $\mathfrak{a}$, applying the Karplevich-Mostow theorem with $f : \mathfrak{a} \to \mathfrak{g}$ being the inclusion map, we have that a Cartan involution ψ on $\mathfrak{a}$ can be extended to a Cartan involution θ on $\mathfrak{g}$. We cover how this result may be generalised for $\mathbb{Z}_2$-graded Lie algebras. This grading is known to be induced from an involutive automorphism, and a natural generalisation would be for the summands of the grading to be stable under both the original and extended Cartan involutions. Since this clearly occurs if and only if the Cartan involutions commute with the involutive automorphism, we arrive at the following theorem. This theorem is from [9, Proposition 2] and refers to [24, Theorem 6] and [23, Chapter IV.2, Theorem 2.1] for a proof, but with no details given. The aim of this section is to provide our own proof for this result.

Theorem 4.11. *Let $\mathfrak{g}$ be a real semisimple Lie algebra, σ an involutive automorphism of $\mathfrak{g}$, and $\mathfrak{a}$ a semisimple subalgebra of $\mathfrak{g}$ that is σ-invariant. Then every Cartan involution ψ of $\mathfrak{a}$ commuting with σ can be extended to a Cartan involution θ of $\mathfrak{g}$ that commutes with σ.*

Proof. First observe that the Cartan involution ψ induces a Cartan decomposition $\mathfrak{a} = \mathfrak{k}_1 \oplus \mathfrak{p}_1$. Since the Killing form on $\mathfrak{k}_1$ is negative definite by definition of the Cartan decomposition, $\mathfrak{k}_1$ is compact. Applying the Karplevich-Mostow theorem given above with the inclusion map $i : \mathfrak{a} \to \mathfrak{g}$, there is a Cartan decomposition $\mathfrak{g} = \mathfrak{k}_2 \oplus \mathfrak{p}_2$ with $\mathfrak{k}_1 \subseteq \mathfrak{k}_2$ and $\mathfrak{p}_1 \subseteq \mathfrak{p}_2$; this is equivalent to the induced Cartan involution $\tilde{\theta}$ extending ψ.

We now follow the proof of a theorem of Loos given in [23, Chapter IV.2, Theorem 2.1] to construct a Cartan involution extending ψ that also commutes with σ. Let $\sigma\tilde{\theta} = \nu$ and recall from Section 2.1 that $h_{\tilde{\theta}}$ is the Hermitian form induced from $\tilde{\theta}$. For $x, y \in \mathfrak{g}$, it is known that

$$h_{\tilde{\theta}}(\nu(x), y) = -\kappa(\nu(x), \tilde{\theta}(y)) = -\kappa(\sigma\tilde{\theta}(x), \tilde{\theta}(y)).$$

Since the Killing form is invariant with respect to automorphisms of $\mathfrak{g}$, applying ν to both terms in the bilinear form and using the fact that $\tilde{\theta}$ and σ are involutive gives

$$-\kappa(\sigma\tilde{\theta}(x), \tilde{\theta}(y)) = -\kappa(x, \tilde{\theta}\sigma\tilde{\theta}(y)) = h_{\tilde{\theta}}(x, \nu(y)).$$

Hence, $h_{\tilde{\theta}}(\nu(x), y) = h_{\tilde{\theta}}(x, \nu(y))$ and the operator ν is symmetric. Therefore, for nonzero $x \in \mathfrak{g}$, we have $h_{\tilde{\theta}}(\nu^2(x), x) = h_{\tilde{\theta}}(\nu(x), \nu(x))$. Now, from the Cartan decomposition on $\mathfrak{g}$ let $\nu(x) = k + p$ with $k \in \mathfrak{k}$ and $p \in \mathfrak{p}$. Since ν is an automorphism, we must have nonzero k or p. Then

$$h_{\tilde{\theta}}(\nu(x), \nu(x)) = h_{\tilde{\theta}}(k + p, k + p) = -\kappa(k + p, k - p) = \kappa(p, p) - \kappa(k, k) > 0,$$

with the last equality following from bilinearity of the Killing form and definition of the Cartan decomposition. Hence, $h_{\tilde{\theta}}(\nu^2(x), x) > 0$ for nonzero $x \in \mathfrak{g}$, so ν^2 is positive definite relative to $\kappa_{\tilde{\theta}}$. By Section 3.2, the exponential map is a bijective map from the symmetric linear transformations to the positive definite linear transformations, so there is a unique symmetric linear transformation A of $\mathfrak{g}$ such that $\nu^2 = \exp(A)$.

We now follow the proof from Loos to show that $\exp(tA)$ is also an automorphism of $\mathfrak{g}$ for all real $t \in \mathbb{R}$. Since ν is symmetric, it follows from [22, pp. 172, Section 7.2] that ν has a basis of eigenvectors, with real eigenvalues. Let $B = \{x_1, ..., x_n\}$ be this basis, so $\nu(x_i) = \lambda_i x_i$ and $\nu^2(x_i) = \lambda_i^2 x_i$ for $\lambda_i \in \mathbb{R}$. Since $\mathfrak{a}$ is invariant under both $\tilde{\theta}$ and σ, it is also invariant under ν. As B is a basis of eigenvectors of ν, we can assume without loss of generality that the first k entries of B form a basis for $\mathfrak{a}$. Furthermore, for $1 \leq i \leq k$, we have $\nu(x_i) = \sigma(\tilde{\theta}(x_i)) = (\sigma \circ \psi)(x_i)$. Since ψ and σ

are commuting involutions that map $\mathfrak{a}$ into itself, $\sigma \circ \psi$ is also an involution mapping $\mathfrak{a}$ into itself. Thus, $\mathfrak{a}$ decomposes into the ± 1-eigenspaces of $\sigma \circ \psi$, so we may further assume without loss of generality that the a_i's form this basis of eigenspaces. Hence, we see that, for $1 \leq i \leq k$, that $\nu^2(x_i) = x_i$, so $\lambda_i^2 = 1$. Returning to an arbitrary $1 \leq i \leq n$, since ν^2 is positive definite, there exists an $a_i \in \mathbb{R}$ such that $\exp(a_i) = \lambda_i^2$. Now, we see $\mathfrak{g}$ has structure constants c_{ij}^k, so $[x_i, x_j] = \sum_{k=1}^n c_{ij}^k x_k$. Since ν^2 is an automorphism, applying it to both sides of this equation gives

$$\nu^2([x_i, x_j]) = \sum_{k=1}^n c_{ij}^k \nu^2(x_k) = \sum_{k=1}^n c_{ij}^k \exp(a_k) x_k =$$

$$[\nu^2(x_i), \nu^2(x_j)] = \exp(a_i)\exp(a_j)[x_i, x_j] = \sum_{k=1}^n c_{ij}^k \exp(a_i)\exp(a_j) x_k.$$

Since $\{x_1, ..., x_n\}$ is a basis, it follows that $c_{ij}^k \exp(a_k) = c_{ij}^k \exp(a_i)\exp(a_j)$ for all $\{1 \leq i, j, k \leq n\}$. Since these are real numbers, it follows that, for any $t \in \mathbb{R}$, we have $(c_{ij}^k \exp(a_i)\exp(a_j))^t = (c_{ij}^k)^t \exp(ta_i)\exp(ta_j) = (c_{ij}^k)^t \exp(ta_k)$. From Section 3.2, the map which has B as eigenvectors, and eigenvalues $\lambda_i = \exp(ta_i)$, is denoted $\exp(tA)$. The above reasoning in reverse then shows that $\exp(tA)$ is also an automorphism. Furthermore, since by definition there is a basis of simultaneous eigenvectors, $\exp(tA)$ commutes with ν. Now observe that

$$\tilde{\theta}\nu^2\tilde{\theta} = \tilde{\theta}(\sigma\tilde{\theta}\sigma\tilde{\theta})\tilde{\theta} = \tilde{\theta}\sigma\tilde{\theta}\sigma = \nu^{-2} = (\nu^2)^{-1}.$$

Thus, $(\tilde{\theta}\nu^2\tilde{\theta})(\tilde{\theta}(x_i)) = \exp(a_i)\tilde{\theta}(x_i)$, so $\tilde{\theta}(x_i)$ is an eigenvector of ν^2 with eigenvector $\exp(-a_i)$. By definition of $\exp(tA)$, we then have that $\tilde{\theta}(x_i)$ is also an eigenvector of $\exp(tA)$ with eigenvalue $\exp(-ta_i)$, and that $(\tilde{\theta}\exp(tA)\tilde{\theta})(\tilde{\theta}(x_i)) = \exp(ta_i)\tilde{\theta}(x_i)$. Thus, $(\tilde{\theta}\exp(tA)\tilde{\theta})(\tilde{\theta}(x_i)) = \exp(-tA)\tilde{\theta}(x_i)$. Since $\tilde{\theta}$ is an automorphism, $\tilde{\theta}(B)$ is also a basis for $\mathfrak{g}$, so we conclude that $\tilde{\theta}\exp(tA)\tilde{\theta} = \exp(-tA)$, for all $t \in \mathbb{R}$.

We now show that, for a Cartan involution ζ and an automorphism L, the conjugate $\zeta^L = L\zeta L^{-1}$ is also a Cartan involution. If ζ has ± 1-eigenspace decomposition $\mathfrak{g} = \mathfrak{k}_\zeta \oplus \mathfrak{p}_\zeta$, then clearly ζ^L has ± 1-eigenspace decomposition $\mathfrak{g} = L(\mathfrak{k}_\zeta) \oplus L(\mathfrak{p}_\zeta)$. Since L is an automorphism and the Killing form is invariant, L preserves positive- and negative-definiteness. By definition, it follows that ζ^L is also a Cartan involution. In particular $\theta = \exp(tA)\tilde{\theta}\exp(-tA)$ is a Cartan involution. We then have that

$$\begin{aligned}\sigma\theta &= \sigma\exp(tA)\tilde{\theta}\exp(-tA) = \sigma\tilde{\theta}(\tilde{\theta}\exp(tA)\tilde{\theta})\exp(-tA)\\ &= \sigma\tilde{\theta}\exp(-2tA) = \nu\exp(-2tA),\end{aligned}$$

and

$$\begin{aligned}\theta\sigma &= \exp(tA)\tilde{\theta}\exp(-tA)\sigma = \exp(tA)(\tilde{\theta}\exp(-tA)\tilde{\theta})\tilde{\theta}\sigma\\ &= \exp(2tA)\tilde{\theta}\sigma = \exp(2tA)\nu^{-1}.\end{aligned}$$

For a basis element x_i, we then have

$$(\sigma\theta)(x_i) = \nu(\exp(-2tA)(x_i)) = \lambda_i\exp(-2ta_i)x_i,$$

$$(\theta\sigma)(x_i) = \exp(2tA)(\nu^{-1}(x_i)) = \exp(2ta_i)\lambda_i^{-1}x_i.$$

Since $\lambda_i^2 = \exp(a_i)$, we know $\lambda_i = \pm\exp(\frac{1}{2}a_i)$. Thus, for $t = \frac{1}{4}$, observe that

$$\begin{aligned}\lambda_i \exp(-2ta_i) &= \pm\exp(\frac{1}{2}a_i)\exp(-\frac{1}{2}a_i) = \pm 1\\ &= \exp(\frac{1}{2}a_i)(\pm\exp(-\frac{1}{2}a_i)) = \exp(2ta_i)\lambda_i^{-1},\end{aligned}$$

so $(\sigma\theta)(x_i) = (\theta\sigma)(x_i)$. Since B forms a basis, this shows that $\sigma\theta = \theta\sigma$, so θ is a Cartan involution commuting with σ. Lastly, for $1 \leq i \leq k$, recall that $\lambda_i^2 = 1$, so $a_i = 0$. As a result, the maps $\exp(tA)$ and $\exp(-tA)$ fix x_i, so they both fix all of $\mathfrak{a}$. Therefore, $\theta|_{\mathfrak{a}} = \tilde{\theta}|_{\mathfrak{a}} = \psi$, showing that θ is our desired extension of ψ. □

Let $\mathfrak{g}$ be a real semisimple $\mathbb{Z}_m$-graded Lie algebra for $m \in \mathbb{N}$ with complexification $\mathfrak{g}^c$ and the grading on $\mathfrak{g}^c$ induced by an automorphism θ^c. From van Le [21, Theorem 3.3], it is known, that for a compact real form $\mathfrak{u}$ of $\mathfrak{g}^c$ invariant under θ^c, there exists an automorphism ϕ of $\mathfrak{g}^c$ that commutes with θ^c, such that $\phi(\mathfrak{u})$ is invariant under both θ^c and the complex conjugation $\tau_{\mathfrak{g}}$. There always exists a compact real form $\mathfrak{u}$ of $\mathfrak{g}^c$ that is invariant under θ^c by [16, Lemma 5.2, Chapter X]. Since $\theta^c(x) = \omega^i x$ for $x \in \mathfrak{g}_i^c$, where ω is a primitive ith root of unity, and the complex conjugate of ω^i is ω^{-i}, it follows that $\theta^c\tau_{\mathfrak{g}} = \tau_{\mathfrak{g}}(\theta^c)^{-1}$. Let τ be the real structure of $\phi(\mathfrak{u})$ constructed before. The conditions on $\mathfrak{u}$ are equivalent to τ commuting with $\tau_{\mathfrak{g}}$, so from Section 2.2, it is known that $\psi = \tau\tau_{\mathfrak{g}}$ is a Cartan involution. Since τ commutes with θ, we then also have that $\theta^c\psi = \psi(\theta^c)^{-1}$. Hence, this shows that there exists a Cartan involution on $\mathfrak{g}$ that reverses the grading, as stated in [21, Theorem 3.3(2)].

Appendix A

Additional background material

A.1 Differential geometry and Lie groups

A.1.1 Background for differential geometry

For the sake of completeness, we briefly cover some basic definitions and theorems of smooth manifolds and Lie groups. The content here may be found in [5] and [18]. A **real manifold** M is a topological space that is Hausdorff, second countable and locally Euclidean (each point has an open neighbourhood homeomorphic to a neighbourhood of $\mathbb{R}^n$ for some $n \in \mathbb{N}$). If the manifold is connected, the value n is identical across all points of the manifold and is the **dimension** of the manifold. The above homeomorphism into Euclidean space is a **chart**, and a union of charts that cover the manifold is an **atlas** for the manifold. For two charts $\phi_1 : U \to \mathbb{R}^n$ and $\phi_2 : V \to \mathbb{R}^n$ and a point $p \in U \cap V$, there is a **transition map** $\phi_2 \circ \phi_1^{-1} : \phi_1(U) \to \phi_2(V)$ relating the two charts. As $\phi_1(U)$ and $\phi_2(V)$ are neighbourhoods of the image of p and are in Euclidean space, tools from the study of Euclidean spaces can be used for the study of manifolds. In particular, **smooth manifolds** are manifolds which have an atlas of charts such that every transition map is smooth, allowing calculus to be used. A **complex manifold** is a smooth manifold, but with charts onto the open unit disc in $\mathbb{C}^n$ and with transition maps being holomorphic.

Again letting $\phi : U \to \mathbb{R}^n$ be a chart and choosing some $p \in U$, let $C^\infty(U, p)$ be the set of smooth, real-valued functions with domain an open subset of U containing p, and let $S(U, p)$ be the set of differentiable curves $s : (a, b) \to U$ where $a < 0 < b$ and $s(0) = p$. Then, for $\alpha, \beta \in S(U, p)$, consider the condition

$$\frac{\mathrm{d}}{\mathrm{d}t} f(\alpha(t))\bigg|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t} f(\beta(t))\bigg|_{t=0} \quad \forall f \in C^\infty(U, p). \qquad (*)$$

Equation (*) is useful in two ways. First, two curves that satisfy the equation are infinitesimally equivalent, and this is an equivalence relation with equivalence class with denoted by $\langle \alpha \rangle_p$. This set of equivalence classes is denoted $T_p(M)$ and is the tangent space of M at p. Second, equation (*) defines an operator $D_{\langle\alpha\rangle_p}$ on $C^\infty(U,p)$. Note that $D_{\langle\alpha\rangle_p}(\phi)$ is a directional derivative defined without use of an explicit coordinate system.

The concept of tangent vectors can also be expressed using the functions in $C^\infty(U,p)$, by generalising the directional derivative. Since this operator only considers the local behaviour of functions, we further specify two functions $f, g \in C^\infty(U,p)$ to be germinally equivalent if they are equal in some neighbourhood of p. This is also an equivalence relation, with the equivalence class being a **germ** $\mathfrak{G}_p$ at p. Then a **derivative operator** on $\mathfrak{G}_p$ is an $\mathbb{R}$-linear map $D : \mathfrak{G}_p \to \mathbb{R}$ that satisfies the Leibnitz property $D(fg) = D(f)g(p) + D(g)f(p)$ for all $f, g \in \mathfrak{G}_p$. The set of all derivative operators is denoted $T(\mathfrak{G}_p)$, but is also called the tangent space at p, since it forms a vector space isomorphic to $T_p(M)$. In both cases, the (vector space) dimension of the tangent space is equal to the dimension of the manifold. Furthermore, the disjoint union of the tangent spaces across all points in the manifold is the **tangent bundle** on M, with a **bundle projection** $\pi : T(M) \to M$ defined by $\pi(X_p) = p$ for $p \in T_p(M)$. A **vector field** on M is a smooth map $X : M \to T(M)$ such that $\pi \circ X = \mathrm{id}_M$; geometrically, a vector field associates a tangent vector to every point on the manifold.

A.1.2 Lie Groups and the adjoint action

A **Lie group** G is a topological group that is also a smooth (real or complex) manifold, as given in [20, Section I.10]. Recall that a vector field on a manifold is a smooth map $V : G \to T(G)$ such that $V(p) \in T_p(G)$. Then the tangent space $T_e(G)$ of G at the identity e in G has a Lie algebra structure given by the bracket $[X(e), Y(e)] = (X \circ Y - Y \circ X)(e)$ for vector fields X and Y (so $X(e), Y(e) \in T_e(G)$). Conversely, every Lie algebra $\mathfrak{g}$ can be found in this way from a Lie group G. For a real or complex Lie algebra $\mathfrak{g}$ with Lie group G, the automorphism group $\mathrm{Aut}\,\mathfrak{g}$ is a Lie group with Lie algebra $\mathrm{Der}\,\mathfrak{g}$.

Recall, for a smooth map $\Phi : U \to V$, that the **differential** of Φ at a point p is a linear isomorphism $\mathrm{d}_p\Phi : T_p(U) \to T_{\Phi(p)}(V)$. In particular, for a Lie group G and $g \in G$, the conjugation map $\alpha_g : h \mapsto ghg^{-1}$ is an automorphism and, since group multiplication is smooth by definition of the Lie group, it is smooth. Hence, there is a map $\mathrm{Ad} : G \to \mathrm{Aut}\,\mathfrak{g}$ defined by $\mathrm{Ad}(g) = \mathrm{d}_e\alpha_g$, called the **adjoint representation of** $\mathfrak{g}$. Since $\mathrm{Aut}\,\mathfrak{g}$ is another Lie group, $d_e\mathrm{Ad}$ is defined, and in fact $d_e\mathrm{Ad} = \mathrm{ad}$, where $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is the adjoint representation of $\mathfrak{g}$. The image $\mathrm{Ad}(G)$ is a normal subgroup of $\mathrm{Aut}\,\mathfrak{g}$; if G is connected, this is the adjoint group of the Lie algebra of G.

The connected component of the identity of $\mathrm{Aut}\,\mathfrak{g}$ with Lie algebra $\mathrm{ad}\,\mathfrak{g}$ is precisely the adjoint group of $\mathfrak{g}$, defined as the image $\mathrm{Ad}(G)$. Analogously, for a real Lie algebra $\mathfrak{g}$, the adjoint group of $\mathfrak{g}$ is the connected Lie subgroup of $\mathrm{Aut}\,\mathfrak{g}$ with Lie algebra $\mathrm{ad}\,\mathfrak{g}$, and is generated by the $\exp(\mathrm{ad}\,x)$ for $x \in \mathfrak{g}$. This is always defined since $\mathrm{ad}\,x$ is an endomorphism on $\mathfrak{g}$ and the exponential series converges absolutely given any linear map, by [5, Example 5.1.22].

A.2 Zariski topology

The topology on graded Lie algebras is generally the Zariski topology. Here, we summarise some key properties. From [18, Section 1.3], for a field K, consider the **affine space** A^n of dimension n over K to comprise the elements of K^n. Then the **Zariski topology** is the topology with closed sets being the common zeroes of a finite number of polynomials over K in A^n, called algebraic sets. Note these zeroes are also the zeroes of the ideal generated by those polynomials. As a trivial example, the closed sets in A^1 are precisely the finite sets of elements in K since polynomials in one variable have finite zeroes by the divisor theorem.

Of particular note is that the open sets in the Zariski topology are dense, so the topology is non-Hausdorff, although it satisfies the ascending chain condition. Each ideal in $K[x_1, ..., x_n]$ is finitely generated, so it corresponds to an algebraic set. Conversely, every radical ideal $\mathfrak{r}$ (an ideal where, if $f^r \in \mathfrak{r}$, then $f \in \mathfrak{r}$) corresponds to a unique algebraic set (this is Hilbert's Nullstellensatz, shown in, for example, [1, Chapter 1]).

An algebraic set is **irreducible** if it cannot be written as the union of two closed and nonempty sets. In particular, A^n is irreducible and every closed set can be expressed as a finite union of irreducible components. Irreducible sets are connected by definition of connectedness, but the converse is not always true. For example, in A^2, the algebraic set of solutions to $x^2 = y^2$ is connected, but it is a union of the solutions to $x = y$ and $x = -y$. These also have an algebraic characterisation: the ideals that correspond to irreducible algebraic sets are precisely the prime ideals of $K[x_1, ..., x_n]$.

A.3 Tensor and exterior algebras

Let V and W be finite-dimensional vector spaces over a field F with dimensions m and n respectively. The **tensor product** $V \otimes_F W$ is the mn-dimensional F-vector space with basis $\{v \otimes w : v \in \mathcal{B}_V, w \in \mathcal{B}_W\}$, where $\mathcal{B}_V$ (resp. $\mathcal{B}_W$) is a basis of V (resp. W). In particular, letting T^nV be the tensor product of V with itself n times, and defining $T^0V = F$ and $T^1V = V$, we get the **tensor algebra** of V as $T(V) = \bigoplus_{n\geq 0} T^nV$, with componentwise addition and multiplication given by the tensor product defined as $\otimes : T^mV \times T^nV \to T^{m+n}V$, $(v_1 \otimes ... \otimes v_m) \otimes (\tilde{v}_1 \otimes ... \otimes \tilde{v}_n) = v_1 \otimes ... \otimes v_m \otimes \tilde{v}_1 \otimes ... \otimes \tilde{v}_n$. Furthermore, as $T(V)$ is an associative algebra, we can consider the two-sided ideal $\mathcal{I}$ generated by $v \otimes v$ for $v \in V$. The resulting quotient space $\bigwedge V = T(V)/\mathcal{I}$ is the **exterior algebra** of V, an associative algebra with multiplication given by the wedge product $\wedge$. In particular, elements of the form $v \wedge w \wedge x$ span a subspace $\bigwedge^3 V$ of the exterior algebra, the space of **trivectors** of V, objects that we focus particularly on in Chapter 4.

Now fixing the vector spaces to be real and letting $V^\star$ be the vector space of linear functionals of V, a (p, q)-tensor T is an element of $V \otimes ... \otimes V \otimes V^\star \otimes ... \otimes V^\star$, where $V^\star$ occurs p times and V occurs q times. The tensor then has **contravariant rank**

p and **covariant rank** q. Taking a basis $\{v_i\}_{1\leq i\leq n}$ of V and a basis $\{\epsilon_i\}_{1\leq i\leq n}$ for $V^\star$ allows the tensor to be expressed in terms of n^{p+q} components as

$$T = \sum_{\substack{i_1 \dots i_p \\ j_1 \dots j_1}} T^{i_1 i_2 \dots i_p}_{j_1 j_2 \dots j_q} v_{i_1} \otimes \dots \otimes v_{i_p} \otimes \epsilon_{j_1} \otimes \dots \otimes \epsilon_{j_q}.$$

Now, for a representation $\psi : G \to \mathrm{GL}(V)$, taking a basis on V and representing $\psi(g)$ as a matrix, the dual representation is $\psi^\star(g) = (\psi(g)^{-1})^\intercal$. This allows the representation to be extended to the (p,q)**-tensor representation**, where $\psi^{(p,q)} = \psi \otimes \psi \otimes \dots \otimes \psi^\star \otimes \dots \otimes \psi^\star$, which defines an action of G on a (p,q)-tensor T by $g \cdot T = \psi^{(p,q)}(g)(T) = \tilde{T}$ which has components

$$\tilde{T}^{i_1 i_2 \dots i_p}_{j_1 j_2 \dots j_q} = \psi(g)^{i_1}_{k_1} \dots \psi(g)^{i_p}_{k_p} \psi(g)^{l_1}_{j_1} \dots \psi(g)^{l_q}_{j_q} T^{k_1 \dots k_p}_{l_1 \dots l_q}.$$

Appendix B

Notation

$\mathbb{N}$	Natural numbers, $\{1, 2, 3, ...\}$
$\mathbb{Z}$, $\mathbb{Z}_\infty$	Integers
$\mathbb{R}$	Real numbers
$\mathbb{C}$	Complex numbers
$\mathbb{Z}_n$	Integers modulo n
$\mathfrak{g}$	Lie algebra
$\mathfrak{g}^c$	Complex Lie algebra
M_{ij}	The entry of the matrix M at row i and column j
e_{ij}	The matrix with entry 1 at (i, j) and 0 elsewhere
$\mathfrak{gl}(n, F)$	The general linear algebra of $n \times n$ matrices with entries in F.
$\mathfrak{sl}(n, F)$	The subalgebra in $\mathfrak{gl}(n, F)$ of matrices with trace zero.
$\mathfrak{n}(n, F)$	The subalgebra in $\mathfrak{gl}(n, F)$ of diagonal matrices.
$\mathfrak{b}(n, F)$	The subalgebra in $\mathfrak{gl}(n, F)$ of upper triangular matrices.
tr	Trace map
det	Determinant map
$A^\intercal$	Matrix transpose of A
$\kappa(x, y)$	The Killing form of $\mathfrak{g}$.
$\{f_\alpha, h_\alpha, e_\alpha\}$	The $\mathfrak{sl}_2$-triple in $\mathfrak{g}$ associated with the root α.
$\bar{\mathfrak{b}}$ ($\mathfrak{b} \leq \mathfrak{g}(\mathbb{C})$)	The complex conjugate $\sigma(\mathfrak{b})$ in $\mathfrak{g}(\mathbb{C})$.
$\bar{\mathfrak{g}}^c$	The complex conjugate Lie algebra.
$\mathrm{Sym}(\Omega)$	The symmetric group of the set Ω

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