Advancements in Functional State Estimation Using Sliding Mode Observers

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ABSTRACT

This thesis explores the development of functional state estimation schemes using sliding mode observers. In many practical situations, the internal state of a system cannot be determined by direct measurement, and an observer may be use to estimate the unavailable states based on the measured outputs and control inputs. Existing work has shown that the functional observer (which estimates only a subset of the states) can be used to eliminate the redundancy possessed in full-state estimation, and also reduce computational burden. The sliding mode observer has been shown to have excellent performance in robust state estimation, and also has the extra ability of estimating the unknown input. Its use for functional state estimation is still very much in its infancy with no significant works being reported. This thesis combines the theories of functional state estimation and sliding mode observer to reduce the observer complexity as well as improve the overall performance of the observer.

The first contribution of this thesis is the reformulation of the system such that the sliding mode observer can be implemented directly to achieve functional state estimation. The next contribution is the development of a two-observer scheme (which uses two observers in cascade) for functional state estimation and the investigation of its existence conditions, which were found to be weaker and more general than previous work. Following that, the work of cascading observers was extended to multiple sliding mode observers is explored, together with the investigation of its existence conditions in terms of the original system matrices; these conditions enable the user to know at the outset whether the observer scheme is applicable to their system or not, and also systematically calculates the number of observers necessary and sufficient. Following that, a new functional state estimation method is introduced, where the functional state is modeled as an unknown input, and then estimated using the ability of the sliding mode observer to estimate unknown inputs.

Finally, the functional observer is implemented on the a class of descriptor systems, and the necessary and sufficient conditions are investigated. The efficiency of the functional observer schemes presented in this thesis are demonstrated by simulation examples.

STATEMENT OF AUTHORSHIP

The thesis contains no material which has been accepted for the award of any other degree or diploma in any university or other institution. I affirm that, to the best of my knowledge, the thesis contains no material previously published or written by another person, except where due reference is made in the text of the thesis.

Signed: .....

Kee Chew Yee

March 2014

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Chapter 1

Research Motive and Overview of Thesis

The theory of automatic control has contributed very much to the advancement of technology, where performance of dynamic engineering systems have been greatly improved which in turn results in greater productivity and reliability. Controllers are usually implemented via feedback, where the output is measured and compared against a reference signal; then the difference will be injected into the controller such that adjustments can be made on the system being controlled in order to achieve optimal performance. The output are a subset of the *states* of the system for which measurement devices have been installed to obtain their signals electrically. Information of the states is very useful as they can be used to provide better control, and also to obtain more information about the system. In most cases, not all states are measured due to various reasons, for example:

- sensors are expensive
- it is not physically feasible to install the sensor
- the state does not have a physical meaning

An observer may be used to overcome these problems, by providing an estimate of the states. Essentially, an observer is a model of the dynamic system, where the input to the system is also injected into the observer; the states of the observer are taken to be the state estimates. Then the system and observer outputs are compared; the difference (known as the 'output estimation error') is injected back into the observer as a 'correction signal' to adjust the state estimates.

However, estimating the entire state vector may not be practical for high-dimensional systems as it requires high computational power and causes the resulting observer to be too

sensitive to measurement noise [1]. In many situations, only part of the state information is required. For example, only the position vectors are required for the feedback controller design in stabilizing the Active Magnetic Bearings (AMB) system [2]. Apart from that, not estimating the entire state was found to be more accurate and less restrictive compared to the conventional observer in the application on a high-dimensional chemical distillation process [1] and linear motor states estimation. [3]. A *functional observer*, which only estimates a function or subset of the state vector, can be implemented to reduce the observer complexity.

A common issue in state estimation is the *unknown input*; these could represent any mismatches between the model and actual system, such as unmodelled dynamics, parameter variations and external disturbances. These unknown inputs could render the state estimate to be inaccurate. An interesting option that addresses this issue is the *sliding mode observer* [4, 5, 6] which utilizes a nonlinear switching term and can produce a state estimate that is robust to a class of unknown inputs. In addition, it was found that suitable processing of the switching term can produce an estimate of the unknown input. Most functional observers have been using linear observers, and there has been very little work done using sliding mode observers for functional state estimation. This thesis presents work on the development of functional state estimation schemes using sliding mode observers. The remainder of this chapter presents an overview of the thesis, as well as its contributions to the field of research.

1.1 Structure of Thesis

Chapter 2 - This chapter provides an overview of the existing works on observers, including linear observers, unknown input observers, functional observers and sliding mode observers. Research results that shows the advantages of functional state estimation over full-state estimation are mentioned and the benefits of sliding mode observer are presented.

Chapter 3 - This chapter describes the sliding mode observer scheme by Edwards & Spurgeon [5] that will be used in the work of this thesis. A systematic design proce-

dure of the observer scheme and the necessary and sufficient existence conditions are presented in this chapter. On top of that, the ability of the Edwards-Spurgeon observer in estimating the unknown input is demonstrated. Then, the use of multiple observers for state estimation is illustrated and the ability of the scheme in relaxing the existence conditions is shown. A new functional observer scheme is introduced in this chapter where the Edwards-Spurgeon sliding mode observer can be implemented directly and the necessary and sufficient conditions for the observer scheme were presented in terms of the original system matrices. The sliding mode functional observer developed in this chapter will serve as the basis for the rest of the works in this thesis

Chapter 4 - This chapter presents a new scheme in functional state estimation, using two cascaded sliding mode observers. The necessary and sufficient conditions are investigated and presented in terms of the original systems matrices. This will enable the user to determine whether or not the scheme is applicable at the outset of the design. In addition, this also provides a basis comparison for future research in the functional state estimation. The work in this chapter has been accepted for publication in the International Journal of Robust and Nonlinear Control with the following details: C.Y. Kee, C.P. Tan, K.Y. Ng and H. Trinh. "New results in robust functional state estimation using two sliding mode observers in cascade," *International Journal of Robust and Nonlinear Control*, Feb. 2013. doi:10.1002/rnc.2973.

Chapter 5 - This chapter will further relax the existence conditions in Chapter 4 by proposing a functional observer scheme that uses more than one sliding mode observers in cascade. The necessary and sufficient condition for the observer scheme to be feasible is investigated and presented in terms of the original system matrices. A suitable algorithm is formulated to calculate the number of the observers required to achieve the functional state estimation. This work is currently under review by the IEEE Transactions on Automatic Control.

Chapter 6 - This chapter explores estimating the functional state using the idea of unknown input estimation, which is one of the characteristics of sliding mode observers. The functional state is modeled as an unknown input, then the ability of Edwards-Spurgeon

observer in estimating unknown input from discontinuous output injection error is exploited to estimate the functional state. The necessary and sufficient conditions in terms of the original system matrices for the new scheme to be feasible is investigated to ensure all functional state components can be accurately estimated.

Chapter 7 - This chapter implements the sliding mode functional observer scheme on a class of descriptor systems. First, the original system matrices are re-casted into a form such that accurate functional state estimation can be achieved. After that, the system and the observer can be placed into a form that is similar to the Edwards-Spurgeon observer. The existence conditions in terms of the original system matrices is investigated.

Chapter 8 - This chapter conclude the research results presented in this thesis and recommends some potential future development in the field on functional state estimation based on the outcomes of this thesis.

Chapter 2

An Introduction to Observers

This chapter will discuss the the development of observers over the years. It begins with the development of Luenberger observer (also known as linear observer) in Section 2.1. Following that, techniques in robust estimation will be discussed in Section 2.1.1. The *functional observer*, which only estimates a function or subset of the state vector will be explored in Section 2.2. Section 2.3 will describe the development of sliding mode observers and finally functional state estimation using the sliding mode observer will be shown Section 2.4.

2.1 Linear Observer

Consider the linear time-invariant sytem described by

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{2.1}$$

where $x \in \mathbb{R}^n$ is the state, $y \in \mathbb{R}^p$ is the output, and $u \in \mathbb{R}^m$ is the input.

The first observer was presented by Luenberger [7] in the following form

$$\dot{\hat{x}} = A\hat{x} + Bu - L(C\hat{x} - y) \tag{2.2}$$

where $\hat{x}$ is the estimate of the state x. A schematic diagram from the system (2.1) and observer (2.2) is shown in Figure 2.1. To analyze the performance of the observer, define $e = \hat{x} - x$ as the state estimation error; then from (2.1) and (2.2), it can be found that

$$\dot{e} = (A - LC)e \tag{2.3}$$

Since there is no 'external input' for the error system in (2.3), the state estimation error, e will converge to zero exponentially and the estimated state, $\hat{x}$ will converge to the actual state x if matrix L is chosen such that $\lambda(A - LC)$ (eigenvalues of $(A - LC)$) has negative real parts.

In the works of Luenberger [8, 7], the focus was only on estimating the system states for continuous-time linear time-invariant systems. Later on, many researchers have extended the observer theory to estimate the state of time-varying systems [9, 10, 11], discrete-time systems [12, 9] and stochastic systems [13, 14].

2.1.1 Robust State Estimation

In reality, all dynamic systems cannot be represented in the ideal state-space representation as in (2.1); there is usually mismatch between the actual system and model (around which the observer is designed about), such as unmodeled dynamics, varying parameters and external disturbances. The dynamic system can then be better represented by:

$$\begin{aligned} \dot{x} &= Ax + Bu + Df \\ y &= Cx \end{aligned} \tag{2.4}$$

where $f \in \mathbb{R}^q$ encapsulates all disturbances within the system (also known as unknown input). A popular observer scheme that can be used to estimate the state is the *Unknown Input Observer* (UIO), which is a linear observer and has been well-researched by [15,

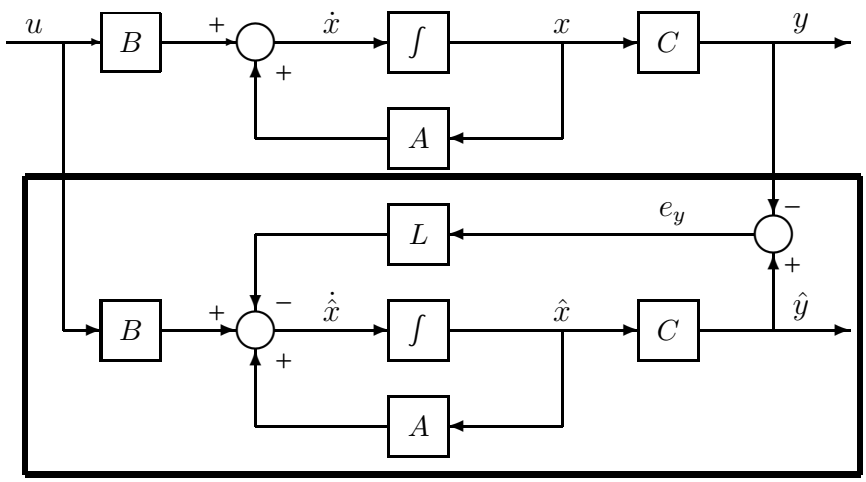

Figure 2.1: Schematic diagram of system (2.1) and observer (2.2).

16, 17, 18, 19, 20, 21, 22]. An UIO by Darouach *et al.* [23] has the following form

$$\begin{aligned}\dot{z} &= Nz + Hy + Gu \\ \hat{x} &= z - Ey\end{aligned} \tag{2.5}$$

where $z \in \mathbb{R}^n$ and $\hat{x} \in \mathbb{R}^n$. Matrices (N, H, G, E) are to be determined later. By subtracting (2.4) from (2.5), the following error system results

$$\dot{e} = Ne + (NP + HC - PA)x + (G - PB)u - PDf \tag{2.6}$$

where $P = I_n + EC$. For the state estimation error e to converge independently of f, the matrix N must be chosen to be stable, where

$$N = PA - KC \tag{2.7}$$

and the rest of terms in (2.6) will have the following solution:

$$\begin{aligned}H &= K(I_p + CE) - PAE \\ E &= -D(CD)^{\dagger} + Y(I_p - (CD)(CD)^{\dagger}); \\ G &= PB \\ PD &= 0\end{aligned} \tag{2.8}$$

where $(CD)^{\dagger}$ is the generalized inverse matrix for CD (where $(CD)^{\dagger}(CD) = I_q$), and Y is an arbitrary matrix of appropriate dimensions. Matrices (N, H, G, E) can be designed to satisfy equations (2.8) and matrix K can be chosen such that N is stable if and only if

- $\mathrm{rank}(CD) = \mathrm{rank}(D)$
- invariant zeros of (A, D, C) must be stable

2.2 Functional State Estimation

Estimating the entire state vector may not be practical for high-dimensional systems as it requires high computational power and causes the resulting observer to be too sensitive to measurement noise [1, 24]. In many situations, only part of the state information is

required. For example, only the position vectors are required for the feedback controller design in stabilizing the AMB system [2]. Apart from that, not estimating the entire state was found to be more accurate and less restrictive compared to the conventional observer in the application on a high-dimensional chemical distillation process [1] and linear motor states estimation. [3].

A *functional observer*, which only estimates a function or subset of the state vector, can be implemented to reduce the observer complexity [25]. The state to be estimated can be modeled in the following form:

$$z = Fx \tag{2.9}$$

where $F \in \mathbb{R}^{r \times n}$ has full row rank. The problem of observing a linear function of the states was first reported in [26], where any single linear combination of the state vector can be estimated by the linear functional observer of order $(\epsilon - 1)$, where ϵ is the observability index, defined as the lest positive integer for which the following condition holds:

- $\operatorname{rank} \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{\epsilon-1} \end{bmatrix} = n$

Following the work of Luenberger [26], significant amounts of work have been done to obtain the minimal-order functional observer and many algorithms have been developed to estimate the functional state [27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37]. Fortmann & Williamson [38] were the first one to derive the necessary and sufficient for the single-output case, and Moore & Lewich [39] extended the result of Fortmann & Williamson to the multiple-output case. However, additional computational effort is required to obtain some of the variables first before the conditions in [38, 39] can be tested. Then, Watson & Grigoriadis [40] developed the necessary and sufficient conditions for the existence of functional observer, which can be written in terms of the original system matrices as follows:

- $\operatorname{rank} \begin{bmatrix} FA \\ C \\ F \end{bmatrix} = \operatorname{rank} \begin{bmatrix} C \\ F \end{bmatrix}$

However, it was proven by Darouach in [41] that these conditions are only sufficient in general and he extended this result to the following necessary and sufficient conditions

- $\text{rank}\begin{bmatrix} FA \\ CA \\ C \\ F \end{bmatrix} = \text{rank}\begin{bmatrix} CA \\ C \\ F \end{bmatrix}$
- $\text{rank}\begin{bmatrix} sF - FA \\ CA \\ C \end{bmatrix} = \text{rank}\begin{bmatrix} CA \\ C \\ F \end{bmatrix}, \ \forall s \in \mathbb{C}, Re(s) \geq 0$

Motivated by the work of Darouach [41], Trinh & Fernando [42] provides the upper bound on the order of the functional observer for the case when the conditions in [41] is not satisfied. Fernando *et al.* [43] then presented the idea of functional observability/detectability where the triplet (A, C, F) could be used to ensure that the system is capable to accurately estimate the linear function of the states from the available the system inputs and outputs. A recent work published by Rotella & Zambettakis [37] provides a relatively simple solution for minimal-order observer design. However, an iterative process is required if the necessary and sufficient conditions are not satisfied. Therefore, this approach is not suitable for a high-dimensional system as it might require a higher number of iteration, which increases the computational complexity significantly. Practically, the conditions presented in [41] are generally accepted by most researchers when constructing a linear functional observer.

2.2.1 Functional State Estimation for Systems with Unknown Inputs

Design of functional observer for systems with unknown input were first considered by Trinh & Ha [44] and Xiong & Saif [45]. Then, Darouach [46] extended the result in [41] for system with unknown input, and the conditions in [41] become

- $\text{rank}\begin{bmatrix} FA & FD \\ CA & CD \\ C & 0 \\ F & 0 \end{bmatrix} = \text{rank}\begin{bmatrix} CA & CD \\ C & 0 \\ F & 0 \end{bmatrix}$
- $\text{rank}\begin{bmatrix} sF - FA & FD \\ CA & CD \\ C & 0 \end{bmatrix} = \text{rank}\begin{bmatrix} CA & CD \\ C & 0 \\ F & 0 \end{bmatrix}, \ \forall s \in \mathbb{C}, Re(s) \geq 0$

Then, Trinh *et al.* [47] developed a scheme that required less restrictive existence conditions to include the case where unknown inputs are more than outputs, which could be used in place of the well-known observer matching condition.

Apart from the general theory development, the problem of estimating a linear function of the states for various systems such as time-delay system [48, 49, 50, 51, 52], discrete-time system [53, 51] and descriptor system [54, 55, 56, 57, 58, 59, 60] were available in the literature. In addition, lab experiments on double inverted pendulum using a linear functional observer done by Furuta & Suzuki [24] were found to be more stable and required less computational time compare to a minimal-order state observer. Besides, functional state estimation for high-dimensional chemical distillation processes [1] and linear motor [3] were found to be more accurate and less restrictive compared to the conventional observer.

2.3 Sliding Mode Observers

Sliding mode observers were first introduced by Utkin [4] in 1977. A nonlinear discontinuous term dependent on the output estimation error, $e_y = C\hat{x} - y$ was injected into the observer to drive the trajectories of the error system onto the sliding surface $\mathcal{S} = \{e : Ce = 0\}$, and force them to remain there. This is unlike the linear observers where $e_y \to 0$ when $t \to \infty$. A reduced-order sliding motion takes place when e_y is forced to zero in *finite time* and the estimate of the non-output states would converge to the actual non-output states. Consider the following linear uncertain system:

$$\begin{aligned} \dot{x} &= Ax + Bu + Df \\ y &= Cx \end{aligned} \tag{2.10}$$

where $x \in \mathbb{R}^n$ is the state, $y \in \mathbb{R}^p$ is the output, $u \in \mathbb{R}^m$ is the input and $f \in \mathbb{R}^q$ is the external disturbance. The observer proposed by Utkin [4] can be expressed as follow:

$$\dot{\hat{x}} = A\hat{x} + Bu + G_n\nu \tag{2.11}$$

where $\hat{x}$ is the estimate of x. Let $e := \hat{x} - x$ be the state estimation error, where

$$\dot{e} = Ae + G_n\nu - Df \tag{2.12}$$

and the nonlinear switching term, ν can be defined as:

$$\nu_i = -\rho\frac{e_{y,i}}{|e_{y,i}|},\ \rho \in \mathbb{R}_+ \tag{2.13}$$

It was found that provided ρ is chosen to be large enough, then all components of e_y would converge to zero in finite time, after which *sliding motion* is said to have occured. During sliding motion, the other components (non-output) of e will experience a first order decay towards zero. Provided certain conditions are met, the state estimation error is not affected by the signal f; this is a typical property of sliding mode observers which makes it very attractive. A larger value of ρ would enlarge the sliding patch (the set of initial conditions of e where sliding motion takes place). However, increasing the value of ρ would increase the chattering, which is highly undesirable in practical systems.

Slotine *et al.* [61] had improved the observer design by feeding back the outputs estimation error, e_y in both linear and discontinuous behavior, where the sliding mode observer can be expressed as:

$$\dot{\hat{x}} = A\hat{x} + Bu - G_l e_y + G_n\nu \tag{2.14}$$

It was found that an appropriate choice of G_l would increase the size of the sliding patch. Then, Walcott & Zak [62] formulated an observer design method where under appropriate assumptions, the error system were proven to be quadratically stable despite the presence of the bounded uncertainties. However, a symbolic manipulation package might be required for a large scale system as the calculation involved is complicated.

Following the work of Walcott & Zak [62], Edwards & Spurgeon [5] formulated a systematic method to improve the observer design presented by Walcott & Zak. The necessary and sufficient conditions for the observer to exist and robustly estimate the state, x independent of the disturbance, f were given in terms of the original system matrices [63] as follows:

- $\mathrm{rank}(CD) = \mathrm{rank}(D) = q$

- invariant zeros of (A, D, C) must be stable

The superb system performance of sliding mode observers has made it a promising approach for observer design. Comparative studies had been carried out by Chen & Dunnigan [64] and Zhang *et al.* [65] to evaluate the performance of Kalman filter, Luenberger observer and sliding mode observer for state estimation in an induction machine. Both studies came to a similar conclusion, where the Luenberger observer and the sliding mode observer were easier to implement while sliding mode observer was more robust towards parameter uncertainties and external noise. Then, Edwards & Tan [66] compared the ability of sliding mode observers in reconstructing faults with the linear unknown input observer and found the sliding mode observer performed better.

On top of that, the sliding mode technique is widely used in industrial systems. One of the most common industrial applications is induction motor control [67, 68, 69]. Also, sliding mode observer is applicable to various type of system such as speed estimation for robotic manipulator [70], heat estimation for state monitoring purpose in a continuous stirred tank [71] and position control for pneumatic actuator [72]. The works reported shared a common outcome, where the sliding mode observer was found to be robust towards parameters uncertainties and insensitive to external noise.

At the same time, it was found that the discontinuous term ν was able to track the unknown input (which could represent disturbances, faults or nonlinearities); this ability has then stimulated the development of fault detection and isolation schemes using sliding mode observers [63, 73, 74, 75, 76, 77, 78]. An extensive review of sliding mode observers is available in [6].

2.3.1 Cascaded Sliding Mode Observers

Following the work of Edwards *et al.* [63], Tan *et al.* [79, 80] proposed a scheme using two sliding mode observers in cascade, where it was found that signals from the first observer were analytically found to be the output of a ‘fictitious’ system, then a second sliding mode observer was implemented on the fictitious system; it was found that the scheme had less restrictive existence conditions compared to Edwards *et al.* [63]. Then, Tan [81,

82] further extended the work by using multiple observers in cascade, and investigated the existence conditions which were found to be more relaxed. This scheme will be expounded in greater detail in Sections 3.2.3 and 3.2.4.

2.4 Functional State Estimation using Sliding Mode Observers

From the survey of literature, the functional observer approach can be used to eliminate redundancy possessed in full state estimation [24, 1], while the sliding mode observer has better performance compare to the linear unknown input observer [66]. Thus, a combination of both observer theories was believed to be able to improve overall performance. However, not much work has been published on this particular topic. The idea of sliding mode functional observer was first presented by Xiong & Saif [83] where they modified the Walcott & Zak [62] sliding mode observer to estimate only the functional state. Then, Fernando *et al.* presented a series of studies about sliding mode functional observers [84, 56, 85, 86, 87]. The proposed observer in [84] had injected the nonlinear switching function directly into the conventional functional observer; and the conditions for the sliding mode functional observer to exist are similar to the one presented by Darouach in [41]. The sliding mode functional observer concept is then extended to a system with unknown input by Nikraz *et al.* [87] and the existence conditions for the observer are given as follow:

- $rank \begin{bmatrix} FA & FD \\ CA & CD \\ C & 0 \\ F & 0 \end{bmatrix} = \text{rank} \begin{bmatrix} CA & CD \\ C & 0 \\ F & 0 \end{bmatrix}$
- $\text{rank} \begin{bmatrix} sF - FA & -FD \\ CA & CD \\ C & 0 \end{bmatrix} = \text{rank} \begin{bmatrix} CA & CD \\ C & 0 \\ F & 0 \end{bmatrix} \forall s \in \mathbb{C}, Re(s) \geq 0$

which is similar to the one in [46] that esitmate the functional states of system with unknown input. Also, a descriptor system approach was employed by Nikraz *et al.* [86] to solve the functional estimation problem with unknown input. At the same time, the theory of sliding mode functional observer has been implemented on nonlinear systems by Fernando *et al.* [85] and descriptor systems by Alahakoon *et al.* [56]

2.5 Conclusion and Direction on Research Thesis

This chapter has presented an introduction of observer and how it estimates the unavailable states. Following that, robust estimation in the presence of unknown inputs was explored. Then, the problem of functional estimation was discussed, where it was found that not estimating the entire state had several advantages such as being less restrictive and consuming less computational power (especially for high-dimensional system). The sliding mode observer was also described in this chapter where it was found that it has better performance than conventional linear observers. As a result, a combination of both functional observer and sliding mode observer was believed to be able to improve overall performance. However, the literature survey shown that not much work has been published on this particular topic. Therefore, this thesis will further explore the use of sliding mode observer schemes for functional state estimation.

Chapter 3

Sliding Mode Observers and its Application to Functional State Estimation

3.1 Introduction

This chapter will describe the sliding mode observer by Edwards & Spurgeon [5] which will be used in the work of this thesis, as well as further developments of its use. Then its application to functional state estimation will be explored. It begins with the observer structure, followed by a systematic design method (Section 3.2.1) and the properties of sliding motion are presented (Section 3.3.2). Following that, in Sections 3.2.3 and 3.2.4 the works of Tan *et al.* [79, 81, 80] are presented, where several observers are placed in cascade to relax one of the existence conditions. Finally, Section 3.3 ends this chapter with demonstrating how the observer can be used for functional state estimation; in addition its existence conditions are investigated and presented in terms of the original system matrices. Numerical examples at various parts of the chapter are used to illustrate the theories presented.

3.2 The Edwards-Spurgeon Observer

Consider the uncertain linear time-invariant system described by

$$\begin{aligned} \dot{x} &= Ax + Bu + Df \\ y &= Cx \end{aligned} \tag{3.1}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^p$, $u \in \mathbb{R}^m$ and $f \in \mathbb{R}^q$ represent the state, output, control input and unknown input respectively. The matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$ and $D \in \mathbb{R}^{n\times q}$

are known and constant; without loss of generality, it is assumed that $\mathrm{rank}(C) = p$ and $\mathrm{rank}(D) = q$ and also assume $q < p \leq n$. The function $f : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^q$ is unknown but bounded such that $\|f\| \leq \alpha$ where $\alpha : \mathbb{R}_+ \times \mathbb{R}^p \to \mathbb{R}_+$ is a known function which in practice can be found empirically to estimate the magnitude of the uncertainty f [62]. Edwards & Spurgeon [5] have proven that if $p > q$ and if system (3.1) satisfies the following conditions

A1. $\mathrm{rank}(CD) = \mathrm{rank}(D) = q$

A2. $\mathrm{rank}\begin{bmatrix} sI - A & D \\ C & 0 \end{bmatrix} = n + \mathrm{rank}(D), \ \forall s \in \mathbb{C}, Re(s) \geq 0$

then there exists a sliding mode observer that can accurately estimate the system state, x as well as the unknown input f. This will be shown in Sections 3.2.1 and 3.2.2.

Lemma 3.1 *[5] If and only if condition A1 is satisfied, a linear state transformation $x \mapsto T_o x$ exist such that the system matrices (A, D, C) from* (3.1) *will have the following structure*

$$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}, \ D = \begin{bmatrix} 0 \\ D_2 \end{bmatrix} \begin{matrix} \updownarrow n-p \\ \updownarrow p \end{matrix}, \ C = \begin{bmatrix} 0 & C_2 \end{bmatrix} \tag{3.2}$$

where $C_2 \in \mathbb{R}^{p \times p}$ is orthogonal and A_3 and D_2 can be further partitioned as follows

$$A_3 = \begin{bmatrix} A_{31} \\ A_{32} \end{bmatrix}, \ D_2 = \begin{bmatrix} 0 \\ D_o \end{bmatrix} \begin{matrix} \updownarrow p-q \\ \updownarrow q \end{matrix}$$

with D_o being full rank. ♯

The observer proposed by Edwards & Spurgeon [5] is as follows:

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bu - G_l e_y + G_n \nu \\ \hat{y} &= C\hat{x} \end{aligned} \tag{3.3}$$

where $(\hat{x}, \hat{y})$ are the estimates of (x, y). The nonlinear discontinuous output error injection vector ν is defined as

$$\nu = -\rho \frac{e_y}{\|e_y\|} \quad \text{if} \quad e_y \neq 0 \tag{3.4}$$

where $e_y = \hat{y} - y$ (the output estimation error) and ρ is a positive scalar dependent on the magnitude of the uncertainty. The gain matrices, $G_l \in \mathbb{R}^{n\times p}$ and $G_n \in \mathbb{R}^{n\times p}$ are to be determined and in particular G_n has the form

$$G_n = \begin{bmatrix} -LC_2^{-1} \\ C_2^{-1} \end{bmatrix} P_o^{-1} \tag{3.5}$$

where $P_o \in \mathbb{R}^{p\times p}$ is a symmetric positive definite matrix and $L = \begin{bmatrix} L_o & 0 \end{bmatrix}$, where $L_o \in \mathbb{R}^{(n-p)\times(p-q)}$ is designed such that $A_1 + L_o A_{31}$ is stable. Define the state estimation error $e := \hat{x} - x$ and subtracting equations (3.1) from (3.3) yields

$$\dot{e} = (A - G_l C)e + G_n \nu - Df \tag{3.6}$$

3.2.1 Design of Observer to Achieve Sliding Motion

Essentially, the design of the observer is to calculate P_o, L_o and G_l such that a stable sliding motion takes place on the surface $e_y = 0$. Various design methods have been published, each with different objectives; in [5] a simple but systematic procedure was proposed, whereas in [88] the observer was designed using optimal observation principles so that the designer could systematically tune the magnitude of the gains, and in [89] the observer was designed for robust fault reconstruction. In this thesis, the design method by Edwards & Spurgeon [5] will be used, and it is described in the following:

After the structure in (3.2) has been achieved, choose L_o such that $A_1 + L_o A_{31}$ is stable (the possibility of doing this will be described later in Remark 3.1). Then introduce a change of coordinates T_L, where

$$T_L = \begin{bmatrix} I_{n-p} & L \\ 0 & C_2 \end{bmatrix} \tag{3.7}$$

such that the system matrices (A, D, C) in (3.2) can be further transformed to become

$$\mathcal{A} = \begin{bmatrix} \mathcal{A}_1 & \mathcal{A}_2 \\ \mathcal{A}_3 & \mathcal{A}_4 \end{bmatrix}, \ \mathcal{D} = \begin{bmatrix} 0 \\ \mathcal{D}_2 \end{bmatrix} \begin{matrix} \updownarrow n-p \\ \updownarrow p \end{matrix}, \ \mathcal{C} = \begin{bmatrix} 0 & I_p \end{bmatrix} \tag{3.8}$$

where $\mathcal{A}_1 = A_1 + L_o A_{31}$ (which is stable) and $\mathcal{D}_2 = C_2 D_2$. Thus, the state estimation error (3.6) in the coordinates of (3.8) becomes

$$\dot{e}_L = \mathcal{A}_o e_L + \mathcal{G}_n \nu - \mathcal{D} f \tag{3.9}$$

where $\mathcal{A}_o = \mathcal{A} - \mathcal{G}_l\mathcal{C}$. The gain $\mathcal{G}_n$ (from (3.5)) in this coordinate system will then have the following structure

$$\mathcal{G}_n = \begin{bmatrix} 0 \\ P_o^{-1} \end{bmatrix} \tag{3.10}$$

Choose $\mathcal{G}_l$ as follows

$$\mathcal{G}_l = \begin{bmatrix} \mathcal{A}_2 \\ \mathcal{A}_4 - \mathcal{A}_4^s \end{bmatrix} \tag{3.11}$$

where $\mathcal{A}_4^s$ is a stable design matrix, and calculate a suitable P_o such that

$$P_o\mathcal{A}_4^s + (\mathcal{A}_4^s)^T P_o < 0 \tag{3.12}$$

Then the error system (3.9) can be partitioned to be

$$\dot{e}_1 = \mathcal{A}_1 e_1 \tag{3.13}$$

$$\dot{e}_y = \mathcal{A}_3 e_1 + \mathcal{A}_4^s e_y + P_o^{-1}\nu - \mathcal{D}_2 f \tag{3.14}$$

Lemma 3.2 *If ρ from* (3.4) *is chosen to be large enough, in particular $\rho > \|CD\|\alpha + \gamma$, $\gamma > 0$, then an ideal sliding motion takes place in finite time on the surface $e_y = 0$.* ♯

Proof

Define Lyapunov function $V = e_y^T P_o e_y$ and differentiate with respect to time to get

$$\begin{aligned} \dot{V} &= \dot{e}_y^T P_o e_y + e_y^T P_o \dot{e}_y \\ &= e_y^T (P_o\mathcal{A}_4^s + (\mathcal{A}_4^s)^T P_o) e_y + 2e_y^T P_o \mathcal{A}_3 e_1 - 2e_y^T P_o D_2 f + 2e_y^T \nu \end{aligned} \tag{3.15}$$

Since from (3.12) P_o has been chosen such that $P_o\mathcal{A}_4^s + (\mathcal{A}_4^s)^T P_o < 0$, then

$$\begin{aligned} \dot{V} &\leq -2\rho\|e_y\| - 2e_y^T P_o D_2 f + 2e_y^T P_o \mathcal{A}_3 e_1 \\ &\leq -2\|e_y\|(\rho - \|P_o CD\|\alpha - \|P_o\mathcal{A}_3 e_1\|) \end{aligned} \tag{3.16}$$

Since $\mathcal{A}_1$ has been designed to be stable, then from (3.13) in finite time $\|P_o\mathcal{A}_3 e_1\| < \gamma$ will be achieved and (3.16) will become

$$\dot{V} \leq -2\mu\|e_y\| \tag{3.17}$$

where $\mu = \rho - \|P_o CD\|\alpha - \gamma > 0$.

Inequality (3.17) is known as the *reachibility condition* which guarantees sliding motion as will be proven as follows:

From Rayleigh's inequality [90], $e_y^T P_o e_y \geq \|e_y\|^2 \lambda_{min}(P_o) \geq 0$ and hence $\|e_y\| \leq \sqrt{\frac{V}{\lambda_{min}(P_o)}}$.

Therefore, (3.17) becomes

$$\begin{aligned}
\frac{dV}{dt} &\leq -2\mu\sqrt{\frac{V}{\lambda_{min}(P_o)}} \\
\frac{1}{\sqrt{V}}\,dV &\leq -2\mu\sqrt{\frac{1}{\lambda_{min}(P_o)}}\,dt \\
\int_{V=V_o}^{V=0} \frac{1}{\sqrt{V}}\,dV &\leq -2\mu \int_{t=t_o}^{t=t_s} \sqrt{\frac{1}{\lambda_{min}(P_o)}}\,dt \\
-2\mu\sqrt{\frac{1}{\lambda_{min}(P_o)}}\,t_s &\leq -2\sqrt{V_o} \\
\therefore \quad t_s &\leq \frac{\sqrt{\lambda_{min}(P_o)V_o}}{\mu}
\end{aligned} \tag{3.18}$$

Notice that t_s is the time taken for V (and hence e_y also) to reach 0. Since $t_s < \infty$, this proves that $V = 0$ (and hence $e_y = 0$) is achieved in finite time. ■

3.2.2 Properties of the Sliding Motion

When sliding motion occurs, then $e_y = \dot{e}_y = 0$ and (3.13) - (3.14) becomes

$$\begin{aligned}
\dot{e}_1 &= \mathcal{A}_1 e_1 \\
0 &= \mathcal{A}_3 e_1 + P_o^{-1}\nu_{eq} - \mathcal{D}_2 f
\end{aligned} \tag{3.19}$$

where ν_{eq} is the equivalent output error injection term that is required to maintain the sliding motion. Notice that the error dynamics have been reduced from order n to order $n - p$; this reduced order motion is a typical property of sliding mode systems. Since L_o is chosen to make $\mathcal{A}_1$ stable, then $e_1 \to 0$ and the entire error vector e will converge asymptotically. The second equation of (3.19) becomes

$$P_o^{-1}\nu_{eq} \to \mathcal{D}_2 f \tag{3.20}$$

Define $\hat{f} = (\mathcal{D}_2^T\mathcal{D}_2)^{-1}\mathcal{D}_2^T P_o^{-1}\nu_{eq}$, then from (3.19) it is clear that $\hat{f} \to f$; hence the observer is also capable of estimating the unknown input [91, 63, 92]. The term ν_{eq} can be

approximated to any degree of accuracy by replacing (3.4) with

$$\nu = -\rho \frac{e_y}{\|e_y\| + \delta} \tag{3.21}$$

where δ is a small positive constant. In this situation, the error trajectories do not slide perfectly on the surface of $\mathcal{S}$, but within a small boundary layer around it [93]. In practice, the choice of δ is a trade-off between ideal performance and maintaining a smooth output error injection.

Remark 3.1 *From [5], condition A1 is necessary and sufficient for the attainment of the coordinate system* (3.8) *which yields a $\mathcal{D}_2$ that is full column rank, which is in turn necessary and sufficient for $(\mathcal{D}_2^T \mathcal{D}_2)^{-1}$ to exist and $\hat{f} \to f$.*
Condition A2 implies that (A, D, C) is minimum phase, implying that its zeros are stable. In [5] it was shown that the invariant zeros of (A, D, C) are the unobservable modes of (A_1, A_{31}), which need to be stable so that $\mathcal{A}_1$ can be stable, failing which e_1 will not converge and (3.20) *cannot be achieved and f cannot be estimated.* ♯

3.2.3 Two Sliding Mode Observers in Cascade

For the case when condition A1 is not satisfied, i.e. $b = \mathrm{rank}(CD) < \mathrm{rank}(D) = q$, the error e_1 will not converge to zero and f cannot be estimated asymptotically. To overcome this constraint, Ng *et al.* [79] proposed a scheme using two sliding mode observers in cascade (which will be discussed in this subsection). Figure 3.1 shows a schematic diagram of the scheme presented in [79]. The primary observer can be designed using the method presented in Section 3.2.1. Then, the term ν from the primary sliding mode observer can be scaled and filtered to produce $\tilde{y}$, which will serve as the input of the secondary observer. The design of the secondary sliding mode observer will be discussed later. Firstly, the following canonical form is introduced.

Lemma 3.3 *[79] There exist linear transformations $x \mapsto T_s x$, $f \mapsto T_f f$ such that the system matrices from* (3.1) *can be transformed into the following structure*

$$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}, \; D = \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} \begin{matrix} \updownarrow n-p \\ \updownarrow p \end{matrix}, \; C = \begin{bmatrix} 0 & C_2 \end{bmatrix} \tag{3.22}$$

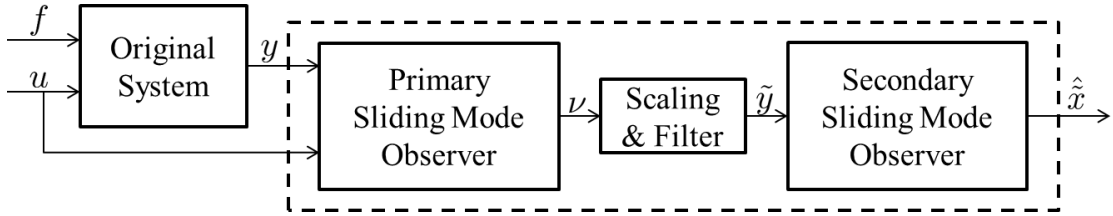

Figure 3.1: Schematic diagram of the two sliding mode observer scheme

where $C_2 \in \mathbb{R}^{p\times p}$ is orthogonal and D_1 and D_2 can be further partitioned into

$$D_1 = \begin{bmatrix} 0 & 0 \\ D_{11} & 0 \end{bmatrix} \begin{matrix} \updownarrow n-p-q+b \\ \updownarrow q-b \end{matrix}, \quad D_2 = \begin{bmatrix} 0 & 0 \\ 0 & D_{22} \end{bmatrix} \begin{matrix} \updownarrow p-b \\ \updownarrow b \end{matrix}$$

where $D_{11} \in \mathbb{R}^{(q-b)\times(q-b)}$ and $D_{22} \in \mathbb{R}^{b\times b}$ are square and invertible. ♯

Notice that if condition A1 is satisfied, then D_{11} does not exist and $D_1 = 0$ and the form in (3.2) is attained. In the coordinate system of (3.22), partition A_3 as

$$A_3 = \begin{bmatrix} A_{31} \\ A_{32} \end{bmatrix} = \begin{bmatrix} A_{311} & A_{312} \\ A_{321} & A_{322} \end{bmatrix} \tag{3.23}$$

where $A_{311} \in \mathbb{R}^{(p-b)\times(n-p-q+b)}$ and $A_{312} \in \mathbb{R}^{(p-b)\times(q-b)}$. Implementing the primary observer as in Section 3.2.1 (but this time L has no specific structure as in (3.5), as long as $A_1 + LA_3$ is stable) and analyzing its performance as in (3.9) - (3.19), the following error system (with similar notation as (3.14)) will result

$$\begin{aligned} \dot{e}_1 &= \mathcal{A}_1 e_1 - \mathcal{D}_1 f \\ \dot{e}_y &= \mathcal{A}_3 e_1 + \mathcal{A}_4^s e_y + P_o^{-1}\nu - \mathcal{D}_2 f \end{aligned} \tag{3.24}$$

where $\mathcal{D}_1 = D_1 + LD_2$.

Lemma 3.4 *Suppose there exist a symmetric positive definite matrix P_1 which is a Lyapunov matrix for $\mathcal{A}_1$, hence*

$$P_1\mathcal{A}_1 + \mathcal{A}_1^T P_1 < 0 \tag{3.25}$$

If $\rho > \frac{2\|P_o\mathcal{A}_3\|\|P_1 D_1\|\alpha}{\mu_1} + \|P_o D_2\|\alpha$, where $\mu_1 = -\lambda_{max}(P_1\mathcal{A}_1 + \mathcal{A}_1^T P_1)$ then sliding motion is achieved. ♯

Proof

Consider the Lyapunov function $V_1 = e_1^T P_1 e_1$ and its derivative

$$\dot{V}_1 = e_1^T (P_1 \mathcal{A}_1 + \mathcal{A}_1^T P_1) e_1 - 2 e_1^T P_1 D_1 f \tag{3.26}$$

From Rayleigh's inequality,

$$e_1^T (P_1 \mathcal{A}_1 + \mathcal{A}_1^T P_1) e_1 \leq \|e_1\|^2 \underbrace{\lambda_{max}(P_1 \mathcal{A}_1 + \mathcal{A}_1^T P_1)}_{-\mu_1} \tag{3.27}$$

where $\mu_1 > 0$. Then, inequality (3.26) becomes

$$\begin{aligned} \dot{V}_1 &< -\mu \|e_1\|^2 + 2\|e_1\| \|P_1 D_1\| \alpha \\ &< -\|e_1\| (\mu \|e_1\| - 2\|P_1 D_1\| \alpha) \end{aligned} \tag{3.28}$$

From Lemma 1 of [89], (3.28) implies that $\|e_1\|$ will be ultimately bounded by

$$\|e_1\| < \frac{2\|P_1 D_1\| \alpha}{\mu} \tag{3.29}$$

Define $V_2 = e_y^T P_o e_y$, and

$$\begin{aligned} \dot{V}_2 &= e_y^T (P_o \mathcal{A}_4^s + (\mathcal{A}_4^s)^T P_o) e_y + 2 e_y^T P_o \mathcal{A}_3 e_1 + 2 e_y^T \nu - 2 e_y^T P_o D_2 f \\ &\leq -2\rho \|e_y\| - 2\|e_y\| \|P_o D_2\| f + 2\|e_y\| \|P_o \mathcal{A}_3\| \|e_1\| \\ &\leq -2\|e_y\| (\rho - \|P_o \mathcal{A}_3\| \|e_1\| - \|P_o D_2\| \alpha) \end{aligned} \tag{3.30}$$

Using (3.29), if $\rho > \frac{2\|P_o \mathcal{A}_3\| \|P_1 D_1\| \alpha}{\mu} + \|P_o D_2\| \alpha$, then the reachability condition (3.17) is achieved and sliding motion takes place. ■

Define $\upsilon := C_2^{-1} P_o^{-1} \nu$, then when sliding motion has taken place, (3.24) can be re-arranged in the coordinates of (3.2) to become:

$$\begin{aligned} \dot{e}_1 &= (A_1 + L A_3) e_1 - (D_1 + L D_2) f \\ \upsilon &= -A_3 e_1 + D_2 f \end{aligned} \tag{3.31}$$

Partition υ according to (3.22) and (3.23) to get

$$\begin{aligned} \upsilon_1 &= -A_{31} e_1 \\ \upsilon_2 &= -A_{32} e_1 + \underbrace{\begin{bmatrix} 0 & D_{22} \end{bmatrix}}_{D_{2,b}} f \end{aligned} \tag{3.32}$$

Filter υ_2 to generate υ_f via $\dot{\upsilon}_f = -A_f\upsilon_f + A_f\upsilon_2$, where $A_f \in \mathbb{R}^{b\times b}$ has positive eigenvalues. Hence by substituting from (3.32), υ_f has the analytical expression

$$\dot{\upsilon_f} = -A_f\upsilon_f - A_f A_{32}e_1 + A_f D_{2,b} f \tag{3.33}$$

Combine (3.31) and (3.33) to get the following dynamic system:

$$\begin{aligned}
\underbrace{\begin{bmatrix} \dot{e}_1 \\ \dot{\upsilon}_f \end{bmatrix}}_{\dot{\tilde{x}}} &= \underbrace{\begin{bmatrix} A_1 + LA_3 & 0 \\ -A_f A_{32} & -A_f \end{bmatrix}}_{\tilde{A}} \underbrace{\begin{bmatrix} e_1 \\ \upsilon_f \end{bmatrix}}_{\tilde{x}} + \underbrace{\begin{bmatrix} -(D_1 + LD_2) \\ A_f D_{2,b} \end{bmatrix}}_{\tilde{D}} f \\
\underbrace{\begin{bmatrix} \bar{\upsilon}_1 \\ \upsilon_f \end{bmatrix}}_{\tilde{y}} &= \underbrace{\begin{bmatrix} -A_{31} & 0 \\ 0 & I_b \end{bmatrix}}_{\tilde{C}} \underbrace{\begin{bmatrix} e_1 \\ \upsilon_f \end{bmatrix}}_{\tilde{x}}
\end{aligned} \tag{3.34}$$

It is obvious that suitable processing of ν has resulted in $\tilde{y}$ which analytically is the output of the system (3.34) that treats f as the unknown input. Hence by applying the standard sliding mode observer theory (conditions A1 and A2 from the previous subsection) on (3.34), if (and only if) the following conditions are satisfied:

B1. $\operatorname{rank}(\tilde{C}\tilde{D}) = \operatorname{rank}(\tilde{D}) = q$

B2. $\operatorname{rank}\begin{bmatrix} sI - \tilde{A} & \tilde{D} \\ \tilde{C} & 0 \end{bmatrix} = n - p + b + \operatorname{rank}(\tilde{D}),\ \forall s \in \mathbb{C},\ Re(s) \geq 0$

then the 'state' $\tilde{x}$ and f can be asymptotically estimated. Define $\hat{e}_1$ as the estimate of e_1, which can be obtained from the estimate of $\tilde{x}$. Then define a measurable signal $\hat{x}_{1,e} = \hat{x}_1 - \hat{e}_1$ where $\hat{x}_1$ is from the first observer. When $\hat{e}_1$ is asymptotically estimated, $\hat{e}_1 \to e_1$ resulting in $\hat{x}_{1,e} \to \hat{x}_1 - e_1 = x_1$. Since sliding motion is achieved in the first observer, the output y is already estimated in finite time and hence the entire state x is esimated asymptotically using this two-observer scheme.

In [79], conditions B1 and B2 can be re-casted in terms of the original sytem matrices as:

C1. $\operatorname{rank}\begin{bmatrix} CAD & CD \\ CD & 0 \end{bmatrix} - \operatorname{rank}(CD) = \operatorname{rank}(D)$

C2. $\operatorname{rank}\begin{bmatrix} sI - A & D \\ C & 0 \end{bmatrix} = n + \operatorname{rank}(D),\ \forall s \in \mathbb{C}, Re(s) \geq 0$

Remark 3.2 *An important implication of this result (which cast the conditions in terms of the original system matrices) is that the existence of the second observer (and feasibility of the whole scheme) does not depend on L which is design freedom in the first observer during sliding motion (see equation* (3.31)*). This means that the first observer can be designed independently without any concern of affecting the feasibility of the overall scheme.* ♯

This scheme will now be illustrated with an example. Consider a second order double integrator with the following matrices:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \; D = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; C = \begin{bmatrix} 1 & 0 \end{bmatrix} \tag{3.35}$$

From the matrices above, it is obvious that $\mathrm{rank}(D) = 1$, $\mathrm{rank}(CD) = 0$ (hence A1 is not satisfied) and

$$\mathrm{rank}\begin{bmatrix} CAD & CD \\ CD & 0 \end{bmatrix} = 1 \tag{3.36}$$

Hence, condition C1 is satisfied and the structure of (3.22) can be achieved with a suitable choice of

$$T_s = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \; T_f = 1 \tag{3.37}$$

System (3.35) under the transformation $x \mapsto T_s x = \begin{bmatrix} x_1 \\ y \end{bmatrix}$ and $f \mapsto T_f f$ becomes

$$T_s A T_s^{-1} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \; T_s D T_f^{-1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \; C T_s^{-1} = \begin{bmatrix} 0 & 1 \end{bmatrix} \tag{3.38}$$

Notice $D_{11} = D_1 = 1$ and $D_{22} = D_2 = 0$. Pick $L = -2$ to get $A_1 + LA_3 = -2$ which will govern the sliding motion of the first observer. Hence T_L in (3.7) becomes

$$T_L = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \tag{3.39}$$

Now, the system matrices from (3.38) are then transformed to become

$$\mathcal{A} = \begin{bmatrix} -2 & -4 \\ 1 & 2 \end{bmatrix}, \; \mathcal{D} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \; \mathcal{C} = \begin{bmatrix} 0 & 1 \end{bmatrix} \tag{3.40}$$

Then, P_o and A_4^s were chosen to be 1 and -0.5 such that $P_o\mathcal{A}_4^s + (\mathcal{A}_4^s)^T P_o < 0$. From (3.10) and (3.11), the gain matrices in the same coordinates are

$$\mathcal{G}_n = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ \mathcal{G}_l = \begin{bmatrix} -4 \\ 2.5 \end{bmatrix} \tag{3.41}$$

The error system from (3.24) becomes

$$\begin{aligned} \dot{e}_1 &= -2e_1 - f \\ \dot{e}_y &= e_1 - 0.5e_y + \nu \end{aligned} \tag{3.42}$$

When sliding motion occurs, $\dot{e}_y = e_y = 0$ and the error system in the coordinates of (3.38) becomes

$$\begin{aligned} \dot{e}_1 &= -2e_1 - f \\ \upsilon &= -e_1 \end{aligned} \tag{3.43}$$

where $\upsilon := C_2^{-1}P_o^{-1}\nu$. Since $b = 0$, the error system (3.43) is already in the form of (3.34) where $\tilde{x}$ and $\tilde{y}$ are e_1 and υ_1 respectively with system matrices $\tilde{A} = -2$, $\tilde{D} = -1$ and $\tilde{C} = -1$. Now, it is obvious that $\mathrm{rank}(\tilde{C}\tilde{D}) = \mathrm{rank}(\tilde{D}) = 1$ and e_1 can be accurately estimated from the secondary observer, and the unavailable state, x_1 can be obtained by subtracting $\hat{e}_1$ (from the second observer) from $\hat{x}_1$ from the first observer. Similar to the first observer design, the gain matrices of the second observer can be designed by placing the observer pole at $\tilde{A}-\tilde{G}_l\tilde{C} = -0.1$ and defining its associated Lyapunov matrix as $\tilde{P}_o = 1$. Then, the gain matrices, in the coordinates of (3.43) are chosen as

$$\tilde{G}_n = -1, \ \tilde{G}_l = 1.9 \tag{3.44}$$

and the resulting closed-loop error system from observer 2 can be written as

$$\dot{\hat{e}}_1 - \dot{e}_1 = -0.1(\hat{e}_1 - e_1) - \nu_2 + f \tag{3.45}$$

where ν_2 is the discontinuous output error injection for observer 2.

The simulation results for system (3.35) are shown in Figures 3.3 - 3.8. The initial condition of the states x were assumed to be $\begin{bmatrix} -0.1 & 0.2 \end{bmatrix}^T$ whilst both observers had zero initial

conditions and $f = 1.83\sin(3.24t - 1.64)$. For both observers, ρ and δ for the nonlinear switching term from (3.21) were selected to be 10 and 1×10^{-2}. Figure 3.2 shows sliding motion occuring almost instantaneously despite the mismatch in initial condition. Figure 3.3 shows that the estimated state $\hat{x}_1$ from the first observer does not track the actual state, x_1 because $\mathrm{rank}(CD) \neq \mathrm{rank}(D)$ and condition A1 is not satisfied. Figure 3.4 shows sliding motion occuring for the second observer, and Figure 3.5 shows the estimate of e_1 from that observer. Then, Figure 3.6 shows $\hat{x}_{1,e} = \hat{x}_1 - \hat{e}_1$ (where $\hat{x}_1$ is from the first observer and $\hat{e}_1$ is from the second observer) that accurately tracks the actual x_1 (and Figure 3.7 is a zoomed-in version). Figure 3.8 shows that the unknown input, $\hat{f}$ can be estimated by the sliding mode observer and able to track the actual unknown input, f.

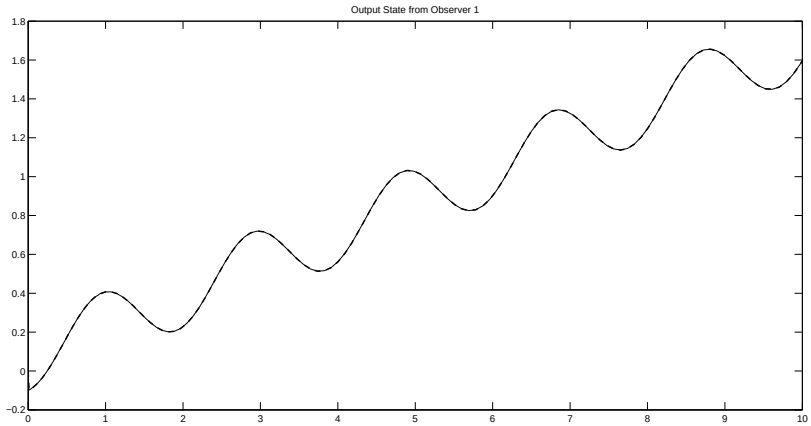

Figure 3.2: The estimated output state $\hat{y}$ from observer 1 (dashed) and the actual output state y (solid).

3.2.4 Multiple Sliding Mode Observers in Cascade

It was shown in the previous section how the addition of an observer in cascade could relax the existence condition (in particular A1 which is the so-called 'matching condition'). Tan & Edwards [81] extended this idea and proposed a scheme of multiple observers in cascade which will be described in this section. However their scheme was proposed for fault reconstruction purpose; in the sequel the scheme will be slightly modified for state estimation. Here the notation will be slightly altered, where $X(i)$ represents the variable

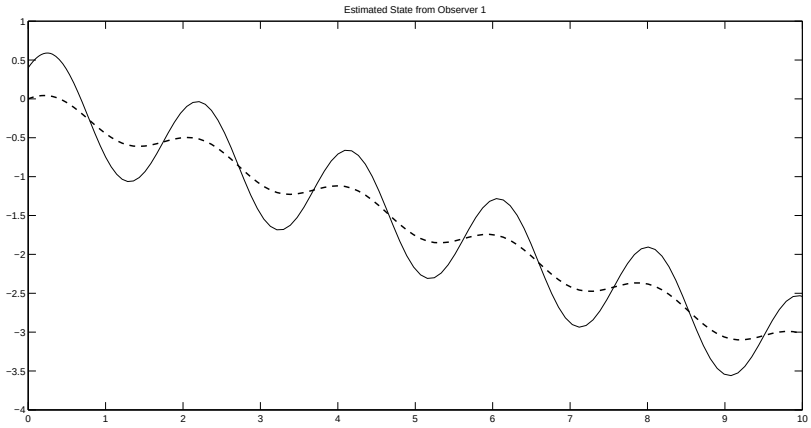

Figure 3.3: The estimated $\hat{x}_1$ (dashed) from observer 1 which does not follow the actual x_1 (solid).

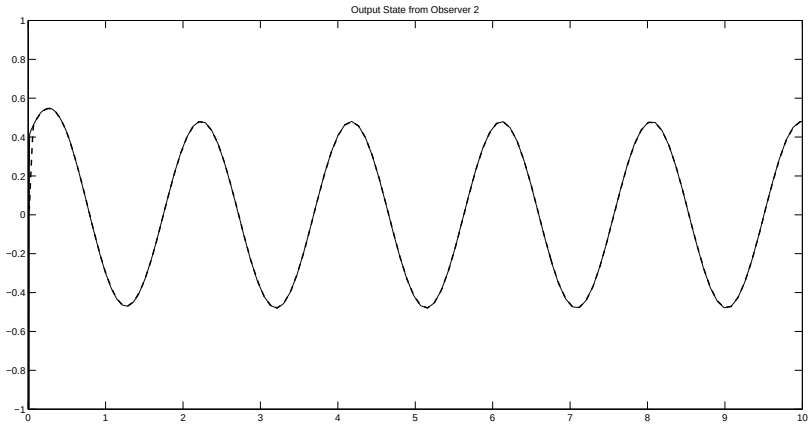

Figure 3.4: The estimated output state $\hat{\tilde{y}}$ from observer 2 (dashed) and the actual output state υ_1 (solid).

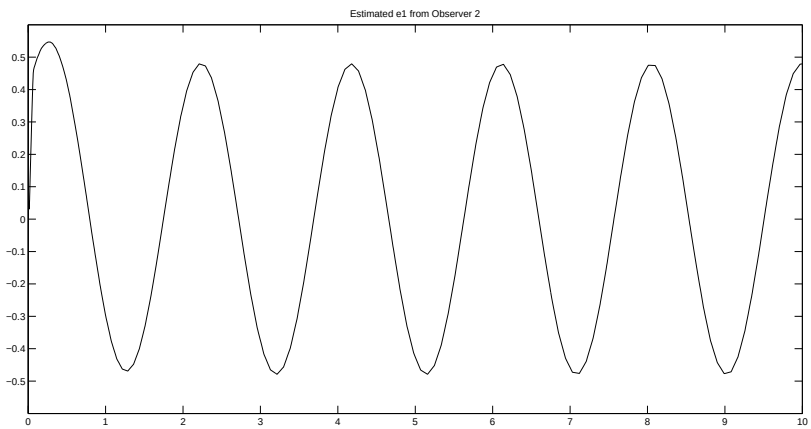

Figure 3.5: The estimated $\hat{e}_1$ from observer 2.

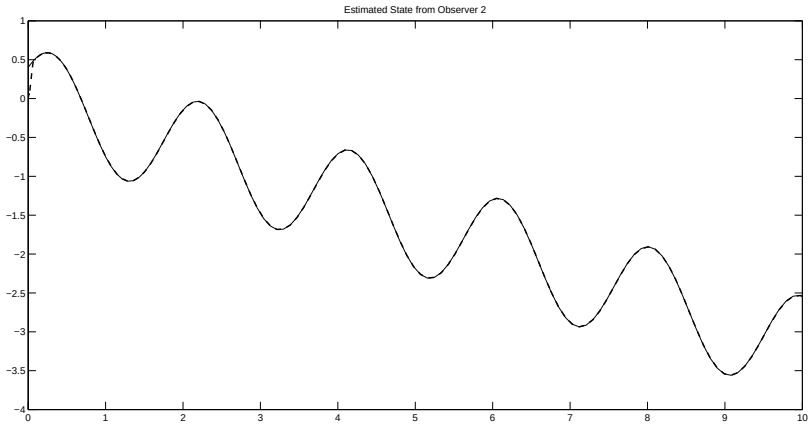

Figure 3.6: The estimated state obtained from $\hat{x}_{1,e} = \hat{x}_1 - \hat{e}_1$ (dashed) converge to the actual x_1 (solid).

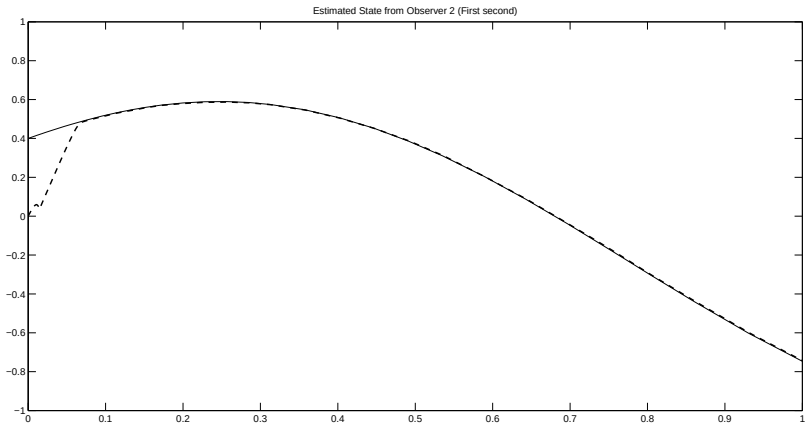

Figure 3.7: First second of Figure 3.6, where sliding motion occurs at around $0.1s$.

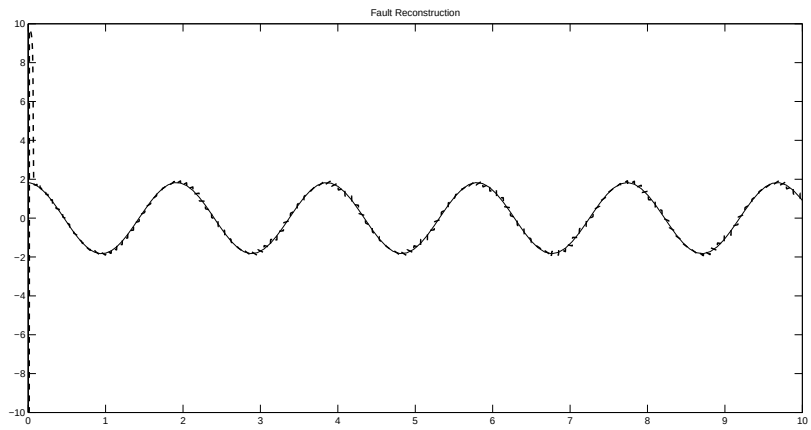

Figure 3.8: The estimated unknown input $\hat{f}$ from observer 2 (dashed) and the actual unknown input f (solid).

X for the ith observer in cascade. Set $i = 1$ and enter the algorithm:

1. The system in consideration is

$$\begin{aligned} \dot{x}(i) &= A(i)x(i) + D(i)f \\ y(i) &= C(i)x(i) \end{aligned} \tag{3.46}$$

where $x(i) \in \mathbb{R}^{n(i)}$,$y(i) \in \mathbb{R}^{p(i)}$ and $f \in \mathbb{R}^{q}$. Partition $x(i)$, $D(i)$ and $C(i)$ as

$$x(i) = \begin{bmatrix} x_1(i) \\ x_2(i) \end{bmatrix}, \; D(i) = \begin{bmatrix} D_1(i) \\ D_2(i) \end{bmatrix}, \; C(i) = \begin{bmatrix} 0 & I_{p(i)} \end{bmatrix} \tag{3.47}$$

   (a) If $\operatorname{rank}(C(i)D(i)) = \operatorname{rank}(D(i)) = q$, then go to Step 5 of the algorithm.

   (b) If $\operatorname{rank}(C(i)D(i)) < q$ and $i = n$, then terminate the algorithm and the state cannot be estimated.

   Otherwise, then define $b(i) = \operatorname{rank}(C(i)D(i))$ and proceed with the algorithm.

2. Define invertible matrices $T_D \in \mathbb{R}^{p(i)\times p(i)}$ and $T_f \in \mathbb{R}^{q\times q}$ such that

$$T_D(i)D_2(i)T_f(i) = \begin{bmatrix} 0 & 0 \\ 0 & D_{22}(i) \end{bmatrix} \tag{3.48}$$

where $D_{22}(i) \in \mathbb{R}^{r(i)\times r(i)}$ is invertible. Then introduce the coordinate transformation $x(i) \mapsto T_{x1}x(i)$, $f \mapsto T_f^{-1}(i)f$ where

$$T_{x1}(i) = \begin{bmatrix} I_{n(i)-p(i)} & 0 \\ 0 & T_D(i) \end{bmatrix} \tag{3.49}$$

and the matrices $D(i)$ and $C(i)$ will be transformed to become

$$\begin{aligned} D(i) &\mapsto \begin{bmatrix} I_{n(i)-p(i)} & 0 \\ 0 & T_D(i) \end{bmatrix} \begin{bmatrix} D_1(i) \\ D_2(i) \end{bmatrix} T_f(i) = \left[\begin{array}{cc} \multicolumn{2}{c}{D_1(i)} \\ \hline 0 & 0 \\ 0 & D_{22}(i) \end{array}\right] \\ C(i) &\mapsto \begin{bmatrix} 0 & I_{p(i)} \end{bmatrix} \begin{bmatrix} I_{n(i)-p(i)} & 0 \\ 0 & T_D^{-1}(i) \end{bmatrix} = \begin{bmatrix} 0 & T_D^{-1}(i) \end{bmatrix} \end{aligned} \tag{3.50}$$

In this coordinate system, let $A(i)$ be partitioned generally as

$$A(i) = \begin{array}{c} \begin{bmatrix} A_1(i) & A_2(i) \\ A_3(i) & A_4(i) \end{bmatrix} \begin{array}{l} \updownarrow n(i)-p(i) \\ \updownarrow p(i) \end{array} \\ \begin{array}{cc} \longleftrightarrow & \longleftrightarrow \\ n(i)-p(i) & p(i) \end{array} \end{array} \tag{3.51}$$

Then partition further

$$A_3(i) = \begin{bmatrix} A_{31}(i) \\ A_{32}(i) \end{bmatrix} \begin{matrix} \updownarrow p(i) - b(i) \\ \updownarrow b(i) \end{matrix} \tag{3.52}$$

Define $p(i+1) = \mathrm{rank}(A_{31}(i)) + b(i) \Rightarrow \mathrm{rank}(A_{31}(i)) = p(i+1) - b(i)$ and invertible matrices $T_{A1}(i) \in \mathbb{R}^{(p(i)-b(i))\times(p(i)-b(i))}$, $T_{A2}(i) \in \mathbb{R}^{(n(i)-p(i))\times(n(i)-p(i))}$ exist such that

$$T_{A1}(i)A_{31}(i)T_{A2}(i) = \breve{A}_{31}(i) = \begin{bmatrix} 0 & 0 \\ 0 & -I_{p(i+1)-b(i)} \end{bmatrix} \tag{3.53}$$

Then define a coordinate transformation

$$T_{x2}(i) = \left[\begin{array}{c|cc} T_{A2}^{-1}(i) & 0 & 0 \\ \hline 0 & T_{A1}(i) & 0 \\ 0 & 0 & I_{b(i)} \end{array}\right] \tag{3.54}$$

and the matrices $A(i)$, $D(i)$, $C(i)$ will be further transformed to become:

$$\begin{aligned} A(i) \mapsto \breve{A}(i) &= T_{x2}(i)A(i)T_{x2}^{-1} \\ &= \left[\begin{array}{cc|c} A_{11}(i) & A_{12}(i) & * \\ \hline 0 & 0 & * \\ 0 & -I_{p(i+1)-b(i)} & * \\ A_{321}(i) & A_{322}(i) & * \end{array}\right] \\ &= \left[\begin{array}{c|c} \breve{A}_1(i) & \breve{A}_2(i) \\ \hline 0 & \\ \breve{A}_{312}(i) & \breve{A}_4(i) \\ \breve{A}_{32}(i) & \end{array}\right] \end{aligned} \tag{3.55}$$

$$D(i) \mapsto \breve{D}(i) = T_{x2}(i)D(i) = \begin{bmatrix} \breve{D}_1(i) \\ \breve{D}_2(i) \end{bmatrix} \tag{3.56}$$

where $\breve{D}_2(i) = \begin{bmatrix} 0 \\ \breve{D}_{2,b}(i) \end{bmatrix}$ and $\breve{D}_{2,b}(i) = \begin{bmatrix} 0 & \breve{D}_{22}(i) \end{bmatrix}$

$$C(i) \mapsto \breve{C}(i) = C(i)T_{x2}^{-1} = \begin{bmatrix} 0 & \breve{C}_2(i) \end{bmatrix} \tag{3.57}$$

where $\breve{C}_2(i) = T_D^{-1}\begin{bmatrix} T_{A1}^{-1}(i) & 0 \\ 0 & I_b(i) \end{bmatrix}$ and the state is transformed to become

$$\begin{aligned} x(i) \mapsto \breve{x}(i) &= T_{x2}(i)x(i) \\ &= \begin{bmatrix} \breve{x}_1(i) \\ \breve{x}_2(i) \end{bmatrix} \begin{matrix} \updownarrow n(i)-p(i) \\ \updownarrow p(i) \end{matrix} \\ &\equiv \begin{bmatrix} \breve{x}_{11}(i) \\ \breve{x}_{12}(i) \\ \breve{x}_2(i) \end{bmatrix} \begin{matrix} \updownarrow n(i)-p(i)-p(i+1)+b(i) \\ \updownarrow p(i+1)-b(i) \\ \updownarrow p(i) \end{matrix} \end{aligned} \tag{3.58}$$

3. Implement a sliding mode observer

$$\begin{aligned} \dot{\hat{\breve{x}}}(i) &= \breve{A}(i)\hat{\breve{x}}(i) - G_l(i)e_y(i) + G_n(i)\nu(i) \\ \hat{y}(i) &= \breve{C}(i)\hat{\breve{x}}(i) \end{aligned} \tag{3.59}$$

where $\nu(i) = -\rho(i)\frac{\breve{e}_y(i)}{\|\breve{e}_y(i)\|}$ and $\breve{e}_y(i) = \breve{C}(i)\hat{\breve{x}}(i) - y(i)$. Notice that $(\breve{D}, \breve{C})$ in (3.56)-(3.57) are in the same form as (D, C) in (3.2). Hence the gains $G_l(i)$, $G_n(i)$ can be designed in the same way as in Section 3.2.1 where

$$G_l(i) = T_L^{-1}(i)\begin{bmatrix} \breve{A}_2(i) \\ \breve{A}_4(i) - \breve{A}_4^s(i) \end{bmatrix}, \; G_n(i) = T_L^{-1}(i)\begin{bmatrix} 0 \\ P_o^{-1}(i) \end{bmatrix}, \tag{3.60}$$

$$T_L = \begin{bmatrix} I_{n(i)-p(i)} & \breve{L}(i) \\ 0 & \breve{C}_2(i) \end{bmatrix}, \; \breve{L}(i) = \begin{bmatrix} \breve{L}_o(i) & 0 \end{bmatrix} \tag{3.61}$$

where $\breve{L}_o \in \mathbb{R}^{(n(i)-p(i))\times(n(i)-p(i))}$ is designed such that $\breve{A}_1(i) + L\breve{(i)}\breve{A}_3(i)$ is stable. Partition

$$\hat{\breve{x}}(i) = \begin{bmatrix} \hat{\breve{x}}_1(i) \\ \hat{\breve{x}}_2(i) \end{bmatrix} \begin{matrix} \updownarrow n(i)-p(i) \\ \updownarrow p(i) \end{matrix} \tag{3.62}$$

and $\breve{e}(i) = \hat{\breve{x}}(i) - \breve{x}(i)$ is the estimation error. If $\rho(i)$ is chosen large enough, then the following error system dynamics of order $n(i) - p(i)$ will result during sliding motion ($\breve{e}_y(i) = \dot{\breve{e}}_y(i) = 0$)

$$\dot{\breve{e}}_1(i) = (\breve{A}_1(i) + \breve{L}\breve{A}_3(i))\breve{e}_1(i) - \breve{D}_1(i)f(i) \tag{3.63}$$

$$0 = \breve{C}_2(i)\breve{A}_3(i)\breve{e}_1(i) - \breve{C}_2(i)\breve{D}_2(i)f(i) + P_o^{-1}(i)\nu_{eq}(i) \tag{3.64}$$

Define $w(i) = \breve{C}_2^{-1}(i)P_o^{-1}(i)\nu_{eq}(i)$ and (3.64) will result in

$$w(i) = -\breve{A}_3(i)\breve{e}_1(i) + \breve{D}_2(i)f \tag{3.65}$$

Partition

$$w(i) = \begin{bmatrix} w_1(i) \\ w_2(i) \\ w_3(i) \end{bmatrix} \begin{matrix} \updownarrow p(i) - p(i+1) \\ \updownarrow p(i+1) - b(i) \\ \updownarrow b(i) \end{matrix} \tag{3.66}$$

and by using (3.55), (3.65) becomes

$$\begin{aligned} w_2(i) &= -\breve{A}_{31}(i)\breve{e}_1(i) \\ w_3(i) &= -\breve{A}_{32}(i)\breve{e}_1 i + \breve{D}_{2,b}(i)f \end{aligned} \tag{3.67}$$

Notice that $w_1(i) = 0$ due to the structure of $\breve{A}_{31}(i)$ and $\breve{D}_2(i)$. Filter $w_3(i)$ to produce $\bar{w}(i)$ as follows

$$\dot{\bar{w}}(i) = -A_f(i)\bar{w}(i) + A_f w_3(i) \tag{3.68}$$

where $-A_f(i)$ is a stable matrix. Then (3.68) analytically becomes:

$$\dot{\bar{w}}(i) = -A_f(i)\bar{w}(i) - A_f(i)\breve{A}_{32}(i)\breve{e}_1(i) + A_f(i)\breve{D}_{2,b}(i)f \tag{3.69}$$

4. Using (3.63) and (3.69), form the following dynamic system with a measurable output signal

$$\begin{aligned} \begin{bmatrix} \dot{\breve{e}}_1(i) \\ \dot{\bar{w}}(i) \end{bmatrix} &= \begin{bmatrix} \breve{A}_1(i) + \breve{L}(i)\breve{A}_3(i) & 0 \\ -A_f(i)\breve{A}_{32}(i) & -A_f(i) \end{bmatrix} \begin{bmatrix} \breve{e}_1(i) \\ \bar{w}(i) \end{bmatrix} + \begin{bmatrix} -\breve{D}_1(i) \\ A_f(i)\breve{D}_{2,b} \end{bmatrix} f \\ \begin{bmatrix} w_2(i) \\ \bar{w}(i) \end{bmatrix} &= \begin{bmatrix} -\breve{A}_{31}(i) & 0 \\ 0 & I_{b(i)} \end{bmatrix} \begin{bmatrix} \breve{e}_1(i) \\ \bar{w}(i) \end{bmatrix} \end{aligned} \tag{3.70}$$

which can be written in the form

$$\begin{aligned} \dot{x}(i+1) &= A(i+1)x(i+1) + D(i+1)f \\ y(i+1) &= C(i+1)x(i+1) \end{aligned} \tag{3.71}$$

where

$$x(i+1) = \begin{bmatrix} \breve{e}_1(i) \\ \bar{w}(i) \end{bmatrix} \begin{matrix} \updownarrow n(i) - p(i) \\ \updownarrow b(i) \end{matrix} = \begin{bmatrix} \breve{e}_{11}(i) \\ \breve{e}_{12}(i) \\ \bar{w}(i) \end{bmatrix} \begin{matrix} \updownarrow n(i) - p(i) - p(i+1) + b(i) \\ \updownarrow p(i+1) - b(i) \\ \updownarrow b(i) \end{matrix} \tag{3.72}$$

Notice that $\breve{e}_{12}(i)$ becomes part of the output $y(i+1)$. Expand $C(i+1)$ using the partition of $\breve{A}_{31}(i)$ to get:

$$C(i+1) = \begin{bmatrix} 0 & I_{p(i+1)-b(i)} & 0 \\ 0 & 0 & I_{b(i)} \end{bmatrix} = \begin{bmatrix} 0 & I_{p(i+1)} \end{bmatrix} \tag{3.73}$$

which is in the identical structure as $C(i)$, and hence the system (3.70) is in the same structure as the original system, and the algorithm can be repeated; return to Step 1 and increase i by 1.

5. Set $k = i$. Implement the Edwards-Spurgeon observer to estimate $x(k)$. Since $\mathrm{rank}(C(k)D(k)) = q$ then the estimate of $x(k)$ will be not affected by f as in Section 3.2.2

6. The estimated state can be extracted from the k observers by the following equation:

$$\hat{x} = T_x^{-1}(1)\hat{x}_o(1) \tag{3.74}$$

where $\hat{x}_o(i)$ is by:

$$\hat{x}_o(i) = \hat{\breve{x}}(i) - \begin{bmatrix} I_{n(i)-p(i)} & 0 \\ 0 & 0 \end{bmatrix} T_x^{-1}(i+1)\hat{x}_o(i+1)\ ,\ i = 1, \ldots, k-1 \tag{3.75}$$

and

$$T_x(i) = \left[\begin{array}{c|c} T_{A2}^{-1}(i) & 0 \\ \hline 0 & \begin{bmatrix} T_{A1}(i) & 0 \\ 0 & I_b(i) \end{bmatrix} T_D(i) \end{array}\right] \tag{3.76}$$

For the special case of $i = k$, $\hat{x}_o(k)$ and $T_x(k)$ are respectively defined to be $\hat{x}_o(k) - \hat{x}(k)$ (which is the estimate of $x(k)$) and $T_x(k) = I_n(k)$.

Remark 3.3 *To show that $\hat{x}$ converges to $x(1)$, the following key relationships are firstly summarized from the algorithm:*

$$n(i+1) = n(i) - p(i) + b(i) \tag{3.77}$$

$$\begin{aligned} x(i+1) = \begin{bmatrix} \breve{e}_1(i) \\ \bar{w}(i) \end{bmatrix} \begin{matrix} \updownarrow n(i)-p(i) \\ \updownarrow b(i) \end{matrix} &= \begin{bmatrix} \breve{e}_{11}(i) \\ \breve{e}_{12}(i) \\ \bar{w}(i) \end{bmatrix} \begin{matrix} \updownarrow n(i)-p(i)-p(i+1)+b(i) \\ \updownarrow p(i+1)-b(i) \\ \updownarrow b(i) \end{matrix} \\ &= \begin{bmatrix} x_1(i+1) \\ y(i+1) \end{bmatrix} \begin{matrix} \updownarrow n(i+1)-p(i+1) \\ \updownarrow p(i+1) \end{matrix} \end{aligned} \tag{3.78}$$

$$\breve{x}(i) = \underbrace{\left[\begin{array}{c|c} T_{A2}^{-1}(i) & 0 \\ \hline 0 & \begin{bmatrix} T_{A1}(i) & 0 \\ 0 & I_b(i) \end{bmatrix} T_D(i) \end{array}\right]}_{T_x(i)} x(i)$$

$$= \begin{bmatrix} \breve{x}_1(i+1) \\ \breve{y}(i+1) \end{bmatrix} = \begin{bmatrix} \breve{x}_{11}(i) \\ \breve{x}_{12}(i) \\ \breve{y}(i) \end{bmatrix} \begin{array}{l} \updownarrow n(i) - p(i) - p(i+1) + b(i) \\ \updownarrow p(i+1) - b(i) \\ \updownarrow p(i) \end{array} \tag{3.79}$$

and

$$\begin{aligned} \breve{e}_{11}(i) &= \hat{\breve{x}}_{11}(i) - \breve{x}_{11}(i) \\ \breve{e}_{12}(i) &= \hat{\breve{x}}_{12}(i) - \breve{x}_{12}(i) \end{aligned} \tag{3.80}$$

Since $x(k)$ *can be estimated, then* $\hat{x}_o(k) \to x(k)$*. Following that from* (3.75)

$$\begin{aligned} \hat{x}_o(k-1) &= \hat{\breve{x}}(k-1) - \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \hat{x}_o(k) \\ &= \begin{bmatrix} \hat{\breve{x}}_{11}(k-1) \\ \hat{\breve{x}}_{12}(k-1) \\ \hat{\breve{y}}(k-1) \end{bmatrix} - \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{\breve{e}}_{11}(k-1) \\ \hat{\breve{e}}_{12}(k-1) \\ \hat{\breve{w}}(k-1) \end{bmatrix} \end{aligned} \tag{3.81}$$

From (3.78), $\breve{e}_{12}(k-1)$ *is part of* $y(k)$ *whose estimate converges in finite time, hence* $\hat{\breve{e}}_{12}(k-1) = \breve{e}(k-1)$ *is achieved in finite time. Then,* $\breve{e}_{11}(k-1)$ *is the non-output state* $x_1(k)$*, and hence* $\hat{\breve{e}}_{11}(k-1) \to \breve{e}_{11}(k-1)$ *asymptotically. Therefore, knowing that* $\hat{\breve{y}}(k-1) = \breve{y}(k-1)$ *in finite time,* $\hat{x}_o(k-1)$ *becomes*

$$\hat{x}_o(k-1) \mapsto \begin{bmatrix} \hat{\breve{x}}_{11}(k-1) - \breve{e}_{11}(k-1) \\ \hat{\breve{x}}_{12}(k-1) - \breve{e}_{12}(k-1) \\ \breve{y}(k-1) \end{bmatrix} = \begin{bmatrix} \breve{x}_{11}(k-1) \\ \breve{x}_{12}(k-1) \\ \breve{y}(k-1) \end{bmatrix} = \breve{x}(k-1) \tag{3.82}$$

Then $\hat{x}_o(k-2)$ *becomes:*

$$\hat{x}_o(k-2) = \hat{\breve{x}}(k-2) - \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T_x^{-1}(k-1)\hat{x}_o(k-1) \tag{3.83}$$

Since $\hat{x}_o(k-1) \to \breve{x}(k-1)$ *and* $x(k-1) = T_x^{-1}(k-1)\hat{x}_o(k-1)$ *(from* (3.79)*) then*

$$\begin{aligned}
\hat{x}_o(k-2) &\mapsto \hat{\breve{x}}(k-2) - \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} x(k-1) \\
&= \begin{bmatrix} \hat{\breve{x}}_{11}(k-2) \\ \hat{\breve{x}}_{12}(k-2) \\ \hat{\breve{y}}(k-2) \end{bmatrix} - \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \breve{e}_{11}(k-2) \\ \breve{e}_{12}(k-2) \\ \bar{w}(k-2) \end{bmatrix} \\
&= \begin{bmatrix} \breve{x}_{11}(k-2) \\ \breve{x}_{12}(k-2) \\ \breve{y}(k-2) \end{bmatrix} = \breve{x}(k-2)
\end{aligned} \tag{3.84}$$

Hence it can be concluded that $\hat{x}_o(i) \to \breve{x}(i)$*, and* $\hat{x}_o(1) \to \breve{x}(1)$*. Then* $\hat{x} = T_x^{-1}(1)\hat{x}_o(1) \to T_x^{-1}(1)\breve{x}(1) = x(1)$ *which shows that* $\hat{x} \to x(1)$*.* ♯

It is clear that for the scheme to be successful, the necessary and sufficient conditions are that $\mathrm{rank}(C(k)D(k)) = q$ and that all observers should have a stable sliding motion, i.e. $(A_1(i), A_{31}(i))$ is detectable for $i = 1, \ldots, k$. Tan & Edwards re-casted those existence conditions, in terms of the original system matrices which could be summarized to be:

D1. $\mathrm{rank}(\Xi(k)) - \mathrm{rank}(\Xi(k-1)) = \mathrm{rank}(D)$, $2 \le k \le n$

D2. $\mathrm{rank}(\Delta) = n + \mathrm{rank}(D)$, $\forall s \in \mathbb{C}, Re(s) \ge 0$

where $\Xi(i)$ and Δ are defined by

$$\Xi(i) = \begin{bmatrix} \Pi(i) & \Pi(i-1) & \ldots & \Pi(1) \\ \Pi(i-1) & \Pi(i-2) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \Pi(1) & 0 & \ldots & 0 \end{bmatrix}, \quad \Delta = \begin{bmatrix} sI - A & D \\ C & 0 \end{bmatrix} \tag{3.85}$$

where $\Pi(i) := CA^{i-1}D$ and k is the number of observers required for accurate state estimation. Condition D1 enables the user to know from the outset, the number of cascaded observers necessary and sufficient to asymptotically estimate the unknown input and state. It can be executed as an algorithm, where i is incremented until it is satisfied.

Remark 3.4

- *Once again, an important implication of conditions D1 and D2 is that the feasibility of the scheme does not rely upon the design freedom of previous observers.*

- *Tan & Edwards [81] have also shown that whilst having more observers may relax the matching condition, this will be effective only up to n observers; this means that if the matching condition is not satisfied for $i = n$, it will never be satisfied for $i = n+1, n+2$ and beyond. This discovery enables the designer to know the limit of the scheme.*

- *The addition of observers in cascade relax the matching condition, but not the 'invariant zero' condition (notice that A2, C2 and D2 are all the same). Alwi* et al. *[94] have proposed a scheme that relaxes that condition, but of the cost of an inaccurate estimate of f that is corrupted by its own time derivative; this may still be acceptable for the case of incipient faults.*

- *To improve the accuracy of the scheme (especially the accuracy of $\nu(i)$), the term ν in* (3.4) *can be replaced by a supertwisting higher order sliding mode algorithm [95] as follows:*

$$\begin{aligned} \nu_j(i) &= -\psi_j(i)\,\mathrm{sign}(e_{y,j}(i))\|e_{y,j}(i)\|^{\frac{1}{2}} + z_{f,j}(i),\ \ j = 1,\ldots,p \\ \dot{z}_{f,j}(i) &= -\beta_j(i)\,\mathrm{sign}(e_{y,j}(i)) - \vartheta_j(i)e_{y,j}(i) \end{aligned} \tag{3.86}$$

where $\psi_j(i)$, $\beta_j(i)$ $\vartheta_j(i)$ are scalars to be selected by the designer. Define $\zeta(i) = \hat{G}(s)f(i+1)$ where

$$\hat{G}(s) = -\begin{bmatrix} C_2(i)D_2(i) & 0 \end{bmatrix} - \mathcal{A}_3(i)(sI - \mathcal{A}_1(i))^{-1}D_1(i) \tag{3.87}$$

Define constant $d_j(i) > \|\dot{\zeta}_j(i)\|$ and choose the gains from (3.86) *as*

$$\begin{aligned} \psi_j(i) > 2\sqrt{d_j(i)},\ \ \lambda_j(i) > 0,\ \ \beta_j(i) > d_j(i) \\ \gamma_j(i) > \frac{\lambda_j^2(i)(\psi_j^3(i) + \frac{5}{4}\psi_j^2(i) + \frac{5}{2}(\beta_j(i) - d_j(i)))}{\psi_j(i)(\beta_j(i) - d_j(i))} \end{aligned} \tag{3.88}$$

♯

3.3 Application of Edwards-Spurgeon Observer on Functional State Estimation

In functional state estimation, only a function or subset of the state vector is estimated. The state to be estimated can be modeled as $z = Fx$ where $z \in \mathbb{R}^r$, $r < n$ is the vector to be estimated and $F \in \mathbb{R}^{r \times n}$ is a known constant matrix.

In this thesis, the Edwards-Spurgeon observer will be used to estimate the functional state, z. The system (3.1) will be transformed and re-formulated into a form where the Edwards-Spurgeon observer presented in Section 3.2 can be applied directly to estimate z. Following that, the existence conditions in terms of the original system matrices will be investigated.

3.3.1 System Formulation

First, re-write system (3.1) as follows to include the functional state

$$\begin{aligned} \dot{x} &= Ax + Df \\ y &= Cx \\ z &= Fx \end{aligned} \tag{3.89}$$

Without loss of generality, it is assumed that $\operatorname{rank}(F) = r < n$ and from the assumption $\operatorname{rank}(C) = p$, it can be assumed that $\operatorname{rank}\begin{bmatrix} F \\ C \end{bmatrix} = r + p$; this is not a restrictive assumption as it simply implies that the 'output states' y will not be estimated since they are already available for measurement.

Lemma 3.5 *There exist a linear state transformation $x \mapsto T_s x$ such that the system matrices (A, C, D, F) from* (3.89) *have the following structure*

$$A = \left[\begin{array}{c|cc} A_1 & A_2 & A_3 \\ \hline A_4 & A_5 & A_6 \\ A_7 & A_8 & A_9 \end{array}\right], \quad D = \left[\begin{array}{c} D_1 \\ \hline D_2 \\ D_3 \end{array}\right] \begin{array}{l} \updownarrow n-p-r \\ \updownarrow r \\ \updownarrow p \end{array}, \qquad \begin{aligned} F &= \left[\begin{array}{c|cc} 0 & I_r & 0 \end{array}\right], \\ C &= \left[\begin{array}{c|cc} 0 & 0 & I_p \end{array}\right] \end{aligned} \tag{3.90}$$

♯

Proof

Since C has full row rank, there exist a nonsingular change of coordinates,

$$T_c = \begin{bmatrix} N_C^T \\ C \end{bmatrix} \tag{3.91}$$

where $N_C \in \mathbb{R}^{n\times(n-p)}$ spans the null space of C, i.e. $CN_C = 0$ such that the system matrices (A, C, D, F) can be transformed to be

$$\begin{aligned} A_c &:= T_c A T_c^{-1} = \begin{bmatrix} A_{c,1} & A_{c,2} \\ A_{c,3} & A_{c,4} \end{bmatrix}, \; D_c := T_c D = \begin{bmatrix} D_{c,1} \\ D_{c,2} \end{bmatrix}, \\ C_c &:= C T_c^{-1} = \begin{bmatrix} 0 & I_p \end{bmatrix}, \; F_c := F T_c^{-1} = \begin{bmatrix} F_{c,1} & F_{c,2} \end{bmatrix} \end{aligned} \tag{3.92}$$

where $F_{c,1} \in \mathbb{R}^{r\times(n-p)}$ has full row rank resulting from the assumption $\operatorname{rank}\begin{bmatrix} F \\ C \end{bmatrix} = r + p$. The matrix C at the bottom of T_c will cause the special structure of C_c above, whilst N_C^T was placed at the top of T_c to ensure that T_c is invertible. In this coordinate system, A_c and D_c have no specific structures. Then define another change of coordinates

$$T_b = \left[\begin{array}{c|c} N_F^T & 0 \\ F_{c,1} & 0 \\ \hline 0 & I_p \end{array}\right] \tag{3.93}$$

where $N_F \in \mathbb{R}^{(n-p)\times(n-p-r)}$ is the null space of $F_{c,1}$. The matrices in (3.92) can then be transformed to become

$$\begin{aligned} A_b &:= T_b A_c T_b^{-1} = \left[\begin{array}{c|cc} A_{b,1} & A_{b,2} & A_{b,3} \\ \hline A_{b,4} & A_{b,5} & A_{b,6} \\ A_{b,7} & A_{b,8} & A_{b,9} \end{array}\right] \begin{array}{l} \updownarrow n-p-r \\ \updownarrow r \\ \updownarrow p \end{array}, \\ D_b &:= T_b D_c = \begin{bmatrix} D_{b,1} \\ D_{b,2} \\ D_{b,3} \end{bmatrix}, \; F_b := F_c T_b^{-1} = \left[\begin{array}{c|cc} 0 & I_r & F_{b,3} \end{array}\right] \end{aligned} \tag{3.94}$$

where $F_{b,3} \in \mathbb{R}^{r\times p}$. The purpose of this transformation is to achieve the special structure of F_b; the top left $(n-p)\times(n-p)$ sub-block of T_b causes the first $n-p$ columns of F_b whereas $F_{b,3}$ has no particular structure. Note that $C_b = C_c T_b^{-1}$ is not affected by this transformation, and A_b, D_b have no particular structures. To eliminate $F_{b,3}$, define another

transformation matrix T_a as

$$T_a = \begin{bmatrix} I_{n-p-r} & 0 & 0 \\ 0 & I_r & -F_{b,3} \\ 0 & 0 & I_p \end{bmatrix} \tag{3.95}$$

and the matrices in (3.94) become

$$A_a := T_a A_b T_a^{-1} = \left[\begin{array}{c|cc} A_{a,1} & A_{a,2} & A_{a,3} \\ \hline A_{a,4} & A_{a,5} & A_{a,6} \\ A_{a,7} & A_{a,8} & A_{a,9} \end{array}\right], \; D_a := T_a D_b = \begin{bmatrix} D_{a,1} \\ D_{a,2} \\ D_{a,3} \end{bmatrix} \begin{array}{l} \updownarrow n-p-r \\ \updownarrow r \\ \updownarrow p \end{array},$$

$$F_a := F_b T_a^{-1} = \left[\begin{array}{c|cc} 0 & I_r & 0 \end{array}\right], \; C_a := C_b T_a^{-1} = \left[\begin{array}{c|cc} 0 & 0 & I_p \end{array}\right] \tag{3.96}$$

The matrices in (3.96) have the same structure as in (3.90) and the proof is complete. ■

In the coordinate system of (3.90), let the system state be transformed and partitioned as follows:

$$x \mapsto T_s x = \begin{bmatrix} x_1 \\ x_2 \\ y \end{bmatrix} \begin{array}{l} \updownarrow n-p-r \\ \updownarrow r \\ \updownarrow p \end{array} \tag{3.97}$$

where $T_s = T_a T_b T_c$. From the structure of F in (3.90), $z = x_2$; therefore, the state components to be estimated would be x_2 while x_1 is the 'unwanted' state, which will be modeled as unknown inputs/disturbances. The system equations (3.89) in the coordinates of (3.90) can be re-written as follows:

$$\underbrace{\begin{bmatrix} \dot{x}_2 \\ \dot{y} \end{bmatrix}}_{\dot{\bar{x}}} = \underbrace{\begin{bmatrix} A_5 & A_6 \\ A_8 & A_9 \end{bmatrix}}_{\bar{A}} \underbrace{\begin{bmatrix} \dot{x}_2 \\ \dot{y} \end{bmatrix}}_{\bar{x}} + \underbrace{\begin{bmatrix} A_4 & D_2 \\ A_7 & D_3 \end{bmatrix}}_{\bar{D}} \underbrace{\begin{bmatrix} x_1 \\ f \end{bmatrix}}_{\bar{f}}$$

$$y = \underbrace{\begin{bmatrix} 0 & I_p \end{bmatrix}}_{\bar{C}} \underbrace{\begin{bmatrix} x_2 \\ y \end{bmatrix}}_{\bar{x}} \tag{3.98}$$

Now, the system in (3.98) is in the same form as the one in (3.1); therefore, the Edwards-Spurgeon observer (3.3) can be implemented on system (3.98) to estimate $\bar{x}$ (and hence x_2). This time however, there is no requirement on the rank of $\bar{D}$ (unlike in Section 3.2, D has full column rank) because the objective is to estimate the functional state x_2 and the unknown inputs x_1 and f are not required to be estimated.

3.3.2 Existence Conditions for Sliding Mode Functional Observation

From Section 3.2, by applying conditions A1-A2, the Edwards-Spurgeon observer can be used to estimate $\bar{x}$ (consisting of x_2 and y) if and only if the following conditions are satisfied

E1. $\operatorname{rank}(\bar{C}\bar{D}) = \operatorname{rank}(\bar{D})$

E2. $\operatorname{rank}\begin{bmatrix} sI - \bar{A} & \bar{D} \\ \bar{C} & 0 \end{bmatrix} = r + p + \operatorname{rank}(\bar{D}), \ \forall s \in \mathbb{C}, Re(s) \geq 0$

However, conditions E1 and E2 are in terms of the transformed system matrices, which require the designer to perform a state transformation first before it can be tested with conditions E1 and E2. It is therefore of interest to recast conditions E1 and E2 in terms of the original system matrices.

Lemma 3.6 *Conditions E1 and E2 can be re-casted in terms of the orginal system matrices as follows*

F1. $\operatorname{rank}\begin{bmatrix} FA & FD \\ CA & CD \\ C & 0 \\ F & 0 \end{bmatrix} = \operatorname{rank}\begin{bmatrix} CA & CD \\ C & 0 \\ F & 0 \end{bmatrix}$

F2. $\operatorname{rank}\begin{bmatrix} sF - FA & -FD \\ CA & CD \\ C & 0 \end{bmatrix} = \operatorname{rank}\begin{bmatrix} FA & FD \\ CA & CD \\ C & 0 \\ F & 0 \end{bmatrix}, \ \forall s \in \mathbb{C}, Re(s) \geq 0$

♯

Proof

Substituting the matrices from (3.98) into condition E1 yields

$$\operatorname{rank}(\bar{C}\bar{D}) = \operatorname{rank}(\bar{D}) \Leftrightarrow \operatorname{rank}\begin{bmatrix} A_4 & D_2 \\ A_7 & D_3 \end{bmatrix} = \operatorname{rank}\begin{bmatrix} A_7 & D_3 \end{bmatrix} \tag{3.99}$$

Let Θ be the LHS (Left Hand Side) of condition F1, where

$$\Theta = \begin{bmatrix} FA & FD \\ CA & CD \\ C & 0 \\ F & 0 \end{bmatrix} \tag{3.100}$$

Substituting the system matrices from (3.90) into Θ will result in

$$\Theta = \begin{bmatrix} A_4 & A_5 & A_6 & D_2 \\ A_7 & A_8 & A_9 & D_3 \\ 0 & I_r & F_3 & 0 \\ 0 & 0 & I_p & 0 \end{bmatrix} \tag{3.101}$$

Therefore, it is straightforward to see that

$$\operatorname{rank}(\Theta) = \operatorname{rank} \underbrace{\begin{bmatrix} A_4 & D_2 \\ A_7 & D_3 \end{bmatrix}}_{\Theta_{a,1}} + p + r \tag{3.102}$$

Then, define Ψ as the RHS (Right Hand Side) of condition F1,

$$\Psi = \begin{bmatrix} CA & CD \\ C & 0 \\ F & 0 \end{bmatrix} \tag{3.103}$$

and Ψ can be written in term of the partitions in (3.90) as follows:

$$\Psi = \begin{bmatrix} A_7 & A_8 & A_9 & D_3 \\ 0 & I_r & F_3 & 0 \\ 0 & 0 & I_p & 0 \end{bmatrix} \tag{3.104}$$

It is then obvious to see that

$$\operatorname{rank}(\Psi) = \operatorname{rank} \underbrace{\begin{bmatrix} A_7 & D_3 \end{bmatrix}}_{\Psi_{a,1}} + p + r \tag{3.105}$$

Hence by using equations (3.100)-(3.102) and (3.103)-(3.105), condition F1 becomes:

$$\operatorname{rank} \begin{bmatrix} A_4 & D_2 \\ A_7 & D_3 \end{bmatrix} = \operatorname{rank} \begin{bmatrix} A_7 & D_3 \end{bmatrix} \tag{3.106}$$

which is equivalent to condition E1 in (3.99).

To show the equivalence of E2 and F2, define Δ_a and Δ_b as follows:

$$\Delta_a = \begin{bmatrix} sF - FA & -FD \\ CA & CD \\ C & 0 \end{bmatrix}, \ \Delta_b = \begin{bmatrix} sI - \bar{A} & \bar{D} \\ \bar{C} & 0 \end{bmatrix} \tag{3.107}$$

Substitute (A, C, D, F) from (3.90) into Δ_a to obtain

$$\Delta_a = \begin{bmatrix} -A_4 & sI_r - A_5 & -A_6 & D_2 \\ -A_7 & -A_8 & sI_p - A_9 & D_3 \\ 0 & 0 & I_p & 0 \end{bmatrix} \tag{3.108}$$

It is then obvious to see that

$$\operatorname{rank}(\Delta_a) = \operatorname{rank} \underbrace{\begin{bmatrix} -A_4 & sI_r - A_5 & D_2 \\ -A_7 & -A_8 & D_3 \end{bmatrix}}_{\Delta_{a,1}} + p \tag{3.109}$$

Substitute $(\bar{A}, \bar{C}, \bar{D})$ from (3.98) into Δ_b to get

$$\Delta_b = \begin{bmatrix} sI_r - A_5 & -A_6 & A_4 & D_2 \\ -A_8 & sI_p - A_9 & A_7 & D_3 \\ 0 & I_p & 0 & 0 \end{bmatrix} \tag{3.110}$$

and it is obvious that

$$\operatorname{rank}(\Delta_b) = \operatorname{rank} \underbrace{\begin{bmatrix} sI_r - A_5 & A_4 & D_2 \\ -A_8 & A_7 & D_3 \end{bmatrix}}_{\Delta_{b,1}} + p \tag{3.111}$$

From (3.109) and (3.111), it is clear that $\Delta_{b,1}$ is simply a re-arrangement of columns of $\Delta_{a,1}$. Hence, Δ_a and Δ_b have equal rank.

It was shown earlier in the proof that the RHS of F2 is equivalent to $\operatorname{rank}(\bar{D}) + p + r$, which is the RHS of E2. Then the LHS of E2 and F2 (Δ_a and Δ_b) was just shown to have equal rank. Hence conditions E2 and F2 are equivalent and the proof is complete. ■

Remark 3.5 *Notice that in order to estimate the entire state, it requires* $\operatorname{rank}(CD) = \operatorname{rank}(D)$*, which in the coordinates of* (3.90) *implies that* D_3 *must be full column rank. However, conditions E1 and F1 do not impose such a constraint. Hence it is shown that estimating only a function of the states might not require condtions as stringent as estimating the entire state.* ♯

Remark 3.6 *The conditions F1-F2 are equivalent to the existence conditons of earlier works on functional state estimation [46, 87, 96, 97].* ♯

3.3.3 An Example

A 7th order aircraft model from Heck *et al.* [98] is used to demonstrate the sliding mode functional observer scheme proposed in this chapter. In the notation of (3.89), the system states and control inputs are the following:

$$x = \begin{bmatrix} \text{bank angle, } \theta_{bank}\ (rad) \\ \text{yaw rate, } \dot{\theta}_{yaw}\ (rad/s) \\ \text{roll rate, } \dot{\theta}_{roll}\ (rad/s) \\ \text{sideslip angle, } \theta_{ss}\ (rad) \\ \text{washout filter state, } W \\ \text{rudder deflection, } \xi_{rudder}\ (rad) \\ \text{aileron deflection, } \xi_{aileron}\ (rad) \end{bmatrix}, \quad u = \begin{bmatrix} \text{rudder command } (rad) \\ \text{aileron command } (rad) \end{bmatrix} \tag{3.112}$$

The matrices A and D are

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -0.154 & -0.0042 & 1.54 & 0 & -0.744 & -0.032 \\ 0 & 0.249 & -1 & -5.2 & 0 & 0.337 & -1.12 \\ 0.0386 & -0.996 & -0.0003 & -2.117 & 0 & 0.02 & 0 \\ 0 & 0.5 & 0 & 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -20 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -25 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 20 & 0 \\ 0 & 25 \end{bmatrix} \tag{3.113}$$

In this study, only the yaw rate, $\dot{\theta}_{yaw}$ and roll rate, $\dot{\theta}_{roll}$ need to be estimated. Thus, the matrix F is defined as

$$F = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.114}$$

Assume that the bank angle, aileron deflection and sideslip angle are measureable, and therefore, the matrix C becomes

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \tag{3.115}$$

It is obvious that $\mathrm{rank}(CD) = 1$ and $\mathrm{rank}(D) = 2$, therefore condition A1 is not satisfied and the observer scheme proposed by Edwards & Spurgeon [5] is not applicable. However, it can be easily verified that condition F1 is satisfied $(\mathrm{rank}(\Theta) = \mathrm{rank}(\Psi) = 7)$ and the sliding mode functional observer scheme proposed in this chapter can be used to accurately estimate the functional state (i.e. yaw rate and roll rate). The structure in (3.90)

can be achieved by performing the state transformation, $x \mapsto T_s x$ in Lemma 3.5. Firstly, to obtain T_c in (3.91), a suitable choice of N_C (such that $CN_C = 0$) is

$$N_C = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.116}$$

Then, F_c in (3.92) becomes

$$F_c = \begin{bmatrix} F_{c,1} & F_{c,2} \end{bmatrix} = \left[\begin{array}{cccc|ccc} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \tag{3.117}$$

To obtain T_b, a suitable choice of N_F (such that $F_{c,1}N_F = 0$) is

$$N_F = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \tag{3.118}$$

Since $F_{c,2}$ is a zero matrix, then $T_a = I_n$. Now, the state transformation $T_s := T_a T_b T_c$ in Lemma 3.5 is

$$T_s = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \tag{3.119}$$

and the matrices from (3.113) - (3.115) after going through the transformation via T_s will achieve the structure in (3.90) as follows

$$A_s = \left[\begin{array}{cc|cc|ccc} -20 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -4 & 0.5 & 0 & 0 & 0 & 0 \\ \hline -0.744 & 0 & -0.154 & -0.0042 & 0 & -0.032 & 1.54 \\ 0.337 & 0 & 0.249 & -1 & 0 & -1.12 & -5.2 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -25 & 0 \\ 0.02 & 0 & -0.996 & -0.0003 & 0.0386 & 0 & -2.117 \end{array}\right], \quad D_s = \left[\begin{array}{cc} 20 & 0 \\ 0 & 0 \\ \hline 0 & 0 \\ 0 & 0 \\ \hline 0 & 0 \\ 0 & 25 \\ 0 & 0 \end{array}\right],$$

$$F_s = \left[\begin{array}{cc|cc|ccc} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right], \; C_s = \left[\begin{array}{cc|cc|ccc} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \tag{3.120}$$

From the structure of C_s and F_s, it can be seen that, in this coordinate system, the last three states are the output states and the 3rd and 4th states are the functional state (state to be estimated).

Next, re-arrange system (3.120) into the form of (3.98) as follows

$$\underbrace{\begin{bmatrix} \dot{x}_2 \\ \dot{y} \end{bmatrix}}_{\dot{\bar{x}}} = \underbrace{\left[\begin{array}{cc|ccc} -0.154 & -0.0042 & 0 & -0.032 & 1.54 \\ 0.249 & -1 & 0 & -1.12 & -5.2 \\ \hline 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -25 & 0 \\ -0.996 & -0.0003 & 0.0386 & 0 & -2.117 \end{array}\right]}_{\bar{A}} \underbrace{\begin{bmatrix} \dot{x}_2 \\ \dot{y} \end{bmatrix}}_{\bar{x}} + \underbrace{\left[\begin{array}{cc|cc} -0.744 & 0 & 0 & 0 \\ 0.337 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 25 \\ 0.02 & 0 & 0 & 0 \end{array}\right]}_{\bar{D}} \underbrace{\begin{bmatrix} x_{12} \\ f \end{bmatrix}}_{\bar{f}}$$

$$y = \underbrace{\left[\begin{array}{cc|ccc} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]}_{\bar{C}} \underbrace{\begin{bmatrix} x_2 \\ y \end{bmatrix}}_{\bar{x}} \tag{3.121}$$

From (3.121), it is obvious that $\mathrm{rank}(\bar{C}\bar{D}) = \mathrm{rank}(\bar{D}) = 2$, which satisfies condition E1. Therefore, the Edwards-Spurgeon observer (3.3) can be implemented directly on system (3.121) to estimate $\bar{x}$.

Using the design procedure in section 3.2, L in (3.5) is selected as

$$L = \begin{bmatrix} 0.0252 & 0 & 70.127 \\ -71 & 0 & 0.25 \end{bmatrix} \tag{3.122}$$

to get $A_1 + LA_3 = \begin{bmatrix} -70 & 0 \\ 0 & -72 \end{bmatrix}$ which will govern the observer sliding motion. Then, P_o and A_4^s are chosen to be I_3 and $diag\{-68, -71, -73\}$. Then gain matrices, G_n and G_l in the same coordinate of (3.121) are

$$G_n = \begin{bmatrix} -0.0252 & 0 & -70.127 \\ 71 & 0 & -0.25 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; G_l = \begin{bmatrix} -2.011 & -0.032 & -5106.9 \\ 4757 & -1.12 & -40.662 \\ 139 & 0 & -0.25 \\ 0 & 46 & 0 \\ 0.0424 & 0 & 140.73 \end{bmatrix} \tag{3.123}$$

Then inital conditions of the states x were assumed to be $\begin{bmatrix} 0.1 & 0 & 0 & -0.2 & 0.3 & -0.1 & 0.4 \end{bmatrix}^T$ whilst the observer had zero intial conditions. The unknown input f were set to be sinusoidal waveforms, where $f_1 = 2.15\sin(1.24t - 0.25)$ and $f_2 = 1.03\sin(1.65t - 0.78)$. The

ρ and δ in (3.21) were selected as 50 and 1×10^{-4}. Figure 3.9 shows the plot of the output estimation error, where sliding motion has taken place at around $0.01s$. Figure 3.10 - 3.11 show the estimated functional state $\hat{x}_2$ can accurately tracks the actual functional state x_2 and Figure 3.12 - 3.13 is the zoom in version of Figure 3.10 - 3.11.

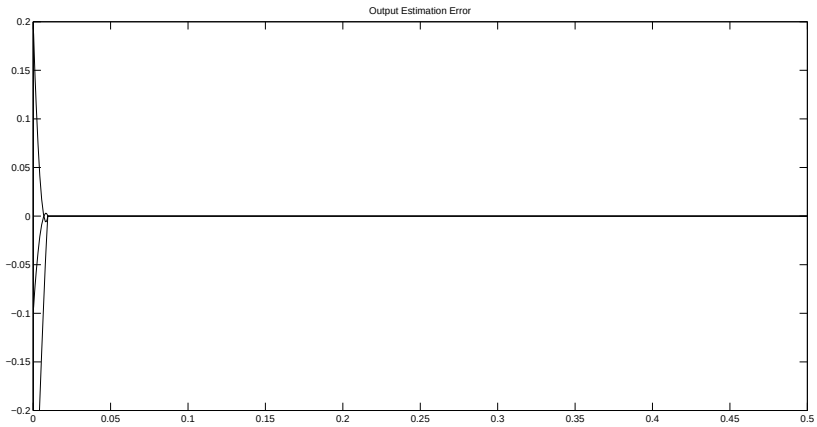

Figure 3.9: Output estimation error $e_y = y - \hat{y}$. All components of e_y are driven to zero at around $t = 0.01s$.

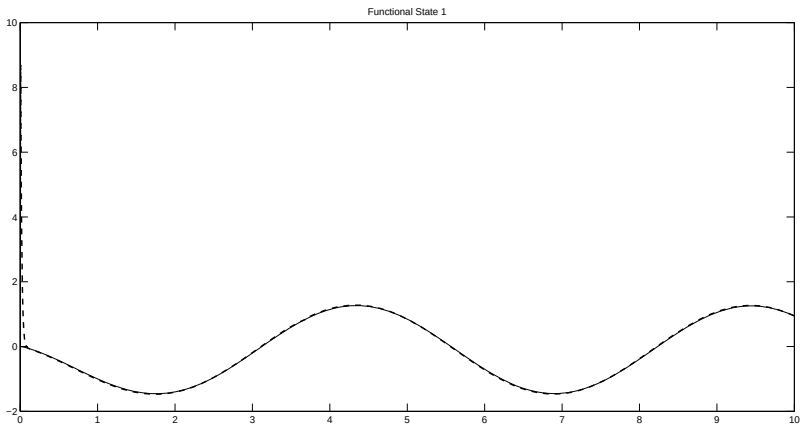

Figure 3.10: The estimated $\hat{\dot{\theta}}_{yaw}$ (dashed) follow the actual $\dot{\theta}_{yaw}$ (solid).

3.4 Conclusion

This chapter has the presented the sliding mode observer scheme by Edwards & Spurgeon together with a systematic method to design it, and also the necessary and sufficient conditions for the observer to be feasible. Following that, it has also shown how cascading observers could relax one of those conditions. Finally it described how the observer

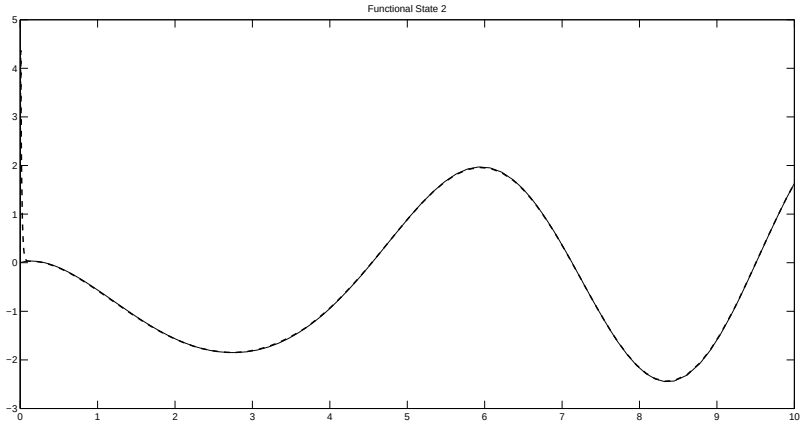

Figure 3.11: The estimated $\hat{\dot{\theta}}_{roll}$ (dashed) follow the actual $\dot{\theta}_{roll}$ (solid).

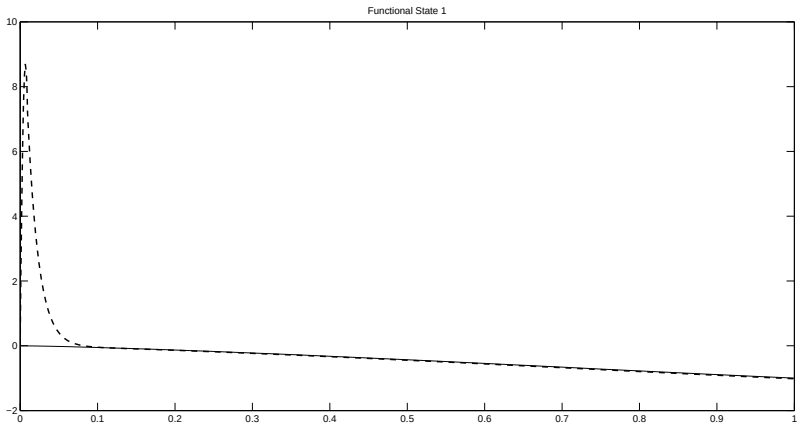

Figure 3.12: First second of Figure 3.10, where the estimated $\hat{\dot{\theta}}_{yaw}$ tracks the actual $\dot{\theta}_{yaw}$ in less than $0.1s$.

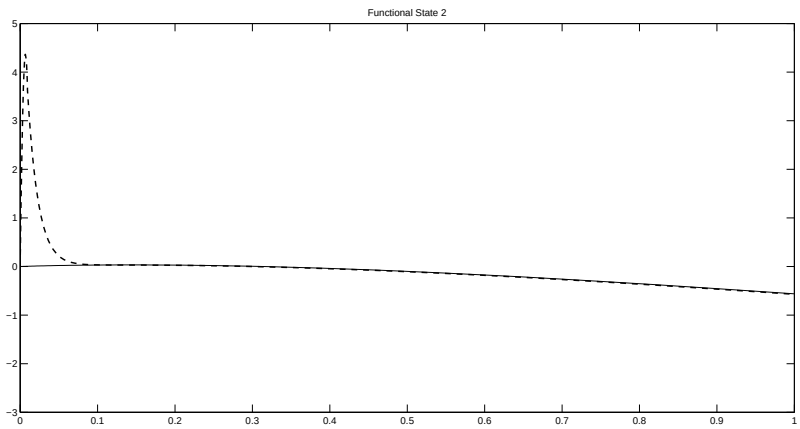

Figure 3.13: First second of Figure 3.11, where the estimated $\hat{\dot{\theta}}_{roll}$ tracks the actual $\dot{\theta}_{roll}$ in less than $0.1s$.

could be used for functional state estimation, and also presented the existence conditions in terms of the original system matrices. Numerical examples were inserted throughout the chapter to illustrate the theories.

Chapter 4

Robust Functional State Estimation using Two Sliding Mode Observers in Cascade

4.1 Introduction

In the previous chapter, it was shown in Sections 3.2.3 and 3.2.4 how cascading sliding mode observers could enable state estimation for a wider class of systems as the matching condition has been relaxed; it was also shown in Section 3.3 how a sliding mode observer design could be modified to achieve functional state estimation. In this chapter, both ideas are combined; using two sliding mode observers in cascade for functional state estimation, and the existence conditions are investigated in terms of the original system matrices.

The original system was firstly recasted into a form suitable for functional state estimation, and the theory of two cascaded sliding mode observers [79] was directly applied to investigate the existence conditions. In Section 4.3, matrix manipulations were performed to recast them in terms of the original system matrice; as expected, the matching condition was found to be less stringent than the case of one observer in Section 3.3 as well as other earlier works on functional state estimation [46, 87, 96, 97]. A simulation example was used to show the effectiveness of this scheme.

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4.2 Problem Statement

Consider the following system

$$\begin{aligned} \dot{x} &= Ax + Df \\ y &= Cx \\ z &= Fx \end{aligned} \tag{4.1}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^p$ and $f \in \mathbb{R}^q$ represent the state, output and unknown input respectively and $z \in \mathbb{R}^r$ is the functional state vector to be estimated. The matrices $A \in \mathbb{R}^{n\times n}$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{n\times q}$ and $F \in \mathbb{R}^{r\times n}$ are known and constant; without loss of generality, it is assumed that $\mathrm{rank}(C) = p$, $\mathrm{rank}(F) = r < n$ and also assume $\mathrm{rank}\begin{bmatrix} F \\ C \end{bmatrix} = r + p$; this is not a restrictive assumption as it simply implies that the 'output states' will not be estimated since they are already available for measurement.

As in Section 3.3, the system (4.1) can be transformed into the structure of (3.90) by using Lemma 3.5,which is

$$A = \left[\begin{array}{c|cc} A_1 & A_2 & A_3 \\ \hline A_4 & A_5 & A_6 \\ A_7 & A_8 & A_9 \end{array}\right], \quad D = \left[\begin{array}{c} D_1 \\ \hline D_2 \\ D_3 \end{array}\right]\begin{array}{l} \updownarrow n-p-r \\ \updownarrow r \\ \updownarrow p \end{array}, \qquad \begin{aligned} F &= \left[\begin{array}{c|cc} 0 & I_r & 0 \end{array}\right], \\ C &= \left[\begin{array}{c|cc} 0 & 0 & I_p \end{array}\right] \end{aligned} \tag{4.2}$$

In the same coordinate system, the state can be transformed and partitioned as follows:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ y \end{bmatrix}\begin{array}{l} \updownarrow n-p-r \\ \updownarrow r \\ \updownarrow p \end{array} \tag{4.3}$$

Then, system (4.1) can be re-arranged to have the structure in (3.98),

$$\begin{aligned} \underbrace{\begin{bmatrix} \dot{x}_2 \\ \dot{y} \end{bmatrix}}_{\dot{\bar{x}}} &= \underbrace{\begin{bmatrix} A_5 & A_6 \\ A_8 & A_9 \end{bmatrix}}_{\bar{A}} \underbrace{\begin{bmatrix} \dot{x}_2 \\ \dot{y} \end{bmatrix}}_{\bar{x}} + \underbrace{\begin{bmatrix} A_4 & D_2 \\ A_7 & D_3 \end{bmatrix}}_{\bar{D}} \underbrace{\begin{bmatrix} x_1 \\ f \end{bmatrix}}_{\bar{f}} \\ y &= \underbrace{\begin{bmatrix} 0 & I_p \end{bmatrix}}_{\bar{C}} \underbrace{\begin{bmatrix} x_2 \\ y \end{bmatrix}}_{\bar{x}} \end{aligned} \tag{4.4}$$

Now, system (4.4) is in the same form as (3.1). Using standard sliding mode observer theory, one of the conditions to estimate $\bar{x}$ whilst rejecting $\bar{f}$ is that $\mathrm{rank}(\bar{C}\bar{D}) = \mathrm{rank}(\bar{D})$

which has been re-casted in terms of the original system matrices as condition F1 in Section 3.3.2. When that condition is not satisfied, a two-observer scheme by Ng *et al.* [79] may be able to estimate $\bar{x}$, subject to the following necessary and sufficient conditions

G1. $\operatorname{rank}\begin{bmatrix} \bar{C}\bar{A}\bar{D} & \bar{C}\bar{D} \\ \bar{C}\bar{D} & 0 \end{bmatrix} = \operatorname{rank}(\bar{C}\bar{D}) + \operatorname{rank}(\bar{D})$

G2. $\operatorname{rank}\begin{bmatrix} sI - \bar{A} & \bar{D} \\ \bar{C} & 0 \end{bmatrix} = p + r + \operatorname{rank}(\bar{D}),\ \forall s \in \mathbb{C}, Re(s) \geq 0$

Since conditions G1 and G2 are in terms of the transformed system matrices, it is therefore of interest to recast conditons G1 and G2 in terms of the original system matrices so that the designer can know at the outset whether the this observer scheme is applicable or not; this is the main contribution of this chapter, which is summarized in Theorem 4.1 below.

Theorem 4.1 *Conditions G1 and G2 can be re-casted in terms of the original system matrices as follows*

H1. $\operatorname{rank}\begin{bmatrix} CA^2 & CA & CAD & CD \\ CA & 0 & CD & 0 \\ C & 0 & 0 & 0 \\ F & 0 & 0 & 0 \\ 0 & C & 0 & 0 \\ 0 & F & 0 & 0 \end{bmatrix} = \operatorname{rank}\begin{bmatrix} FA & FD \\ CA & CD \\ C & 0 \\ F & 0 \end{bmatrix} + \operatorname{rank}\begin{bmatrix} CA & CD \\ C & 0 \\ F & 0 \end{bmatrix}$

H2. $\operatorname{rank}\begin{bmatrix} sF - FA & -FD \\ CA & CD \\ C & 0 \end{bmatrix} = \operatorname{rank}\begin{bmatrix} FA & FD \\ CA & CD \\ C & 0 \\ F & 0 \end{bmatrix},\ \forall s \in \mathbb{C}, Re(s) \geq 0$

□

4.3 Main Result: Proof of Theorem 4.1

The remainder of this chapter provides a constructive proof of Theorem 4.1.

First, to recast G1 and G2 in terms of the original matrices, a further transformation is performed on the matrices in (4.2) as detailed in the following lemma.

Lemma 4.1 *There exist a nonsingular transformation for x and f such that $x \mapsto T_r x$,*

$f \mapsto T_q f$ and the system matrices in (4.2) *will have the following structure*

$$A_r = \left[\begin{array}{ccc|cc|ccc} * & * & * & * & * & * & * & * \\ \hline 0 & A_{411} & 0 & A_{51} & A_{52} & A_{61} & A_{62} & A_{63} \\ 0 & A_{412} & 0 & A_{53} & A_{54} & A_{64} & A_{65} & A_{66} \\ \hline 0 & 0 & 0 & A_{81} & A_{82} & A_{91} & A_{92} & A_{93} \\ 0 & 0 & 0 & A_{83} & A_{84} & A_{94} & A_{95} & A_{96} \\ 0 & 0 & A_{711} & A_{85} & A_{86} & A_{97} & A_{98} & A_{99} \end{array}\right], \quad D_r = \left[\begin{array}{ccc} D_{11} & D_{12} & D_{13} \\ \hline 0 & 0 & 0 \\ 0 & D_{21} & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & D_{31} \\ D_{32} & D_{33} & 0 \end{array}\right] \begin{array}{l} \updownarrow n-p-r \\ \updownarrow r-g \\ \updownarrow g \\ \updownarrow p-b \\ \updownarrow b-c \\ \updownarrow c \end{array},$$

$$F_r = \begin{bmatrix} 0 & F_2 & 0 \end{bmatrix}, \; C_r = \begin{bmatrix} 0 & 0 & C_3 \end{bmatrix} \tag{4.5}$$

where (*) are matrices that play no role in the proceeding analysis, and A_{51}, A_{54}, A_{91}, A_{95}, A_{99} are square matrices, and $A_{711} \in \mathbb{R}^{c\times c}$, $D_{21} \in \mathbb{R}^{g\times g}$, $D_{31} \in \mathbb{R}^{(b-c)\times(b-c)}$, $F_2 \in \mathbb{R}^{r\times r}$ and $C_3 \in \mathbb{R}^{p\times p}$ are square and invertible. ♯

Proof

In the coordinates of (4.2), define $h = \operatorname{rank}\begin{bmatrix} A_4 \\ A_7 \end{bmatrix}$, where $h \le min\{r+p, n-p-r\}$ since $\begin{bmatrix} A_4 \\ A_7 \end{bmatrix} \in \mathbb{R}^{(r+p)\times(n-p-r)}$. Let R_1 be an orthogonal matrix such that

$$\begin{bmatrix} A_4 \\ A_7 \end{bmatrix} R_1^{-1} = \begin{bmatrix} 0 & A_{m,41} \\ 0 & A_{m,71} \end{bmatrix} \tag{4.6}$$

where $A_{m,41} \in \mathbb{R}^{r\times h}$, $A_{m,71} \in \mathbb{R}^{p\times h}$, and $\begin{bmatrix} A_{m,41} \\ A_{m,71} \end{bmatrix}$ has full column rank. The matrix R_1 can be calculated using the QR decomposition of $\begin{bmatrix} A_4 \\ A_7 \end{bmatrix}$ which can be easily performed in MATLAB. Define

$$T_m = \begin{bmatrix} R_1 & 0 \\ 0 & I_{r+p} \end{bmatrix} \tag{4.7}$$

then the matrices in (4.2) can be transformed to have the following structures

$$A_m := T_m A T_m^{-1} = \left[\begin{array}{cc|cc} A_{m,11} & A_{m,12} & A_{m,2} & A_{m,3} \\ \hline 0 & A_{m,41} & A_{m,5} & A_{m,6} \\ 0 & A_{m,71} & A_{m,8} & A_{m,9} \end{array}\right], \; D_m := T_m D = \left[\begin{array}{c} D_{m,1} \\ \hline D_{m,2} \\ D_{m,3} \end{array}\right],$$

$$F_m := F T_m^{-1} = \left[\begin{array}{c|cc} 0 & I_r & 0 \end{array}\right], \; C_m := C T_m^{-1} = \left[\begin{array}{c|cc} 0 & 0 & I_p \end{array}\right] \tag{4.8}$$

Then, let $\mathrm{rank}(A_{m,71}) = c$, where $c \leq h < p$ while R_2 and R_3 are orthogonal matrices such that

$$R_2 A_{m,71} R_3^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & A_{711} \end{bmatrix} \tag{4.9}$$

where $A_{711} \in \mathbb{R}^{c\times c}$ is full rank. Then, partition $A_{m,41} = \begin{bmatrix} A_{n,41} & A_{n,42} \end{bmatrix}$ in which $A_{n,41} \in \mathbb{R}^{r\times(h-c)}$. Define T_n to be

$$T_n = \left[\begin{array}{c|c|cc} I_{n-p-r} & 0 & 0 & 0 \\ \hline 0 & I_r & 0 & -A_{n,42}(A_{711})^{-1} \\ \hline 0 & 0 & I_{p-c} & 0 \\ 0 & 0 & 0 & I_c \end{array}\right] \left[\begin{array}{cc|c|c} I_{n-p-r-h} & 0 & 0 & 0 \\ 0 & R_3 & 0 & 0 \\ \hline 0 & 0 & I_r & 0 \\ \hline 0 & 0 & 0 & R_2 \end{array}\right] \tag{4.10}$$

Hence, the matrix A_m in (4.8) can be transformed to have the structure in (4.5) as follows

$$A_n := T_n A_m T_n^{-1} = \left[\begin{array}{ccc|c|cc} * & * & * & * & * & * \\ \hline 0 & A_{n,41} & 0 & A_{n,5} & * & * \\ \hline 0 & 0 & 0 & A_{n,81} & * & * \\ 0 & 0 & A_{711} & A_{n,82} & * & * \end{array}\right] \begin{array}{l} \updownarrow n-p-r \\ \updownarrow r \\ \updownarrow p-c \\ \updownarrow c \end{array} \tag{4.11}$$

Then, partition generally $D_n = T_n D$ as

$$D_n = \left[\begin{array}{c} D_{n,1} \\ \hline D_{n,2} \\ \hline D_{n,31} \\ D_{n,32} \end{array}\right] \begin{array}{l} \updownarrow n-p-r \\ \updownarrow r \\ \updownarrow p-c \\ \updownarrow c \end{array} \tag{4.12}$$

To achieve the strucutre of D_r in (4.5), define $\mathrm{rank}(\bar{C}\bar{D}) = b$ and then from (4.4) it is clear that $\mathrm{rank}\begin{bmatrix} A_7 & D_3 \end{bmatrix} = b$. By analyzing the coordinate transform of T_m and T_n on A, it can be seen that

$$R_2 \begin{bmatrix} A_7 & D_3 \end{bmatrix} \begin{bmatrix} R_1^{-1} & 0 \\ 0 & I_q \end{bmatrix} \begin{bmatrix} I_{n-p-r-h} & 0 & 0 \\ 0 & R_3^{-1} & 0 \\ 0 & 0 & I_q \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & D_{n,31} \\ 0 & 0 & A_{711} & D_{n,32} \end{bmatrix} \tag{4.13}$$

which has rank of b. Since $\mathrm{rank}(A_{711}) = c$, hence $\mathrm{rank}(D_{n,31}) = b - c$ and there exist orthogonal matrices R_4 and R_5 such that

$$R_4 D_{n,31} R_5^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & D_{31} \end{bmatrix} \tag{4.14}$$

where $D_{31} \in \mathbb{R}^{(b-c)\times(b-c)}$ is full rank. Then, partition generally

$$D_{n,2}R_5^{-1} = \begin{bmatrix} D_{n,21} & D_{n,22} \end{bmatrix}, \quad D_{n,32}R_5^{-1} = \begin{bmatrix} D_{n,321} & D_{n,322} \end{bmatrix} \tag{4.15}$$

where $D_{n,21} \in \mathbb{R}^{r\times(q-b+c)}$ and $D_{n,321} \in \mathbb{R}^{c\times(q-b+c)}$. Then, let $g = \mathrm{rank}(D_{n,21})$ and define orthogonal matrices R_6 and R_7 such that

$$R_6 D_{n,21} R_7^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & D_{21} \end{bmatrix} \tag{4.16}$$

where $D_{21} \in \mathbb{R}^{g\times g}$ has full rank. Define state transformation T_p and unknown input transformation T_q as follows

$$T_p = \left[\begin{array}{c|c|cccc} I_{n-p-r} & 0 & 0 & 0 & 0 \\ \hline 0 & I_r & 0 & R_8 & 0 \\ \hline 0 & 0 & I_{p-b} & 0 & 0 \\ 0 & 0 & 0 & I_{b-c} & 0 \\ 0 & 0 & 0 & R_9 & I_c \end{array}\right] \left[\begin{array}{c|c|c} I_{n-p-r} & 0 & 0 \\ \hline 0 & R_6 & 0 \\ \hline 0 & 0 & I_p \end{array}\right] \left[\begin{array}{c|c|cc} I_{n-p-r} & 0 & 0 & 0 \\ \hline 0 & I_r & 0 & 0 \\ \hline 0 & 0 & R_4 & 0 \\ 0 & 0 & 0 & I_c \end{array}\right] \tag{4.17}$$

$$T_q = R_5 \begin{bmatrix} R_7 & 0 \\ 0 & I_{b-c} \end{bmatrix} \tag{4.18}$$

where $R_8 = -D_{n,22}(D_{31})^{-1}$ and $R_9 = -D_{n,322}(D_{31})^{-1}$. The system matrices in the coordinates of (4.11) under the transformations $x_n \mapsto x_p = T_p x_p$ and $f \mapsto f_p = T_q f_p$ will have the following structures,

$$A_p := T_p A_n T_p^{-1} = \left[\begin{array}{ccc|cc|ccc} * & * & * & * & * & * & * & * \\ \hline 0 & A_{p,411} & 0 & A_{p,51} & A_{p,52} & A_{p,61} & A_{p,62} & A_{p,63} \\ 0 & A_{p,412} & 0 & A_{p,53} & A_{p,54} & A_{p,64} & A_{p,65} & A_{p,66} \\ \hline 0 & 0 & 0 & A_{p,81} & A_{p,82} & A_{p,91} & A_{p,92} & A_{p,93} \\ 0 & 0 & 0 & A_{p,83} & A_{p,84} & A_{p,94} & A_{p,95} & A_{p,96} \\ 0 & 0 & A_{711} & A_{p,85} & A_{p,86} & A_{p,97} & A_{p,98} & A_{p,99} \end{array}\right] \begin{array}{l} \updownarrow n-p-r \\ \updownarrow r-g \\ \updownarrow g \\ \updownarrow p-b \\ \updownarrow b-c \\ \updownarrow c \end{array},$$

$$D_p := T_p D_n T_q^{-1} = \left[\begin{array}{ccc} D_{p,11} & D_{p,12} & D_{p,13} \\ \hline 0 & 0 & 0 \\ 0 & D_{21} & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & D_{31} \\ D_{p,32} & D_{p,33} & 0 \end{array}\right] \begin{array}{l} \updownarrow n-p-r \\ \updownarrow r-g \\ \updownarrow g \\ \updownarrow p-b \\ \updownarrow b-c \\ \updownarrow c \end{array},$$

$$F_p := F_n T_p^{-1} = \begin{bmatrix} 0 & F_2 & 0 \end{bmatrix}, \; C_p := C_n T_p^{-1} = \begin{bmatrix} 0 & 0 & C_3 \end{bmatrix} \tag{4.19}$$

This transformation of T_p does not affect the special structure of A_n in (4.11) (i.e. A_{711} and the block of zeros are unaltered). Through the transformation of $T_p T_n T_m$ the matrix C will experience a non-singular scaling in the last p columns to produce C_3 that is non-singular whilst the block of zeros in the first $n-p$ columns are unaltered. Then, the matrix F will have a block of zeros in the last p columns whilst the preceeding r columns are scaled non-singularly to produce a non-singular F_2 and the other $n-p-r$ columns of zeros is unaffected by this transformation. Now, A_p, C_p, D_p and F_p in (4.19) are in the structure of (4.5) and the proof is complete. ■

Lemma 4.2 *Conditions G1 and H1 are equivalent.* ♯

Proof

To firstly investigate the implication of G1, the matrices in (4.4) are written in the coordinates of (4.5) as follows:

$$\bar{A} = \left[\begin{array}{cc|ccc} A_{51} & A_{52} & A_{61} & A_{62} & A_{63} \\ A_{53} & A_{54} & A_{64} & A_{65} & A_{66} \\ \hline A_{81} & A_{82} & A_{91} & A_{92} & A_{93} \\ A_{83} & A_{84} & A_{94} & A_{95} & A_{96} \\ A_{85} & A_{86} & A_{97} & A_{98} & A_{99} \end{array}\right] = \begin{array}{c} \begin{bmatrix} \bar{A}_1 & \bar{A}_2 \\ \bar{A}_3 & \bar{A}_4 \end{bmatrix} \end{array} \begin{array}{l} \updownarrow r \\ \updownarrow p \end{array},$$

$$\bar{D} = \left[\begin{array}{ccc|ccc} 0 & A_{411} & 0 & 0 & 0 & 0 \\ 0 & A_{412} & 0 & 0 & D_{21} & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & D_{31} \\ 0 & 0 & A_{711} & D_{32} & D_{33} & 0 \end{array}\right] = \begin{bmatrix} \bar{D}_1 \\ \bar{D}_2 \end{bmatrix} \begin{array}{l} \updownarrow r \\ \updownarrow p \end{array},$$

$$\bar{F} = \begin{bmatrix} F_2 & 0 \end{bmatrix}, \; \bar{C} = \begin{bmatrix} 0 & C_3 \end{bmatrix} \tag{4.20}$$

Substituting the matrices from (4.20) into $\bar{C}\bar{D}$ yields

$$\bar{C}\bar{D} = C_3 \left[\begin{array}{ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & D_{31} \\ 0 & 0 & A_{711} & D_{32} & D_{33} & 0 \end{array}\right] \tag{4.21}$$

Because A_{711}, D_{31} and D_{21} are full rank matrices of rank c, $b-c$ and g respectively, it is straightforward to see that

$$\begin{aligned}\Rightarrow \operatorname{rank}(\bar{C}\bar{D}) + \operatorname{rank}(\bar{D}) &= \operatorname{rank}(A_{711}) + \operatorname{rank}(D_{31}) + \operatorname{rank}(A_{711}) + \operatorname{rank}(D_{31}) \\ &\quad + \operatorname{rank}(D_{21}) + \operatorname{rank}(A_{411}) \\ &= 2b + g + \operatorname{rank}(A_{411})\end{aligned} \tag{4.22}$$

Now, define

$$\Pi = \begin{bmatrix} \bar{C}\bar{A}\bar{D} & \bar{C}\bar{D} \\ \bar{C}\bar{D} & 0 \end{bmatrix} \tag{4.23}$$

Substitute the system matrices from (4.20) into Π_1 to get

$$\Pi = \begin{bmatrix} C_3 & 0 \\ 0 & C_3 \end{bmatrix} \left[\begin{array}{cccccc|ccccc} 0 & X_1 & X_2 & X_3 & X_4 & * & 0 & 0 & 0 & 0 & 0 \\ 0 & * & * & * & * & * & 0 & 0 & 0 & 0 & D_{31} \\ 0 & * & * & * & * & * & 0 & A_{711} & D_{32} & D_{33} & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & D_{31} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_{711} & D_{32} & D_{33} & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \tag{4.24}$$

where $X_1 = A_{81}A_{411} + A_{82}A_{412}$, $X_2 = A_{93}A_{711}$, $X_3 = A_{93}D_{32}$ and $X_4 = A_{82}D_{21} + A_{93}D_{33}$. Since A_{711} and D_{31} are full rank, it is straightforward to see that

$$\begin{aligned}\operatorname{rank}(\Pi) &= \operatorname{rank}(A_{711}) + \operatorname{rank}(D_{31}) + \operatorname{rank}(D_{31}) + \operatorname{rank}\underbrace{\begin{bmatrix} X_1 & X_2 & X_3 & X4 \\ 0 & A_{711} & D_{32} & D_{33} \end{bmatrix}}_{\Pi_1} \\ &= \operatorname{rank}(\Pi_1) + 2b - c\end{aligned} \tag{4.25}$$

Then, post-multiply Π_1 with an invertible matrix to get Π_2 as follows

$$\begin{aligned}\Pi_2 &= \begin{bmatrix} X_1 & X_2 & X_3 & X4 \\ 0 & A_{711} & D_{32} & D_{33} \end{bmatrix} \begin{bmatrix} I_r & 0 & 0 & 0 \\ 0 & I_{p-b} & -A_{711}^{-1}D_{32} & -A_{711}^{-1}D_{33} \\ 0 & 0 & I_{b-c} & 0 \\ 0 & 0 & 0 & I_c \end{bmatrix} \\ &= \begin{bmatrix} A_{81}A_{411} + A_{82}A_{412} & A_{93}A_{711} & 0 & A_{82}D_{21} \\ 0 & A_{711} & 0 & 0 \end{bmatrix}\end{aligned} \tag{4.26}$$

Thus,

$$\operatorname{rank}(\Pi_2) = \operatorname{rank}\underbrace{\begin{bmatrix} A_{81}A_{411} + A_{82}A_{412} & A_{82}D_{21} \end{bmatrix}}_{\Pi_3} + c \tag{4.27}$$

and from (4.26), knowing that $\operatorname{rank}(\Pi_1) = \operatorname{rank}(\Pi_2)$, then (4.25) can be written as

$$\operatorname{rank}(\Pi) = \operatorname{rank}(\Pi_3) + 2b \tag{4.28}$$

Therefore, from (4.22) and (4.23) - (4.28), condition G1 implies

$$\begin{aligned} &\operatorname{rank}(\Pi) = \operatorname{rank}(\bar{C}\bar{D}) + \operatorname{rank}(\bar{D}) \\ &\Rightarrow \operatorname{rank}(\Pi_3) + 2b = 2b + g + \operatorname{rank}(A_{411}) \\ &\Rightarrow \operatorname{rank}\begin{bmatrix} A_{81} & A_{82} \end{bmatrix}\begin{bmatrix} A_{411} & 0 \\ A_{412} & D_{21} \end{bmatrix} = g + \operatorname{rank}(A_{411}) \end{aligned} \tag{4.29}$$

To investigate the implication of H1, define $K = -\begin{bmatrix} A_3 \\ A_6 \\ A_9 \end{bmatrix}$ and hence in the coordinates of (4.5),

$$A_o = A + KC = \left[\begin{array}{ccc|cc|c} * & * & * & * & * & 0 \\ \hline 0 & A_{411} & 0 & A_{51} & A_{52} & 0 \\ 0 & A_{412} & 0 & A_{53} & A_{54} & 0 \\ \hline 0 & 0 & 0 & A_{81} & A_{82} & 0 \\ 0 & 0 & 0 & A_{83} & A_{84} & 0 \\ 0 & 0 & A_{711} & A_{85} & A_{86} & 0 \end{array}\right] \tag{4.30}$$

Then, define

$$\Psi_a = \begin{bmatrix} CA^2 & CA & CAD & CD \\ CA & 0 & CD & 0 \\ C & 0 & 0 & 0 \\ F & 0 & 0 & 0 \\ 0 & C & 0 & 0 \\ 0 & F & 0 & 0 \end{bmatrix} \tag{4.31}$$

and pre-multiply Ψ_a with an invertible matrix, K_1 to obtain $\Psi_{o,a}$ as follows,

$$
\begin{aligned}
\Psi_{o,a} &= K_1 \Psi_a \\
&= \begin{bmatrix} I_n & CK & (CA+CKC)K & 0 & CK & 0 \\ 0 & I_n & CK & 0 & 0 & 0 \\ 0 & 0 & I_p & 0 & 0 & 0 \\ 0 & 0 & 0 & I_r & 0 & 0 \\ 0 & 0 & 0 & 0 & I_p & 0 \\ 0 & 0 & 0 & 0 & 0 & I_r \end{bmatrix} \begin{bmatrix} CA^2 & CA & CAD & CD \\ CA & 0 & CD & 0 \\ C & 0 & 0 & 0 \\ F & 0 & 0 & 0 \\ 0 & C & 0 & 0 \\ 0 & F & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} CA_o^2 & CA_o & CA_oD & CD \\ CA_o & 0 & CD & 0 \\ C & 0 & 0 & 0 \\ F & 0 & 0 & 0 \\ 0 & C & 0 & 0 \\ 0 & F & 0 & 0 \end{bmatrix}
\end{aligned} \tag{4.32}
$$

Then, substitute for A_o from (4.30) and C and F from (4.5) such that

$$
\Psi_{o,a} = H_1 \left[\begin{array}{ccccc|ccccc|ccc|ccc}
0 & X_1 & 0 & * & 0 & 0 & 0 & 0 & * & 0 & 0 & A_{82}D_{21} & 0 & 0 & 0 & 0 \\
0 & * & 0 & * & 0 & 0 & 0 & 0 & * & 0 & 0 & * & 0 & 0 & 0 & D_{31} \\
* & * & * & * & 0 & 0 & 0 & A_{711} & * & 0 & * & * & * & D_{32} & D_{33} & 0 \\
\hline
0 & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & D_{31} & 0 & 0 & 0 \\
0 & 0 & A_{711} & * & 0 & 0 & 0 & 0 & 0 & 0 & D_{32} & D_{33} & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & I_p & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & I_r & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_p & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_r & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right] \tag{4.33}
$$

where $H_1 = diag\{C_3, C_3, C_3, F_2, C_3, F_2\}$ which is invertible. Both A_{711} and D_{31} are full rank matrices, therefore the LHS of H1 becomes

$$
\begin{aligned}
\mathrm{rank}(\Psi_{o,a}) = \mathrm{rank}(\Psi_a) &= \mathrm{rank}\begin{bmatrix} X_1 & A_{82}D_{21} \end{bmatrix} + 2b + 2p + 2r \\
&= \mathrm{rank}\begin{bmatrix} A_{81} & A_{82} \end{bmatrix} \begin{bmatrix} A_{411} & 0 \\ A_{412} & D_{21} \end{bmatrix} + 2b + 2p + 2r
\end{aligned} \tag{4.34}
$$

where X_1 has been defined in (4.24). To find the expression for the RHS of H1, now

define

$$\Theta = \begin{bmatrix} FA & FD \\ CA & CD \\ C & 0 \\ F & 0 \end{bmatrix}, \quad \Psi_b = \begin{bmatrix} CA & CD \\ C & 0 \\ F & 0 \end{bmatrix} \tag{4.35}$$

and pre-multiply with invertible matrices K_2 and K_3 respectively to get

$$\Theta_o = K_2\Theta = \underbrace{\begin{bmatrix} I_r & 0 & FK & 0 \\ 0 & I_p & CK & 0 \\ 0 & 0 & I_p & 0 \\ 0 & 0 & 0 & I_r \end{bmatrix}}_{K_2} \begin{bmatrix} FA & FD \\ CA & CD \\ C & 0 \\ F & 0 \end{bmatrix} = \begin{bmatrix} FA_o & FD \\ CA_o & CD \\ C & 0 \\ F & 0 \end{bmatrix} \tag{4.36}$$

$$\Psi_{o,b} = K_3\Psi_b = \underbrace{\begin{bmatrix} I_p & CK & 0 \\ 0 & I_p & 0 \\ 0 & 0 & I_r \end{bmatrix}}_{K_3} \begin{bmatrix} CA & CD \\ C & 0 \\ F & 0 \end{bmatrix} = \begin{bmatrix} CA_o & CD \\ C & 0 \\ F & 0 \end{bmatrix} \tag{4.37}$$

Then, substitute for A_o from (4.30) and matrices C and F from (4.5) into Θ_o and $\Psi_{o,b}$ to get

$$\Theta_o = \left[\begin{array}{c|c|cc} R_F & 0 & 0 & 0 \\ \hline 0 & C_3 & 0 & 0 \\ \hline 0 & 0 & C_3 & 0 \\ 0 & 0 & 0 & F_2 \end{array}\right] \left[\begin{array}{ccccc|ccc} 0 & A_{411} & 0 & * & 0 & 0 & 0 & 0 \\ 0 & A_{412} & 0 & * & 0 & 0 & D_{21} & 0 \\ \hline 0 & 0 & 0 & * & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & * & 0 & 0 & 0 & D_{31} \\ 0 & 0 & A_{711} & * & 0 & D_{32} & D_{33} & 0 \\ \hline 0 & 0 & 0 & 0 & I_p & 0 & 0 & 0 \\ 0 & 0 & 0 & I_r & 0 & 0 & 0 & 0 \end{array}\right] \tag{4.38}$$

$$\Psi_{o,b} = \left[\begin{array}{c|cc} C_3 & 0 & 0 \\ \hline 0 & C_3 & 0 \\ 0 & 0 & F_2 \end{array}\right] \left[\begin{array}{ccccc|ccc} 0 & 0 & 0 & * & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & * & 0 & 0 & 0 & D_{31} \\ 0 & 0 & A_{711} & * & 0 & D_{32} & D_{33} & 0 \\ \hline 0 & 0 & 0 & 0 & I_p & 0 & 0 & 0 \\ 0 & 0 & 0 & I_r & 0 & 0 & 0 & 0 \end{array}\right] \tag{4.39}$$

Because A_{711}, D_{21} and D_{31} are full rank matrices, it is straightforward to see that

$$\operatorname{rank}(\Theta_o) = p + r + b + g + \operatorname{rank}(A_{411}) \tag{4.40}$$

and

$$\operatorname{rank}(\Psi_{o,b}) = p + r + b \tag{4.41}$$

Therefore, condition H1 can be written as

$$\begin{aligned} &\operatorname{rank}(\Psi_a) = \operatorname{rank}(\Psi_b) + \operatorname{rank}(\Theta) \\ \Rightarrow &\operatorname{rank}(\Psi_{o,a}) = \operatorname{rank}(\Psi_{o,b}) + \operatorname{rank}(\Theta_o) \end{aligned} \tag{4.42}$$

which from (4.34), (4.40) and (4.41) becomes

$$\Rightarrow \operatorname{rank}\begin{bmatrix} A_{81} & A_{82} \end{bmatrix}\begin{bmatrix} A_{411} & 0 \\ A_{412} & D_{21} \end{bmatrix} + 2b + 2p + 2r = 2p + 2r + 2b + g + \operatorname{rank}(A_{411}) \tag{4.43}$$

which simplifies to condition G1 in (4.29) and the proof is complete. ■

Lemma 4.3 *Conditions G2 and H2 are equivalent.* ♯

Proof

Define Δ_a and Δ_b as follow

$$\Delta_a = \begin{bmatrix} sF - FA & -FD \\ CA & CD \\ C & 0 \end{bmatrix}, \quad \Delta_b = \begin{bmatrix} sI - \bar{A} & \bar{D} \\ \bar{C} & 0 \end{bmatrix} \tag{4.44}$$

To investigate condition H2, substitute for the matrices from (4.5) into Δ_a to get

$$\Delta_a = \begin{bmatrix} F_2 & 0 & 0 \\ 0 & C_3 & 0 \\ 0 & 0 & C_3 \end{bmatrix}\left[\begin{array}{cccccc|ccc} 0 & -A_{411} & 0 & sI_{r-g} - A_{51} & -A_{52} & * & 0 & 0 & 0 \\ 0 & -A_{412} & 0 & * & * & * & 0 & -D_{21} & 0 \\ \hline 0 & 0 & 0 & A_{81} & A_{82} & * & 0 & 0 & 0 \\ 0 & 0 & 0 & * & * & * & 0 & 0 & D_{31} \\ 0 & 0 & A_{711} & * & * & * & D_{32} & D_{33} & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & I_p & 0 & 0 & 0 \end{array}\right] \tag{4.45}$$

Because I_p, A_{711}, D_{21} and D_{31} are full rank matrices, it is straightforward to see that

$$\operatorname{rank}(\Delta_a) = \operatorname{rank}(\Delta_{a,1}) + p + g + b \tag{4.46}$$

where

$$\Delta_{a,1} = \begin{bmatrix} A_{411} & sI_{r-g} - A_{51} & -A_{52} \\ 0 & A_{81} & A_{82} \end{bmatrix} \tag{4.47}$$

From (4.40), it can be seen that the RHS of condition H2 has rank of $p + r + b + g + \mathrm{rank}(A_{411})$. Hence condition H2 implies that

$$\mathrm{rank}(\Delta_{a,1}) = r + \mathrm{rank}(A_{411}) \tag{4.48}$$

To investigate G2, expand Δ_b using (4.20) to get

$$\Delta_b = \begin{bmatrix} I_{r+p} & 0 \\ 0 & C_3 \end{bmatrix} \left[\begin{array}{ccc|ccc|ccc} sI_{r-g} - A_{51} & -A_{52} & * & 0 & A_{411} & 0 & 0 & 0 & 0 \\ * & * & * & 0 & A_{412} & 0 & 0 & D_{21} & 0 \\ \hline -A_{81} & -A_{82} & * & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & 0 & 0 & 0 & 0 & 0 & D_{31} \\ * & * & * & 0 & 0 & A_{711} & D_{32} & D_{33} & 0 \\ \hline 0 & 0 & I_p & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \tag{4.49}$$

Since I_p, A_{711}, D_{21} and D_{31} are full rank matrices, it is straightforward to see that

$$\mathrm{rank}(\Delta_b) = \mathrm{rank}(\Delta_{b,1}) + p + g + b \tag{4.50}$$

where

$$\Delta_{b,1} = \begin{bmatrix} sI_{r-g} - A_{51} & -A_{52} & A_{411} \\ -A_{81} & -A_{82} & 0 \end{bmatrix} \tag{4.51}$$

Pre-multiply and post-multiply $\Delta_{b,1}$ with the following full-rank matrices respectively

$$K_4 = \begin{bmatrix} I_{r-g} & 0 \\ 0 & -I_{p-b} \end{bmatrix}, \quad K_5 = \begin{bmatrix} 0 & I_{r-g} & 0 \\ 0 & 0 & I_g \\ I_{h-c} & 0 & 0 \end{bmatrix} \tag{4.52}$$

to yield

$$\Delta_{b,2} = K_4 \Delta_{b,1} K_5 = \begin{bmatrix} A_{411} & sI_{r-g} - A_{51} & -A_{52} \\ 0 & A_{81} & A_{82} \end{bmatrix} \tag{4.53}$$

which is $\Delta_{a,1}$ in (4.47). From (4.4) and (4.20), it can be established that

$$\bar{D} = \left[\begin{array}{cc|ccc} A_{411} & 0 & 0 & 0 & 0 \\ A_{412} & 0 & 0 & D_{21} & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & D_{31} \\ 0 & A_{711} & D_{32} & D_{33} & 0 \end{array}\right] \tag{4.54}$$

and from (4.29), $\mathrm{rank}(\bar{D}) = \mathrm{rank}(A_{411}) + b + g$. Hence condition G2 implies

$$\mathrm{rank}(\Delta_{b,2}) = \mathrm{rank}(A_{411}) + r \tag{4.55}$$

which is equivalent to H2. ■

Hence Theorem 4.1 is proven. Lemmas 4.2 and 4.3 have proven that conditions H1 and H2 are equivalent to G1 and G2. Hence if the original system matrices satisfy conditions H1 and H2, then a system in the form of (4.4) that satisfies conditions G1 and G2 can be obtained and functional state estimation can be achieved using the two-observer method described in Section 3.2.3.

4.4 Numerical Examples

In this section, a 7th order aircraft model [98] shown in Section 3.3.3 and a 2-cart system in [81] will be used to show that functional state estimation can be achieved by using two sliding mode observer in cascade.

The system states and control inputs for the aircraft model are

$$x = \begin{bmatrix} \text{bank angle, } \theta_{bank}\ (rad) \\ \text{yaw rate, } \dot{\theta}_{yaw}\ (rad/s) \\ \text{roll rate, } \dot{\theta}_{roll}\ (rad/s) \\ \text{sideslip angle, } \theta_{ss}\ (rad) \\ \text{washout filter state, } W \\ \text{rudder deflection, } \xi_{rudder}\ (rad) \\ \text{aileron deflection, } \xi_{aileron}\ (rad) \end{bmatrix}, \quad u = \begin{bmatrix} \text{rudder command } (rad) \\ \text{aileron command } (rad) \end{bmatrix} \tag{4.56}$$

In the notation (4.1), the matrices A and D are

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -0.154 & -0.0042 & 1.54 & 0 & -0.744 & -0.032 \\ 0 & 0.249 & -1 & -5.2 & 0 & 0.337 & -1.12 \\ 0.0386 & -0.996 & -0.0003 & -2.117 & 0 & 0.02 & 0 \\ 0 & 0.5 & 0 & 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -20 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -25 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 20 & 0 \\ 0 & 25 \end{bmatrix} \tag{4.57}$$

The functional states are $z = \begin{bmatrix} \dot{\theta}_{yaw} \\ \dot{\theta}_{roll} \end{bmatrix}$. Thus, the matrix F is defined as

$$F = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{4.58}$$

Assume that the measurement of the aileron deflection is not available. Therefore, only the bank angle and sideslip angles are measureable, and the matrix C becomes the following:

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \tag{4.59}$$

It can be easily verified that the system matrices in (4.57) - (4.59) satisfy condition H1 and H2. However, conditions A1, C1 and F1 are not satisfied, which means the observer designs presented in Chapter 3 are not applicable. The aileron deflection measurement is required to satisfy condition F1. Hence, this confirms that the two-observer method in this chapter can be used to achieve robust funcitonal state estimation with fewer sensors. First, perform the state transformation $x \mapsto T_s x$ in Lemma 4.1 to achieve the structure in (4.5), where

$$T_s = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \tag{4.60}$$

and the matrices from (4.57) - (4.59) after going through the transformation T_s will achieve the structure in (4.5) as follows

$$A_s = \left[\begin{array}{rrr|rr|rr} -4 & 0 & 0 & 0.5 & 0 & 0 & 0 \\ 0 & -20 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -25 & 0 & 0 & 0 & 0 \\ \hline 0 & -0.744 & -0.032 & -0.154 & -0.004 & 0 & 1.54 \\ 0 & 0.337 & -1.12 & 0.249 & -1 & 0 & -5.2 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0.02 & 0 & -0.996 & 0 & 0.039 & -2.117 \end{array}\right],$$

$$D_s = \left[\begin{array}{cc} 0 & 0 \\ 20 & 0 \\ 0 & 25 \\ \hline 0 & 0 \\ 0 & 0 \\ \hline 0 & 0 \\ 0 & 0 \end{array}\right], \qquad F_s = \left[\begin{array}{ccc|cc|cc} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right], \quad C_s = \left[\begin{array}{ccc|cc|cc} 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \tag{4.61}$$

From the structure of C_s and F_s, it can be seen that, in this coordinate system, the last two states are the output states and the 4th and 5th states are the functional state (state to be estimated).

Next, re-arrange system (4.61) into the form of (4.4) as follows

$$\underbrace{\begin{bmatrix} \dot{x}_2 \\ \dot{y} \end{bmatrix}}_{\dot{\bar{x}}} = \underbrace{\left[\begin{array}{cc|cc} -0.154 & -0.004 & 0 & 1.54 \\ 0.249 & -1 & 0 & -5.2 \\ \hline 0 & 1 & 0 & 0 \\ -0.996 & 0 & 0.039 & -2.117 \end{array}\right]}_{\bar{A}} \underbrace{\begin{bmatrix} \dot{x}_2 \\ \dot{y} \end{bmatrix}}_{\bar{x}} + \underbrace{\left[\begin{array}{ccc|cc} 0 & -0.744 & -0.032 & 0 & 0 \\ 0 & 0.337 & -1.12 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0.02 & 0 & 0 & 0 \end{array}\right]}_{\bar{D}} \underbrace{\begin{bmatrix} x_{12} \\ f \end{bmatrix}}_{\bar{f}}$$

$$y = \underbrace{\left[\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]}_{\bar{C}} \underbrace{\begin{bmatrix} x_2 \\ y \end{bmatrix}}_{\bar{x}} \tag{4.62}$$

Now, conditon G1 is satisfied by system (4.62). Therefore, the two-sliding mode observer design [79] can be implemented directly on system (4.62) to estimate the functional state, x_2.

Using the two observer design procedure in Section 3.2.3, L_1 in (3.5) is selected as

$$L_1 = \begin{bmatrix} 0.004 & 17.918 \\ -19 & 0.25 \end{bmatrix} \tag{4.63}$$

to get $A_1 + L_1 A_3 = \begin{bmatrix} -18 & 0 \\ 0 & -20 \end{bmatrix}$ which will govern the observer sliding motion. Then, P_o and A_4^s are chosen to be I_3 and $diag\{-19, -17\}$. The gain matrices, $G_{n,1}$ and $G_{l,1}$ in the same coordinate of (4.62) are

$$G_{n,1} = \begin{bmatrix} -0.004 & -17.918 \\ 19 & -0.25 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad G_{l,1} = \begin{bmatrix} -0.151 & -300.3 \\ 342 & -13.662 \\ 38 & -0.25 \\ 0.043 & 32.729 \end{bmatrix} \tag{4.64}$$

As a result, the dynamical system (3.34) resulting from the sliding motion of the primary observer is as follows:

$$\underbrace{\begin{bmatrix} \dot{e}_2 \\ \dot{\upsilon}_f \end{bmatrix}}_{\dot{\tilde{x}}} = \underbrace{\begin{bmatrix} -18 & 0 & 0 \\ 0 & -20 & 0 \\ 0.996 & 0 & -1 \end{bmatrix}}_{\tilde{A}} \underbrace{\begin{bmatrix} e_2 \\ \upsilon_f \end{bmatrix}}_{\tilde{x}} + \underbrace{\left[\begin{array}{ccc|cc} 0 & 0.744 & 0.032 & 0 & 0 \\ 0 & -0.337 & 1.12 & 0 & 0 \\ 0 & 0.02 & 0 & 0 & 0 \end{array}\right]}_{\tilde{D}} \tilde{f}$$

$$\underbrace{\begin{bmatrix} \bar{\upsilon}_1 \\ \upsilon_f \end{bmatrix}}_{\tilde{y}} = \underbrace{\begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\tilde{C}} \underbrace{\begin{bmatrix} e_2 \\ \upsilon_f \end{bmatrix}}_{\tilde{x}} \tag{4.65}$$

where e_2 is the estimation error of x_2 from the primary observer. It is obvious that $\mathrm{rank}(\tilde{C}\tilde{D}) = \mathrm{rank}(\tilde{D})$, a secondary Edwards & Spurgeon observer [5] can be implemented on system (4.65) to estimate e_2, and the state to be estimated, x_2 can be obtained by substracting the estimated x_2 from the primary observer and e_2 from the secondary observer. From (4.65), it is clear that the number of states is equal to the number of outputs, and hence there will be no reduced order sliding motion. Similar to the first observer design, the poles of the observer are placed at $\{-21, -19\}$ and its associated Lyapunov matrix is defined as $P_{o,2} = I_2$. The gain matrices, $G_{n,2}$ and $G_{l,2}$ for the second observer in the coordinate system of (4.65) are

$$G_{n,2} = \begin{bmatrix} 0 & 2.008 \\ -1 & 0 \\ 0 & 1 \end{bmatrix}, \; G_{l,2} = \begin{bmatrix} 0 & 2.008 \\ -1 & 0 \\ 0 & 20 \end{bmatrix} \tag{4.66}$$

The simulation results are shown in Figures 4.1 - 4.6. The inital conditions of the states x were assumed to be $\begin{bmatrix} 0.1 & 0 & 0 & -0.2 & 0.3 & -0.1 & 0.4 \end{bmatrix}^T$ whilst the observer had zero intial conditions. The unknown input f were set to be sinusoidal waveforms, where $f_1 = 0.54\sin(1.24t - 0.25)$ and $f_2 = 0.65\sin(1.65t - 0.78)$. The ρ and δ in (3.21) were selected as 10 and 1×10^{-2} for both observers.

Figures 4.1 and 4.4 show the plot of the output estimation error, e_y from observer 1 and observer 2 respectively. Figures 4.2 and 4.3 show the estimation of the functional state $\hat{z} = \begin{bmatrix} \hat{\dot{\theta}}_{yaw} \\ \hat{\dot{\theta}}_{roll} \end{bmatrix}$ and the actual z. It can be seen from Figure 4.2 that the estimation of $\hat{\dot{\theta}}_{yaw}$ can accurately tracks the actual $\dot{\theta}_{yaw}$ while Figure 4.3 shows that $\hat{\dot{\theta}}_{roll}$ does not track the actual $\dot{\theta}_{roll}$. Figure 4.5 and Figure 4.6 show the estimation of the functional state $\hat{\tilde{z}}$ obtained

from $\hat{\bar{z}}_i = \hat{z}_i - \hat{e}_{2,i}$, $i = 1, 2$ can accurately track the actual z, where $\hat{z}_i$ are from observer 1 and $\hat{e}_{2,i}$ are from observer 2.

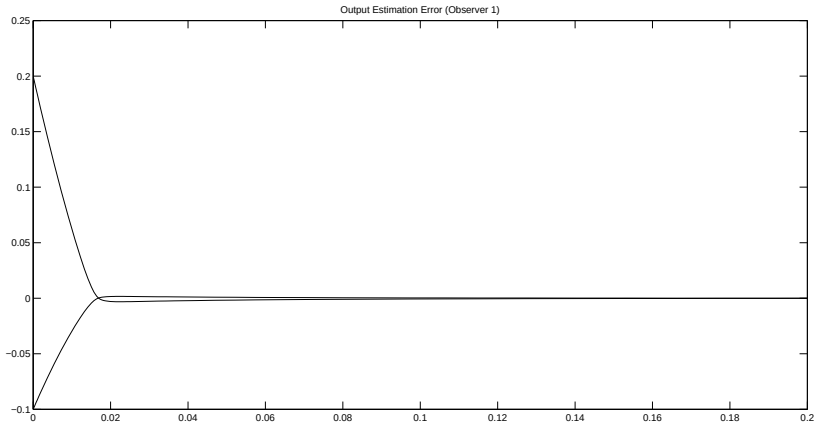

Figure 4.1: Output estimation error $e_y = y - \hat{y}$ from Observer 1. All components of e_y are driven to zero at around $0.12s$.

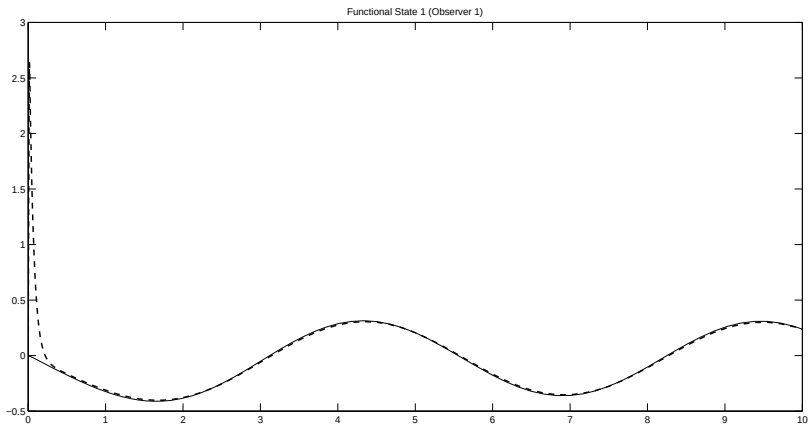

Figure 4.2: The estimated $\hat{\dot{\theta}}_{yaw}$ (dashed) from observer 1 follows the actual, $\dot{\theta}_{yaw}$ (solid).

The two-carts system in Figure 4.7 [81] is also used to show the effectiveness of the functional state estimation scheme using two-observers in this chapter. The first cart is connected to a rigid wall via a damper, and a second cart is connected to the first cart by a spring. An external force is then applied to the second cart via an actuator. The dynamics of the two-carts system is described by

$$
\begin{aligned}
m_1\ddot{\theta}_1 &= -b_1\dot{\theta}_1 - k(\theta_1 - \theta_2) \\
m_2\ddot{\theta}_2 &= u + k(\theta_1 - \theta_2)
\end{aligned}
\tag{4.67}
$$

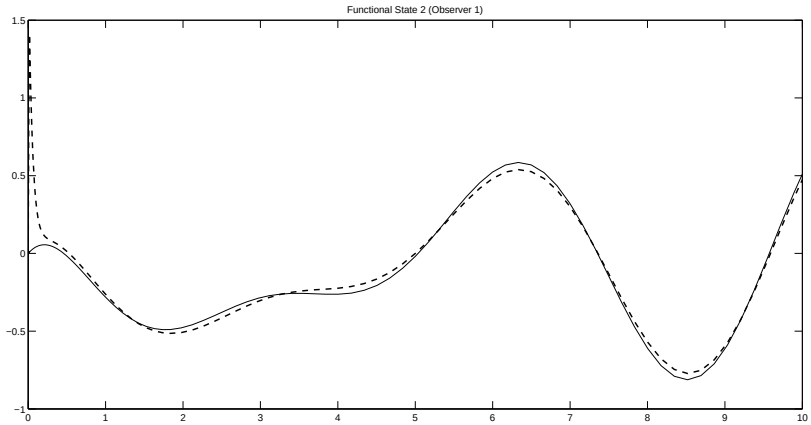

Figure 4.3: The estimated $\hat{\dot{\theta}}_{roll}$ (dashed) from observer 1 does not follow the actual, $\dot{\theta}_{roll}$ (solid) because conditions E1 and F1 are not satisfied.

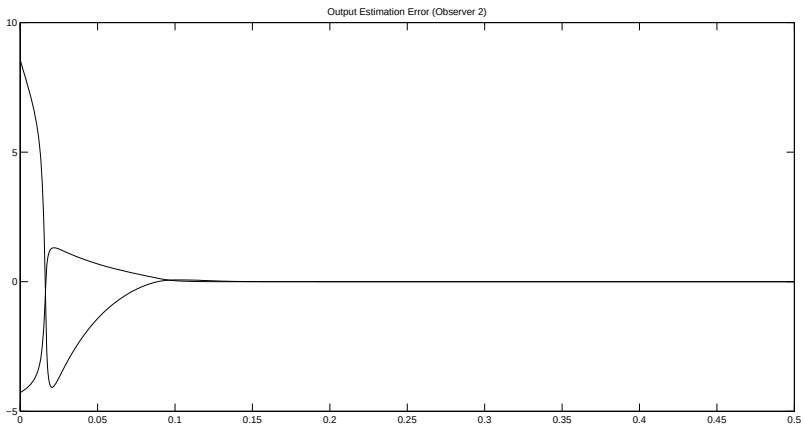

Figure 4.4: Output estimation error $e_{y,2} = y - \hat{y}$ from Observer 2.

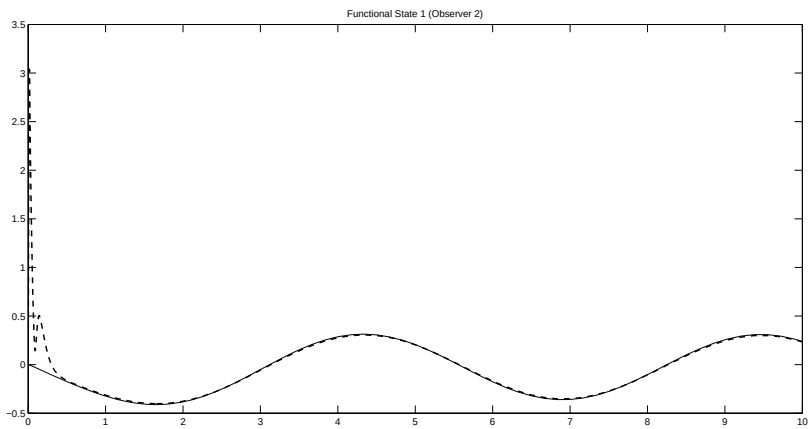

Figure 4.5: The estimated $\hat{\dot{\theta}}_{yaw}$ (dashed) from observer 2 obtained from $\hat{\bar{z}}_1 = \hat{z}_1 - \hat{e}_{2,1}$ follows the actual, $\dot{\theta}_{yaw}$ (solid).

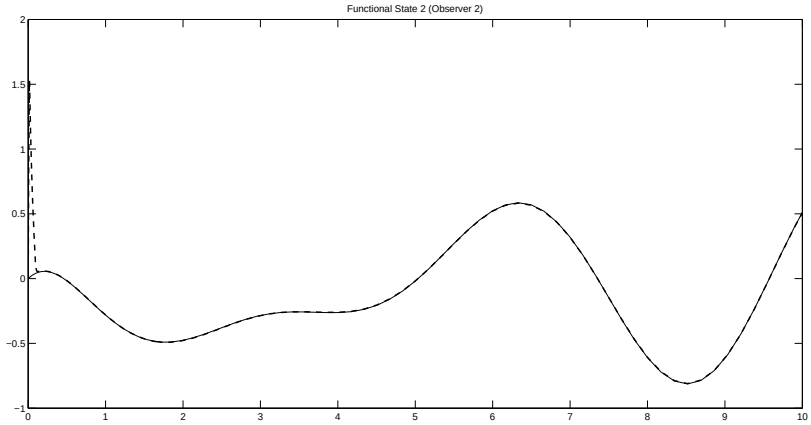

Figure 4.6: The estimated $\hat{\dot{\theta}}_{roll}$ (dashed) from observer 2 obtained from $\hat{\bar{z}}_2 = \hat{z}_2 - \hat{e}_{2,2}$ follows the actual, $\dot{\theta}_{roll}$ (solid).

Assume that the control input is the force command and the force on the second cart, u, is achieved from the force command, u_c via an actuator modelled as

$$\tau \dot{u} + u = u_c \tag{4.68}$$

where $\tau > 0$. Define the system states as

$$x = \begin{bmatrix} u\ (N) \\ \dot{\theta}_1\ (m/s) \\ \dot{\theta}_2\ (m/s) \\ \theta_1\ (m) \\ \theta_2\ (m) \end{bmatrix} \tag{4.69}$$

then in the notation of (4.1), the state equation can be written as

$$\frac{d}{dt}\begin{bmatrix} u \\ \dot{\theta}_1 \\ \dot{\theta}_2 \\ \theta_1 \\ \theta_2 \end{bmatrix} = \underbrace{\begin{bmatrix} -\frac{1}{\tau} & 0 & 0 & 0 & 0 \\ 0 & -\frac{b}{m_1} & 0 & -\frac{k}{m_1} & \frac{k}{m_1} \\ \frac{1}{m_2} & 0 & 0 & \frac{k}{m_2} & -\frac{k}{m_2} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}}_{A} \begin{bmatrix} u \\ \dot{\theta}_1 \\ \dot{\theta}_2 \\ \theta_1 \\ \theta_2 \end{bmatrix} + \underbrace{\begin{bmatrix} \frac{1}{\tau} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{D} \begin{bmatrix} \varepsilon_u \\ \varepsilon_1 \\ \varepsilon_2 \end{bmatrix} \tag{4.70}$$

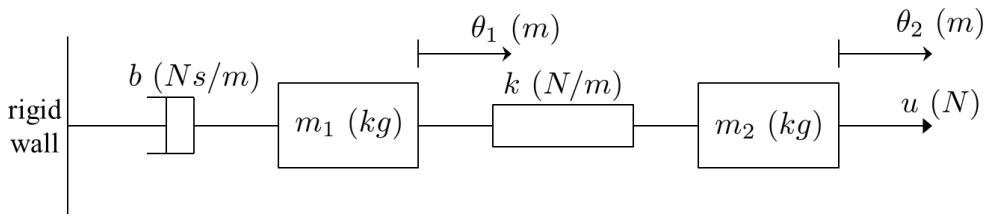

Figure 4.7: Schematic diagram of the two-cart system.

where ε_u is the nonlinearities from the actuator while ε_1 and ε_2 are the uncertainties present in (4.67) (which could be external unknown inputs or parameter variations). Assume the positions of both carts are measurable, then $y = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}$ and matrix C becomes

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{4.71}$$

In this example, the state to be estimated are $z = \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$. Therefore, matrix F can be defined as

$$F = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \tag{4.72}$$

Assume both carts have a nominal mass of $m_1 = m_2 = 1kg$, the damper has a nominal constant of $b = 1Ns/m$ and the spring has a nominal constant of $c = 1N/m$. Then, it can be easily verified that $\mathrm{rank}(\Psi_a) = 10$, $\mathrm{rank}(\Psi_b) = 4$ and $\mathrm{rank}(\Theta) = 6$. Therefore, condition H1 is satisfied and functional state estimation can be achieved by using two sliding mode observer in cascade. However, condition F1 is not satisfied since $\mathrm{rank}(\Psi_b) \neq \mathrm{rank}(\Theta)$ and one observer is insufficient to estimate the functional state. In addition, it can be verified that $\mathrm{rank}(CD) = 0$, $\mathrm{rank}(D) = 3$ and $\mathrm{rank}\begin{bmatrix} CAD & CD \\ CD & 0 \end{bmatrix} = 2$, which means conditions A1 and C1 are not satisfied, and full state estimation cannot be achieved. First, a suitable choice of T_s is

$$T_s = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{4.73}$$

and the structure in (4.5) is achieved as follows

$$A_s = \left[\begin{array}{c|cc|cc} -1 & 0 & 0 & 0 & 0 \\ \hline 0 & -1 & 0 & 1 & -1 \\ 1 & 0 & 0 & -1 & 1 \\ \hline 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{array}\right], \quad D_s = \left[\begin{array}{ccc} 1 & 0 & 0 \\ \hline 0 & 1 & 0 \\ 0 & 0 & 1 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right], \qquad \begin{array}{l} F_s = \left[\begin{array}{c|cc|cc} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right], \\ C_s = \left[\begin{array}{c|cc|cc} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] \end{array} \tag{4.74}$$

In the coordinates of (4.74), re-arrange the system matrices into the form of (4.4) as follows

$$\underbrace{\frac{d}{dt}\begin{bmatrix}\dot\theta_1\\ \dot\theta_2\\ \theta_2\\ \theta_1\end{bmatrix}}_{\dot{\bar{x}}} = \underbrace{\left[\begin{array}{cc|cc}-1 & 0 & 1 & -1\\ 0 & 0 & -1 & 1\\ \hline 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\end{array}\right]}_{\bar{A}} \underbrace{\begin{bmatrix}\dot\theta_1\\ \dot\theta_2\\ \theta_2\\ \theta_1\end{bmatrix}}_{\bar{x}} + \underbrace{\left[\begin{array}{c|ccc}0 & 0 & 1 & 0\\ 1 & 0 & 0 & 1\\ \hline 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{array}\right]}_{\bar{D}} \underbrace{\begin{bmatrix}u\\ \varepsilon_u\\ \varepsilon_1\\ \varepsilon_2\end{bmatrix}}_{\bar{f}}$$

$$\underbrace{\begin{bmatrix}\theta_2\\ \theta_1\end{bmatrix}}_{y} = \underbrace{\left[\begin{array}{cc|cc}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{array}\right]}_{\bar{C}} \underbrace{\begin{bmatrix}\dot\theta_1\\ \dot\theta_2\\ \theta_2\\ \theta_1\end{bmatrix}}_{\bar{x}} \tag{4.75}$$

It is obvious that system (4.75) satisfied condition G1, and the two-sliding mode observer design [79] can be used to accurately estimate x_2.

Using the two observer design procedure in Section 3.2.3, L_1 is selected such that $A_1 + L_1 A_3 = \begin{bmatrix}-15 & 0\\ 0 & -18\end{bmatrix}$. Then, P_o and A_4^s are chosen to be I_2 and $diag\{-17, -14\}$. The gain matrices, $G_{n,1}$ and $G_{l,1}$ in the same coordinates of (4.75) are

$$G_{n,1} = \begin{bmatrix}0 & 14\\ 18 & 0\\ 1 & 0\\ 0 & 1\end{bmatrix}, \quad G_{l,1} = \begin{bmatrix}1 & 181\\ 305 & 1\\ 35 & 0\\ 0 & 28\end{bmatrix} \tag{4.76}$$

As a result, the dynamical system (3.34) resulting from the sliding motion of the primary observer is as follows:

$$\underbrace{\begin{bmatrix}\dot e_2\\ \dot\upsilon_f\end{bmatrix}}_{\dot{\tilde{x}}} = \underbrace{\begin{bmatrix}-15 & 0\\ 0 & -18\end{bmatrix}}_{\tilde{A}} \underbrace{\begin{bmatrix}e_2\\ \upsilon_f\end{bmatrix}}_{\tilde{x}} + \underbrace{\left[\begin{array}{c|cc}0 & 0 & -1 & 0\\ -1 & 0 & 0 & -1\end{array}\right]}_{\tilde{D}} \tilde{f}$$

$$\underbrace{\begin{bmatrix}\bar\upsilon_1\\ \upsilon_f\end{bmatrix}}_{\tilde{y}} = \underbrace{\begin{bmatrix}0 & -1\\ -1 & 0\end{bmatrix}}_{\tilde{C}} \underbrace{\begin{bmatrix}e_2\\ \upsilon_f\end{bmatrix}}_{\tilde{x}} \tag{4.77}$$

Now, it is obvious that $\mathrm{rank}(\tilde{C}\tilde{D}) = \mathrm{rank}(\tilde{D})$, a secondary Edwards-Spurgeon observer [5] can be implemented on system (4.77) to estimate e_2. Since the number of states is equal to the number of outpus, there will be no reduced order sliding motion. The poles of the observer are placed at $\{-22, -23\}$ and its associated Lyapunov matrix is defined

as $P_{o,2} = I_2$. The gain matrices, $G_{n,2}$ and $G_{l,2}$ for the second observer in the coordinate system of (4.77) are

$$G_{n,2} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \; G_{l,2} = \begin{bmatrix} 0 & -8 \\ -4 & 0 \end{bmatrix} \tag{4.78}$$

The simulation results are shown in Figures 4.8 - 4.13. The inital conditions of the states x were assumed to be $\begin{bmatrix} 0.1 & 0 & 0 & -0.2 & 0.3 \end{bmatrix}^T$ whilst the observer had zero intial conditions. The unknown input f were set to be sinusoidal waveforms, where $f_1 = 2.15\sin(1.24t - 0.25)$, $f_2 = 1.03\sin(1.65t - 0.78)$ and $f_3 = 0.43\sin(1.52t - 0.31)$. The ρ and δ in (3.21) for both observers were selected as 10 and 1×10^{-2}. Figures 4.8 and 4.11 show the plot of the output estimation error, e_y from observer 1 and observer 2 respectively. Figures 4.9 and 4.10 show the estimation of the functional state, $\hat{z} = \begin{bmatrix} \hat{\theta}_1 \\ \hat{\theta}_2 \end{bmatrix}$ from oberver 1 does not track the actual z. Figures 4.12 and 4.13 show that the estimated state, $\hat{\bar{z}}$ from observer 2 can accurately track the actual z, where $\hat{\bar{z}}$ is obtained from $\hat{\bar{z}}_i = \hat{z}_i - \hat{e}_{2,i}$, $i = 1, 2$. $\hat{z}_i$ are the estimated state from observer 1 while $\hat{e}_{2,i}$ are the estimated state from observer 2.

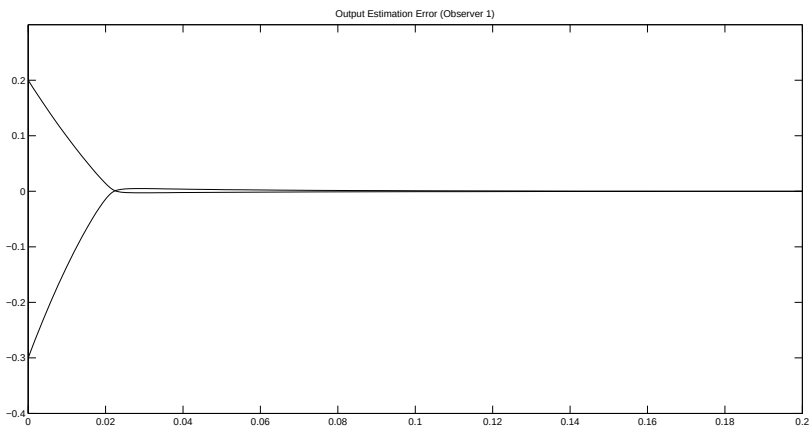

Figure 4.8: Output estimation error $e_y = y - \hat{y}$ from Observer 1. All components of e_y are driven to zero at around $0.12s$.

4.5 Conclusion

This chapter has presented a new scheme in functional state estimation, using two sliding mode observers in cascade. First, the original system was expressed in a form such that existing sliding mode observer theory could be implemented to achieve functional state

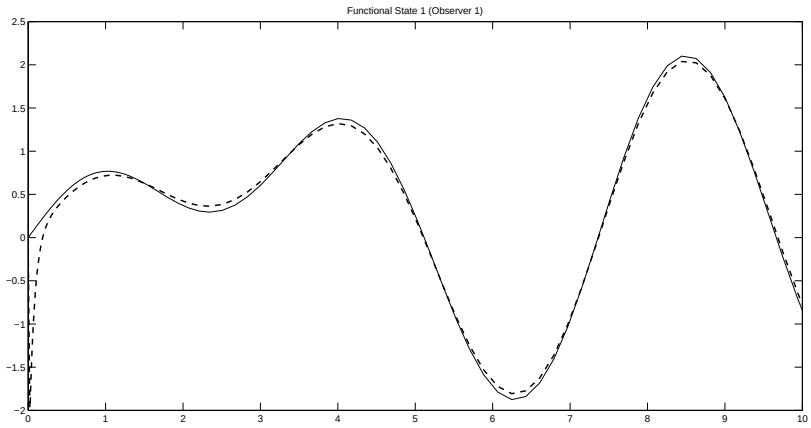

Figure 4.9: The estimated $\hat{\dot{\theta}}_1$ (dashed) does not follow the actual $\dot{\theta}_1$ (solid)

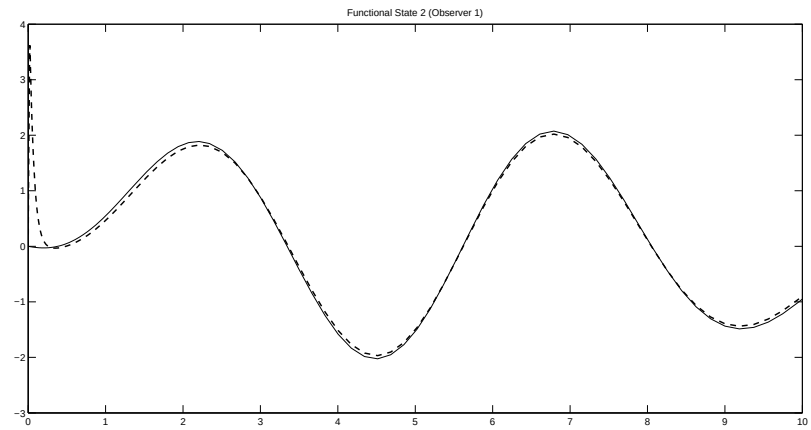

Figure 4.10: The estimated $\hat{\dot{\theta}}_2$ (dashed) does not follow the actual $\dot{\theta}_1$ (solid)

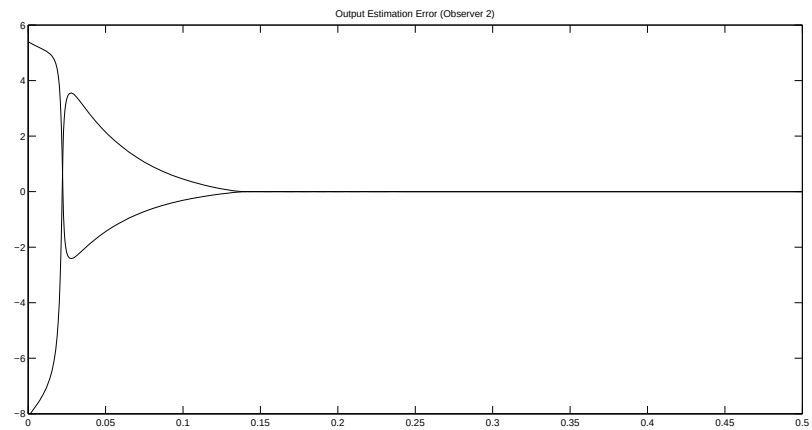

Figure 4.11: Output estimation error $e_{y2} = y - \hat{y}$ from Observer 2.

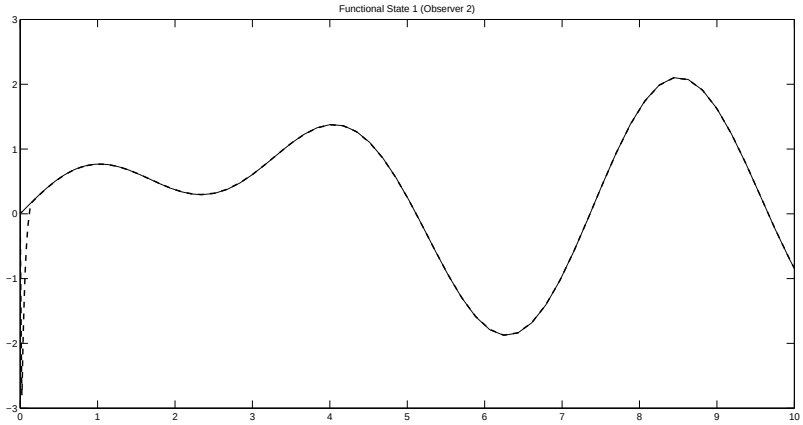

Figure 4.12: The estimated $\hat{\dot{\theta}}_1$ (dashed) obtained from $\hat{\bar{z}}_1 = \hat{z}_1 - \hat{e}_{2,1}$ converge to the actual $\dot{\theta}_1$ (solid).

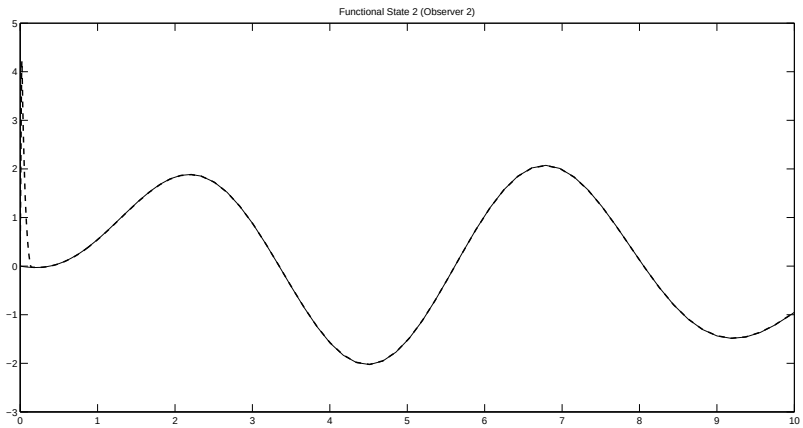

Figure 4.13: The estimated $\hat{\dot{\theta}}_2$ (dashed) obtained from $\hat{\bar{z}}_2 = \hat{z}_2 - \hat{e}_{2,2}$ converge to the actual $\dot{\theta}_2$ (solid).

estimation. Then, the necessary and sufficient conditions in terms of the original system matrices were investigated and were found to be less stringent than existing sliding mode functional estimation work, which could mean that functional state observation can be achieved with fewer sensors thus saving cost and reducing complexity. The investigated conditions are in terms of the original system matrices, which enable the user to know from the outset whether or not the scheme is applicable for their system, and also provides a basis for comparison for further developments in functional state estimation. Finally, simulation results showed the effectiveness of the functional observer scheme proposed in this chapter.

Chapter 5

Robust Functional State Estimation using Multiple Sliding Mode Observers in Cascade

5.1 Introduction

It was described in Chapter 3 how adding more sliding mode observers in cascade will enable state estimation under less restrictive conditions. Chapter 4 described the use of two cascaded observers for functional state estimation, and as expected, found that functional state estimation could be achieved with weaker conditions. This chapter further improves the work in Chapter 4 by proposing the theory of multiple (more than two) observers in cascade for functional state estimation, and investigating the necessary and sufficient conditions in terms of the original system matrices; this will lead to the development of a systematic algorithm that calculates the necessary and sufficient number of observers for functional state estimation, without having to first enter the design process. As expected, it is found that the existence conditions discussed in this chapter are less stringent compared to conditions F1 and H1 for one and two sliding mode observers, respectively. From a physical viewpoint, this could mean that less outputs (measurement sensors) are required for accurate functional state estimation which could reduce cost and complexity of the overall system.

This chapter begins with the problem statement and followed by the main result. Then, the original system will be transformed and re-arranged into a form such that the Edwards-Spurgeon observer can be directly implemented to achieve accurate functional state estimation. A constructive proof of the necessary and sufficient conditions is also provided. Finally the observer scheme is validated with a numerical example.

5.2 Problem Statement and Main Result

Consider the linear time-invariant system described by

$$\begin{aligned} \dot{x} &= Ax + Df \\ y &= Cx \\ z &= Fx \end{aligned} \tag{5.1}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^p$ and $f \in \mathbb{R}^q$ represent the state, output and unknown input respectively and $z \in \mathbb{R}^r$ is the vector to be estimated and $A \in \mathbb{R}^{n\times n}$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{n\times q}$ and $F \in \mathbb{R}^{r\times n}$ are known constant matrices; Without loss of generality, it is assumed that $\mathrm{rank}(C) = p$, $\mathrm{rank}(F) = r < n$. Also assume that $\mathrm{rank}\begin{bmatrix} F \\ C \end{bmatrix} = r+p$; this is not a restrictive assumption as it simply implies that the 'output states' will not be estimated since they are already available from the measurement.

The objective is to asymptotically estimate the functional state z and hence achieve *functional state estimation*. In Chapter 4, this was done using two sliding mode observers in cascade and found that functional state estimation could be achieved under less restrictive conditions than if only one observer was used. This chapter extends the work discussed in Chapter 4 by proposing a scheme that is even less restrictive, using more than two sliding mode observers in cascade.

Theorem 5.1 *The functional state z can be accurately estimated using k sliding mode observers in cascade if and only if the following conditions are satisfied:*

J1. $\mathrm{rank}(\Psi(k)) - \mathrm{rank}(\Psi(k-1)) = \mathrm{rank}(\Theta),\ 2 \le k \le r+p$

J2. $\mathrm{rank}(\Delta_a) = \mathrm{rank}(\Theta),\ \forall s \in \mathbb{C}, Re(s) \ge 0$

where

$$\Psi(i) = \left[\begin{array}{c|c} \bar{\Phi}(i) & \bar{\Gamma}(i) \\ \hline \bar{\Upsilon}(i) & 0 \end{array}\right] \tag{5.2}$$

$$= \left[\begin{array}{cccc|cccc} \Phi(i) & \Phi(i-1) & \dots & \Phi(1) & \Gamma(i) & \Gamma(i-1) & \dots & \Gamma(1) \\ \Phi(i-2) & \Phi(i-2) & \dots & 0 & \Gamma(i-1) & \Gamma(i-2) & \dots & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots & \vdots & \iddots & \vdots \\ \Phi(1) & 0 & \dots & 0 & \Gamma(1) & 0 & \dots & 0 \\ \hline \Upsilon & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & \Upsilon & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \Upsilon & 0 & 0 & \dots & 0 \end{array}\right],$$

$$\Theta = \begin{bmatrix} FA & FD \\ CA & CD \\ C & 0 \\ F & 0 \end{bmatrix}, \ \Delta_a = \begin{bmatrix} sF - FA & -FD \\ CA & CD \\ C & 0 \end{bmatrix} \tag{5.3}$$

and $\Phi(i) := CA^i$, $\Gamma(i) := CA^{i-1}D$ *and* $\Upsilon := \begin{bmatrix} F \\ C \end{bmatrix}$. □

The following section provides a constructive proof of Theorem 5.1. Notice that J1 and J2 are in terms of the original system matrices, system (5.2) will be re-casted into a form such that the multiple observer scheme in Section 3.2.4 can be implemented directly.

5.3 Proof of Theorem 5.1

Using Lemma 3.5, system (5.1) can be transformed into the following structure

$$A = \left[\begin{array}{c|cc} A_1 & A_2 & A_3 \\ \hline A_4 & A_5 & A_6 \\ A_7 & A_8 & A_9 \end{array}\right], \ D = \left[\begin{array}{c} D_1 \\ \hline D_2 \\ D_3 \end{array}\right] \begin{array}{l} \updownarrow n-p-r \\ \updownarrow r \\ \updownarrow p \end{array}, \qquad \begin{array}{l} F = \left[\begin{array}{c|cc} 0 & I_r & 0 \end{array}\right], \\ C = \left[\begin{array}{c|cc} 0 & 0 & I_p \end{array}\right] \end{array} \tag{5.4}$$

The system states, x, in the coordinate system of (5.4) can be transformed and partitioned as follows:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ y \end{bmatrix} \begin{array}{l} \updownarrow n-p-r \\ \updownarrow r \\ \updownarrow p \end{array} \tag{5.5}$$

From the structure of F in (5.4), $z = x_2$; therefore, the state components to be estimated would be x_2 while x_1 (which is not required to be estimated), will be modeled as unknown inputs/disturbances. The system equations (5.1) in the coordinates of (5.4) can therefore be re-written as a system of order $\bar{n} := r + p$, as follows:

$$\underbrace{\begin{bmatrix} \dot{x}_2 \\ \dot{y} \end{bmatrix}}_{\dot{\bar{x}}} = \underbrace{\begin{bmatrix} A_5 & A_6 \\ A_8 & A_9 \end{bmatrix}}_{\bar{A}} \underbrace{\begin{bmatrix} \dot{x}_2 \\ \dot{y} \end{bmatrix}}_{\bar{x}} + \underbrace{\begin{bmatrix} A_4 & D_2 \\ A_7 & D_3 \end{bmatrix}}_{\bar{D}} \underbrace{\begin{bmatrix} x_1 \\ f \end{bmatrix}}_{\bar{f}}$$

$$y = \underbrace{\begin{bmatrix} 0 & I_p \end{bmatrix}}_{\bar{C}} \underbrace{\begin{bmatrix} x_2 \\ y \end{bmatrix}}_{\bar{x}} \tag{5.6}$$

Notice that the system in (5.6) is in the same form as (3.1) and hence a cascade observer scheme [81, 95] described in Section 3.2.4 can be used to estimate z. Using the results of Tan & Edwards [81] (conditions D1-D2 in Section 3.2.4), asymptotic estimation of $\bar{x}$ (and hence z also) can be achieved by k sliding mode observers in cascade if and only if the following conditions hold:

K1. $\mathrm{rank}(\Xi(k)) - \mathrm{rank}(\Xi(k-1)) = \mathrm{rank}(\bar{D}),\ 2 \le k \le \bar{n}$

K2. $\mathrm{rank}(\Delta_b) = \bar{n} + \mathrm{rank}(\bar{D}),\ \forall s \in \mathbb{C}, Re(s) \ge 0$

where $\Xi(i)$ and Δ_b are defined by

$$\Xi(i) = \begin{bmatrix} \Pi(i) & \Pi(i-1) & \dots & \Pi(1) \\ \Pi(i-1) & \Pi(i-2) & \dots & 0 \\ \vdots & \vdots & \iddots & \vdots \\ \Pi(1) & 0 & \dots & 0 \end{bmatrix},\ \Delta_b = \begin{bmatrix} sI - \bar{A} & \bar{D} \\ \bar{C} & 0 \end{bmatrix} \tag{5.7}$$

where $\Pi(i) := \bar{C}\bar{A}^{i-1}\bar{D}$. To prove Theorem 5.1, it will be shown that J1 and J2 are equivalent to K1 and K2 respectively. In the following, Lemmas 5.1-5.4 will be established, which will be used to show the equivalence of J1 and K1.

Lemma 5.1 *Let* $\bar{K} = -\begin{bmatrix} A_6 \\ A_9 \end{bmatrix}$ *and define* $\bar{A}_o := \bar{A} + \bar{K}\bar{C}$. *Therefore,*

$$\bar{A}_o = \begin{bmatrix} A_5 & 0 \\ A_8 & 0 \end{bmatrix} \tag{5.8}$$

Define $\Pi_o(i) := \bar{C}\bar{A}_o^{i-1}\bar{D}$ *and*

$$\Xi_o(i) = \begin{bmatrix} \Pi_o(i) & \Pi_o(i-1) & \dots & \Pi_o(2) & \Pi_o(1) \\ \Pi_o(i-1) & \Pi_o(i-2) & \dots & \Pi_o(1) & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ \Pi_o(2) & \Pi_o(1) & \dots & 0 & 0 \\ \Pi_o(1) & 0 & \dots & 0 & 0 \end{bmatrix} \tag{5.9}$$

The matrices $\Xi(i)$ *from* (5.7) *and* $\Xi_o(i)$ *from* (5.8) *have equal rank.* ♯

Proof

Define $\bar{\Sigma}(i) := -\bar{C}(\bar{A} + \bar{K}\bar{C})^i\bar{K}$ and

$$\bar{J}(i) = \begin{bmatrix} I_p & \bar{\Sigma}(0) & \bar{\Sigma}(1) & \dots & \bar{\Sigma}(i-2) \\ 0 & I_p & \bar{\Sigma}(0) & \dots & \bar{\Sigma}(i-3) \\ 0 & 0 & I_p & \dots & \bar{\Sigma}(i-4) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & I_p \end{bmatrix} \tag{5.10}$$

which by construction is square and invertible. It can be shown that $\Xi(i) = \bar{J}(i)\Xi_o(i)$ and hence $\mathrm{rank}(\Xi) = \mathrm{rank}(\Xi_o)$ since $\bar{J}(i)$ is full rank. ■

Lemma 5.2 *Define* $\Phi_o(i) := CA_o^i$ *and* $\Gamma_o(i) := CA_o^{i-1}D$*, then define the following matrix:*

$$\Psi_o(i) = \left[\begin{array}{c|c} \bar{\Phi}_o(i) & \bar{\Gamma}_o(i) \\ \hline \bar{\Upsilon}(i) & 0 \end{array}\right]$$

$$= \left[\begin{array}{cccc|cccc} \Phi_o(i) & \Phi_o(i-1) & \dots & \Phi_o(1) & \Gamma_o(i) & \Gamma_o(i-1) & \dots & \Gamma_o(1) \\ \Phi_o(i-2) & \Phi_o(i-2) & \dots & 0 & \Gamma_o(i-1) & \Gamma_o(i-2) & \dots & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots & \vdots & \iddots & \vdots \\ \Phi(1) & 0 & \dots & 0 & \Gamma_o(1) & 0 & \dots & 0 \\ \hline \Upsilon & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & \Upsilon & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \Upsilon & 0 & 0 & \dots & 0 \end{array}\right] \tag{5.11}$$

where $A_o := A + KC$ *and* $K = -\begin{bmatrix} A_3 \\ A_6 \\ A_9 \end{bmatrix}$. *Hence,* A_o *can be written as*

$$A_o = \left[\begin{array}{c|cc} A_1 & A_2 & 0 \\ \hline A_4 & A_5 & 0 \\ A_7 & A_8 & 0 \end{array}\right] \tag{5.12}$$

The matrices $\Psi(i)$ *from* (5.2) *and* $\Psi_o(i)$ *from* (5.11) *have equal rank.* ♯

Proof

Define $\Sigma(i) := C(A + KC)^i K$ and $\Sigma_o(i) := \begin{bmatrix} \Sigma(i) & 0 \end{bmatrix}$. Then, an invertible matrix can be introduced as follow

$$J(i) = \left[\begin{array}{cccc|cccc} I_p & \Sigma(0) & \dots & \Sigma(i-2) & \Sigma_o(i-1) & \Sigma_o(i-2) & \dots & \Sigma(0) \\ 0 & I_p & \dots & \Sigma(i-3) & \Sigma_o(i-2) & \Sigma_o(i-3) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \iddots & \vdots \\ 0 & 0 & \dots & I_p & \Sigma_o(0) & 0 & \dots & 0 \\ \hline 0 & 0 & \dots & 0 & I_{r+p} & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & I_{r+p} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & I_{r+p} \end{array}\right] \tag{5.13}$$

It can be shown that $\Psi_o(i) = J(i)\Psi(i)$, and hence $\mathrm{rank}(\Psi_o(i)) = \mathrm{rank}(\Psi(i))$ since $J(i)$ is invertible. ■

Lemma 5.3 *The following is true:*

$$\mathrm{rank}(\Psi_o(i)) = \mathrm{rank}(\Xi_o(i)) + i(p + r) \tag{5.14}$$

♯

Proof

Firstly, iterative expressions for the partitions of A_o^i will be established. Write A_o^i as follows:

$$A_o^i = \begin{bmatrix} M_1(i) & 0 \\ M_2(i) & 0 \end{bmatrix} \begin{array}{l} \updownarrow n-p \\ \updownarrow p \end{array} \tag{5.15}$$

where $M_1(i)$ and $M_2(i)$ can be derived from (5.12) to be

$$M_1(i) = \begin{bmatrix} A_1 & A_2 \\ A_4 & A_5 \end{bmatrix}^i, \; M_2(i) = \begin{bmatrix} A_7 & A_8 \end{bmatrix} \begin{bmatrix} A_1 & A_2 \\ A_4 & A_5 \end{bmatrix}^{i-1} \tag{5.16}$$

Then, partition $M_2(i) = \begin{bmatrix} M_{21}(i) & M_{22}(i) \end{bmatrix}$, where $M_{21}(i) \in \mathbb{R}^{p\times(n-p-r)}$ and $M_{22}(i) \in \mathbb{R}^{p\times r}$. Hence,

$$CA_o^i = \begin{bmatrix} M_{21}(i) & M_{22}(i) & 0 \end{bmatrix} \Rightarrow CA_o^i D = M_{21}(i)D_1 + M_{22}(i)D_2 \tag{5.17}$$

Then, to further expand $M_{21}(i)$ and $M_{22}(i)$, define

$$M_1(j) = \begin{bmatrix} A_1 & A_2 \\ A_4 & A_5 \end{bmatrix}^j = \begin{bmatrix} V_1(j) & W_1(j) \\ V_2(j) & W_2(j) \end{bmatrix}, \text{ for } 0 \le j \le i \tag{5.18}$$

It is obvious that $V_1(0) = I_{n-p-r}$, $W_1(0) = 0$, $V_2(0) = 0$ and $W_2(0) = I_r$. Then by using $M_1(j) = M_1(j-1)M_1(1)$ and expanding it, it can be shown that for $1 \le j \le i$

$$\begin{aligned} V_1(j) &= V(j-1)A_1 + W_1(j-1)A_4 \\ V_2(j) &= V_2(j-1)A_1 + W_2(j-1)A_4 \\ W_1(j) &= V_1(j-1)A_2 + W_1(j-1)A_5 \\ W_2(j) &= V_2(j-1)A_2 + W_2(j-1)A_5 \end{aligned} \tag{5.19}$$

Therefore, by using the expression for $M_2(i)$ in (5.16) together with (5.18), $M_{21}(i)$ and $M_{22}(i)$ can be re-written as follows:

$$\begin{aligned} M_{21}(i) &= A_7V_1(i-1) + A_8V_2(i-1) \\ M_{22}(i) &= A_7W_1(i-1) + A_8W_2(i-1) \end{aligned} \tag{5.20}$$

By substituting iteratively for $W_1(j-1)$, $j = 1, \ldots, i$ into the expressions of $V_1(j)$ and $W_1(j)$ from (5.19) it can be shown that

$$\begin{aligned} W_1(i) &= \sum_{j=0}^{j=i-1} V_1(j)A_2A_5^{i-1-j} \\ V_1(i) &= V_1(i-1)A_1 + \sum_{j=0}^{j=i-2} V_1(j)A_2A_5^{i-2-j}A_4 \end{aligned} \tag{5.21}$$

Likewise, by doing the same with $W_2(j)$ and $V_2(j)$ respectively it can also be shown that

$$\begin{aligned} W_2(i) &= A_5^i + \sum_{j=0}^{j=i-1} V_2(j) A_2 A_5^{i-1-j} \\ V_2(i) &= V_2(i-1)A_1 + A_5^{i-1}A_4 + \sum_{j=0}^{j=i-2} V_2(j) A_2 A_5^{i-2-j} A_4 \end{aligned} \tag{5.22}$$

The expressions in (5.21)-(5.22) are the iterative expressions for the partitions of A_o^i. Since $\Upsilon := \begin{bmatrix} F \\ C \end{bmatrix}$, using the structures of C and F in (5.4) as well as $\Phi_o(i) = CA_o^i$ in (5.17), it can be shown that $\mathrm{rank}(\Psi_o(i)) = \mathrm{rank}\underbrace{\begin{bmatrix} M_{21,o}(i) & \bar{\Gamma}_o(i) \end{bmatrix}}_{\Phi_{o,2}(i)} + ip + ir$, where

$$M_{21,o}(i) = \begin{bmatrix} M_{21}(i) & M_{21}(i-1) & \dots & M_{21}(1) \\ M_{21}(i-1) & M_{21}(i-2) & \dots & 0 \\ \vdots & \vdots & \iddots & \vdots \\ M_{21}(1) & 0 & \dots & 0 \end{bmatrix} \tag{5.23}$$

Define $\Lambda_a(i) := A_2 A_5^{i-3} A_4$ and $\Lambda_b(i) := A_2 A_5^{i-3} D_2$. By substituting (5.21) and (5.22) into $M_{21}(i)$ and $M_{22}(i)$ in (5.20), it can be shown that $M_{21}(i)$ and $\Gamma_o(i)$ (using (5.17)) are

$$\begin{aligned} M_{21}(i) &= \underbrace{A_7 V_1(i-2)A_1 + A_8 V_2(i-2)A_1}_{M_{21}(i-1)A_1} + A_8 A_5^{i-2} A_4 \\ &\quad + A_7 \sum_{j=0}^{j=i-3} V_1(j) A_2 A_5^{i-3-j} A_4 + A_8 \sum_{j=0}^{j=i-3} V_2(j) A_2 A_5^{i-3-j} A_4 \\ \Gamma_o(i) &= \underbrace{A_7 V_1(i-2)D_1 + A_8 V_2(i-2)D_1}_{M_{21}(i-1)D_1} + A_8 A_5^{i-2} D_2 \\ &\quad + A_7 \sum_{j=0}^{j=i-3} V_1(j) A_2 A_5^{i-3-j} D_2 + A_8 \sum_{j=0}^{j=i-3} V_2(j) A_2 A_5^{i-3-j} D_2 \end{aligned} \tag{5.24}$$

and the following will result:

$$\begin{aligned} A_8 A_5^{i-2} A_4 &= M_{21}(i) - M_{21}(i-1)A_1 - \sum_{j=1}^{j=i-2} M_{21}(j)\Lambda_a(i-j+1) \\ A_8 A_5^{i-2} D_2 &= \Gamma_o(i) - M_{21}(i-1)D_1 - \sum_{j=1}^{j=i-2} M_{21}(j)\Lambda_b(i-j+1) \end{aligned} \tag{5.25}$$

Then, define an invertible matrix $\Omega(i) \in \mathbb{R}^{i(n-p-r+q)\times i(n-p-r+q)}$ as

$$\Omega(i) = \begin{bmatrix} \Omega_a(i) & \Omega_b(i) \\ 0 & I_{iq} \end{bmatrix} \tag{5.26}$$

where

$$\Omega_a(i) = \begin{bmatrix} I_{n-p-r} & -A_1 & -\Lambda_a(3) & \dots & -\Lambda_a(i-2) & -\Lambda_a(i-1) & -\Lambda_a(i) \\ 0 & I_{n-p-r} & -A_1 & \dots & -\Lambda_a(i-3) & -\Lambda_a(i-2) & -\Lambda_a(i-1) \\ 0 & 0 & I_{n-p-r} & \dots & -\Lambda_a(i-4) & -\Lambda_a(i-3) & -\Lambda_a(i-2) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & I_{n-p-r} & -A_1 & -\Lambda_a(3) \\ 0 & 0 & 0 & \dots & 0 & I_{n-p-r} & -A_1 \\ 0 & 0 & 0 & \dots & 0 & 0 & I_{n-p-r} \end{bmatrix} \tag{5.27}$$

and

$$\Omega_b(i) = \begin{bmatrix} 0 & -D_1 & -\Lambda_b(3) & \dots & \Lambda_b(i-2) & \Lambda_b(i-1) & \Lambda_b(i) \\ 0 & 0 & -D_1 & \dots & \Lambda_b(i-3) & \Lambda_b(i-2) & \Lambda_b(i-1) \\ 0 & 0 & 0 & \dots & \Lambda_b(i-4) & \Lambda_b(i-3) & \Lambda_b(i-2) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & -D_1 & \Lambda_b(3) \\ 0 & 0 & 0 & \dots & 0 & 0 & -D_1 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \end{bmatrix} \tag{5.28}$$

It is obvious that $M_{21}(1) = A_7$ and $\Gamma_o(1) = D_3$. From (5.8), $\bar{A}_o^i$ can be written as

$$\bar{A}_o^i = \begin{bmatrix} A_5^i & 0 \\ A_8 A_5^{i-1} & 0 \end{bmatrix} \tag{5.29}$$

and from (5.6)

$$\begin{aligned} \Pi_o(i+1) &= \bar{C}\bar{A}_o^i\bar{D} \\ &= \begin{bmatrix} A_8 A_5^{i-1} A_4 & A_8 A_5^{i-1} D_2 \end{bmatrix} \\ &= \begin{bmatrix} \Pi_1(i) & \Pi_2(i) \end{bmatrix} \end{aligned} \tag{5.30}$$

with the exception of the case $\Pi_o(1) = \begin{bmatrix} A_7 & D_3 \end{bmatrix}$. Therefore, $\Xi_o(i)$ from (5.9) can be written as

$$\Xi_o(i) = \left[\begin{array}{cc|cc|c|cc|cc} \Pi_1(i-1) & \Pi_2(i-1) & \Pi_1(i-2) & \Pi_2(i-2) & \dots & \Pi_1(2) & \Pi_2(2) & A_7 & D_3 \\ \Pi_1(i-2) & \Pi_2(i-2) & \Pi_1(i-3) & \Pi_2(i-3) & \dots & A_7 & D_3 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \iddots & \vdots & \vdots & \vdots & \vdots \\ \Pi_1(2) & \Pi_2(2) & A_7 & D_3 & \dots & 0 & 0 & 0 & 0 \\ A_7 & D_3 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \end{array}\right] \tag{5.31}$$

It can be shown using (5.24) and (5.25) that $\Psi_{o,2}(i)\Omega(i) = \begin{bmatrix}\Pi_a(i) & \Pi_b(i)\end{bmatrix}$, where

$$\Pi_a(i) = \begin{bmatrix} \Pi_1(i-1) & \Pi_1(i-2) & \dots & \Pi_1(2) & A_7 \\ \Pi_1(i-2) & \Pi_1(i-3) & \dots & A_7 & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ \Pi_1(2) & A_7 & \dots & 0 & 0 \\ A_7 & 0 & \dots & 0 & 0 \end{bmatrix} \tag{5.32}$$

and

$$\Pi_b(i) = \begin{bmatrix} \Pi_2(i-1) & \Pi_2(i-2) & \dots & \Pi_2(2) & D_3 \\ \Pi_2(i-2) & \Pi_2(i-3) & \dots & D_3 & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ \Pi_2(2) & D_3 & \dots & 0 & 0 \\ D_3 & 0 & \dots & 0 & 0 \end{bmatrix} \tag{5.33}$$

It is clear that $\Omega(i)$ is square and invertible, and from (5.33), $\Psi_{o,2}(i)\Omega(i)$ is simply a re-arrangement of columns of $\Xi_o(i)$ in (5.31). Hence, $\Psi_{o,2}$ and $\Xi_o(i)$ have equal rank and the proof is complete. ■

Lemma 5.4 *The following is true:*

$$\mathrm{rank}(\Theta) = \mathrm{rank}(\bar{D}) = p + r. \tag{5.34}$$

♯

Proof

Substitute matrices A, C, D and F from (5.1) into Θ to get

$$\Theta = \begin{bmatrix} A_4 & A_5 & A_6 & D_2 \\ A_7 & A_8 & A_9 & D_3 \\ 0 & 0 & I_p & 0 \\ 0 & I_r & 0 & 0 \end{bmatrix} \tag{5.35}$$

and it is straightforward to see that

$$\mathrm{rank}(\Theta) = \mathrm{rank}\begin{bmatrix} A_4 & D_2 \\ A_7 & D_3 \end{bmatrix} + p + r \tag{5.36}$$

From (5.6), $\mathrm{rank}(\Theta)$ is equivalent to $\mathrm{rank}(\bar{D}) + p + r$. ■

By using Lemma 5.2, the LHS of J1 becomes $\mathrm{rank}(\Psi_o(k)) - \mathrm{rank}(\Psi_o(k-1))$. Then by using Lemma 5.3, it becomes $\mathrm{rank}(\Xi_o(k)) - \mathrm{rank}(\Xi_o(k-1)) + p + r$ which then becomes $\mathrm{rank}(\Xi(k)) - \mathrm{rank}(\Xi(k-1)) + p + r$ using Lemma 5.1. Then by using Lemma 5.4, the right-hand-side of J1 becomes $\mathrm{rank}(\bar{D}) + p + r$. By combining the left and right-hand-sides of J1, it is clear that it is equivalent to K1.

Lemma 5.5 *Matrices* Δ_a *and* Δ_b *from* (5.3) *and* (5.7) *have equal rank.* ♯

Proof

Substitute for (A, C, D, F) from (5.4) into Δ_a to get

$$\Delta_a = \begin{bmatrix} -A_4 & sI_r - A_5 & -A_6 & -D_2 \\ A_7 & A_8 & A_9 & D_3 \\ 0 & 0 & I_p & 0 \end{bmatrix} \tag{5.37}$$

and it is straightforward to see that

$$\mathrm{rank}(\Delta_a) = \mathrm{rank} \underbrace{\begin{bmatrix} -A_4 & sI_r - A_5 & -D_2 \\ A_7 & A_8 & D_3 \end{bmatrix}}_{\Delta_{a,1}} + p \tag{5.38}$$

Then, substitute the matrices $(\bar{A}, \bar{C}, \bar{D})$ from (5.6) into Δ_b, where

$$\Delta_b = \begin{bmatrix} sI_r - A_5 & -A_6 & A_4 & D_2 \\ A_8 & A_9 & A_7 & D_3 \\ 0 & I_p & 0 & 0 \end{bmatrix} \tag{5.39}$$

and it is obvious that

$$\mathrm{rank}(\Delta_b) = \mathrm{rank} \underbrace{\begin{bmatrix} sI_r - A_5 & A_4 & D_2 \\ -A_8 & A_7 & D_3 \end{bmatrix}}_{\Delta_{b,1}} + p \tag{5.40}$$

From (5.38) and (5.40), it is clear that $\Delta_{b,1}$ is simply a re-arrangement of columns of $\Delta_{a,1}$. Hence, Δ_a and Δ_b have equal rank and the proof is complete. ■

From Lemma 5.4, it is proven that the RHS of J2 is equivalent to $\mathrm{rank}(\bar{D}) + p + r$, which is the RHS of K2. The LHS of J2 and K2 are proven to have equal rank in Lemma 5.5. Thus, it is clear that condition J2 and K2 are equivalent and the proof of Theorem 5.1 complete.

5.4 Numerical Example

A 3rd order water tank modified from [82] is used as an example to validate the scheme presented in this chapter. The dynamics of the water tank system is described by $\dddot{\theta} + 10\ddot{\theta} + 31\dot{\theta} + 30\theta = u$ where θ is the water height and u is the inlet flow rate, which are measurable. It is assumed that u is generated by a first order device where $\dot{u} = -u + u_c$, in which u_c is the flow command. Suppose the sensor for u is faulty. Denote θ_m and u_m to be the measurements for θ and u, respectively, hence $\theta_m = \theta$, $u_m = u + f$, where f encapsulates potential faults. Then z_f is obtained by filtering u_m where $\dot{z}_f = -z_f + u_m$. In the notation of (5.1), the state equation that represent the dynamics of the water tank system is

$$\frac{d}{dt}\begin{bmatrix} \ddot{\theta} \\ \dot{\theta} \\ u \\ \theta \\ z_f \end{bmatrix} = \underbrace{\begin{bmatrix} -10 & -31 & 1 & -30 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 \end{bmatrix}}_{A} \begin{bmatrix} \ddot{\theta} \\ \dot{\theta} \\ u \\ \theta \\ z_f \end{bmatrix} + \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{D} \begin{bmatrix} \varepsilon_d \\ f \\ \varepsilon_c \end{bmatrix} \tag{5.41}$$

where ε_d is the uncertainty presents in the dynamic equation while ε_c is the nonlinearities from the sensor used to measure u. The measurable outputs are $y = \begin{bmatrix} \theta_m \\ z_f \end{bmatrix}$ and the states to be estimated are $z = \begin{bmatrix} \ddot{\theta} \\ \dot{\theta} \end{bmatrix}$, hence matrices C and F can be defined as

$$C = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \; F = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \tag{5.42}$$

It can be easily verified that $\mathrm{rank}(\Psi(3)) = 16$, $\mathrm{rank}(\Psi(2)) = 10$, $\mathrm{rank}(\Psi(1)) = 5$ and $\mathrm{rank}(\Theta) = 6$, which follows that $\mathrm{rank}(\Psi(3)) - \mathrm{rank}(\Psi(2)) = \mathrm{rank}(\Theta)$ and condition J1 is satisfied for $k = 3$. Hence, robust functional state estimation can be achieved by using 3 cascaded sliding mode observers. Also, it can be shown that J1 is not satisfied for $k = 1, 2$ and hence 1 and 2 observers are not sufficient for functional state estimation. The two-observer scheme in [99] can accurately estimate $\dot{\theta}$ but not $\ddot{\theta}$. The single observer design by [87] is not applicable to this system. In addition, it can be verified that the full-state estimation scheme by Tan & Edwards [81] is not applicable because $\mathrm{rank}(D) > \mathrm{rank}(C)$ which means there are more unknown inputs than outputs.

The structure in (5.4) can be achieved by a suitable choice of T_s

$$T_s = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{5.43}$$

and system (5.6) can be formed as follows:

$$\underbrace{\frac{d}{dt}\begin{bmatrix} \ddot{\theta} \\ \dot{\theta} \\ \theta \\ z_f \end{bmatrix}}_{\dot{\bar{x}}(1)} = \underbrace{\left[\begin{array}{cc|cc} -10 & -31 & -30 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{array}\right]}_{\bar{A}(1)} \underbrace{\begin{bmatrix} \ddot{\theta} \\ \dot{\theta} \\ \theta \\ z_f \end{bmatrix}}_{\bar{x}(1)} + \underbrace{\left[\begin{array}{c|ccc} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right]}_{\bar{D}(1)} \underbrace{\begin{bmatrix} u \\ \varepsilon_d \\ f \\ \varepsilon_c \end{bmatrix}}_{\bar{f}(1)}$$

$$\underbrace{\begin{bmatrix} \theta \\ z_f \end{bmatrix}}_{\bar{y}(1)} = \underbrace{\left[\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]}_{\bar{C}(1)} \underbrace{\begin{bmatrix} \ddot{\theta} \\ \dot{\theta} \\ \theta \\ z_f \end{bmatrix}}_{\bar{x}(1)} \tag{5.44}$$

It can be seen that $\mathrm{rank}(\Xi(3)) = 4$, $\mathrm{rank}(\Xi(2)) = 2$ and $\mathrm{rank}(\bar{D}(1)) = 2$. Hence, $\mathrm{rank}(\Xi(3)) - \mathrm{rank}(\Xi(2)) = \mathrm{rank}(\bar{D}(1))$ and the system in the form of (5.6) satisfy condition K1, which confirms that three cascaded sliding mode observers can be used to accurately estimate $\bar{x}(1)$ (and hence z) despite the disturbances $\bar{f}(1)$.

The observers used in this chapter is the one by Tan & Edwards [81]. The design algorithm given in Chapter 3 will be used in this chapter to design the observers. The first observer is designed on system (5.44), where $\bar{n}(1) = 4$, $\bar{p}(1) = 2$ and $L(1)$ is chosen as

$$L(1) = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \tag{5.45}$$

such that $A_1(1) + L(1)A_3(1) = \begin{bmatrix} -4 & 0 \\ 0 & -5 \end{bmatrix}$. Then, $P_o(1)$ and $A_4^s(1)$ are chosen to be I_2 and $diag\{-2, -3\}$. The gain matrices, $G_n(1)$ and $G_l(1)$ in the coordinates of (5.44) are

$$G_n(1) = \left[\begin{array}{cc} -1 & 0 \\ -1 & 0 \\ \hline 1 & 0 \\ 0 & 1 \end{array}\right], \quad G_l(1) = \left[\begin{array}{cc} 9 & 0 \\ -3 & 0 \\ \hline 1 & 0 \\ 0 & 2 \end{array}\right] \tag{5.46}$$

As a result, a system of order $\bar{n}(2) := 3$ and $\bar{p}(2) = 2$ can be formed, where

$$\bar{A}(2) = \left[\begin{array}{c|cc} -10 & -30 & 0 \\ \hline 1 & 1 & 0 \\ 0 & 0 & -1 \end{array}\right], \ \bar{D}(2) = \left[\begin{array}{c|ccc} -1 & -1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right], \ \bar{C}(2) = \left[\begin{array}{c|cc} 0 & -1 & 0 \\ 0 & 0 & 1 \end{array}\right] \tag{5.47}$$

Select $L(2) = \begin{bmatrix} 9 & 0 \end{bmatrix}$ to get $A_1(2) + L(2)A_3(2) = -1$. Then, define the Lyapunov matrix as $P_o(2) = I_2$ and $A_4^s(2)$ is chosen to be $diag\{-2, -3\}$ such that

$$G_n(2) = \left[\begin{array}{cc} 9 & 0 \\ \hline -1 & 0 \\ 0 & 1 \end{array}\right], \ G_l(2) = \left[\begin{array}{cc} -51 & 0 \\ \hline 7 & 0 \\ 0 & 1 \end{array}\right] \tag{5.48}$$

Now, the system for observer 3 can be obtained, where

$$\bar{A}(3) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \ \bar{D}(3) = \left[\begin{array}{c|ccc} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right], \ \bar{C}(3) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \tag{5.49}$$

It is straightforward to see that $\mathrm{rank}(\bar{C}(3)\bar{D}(3)) = \mathrm{rank}(\bar{D}(3)) = 2$. Hence, condition J1 for $k = 3$ is verified, that 3 observers are necessary and sufficient to accurately estimate the functional state. The number of states is equal to the number of outputs, and hence there will be no reduced order sliding motion. The poles of observer 3 are selected to be $A_4^s(3) = diag\{-0.1, -0.3\}$ and its associated Lyapunov matrix is defined as $P_o(3) = I_2$. The gain matrices in the coordinate of (5.49) are

$$G_n(3) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \ G_l(3) = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \tag{5.50}$$

The initial conditions of the states are assumed to be

$$x(0) = \begin{bmatrix} 0.2 & -0.3 & 0.3 & 0.1 & 0.8 \end{bmatrix}^T \tag{5.51}$$

and f were set to be sinusoidal waveforms, where $f_1(t) = 0.58\sin(1.21t - 0.26)$, $f_2(t) = 0.64\sin(1.63t + 0.12)$ and $f_3(t) = 0.75\sin(1.76t + 0.12)$. Figures 5.1 - 5.9 shows the response of the functional state, $z = \begin{bmatrix} \ddot{\theta} \\ \dot{\theta} \end{bmatrix}$ and its estimates obtained from the 3-observer scheme which shows good estimation results. Figures 5.1, 5.4 and 5.7 show the plot of the output estimation error, e_y from observer 1, 2 and 3 respectively. Figures 5.2 and 5.3 show that $\hat{z}(1) = \begin{bmatrix} \hat{\ddot{\theta}}(1) \\ \hat{\dot{\theta}}(1) \end{bmatrix}$ from observer 1 does not track the actual z. Figures 5.5 and 5.6 shows that two sliding mode observer is sufficient to estimate $\dot{\theta}$, but not $\ddot{\theta}$. The estimation

of $\hat{z}(2) = \begin{bmatrix} \hat{\ddot{\theta}}(2) \\ \hat{\dot{\theta}}(2) \end{bmatrix}$ are obtained from $\hat{z}_i(2) = \hat{z}_i(1) - \hat{e}_{2,i}(1)$, $i = 1, 2$ where $\hat{z}_i(1)$ is from observer 1 and $\hat{e}_{2,i}(1)$ is the estimated state from observer 2. Figures 5.8 and 5.9 show that $\hat{z}(3)$ obtained from $\hat{z}_i(3) = \hat{z}_i(2) - \hat{e}_{2,i}(2)$ (where $\hat{e}_{2,i}(2)$ is the estimated state from observer 3) can be accurately track the actual z.

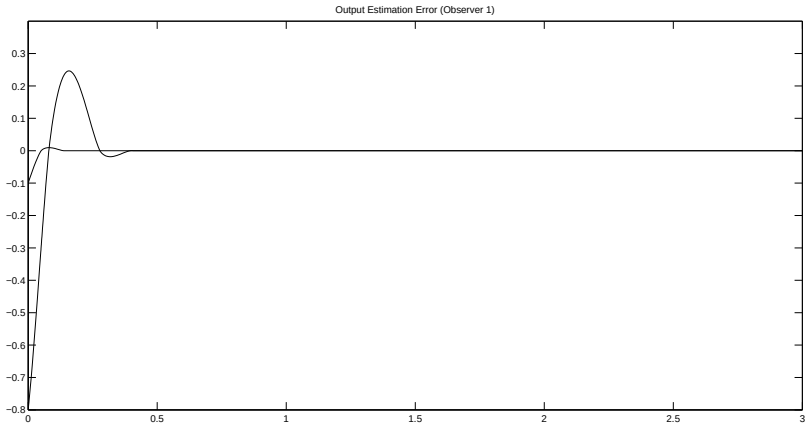

Figure 5.1: Output estimation error $e_y = y - \hat{y}$ from Observer 1. All components of e_y are driven to zero at around $0.5s$.

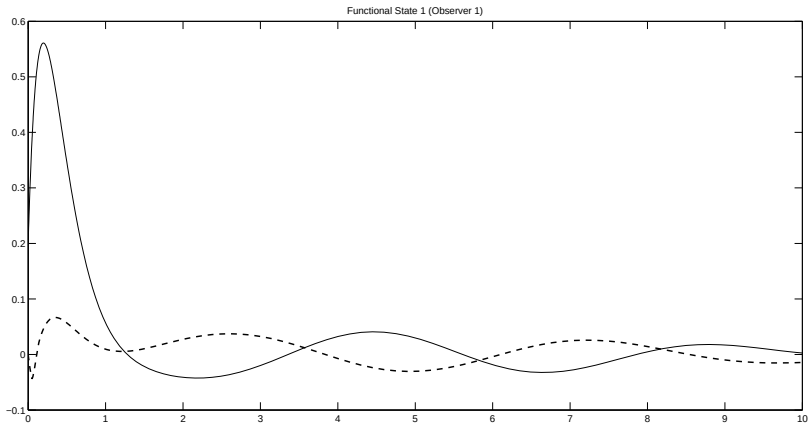

Figure 5.2: The estimated $\hat{\ddot{\theta}}(1)$ (dashed) from observer 1 does not follow the actual $\ddot{\theta}$ (solid).

5.5 Conclusion

This chapter presents a functional state estimation scheme using cascaded sliding mode observers, which generalizes the observer scheme discussed in the previous chapters. The

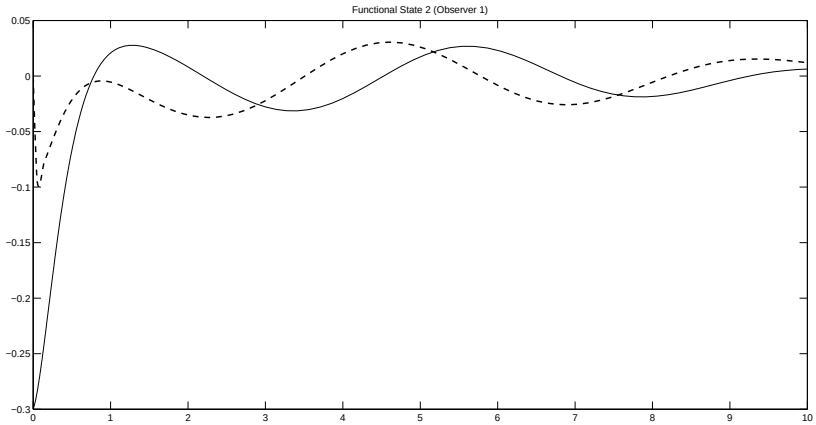

Figure 5.3: The estimated $\hat{\dot{\theta}}(1)$ (dashed) from observer 1 does not follow the actual $\dot{\theta}$ (solid).

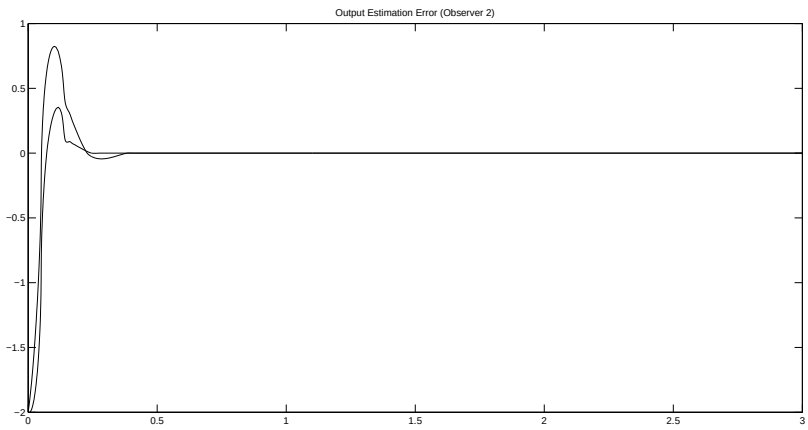

Figure 5.4: Output estimation error from Observer 2.

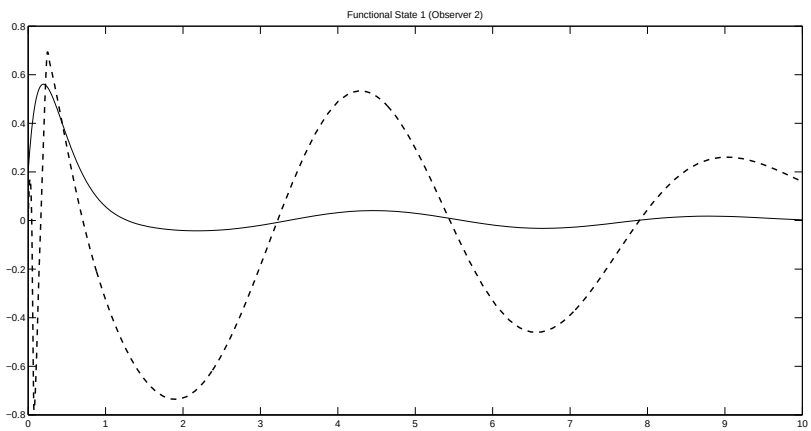

Figure 5.5: The estimated $\hat{\ddot{\theta}}(2)$ (dashed) obtained from $\hat{z}_1(2) = \hat{z}_1(1) - \hat{e}_{2,1}(1)$ and the actual $\ddot{\theta}$ (solid), where $\hat{z}_1(1)$ is from observer 1 and $\hat{e}_{2,1}(1)$ is from observer 2.

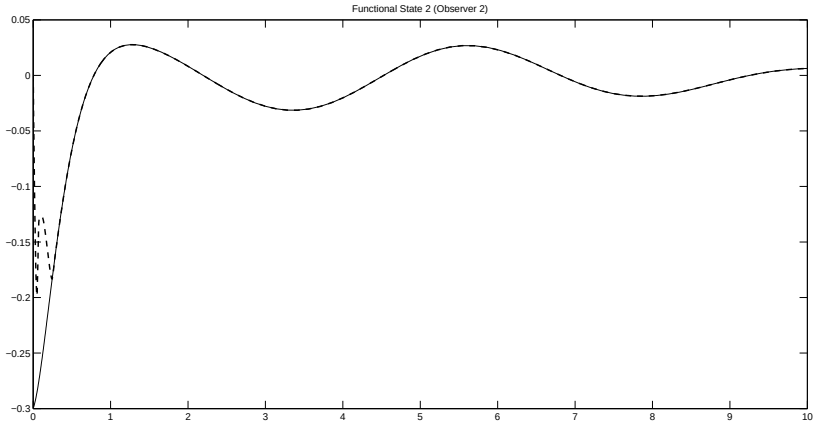

Figure 5.6: The estimated $\hat{\dot{\theta}}(2)$ (dashed) obtained from $\hat{z}_2(2) = \hat{z}_2(1) - \hat{e}_{2,2}(1)$ and the actual $\dot{\theta}$ (solid), where $\hat{z}_2(2)$ is from observer 1 and $\hat{e}_{2,2}(1)$ is from observer 2.

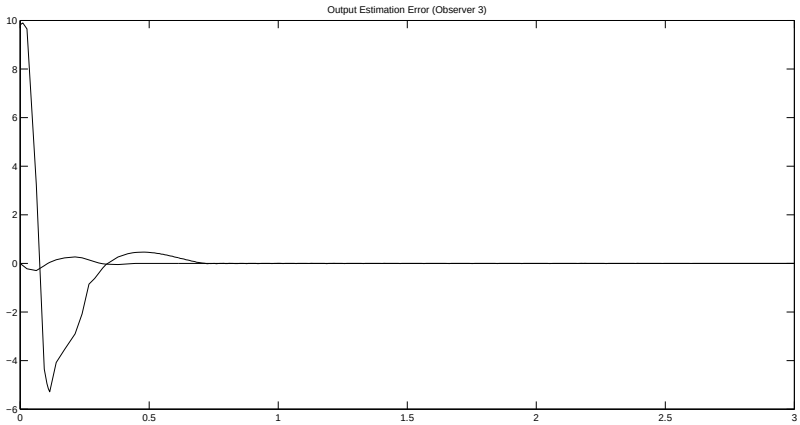

Figure 5.7: Output estimation error from Observer 3.

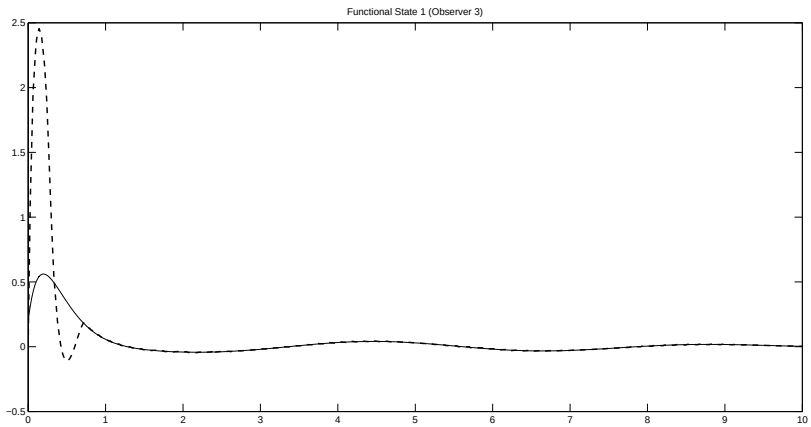

Figure 5.8: The estimated $\hat{\ddot{\theta}}(3)$ (dashed) obtained from $\hat{z}_1(3) = \hat{z}_1(2) - \hat{e}_{2,1}(2)$ and the actual $\ddot{\theta}$ (solid), where $\hat{e}_{2,1}(2)$ is from observer 3.

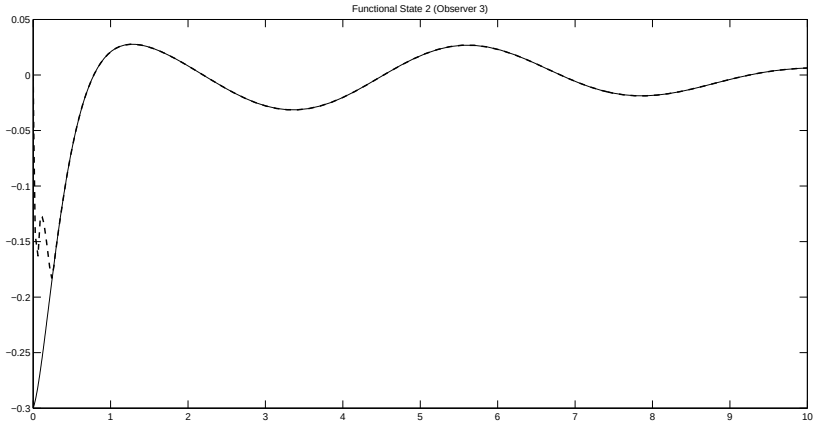

Figure 5.9: The estimated $\hat{\dot{\theta}}(3)$ (dashed) obtained from $\hat{z}_2(3) = \hat{z}_2(2) - \hat{e}_{2,2}(2)$ and the actual $\dot{\theta}$ (solid), where $\hat{e}_{2,2}(2)$ is the state estimated from observer 3.

main contribution is the investigation of the necessary and sufficient conditions in terms of the original matrices and the formulation of a suitable algorithm that calculates the number of observers needed to achieve functional state estimation. The conditions in this chapter are found to be less stringent compare to the observer scheme presented in Chapters 3 and 4, as well as existing sliding mode functional observation work. The physical implication of less stringent existence conditions is that less output sensors are required, thus saving cost and reducing system complexity. Furthermore, this chapter also presents the calculation of the number of non-functional states and unknown inputs that can be estimated. The effectiveness of the scheme is shown by the simulation result of a 3rd order water tank system.

Chapter 6

Robust Functional State Estimation using Unknown Input Estimation Techniques

6.1 Introduction

It was shown in Chapter 3 that functional state estimation can be achieved by first re-arranging the system into a form where existing sliding mode observer theory could be applied directly and the functional state can be estimated from the observer state; this technique was used in Chapters 4 and 5. It was also shown in Section 3.2.2 that the unknown input/disturbances can be estimated from the output injection signal, ν. This chapter presents a novel alternative (to the method in Section 3.3 which obtains the estimate from the observer state) for functional state estimation, where the functional state is modeled as an unknown input and subsequently estimated using the ability of the sliding mode observer to estimate unknown inputs.

First, the original system was transformed and reformulated into a form where the Edwards-Spurgeon observer [5] can be applied directly but this time the functional state and (original) unknown input are combined to form an 'augmented unknown input'. Then, in Section 6.3 the existence conditions were investigated and recasted in terms of the original system matrices. Finally in Section 6.4 a simulation example was used to show the effectiveness of this scheme.

6.2 Problem Statement and System Formulation

Consider the following system

$$\begin{aligned} \dot{x} &= Ax + Df \\ y &= Cx \\ z &= Fx \end{aligned} \tag{6.1}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^p$ and $f \in \mathbb{R}^q$ represent the state, output and unknown input respectively and $z \in \mathbb{R}^r$ is the vector to be estimated. The matrices $A \in \mathbb{R}^{n\times n}$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{n\times q}$ and $F \in \mathbb{R}^{r\times n}$ are known and constant; without loss of generality, it is assumed that $\mathrm{rank}(C) = p$, $\mathrm{rank}(F) = r < n$ and also assume $\mathrm{rank}\begin{bmatrix} F \\ C \end{bmatrix} = r + p$; this is not a restrictive assumption as it simply implies that the 'output states' will not be estimated since they are already available for measurement.

In this chapter, system (6.1) will be formulated in a way such that the functional state estimation can be achieved by using unknown input (fault) estimation paradigms as in [5, 63, 88, 89]. First, system (6.1) is transformed into the structure of (3.90) by using Lemma 3.5, where

$$A = \left[\begin{array}{c|cc} A_1 & A_2 & A_3 \\ \hline A_4 & A_5 & A_6 \\ A_7 & A_8 & A_9 \end{array}\right], \quad D = \left[\begin{array}{c} D_1 \\ \hline D_2 \\ D_3 \end{array}\right]\begin{array}{l} \updownarrow n-p-r \\ \updownarrow r \\ \updownarrow p \end{array}, \qquad \begin{array}{l} F = \left[\begin{array}{c|cc} 0 & I_r & 0 \end{array}\right], \\ C = \left[\begin{array}{c|cc} 0 & 0 & I_p \end{array}\right] \end{array} \tag{6.2}$$

Then, the system states, x, in this coordinate system can be transformed and partitioned as follows:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ y \end{bmatrix}\begin{array}{l} \updownarrow n-p-r \\ \updownarrow f \\ \updownarrow p \end{array} \tag{6.3}$$

Hence, $z = Fx$ will become $z = x_2$, where x_2 is the state components to be estimated while the estimation of x_1 is not required. In the coordinates of (6.2), system (6.1) can be

re-written as follows:

$$\underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{y} \end{bmatrix}}_{\dot{\bar{x}}} = \underbrace{\begin{bmatrix} A_1 & A_3 \\ A_7 & A_9 \end{bmatrix}}_{\bar{A}} \underbrace{\begin{bmatrix} x_1 \\ \dot{y} \end{bmatrix}}_{\bar{x}} + \underbrace{\begin{bmatrix} A_2 & D_1 \\ A_8 & D_3 \end{bmatrix}}_{\bar{D}} \underbrace{\begin{bmatrix} x_2 \\ f \end{bmatrix}}_{\bar{f}}$$

$$y = \underbrace{\begin{bmatrix} 0 & I_p \end{bmatrix}}_{\bar{C}} \underbrace{\begin{bmatrix} x_1 \\ y \end{bmatrix}}_{\bar{x}} \tag{6.4}$$

where $\bar{f} \in \mathbb{R}^{q+r}$ is an 'augmented unknown input' of order $q+r$. This is different from the formulation in (3.98). In addition, the rank of $\bar{D}$ must now be considered since x_2 (which is to be estimated) is part of the unknown input; this will be addressed later. Notice that the system in (6.4) is in the same form as the one in (3.1); therefore, the Edwards-Spurgeon observer (3.3) can be implemented on system (6.4) to estimate $\bar{x}$ and x_2 (which is part of the unknown input $\bar{f}$).

One of the conditions to obtain a sliding motion independent of $\bar{f}$ is that

$$\operatorname{rank}(\bar{C}\bar{D}) = \operatorname{rank}(\bar{D}) \Rightarrow \operatorname{rank}\begin{bmatrix} A_8 & D_3 \end{bmatrix} = \operatorname{rank}\begin{bmatrix} A_2 & D_1 \\ A_8 & D_3 \end{bmatrix} \tag{6.5}$$

Suppose that

$$\begin{bmatrix} A_8 & D_3 \end{bmatrix} = U_1 \begin{bmatrix} 0 & 0 \\ 0 & \Gamma_2 \end{bmatrix} U_2 \tag{6.6}$$

where U_1, U_2 are orthogonal, and Γ_2 is a square and invertible matrix which results in $\operatorname{rank}\begin{bmatrix} A_8 & D_3 \end{bmatrix} = \operatorname{rank}(\Gamma_2)$. The condition in (6.5) therefore implies that $\begin{bmatrix} A_2 & D_1 \end{bmatrix}$ can be decomposed into the form similar to (6.6) as

$$\begin{bmatrix} A_2 & D_1 \end{bmatrix} = \begin{bmatrix} 0 & \Gamma_1 \end{bmatrix} U_2 \tag{6.7}$$

where Γ_1 and Γ_2 have the same number of columns. Introduce a change of coordinates

$$T_g = \begin{bmatrix} I_r & -\begin{bmatrix} A_2 & D_1 \end{bmatrix}\begin{bmatrix} A_8 & D_3 \end{bmatrix}^{\dagger} \\ 0 & I_p \end{bmatrix} \tag{6.8}$$

where $\begin{bmatrix} A_8 & D_3 \end{bmatrix}^{\dagger}$ is the Moore-Penrose inverse of $\begin{bmatrix} A_8 & D_3 \end{bmatrix}$ which will have the form

$$\begin{bmatrix} A_8 & D_3 \end{bmatrix}^{\dagger} = U_2^{-1} \begin{bmatrix} 0 & 0 \\ 0 & \Gamma_2^{-1} \end{bmatrix} U_1^{-1} \tag{6.9}$$

Therefore by using (6.6)-(6.9) the matrices in (6.4) will be transformed to become

$$\bar{D} \mapsto T_g\bar{D} = \begin{bmatrix} 0 & 0 \\ A_8 & D_3 \end{bmatrix}, \ \bar{C} \mapsto \bar{C}T_g^{-1} = \begin{bmatrix} 0 & I_p \end{bmatrix}, \ \bar{A} \mapsto T_g\bar{A}T_g^{-1} = \begin{bmatrix} \bar{A}_1 & \bar{A}_2 \\ \bar{A}_3 & \bar{A}_4 \end{bmatrix} \tag{6.10}$$

Lemma 6.1 *The state x_2 can be fully estimated using the unknown input estimation method of a sliding mode observer if and only if* $\operatorname{rank}\begin{bmatrix} A_8 & D_3 \end{bmatrix} - \operatorname{rank}(D_3) = r$ ♯

Proof

Consider $\bar{D}\bar{f}$ in the coordinate of (6.10), which is

$$\bar{D}\bar{f} = \begin{bmatrix} 0 & 0 \\ A_8 & D_3 \end{bmatrix}\begin{bmatrix} x_2 \\ f \end{bmatrix} \tag{6.11}$$

Let $\operatorname{rank}(\bar{D}) = r_{\bar{D}}$, $\operatorname{rank}(A_8) = r_{\bar{D}1}$, $\operatorname{rank}(D_3) = r_{\bar{D}2}$ and $x_2 \in \mathbb{R}^{n_{\bar{D}1}}$, $f \in \mathbb{R}^{n_{\bar{D}2}}$.

Let there be orthogonal matrices W_2 and V_2 such that

$$W_2D_3V_2 = \begin{bmatrix} 0 & 0 \\ 0 & D_{32} \end{bmatrix} \tag{6.12}$$

where $D_{32} \in \mathbb{R}^{r_{\bar{D}2}\times r_{\bar{D}2}}$ is invertible. Suppose that

$$W_2A_8 = \begin{bmatrix} A_{81} \\ A_{82} \end{bmatrix}\begin{matrix} \updownarrow p - r_{\bar{D}2} \\ \updownarrow r_{\bar{D}2} \end{matrix} \tag{6.13}$$

Then let there be orthogonal matrices W_1 and V_1 such that

$$W_1A_{81}V_1 = \begin{bmatrix} 0 & 0 \\ 0 & A_{812} \end{bmatrix} \tag{6.14}$$

where $A_{812} \in \mathbb{R}^{r_x\times r_x}$ is square and invertible. Now introduce a further state transformation

$$\bar{T}_{g1} = \left[\begin{array}{c|cc} I_{\bar{n}-\bar{p}} & 0 & 0 \\ \hline 0 & W_1 & 0 \\ 0 & 0 & r_{\bar{D}2} \end{array}\right]\begin{bmatrix} I_{\bar{n}-\bar{p}} & 0 \\ 0 & W_2 \end{bmatrix} \tag{6.15}$$

such that $\bar{x} \mapsto \bar{T}_{g1}\bar{x}$, and a transformation

$$\bar{T}_{g2} = \begin{bmatrix} V_1 & 0 \\ 0 & V_2 \end{bmatrix} \tag{6.16}$$

such that $\bar{f} \mapsto \bar{T}_{g2}^{-1}\bar{f}$ and let $A_{82}V_1 = \begin{bmatrix} A_{821} & A_{822} \end{bmatrix}$; hence the matrix $\bar{D}$ will be transformed to become

$$\bar{D} \mapsto \bar{T}_{g1}\bar{D}\bar{T}_{g2} = \begin{array}{c} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & A_{812} & 0 & 0 \\ A_{821} & A_{822} & 0 & D_{32} \end{bmatrix} \end{array} \begin{array}{l} \updownarrow \bar{n} - \bar{p} \\ \updownarrow \bar{p} - r_x - r_{\bar{D}2} \\ \updownarrow r_x \\ \updownarrow r_{\bar{D}2} \end{array} \tag{6.17}$$

and $\bar{f}$ will become

$$\bar{f} \mapsto \bar{T}_{g2}^{-1}\bar{f} = \begin{bmatrix} V_1^{-1}x_2 \\ V_2^{-1}f \end{bmatrix} \begin{bmatrix} x_{21} \\ x_{22} \\ f_1 \\ f_2 \end{bmatrix} \tag{6.18}$$

Introduce another transformation for $\bar{f}$ as

$$\bar{T}_{g3} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -A_{821}D_{32}^{-1} & -A_{822}D_{32}^{-1} & 0 & 1 \end{bmatrix} \tag{6.19}$$

where

$$\bar{f} \mapsto \bar{T}_{g3}^{-1}\bar{f} = \begin{bmatrix} x_{21} \\ x_{22} \\ f_1 \\ A_{821}D_{32}^{-1}x_{21} + A_{822}D_{32}^{-1}x_{22} + f_2 \end{bmatrix} \tag{6.20}$$

and

$$\bar{D} \mapsto \bar{D}\bar{T}_{g3} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & A_{812} & 0 & 0 \\ 0 & 0 & 0 & D_{32} \end{bmatrix} \tag{6.21}$$

and $\bar{D}\bar{f}$ can now be equivalently re-expressed as

$$\bar{D}\bar{f} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ A_{812} & 0 \\ 0 & D_{32} \end{bmatrix} \begin{bmatrix} x_{22} \\ \tilde{f}_2 \end{bmatrix} \tag{6.22}$$

where $\tilde{f}_2 = A_{821}D_{32}^{-1}x_{21} + A_{822}D_{32}^{-1}x_{22} + f_2$, which is in the same form as (3.2).

Notice that the state transformation $\bar{T}_{g1}$ does not affect the structure of $\bar{C}$ as $\bar{C}\bar{T}_{g1}^{-1} = \begin{bmatrix} 0 & \bar{C}_2 \end{bmatrix}$, $\bar{C}_2 \in \mathbb{R}^{\bar{p}\times\bar{p}}$ where

$$\bar{C} = W_2^{-1}\begin{bmatrix} W_1^{-1} & 0 \\ 0 & I_{r_{\bar{D}2}} \end{bmatrix} \tag{6.23}$$

which is still in the same form as (3.2). Hence by using the theory in Section 3.2.2, in particular (3.20), the vector x_{22} and $\tilde{f}_2$ can be estimated using the equivalent output error injection signal. However, from (6.22), $\tilde{f}_2$ will always be affected by f_2; hence in order for all components of x_2 to be estimated independently from f_2, it will require that all components of x_2 to be encapsulated in x_{22} and therefore x_{21} does not exist. This in turn requires $\mathrm{rank}(A_{812}) = n_{\bar{D}1}$. Recall that $\mathrm{rank}(\bar{D}) = r_{\bar{D}}$, which then from (6.17) results in $r_{\bar{D}} = r_x + r_{\bar{D}2}$. As $r_x = \mathrm{rank}(A_{812})$, it can then be concluded that in order for all components of x_2 to be successfully estimated independently from f, it requires

$$r_{\bar{D}} = n_{\bar{D}1} + r_{\bar{D}2} \Rightarrow \mathrm{rank}\begin{bmatrix} A_8 & D_3 \end{bmatrix} - \mathrm{rank}(D_3) = r \tag{6.24}$$

and the proof is complete. ■

6.3 Existence Conditions

The Edwards-Spurgeon observer [5] can achieve sliding motion independently of $\bar{f}$ and estimate x_2 if and only if the following conditions are satisfied

M1. $\mathrm{rank}(\bar{C}\bar{D}) = \mathrm{rank}(\bar{D})$

M2. $\mathrm{rank}\begin{bmatrix} sI - \bar{A} & \bar{D} \\ \bar{C} & 0 \end{bmatrix} = \mathrm{rank}(\bar{D}) + p + r, \ \forall s \in \mathbb{C}, Re(s) \geq 0$

M3. $\mathrm{rank}\begin{bmatrix} A_8 & D_3 \end{bmatrix} - \mathrm{rank}(D_3) = r$

Conditions M1 - M2 are the standard conditions of the Edwards-Spurgeon observer, similar to A1-A2 (Section 3.2) for the purpose of obtaining a stable sliding motion whilst rejecting $\bar{f}$. Condition M3 simply satisfies the condition of Lemma 6.1. However, conditions M1 - M3 are in terms of the partitions of the transformed system matrices, which

require the designer to perform a state transformation first before it can be tested. It is therefore of interest to recast them in terms of the original system matrices.

Theorem 6.1 *Conditions M1, M2 and M3 can be re-casted respectively in terms of the orginal system matrices as follows*

N1. $\operatorname{rank}\begin{bmatrix} AF^T & D & F^T \end{bmatrix} = \operatorname{rank}\begin{bmatrix} CAF^T & CD \end{bmatrix} + r$

N2. $\operatorname{rank}\begin{bmatrix} sI - A & AF^T & D & F^T \\ F & 0 & 0 & 0 \\ C & 0 & 0 & 0 \end{bmatrix} = \operatorname{rank}\begin{bmatrix} AF^T & D & F^T \end{bmatrix} + 2r + p, \ \forall s \in \mathbb{C}, Re(s) \geq 0$

N3. $\operatorname{rank}\begin{bmatrix} CAF^T & CD \end{bmatrix} - \operatorname{rank}\begin{bmatrix} CD \end{bmatrix} = r$

□

To prove Theorem 6.1, it will be shown that M1, M2 and M3 are respectively equivalent to N1, N2 and N3.

Lemma 6.2 *Conditions M1 and N1 are equivalent.* ♯

Proof

Substitute the matrices from (6.4) into condition M1 to get

$$\operatorname{rank}(\bar{C}\bar{D}) = \operatorname{rank}(\bar{D}) \Leftrightarrow \operatorname{rank}\begin{bmatrix} A_8 & D_3 \end{bmatrix} = \operatorname{rank}\begin{bmatrix} A_2 & D_1 \\ A_8 & D_3 \end{bmatrix} \tag{6.25}$$

Define Φ and Σ as follow:

$$\Phi = \begin{bmatrix} AF^T & D & F^T \end{bmatrix}, \ \Sigma = \begin{bmatrix} CAF^T & CD \end{bmatrix} \tag{6.26}$$

In the coordinates of (6.2), expand Φ and Σ to get

$$\Phi = \begin{bmatrix} A_2 & D_1 & 0 \\ A_5 & D_2 & I_r \\ A_8 & D_3 & 0 \end{bmatrix}, \ \Sigma = \begin{bmatrix} A_8 & D_3 \end{bmatrix} \tag{6.27}$$

Then, it is straightforward to see that

$$\operatorname{rank}(\Phi) = \operatorname{rank}\underbrace{\begin{bmatrix} A_2 & D_1 \\ A_8 & D_3 \end{bmatrix}}_{\Phi_1} + r \tag{6.28}$$

Hence, condition N1 becomes

$$\operatorname{rank}\begin{bmatrix} A_2 & D_1 \\ A_8 & D_3 \end{bmatrix} + r = \operatorname{rank}\begin{bmatrix} A_8 & D_3 \end{bmatrix} + r \tag{6.29}$$

which is equivalent to condition M1 in (6.25). ■

To show the equivalence of conditions M2 and N2, define Δ_b and Δ_c as follow:

$$\Delta_b = \begin{bmatrix} sI - \bar{A} & \bar{D} \\ \bar{C} & 0 \end{bmatrix}, \ \Delta_c = \begin{bmatrix} sI - A & AF^T & D & F^T \\ F & 0 & 0 & 0 \\ C & 0 & 0 & 0 \end{bmatrix} \tag{6.30}$$

Lemma 6.3 *The following is true*

$$\operatorname{rank}(\Delta_c) = \operatorname{rank}(\Delta_b) + 2r \tag{6.31}$$

♯

Proof

Substitute (A, C, D, F) from (6.2) into Δ_c to get

$$\Delta_c = \begin{bmatrix} sI_{(n-p-r)} - A_1 & -A_2 & -A_3 & A_2 & D_1 & 0 \\ -A_4 & sI_r - A_5 & -A_6 & A_5 & D_2 & I_r \\ -A_7 & -A_8 & sI_p - A_9 & A_8 & D_3 & 0 \\ 0 & I_r & 0 & 0 & 0 & 0 \\ 0 & 0 & I_p & 0 & 0 & 0 \end{bmatrix} \tag{6.32}$$

$$\Rightarrow \operatorname{rank}(\Delta_c) = \operatorname{rank}\underbrace{\begin{bmatrix} sI_{(n-p-r)} - A_1 & A_2 & D_1 \\ -A_7 & A_8 & D_3 \end{bmatrix}}_{\Delta_{c,1}} + p + 2r \tag{6.33}$$

In the coordinates of (6.4), Δ_b becomes

$$\Delta_b = \begin{bmatrix} sI_{(n-p-r)} - A_1 & -A_3 & A_2 & D_1 \\ -A_7 & sI_p - A_9 & A_8 & D_3 \\ 0 & I_p & 0 & 0 \end{bmatrix} \tag{6.34}$$

$$\Rightarrow \Delta_b = \underbrace{\begin{bmatrix} sI_{(n-p-r)} - A_1 & A_2 & D_1 \\ -A_7 & A_8 & D_3 \end{bmatrix}}_{\Delta_{b,1}} + p \tag{6.35}$$

From (6.33) and (6.35), it is clear that $\Delta_{b,1} = \Delta_{c,1}$. Hence, $\operatorname{rank}(\Delta_c) = \operatorname{rank}(\Delta_b) + 2r$ and Lemma 6.3 is proven. ■

By using equation (6.28) , it is proven that $\mathrm{rank}(\Phi) = \mathrm{rank}(\bar{D}) + r$. Therefore, the RHS of condition M2 is equivalent to $\mathrm{rank}(\Phi) + p$. By using Lemma 6.3, the LHS of condition N2 is equivalent to $\mathrm{rank}(\Delta_b) + 2r$. Hence, it is proven that conditions M2 and N2 are equivalent.

Lemma 6.4 *Conditions M3 and N3 are equivalent.* ♯

Proof

In the coordinates of (6.2), it is obvious that

$$\mathrm{rank}\begin{bmatrix} CAF^T & CD \end{bmatrix} - \mathrm{rank}(CD) \Rightarrow \mathrm{rank}\begin{bmatrix} A_8 & D_3 \end{bmatrix} - \mathrm{rank}(D_3) \tag{6.36}$$

Therefore, condition N3 implies that $\mathrm{rank}\begin{bmatrix} A_8 & D_3 \end{bmatrix} - \mathrm{rank}(D_3) = r$ which is equivalent to M3. Hence, Lemma 6.4 is proven. ■

Condition M3 implies that A_8 has full column rank of r (necessary and sufficient so that all components of x_2 can be estimated) and that its columns are independent of D_3 (necessary and sufficient so that x_2 can be decoupled from any components of f).

6.4 Numerical Example

The two-carts system shown in Section 4.4 will be used to demonstrate the observer scheme proposed in this chapter. From (4.70)-(4.72), the system matrices are

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 1 \\ 1 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \; C = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \; F = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \tag{6.37}$$

Assume that the actuator is not faulty, then matrix D can be modified to be

$$D = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{6.38}$$

Condition F1 (Lemma 3.6) is not satisfied by the system, but condition H1 (Theorem 4.1) can be satisfied; therefore, x_2 only can be estimated by using the two-observers scheme

in Chapter 4. It can be verified that conditions N1, N2 and N3 proposed in this chapter is satisfied and hence x_2 can be estimated using one observer (hence simpler) by modeling it as an unknown input and using the observer scheme in this chapter.

First, apply transformation T_s to achieve the structure in (6.2), where

$$T_s = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{6.39}$$

and system (6.37)-(6.38) becomes

$$A_s = \left[\begin{array}{c|cc|cc} -1 & 0 & 0 & 0 & 0 \\ \hline 0 & -1 & 0 & 1 & -1 \\ 1 & 0 & 0 & -1 & 1 \\ \hline 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{array}\right], \quad D_s = \left[\begin{array}{cc} 0 & 0 \\ \hline 1 & 0 \\ 0 & 1 \\ \hline 0 & 0 \\ 0 & 0 \end{array}\right], \qquad \begin{array}{l} F_s = \left[\begin{array}{c|cc|cc} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right], \\ C_s = \left[\begin{array}{c|cc|cc} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] \end{array} \tag{6.40}$$

In the coordinates of (6.40), re-arrange the system matrices into the form of (6.4) as follows:

$$\underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{y} \end{bmatrix}}_{\dot{\bar{x}}} = \underbrace{\left[\begin{array}{c|cc} -1 & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]}_{\bar{A}} \underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{y} \end{bmatrix}}_{\bar{x}} + \underbrace{\left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right]}_{\bar{D}} \underbrace{\begin{bmatrix} x_2 \\ f \end{bmatrix}}_{\bar{f}}$$

$$y = \underbrace{\left[\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]}_{\bar{C}} \underbrace{\begin{bmatrix} x_1 \\ y \end{bmatrix}}_{\bar{x}} \tag{6.41}$$

where x_1 and y are the state components while x_2 and f are the unknown inputs, which can be estimated from the output injection error ν using the method presented in Section 3.2.2. It is obvious that $\mathrm{rank}(\bar{C}\bar{D}) = \mathrm{rank}(\bar{D}) = 2$, which satisfy condition M1 and the Edwards-Spurgeon observer can be implemented directly on system (6.41). From $\bar{D}$ in (6.41), it is clear that A_8 (in the notation of (6.4)) has full column rank and hence condition M3 is satisfied and all components of x_2 can be estimated using the Edwards-Spurgeon observer.

Using the design procedure in Section 3.2, L is selected as

$$L = \begin{bmatrix} -0.1 & 0 \end{bmatrix} \tag{6.42}$$

Then, P_o and A_4^s are chosen to be I_2 and $diag\{-0.1, -0.2\}$. Then gain matrices, G_n and G_l in the same coordinates as (6.41) are

$$G_n = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \; G_l = \begin{bmatrix} -0.9 & -0.8 \\ 0.1 & 0 \\ 0 & 0.2 \end{bmatrix} \tag{6.43}$$

The simulation results are shown in Figures 6.1 - 6.5. Then inital conditions of the states x were assumed to be $\begin{bmatrix} 0.1 & -0.6 & 0.9 & -0.2 & 0.3 \end{bmatrix}^T$ whilst the observer had zero intial conditions. The unknown input f were set to be sinusoidal waveforms, where $f_1 = 2.15\sin(1.24t - 0.25)$ and $f_2 = 1.03\sin(1.65t - 0.78)$. The ρ and δ in (3.21) were selected as 20 and 1×10^{-2}. Figure 6.1 shows the plot of the output estimation error, where sliding motion occurs in less than $0.05s$. Then, Figures 6.2 and 6.4 show the components of the functional state $\hat{x}_2$ estimated from the discontinuous output injection error ν accurately track the actual x_2. Figures 6.3 and 6.5 are the zoomed-in versions of Figures 6.2 and 6.4.

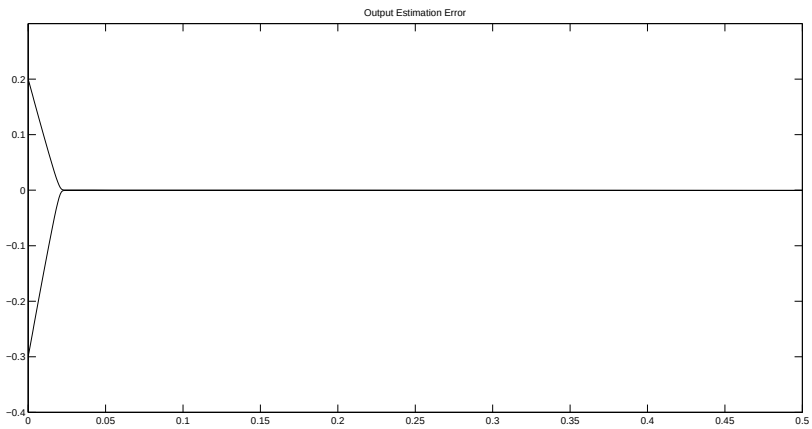

Figure 6.1: Output estimation error $e_y = y - \hat{y}$.

6.5 Conclusion

This chapter has presented a new scheme in functional state estimation using unknown input estimation paradigms, where the functional state is modeled as an unknown input and estimated by exploiting the ability of the Edwards-Spurgeon observer in estimating unknown inputs from the discontinuous output injection error. First, the system was expressed in a form such that the functional state is modeled as an unknown input, where

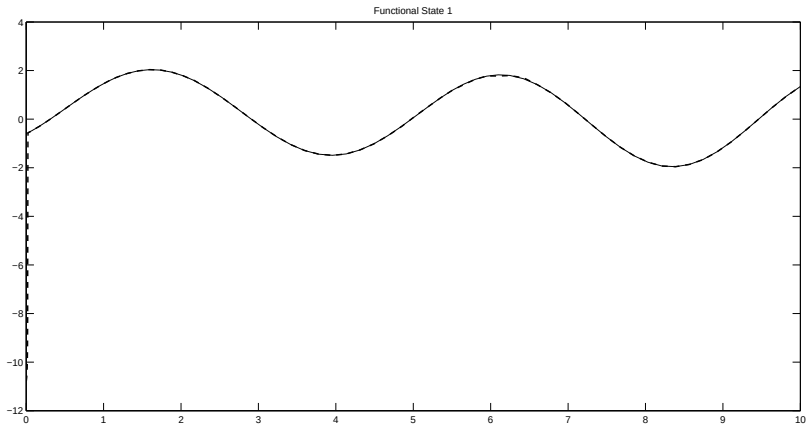

Figure 6.2: The first component of functional state, $\hat{x}_{2,1}$ (dashed) estimated from the discontinuous output injection error ν can accurately tracks the actual $x_{2,1}$ (solid).

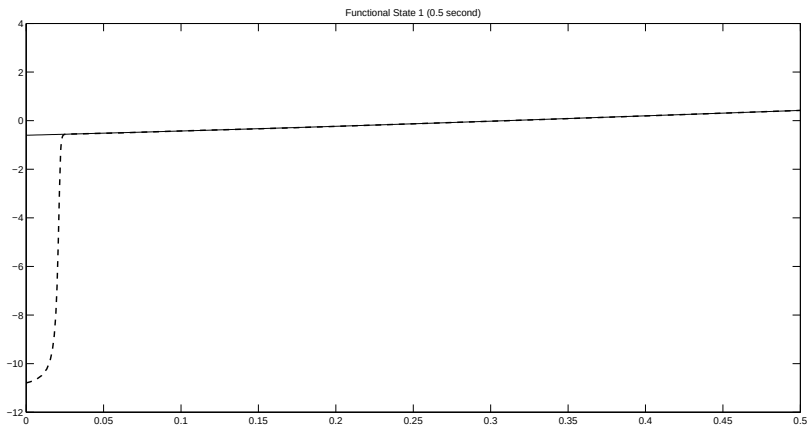

Figure 6.3: First 0.5 seconds of Figure 6.2. The estimated $\hat{x}_{2,1}$ can accurately tracks the actual $x_{2,1}$ in less than $0.05s$.

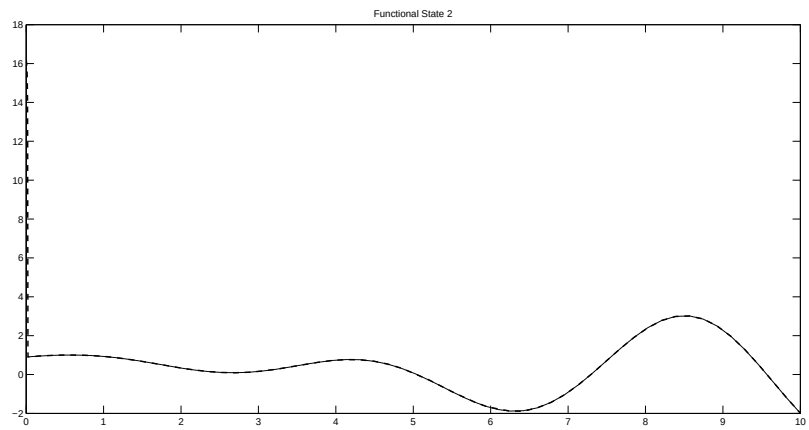

Figure 6.4: The second component of functional state, $\hat{x}_{2,2}$ (dashed) estimated from the discontinuous output injection error ν can accurately tracks the actual $x_{2,2}$ (solid).

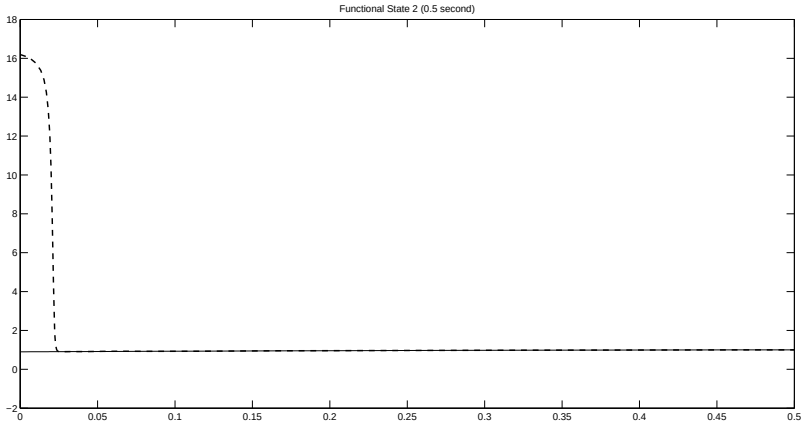

Figure 6.5: First 0.5 seconds of Figure 6.4. The estimated $\hat{x}_{2,2}$ can accurately tracks the actual $x_{2,2}$ in less than $0.05s$.

the ability of the Edwards-Spurgeon observer to estimate unknown inputs is exploited to estimate it. Then, the necessary and sufficient conditions in terms of the original system matrices were investigated to ensure all functional state components can be accurately estimated. Finally, simulation results of a two-cart system were used to show the effectiveness of the scheme, where compared to the work in earlier parts of this thesis, functional state estimation could be achieved using less observers thus reducing the complexity.

Chapter 7

Robust Functional State Estimation for a Class of Descriptor Systems

7.1 Introduction

Descriptor systems [100] are a special class of systems, where in the state-space representation, the state derivative is pre-multiplied with a rank deficient matrix. Compared to the standard state-space system (that has been studied thus far in this thesis), the descriptor system is more general, and has been able to better represent various systems such as gas turbines [101], mechanical systems [102], wastewater treatment plant [103], and even economic systems [104, 105] just to name a few. In addition, certain problems in fault detection were more conveniently solved when the faulty system was reformulated as a descriptor system [106, 86, 107, 108, 25]

The use of observers for estimating states in descriptor systems has been widely studied [109, 110, 111, 112, 113, 100, 114] however functional state estimation for descriptor systems has not received much attention to-date. In addition, most of the studies available in the literature do not consider the presence of unknown input in the system [55, 56, 57, 58, 59, 60] except Feng [115] and Ezzine [116] that use finite time observers and unknown input observers respectively. As mentioned earlier in this thesis, it is known that sliding mode observers are capable of performing better than linear observers [66], and in addition cascading sliding mode observers can relax certain existence conditions [81, 82]. This chapter seeks to explore the use of sliding mode observers for functional state estimation for a class of descriptor systems.

The organization of this chapter is as follows: In Section 7.2 the sliding mode observer

to be used [112] will be described, together with the investigation of its existence conditions. Then in Section 7.3 the original descriptor system is recasted into a form such the sliding mode observer by [112] was directly applied to estimate the functional state. The existence conditions in terms of the original system matrices are investigated in Section 7.3.1, and finally a simulation example in Section 7.4 is used to show the effectiveness of this observer scheme.

7.2 Preliminaries: Sliding Mode Observers for Descriptor System

Consider the following descriptor system

$$\begin{aligned} E\dot{x} &= Ax + Df \\ y &= Cx \end{aligned} \tag{7.1}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^p$ and $f \in \mathbb{R}^q$ represent the state, output and unknown input respectively. The matrices $A \in \mathbb{R}^{n\times n}$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{n\times q}$ and $E \in \mathbb{R}^{n\times n}$ are known and constant; without loss of generality, it is assumed that $\mathrm{rank}(C) = p$ and $\mathrm{rank}(E) = k < n$. Pre-multiply the state equation in (7.1) with a non-singular matrix R and add $V\dot{y}$ to both sides will result in

$$RE\dot{x} + V\dot{y} = RAx + RDf + V\dot{y} \tag{7.2}$$

If R and V are chosen such that $RE + VC = I_n$, then (7.2) becomes

$$\dot{x} = RAx + RDf + V\dot{y} \tag{7.3}$$

The sliding mode observer proposed by Yeu & Kawaji [112] for system (7.3) is as follows:

$$\dot{w} = (RA - G_lC)w + (-G_l(I_p - CV) - RAV)y - G_n\nu \tag{7.4}$$

$$\hat{x} = Vy - w \tag{7.5}$$

where $e_y = C\hat{x} - y$ and $\nu = -\rho\frac{e_y}{\|e_y\|}$. The gain matrices G_n and G_l can be designed in the same way as in Section 3.2.1. Substitute for w from (7.5) into (7.4) yields

$$\dot{\hat{x}} = RA\hat{x} + G_n\nu - G_le_y + V\dot{y} \tag{7.6}$$

Define the state estimation error $e := \hat{x} - x$ and substracting equation (7.3) from (7.6) yields

$$\dot{e} = (RA - G_l C)e + G_n \nu - RDf \tag{7.7}$$

Equations (7.3) and (7.6) are in the same form as (3.1) and (3.3), where $V\dot{y}$ is the 'known input' whilst RDf is the 'unknown input'; therefore, the design method for Edwards-Spurgeon observer in Section 3.2.1 can be implemented directly to design G_l and G_n based on the system matrices (RA, RD, C); following that it can also be shown that sliding motion takes place and the state x can be estimated independently of f.

7.2.1 Existence Conditions

This scheme is applicable if and only if the following conditions are satisfied

P1. $\mathrm{rank}(CRD) = \mathrm{rank}(RD)$

P2. $\mathrm{rank}\begin{bmatrix} sI_n - RA & RD \\ C & 0 \end{bmatrix} = \mathrm{rank}(RD) + n, \ \forall s \in \mathbb{C}, Re(s) \geq 0$

P3. There exist matrices R and V such that $RE + VC = I_n$.

By comparing systems (3.1) and (7.3), it can be seen that conditions P1 - P2 are direct application of conditions A1 - A2 on system (7.3) while condition P3 is necessary and sufficient to place system (7.1) into the form of (7.3). However, conditions P1 - P3 require the designer to calculate the matrices R and V first before it can be tested. Therefore, it is of interest to re-cast conditions P1 - P3 in terms of the original system matrices; this was achieved by Yu & Liu [114] who re-casted them to be:

Q1. $\mathrm{rank}\begin{bmatrix} E & D \\ C & 0 \end{bmatrix} = \mathrm{rank}(D) + n$

Q2. $\mathrm{rank}\begin{bmatrix} sE - A & D \\ C & 0 \end{bmatrix} = \mathrm{rank}(D) + n, \ \forall s \in \mathbb{C}, Re(s) \geq 0$

In the following, Lemmas 7.1 - 7.4 will be established, to show the equivalence of conditions P1-P3 and conditions Q1-Q2. First, system (7.1) will be transformed into a structure similar to (3.2) by using Lemma 7.1. Then, the structure of matrix R will be established in Lemma 7.2. Conditions P1-P3 and Q1-Q2 will be proven to be equivalent in Lemmas

7.3-7.4. It will be assumed that $\operatorname{rank}\begin{bmatrix} E \\ C \end{bmatrix} = I_n$; this does not impose any restriction as it is necessary for P3 to be solvable.

Lemma 7.1 *There exist a linear state transformation $x \mapsto T_w x$ and a state equation transformation, T_v such that system* (7.1) *becomes*

$$\begin{aligned} T_v E T_w^{-1} \dot{x} &= T_v A T_w^{-1} x + T_v D f \\ y &= C T_w^{-1} x \end{aligned} \tag{7.8}$$

where matrices (E, A, D, C) will be transformed into the following structures

$$E \mapsto T_v E T_w^{-1} = \begin{bmatrix} I_{n-p} & 0 \\ 0 & E_4 \end{bmatrix}, \; A \mapsto T_v A T_w^{-1} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{matrix} \updownarrow n-p \\ \updownarrow p \end{matrix},$$

$$D \mapsto T_v D = \begin{bmatrix} D_1 \\ D_2 \end{bmatrix}, \; C \mapsto C T_w^{-1} = \begin{bmatrix} 0 & I_p \end{bmatrix} \tag{7.9}$$

♯

Proof

Since $\operatorname{rank}(C) = p$, there exist a nonsingular change of coordinates,

$$T_{w2} = \begin{bmatrix} N_C^T \\ C \end{bmatrix} \tag{7.10}$$

where $N_C \in \mathbb{R}^{n \times (n-p)}$ spans the null space of C, i.e. $C N_C = 0$ such that the matrix C can be transformed to be

$$C_w := C T_{w2}^{-1} = \begin{bmatrix} 0 & I_p \end{bmatrix} \tag{7.11}$$

In this coordinate system, matrices A and E can be partitioned generally as

$$A_w := A T_{w2}^{-1} = \begin{bmatrix} \underset{n-p}{\underset{\longleftrightarrow}{A_{w,1}}} & \underset{p}{\underset{\longleftrightarrow}{A_{w,2}}} \end{bmatrix}, \; E_w := E T_{w2}^{-1} = \begin{bmatrix} \underset{n-p}{\underset{\longleftrightarrow}{E_{w,1}}} & \underset{p}{\underset{\longleftrightarrow}{E_{w,2}}} \end{bmatrix}, \tag{7.12}$$

From (7.8), it can be seen that matrix D is not affected by this transformation. Therefore, $D_w := D$ has no specific structure. From the assumption $\operatorname{rank}\begin{bmatrix} E \\ C \end{bmatrix} = n$ and the structure

of C_w, it is then obvious that $\mathrm{rank}(E_{w,1}) = n - p$ (which is full column rank). Then, an orthogonal matrix, U_1 can be introduced such that

$$U_1^{-1} E_{w,1} = \begin{bmatrix} E_{o,1} \\ 0 \end{bmatrix} \tag{7.13}$$

where $U_1 \in \mathbb{R}^{n\times n}$ and $E_{o,1} \in \mathbb{R}^{(n-p)\times(n-p)}$ are orthogonal. The matrix U_1 and $E_{o,1}$ can be calculated using QR decomposition of $E_{w,1}$ which can be easily performed in MATLAB. Then, let

$$U_1^{-1} E_{w,2} = \begin{bmatrix} E_{w,21} \\ E_{w,22} \end{bmatrix} \begin{matrix} \updownarrow n-p \\ \updownarrow p \end{matrix} \tag{7.14}$$

and define a state equation transformation, T_v (i.e. to pre-multiply the state equation with) and a state transformation, T_{w1} as

$$T_v = \begin{bmatrix} E_{o,1}^{-1} & 0 \\ 0 & I_p \end{bmatrix} U_1^{-1}, \; T_{w1} = \begin{bmatrix} I_{n-p} & -E_{o,1}^{-1} E_{w,21} \\ 0 & I_p \end{bmatrix} \tag{7.15}$$

Now, the system matrices in the coordinate of (7.11) - (7.12) can be transformed to have the following structures

$$E_v := T_v E_w T_{w1}^{-1} = \begin{bmatrix} I_{n-p} & 0 \\ 0 & E_{v,4} \end{bmatrix}, \; A_v := T_v A_w T_{w1}^{-1} = \begin{bmatrix} A_{v,1} & A_{v,2} \\ A_{v,3} & A_{v,4} \end{bmatrix} \begin{matrix} \updownarrow n-p \\ \updownarrow p \end{matrix},$$

$$D_v := T_v D_w = \begin{bmatrix} D_{v,1} \\ D_{v,2} \end{bmatrix} \begin{matrix} \updownarrow n-p \\ \updownarrow p \end{matrix}, \; C_v := C_w T_{w1}^{-1} = \begin{bmatrix} 0 & I_p \end{bmatrix} \tag{7.16}$$

From (7.16), it can be seen that the special structure of matrix C is not affected by this transformation, and A_v, D_v have no particular structures. Therefore, under the state transformation of $T_w = T_{w1} T_{w2}$ and the state equation transformation of T_v, the matrices in (7.16) have the same structures as (7.9) and the proof is complete. ■

Lemma 7.2 *If condition P3 is satisfied, then matrix R will have the following structure:*

$$R = \begin{bmatrix} I_{n-p} & R_2 \\ 0 & R_4 \end{bmatrix}, \; |R_4| \neq 0 \tag{7.17}$$

where $R_4 \in \mathbb{R}^{p\times p}$ is square and invertible. ♯

Proof

Re-write condition P3 as follow

$$\begin{bmatrix} R & V \end{bmatrix} \begin{bmatrix} E \\ C \end{bmatrix} = I_n \tag{7.18}$$

Then, partition R and V generally into

$$R = \begin{bmatrix} R_1 & R_2 \\ R_3 & R_4 \end{bmatrix}, \; V = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \begin{matrix} \updownarrow n-p \\ \updownarrow p \end{matrix} \tag{7.19}$$

Substitute matrices E, C from (7.9) and R, V from (7.19) into (7.18) yields

$$\begin{bmatrix} R_1 & R_2 & V_1 \\ R_3 & R_4 & V_2 \end{bmatrix} \begin{bmatrix} I_{n-p} & 0 \\ 0 & E_4 \\ 0 & I_p \end{bmatrix} = \begin{bmatrix} I_{n-p} & 0 \\ 0 & I_p \end{bmatrix} \tag{7.20}$$

Solving equation (7.20) will result in $R_1 = I_{n-p}$, $R_3 = 0$ and

$$R_2 E_4 + V_1 = 0 \tag{7.21}$$

$$R_4 E_4 + V_2 = I_p \tag{7.22}$$

From (7.20), it can be seen that matrices R_2 and V_1 have no specific structure. Since matrix R is orthogonal, then R_4 is a square and invertible matrix and $V_2 = -R_4 E_4$. ■

Lemma 7.3 *Conditions P1 and Q1 are equivalent.* ♯

Proof

Expand condition P1 using (7.9) and (7.17) to get

$$\operatorname{rank}(CRD) = \operatorname{rank}(RD) \Leftrightarrow \operatorname{rank}(R_4 D_2) = \operatorname{rank} \begin{bmatrix} D_1 + R_2 D_2 \\ R_4 D_2 \end{bmatrix} \tag{7.23}$$

Since R is square and invertible, and $RE + VC = I_n$, then the LHS of condition Q1 becomes,

$$\begin{aligned} \operatorname{rank} \begin{bmatrix} E & D \\ C & 0 \end{bmatrix} &= \operatorname{rank} \begin{bmatrix} R & V \\ 0 & I_p \end{bmatrix} \begin{bmatrix} E & D \\ C & 0 \end{bmatrix} \\ &= \operatorname{rank} \begin{bmatrix} I_n & RD \\ C & 0 \end{bmatrix} \end{aligned} \tag{7.24}$$

Substitute matrices C, D from (7.9) and R from (7.17) into (7.24) yields

$$\operatorname{rank}\begin{bmatrix} I_n & RD \\ C & 0 \end{bmatrix} = \operatorname{rank}\left[\begin{array}{cc|c} I_{n-p} & 0 & D_1 + R_2 D_2 \\ 0 & I_p & R_4 D_2 \\ 0 & I_p & 0 \end{array}\right]$$
$$= n + \operatorname{rank}(R_4 D_2) \tag{7.25}$$

which is equivalent to the $n + \operatorname{rank}(CRD)$. Since R is square and invertible, then $\operatorname{rank}(RD) = \operatorname{rank}(D)$. By combining the LHS and RHS of Q1, it is obvious that condition Q1 becomes $\operatorname{rank}(CRD) = \operatorname{rank}(RD)$ which is in fact P1. ■

Lemma 7.4 *Conditions P2 and Q2 are equivalent.* ♯

Proof

If condition P3 holds, then the LHS of condition Q2 becomes

$$\operatorname{rank}\begin{bmatrix} sE - A & D \\ C & 0 \end{bmatrix} = \operatorname{rank}\begin{bmatrix} R & sV \\ 0 & I_p \end{bmatrix}\begin{bmatrix} sE - A & D \\ C & 0 \end{bmatrix}$$
$$= \operatorname{rank}\begin{bmatrix} sI_n - RA & RD \\ C & 0 \end{bmatrix} \tag{7.26}$$

which is equivalent to the LHS of condition P2. As mentioned in Lemma 7.3, $\operatorname{rank}(RD) = \operatorname{rank}(D)$ because R is a square and invertible matrix. Therefore, it is proven that conditions P2 and Q2 are equivalence. ■

As for condition P3, there exist R and V to satisfy $RE + VC = I_n$ which can be re-written as

$$\begin{bmatrix} R & V \end{bmatrix}\begin{bmatrix} E \\ C \end{bmatrix} = I_n \tag{7.27}$$

In order for (7.27) to hold, then $\operatorname{rank}\begin{bmatrix} E \\ C \end{bmatrix} = n$ must be satisfied; this however is already implicit in condition Q1 which implies that the matrix $\begin{bmatrix} E & D \\ C & 0 \end{bmatrix}$ must be full column rank.

7.3 Main Result: Functional State Estimation for Descriptor Systems

Consider the following descriptor system for functional state estimation

$$\begin{aligned} E\dot{x} &= Ax + Df \\ y &= Cx \\ z &= Fx \end{aligned} \tag{7.28}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^p$ and $f \in \mathbb{R}^q$ represent the state, output and unknown input respectively and $z \in \mathbb{R}^r$ is the vector to be estimated. The matrices $A \in \mathbb{R}^{n\times n}$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{n\times q}$, $E \in \mathbb{R}^{n\times n}$ and $F \in \mathbb{R}^{r\times n}$ are known and constant; without loss of generality, it is assumed that $\mathrm{rank}(C) = p$, $\mathrm{rank}(F) = r < n$ and also assume $\mathrm{rank}\begin{bmatrix} F \\ C \end{bmatrix} = r + p$; this is not a restrictive assumption as it simply implies that the 'output states' will not be estimated since they are already available for measurement. Let $\mathrm{rank}\begin{bmatrix} E \\ C \end{bmatrix} = n$, and the following lemma can be established.

Lemma 7.5 *There exist a linear state transformation $x \mapsto T_z x$ and a state equation transformation, T_y such that system* (7.28) *becomes*

$$E \mapsto T_y E T_z^{-1} = \left[\begin{array}{c|cc} I_{n-p-r} & 0 & 0 \\ \hline 0 & I_n & 0 \\ 0 & 0 & E_9 \end{array}\right], \quad A \mapsto T_y A T_z^{-1} = \left[\begin{array}{c|cc} A_1 & A_2 & A_3 \\ \hline A_4 & A_5 & A_6 \\ A_7 & A_8 & A_9 \end{array}\right] \begin{array}{l} \updownarrow n-p-r \\ \updownarrow r \\ \updownarrow p \end{array},$$

$$D \mapsto T_y D = \left[\begin{array}{c} D_1 \\ \hline D_2 \\ D_3 \end{array}\right], \quad F \mapsto F T_z^{-1} = \left[\begin{array}{c|cc} 0 & I_r & 0 \end{array}\right], \quad C \mapsto C T_z^{-1} = \left[\begin{array}{c|cc} 0 & 0 & I_p \end{array}\right] \tag{7.29}$$

♯

Proof

First, the state transformation $x \mapsto T_s x$ in Lemma 3.5 can be used to transform the original system matrices into the following structure

$$A_s := A T_s^{-1} = \left[\begin{array}{ccc} \underbrace{A_{s,1}}_{n-p-r} & \underbrace{A_{s,2}}_{r} & \underbrace{A_{s,3}}_{p} \end{array}\right], \quad E_s := E T_s^{-1} = \left[\begin{array}{ccc} \underbrace{E_{s,1}}_{n-p-r} & \underbrace{E_{s,2}}_{r} & \underbrace{E_{s,3}}_{p} \end{array}\right],$$

$$F_s := FT_s^{-1} = \begin{bmatrix} 0 & I_r & 0 \end{bmatrix},\ C_s := CT_s^{-1} = \begin{bmatrix} 0 & 0 & I_p \end{bmatrix} \tag{7.30}$$

Matrix D is not affected by this transformation. Therefore, $D_s := D$ has no specific structure. From the assumption $\operatorname{rank}\begin{bmatrix} E \\ C \end{bmatrix} = n$ and the structure of C_s, it is obvious that $\operatorname{rank}\begin{bmatrix} E_{s,1} & E_{s,2} \end{bmatrix} = n - p$ (which is full column rank). Notice that $E_s = \left[\begin{array}{cc|c} E_{s,1} & E_{s,2} & E_{s,3} \end{array}\right]$ is in the same structure as $E_w = \begin{bmatrix} E_{w,1} & E_{w,2} \end{bmatrix}$ in Lemma 7.1, then similar design technique can be used to achieve the structure in (7.29). Define an orthogonal matrix, U_2 such that

$$U_2^{-1}\begin{bmatrix} E_{s,1} & E_{s,2} \end{bmatrix} = \begin{bmatrix} E_{s,10} \\ 0 \end{bmatrix} \tag{7.31}$$

where $U_2 \in \mathbb{R}^{n\times n}$ and $E_{s,10} \in \mathbb{R}^{(n-p)\times(n-p)}$ are orthogonal. The matrix U_2 and $E_{s,10}$ can be obtained from the QR decomposition of $\begin{bmatrix} E_{s,1} & E_{s,2} \end{bmatrix}$ in MATLAB. Then, let

$$U_2^{-1}E_{s,3} = \begin{array}{c} \begin{bmatrix} E_{s,31} \\ E_{s,32} \end{bmatrix} \end{array} \begin{array}{l} \updownarrow n-p \\ \updownarrow p \end{array} \tag{7.32}$$

and define a state equation transformation, T_y (i.e. to pre-multiply the state equation with) and a state transformation, T_{z1} as

$$T_y = \begin{bmatrix} E_{s,10}^{-1} & 0 \\ 0 & I_p \end{bmatrix} U_2^{-1},\ T_{z1} = \begin{bmatrix} I_{n-p} & -E_{s,10}^{-1}E_{s,31} \\ 0 & I_p \end{bmatrix} \tag{7.33}$$

Under the state transformation of $T_z = T_{z1}T_s$ and the state equation transformation of T_y, system (7.30) can be transformed to be

$$A_y := T_yA_sT_{z1}^{-1} = \left[\begin{array}{c|cc} A_{y,1} & A_{y,2} & A_{y,3} \\ \hline A_{y,4} & A_{y,5} & A_{y,6} \\ A_{y,7} & A_{y,8} & A_{y,9} \end{array}\right],\ D_y := T_yD_s = \begin{array}{c} \begin{bmatrix} D_{y,1} \\ \hline D_{y,2} \\ D_{y,3} \end{bmatrix} \end{array} \begin{array}{l} \updownarrow n-p \\ \updownarrow r \\ \updownarrow p \end{array},$$

$$E_y := T_yE_sT_{z1}^{-1} = \left[\begin{array}{c|cc} I_{n-p-r} & 0 & 0 \\ \hline 0 & I_n & 0 \\ 0 & 0 & E_{y,9} \end{array}\right], \qquad \begin{array}{l} C_y := C_sT_{z1}^{-1} = \left[\begin{array}{c|cc} 0 & 0 & I_p \end{array}\right], \\ F_y := F_sT_{z1}^{-1} = \left[\begin{array}{c|cc} 0 & I_r & 0 \end{array}\right] \end{array} \tag{7.34}$$

Notice that the special structures of C and F are not affected by the transformation. Now, system (7.34) is in the same form as (7.29) and the proof is complete. ■

In the coordinates of (7.29), the system states can be transformed and partitioned as follows:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ y \end{bmatrix} \begin{matrix} \updownarrow n-p-r \\ \updownarrow r \\ \updownarrow p \end{matrix} \tag{7.35}$$

From the structure of F in (7.29), $z = Fx$ will become $z = x_2$, where x_2 is the state to be estimated and x_1 is the 'unwanted' state, which will be modeled as unknown inputs/disturbances. The system equation (7.28) in the coordinates of (7.29) can therefore be re-written as follows as a system of order $\bar{n} = r + p$:

$$\underbrace{\begin{bmatrix} I_r & 0 \\ 0 & E_9 \end{bmatrix}}_{\bar{E}} \underbrace{\begin{bmatrix} \dot{x}_2 \\ \dot{y} \end{bmatrix}}_{\dot{\bar{x}}} = \underbrace{\begin{bmatrix} A_5 & A_6 \\ A_8 & A_9 \end{bmatrix}}_{\bar{A}} \underbrace{\begin{bmatrix} x_2 \\ y \end{bmatrix}}_{\bar{x}} + \underbrace{\begin{bmatrix} A_4 & D_2 \\ A_7 & D_3 \end{bmatrix}}_{\bar{D}} \underbrace{\begin{bmatrix} x_1 \\ f \end{bmatrix}}_{\bar{f}}$$

$$y = \underbrace{\begin{bmatrix} 0 & I_p \end{bmatrix}}_{\bar{C}} \underbrace{\begin{bmatrix} x_2 \\ y \end{bmatrix}}_{\bar{x}} \tag{7.36}$$

Notice that the system (7.36) is in the same form as in (7.1) and $\operatorname{rank}\begin{bmatrix} \bar{E} \\ \bar{C} \end{bmatrix} = \bar{n}$; therefore, the observer scheme in [112] (described in Section 7.2) and the existence conditions in [114] (described in Section 7.2.1) can be implemented on system (7.36) to estimate $\bar{x}$, and hence x_2.

7.3.1 Existence Conditions

By applying conditions Q1 - Q2 in Section 7.2, the observer by Yue & Kawaji in (7.4) - (7.5) can be used to estimate $\bar{x}$ if and only if system (7.36) satisfy the following conditions:

R1. $\operatorname{rank}\begin{bmatrix} \bar{E} & \bar{D} \\ \bar{C} & 0 \end{bmatrix} = \operatorname{rank}(\bar{D}) + \bar{n}$

R2. $\operatorname{rank}\begin{bmatrix} s\bar{E} - \bar{A} & \bar{D} \\ \bar{C} & 0 \end{bmatrix} = \operatorname{rank}(\bar{D}) + \bar{n}, \ \forall s \in \mathbb{C}, Re(s) \geq 0$

However, conditions R1 - R2 are in terms of the transformed system matrices, which required the designer to transform the original system matrices first before it can be tested.

Therefore, it is of interest to re-cast conditions R1 - R2 in terms of the original system matrices.

Theorem 7.1 *Conditions R1 and R2 can be re-casted in terms of the orginal system matrices as follows*

S1. $\operatorname{rank}\begin{bmatrix} E & EA & ED \\ C & 0 & 0 \\ 0 & CA & CD \\ 0 & C & 0 \\ 0 & F & 0 \end{bmatrix} = \operatorname{rank}\begin{bmatrix} FA & FD \\ CA & CD \\ C & 0 \\ F & 0 \end{bmatrix} + n$

S2. $\operatorname{rank}\begin{bmatrix} sF - FA & -FD \\ CA & CD \\ C & 0 \end{bmatrix} = \operatorname{rank}\begin{bmatrix} FA & FD \\ CA & CD \\ C & 0 \\ F & 0 \end{bmatrix}, \ \forall s \in \mathbb{C}, Re(s) \geq 0$

□

The remainder of this section provides a constructive proof of Theorem 7.1.

Lemma 7.6 *Define*

$$W_a = \begin{bmatrix} \bar{E} & \bar{D} \\ \bar{C} & 0 \end{bmatrix}, \ Y_a = \begin{bmatrix} E & EA & ED \\ C & 0 & 0 \\ 0 & CA & CD \\ 0 & C & 0 \\ 0 & F & 0 \end{bmatrix} \tag{7.37}$$

and the following is true:

$$\operatorname{rank}(Y_a) = \operatorname{rank}(W_a) + n \tag{7.38}$$

♯

Proof

Substitute the matrices from (7.29) into Y_a will result in

$$Y_a = \begin{bmatrix} I_{n-p-r} & 0 & 0 & A_1 & A_2 & A_3 & D_1 \\ 0 & I_r & 0 & A_4 & A_5 & A_6 & D_2 \\ 0 & 0 & E_9 & E_9A_7 & E_9A_8 & E_9A_9 & E_9D_3 \\ 0 & 0 & I_p & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & A_7 & A_8 & A_9 & D_3 \\ 0 & 0 & 0 & 0 & 0 & I_p & 0 \\ 0 & 0 & 0 & 0 & I_r & 0 & 0 \end{bmatrix} \tag{7.39}$$

Then, it is straightforwad to see that

$$\operatorname{rank}(Y_a) = \operatorname{rank}\underbrace{\begin{bmatrix} E_9A_7 & E_9D_3 \\ A_7 & D_3 \end{bmatrix}}_{Y_{a,1}} + n + p + r \tag{7.40}$$

Pre-multiply $Y_{a,1}$ with the a full-rank matrix yields

$$\begin{aligned} \operatorname{rank}(Y_{a,1}) &= \operatorname{rank}\begin{bmatrix} I_p & -E_9 \\ 0 & I_p \end{bmatrix}\begin{bmatrix} E_9A_7 & E_9D_3 \\ A_7 & D_3 \end{bmatrix} \\ &= \operatorname{rank}\begin{bmatrix} A_7 & D_3 \end{bmatrix} \end{aligned} \tag{7.41}$$

Hence,

$$\operatorname{rank}(Y_a) = \operatorname{rank}\begin{bmatrix} A_7 & D_3 \end{bmatrix} + n + p + r \tag{7.42}$$

Write W_a in terms of (7.36) to get

$$W_a = \begin{bmatrix} I_r & 0 & A_4 & D_2 \\ 0 & E_9 & A_7 & D_3 \\ 0 & I_p & 0 & 0 \end{bmatrix} \tag{7.43}$$

and it is obvious that

$$\operatorname{rank}(W_a) = \operatorname{rank}\begin{bmatrix} A_7 & D_3 \end{bmatrix} + p + r \tag{7.44}$$

Combining (7.42) and (7.44) results in $\operatorname{rank}(Y_a) = \operatorname{rank}(W_a) + n$ and the proof is complete. ■

Lemma 7.7 *Define* Θ *as*

$$\Theta = \begin{bmatrix} FA & FD \\ CA & CD \\ C & 0 \\ F & 0 \end{bmatrix} \tag{7.45}$$

and the following is true:

$$\operatorname{rank}(\Theta) = \operatorname{rank}(\bar{D}) = p + r. \tag{7.46}$$

♯

Proof

Substitute the matrices from (7.29) into Θ to get

$$\Theta = \begin{bmatrix} A_4 & A_5 & A_6 & D_2 \\ A_7 & A_8 & A_9 & D_3 \\ 0 & 0 & I_p & 0 \\ 0 & I_r & 0 & 0 \end{bmatrix} \tag{7.47}$$

and it is straightforward to see that

$$\operatorname{rank}(\Theta) = \operatorname{rank} \underbrace{\begin{bmatrix} A_4 & D_2 \\ A_7 & D_3 \end{bmatrix}}_{\bar{D}} + p + r \tag{7.48}$$

By using (7.36), $\operatorname{rank}(\Theta)$ is equivalent to $\operatorname{rank}(\bar{D}) + p + r$ and the proof is complete. ■

It is proven in Lemma 7.6 that the LHS of S1 is equivalent to $\operatorname{rank}(W_a) + n$ and Lemma 7.7 shows that the RHS of S1 becomes $\operatorname{rank}(\bar{D}) + n + p + r$. By combining both LHS and RHS of S1, it is clear that condition S1 becomes $\operatorname{rank}(W_a) = \operatorname{rank}(\bar{D}) + p + r$ which is in fact condition R1.

To prove the equivalence of conditions S2 and R2, define Δ_a and Δ_b as

$$\Delta_a = \begin{bmatrix} sF - FA & -FD \\ CA & CD \\ C & 0 \end{bmatrix}, \ \Delta_b = \begin{bmatrix} s\bar{E} - \bar{A} & \bar{D} \\ \bar{C} & 0 \end{bmatrix} \tag{7.49}$$

Lemma 7.8 *Matrices Δ_a and Δ_b from* (7.49) *have equal rank.*

Proof

In the coordinates of (7.29), Δ_a can be written as

$$\begin{aligned} \operatorname{rank}(\Delta_a) &= \operatorname{rank} \begin{bmatrix} -A_4 & sI_r - A_5 & -A_6 & -D_2 \\ A_7 & A_8 & A_9 & D_3 \\ 0 & 0 & I_p & 0 \end{bmatrix} \\ &= \operatorname{rank} \underbrace{\begin{bmatrix} -A_4 & sI_r - A_5 & -D_2 \\ A_7 & A_8 & D_3 \end{bmatrix}}_{\Delta_{a,1}} + p \end{aligned} \tag{7.50}$$

Then, write Δ_b in terms of (7.36) to get

$$\begin{aligned}\operatorname{rank}(\Delta_b) &= \operatorname{rank}\begin{bmatrix} sI_r - A_5 & -A_6 & A_4 & D_2 \\ -A_8 & sE_9 - A_9 & A_7 & D_3 \\ 0 & I_p & 0 & 0 \end{bmatrix} \\ &= \operatorname{rank}\underbrace{\begin{bmatrix} sI_r - A_5 & A_4 & D_2 \\ -A_8 & A_7 & D_3 \end{bmatrix}}_{\Delta_{b,1}} + p \end{aligned} \tag{7.51}$$

By comparing (7.50) and (7.51), it is clear that $\Delta_{b,1}$ is simply a re-arrangement of columns of $\Delta_{a,1}$. Hence, it is proven that Δ_a and Δ_b have equal rank. ■

It is proven in Lemma 7.7 that the RHS of S2 is equivalent to $\operatorname{rank}(\bar{D}) + p + r$, which is the RHS of R2. Then, the LHS of S2 and R2 are proven to have equal rank in Lemma 7.8. Thus, it is clear that conditions S2 and R2 are equivalent and the proof of Theorem 7.1 is complete.

7.4 Numerical Example

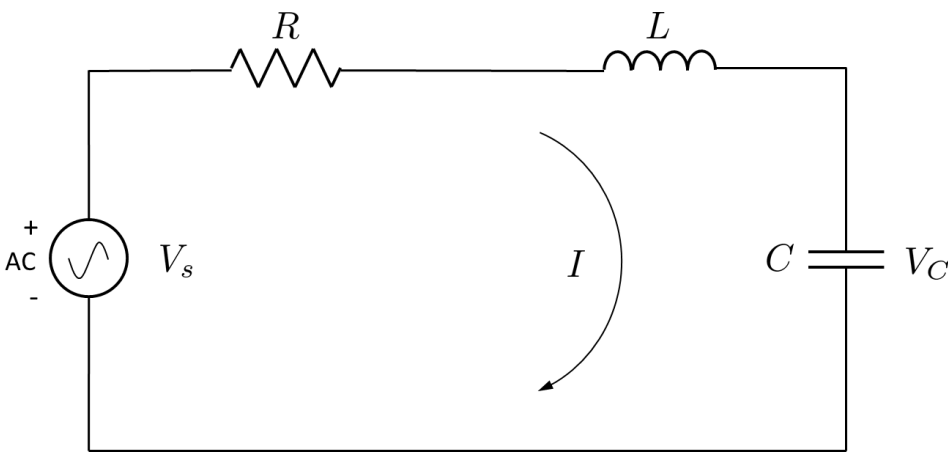

Figure 7.1: Schematic diagram of a single-loop circuit network.

The observer scheme proposed in this chapter will be illustrated with a single-loop circuit network [100] shown in Figure 7.1. By using basic circuit theory and Kirchoff's law, the system can be described by the following equations:

$$\begin{aligned} L\dot{I} = V_L,\ \dot{V}_C = \frac{1}{C}I,\ RI = V_R, \\ V_L + V_C + V_R = V_S \end{aligned} \tag{7.52}$$

where R, L and C are the resistance, inductance and capacitance, respectively, and their voltages are denoted by V_R, V_L and V_C respectively. V_S is the voltage source achieved from the control input, V_{in}. Assume that the voltage source is faulty and there are some delays between V_{in} and V_S, hence, the input equation can be written as

$$\dot{V}_S = \frac{1}{\tau}V_S + \frac{1}{\tau}V_{in} + f \tag{7.53}$$

where f encapsulates potential faults which will be modeled as an unknown input. In the coordinates of (7.28), the state equation can be written as

$$\underbrace{\begin{bmatrix} L & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{E} \underbrace{\begin{bmatrix} \dot{I} \\ \dot{V}_L \\ \dot{V}_C \\ \dot{V}_R \\ \dot{V}_S \end{bmatrix}}_{\dot{x}} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ \frac{1}{C} & 0 & 0 & 0 & 0 \\ -R & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & -1 \\ 0 & 1 & 0 & 0 & -\frac{1}{\tau} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} I \\ V_L \\ V_C \\ V_R \\ V_S \end{bmatrix}}_{x} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{\tau} \end{bmatrix}}_{B} V_{in} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{D} f \tag{7.54}$$

where $R = 50\Omega$, $L = 7H$, $C = 2F$ and $\tau = 5$. Assume that V_L and V_R are measurable, then $y = \begin{bmatrix} V_L \\ V_R \end{bmatrix}$ and matrix C becomes

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{7.55}$$

In this example, only I need to be estimated, where $z = I$. Therefore, matrix F can be defined as

$$F = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{7.56}$$

It can be easily verified that $\operatorname{rank}\begin{bmatrix} E & D \\ C & 0 \end{bmatrix} = 5$ and $n + q = 6$. Hence, conditions Q1 - Q2 are not satisfied and the observer scheme by [112] described in Section 7.2 cannot be used to estimate the entire system state. However, system (7.54) - (7.56) can satisfy condition S1, where $\operatorname{rank}(Y_a) = 9$ and $\operatorname{rank}(\Theta) + n = 9$. Therefore, the observer scheme proposed in this chapter can be used to accurately estimate I.

A suitable choice of T_y and T_z are

$$T_z = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad T_y = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0.143 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \tag{7.57}$$

and the structure of (7.29) is achieved as follows

$$E_y := T_y E T_z^{-1} = \left[\begin{array}{cc|c|cc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right], \quad A_y := T_y A T_z^{-1} \left[\begin{array}{cc|c|cc} 0 & 0 & 0.5 & 0 & 0 \\ 0 & -0.2 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0.143 & 0 \\ \hline 1 & -1 & 0 & 1 & 1 \\ 0 & 0 & -50 & 0 & 1 \end{array}\right],$$

$$B_y := T_y B = \left[\begin{array}{c} 0 \\ 0.2 \\ \hline 0 \\ \hline 0 \\ 0 \end{array}\right], \quad D_y := T_y D = \left[\begin{array}{c} 0 \\ 1 \\ \hline 0 \\ \hline 0 \\ 0 \end{array}\right], \qquad \begin{array}{l} C_y := C T_z^{-1} = \left[\begin{array}{cc|c|cc} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right], \\ F_y := C T_z^{-1} = \left[\begin{array}{cc|c|cc} 0 & 0 & 1 & 0 & 0 \end{array}\right] \end{array} \tag{7.58}$$

Then, re-arrange the system (7.58) into the form of (7.36) as follows

$$\underbrace{\left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]}_{\bar{E}} \underbrace{\begin{bmatrix} \dot{I} \\ \dot{V}_L \\ \dot{V}_R \end{bmatrix}}_{\dot{\bar{x}}} = \underbrace{\left[\begin{array}{c|cc} 0 & 0.143 & 0 \\ \hline 0 & 1 & 1 \\ -50 & 0 & 1 \end{array}\right]}_{\bar{A}} \underbrace{\begin{bmatrix} I \\ V_L \\ V_R \end{bmatrix}}_{\bar{x}} + \underbrace{\left[\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]}_{\bar{D}} \underbrace{\begin{bmatrix} V_C \\ V_S \\ f \end{bmatrix}}_{\bar{f}},$$

$$y = \underbrace{\left[\begin{array}{c|cc} 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]}_{\bar{C}} \underbrace{\begin{bmatrix} I \\ V_L \\ V_R \end{bmatrix}}_{\bar{x}} \tag{7.59}$$

From (7.59), it is obvious that $\operatorname{rank}\begin{bmatrix} \bar{E} & \bar{D} \\ \bar{C} & 0 \end{bmatrix} = 4$ and $\operatorname{rank}(\bar{D}) + \bar{n} = 4$. Therefore, condition R1 is satisfied and matrices $\bar{R}$ and $\bar{V}$ can be chosen as follows

$$\bar{R} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \bar{V} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{7.60}$$

such that $\bar{R}\bar{E} + \bar{V}\bar{C} = I_{\bar{n}}$. Using the design procedure in section 3.2, L can be selected as

$$L = \begin{bmatrix} 0 & 0.4 \end{bmatrix} \tag{7.61}$$

Then, P_o and A_4^s are chosen to be I_2 and $diag\{-18, -19\}$. Then, the gain matrices, G_n and G_l in the same coordinates of (7.59) are

$$G_n = \begin{bmatrix} 0 & -0.4 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad G_l = \begin{bmatrix} 0.143 & -7.6 \\ 19 & 1 \\ 0 & 40 \end{bmatrix} \tag{7.62}$$

The simulation results are shown in Figures 7.2 - 7.4. Then inital conditions of the states x were assumed to be $\begin{bmatrix} 2 & 0 & 3 & 0 & 0 \end{bmatrix}^T$ whilst the observer had zero intial conditions. The control input V_{in} and the unknown input f were set to be sinusoidal waveforms, where $f = 0.81\sin(\frac{\pi}{2}t - 0.25)$ and $V_{in} = 5\sin(\pi t)$. The ρ and δ in (3.21) were selected as 30 and 1×10^{-3}. Figure 7.2 shows the plot of the output estimation error, where sliding motion occurs at $0.25s$. Then, Figure 7.3 shows that the estimated functional state $\hat{I}$ can accurately tracks the actual I and Figure 7.4 is the zoomed-in version of Figures 7.3.

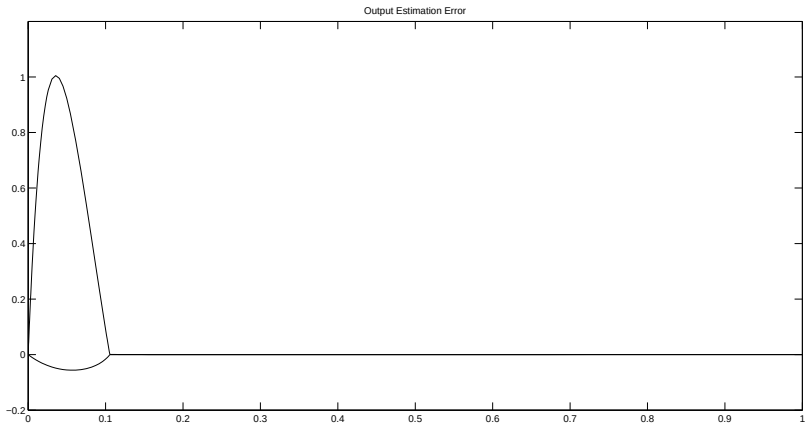

Figure 7.2: Output estimation error $e_y = y - \hat{y}$. All components of e_y are driven to zero at around $0.25s$

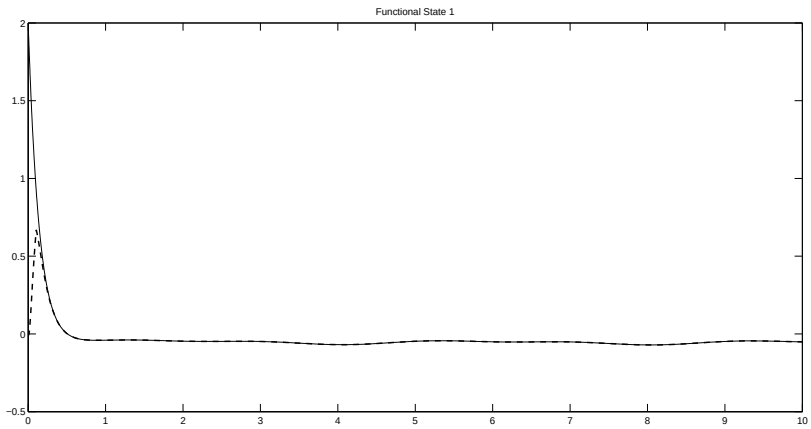

Figure 7.3: The estimated functional state, $\hat{I}$ (dashed) follows the actual I (solid)

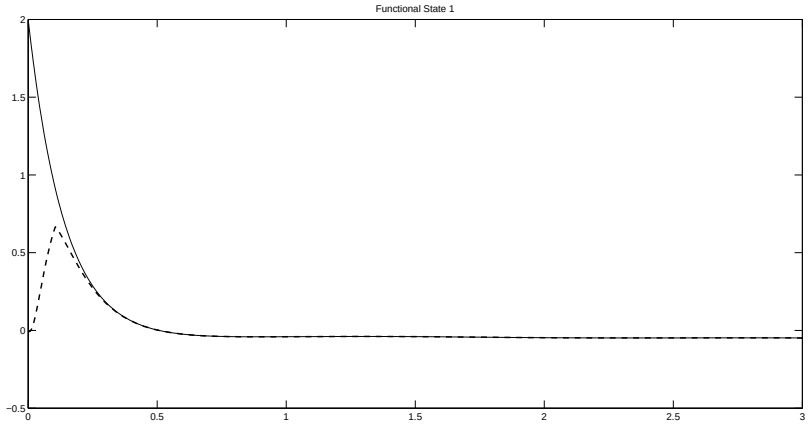

Figure 7.4: First 3 seconds of Figure 7.3. The estimated $\hat{I}$ accurately tracks the actual I in less than $0.5s$.

7.5 Conclusion

This chapter has discussed the problem of functional state estimation for a class of descriptor system using sliding mode observers, as well as the investigation of the necessary and sufficient conditions in terms of the original system matrices. First, the original system was recasted into a form such that the observer by Yue & Kawaji [112] could be implemented directly to estimate the functional state. Then, by using the method in [112], the system and the observer can be placed into a form that is similar to the Edwards-Spurgeon observer, and design techniques by Edwards & Spurgeon can be used to design the observer's parameter. Following that, the existence conditions were investigated. Finally, simulation results of a simple circuit network were used to show the effectiveness of the scheme, where compared to the work in [114], functional state estimation can be used to estimate only the required state, when full-state estimation cannot be achieved.

Chapter 8

Conclusion and Recommendations for Future Research

This thesis has presented the advancements on functional state estimation using sliding mode observers.

Chapter 2 has provided an introduction to the developments of observers and its use for robust state estimation. Starting from the one introduced by Luenberger, observer theory was developed to enable state estimation despite the presence of unknown inputs, most often using the unknown-input observer (UIO). Then, it was found that functional observers can be used to eliminate the redundancy possess in full-state estimation. The sliding mode observer was also described in this chapter. From earlier works which found that sliding mode observer has better performance than linear UIOs, this thesis seeks to combine both theories of functional estimation and sliding mode observer theories to reduce the observer complexity as well as improve the overall performance of the observer.

Chapter 3 has described the sliding mode observer scheme by Edwards & Spurgeon which is the observer used in this thesis. In this chapter, a systematic design method of the observer scheme as well as the necessary and sufficient conditions for the observer to be feasible were presented. In addition, it was also demonstrated how the observer could be used to estimate the unknown input. Then, it was shown in this chapter how one of the existence conditions (known as the matching condition) could be relaxed by placing observers in cascade. Finally, the application of the sliding mode observer for functional state estimation was discussed and the necessary and sufficient conditions for the observer scheme were presented in terms of the original system matrices.

Chapter 4 has proposed a new scheme in functional state estimation, using two cascaded sliding mode observers. The original system matrices were first transformed into a form

such that the existing observer scheme can be used to estimate the functional state. After that, the necessary and sufficient conditions in terms of the original system matrices were investigated and were found to be weaker compared to the existing sliding mode functional observer described in the previous chapter, which could mean that functional state estimation can be achieved with fewer sensors, hence reducing complexity and saving cost. Apart from that, the existence conditions were presented in terms of the original system matrices, which will allow the user to determine whether or not the scheme is applicable at the outset of the design; this also provides a basis for comparison for future research in functional state estimation.

Chapter 5 presented a functional observer scheme using multiple sliding mode observers in cascade, which generalizes the observer scheme discussed in the previous chapters. The main contribution is the investigation of the necessary and sufficient conditions in terms of the original system matrices. By using these conditions, a suitable algorithm can be formulated to calculate the number of observers necessary and sufficient to achieve functional state estimation. At the same time, it was proven that the conditions in this chapter are less stringent compare to the observer scheme presented in the previous chapters. The physical implication of less stringent existence conditions is that less output measurement is required, thus reducing the number of sensors needed and saving cost.

Chapter 6 has proposed a new scheme in functional state estimation. The functional state is modeled as an unknown input and estimated by exploiting the ability of Edwards-Spurgeon observer in estimating unknown input from the discontinuous output injection error. Following that, the necessary and sufficient conditions in terms of the original system matrices were investigated to ensure all functional state components can be accurately estimated. Finally, it was shown in the simulation that functional state estimation could be achieved using less observers compare to the work in earlier parts of this thesis. Therefore, the functional estimation scheme proposed in this chapter can be a good alternative to conventional estimation methods.

Chapter 7 has considered the problem of functional state estimation for descriptor system using sliding mode observers. The main contribution in this chapter is the reformulation

of a descriptor system such that functional state estimation concepts can be applied to it, followed by the investigation of the necessary and sufficient conditions in terms of the original system matrices. First, the original system matrices were re-casted into a form such that accurate functional state estimation can be achieved by direct application of the Yue & Kawaji observer [112]. After that, the system and observer can be placed into a form that is similar to the Edwards-Spurgeon observer by using the method in [112]. Following that the existence conditions in terms of the original system matrices were investigated. Finally, it was shown in the simulation results that functional state estimation can be used to estimate only the required state, when the full-state estimation cannot be achieved.

8.1 Recommendations for Future Work

The work presented in this thesis can be further improved to enhance the research results in the area of functional state estimation. This section will present some potential future developments that can be done based on the outcomes of this thesis.

- The sliding mode functional observer schemes presented in this thesis only focused on matched unknown inputs. Under the presence of unmatched unknown input, the observer will not give an accurate estimate for the 'non-output' states. Therefore, further research can be done in this area. Tan & Edwards [89] have developed a scheme of minimizing the $\mathcal{H}_\infty$ norm from the unmatched unknown input to the estimate of the matched unknown input. It would be of interest to modify this scheme for the case of robust state estimation, where the $\mathcal{H}_\infty$ norm from the unmatched unknown input to the state estimate is minimized, especially for the cases of functional state estimation, and also for multiple observers.

- In Chapter 6, a new paradigm of functional state estimation was presented, where the functional state was estimated as an 'unknown input'; an interesting alternative to the conventional observer schemes. These schemes could be further relaxed by using two or more observers in cascade. In addition, the case of unmatched

unknown input can also be explored.

- Then in Chapter 7, the problem of functional state estimation for a class of descriptor system was discussed. However, it is assumed in this chapter that $\mathrm{rank}\begin{bmatrix} E \\ C \end{bmatrix} = n$. Further research can be done to develop observer schemes for different class of descriptor system such as $\mathrm{rank}\begin{bmatrix} E \\ C \end{bmatrix} < n$. The two-observer scheme in Chapter 4 and multiple observer scheme in Chapter 5 can be applied to descriptor system to further relax its existence conditions. In addition to that, the concepts in Chapter 6 can be used on descriptor system as well.

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