The Effectiveness of Mathematics in Physics

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Abstract

In this thesis I argue that many problems in the philosophy of science and mathematics (in particular, the unreasonable effectiveness of mathematics in physics) can only be addressed within a broader metaphysical framework which provides a coherent world view. I attempt to develop such a framework and draw out its consequences. The attempt is in two parts: first, I develop a speculative framework based on an analogy to set theory; then I combine elements of the framework with ideas from Leibnizian monadology and consistent histories quantum theory to introduce (what I call) quantum monadology. The two parts focus on different aspects of the problem and should be viewed as stages on the way to a final formulation. The inspiration for the thesis came from Plato's *Timaeus* and Wigner's comments on quantum mechanics. As it turned out, Leibniz's *Monadology* became a third key source.

Declaration

This thesis contains no material which has been accepted for the award of any other degree or diploma at any university or equivalent institution and that, to the best of my knowledge and belief, this thesis contains no material previously published or written by another person, except where due reference is made in the text of the thesis.

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Any and all mistakes or omissions are mine alone.

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Don’t be surprised then, Socrates, if it turns out repeatedly that we won’t be able to produce accounts on a great many subjects — on gods or the coming to be of the universe — that are completely and perfectly consistent and accurate. Instead, if we can come up with accounts no less likely than any, we ought to be content, keeping in mind that both I, the speaker, and you, the judges, are only human.

Plato, *Timaeus* 29c

Chapter 1. Introduction

In his article ‘The Unreasonable Effectiveness of Mathematics in the Natural Sciences’, the physicist Eugene Wigner asks "What is mathematics?", "What is physics?", “How does mathematics enter physical theories?” and “Why is mathematics so unreasonably effective in physics?” These are the key questions addressed in my thesis.

Depending on which philosophical school one adheres to, mathematics might be a form of logic (logicism), a construction of the human mind (intuitionism), a game played with symbols (formalism), a language describing the properties of real, abstract entities (platonism), or a language describing fictional entities (fictionalism). One thing which all schools have to explain is the effectiveness of mathematics in physics. It is a striking feature of mathematics that it allows us to model and predict the behaviour of physical systems to an amazing degree of accuracy. For example, the predicted magnetic moment of the electron agrees with the current experimental value to an accuracy of one part in a trillion.

One of the oldest explanations of the effective of mathematics in physics is that, in some profound way, the structure of the world is mathematical. The Pythagoreans believed that “everything is number”. If we interpret this explanation as saying that mathematical structure is the first principle of the universe, then it is still current. Theoretical physicist Max Tegmark claims that our external physical reality is a mathematical structure. In part, his reasoning is based on the trajectory of modern fundamental physics as it becomes increasingly dependent on mathematics and divorced from an ordinary understanding of things like space, time and matter.

The explanation given in my thesis is not too far from the Pythagorean one. However, I add a second principle of the universe: mind. I defend the proposition that mind and mathematical structure are the grounds of reality. The argument takes into account recent developments in mathematics and physics, as well as centuries of debate in philosophy on issues to do with the problems of existence.

The layout of the thesis is as follows. Chapter 2 is an examination of Wigner’s paper on the unreasonable effectiveness of mathematics in the natural sciences. Chapter 3 is the story of the success of Pythagorean metaphysics in developing the standard models of physics. Chapter 4 focuses on set theory. It investigates whether it is feasible to hold a Pythagorean view in the wake of Gödel's Incompleteness Theorems and the threat of a mathematical multiverse. In Chapter 5 I argue that philosophers should not abandon the quest of integrating mathematics and physics into a coherent world view. I venture to embed mathematics into a broader metaphysical framework and to draw out the consequences for the philosophy of mathematics. In Chapter 6 I combine the elements of the framework with ideas from Leibnizian monadology and consistent histories quantum theory in order to explore the relationship between mathematics (potentiality) and physics (actuality). The result is what I call 'quantum monadology'.

Chapter 2. The Applicability of Mathematics

2.1 Overview

Since its publication in 1960, Wigner's paper 'The Unreasonable Effectiveness of Mathematics in the Natural Sciences' has attracted comment from scientists, applied mathematicians and philosophers keen to give their take on what Wigner calls "the miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics" [Wigner, 1960: 14]. There is much disagreement concerning both the nature of the miracle and, more fundamentally, whether there *is* a miracle, or, indeed, anything mysterious at all. Theoretical physicists such as David Gross tend to agree that it is "something of a miracle that we are able to devise theories that allow us to make incredibly precise predictions regarding physical phenomena" [Gross, 1988: 8372]. On the other hand, applied mathematicians such as Jack Schwartz speak of "the pernicious influence of mathematics on science" [Schwartz, 2006: 231] and emphasise how tough it can be to find mathematical solutions for real world problems. Biologists, economists and social scientists[1] are of a similar mind to Schwartz and tend to see mathematics as a sometimes useful tool. Needless to say, philosophers are divided in their opinion. Ernst Nagel thinks that:

> It is no mystery … that pure mathematics can so often be applied… because the symbolic structures it studies are all suggested by the natural structures discovered in the flux of things. [Nagel, 1979: 194].

On the other hand, Mark Steiner thinks that there is a mystery to be explained and that it concerns "the relation between Mind and the Cosmos" [Steiner, 1995: 152].

Part of the confusion is caused by individuals' different philosophies of mathematics and their different accounts of the applicability of mathematics. My main purpose in this chapter is to separate the general proposition:

> P1: That there is something mysterious, or *unreasonable*, about the fact that mathematics is an effective tool for physical applications;

from another, related, but more specific proposition:

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[1] See e.g. [Lesk, 2000] and [Velupillai, 2005].

> P2: That mathematics is *unreasonably effective* in certain applications; specifically, in applications at the cutting-edge of fundamental physics.

Because of Wigner's exposition it is not entirely clear what proposition he is arguing for and he tends to conflate the issues. I shall argue that the second proposition is genuinely mysterious, whatever one's philosophy of mathematics. Furthermore, I shall argue that the mystery can only be explained by some fundamental relationship between mathematics and reality. In later chapters, I shall elaborate on what I think that relationship is. Portentously, I see the unreasonable effectiveness of mathematics in fundamental physics as being the key to the nature of reality itself.

In Section 2.2 I give an extended discussion of Wigner's paper. To sort out the issues I spend some time discussing different philosophies of mathematics and their approach to the question of the applicability of mathematics. Section 2.2.1 focuses on Wigner's philosophy of mathematics as revealed in his paper. Wigner thinks that modern mathematics has outgrown its empirical roots and become like an art form, or intellectual game. Section 2.2.2 examines how this influences his view of the role of mathematics in physics. In Section 2.2.3 it is argued that mathematics never escapes its empirical roots. There are many ways in which physical intuition interacts with mathematical intuition and, whilst mathematicians may enjoy flights of abstraction, they are always pulled back towards physical reality. I discuss the second of Wigner's two main points about scientific theories (which is often neglected in the excitement over his first point about their "unreasonable effectiveness"):

> ... we cannot know whether a theory formulated in terms of mathematical concepts is uniquely appropriate. We are in a position similar to that of a man who was provided with a bunch of keys and who, having to open several doors in succession, always hit on the right key on the first or second trial. He became skeptical concerning the uniqueness of the coordination between keys and doors. [Wigner, 1960: 2]

The point here is that, if we don't really know why our theories work so well, then there is nothing to support the faith of theoretical physicists in their search for a unique, coherent, reductionist picture of the world, as encapsulated in the grandiose (albeit tongue-in-cheek) moniker of a Theory of Everything (or, TOE). Rather than there being a unique TOE, we might very possibly end up with empirically excellent theories in different domains which have nothing in common with each other, or which actually contradict each other. Indeed, this would seem to be quite a likely outcome if we adopt an instrumentalist view of science (i.e. if we think that there is no deep reason why our theories work so well; that they do not tell us anything fundamental about the nature of reality; that they merely act as instruments to enable us to organise sets of phenomena and make predictions from them). The historical view of science leads us to hope that any such impasse in scientific theory-making would act as a catalyst in the formation of an entirely new theory which would reconcile and subsume the incompatible theories by ascending to a higher-level perspective. But Wigner is right to point out that this is more an “article of faith” [Wigner, 1960: 10] than a rational, inductive argument.

In Section 2.3 I examine the rationale underlying physicist’s search for the final theory. What properties would the final theory have? How would we know when we had found it? What do physicists’ mean when they say that a theory is beautiful and that this is a guide to truth? Doubtless, some form of Pythagorean metaphysics inspired many great physicists and led to heuristics which proved very successful in penetrating the depths of quantum reality, but to what extent do the successes support the motivating metaphysics? I defend the Pythagorean view in fundamental physics against supposed failures which I argue are due to misunderstandings about its proper scope. To a certain extent, simplicity is always a goal and guide in the development of physical theories. In Section 2.3.5 I question whether this is because reality is really describable in terms of simple mathematics or because human minds are limited in their understanding (i.e. we only understand theories if they are sufficiently simple). I touch on the views of various physicists and mathematicians regarding the ultimate nature of reality and the ability of the human mind to comprehend it. In Section 2.3.6 I outline a consistent Pythagorean metaphysics which I shall be expounding over the course of the thesis.

In Section 2.4 I return to Wigner’s paper and examine various claims that arise from it in the light of previous discussion. I question what, if anything, is mysterious about the applicability of mathematics and in what sense mathematics can be held to be unreasonably effective in the natural sciences. I conclude, with Wigner, that there are mysteries to do with the existence of laws of nature and the ability of the human mind to understand them. The key to these mysteries is the unreasonable effectiveness of mathematics in applications at the cutting-edge of fundamental physics. This sets the stage for the rest of my thesis.

2.2 The Unreasonable Effectiveness of Mathematics in the Natural Sciences

2.2.1 Wigner's philosophy of mathematics

Wigner contributes to the confusion surrounding his article by conflating the distinct propositions:

> P1: That there is something mysterious, or *unreasonable*, about the fact that mathematics is an effective tool for physical applications; and
>
> P2: That mathematics is *unreasonably* effective in certain applications; specifically, in applications at the cutting-edge of fundamental physics.

From the article, it appears that he holds both propositions, although the bulk of the detailed argument is directed towards establishing the narrower Proposition 2. In regards to Proposition 1, he makes the claim that "the enormous usefulness of mathematics in the natural sciences is something bordering on the mysterious and that there is no rational explanation for it" [Wigner, 1960: 2]. His reasons for making this claim are grounded in his philosophy of mathematics which is revealed in his answer to the question 'What is Mathematics?' [Wigner, 1960: 2-3]. According to Wigner, "mathematics is the science of skillful operations with concepts and rules invented just for this purpose" [Wigner, 1960: 2]. He concedes that the origins of mathematics (e.g. elementary geometry) lie in concepts abstracted from entities in the real world, but thinks that modern mathematicians have exhausted the interesting theorems derivable in terms of those concepts and simply invent new ones so as to have a more interesting game to play, or, as Wigner puts it, so that they can "demonstrate [their] ingenuity and sense of formal beauty" [Wigner, 1960: 2]. The picture of mathematics which emerges is of a competitive game played by highly intelligent people seeking to impress one another and satisfy their aesthetic sense. The concepts and rules of the game are freely invented subject to internal aesthetic criteria (i.e. internal to the community of mathematicians). The criteria are not well-defined and evolve culturally in particular areas of specialisation. Loosely, mathematicians may be described as striving to obtain surprising results of great generality by using ingenious, logical, deductive arguments with a few simple premises based on well-defined concepts.

Wigner's philosophy of mathematics is related to *formalism*. Formalism views mathematics as a combinatorial game played with meaningless symbols following the rules of "proof" [Shapiro, 2000: 144-148]. Under this view, applications of mathematics can seem mysterious because the game has no content and, hence, cannot tell us true things about the world, except by coincidence. If the game's axioms happened to be true for some structures in the real world then the logical consequences of those axioms would also be true for those structures and, potentially, applicable; but that would be rather like the game of chess turning out to have real world applications (as noted by Frege in his attack on formalism [Dummett, 1991: 254]).

Probably Wigner does not hold a strict formalist view. His view may be closer to *mathematical pluralism*, as expounded by Priest [2013]. Priest argues that the pluralism evident in mathematical practice is directly analogous to the plurality of games and that mathematics is like a family of games. However, he differs from formalism in his assertion that mathematics has content. He writes:

> When one learns a branch of mathematics initially, one may be doing little more than operating on symbols according to rules; but the phenomenology of a fully fledged mathematical practice is exactly one of acquaintance with the objects that the symbols (noun phrases) refer to. [Priest, 2013: 7].

Priest is not dogmatic about the source of mathematical content provided it is accepted that all mathematical games are equally valid. He discusses several possible interpretations of the objects referred to by mathematical theories, as outlined in the following paragraphs.

It could be that the objects referred to by mathematical theories exist independently of the human mind. This is the view of 'really full-blooded platonism' [Beall, 1999]. In regards to applications, this view shares the problem of all forms of platonism; i.e., if mathematical objects exist in an abstract world, outside of space and time, then how do we explain their relationship to the physical world? One possible answer is that the world is intrinsically mathematical, but that would need to be established. Furthermore, an account would have to be given of how we come to know about these non-causal, non-spatio-temporal objects. Neither challenge has been convincingly answered by traditional platonists.

It could be that the existence and meaning of mathematical objects is determined by convention, as suggested by Carnap [1950]. According to conventionalism, mathematicians set up theoretical frameworks within which they agree on what objects exist and what their properties are. Outside a framework, questions of existence are meaningless. If scientists find these frameworks useful then they are welcome to incorporate them into their theories of the world, just as they might use any other tool. This view naturally leads to a form of instrumentalism about scientific applications.

It could be that mathematical objects are like fictional objects. Field [1989] advocates a philosophy of fictionalism about mathematics which holds that mathematical statements are literally false, but are 'true-in-a-story'; much in the same way that the statement "Sherlock Holmes lived on Baker Street" is literally false, but is true in the context of Conan-Doyle's detective story. Field is a nominalist who does not accept the existence of mathematical objects. He argues that mathematical existence claims do not have to be true in order for mathematics to be useful in scientific applications. The only requirement is that mathematics be *conservative*; i.e., "any inference from nominalistic premises to a nominalistic conclusion that can be made with the help of mathematics could be made without it" [Field, 1980: x]. He sees the role of mathematics in science as being that of a useful deductive tool which preserves truth among nominalistically-stated claims, and which aids clear thinking and shortens derivations. In his view, mathematics does not lead to genuinely new conclusions and so mathematical objects are, in principle, dispenable.

Wigner's view of mathematics might not be as all-inclusive as that of Priest (e.g. it might not stretch to paraconsistent logic) but the general picture of mathematics as a family of games with meaningful content seems to fit his comments. Wigner speaks of people "inventing" mathematical concepts, so it is fair to assume that he thinks of mathematical objects as being a construction of the human mind. This makes him a type of constructivist. In contrast, platonists think of mathematical objects as existing independently of the human mind, so a platonist would speak of people "discovering" mathematical concepts. Both groups acknowledge the role of mathematics as a human cultural product but platonists, in addition, think that there is an objective mathematical realm — a true Mathematics, so to speak — which is the ultimate measure of the truth of human mathematics. Thus, for a platonist, human mathematics could be wrong in ways which would not be acknowledged by constructivists (for whom human mathematics is the only measure). This distinction will become important in Chapters 4 and 5 of my thesis.

Consistent with the interpretation of Wigner as a constructivist, he relates an apocryphal story at the beginning of his article in which he expresses sympathy with a classmate's puzzlement over the fact that the number pi, which was defined by Ancient Greek mathematicians in their investigations of elementary geometry, could have anything to do with the statistics of human population trends. For constructivists, the question arises as to how something which we invented and pursued for our own subjective purposes (e.g. aesthetic pleasure or intellectual display) can have any predictive power in scientific theories which purport to describe a mind-independent world (i.e. the typical assumption of scientific realism). So it is reasonable to suppose that at least part of the mystery which Wigner finds in the applicability of mathematics derives from his constructivist philosophy. However, I will argue that he finds that mathematics is unreasonably effective in fundamental physics for reasons which transcend any particular philosophy.

2.2.2 The role of mathematics in physics

Unlike Field, Wigner sees the role of mathematics in fundamental physics as going beyond that of a tool. He writes:

> … the role of evaluating the consequences of already established theories is not the most important role of mathematics in physics… Mathematics does play … a more sovereign role. [Wigner, 1960: 6].

He goes on to describe the role of mathematics in formulating the laws of nature. He explicitly mentions the axioms of quantum mechanics which involve state vectors in a complex Hilbert space[2]:

> … the use of complex numbers is in this case not a calculational trick of applied mathematics but comes close to being a necessity in the formulation of the laws of quantum mechanics. [Wigner, 1960: 7].

His point is that physicists *think* in mathematical terms. It often happens in the course of their work that they discover physical truths that require innovative mathematical concepts for their formulation and then find, to their surprise, that those very same concepts have already been discovered by mathematicians (indeed, it will often require a "bridging" person of very special talents to recognise in the physicists' work an application of esoteric mathematical concepts expressed in a totally different language). Physicists do not just pick useful concepts out of the mathematician's expansive toolbox and apply them; in the way a biologist, economist or social scientist would do. It seems that the laws of nature are written in the language of mathematics and that mathematics is essential for their formulation (as Galileo suggested) [Wigner, 1960: 6]. As partial explanation, Wigner cites Einstein's view that physicists seek beautiful theories and, so, are driven by aesthetic criteria similar to those employed by mathematicians [Wigner, 1960: 7]. However, he emphasises that this in no way explains the amazing empirical accuracy of physical theories.

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[2] It is not clear whether or not quantum mechanics can be explained in nominalistic terms as required by Field: Malament [1982] suggests not, but Balaguer [1996] makes an attempt.

Wigner thinks that the degree of accuracy of the predictions of our physical theories indicates that mathematics really is the correct language to use in describing laws of nature. It is evidence that physicists, in constructing theories, are not just using mathematics because they have been trained to do so and are primed to spot mathematical analogies in experimental data. If that were the case, the situation would be more like that of a biologist who spots a linear trend between two correlates: the trend might be a useful predictive tool but could hardly be expected to be accurate, and could sometimes be quite misleading, depending on the underlying explanation for the trend. The mathematical relationships discovered by physicists are remarkably accurate and earn the status of "laws of nature" through their (as far as we know) universal application. Even when superseded, as Newtonian gravity was superseded by Einstein's General Relativity, they remain applicable (and applied) in their original domain. Wigner gives an example from quantum electrodynamics (i.e., the prediction of the Lamb shift), quoting an agreement with experiment better than one part in a thousand [Wigner, 1960: 10]. Since Wigner's time that theory has been subject to much greater experimental scrutiny and it has passed every test with flying colours. It predicts the magnetic moment of the electron to an accuracy of one part in a trillion [Gabrielse, 2013: 64].

In *Principia Mathematica*, Newton was able to express his laws in terms of elementary geometry even though he had had to invent calculus in order to discover them. The mathematics required for quantum field theory (e.g. Hilbert spaces and non-Abelian gauge groups) is far beyond that level and has no obvious connection with experience. The mathematics used in String Theory (e.g. Calabi-Yau manifolds and specialised algebraic geometry) is beyond all but the most mathematically able contemporary physicists. It is a mystery why such deep mathematics is needed to formulate fundamental physics. Priest says:

> There would seem to be no *a priori* reason why such parts of mathematics sometimes find application. Perhaps we just have to accept this as a contingent feature of the world in which we live. [Priest, 2013: 11].

The sense of mystery is exacerbated by the way in which esoteric mathematics is developed. Von Neumann describes the salient features of the process and its concomitant dangers:

> As a mathematical discipline travels far from its empirical source, or still more, if it is a second and third generation only indirectly inspired by ideas coming from "reality," it is beset with very grave dangers. It becomes more and more purely aestheticising, more and more purely l'art

pour l'art. This need not be bad, if the field is surrounded by correlated subjects, which still have closer empirical connections, or if the discipline is under the influence of men with an exceptionally well-developed taste. But there is a grave danger that the subject will develop along the line of least resistance, that the stream, so far from its source, will separate into a multitude of insignificant branches, and that the discipline will become a disorganised mass of details and complexities. In other words, at a great distance from its empirical source, or after much "abstract" inbreeding, a mathematical subject is in danger of degeneration. At the inception the style is usually classical; when it shows signs of becoming baroque, then the danger signal is up. [Von Neumann, 1947: 196].

Although the danger which Von Neumann presages is real, it is a remarkable historical fact that the results of mathematical inbreeding have often found important applications years after their invention. It would be a bold person who would be prepared to write off any mathematical theory as inapplicable. It is a mystery as to why that should be so.

2.2.3 The empirical foundations of mathematics and physics

A common response to Wigner's mystery is to question whether the mathematics which finds application in physical theories is ever really of the "baroque" variety described by Von Neumann. Its development might be more in line with the following quote from Courant:

> The flight into abstraction must be something more than a mere escape; start from the ground and re-entry are both indispensable, even if the same pilot cannot always handle all phases of the trajectory. The substance of the purest mathematical enterprise may often be provided by tangible physical reality. [Courant, 1964: 48].

In other words, mathematics never really escapes from its grounding in empirical concepts. This seems to be the substance of Nagel's response (as quoted in Section 2.1). It is also part of the responses given by Azzouni [2000] and Grattan-Guiness [2008].

Azzouni argues that mathematical concepts are often implicationally connected to other mathematical concepts which are not remote from physical applications; and that implicational opacity in respect to pure mathematics prevents us from seeing this clearly. There are also strong cultural links between mathematics and physics. They have an overlapping history and are undertaken by similar types of people with similar types of training and plenty of opportunities to interact and exchange ideas.

Grattan-Guiness points out that there are strong methodological links between mathematics and physics. Mathematicians and physicists build their theories in a similar way; using intuition, conjecture and verification. Much of their creative activity involves making analogies to established concepts. In doing this, they draw on a ubiquitous stock of common notions such as linearity, symmetry, and periodicity [Grattan-Guiness, 2008: 10]. Some mathematical theories draw on *direct* analogies to systems in the real world (e.g. knot theory and real-world knots) but, even in more apparently baroque cases, where the links are *indirect*, the intuition is ultimately grounded in such common notions. Thus, one might justifiably be suspicious about claims that advanced mathematical concepts were invented independently of influences arising from the contemplation of the physical world.

One suspects that a mathematical subject cannot become too baroque lest it lose its interest to students and dies off. It will always be pulled back to the real world to be reinvigorated with new concepts. Indeed, following his earlier quote, Von Neumann goes on to say "[W]henever this stage is reached, the only remedy seems … to be a rejuvenating return to the source: the reinjection of more or less directly empirical ideas" [von Neumann, 1947: 196]. A recent example of this is the rejuvenation of geometry with ideas drawn from the physics of String Theory (e.g., Witten's representation of the Jones invariants of knots using Chern-Simons field theory, for which he won the 1990 Fields Medal). Also, Dyson [1972] documents many historical examples of fruitful interaction between mathematicians and physicists and speculates on some missed opportunities.

It is interesting to consider whether physics could ever be in danger of becoming baroque in a sense analogous to that used by Von Neumann. On the common view of scientific practice this would be impossible since scientific theorising is supposedly driven by experimental data. However, fundamental physics is currently at a stage in which its two most successful theories, General Relativity and the Standard Model of Particle Physics, whilst demonstrating remarkable predictive accuracy in their respective domains, nevertheless have manifest inadequacies, and there is insufficient experimental data to indicate how they should be transcended. The main problem is that the formulation of General Relativity is incompatible with quantum mechanics so the two theories cannot easily be unified. General Relativity breaks down at very small length scales (i.e. in the order of the Planck length of 10^{-35}m) beyond the reach of standard experiments. Different cosmological models derived from General Relativity have been tested using data from satellites and from astronomical observations, but this has not required changes to the fundamental equations of the theory. Similarly, the Standard Model has been tested up to energy levels of 4TeV per beam at the Large Hadron Collider and has passed every test without the need for new physics.

Physicists are not short of ideas for how their subject might progress. A number of different avenues are being investigated. There are fundamentally new theories such as String Theory, supersymmetry, and loop quantum gravity. There are bottom-up models based on particular assumptions which extend known theories in controlled ways and target potential outcomes of planned experiments. There are computer experiments which have become more and more sophisticated and probing. Many imaginative new concepts are being developed in thought experiments. In String Theory, as evidenced by Witten's receipt of the Fields Medal, these new concepts verge on pure mathematics. Notwithstanding these efforts, it is a fact that real experimental data are needed to show the way forward and, in their absence, physics is beginning to reveal baroque tendencies (witness the detailed discussions of different levels of infinite multiverses; or talk of a mirror universe of dark matter and dark forces; or Boltzmann brains materialising out of the quantum vacuum).

Most likely, a breakthrough will be achieved in the near future and angst over the state of physics will diminish for a while. However, there remains the question of the future evolution of scientific theories. In the last section of his paper, Wigner speculates on a situation in which "some of the laws of nature will be in conflict with each other in their implications, but each convincing enough in its own domain so that we may not be willing to abandon any of them" [Wigner, 1960: 11]. In his scenario, such a situation arises because we develop a growing number of theories which are "proved" in their domain of phenomena but which are ultimately incompatible. A more encompassing theory might exist which would resolve their conflicts but it is beyond our means of discovery. Alternatively, there may be no such encompassing theory. Since we don't really understand why our theories work so well there is no reason to expect them to be sign-posts on the way to a coherent, unified understanding of reality. This is a nightmare scenario for a theoretical physicist. In the next section, I shall defend an alternative view — the Pythagorean view — which holds that mathematical intuition can guide us towards a deep understanding of our fundamental physical theories.

2.3 The Pythagorean View

2.3.1 Truth and beauty

Traditionally, many great theoretical physicists have been driven in their work by an underlying belief in the rationality of the universe and the ability of the human mind to comprehend it. Beyond this, to various degrees, they have believed in the existence of an underlying mathematical explanation of reality: an explanation of great simplicity and beauty and sense of inevitability. Weinberg [1994] speaks of the beauty of present theories as being an anticipation, a premonition, of the beauty of the final theory. In a much quoted sentence, he expresses his dream:

> We are discovering laws that are becoming increasingly coherent and universal, and beginning to suspect that this is not merely an accident, that there is a beauty in these laws that mirrors something that is built into the structures of the universe at a very deep level. [Weinberg, 1994: 243].

To the extent that this dream is alive, it is not too much of a stretch to see theoretical physics as harbouring an elite brotherhood of Pythagoreans within an increasingly relativised world. Amongst physicists who acknowledge being motivated by the Pythagorean view, we can count Einstein, Heisenberg, Schroedinger, Dirac, Weyl, Gell-Mann, Weinberg, Gross and Wilczek. This is an impressive group and it is worthwhile examining what they have to say about their discoveries.

Bangu [2006a] explores the Pythagorean heuristic in physics. He looks for evidence to test the thesis that believing in a Pythagorean metaphysics has led to successful strategies in the formulation of physical theories. The motivating belief can be summarised in the aphorism: "Beauty is truth, truth beauty". On the face of it, there are at least two basic problems with this in the context of scientific theories. Firstly, scientific theories are supposed to be objective and beauty is a subjective criterion. Secondly, it doesn't always hold.

Take the first point. We believe in our scientific theories (to the extent that we do) because they explain and predict phenomenological data that are produced in repeatable laboratory experiments. The criteria for acceptance are objective. Any competent scientist with the requisite equipment, anywhere in the world, can verify them. On the other hand, people do not generally agree on the criteria for beauty. Their aesthetic tastes depend on such things as education and culture. Dirac accepted that this is so for certain kinds of beauty (e.g. literature and poetry) but held that mathematical beauty is different:

> …it is of a completely different kind and transcends these personal factors. It is the same in all countries and at all periods of time. (Dirac Quoted in [Dyson, 1986: 102])

According to Dirac, mathematical beauty is largely objective. The criteria expand over time and change in detail as our knowledge grows (e.g., we can now appreciate the beauty in non-Euclidean geometries) but we still recognise the beauty in ancient mathematics. It is not superseded in that sense. Mathematics has a timeless beauty linked to ideas of simplicity, symmetry, generality, ingenuity and necessary truth. This is in contrast to many ideas in ancient biology, for example, which seem absurd in the light of later knowledge. Still, one might think that experience and training does affect the aesthetic criteria of individual physicists and that this biases their evaluation of particular mathematical theories. For example, Einstein was initially biased in favour of a steady-state universe and this led him to favour a particular form of the equations of General Relativity. He was only persuaded of the correctness of the Big Bang model by later theoretical and experimental work. Similarly, many physicists initially found the methods of renormalisation repugnant and were later persuaded by its experimental efficacy.

European physicists with a pre-war education might have been inculcated with Platonic philosophy and have favoured the notion of mathematical forms as a guide to truth and beauty, but why should that have helped them in their work in physics and why should such a notion survive in the more American-dominated world of physics today? Wouldn't one expect American physicists to have a more pragmatic view (i.e. "whatever works") in line with their own culture? The fact is that physicists combine philosophical and pragmatic aspects in their thinking to varying degrees, depending on their inclinations and talents. Both aspects are important in their work. Some physicists value mathematical beauty over pragmatism and they will be attracted to the more pure mathematical parts of their subject (currently, String Theory). Other physicists are more pragmatic and work on models closely aligned with experiments. All physicists appreciate the mathematical beauty of our best theories. Few would advocate a beautiful theory in contradiction to experimental data.

Strangely, Dirac came close to endorsing mathematical beauty as a superior criterion to empirical adequacy. For example, he said:

> A theory with mathematical beauty is more likely to be correct than an ugly one that fits some experimental data. [Dirac 1970: 29].

Even more intriguingly:

> There are occasions when mathematical beauty should take priority over agreement with experiment. (Quoted in [Bangu, 2006a: 391]).

Bangu rightly dismisses the claim that mathematical beauty can somehow trump experimental data:

> Historical evidence supporting the strong claim — that despite systematic and consistent experimental refutations a beautiful theory was still accepted — is, it seems, simply nil. [Bangu, 2006a: 392].

However, I maintain that there is some kernel of truth in what Dirac says, as evidenced by the fact that it is difficult to kill off a beautiful theory.

String Theory is a good example of this phenomenon. It encompasses ideas from failed early attempts to unify field theories by physicists such as Einstein, Kaluza and Klein. It was originally developed as a theory of hadrons then as a string-based description of the strong force. As such, it made many predictions that directly contradicted experiments, leading to it being ignored by the mainstream. Nevertheless, it always had a group of talented devotees who progressed its conceptual base. In 1974, the discovery was made that all known string theories include a particle that has the properties of a graviton (i.e. the, as yet undetected, force-carrying particle of the gravitational field) and the decisive step was made to reinterpret String Theory as a fundamental theory of quantum gravity. Further momentum came with the realisation that it had the ability to describe all the elementary particles and their interactions, so it had the potential to unify all known forces in a natural way. Today it is the mainstream pathway to the goal of unification in physics and has many devotees.

It is tempting to read this history as the story of a small band of Pythagoreans who believed that a theory was so beautiful that it must be a theory of something — i.e., it must be pointing to a deeper truth — and who inadvertently discovered a candidate Theory of Everything. However, until String Theory has experimental verification it cannot be accepted as a true fundamental theory. Without the potential to consistently describe known physics it would never even have taken off. Beautiful mathematical theories give a conceptual understanding which transcends their link to experimental phenomena but physicists cannot accept a theory which contradicts experiment. Both the philosophical and the pragmatic aspect are needed for a theory to really thrive. In the end, experiment rules.

Weinberg marvels that the formal structure of beautiful theories often survives even when the underlying concepts do not (e.g. from relativistic wave mechanics to quantum field theory). He says that the beautiful mathematical structures themselves have “an odd kind of portability [and] can be carried over from one conceptual environment to another and serve many different purposes" [Weinberg, 1994: 152]. Commenting on the same phenomenon, Dirac had previously written:

> One may describe this situation by saying that the mathematician plays a game in which he himself invents the rules while the physicist plays a game in which the rules are provided by Nature, but as time goes on it becomes increasingly evident that the rules which the mathematician finds interesting are the same as those which Nature has chosen. [Dirac, 1939: 124].

This brings us directly back to the theme of Wigner's paper: the unreasonable effectiveness of mathematics in fundamental physics. Recall that Wigner had said that, to his knowledge, the observation which comes closest to an explanation for the mathematical concepts cropping up in physics is Einstein's statement that the only physical theories which we are willing to accept are the beautiful ones [Wigner, 1960: 7]. Dirac seems to go further than Wigner or Weinberg in explicitly embracing a Pythagorean explanation. He has a deep conviction that beautiful mathematics is a guide to truth. A beautiful mathematical theory will win out ultimately, though not necessarily in the short term. An ugly theory which manages to explain a set of seemingly unrelated phenomena and to match the experimental data is useful in the short term, allowing us to uncover deeper phenomena, but can only ever be a stepping stone. Mathematical beauty is a better guide to ultimate truth and there are occasions when we need to keep working on a beautiful theory, even when it does not agree with experiment, in order to learn what it is trying to tell us. It might reveal a deeper truth and we need to keep an open mind.

2.3.2 The Pythagorean heuristic

As physics probes deeper and deeper into the fundamental nature of reality and reaches realms where the intuitions developed in our everyday world no longer apply, there are two broad strategies which can be employed to further our understanding:

(1) Use models to extrapolate from known phenomena to unknown phenomena; testing the boundaries but keeping close to experimentally verifiable phenomena. Here the focus is on developing theoretical and experimental tools in tandem to discover new phenomena. The theoretical tools may have ugly features but they will help discriminate between the beautiful theories being developed in Strategy (2).

(2) Broaden and deepen the conceptual basis of mathematically beautiful theories to eliminate the known ugly bits and to encompass more phenomena. Here the focus is on developing a unified basis for all of physics; resolving the theoretical contradictions whilst retaining beautiful features. Experimental verification is secondary. It is even permissible to work on theories which contradict experimental data if they contain potentially interesting ideas.

Both these strategies are vigorously pursued by physicists today. Neither one, on its own, can lead to complete success. Dirac had the view that Strategy (2) would be the best path to new insights in his time, when the conceptual difficulties of quantum physics had stymied the traditional use of thought experiments and analogical reasoning. He wrote:

> The theoretical worker in the future will therefore have to proceed in a more indirect way. The most powerful method of advance that can be suggested at present is to employ all the resources of pure mathematics in attempts to perfect and generalise the mathematical formalism that forms the existing basis of theoretical physics, and after each success in this direction, to try to interpret the new mathematical features in terms of physical entities. [Dirac, 1931: 60].

His conviction regarding the relationship between truth and beauty was an article of faith, like a religion. It was reinforced in his mind by its success, as the following comment shows:

> It was a very profitable religion to hold and can be considered as the basis of much of our success. [Dirac, 1977: 156]

In these two quotes, Dirac is acknowledging that his beliefs form the basis of a heuristic strategy which should be judged by its success. This is what Bangu is most interested in investigating in his paper. He calls it "the heuristic interpretation of the C-thesis (i.e., taking "elegance" as a guide when trying to guess the correct physical theory)" [Bangu, 2006a: 395].

First, Bangu records the successes of the Pythagorean heuristic. These are many and impressive. The flagship is Einstein's theory of general relativity which was cited by Dyson as being "the most spectacular example in physics of the successful use of mathematical imagination" [Dyson, 1964: 131]. Then there is Schroedinger's wave equation; Dirac's spinor theory (which led him to predict the existence of positrons); and the electro-weak theory developed by Glashow, Weinberg and Salam. In hindsight, there are also many missed opportunities in which mathematics could have prefigured later physical discoveries were the communication between the mathematical and physical minds better coordinated [Dyson, 1972]. Nevertheless, Bangu finds the evidence of success to be equivocal. It may just be that some great physicists of a certain era monomaniacally insisted on using a Pythagorean heuristic and sometimes their guesses paid off [Bangu, 2006a: 410]. Their successes may be due more to their own brilliance than to any heuristic they used.

Next, Bangu records the failures of the Pythagorean heuristic (N.B. I shall have more to say about this in next section). He gives examples of scientists who adhered to the beauty principle in their work and were ultimately proved wrong; e.g., Ptolemy. Then there are examples of physicists who didn't adhere to the beauty principle but were very successful; e.g., Bohr and Feynman. Finally, there is the notorious case of Dirac dismissing the theory of quantum electrodynamics because it is too ugly [Bangu, 2006a: 392]. Bangu concludes that:

> Several wonderful successes are systematically mixed up with delusions, so one may begin to think that the usefulness of this guide is rather limited, a matter of good fortune. [Bangu, 2006a: 406].

He sees the Pythagorean heuristic as being a useful strategy to move research in physics forward at times when other pathways (both experimental and theoretical) cannot provide guidance. Physicists' historical adherence to it is understandable in that context.

Bangu thinks that the failures of the Pythagorean heuristic speak against a Pythagorean metaphysics. Whereas a consistent record of successes for the Pythagorean heuristic would merit an investigation into the beliefs underlying it — and a potential explanation in terms of a relation of structural similarity between the physical world and certain mathematical formalisms — its failures mean that "the metaphysics in question is prima facie dubious too, and should perhaps be abandoned" [Bangu, 2006a: 412]. Bangu notes that physicists did not abandon Pythagorean metaphysics and he himself is not prepared to abandon it either since, despite his equivocation, he finds that "the success of the heuristic based on beautiful mathematics is still mysterious" [Bangu, 2006a: 412]. In the next section I will reexamine the failures of the Pythagorean heuristic in an attempt to defend the Pythagorean view in fundamental physics.

2.3.3 A defence of the Pythagorean view in fundamental physics

Bangu sees the many failures of the Pythagorean heuristic as counting against a Pythagorean metaphysics. I argue that a Pythagorean would view such cases differently. No doubt, deifying the number 10, as Pythagoras did, would be considered eccentric these days, but the underlying insight that 'everything is number' was, at the least, a prescient portent of the fundamental role that mathematics would come to play in the description of reality and, at the best, may be essentially *right*, in that we may yet discover that reality is discrete and governed by number. Similarly, reprising the geometric tradition, Plato's reduction of reality to triangles does not literally fit with modern science, yet one of our most beautiful fundamental theories — the theory of the strong force of quarks and gluons — is based on a mathematical three-fold symmetry.

In outlining failure cases, Bangu cites Dyson [1964]. Dyson deems Ptolemy's planetary model, which restricted heavenly bodies to move on spheres and circles, to be "the worst of all the historic setbacks of physical science" [Dyson, 1964: 132]. He attributes this setback to conservative mathematical intuition, but that is very tough. The root of the problem was the acceptance of Aristotle's view of an earth-centred universe divided into a sub-lunar realm of change and an unchanging celestial supra-lunar realm. Given that restriction, and the mathematics of the day, Ptolemy's epicycles were an ingenious fit to the observed data. The dominance of Aristotle's philosophy was due to historical and religious factors, not conservative mathematical intuition.

The essential Pythagorean intuition of the heavenly bodies being ruled by mathematical law was largely vindicated by Newton and Einstein. Newton expressed his celestial laws in the language of classical Greek geometry. Einstein was driven to use non-Euclidean geometry in order to match his intuition of locally Lorentz-invariant laws. He adhered to the belief that the fundamental structure of the universe is geometric, only the geometrical concepts available to him were far more sophisticated than those available to Newton or Ptolemy. Following the development of Einstein's theory of general relativity, there were many attempts to unify all known physical laws into a single geometrical representation (e.g., by Weyl, Eddington, Kaluza and Einstein himself). These attempts were unsuccessful but are ongoing. There is a deep connection between concepts in modern physics and geometry (e.g. the relationship between gauge theory and fibre bundle theory; the connection between quantum numbers and the characteristic classes of global differential geometry; the relationship between String Theory, Calabi-Yau manifolds and knot theory). Physicists can still believe, with Plato, that "geometry draws the soul toward truth."

Bangu's other failure cases are equivocal. Dirac's work on magnetic monopoles was important and may yet be experimentally vindicated if it can be shown that monopoles played a crucial role in the evolution of the early universe. Dirac's dismissal of quantum electrodynamics was wrong but, in the bigger picture, his instinctive dislike of the use of renormalisation methods to contain infinities in a theory might prove prescient if the ultimate theory doesn't require such tricks. In this vein, Weinberg [1997: 12] thinks that quantum field theory may be nothing more than a low energy approximation to a much deeper and quite different underlying field theory which has renormalised couplings that do not run off to infinity with increasing energy.

Arguably, Dirac fixated on particular criteria of formal beauty that blinded him to some useful techniques but his overall record was astonishing. On the converse side, the existence of many outstanding physicists who disparage the Pythagorean view and achieve their success using other strategies does nothing to resolve the matter one way or the other.

I take Bangu's general point that not all beautiful mathematical theories are true and that an individual's subjective aesthetic criteria can lead them astray. Furthermore, whilst a Pythagorean metaphysics tells us something about the goal of physical theories, a Pythagorean heuristic might not always be the best way forward in the short-term. As Von Neumann pointed out [1955: 492], physicists are opportunists and will try any heuristic which gives them a chance of making progress. Ad hoc models, such as Bohr's atomic model, play a vital role in advancing physics. There is an assumption that any method that works (i.e. any method that covers many complicated and heterogeneous phenomena and leads to accurate predictions) must contain some clue to the truth. However, physicists would consider ad hoc models to be only temporary and would always be looking for the general principles and coherent mathematical theory which supersedes them. According to Dyson:

> The material at hand for theoretical work consists of fragments of mathematics, cookbook rules of calculation and a few general principles surviving from earlier days. [Dyson, 1964: 133].

He discusses the three main methods at the cutting-edge of theoretical physics in his day — field theory, S-matrix theory and group theory. He considers the approach of field theory to embody the prejudice that "deep physical understanding and deep mathematics ought to go together" [Dyson, 1964: 134]. S-matrix theory, on the other hand, is considered to be a process of trial and error which is very successful in interpreting experiments and in giving guidance to experimenters. Finally, group theory is considered to be an established, deep mathematical theory which is very powerful as a classificatory tool but too abstract to give deep physical understanding. None of them satisfy his aesthetic criteria for what a theory ought to be because they are "too vague, too partial or too fragmentary" [Dyson, 1964: 146].

With hindsight we can say that the approaches of field theory and group theory have led to the Standard Model of particle physics whilst S-matrix theory has been consigned to the annals of history. It served its purpose but has been superseded by a more beautiful mathematical theory. No doubt Dyson would confer theory status on the Standard Model whilst recognising its inadequacies. It provides deep physical understanding through its use of symmetry principles, renormalisation, mediated forces and spontaneous symmetry breaking (to highlight a few of its components). Ad hoc models can be useful stepping stones in developing theories but they don't provide deep understanding. Ultimately, true theories must be mathematically beautiful.

The Standard Model is not the ultimate theory. It has some ugly features and is incompatible with General Relativity. In applying Strategy (2) current theorists try to supersede these defects. Supersymmetric theories embed the group structure of the Standard Model — i.e., SU(3) X SU(2) X U(1) — into a more encompassing group structure — e.g., SO(10) — thereby allowing for all the particle forces to be unified at high energies and for a fundamental fermion-boson symmetry relation to be restored. In supersymmetric theories, all elementary particles can be thought of as irreducible representations of one supersymmteric group. It will be very disappointing for Pythagoreans if the beautiful features of supersymmetric theories are not supported by experimental data in the near future. Moreover, it would be a major setback for theoretical physics because only supersymmetric versions of String Theory incorporate fermionic particles in a natural way and avoid the problems of tachyonic modes (i.e. particles that travel faster than the speed of light). The search for a grand unified theory of physics reconciling quantum theory and General Relativity would still progress, but radically new ideas would be needed.

One can view physical theories as involving three levels in the direction of increasing understanding:

(1) phenomenological laws from which direct predictions are made for particular experiments;

(2) frame laws from which phenomenological laws are derived for particular cases; and

(3) mathematical forms embodying principles from which frame laws are derived.

Historically, it has only been in rare cases that a pure Pythagorean heuristic involving the sequence (3) to (2) to (1) has been made to work. Einstein's development of the theory of general relativity was such a case. The more usual route has been to progress from phenomenological data (e.g. on moving bodies) to phenomenological laws (e.g. Galileo's laws) to frame laws (e.g. Newton's Laws of Motion) to principles (e.g. invariance of laws under a particular symmetry group — the Galilei group in the case of Newton's Laws of Motion — and associated conservation principles). Really, it was through historical repetitions of the sequence (1) to (2) to (3) (e.g. Newtonian mechanics, gravity, thermodynamics, electromagnetism, statistical mechanics) and through analogies between these instances that physicists were able to inductively build up an understanding of the mathematical form of fundamental principles. That, in turn, put them in a position to more successfully apply a Pythagorean heuristic. Modern field theories were inspired by the form of Maxwell's equations and by the principles of symmetry and locality. Progress often comes from embedding a known mathematical structure in an even simpler, more encompassing, more beautiful mathematical structure (e.g. embedding the Galilei group in the Lorentz group [Dyson, 1972: 640-641], or embedding the Standard Model in a supersymmetric model). Thus, beautiful mathematics has become a useful guide in developing new theory. String Theory is the modern theory which most closely embodies the prejudice that deep physical understanding and deep mathematics ought to go together. Indeed, it is so deeply mathematical that it is hard to see it progressing without a Pythagorean heuristic. We must hope that a Pythagorean metaphysics is a reliable guide to truth.

2.3.4 Is mathematical intuition really effective in the natural sciences?

Dyson [1964: 132] speaks of the double-edged nature of mathematical intuition in that it can sometimes be more hampering than liberating. Bangu [2006a: 404-408] picks up on this point and on the semi-serious view expressed by the contemporary mathematician Schwartz [2006] that mathematics has a "pernicious influence" on science. There are several aspects to this. Firstly, what is considered mathematically beautiful in one place or in one era (e.g. circles, spheres and regular polyhedra in Ancient Greek philosophy) may become an inappropriate straightjacket for thinking. This returns us to the issue of the subjectivity of aesthetic criteria in mathematics which was discussed in Section 2.3.1. As I argued there, despite local and individual biases, our experience with mathematics is now sufficiently mature for objective criteria to have emerged. Mathematics has become very sophisticated. The essential components of its beauty have been distilled over time and redeployed in various forms. What was considered beautiful in regular polyhedra — properties such as simplicity, symmetry, formalrelatedness — manifest in different ways in the forms of abstract algebra, for example. Moreover, mathematics is now a global operation. What is considered beautiful and important by mathematicians in Russia would also be appreciated by mathematicians in the United States, or anywhere else in the world for that matter.

A second, related concern is that our thinking will be straightjacketed by mathematical forms with which we have become familiar. Consider the Lagrangian formulation. It has proven fruitful in many different areas in physics and has become a "standard" formulation. From one perspective, this illustrates the great power of mathematics in crystallising the formal similarities between seemingly disparate problems; unifying and connecting them in ways which allow insights from one to be applied to the other. From another perspective, it means that there is an automatic tendency to express new problems in Lagrangian form when perhaps a different structural approach would provide new insights. This is the predicament, not specific to mathematics, of developing powerful tools and then trying to solve all problems using those tools. It is pragmatism at work. It may hamper the development of new tools but there will always be talented individuals who relish the challenge of breaking paradigms and who have the ability to structure problems in a totally new way. That is what it means to be at the cutting-edge of fundamental physics.

Thirdly, mathematics is a formal subject with internal criteria of rigour which can conflict with the more fluid thinking required for developing physical theories. This is certainly an issue. After Witten was awarded the Fields Medal in 1990, some mathematicians objected that his conjectural results should not be considered proper mathematics until they were rigorously proven and should more appropriately be termed "theoretical mathematics" and treated with care by mathematical journals [Stoltzner, 2005: 199]. However, other mathematicians objected to the characterisation of their subject as necessarily defined by formal proof. Atiyah protested:

> … if mathematics is to rejuvenate itself and break new ground it will have to allow for the exploration of new ideas and techniques which, in their creative phase, are likely to be dubious as in some of the great eras of the past. [Atiyah et al., 1994: 178].

Great mathematicians and theoretical physicists have always pushed the envelope in their exploration of new concepts, not always bothering to slow down their stream of ideas by rigorously proving each step. Witness the intuitive methods of Euler, Cantor and Ramanujan. Chaitin bemoans the 20th century development of “a lawyer’s vision of maths, where the main goal is the nit-picking avoidance of mistakes” [Chaitin, 2007: 49]. Mathematics need not, and should not, be an intellectual straightjacket.

Fourthly, Schwartz makes the general point that mathematics must deal with well-defined situations [Schwartz, 2006: 232]. This means that, in formulating an application as a solvable mathematical problem, many messy details will have to be left out. Then there is a danger that the simplified problem will be taken too literally and its efficacy overestimated, or that the details will be inappropriately neglected. One example which Schwartz gives is taken from the social sciences [Schwartz, 2006: 233]. For many years, economic analysis was simplified in the form of linear systems of equations. That is still a standard approach, though more attention is now being paid to the effects of non-linearities. Economists understood that their subject had essential non-linearities but they were seduced by the intellectual attractiveness of mathematical argument and by all the wonderful results that could be calculated using linear systems. Whilst linearisation is appropriate for highlighting important features of economic problems, there is no doubt that some economists took their models too literally and lost the capacity to think through all the complicated interdependencies of economic variables. The real world is a complicated place and mathematics is not always the most appropriate tool to describe it. In subject areas which have not traditionally relied on mathematical expertise, it can be difficult for practitioners to know what is appropriate and what is not.

There are related issues in physics. Typically, physicists will take care to evaluate the effects of details which they have left out of idealised mathematical formulations of real world problems. However, having the sophistication to use mathematics appropriately means that they tend to select problems which are amenable to mathematical analysis in the first place. Thus, there is a selection effect whereby they tend to concentrate on problems they can solve. This is a response which is often made to Wigner’s paper: physicists think that mathematics is unreasonably effective in the natural sciences because they choose problems for which it is effective[3]. There is a lot of truth in this. On the one hand, physicists and mathematicians are always trying to push the boundaries. For example, traditional mathematical tools could only scratch the surface of fluid dynamics problems involving turbulence or systems far from equilibrium (common features in the real world) so new tools were invented to explore chaotic and non-equilibrium systems. On the other hand, the detailed complexity of the real world will always be beyond our most sophisticated mathematical techniques. The subset of applications which

[3] See e.g. [Grattan-Guiness, 2008], [Azzouni, 2000] and [Wilczek, 2006].

are amenable to interesting mathematical analysis of the sort favoured by Dirac and Einstein is small. By far the most efficacious techniques in applied mathematics are brute force computer models which use finite difference or finite element methods to divide the domain of interest into a grid of smaller problems within which the evolutive equations can be solved sequentially and patched together at the boundaries. These models are limited by their essential linearity and by the size of the grid which can be accommodated by current computers.

Wilson [2000] argues that, with the improvement in computing techniques and computing resources, very few practical problems now defeat a mathematical formulation altogether. For example, we have weather models which give some sort of predicative ability over a seven day timeframe. It is even possible to calculate the mass of the proton from first principles using the equations of the Standard Model of particle physics. The mathematical formulation will typically need to be simplified for practical convenience, but there is no "*qualitatively unavoidable mismatch* that would persist even if all simplifications for the sake of practical convenience were eschewed" [Wilson, 2000: 300]. Wilson sees this as being a vindication of *mathematical optimism* which he defines as the doctrine that "every real-life physical structure can be expected to possess a suitable direct representative within the world of mathematics" [Wilson, 2000: 297].

Wilson differentiates between two forms of mathematical optimism: *lazy optimism* and *honest optimism*. Lazy optimism simply asserts that any physical structure can be represented as a mathematical structure. In this spirit, Quine [1976] demonstrated how statements of physical theory can be reinterpreted as statements about pure sets. That is a trivial change in notation which does not lead to new understanding — understanding comes from the *interpretive scheme* required to effect the translation in the first place [Wilson, 2000: 304-305]. In contrast, honest optimism is based on Euler's achievement of showing how physical problems can be formulated in terms of evolutive differential equations which, given initial and boundary conditions, can be iterated to reveal how the physics unfolds. Evolutive equations compress the information required to represent physical reality, rather than simply relabeling it. The physicists' Theory of Everything will presumably lead to some form of evolutive equations which, together with the initial and boundary conditions, will encode all of reality.

Wilson suggests that the potential weak point in this sort of optimism lies, not in the equations themselves, but in the mathematical specifiability of the initial and boundary conditions. His examples come from applied mathematics (e.g. modeling fractures and cavitation [Wilson, 2000: 312]) but his concern is equally relevant to general relativity (e.g. the initial condition of the wave-function of the universe [Hartle, 1993]) and particle physics (e.g. problems with renormalisation in field theories [Weinberg, 1997]). Wigner emphasises the same point about initial and boundary conditions in his paper:

> … the laws of nature can be used to predict future events only under exceptional circumstances when all the relevant determinants of the present state of the world are known. [Wigner, 1960: 5]

Nevertheless, it is a fact that computer simulations now enable us to investigate possible solutions of a huge range of physical problems. Complicating features such as air resistance and non-linear feedback mechanisms can be modelled to any degree of accuracy in principle. This is a remarkable recent development which blunts criticism of Wigner's paper along the lines that mathematics frequently fails to find practical solutions to physical problems. However, it still leaves us with an issue in regards to the efficacy of *beautiful* mathematics. Computational simulations have their own special interest and challenges for mathematicians and physicists, but they are not mathematically beautiful in the sense of Dirac.

2.3.5 Is simplicity the key?

Another way in which mathematics facilitates explanation in science is through the simplicity of its concepts and its lack of ambiguity. There are two sides to this. Einstein thought that nature realises the simplest conceivable ideas and that the key to the understanding of natural phenomena is through using purely mathematical constructions to discover those underlying concepts and the laws connecting them with each other [Einstein, 1954: 274]. On the other side, it is argued that we look for simple, mathematicised theories because they are the ones we have the best chance of understanding and not because of anything to do with the fundamental nature of reality[4].

According to the second view, mathematics is effective in physics because it models an idealised system about which the physicist can think clearly [Dyson, 1964: 133]. It enables the physicist to focus on essential features and to predict their behaviour with sufficient accuracy for practical purposes, without being overwhelmed by all the potential complexities. Physical theories do not describe reality at all but merely an idealised system which we construct using simple mathematical concepts chosen because we understand them and they match reality closely enough for our purposes (this is a

[4] See e.g. Carrier [2003]

contingent fact which we find out by experimentation). The real world is a messy, complicated place that cannot be described by mathematics.

This leaves us wondering why simple mathematicised theories work at all. In a naturalist view of the world, it is because our experience of the universe can be predicted in terms of filtered, regular structures (otherwise, we wouldn't be able to survive) and we have developed mathematics expressly for the purpose of describing such structures. Barrow [2007: 231-233] explores this account using the paradigm of a computer to describe the universe. The computer hardware corresponds to the fundamental components of reality and the software corresponds to the laws of nature which act to compress the information required to update the state of the universe at any given time. This is a process view, not dissimilar to the biological paradigm in which the universe is seen as an organism which develops from a seed. The existence of laws of nature (or, equivalently, the algorithmic compressibility of the universe) is taken as a brute fact. Searching for the final theory is like trying to work out the shortest program capable of encoding the information required to construct the universe from scratch.

Barrow notes the strange mirroring of the process by our minds. Our minds filter the overwhelming amount of information gathered by our sensory organs, then compress it and interpret it according to learned heuristics. We use these heuristics to create predictive scenarios — i.e. alternative possible worlds. We do this naturally, as a consequence of natural selection, because our ability to tailor responses to alternative possible worlds promotes our survival under uncertainty. The better we get at this process, the greater will be the rewards we reap as a species. According to Barrow:

> Since the physical world is algorithmically compressible, mathematics is useful to describe it because it is the language of the abbreviation of sequences. (As quoted in [Dorato, 2005: 141-142]).

So, we use mathematics to help us make better predictions.

However, if natural selection is the mechanism by which we have learned to recognise useful structures, it is difficult to explain how we have developed the ability to describe deep structures of reality which have nothing to do with evolutionary fitness (e.g. quantum field theories). Barrow expounds on this mystery:

> Our minds are the products of the laws of nature, yet they are in a position to reflect upon them. How fortuitous that our minds should be poised to fathom the depth of nature's secrets. Why should we be clever enough to find TOE? Our brains have limits. Why should its categories of thought and understanding be able to cope with the scope and nature of the real world? Why should TOE be written in a language that our minds can decode? Why has the process of natural selection so over-endowed us with mental faculties that we can understand the whole fabric of the universe far beyond anything required for our past and present survival? [Barrow, 2007: 203].

He summarises the dilemma in the memorable aphorism: "A universe simple enough to be understood is too simple to produce a mind capable of understanding it" [Barrow, 1990: 342-343]. According to this view, if we think we have found the final theory then we are kidding ourselves — we have oversimplified the problem and do not really understand it. If we approach the problem scientifically then the more we find out about the universe, the more complexity will be revealed to us, and our only hope of progress is to evolve to be more complex ourselves. Science is an asymptotic process with no final revelation.

Wolfram [2002] has an interesting variation on the computer paradigm. He considers the universe to be a discrete system of binary information updated by simple rules. His work on cellular automata has revealed that a remarkable range of complex behaviours arise in systems which have very simple, nearest neighbour rules[5]. Moreover, the maximum level of complexity achievable by such systems is reached very quickly as a sort of self-organising behaviour. This leads him to hypothesise that the process of evolution is not as vital to the development of complexity as is usually supposed:

> Whenever natural selection is an important determining factor, I suspect that one will inevitably see many of the same simplifying features as in systems created through engineering. And only when natural selection is not crucial, therefore, will biological systems be able to exhibit the same level of complexity that one observes for example in many systems in physics. [Wolfram, 2002: 396].

Wolfram thinks that human minds are maximally complex structures which have developed inside a discrete system. We naturally judge our environment to be complex because it contains many structures which are at the same level of complexity as ourselves:

[5] Of particular interest is the rule 110 cellular automaton which emulates a universal Turing machine and so can do any computation that can be done by any computer.

> Observers will tend to be computationally equivalent to the systems they observe – with the inevitable consequence that they will consider the behaviour of such systems complex. [Wolfram, 2002: 737].

According to Wolfram, the human mind is not special. It operates within the system it observes and is not more complex than it.

Weinberg [2002] points out that, whilst these ideas are intriguing, Wolfram's work has not led to any fundamentally new understanding of physical phenomena. It is true that cellular automata show behaviour which mimics the behaviour of some physical systems, but that may be a superficial resemblance and not due to any fundamental similarity in the underlying dynamics. If simple rules can produce a cellular automaton that looks like a growing snowflake does that really help us understand the physics of snowflakes (compared to, say, the traditional approach of describing heat and solute flow using differential equations)? Until we can use cellular automata to make accurate predictions about the world they cannot be considered truly fundamental. As it is, many experts have argued that there are details of known science which appear to thwart Wolfram's vision of the universe as a discrete system governed by simple, local rules (e.g. non-local effects due to entanglement in quantum mechanics; complexity arising in evolutionary biology) [6].

In science, computer simulations cannot provide a deep understanding of underlying principles. Human minds must first come up with those principles, then program computers to explore their consequences. Within that context, computers can lead human minds to a new understanding of particular phenomena and predictions of new phenomena. Similarly, in mathematics, computers can display theorems which are surprising and interesting to mathematicians but they cannot give the same sense of understanding as a cleverly constructed human proof. This is because they are operating at the logical, rather than the conceptual, level. Gödel thought that the rational nature of the world and of the human mind is such that reason must have the ability to answer any question which it can clearly pose to itself [Wang, 1996: 317]. In particular, he thought that clearly posed number-theoretic problems are decidable by the human mind and that, therefore, the human mind is superior to any computer (for which there must be undecidable number-theoretic problems as shown by Gödel's First Incompleteness Theorem). According to Gödel, human minds are special and can understand truths that no computer ever will.

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[6] See e.g. [Aaronson, 2002], [Barrow, 2007], [Berry et al., 2002], [Lavers, 2002], [Rucker, 2003], [Weinberg, 2002].

Fundamental physicists "pursue the old reductionist dream of finding laws of nature that are not explained by anything else, but that lie at the roots of all chains of explanation" [Weinberg, 2002: 2]. They search for the final theory which will directly represent fundamental physical reality in mathematical equations without any need for idealisation or simplification. Their rationalist belief in our ability to find the final theory leads them to look for theories which are in some sense simple, so they agree that simplicity is one of the aesthetic criteria to be desired in scientific theories. However, this doesn't mean that the underlying mathematics will be simple in the sense of classical mechanics or Wolfram's cellular automata. Sometimes esoteric mathematics is needed to express simple ideas of symmetry (e.g. group theory) or topology (e.g. knot theory) or aggregation (e.g. set theory).

Wigner marvels at "the two miracles of the existence of laws of nature and of the human mind's capacity to divine them" [Wigner, 1960: 7]. In the Pythagorean view, these miracles are explained by a harmony between our minds and the laws of nature which are written in the language of mathematics. It is that harmony which we come to appreciate over time and then codify as 'mathematical beauty'. In his discussion of aesthetics in science, McAllister [1998] argues that the process of science is one of learning to recognise which aesthetic properties are reliable signs of truth. Scientists do this by selecting theories which have the best empirical performance, then abstracting their mathematical properties and elevating them to the status of principles. Principles are treated conservatively in the framing of scientific theories and revolutions occur when empirical data overthrow an existing principle and force us to update our aesthetic criteria. McAllister writes:

> If an aesthetic property were a sign of truth, a theory that exhibited that property would necessarily be true, and would therefore show the best empirical performance conceivable. [McAllister, 1998: 177].

As time goes by, we learn to screen out the contingent aesthetic properties associated with empirically successful theories (e.g. sphericity) and focus on necessary aesthetic properties (no definitive example can be given here but, perhaps, some broad symmetry principle will stand the test of time). McAllister suggests that scientific understanding should be viewed partly as an aesthetic attainment:

> It may be, for example, that theories strike us as beautiful to the extent to which they provide understanding, or that the sensation of understanding is elicited by perceiving a theory with particular aesthetic features. [McAllister, 2002: 9]

Pythagoreans will understand when they have found the final theory because of its beauty, simplicity, logical inevitability and explanatory power.

2.3.6 Towards a consistent Pythagorean metaphysics

I accept the limitations of mathematics in solving real world problems but I argue, against Bangu [2006a, 407-409], that this has little to do with the Pythagorean metaphysics of theoretical physicists (or, for that matter, that it really impacts the core of Wigner's mystery). The Pythagoreans' claim is about the essential mathematical nature of the underlying laws of physics; not about the messy, complicated world which evolves from them. The claim is not that "the world instantiates a beautiful, harmonious pattern" [Bangu, 2006a: 408] or that physical processes are always describable by beautiful mathematics. It is that there is a beautiful, unified theory of physics which exists outside of spacetime and which, through a process of actualisation (involving compactification of extra dimensions in the case of String Theory or spontaneous symmetry-breaking in the case of gauge theories), leads to the emergence of spacetime and, ultimately, all the complexity we see in the world today. To discover this theory is the goal of Pythagoreans in fundamental physics. Remnants of the theory are detectable in the world and reflect its beauty in varying degrees. At the very high energy levels achievable in particle accelerators, approaching those which existed moments after the Big Bang, we get as close as we can to instantiating the theory itself. That is still not very close. At those levels, we are already working in the four dimensions of spacetime; gravity has split off from the fundamental particle forces; and the strong force has split off from the electroweak force. The component theories each have their own beauty, as seen in the emergence of the elementary particles as nearly ideal embodiments of their abstract symmetry principles. However, they are merely effective theories which apply in a limited energy range, smoothing out the effects of higher energy levels. They are not isomorphic to the fundamental structure of reality.

It is a fact that, in order for interesting physical structures to form, the perfect symmetry of the underlying theory must be broken. For example, the weak force violates parity symmetry which means that left-right equivalence is broken. This might be considered ugly from one perspective but, as Randall [2011: 263] points out, it is necessary so that the elementary particles can acquire their range of masses and act as the basic building blocks of our world. It is one of many steps required "to connect [the ultimate theory] to the fascinating and complex phenomena we see in our world" Randall [2011: 270]. The really amazing thing is that, as we scale up from the micro-world to the macro-world, we see repeating patterns of behavior which can be described using the same mathematics [Wilczek, 1998].

Examples abound: mathematics which is used to describe scaling behavior near the phase transition of a magnetic material can also be used to describe scaling behavior in the deep interior of a proton; Bohr's planetary model of the atom was a useful tool in the early days of quantum mechanics; Einstein applied Planck's formula for excitations of the electromagnetic field to vibrations of a crystal. Without these links between the micro-world and the macro-world we would never have been able to begin to explore the quantum realm.

It is not at all obvious why the laws of physics at one scale should show any resemblance to those at another scale. Wilczek gives the following explanation:

> …the most basic conceptual principles governing physics as we know it — the principle of locality and the principle of symmetry — are upwardly heritable. [Wilczek, 1998: 13].

In other words, the principles governing the elementary units apply to appropriately constructed assemblies of units. Moreover:

> … it is because coarse-grained versions of local and symmetric equations remain local and symmetric that even approximate forms of the laws retain much of their elegance. Physics had beautiful equations long before the emergence of the standard model; and the standard model has beautiful equations even though it is surely not the ultimate truth. [Wilczek, 2006: 9].

This is just a sketch of an explanation. Clearly, there are features of the micro-world that do not resemble anything in the macro-world (witness the non-intuitive marvels of quantum mechanics). Also, there are symmetries of the micro-equations which are not manifested at the macro-level. What Wilczek is doing here is acknowledging a remarkable feature of nature — that beautiful mathematical equations are unreasonably effective in describing its behavior — and attempting a rational explanation, rather than leaving it as a mystery. His explanation is reductionist. Beauty is grounded in "ultimate truth", in the yet to be discovered ultimate theory of fundamental physics, and it is communicated upwards through the process of actualisation by features which are retained under coarse-graining. In an analogous way, the underlying properties of DNA lead to resemblances amongst life forms at all scales. So, in this view, the ultimate theory is like a set of instructions which directs the creation and construction of the world. It is beautiful and its beauty is reflected in nature. Weinberg agrees that "the beauty of our present theories is an anticipation, a premonition, of the beauty of the final theory" [Weinberg, 1994: 165]. It is evidence that we are moving in the right direction.

Wilczek's explanation is the basis of a consistent Pythagorean view. It explains both the successes of the Pythagorean heuristic — which would otherwise seem unreasonable — and the failures. The world unfolds from the ultimate theory and we are part of that process. The clues to the ultimate theory are all around us in nature. As time goes by, our knowledge of nature grows and our aesthetic criteria of discovery are refined, leading to improvements in the tools which we use to increase our knowledge. Our development of mathematics is an essential part of the process. True, the Pythagorean heuristic will have failures. That is the essence of any heuristic: trial and error. We make mistakes as we learn more about what the nature of true beauty is. Nevertheless, truth and beauty are our best guides and mathematics is the pathway.

If this analysis correctly reflects the view of the Pythagoreans then Dirac was right to compare it to a religion. It is a long way from Bangu's more pragmatic analysis:

> True, looking for mathematical beauty and mathematical simplicity does not always work, but when no other strategy looks credible, following the lead of mathematical beauty and simplicity in inventing physical hypotheses might help. [Bangu, 2006a: 409].

Such pragmatism does not get to the heart of Wigner's mystery. It does not explain the unreasonable effectiveness of mathematics in the natural sciences. In my thesis, I will be defending the viability of the Pythagorean view, arguing that it is supported by the historical development of mathematics and physics, and that it gives the best answer to the questions which Wigner raised.

2.4 Claim and Counter-Claim: What of Wigner's Mystery?

In this chapter, I have explored issues raised by Wigner's paper 'The Unreasonable Effectiveness of Mathematics in the Natural Sciences'. The discussion has touched on the history and philosophy of mathematics, the relationship of mathematics to the world, the nature of scientific theories and the role of mathematics in explanation. In this section, I will isolate and examine various claims that arise from the discussion — i.e. claims about the unreasonable effectiveness, or mysteriousness, of mathematics in applications — to see what philosophical issues remain. I will conclude that whilst many issues surrounding the applicability of mathematics disappear upon examination (depending, to some extent, on one's philosophy of mathematics) there are some core issues to do with the application of mathematics at the cutting-edge of fundamental physics which remain (and these are, to a large extent, independent of one's philosophy of mathematics). These core issues will be the focus of my thesis, to be explored in the following chapters.

Claim 1. There is something mysterious about the applicability of mathematics in general.

Not really.

Philosophers will find the issue of the applicability of mathematics to be more or less problematic, depending on their philosophy of mathematics, but they will all be able to give an account one way or another[7]. Strict formalists will have the hardest time of this. Their account will make the applicability of mathematics depend on finding interpretations that make its axioms true. Fictionalists, on the other hand, will say that mathematics is literally false but that it functions as a useful deductive tool in applications. Platonists will face a question about how acausal mathematical objects can relate to the objects in the world. How should we analyse mixed sentences containing mathematical and non-mathematical terms? For example, in the sentence 'There are two roses in the garden' — what does the number 'two' have to do with 'roses in the garden'? Fortunately, as far as applications go, this question was answered by the work of Frege[8]. It will be worthwhile to review Frege's approach in some detail.

Frege considered applicability to be essential to mathematics. He said:

> It is application alone that elevates arithmetic beyond a game to the rank of a science. So applicability necessarily belongs to it. (Frege as quoted in [Pincock, 2010: 2]).

He treated mathematical sentences as sentences of a language which express meaningful claims. For mixed sentences, he analysed the mathematical terms as contributing to the overall semantics of the sentence in the same way as the non-mathematical terms (e.g. they can function as subject or predicate or higher-order concept). In the sentence 'There are two roses in the garden', the mathematical term 'two' is analysed as the second-order concept 'being a concept under which two objects fall'. The whole sentence is interpreted as telling us something meaningful about the non-mathematical concept 'roses in the garden'; i.e. that it falls under the second-order concept 'two'. Any apparent circularity is avoided by defining the concept 'being a concept under which two objects fall' in purely logical terms without

[7] See e.g. [Shapiro, 2000]. If all else fails, the applicability of mathematics can be taken as a brute fact.
[8] See e.g. [Dummett, 1991] and [Steiner, 1995: 132-138]

appealing to the concept 'two'. Thus, '*F* is a concept under which two objects fall' is defined as any concept *F* that satisfies the following condition:

> There are distinct things *x* and *y* that fall under the concept *F* and anything else that falls under the concept *F* is identical to either *x* or *y*. [Zalta, 2013: §2.5].

This is a purely logical statement. In general, Frege analysed statements about number as higher-order logical statements about concepts.

Frege treated all applications of arithmetic as instantiations of highly general truths of logic by specific concepts and relations [Dummett, 1991: 303]. If the subject of an arithmetical statement is a non-mathematical concept like 'roses in the garden' then there is no need to make direct reference to mathematical objects as such, so there is no need to assume the existence of mathematical objects. On the other hand, if the subject of the statement is a mathematical concept like 'prime numbers between 10 and 15' then we must assume the existence of this mathematical object if we are to accept the truth of the statement. Therefore, the existence of mathematical objects is required by applications of mathematical theorems internal to mathematics (i.e. by pure mathematics). Frege was a platonist who believed in the existence of mathematical objects. For him, the very wide application of arithmetical statements is a consequence of the fact that they express a thought, which can be a thought about anything numerable. He wrote:

> The basis of arithmetic lies deeper, it seems, than that of any of the empirical sciences, and even than that of geometry. The truths of arithmetic govern all that is numerable. This is the widest domain of all; for to it belongs not only the actual, not only the intuitable, but everything thinkable. Should not the laws of number, then, be connected very intimately with the laws of thought? [Frege, 1884: §14]

Frege's ambition was to develop all of mathematics so that it could be understood as reasoning about the contents of thought rather than requiring any intuition of mathematical objects. He wanted to show that mathematical concepts can be defined in terms of purely logical concepts and that mathematical principles can be derived from the laws of logic alone [Shapiro, 2000: 108-115]. His philosophy of mathematics is called *logicism*.

Frege developed a system of arithmetic (equivalent to standard Peano arithmetic) using only second-order logic and a principle known as Hume's Principle [Shapiro, 2000: 135-138]:

> Hume's Principle: The number of *F*s is identical to the number of *G*s if and only if the objects falling under the concept *F* can be put in one-one correspondence with the objects falling under the concept *G*.

This was a great achievement but not, as it turned out, a vindication of logicism. Hume's Principle is not accepted as an analytic principle of logic (though its status is still a matter of debate[9]). Also, Frege's work assumes the existence of numbers and, as a general rule, existential claims cannot be substantiated by logic alone. Furthermore, his work fails to give a criterion for determining which objects are numbers (i.e. when considering objects chosen from the domain of all logical and non-logical objects). Therefore, his attempt to reduce arithmetic to logic is generally considered to have failed (even neglecting issues to do with Russell's Paradox which arose as a consequence of assumptions which Frege made in trying to establish Hume's Principle in logic). Nevertheless, his work demonstrates the importance of logic in mathematics. Today it is thought that set theory (and, hence, almost all of mathematics) is reducible to logic supplemented by the non-logical concept of set membership.

Frege's project of logicism may have failed yet, in regards to applications, he showed how to interpret some mixed arguments containing mathematical statements in a way which avoids reference to mathematical objects altogether and so is compatible with non-platonistic philosophies. In particular, he showed how to do this for arguments containing number statements. That is sufficient to explain applications of mathematics to accounting, for example. Steiner [1995] discusses how Frege's insights can be adapted to a set-theoretical context:

> … numbers characterise sets, not physical objects; while sets can contain, of course, physical bodies. Set theory is applicable … simply because physical objects can be members of sets… To 'apply' set theory to physics, one need only add special functions from physical to mathematical objects (such as the real numbers). Functions themselves can be sets (of ordered pairs, in fact). As a result, modern—Fregean—logic shows that *the only relation between a physical and a mathematical object we need recognise is that of set membership*. [Steiner, 1995: 136-137] (My italics).

This interpretation of physical theories in terms of set-theoretical models is further developed by Suppes [1960]. He discusses how to axiomatise classical particle mechanics, electromagnetic theory and

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[9] See e.g. [Hale and Wright, 2001]

statistical mechanics. He suggests that there is no systematic difference between the axiomatic formulation of theories in physics and in pure mathematics, and that model techniques used in pure mathematics (e.g. embedding theorems) can be extended to empirical applications. He pays particular attention to the relationship between the set-theoretical model of the physical theory (which typically contains elements such as continuous functions and infinite sequences) and the finitistic set-theoretical models of the data. His approach formalises the insights given by mapping accounts in philosophy.

Responses to Wigner's paper in the Fregean tradition treat the effectiveness of mathematics as a particular instance of the effectiveness of natural languages: "mathematics is just a more rigorous, less ambiguous, and formally organised language" [Dorato, 2005: 138]. It is easy to see how this explains the descriptive and conceptual roles of mathematics in physical theories but it is harder to see how it explains its predictive accuracy. It seems that we can use English for describing and conceptualising the world but not for making predictions.

To summarise my response to Claim 1: there is more to the effectiveness of mathematics in physics than can be addressed by a Fregean approach, but Frege succeeded in demonstrating that there is nothing mysterious about the applicability of mathematics in general.

Claim 2. There is something mysterious about the applicability of modern mathematics.

Not as such.

As discussed in Section 2.2, Wigner defends a version of this claim based on his view of modern mathematics as a game invented by highly intelligent people for aesthetic pleasure and intellectual display, divorced from empirical considerations. Under this view, applications of modern mathematics seem to imply some mysterious connection between human aesthetic criteria and fundamental structures of reality — i.e., a universe in harmony with our minds, or a "relation between Mind and the Cosmos" [Steiner, 1995: 152]. However, I have argued that mathematics can never be divorced from its empirical foundations: it may be led into flights of abstraction but it is always pulled back to physical reality as a source of new ideas. Even in abstraction, we underestimate the implicational connections to other mathematical concepts and the role that analogy plays in conceptual thinking. On top of this, there are strong methodological and cultural links between mathematics and physics which keep them loosely related. In short, mathematics generates a well-stocked warehouse of abstract structures which have some grounding in physical reality so it is not too surprising that a subset of them finds application to real world problems[10].

Steiner [1995: 152-153] constructs the following argument from Wigner's paper:

P1 Mathematical concepts arise from the aesthetic impulse in humans.

P2 It is unreasonable to expect that what arises from the aesthetic impulse in humans should be significantly effective in physics.

P3 Nevertheless, a *significant* number of these concepts are *significantly* effective in physics.

C Hence, mathematical concepts are unreasonably effective in physics.

The weak points of this argument are Propositions 1 and 2. As discussed above, mathematical concepts cannot be divorced from their empirical foundations. Furthermore, as I argued in Section 2.3, our aesthetic criteria evolve together with our understanding of the universe and are reinforced by the structures that we find in nature. Dirac thought that aesthetic criteria in mathematics are largely objective. Platonists would think that beauty in mathematics is something we discover, rather than construct, and that Steiner is wrong to say that "the concept of mathematics is itself species-specific" [Steiner, 1998: 6]. Many influential mathematicians have explicitly rejected the anthropocentric view [Bangu, 2006b]. The fairest stance is to acknowledge that mathematics is driven by many competing influences: as a cultural product, it does have anthropocentric influences but, on some realist views, it is not just a construction of the human mind. Rather than being species-specific, it may be determined by the structure of the world, or it may be a first principle prior to physical reality. These issues depend on one's philosophy of mathematics and put Steiner's conclusion in doubt.

Claim 3. It is mysterious how mathematics can be used to make predictions about the world.

No.

[10] At the moment, only a subset of mathematics is applied, but the true extent of the applicability of mathematics is not known, so nothing can be inferred from this.

The explanation of our ability to use mathematics to make predictions about the world properly requires a historical/evolutionary/genetic approach which incorporates the development of mathematics as a language. This approach was touched on in Section 2.3.5 and in my response to Claim 1. We start with the assumption of structure in the world. This is not to say that reality is *necessarily* structured but that, without some structural permanence, human beings would not have evolved and would not be in a position to describe the world at all. Then, according to Longo [2005], we are presented with structured phenomena which we try to make intelligible in order to learn about our environment and thrive in it. As Kant suggested, it is not the case that we are presented with pre-organised phenomena which we describe independently. Rather, our minds project *a priori* forms, constructions and categories of our own onto our experience of nature. Also, we construct special tools to organise phenomena in particular ways. Natural language is a tool for organising reality as it is filtered through our five senses. Mathematics is a tool for organising phenomena in different ways, with an emphasis on abstracting out permanent, structural relationships in nature.

Some form of quantitative understanding develops at the very beginning of human evolution and is selected for because of the advantages which it confers for survival. For example, it is advantageous to be able to distinguish how many predators are present, or what quantity of food is available relative to the number of consumers. Arithmetic develops from these beginnings. It is a cultural construction which augments our natural quantitative abilities and confers additional evolutionary advantages. Physics and mathematics develop together as a way of making more precise predictions about the world. More sophisticated mathematical tools are required as more sophisticated experimental tools are constructed to probe the structure of reality. Because of the flexibility and freedom of mathematics we are always able to use it to describe any structure we discover. Either we invent new mathematics for the purpose or, as frequently happens, physicists find that the structure they need has already been described by mathematicians before being observed in nature. It also happens, as discussed in Section 2.3.4, that physicists actively select problems to which they can apply known mathematics, or idealise and simplify problems in order to facilitate applications.

Longo considers mathematics to be the result of a knowledge process:

> We cannot separate Mathematics from the understanding of reality itself; even its autonomous, "autogenerative" parts, are grounded on key regularities of the world, the regularities “we see” and develop by language and gestures. [Longo, 2005: 353].

To fully understand any particular application of mathematics we must be able to tell a story about how the relevant concepts were developed from our experience of the world. Steiner does this for the concepts of 'number', 'addition', 'multiplication', 'symmetry' and 'linearity' [Steiner, 1995: 138-144]. Grattan-Guiness [2008: 10] provides a table linking twenty-two concepts from mathematics with concepts from the sciences and/or the world.

In Longo's view, there are limits to the effectiveness of mathematics in making predictions about the world. In particular, he thinks that it will have limited effectiveness in sciences where invariance and stability are not at the core; e.g., in biology [Longo, 2005: 376-377]. This is debatable. However, it is clear that there are limitations on mathematics, as evidenced by Gödel's theorems (the existence of true statements that cannot be proved formally), the concepts of computational complexity (the existence of many natural problems that cannot be solved by practical algorithms) and chaos (the existence of natural equations that cannot be solved systematically) [Wilczek, 2006: 8]. Barrow says that mathematics is useful in making predictions about the world because so many natural sequences are algorithmically compressible and mathematics is the language of the abbreviation of sequences (see Section 2.3.5). For sequences in nature which cannot be compressed (e.g. particular chaotic sequences), mathematics is ineffective.

I agree that some sort of Kantian approach supplemented by an evolutionary story is adequate to explain how we use mathematics to make predictions about the structures we discover in the world. However, it doesn't explain the *active* role which mathematics played in the discovery of the structures of the quantum world (see Section 2.3). Intuitions developed to promote our evolutionary survival do not apply to the quantum world. Nevertheless, physicists were able to penetrate its depths using Pythagorean or formal analogies developed in mathematics. The success of the Pythagorean heuristic in theoretical physics seems to point to some deeper connection between mathematics and reality, independent of the human mind. So we are not yet finished with Wigner's mystery.

Claim 4. Mathematics is unreasonably effective in applications in general.

No.

I will summarise the responses here, since we have met them before.

- Mathematics has many failures; e.g., it is not effective in describing natural sequences which are not compressible.

- The application of mathematics to real world problems often requires idealisation and simplification or, in other words, a certain amount of shoe-horning.

- There is such an abundance of mathematical structures that it is not surprising that they are often effective in describing physical structures.

- Mathematics is effective because it is a good, flexible tool.

- Mathematics is effective because we cognitively constitute the world in a mathematical manner.

- Mathematics is effective because it sources concepts in the empirical world.

Claim 5. It is mysterious that mathematics which was developed for one purpose can find application in an entirely different context.

Possibly (it depends on the context).

It is not mysterious that there are some unexpected connections. The ubiquity of pi is a case in point. Consider the following passage from Newsweek 1998:

> Humans invent abstract mathematics, basically making it up out of their imaginations, yet math magically turns out to describe the world. Greek mathematicians divided the circumference of a circle by its diameter, for example, and got the number pi, 3.14159... . Pi turns up in equations that describe subatomic particles, light and other quantities that have no obvious connections to circles.
>
> This points … "to a very deep fact about the nature of the universe," namely, that our minds, which invent mathematics, conform to the reality of the cosmos. We are somehow tuned in to its truths. [Begley, 1998].

The boring fact is that, one way or another, when pi turns up in equations it is because of a connection to circles, whether or not the connection is obvious. Clearly, any physical problem involving spherical symmetry or spherical coordinates will include factors of pi. Also, complex numbers are usually expressed in a polar coordinate system consisting of amplitude and phase (i.e. one coordinate is the radius of a circle centred on the origin of the complex plane and the other coordinate is the angle representing a counter-clockwise rotation from the positive real line). Trigonometric functions are expressed in radians which are defined so that a complete circle spans an angle of 2π radians. Common trigonometric functions have periods that are multiples of pi. Moreover, pi appears in formulas for areas and volumes of geometrical shapes based on circles such as ellipses, spheres, and cones. Hence, it appears in definite integrals involving these sorts of shapes. Ultimately, this is why it appears in population statistics (see [Wigner, 1960: 1]) — because the area under the graph of the normal distribution is given by the integral of the Gaussian function. Spherical coordinate systems, complex numbers, trigonometric functions and Gaussian distributions are ubiquitous in physics so the ubiquity of pi is not a mystery.

Unexpected connections in mathematics need to be explained on a case by case basis. Sometimes, as above, they are because of implicational opacity. Sometimes they are because of analogies between systems which are not fundamentally mathematical analogies but are describable in mathematical terms (e.g. linearity, symmetry, periodicity). Sometimes they are due to common features which dominate systems in the asymptotic limit. Such unexpected connections are surprising but they are not mysterious.

In its role as a language, mathematics naturally adapts to analogous situations, transferring from one field to another. Because of the formalist construction of modern mathematics, it has no particular empirical subject matter, so techniques and results from one application can be applied to others with the same formal structure even when they are quite different physically. This means that, when we have a physical explanation of analogous behaviour between a mathematically-characterised system and an uncharacterised system, we can transfer the mathematical formulation from the characterised to the uncharacterised system. More than this, when we observe partially analogous behaviour between a mathematically-characterised system and an unknown system, it is reasonable to *try* to transfer the mathematical formulation from the known to the unknown system, even when we have no physical explanation of the underlying analogy (especially if we can't think of anything else to try). Of course, it is unreasonable to expect such unfounded analogies to be very successful. That is what is so surprising about the use of the Pythagorean heuristic in fundamental physics: its repeated success. In the development of quantum field theory, mathematical formulations from areas such as classical field theory, matrix theory and group theory were tried in an effort to match experimental results and predict new phenomena, with remarkable success. The physical theory emerged as part of the process. The physical theory then provided an explanation of why the mathematical analogies worked in the first place, but this was after the fact. I am reminded of Wigner's metaphor about the keys (see Section 1): a man has a bunch of keys and finds that when he tries to open a succession of doors he always hits on the right key on the first or second trial. There is something mysterious about this success but it is not entirely clear what it means.

In the case of quantum field theory, why should mathematics developed in the macro-world tell us anything about the micro-world? In Section 2.3.5, I gave Wilczek's reductionist explanation for resemblances between the laws of physics at different scales. He argues that the properties of locality and symmetry are inherent to the fundamental nature of reality and that it is because coarse-grained versions of local and symmetric equations *remain* local and symmetric that even approximate forms of the underlying laws of nature retain much of their beauty.

This is not a full explanation since we are not in possession of the underlying laws. However, it is the basis of a rational explanation of what would otherwise be the *unreasonable* effectiveness of mathematics in fundamental physics. It presupposes something about the nature of reality. For one thing, it presupposes some special features (i.e., locality and symmetry). More importantly, it presupposes a reductionist view of reality in which all the patterns and structures in nature are ultimately explainable in terms of one mathematical structure (i.e., the ultimate theory, TOE). This is a form of Pythagoreanism. It goes beyond the common structuralist view of the effectiveness of mathematics (i.e., that reality must contain some structure and we are biologically adapted to interpret it) to explain why some simple structures in nature are so pervasive and far-reaching. As pointed out by Barrow [2007: 147], chaotic solutions of equations are the norm rather than the exception in mathematics so an explanation is needed of why there are so many linear and simple phenomena in nature.

Claim 6. It is mysterious that mathematics can be used to discover things about the world.

Sometimes.

It is not mysterious when the discovery is based on the deductive consequences of mathematics. We only have a limited natural ability to deduce consequences so we can discover surprising things about mathematicised systems by making the consequences explicit (e.g. by running a computer program).

It is not mysterious when mathematics leads to discovery by providing a different way of viewing phenomena (e.g. by using a momentum space representation; or applying other, specialised mathematical filters). We use mathematics in designing instruments and then in interpreting the results. In this way, we use mathematics to discover features which would otherwise remain hidden. Quarks could not have been discovered before the mathematicised theory of the strong force was developed, and very complicated instruments designed to produce and interpret its predictions. Mathematical concepts enable us to conceive and propose new physical objects which are sometimes verified experimentally.

As discussed under Claim 5, it is not mysterious when we discover things about an unknown structure by applying mathematical techniques from an analogous, known structure, provided we can explain the underlying analogy. However, when we cannot explain the underlying analogy then it *is* mysterious. There is no reason why irrelevant analogies should lead to discoveries so there is a missing explanation which needs to be found. Steiner [1989] argues that many of the discoveries in fundamental physics over the past hundred years or so have relied on formal mathematical analogies which had no physical basis and were therefore irrelevant (at least, as far as anyone knew at the time). He gives many examples: Schroedinger's discovery of wave mechanics [1989: 457]; Dirac's equation for relativistic quantum mechanics (which led to the discovery of the positron) [1989: 459]; the Schwarzchild solution for the equations of General Relativity (which led to black holes) [1989: 459]; isospin symmetry in nuclear physics [1989: 461]; unitary symmetry in particle physics (which led to the discovery of the omega-minus particle) [1989: 462]; Heisenberg, Born and Jordan's derivation of matrix mechanics [1989: 467]; and the Klein-Gordon equation [1989: 469].

Steiner [1989: 455] acknowledges that many of the formal mathematical analogies were later grounded physically. Indeed, according to the reductionist view of physics, it is only to be expected that the physical explanation of the formal analogy will gradually emerge as the underlying theory is investigated experimentally. The surprising thing is that it works in the first place. Why, for example, did unitary symmetry prove to be a useful analogy when it doesn't correspond to any physical symmetry of any real space of any dimension (rather, it relates to transformations in a complex space)? Why was the Pythagorean heuristic so often successful? It is not a fail-safe heuristic, since it has had many failures, but it has been remarkably successful in the hands of great physicists.

Steiner argues that "even one successful use of irrelevant analogies to make major discoveries in physics is surprising" [1989: 456]. The predictions of theoretical physics are so precise that when they are verified it is significant and requires explanation. I find his many examples impressive but distracting. They seek to establish a pattern but they leave one wondering about the details of each case. I wonder whether the analogies really are "irrelevant". In the next chapter I will focus on the example of non-abelian gauge theories; in particular, on the theory of quarks and gluons. I will examine this case in detail to flesh out the thesis of the successful use of irrelevant formal mathematical analogies in fundamental physics. If substantiated, it is mysterious and points to the truth of the underlying Pythagorean metaphysics.

Claim 7. It is mysterious why such deep mathematics is needed to describe fundamental physics.

Yes.

David Gross makes this point:

> The mystery of the effectiveness of mathematics in fundamental physics is much deeper than just the miracle of its astonishing utility. After all, it is no surprise that we need mathematics to deal with complicated situations involving systems composed of many parts, all of which are in themselves simple. We have also learned that even simple systems whose microscopic laws of evolution are easy to describe can exhibit extremely complex behaviour. However, we might expect to be able to describe the microscopic laws in terms of simple mathematics. The strangest thing is that for the fundamental laws of physics we still need deep mathematics and that as we probe deeper to reveal the ultimate microscopic simplicity, we require deeper and deeper mathematical structures. Even more, these mathematical structures are not just deep but they are also interesting, beautiful, and powerful. [Gross, 1988: 8372].

It seems that the world could have been describable in terms of the mathematics of cellular automata or Newtonian mechanics, for example. However, what we have discovered in physics is that, whilst the underlying principles may be simple (e.g. locality and symmetry), the mathematics is at the limit of our capabilities. Both General Relativity and the Standard Model of particle physics explain the forces of nature as arising from local symmetries. In General Relativity, this leads to a description in terms of Riemannian geometry. In the Standard Model, it leads to quantum field theory and non-abelian group representations of particles. Gross [1988: 8373] says that the ugly parts of the Standard Model are those that describe the strange spectrum of matter (e.g. the parts involving spontaneous symmetry-breaking). In a similar vein, Einstein said that the beautiful part of General Relativity was the geometric part and the ugly part was the energy–momentum tensor [Ryckman, 2005: 219]. This is consistent with the Pythagorean view of a beautiful mathematical theory underlying reality and an actualisation process which breaks the symmetry and leads to complicated outcomes (see Section 2.3.5).

To some extent, physics and mathematics evolve together. Deep mathematical theories find application in fundamental physics and lead to new empirical discoveries which, in turn, feed back into the development of new concepts in mathematics. Progress is often made through a systematic examination of the singularities and infinities arising at the limits of our fundamental physical theories. As our intelligence evolves, more complexity is revealed to us. This seems to be a never-ending process of which mathematics is an essential part.

Claim 8. It is mysterious that there are laws of nature written in the language of mathematics and our minds can fathom them.

Yes.

As discussed in Section 2.3.3, physicists develop phenomenological laws to predict future phenomena, then embed those phenomenological laws in frame laws and, finally, derive the frame laws from mathematical forms embodying general principles. For some, the ultimate aim is to derive all the frame laws from one mathematical form, TOE. The concept of TOE is of a permanent mathematical structure underlying reality which, when combined with appropriate physical bridge principles, provides a mathematical explanation of all physical phenomena.

There is plenty of room to question this paradigm. We can never experimentally verify the universality of the laws of nature: maybe their form (or their assumed constants) vary with time, or vary from one part of the universe to another, or are different in different multiverses? If we do find TOE, then in what sense will it provide an explanation: will its form self-verify its necessity and finality? Will we really be able to use it to derive all physical phenomena (at least, in theory) or will some things remain irreducible? These are weighty problems. Nevertheless, considering how great is the distance between 'the existence of some structure in the universe' and 'the existence of TOE', it is mysterious how physicists have been able to travel so far along this path already. We even have a candidate for TOE — String Theory[11]. If TOE doesn't exist, it is mysterious how we have come so tantalisingly close to it.

Over the course of history, our physical theories have suffered from our human biases (e.g., our limited sensory abilities; our experience of a locally Euclidean spacetime; the low energies available in our contemporary world; our intuitive attachment to three dimensions of space and one of time). Physicists have consciously tried to over-ride these biases by appealing to non-anthropomorphic, general principles to ground their theories. This has led to the candidates for TOE becoming more general, more symmetric, bigger mathematical forms. Does this process of generalisation have a stopping point? Does the "surplus" structure in TOE (i.e. surplus to our requirements for describing known phenomena) always have some physical interpretation; perhaps requiring a more powerful, rational mind than ours? If we follow this line of reasoning then mathematical and physical reality converges. Something like this scenario is being explored by Max Tegmark with his Mathematical Universe Hypothesis:

> Our external physical reality is a mathematical universe. [Tegmark, 2008: 102].

Then, the effectiveness of mathematics in the natural sciences is explained by the world being intrinsically mathematical. This is the essence of Pythagorean metaphysics.

The existence and form of the laws of nature is linked to our ability to discover them. It is because nature is compressible into local and symmetric mathematical forms that it is intelligible to us[12]. If we couldn't discover the laws of nature by doing local experiments — if, for example, we needed to understand the whole system before we could understand any part of it — then we would never be able to discover them at all. Similarly, if there were no stable structures (i.e. structures which are invariant under translations in space and time) then there would be no laws of nature.

There seems to be a curious match between our minds and the laws of nature. If our minds were simpler then we wouldn't be able to fathom the laws of nature; we would just operate like automata in response to learned brain processes. If our minds were too complex, our brain signals would degenerate into chaos. Our minds seem to be just complex enough for us to discover the

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[11] Despite recent travails in the development of String Theory and the resort of some physicists to a multiverse explanation, I still count it as a candidate for TOE.

[12] To use Barrow's terminology, see Section 2.3.5.

underlying mathematical structures of nature. Evolution can explain some of our capacity to discern such mathematical structures but it doesn't explain how we came to understand the laws of the quantum world which have had no direct impact on human evolutionary fitness. Steiner claims that we discovered the laws of the quantum world by using irrelevant formal analogies from mathematical theories which themselves were developed using species-specific criteria of beauty and convenience. Hence, he concludes that the universe is "user friendly" [Steiner, 1998: 176] and that "the human species has a special place in the scheme of things" [Steiner, 1998: 5]. I dispute that mathematics is predominantly anthropomorphic so I dispute his anthropocentric conclusions. Nevertheless, I agree with much of his analysis. I take the more traditional line that it points to the universe being rational: the laws of nature are written in the language of mathematics which is linked to the language of thought. Moreover, I think it indicates that there is something special about the mind: the mind is more than a computer; it can understand and assign meaning to mathematical forms.

Claim 9. Mathematics is unreasonably effective in applications at the cutting-edge of fundamental physics.

Yes.

This is the claim that I shall be defending in my thesis. It is suggested by the testimony of many great physicists (including, of course, Wigner himself). It is suggested by the existence and form of the laws of nature and our ability to fathom them. It is suggested by the many examples which Steiner gives of irrelevant formal mathematical analogies leading to discoveries in fundamental physics. It is mysterious that formal mathematics analogies can be used to make such accurate predictions of new phenomena. As we probe deeper, our theories become more accurate; closer and closer to an isomorphism with the objects and relations of ultimate reality. The objects themselves become less intuitive; less like familiar macro-objects, more like physico-mathematical signs.

Looking back on our review of mathematical applications, there are some broad categories which can be identified (with no suggestion that these categories are mutually exclusive or exhaustive).

Frege identified a class of applications in which mathematics predominantly functions as logical reasoning about the contents of thought. Applications are interpreted as being instantiations of highly general truths of logic by specific concepts and relations. If the ultimate referents of applied mathematical theories are non-mathematical objects then we end up with a nominalistic, indexing account of applications.

Structuralists identify a class of applications which can loosely be defined as pattern-matching. For this class, mathematics provides a well-stocked warehouse of potentially useful structures and application involves shoe-horning the physical problem into a suitably chosen mathematical structure by simplification, idealisation and inferential mapping.

In trying to penetrate the “next level” of reality, fundamental physicists are in the position of not knowing about the structure they are trying to characterise. For this (third, and final) class of applications, mathematics is not just a logical, descriptive or predictive tool; it plays an active role in the discovery process, actively structuring the way we do and interpret experiments.

It often seems as though mathematical thought mysteriously creates physical objects which are then discovered. I see this as being an extrapolation of the way we conceptualise the world in general. Let me elaborate. If you consider the human brain as being like a Bayesian neural network then what it is doing (as a matter of everyday survival) is taking inputs from our sensory apparatus, applying learned mathematical filters, and creating a predictive model of the world which gets post-dated with actual data. For example, in following the trajectory of a ball in flight, it will always strive to be one step ahead, filling informational gaps with predictions based on past experience and hard-wired interpretative tools. We never “see” what is really there; only an internal representation created by our minds.

Now, our minds have the remarkable capability of understanding something about the way our brains work, thus permitting us to model the brain’s behaviour in a computer. This makes the mathematical functioning of our brain apparent to us. It makes us realise (as computer scientists) how difficult the problem of identifying and representing objects is in practice. What we instinctively interpret as physical objects are really invariant contours of processed data. In conceiving of some mathematical objects, such as triangles, we need only use the natural processes of our brains in a conscious way to abstract out important features of objects. In this way, it is easy to construct a triangle, say, by conceiving of three “points” and filling in the gaps with “straight lines” because our natural brain machinery takes input like this and fills in the gaps to represent physical triangular shapes. Where there is no information it naturally fills in gaps based on some sort of minimisation principle, leading to artefacts like straight lines.

Now, some physical objects cannot be constructed without the conscious use of mathematics. They are physico-mathematical objects. Electromagnetic fields are a case in point. We can see 'light' and feel 'heat' but electromagnetic fields, as such, cannot be observed without using the mathematical theory which describes them. We need to construct new mathematical filters (not inherent in our brains) to interpret phenomena in terms of electromagnetic fields.

At the next level, in fundamental physics, we are dealing with objects which we would have no intuition of at all without mathematics. Mathematics allows us to discover them. It enables us to construct filters which we can then use to interpret the quantum world in terms of physico-mathematical objects.

The philosophical question arises as to whether or not there is any fundamental difference between physical objects, physico-mathematical objects, and mathematical objects[13]. If we identify physical objects with "things in themselves" then we can never know them. We are really always dealing with physico-mathematical objects which we construct from phenomena using mathematical filters. The main difference between them and mathematical objects is that mathematical objects can be conceptualised anywhere anytime without phenomenological input. Moreover, it may be that there is no possible phenomenological input which would be interpretable as some given mathematical object by any rational mind, although we cannot be definitive about that.

People ask: Are electromagnetic fields real? Are quantum fields real? Are quarks real? They are as real as each other and as real as everyday objects. They are ways we represent reality after applying mathematical filters to structure the underlying phenomena. There is nothing particularly different about quantum objects in this respect except that they obey mathematical rules different from the ones hard-wired into our brains.

In the following chapters I will be exploring the mysteries outlined in this section: the existence of laws of nature written in the language of mathematics and our minds ability to fathom them; the use of formal mathematics in discovering things about the quantum (i.e. non-Darwinian) world; the fact that deep mathematics is needed to describe fundamental physics; the asymptotic nature of the quest for knowledge. Key to these mysteries is the unreasonable effectiveness of mathematics in applications at the cutting-edge of fundamental physics. This, in essence, was Wigner's argument, but he overestimated

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[13] See e.g. [Resnik, 1991], [Hale, 1987], [Zolta, 1983]

the ability of the mathematical mind to distance itself from reality. Mind and reality are inextricably linked. It will be my thesis that the mysteries which I have outlined are explainable within a metaphysical framework that takes mind and mathematics as its fundamental principles.

Chapter 3. The Role of Mathematics in Fundamental Physics

3.1 Overview

In this chapter I argue that the unreasonable effectiveness of mathematics in fundamental physics has become evident through our growing understanding of the significance of symmetry principles in nature and the ever-expanding repertoire of mathematical concepts and tools developed to extend that understanding. Symmetry principles are not inherently mathematical — they have formed the conceptual basis of philosophical arguments since earliest times. However, in the last hundred years or so they have been taken to a new level of sophistication, enabling us to explore unobservable physical realms where the concepts are, of necessity, *physico-mathematical*, in the sense that the unobservable physical entities are directly represented by the mathematical forms and our only knowledge of them is implicit in the mathematics. For example, consider the modern gauge theory of quantum chromodynamics (QCD) which describes the dynamics of quarks and gluons inside nucleons. The forces and particles which it contains are determined by the invariance of its equations under the local symmetry group SU(3). Wilczek, one of the discoverers of QCD, writes:

> Quarks and gluons are … not "just another layer"… To an extent that is qualitatively new in science, they are *embodied ideas*… [Their fields] are mathematically complete and perfect objects. You can describe their properties completely using concepts alone, without having to supply samples or make any measurements… Gluons are the objects that obey the equations of gluons. [Wilczek, 2008: 33-34].

QCD is a great achievement of modern physics and it gives us a glimpse of what a true Pythagorean theory would look like.

It is not my intention to tell the story of symmetry from the point of view of physics — that has been done elsewhere[14]. I am interested in what the discoveries of physics tell us about the nature of ultimate reality and its link to mathematics. Therefore I shall often state the consequences of symmetry principles for physical theories without giving full technical justification. The reader is encouraged to consult the technical literature to fill in (or expose) any gaps in my reasoning.

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[14] I refer the reader to Zee's popular account [Zee, 1986] and to technical books such as [Cheng and Li, 1984].

In Section 3.2 it is shown how the search for an impersonal viewpoint in physics has led to the increasing use of local symmetry principles in fundamental theories. 20th century physics started with an emphasis on *global* symmetries (e.g. the Lorentz symmetry of Special Relativity). However, it was realised that global symmetries need to be defined relative to an absolute background and that any such absolute structure in our theories is undesirable because it exists "outside" the system. A relational strategy was followed to identify and remove absolute structure and replace it with relations which evolve subject to dynamical law. Part of this strategy involved removing global symmetries and making them local (i.e. making them depend on local coordinates). Local symmetry principles led to the General Theory of Relativity and gauge theories of particle physics — the two pillars of modern physics.

The brief exposition of Section 3.2 serves to underline the importance of symmetry principles but it threatens to make their discovery seem all too easy. In Section 3.3 I look at some historical case studies to give a sense of the struggle which went into developing our (now) standard theories. I ask why it isn't easy to construct a unified theory of physics. It turns out to be very difficult to satisfy the twin requirements of consistency and backwards compatibility. Just adding new symmetry doesn't work because a mechanism must be given to explain how that symmetry is broken or hidden in the real world. I discuss some of the problems which plagued quantum field theory before an understanding of spontaneous symmetry breaking was developed. Then I ask whether the discovery of local symmetry principles in physics in the 20th century depended on the unreasonable effectiveness of mathematics. I consider the case of the introduction of the SU(3) symmetry group. I conclude that this was not unreasonable, despite the fact that SU(3) applies to an abstract, three-dimensional, complex space with no perceptual links to any real physical symmetry. Rather, the introduction of SU(3) followed a sequence of reasonable heuristic moves in the analysis of data from scattering experiments. Nevertheless, I argue that there is something unreasonable about the overall success of the group theory strategy in fundamental physics — the heuristic moves led to the uncovering of the beautiful, underlying mathematical theory of QCD. QCD is 'almost Pythagorean' in the sense that it is consistent, it essentially has no adjustable parameters, and it is well-behaved at arbitrarily high energies (unlike other quantum field theories which break down at high energies and require the imposition of a high-energy cutoff in order to prevent infinities appearing). In retrospect, it appears to be a milestone on an asymptotic journey towards the ultimate, true theory.

In Section 3.4 I explore the relationship between symmetry principles and Pythagoreanism. First I point out that the increasing importance of symmetry principles in fundamental physics has led to an enormous increase in the level of mathematical sophistication required to work at the cutting-edge. As physicists probe deeper and deeper into the ultimate nature of reality, using symmetry principles as their guide, mathematics becomes ever more effective. I outline a possible narrative for physics which incorporates the struggles of great thinkers such as Ptolemy, Kepler, Newton, Einstein, Dirac and Witten into a story of the triumph of Pythagoreanism culminating in the development of the Theory of Everything (perhaps some extension of String Theory). Present projections of this narrative are in danger of being derailed by future experiments. I ask whether nature is still following the script. I discuss the challenges presented by attempts to link String Theory with phenomenology; in particular, the idea of the multiverse, which is currently gaining traction. Many physicists now believe that there are multiple possible ground states that can condense out of String Theory and that each of these corresponds to a different universe with different dimensionless fundamental constants. Then the detailed structure of our world would be contingent and physics would be like an environmental science investigating the properties of our little patch of reality. I argue that the evidence is not in yet. It is possible that a deep conceptual principle will be found which confers uniqueness on the ultimate theory. Regardless, we have already learned that the unreasonable effectiveness of mathematics in fundamental physics is due to the fact that as we probe the structure of nature at smaller and smaller length scales we discover more and more symmetry. I return to Steiner's argument for the unreasonable effectiveness of mathematics in physics and discuss similarities and differences with my own approach. The fundamental difference is that his approach is epistemological — he argues that Pythagorean strategies have played an essential role in discovery in physics but doesn't draw any metaphysical conclusions — whereas I embrace a form of metaphysical Pythagoreanism, arguing that the reason why Pythagorean strategies have played an essential role in discovery is because the underlying structure of reality is deeply mathematical and is describable by a Pythagorean theory.

In Section 3.5 I am more explicit about what I mean by a Pythagorean theory. I differentiate my ideas from Ancient Pythagoreanism and from ontic structural realism. I define a Pythagorean theory as one which satisfies two criteria: (i) it is a *principled* theory in which the consequences flow logically from a few principles without further empirical input; and (ii) the principles on which it is based are synthetic *a priori* principles shared by all rational beings. I consider the extent to which known theories of physics are Pythagorean. Examples are given of metaphysical principles which could be considered synthetic *a priori*: principles of logic, of symmetry and (by way of introduction) the principle of the One and the Many. I note the similarity between the principle of the One and the Many and Russell's paradox in set theory, then raise the issue of the extent to which the axioms of set theory are innate. Some preliminary discussion of set theory is given in preparation for a more detailed exposition in the next chapter. Connections between set theory and physics are explored in the context of the effective field theory view of physics. I speculate that any "missing" principles in physics will turn out to be innate, mathematical principles with which we are already familiar under a different guise. I summarise the consequences of my metaphysical views for the philosophical problem of the applicability of mathematics.

The last section, Section 3.6, contains my closing thoughts on the principles which underlie metaphysics, mathematics and fundamental physics. I note that thinking about the origins of the universe acts to focus the mind on contentious distinctions common to all three subject areas (e.g. the difference between the possible and the actual; the abstract and the concrete; the necessary and the contingent; mathematical structure and physical structure) without introducing the complexity of the later universe. Referring to the creation story in Plato's *Timaeus*, I argue for the unreasonable effectiveness of ancient metaphysics in dealing with fundamental concepts such as time and the ultimate structure of reality. Reviewing recent work in cosmology and particle physics, I identify a thought pattern which drives physicists to contemplate the extremes of infinity and nothingness, and to oscillate from one extreme to the other. This is a thought pattern familiar to metaphysicians and mathematicians and it leads in to the discussion of set theory in the next chapter.

3.2 The Development of an Impersonal Viewpoint

3.2.1 Removing fixed background: global *versus* local symmetries

The pre-Socratic philosopher Anaximander argued that the Earth floats in the centre of infinite space, not supported by anything, because all the forces on it are symmetrical and it has no reason to move in one direction rather than in another. This argument embodies two principles which are still influential today: (i) the principle of the homogeneity and isotropy of space; and (ii) the 'principle of indifference' whereby a system in equilibrium will not move out of equilibrium without some reason. The second principle is related to Leibniz's Principle of Sufficient Reason which he used to argue for the relationalism of space and time. If, contrary to relationalism, space and time were absolute, then one could ask questions such as 'why did God create the universe where it is rather than displaced by thirty kilometres?' and 'why did God create the universe when he did and not a thousand years earlier?' Leibniz argued that no sufficient reason could differentiate between such alternatives and, since God always acts with sufficient reason, space and time cannot be absolute but must be relational.

More recently, Smolin [2005] has used Leibniz's principle to argue that fixed background spacetimes should be eliminated from the formulation of physical law. He advocates a form of relationalism in which fundamental kinematics and dynamics depend only on the relations between entities and nothing exists outside of these relations. A consequence of his view is that physical theories should not have global symmetries. Thus, he states:

> A cosmological theory should not have global symmetries, for they generate motions and charges that could only be measured by an observer at infinity, who is hence not part of the universe. [Smolin, 2005: 7].

Global symmetries are the ones with which we are most familiar — they include the invariance of physical laws under rotations and translations of the reference frame. Without them, it would be difficult to do physics at all — experiments might give different outcomes depending on where they were performed or on the orientation of the laboratory. Nevertheless, in calling for their elimination, Smolin is simply following the trend in modern physics which has been to explain the existence of global symmetries as a dynamical consequence of fundamental equations which are themselves completely local. This trend is motivated by considerations of locality (i.e. that the laws of physics should involve only local measurements and not action at a distance) and relativity (i.e. that the laws of physics should be the same for all observers regardless of the state of motion they are in). Making symmetries local goes part of the way towards removing biases due to a particular perspective.

Global symmetries need to be defined relative to a fixed background spacetime. Wigner [1979: 17] calls them "geometrical" symmetry principles because they are a feature of spacetime itself. Their invariants are defined in terms of active transformations applied to physical events. Global symmetries were the focus of early 20th century physics. Einstein extracted the global Lorentz symmetries from Maxwell's equations of electromagnetism and made them the principles of his Special Theory of Relativity. The full symmetry group of Special Relativity (and, hence, of Minkowski spacetime) is the Poincaré group — which is a combination of the Lorentz group with the group of translations. Wigner identified the elementary particles of quantum mechanics as irreducible representations of the Poincaré group and provided a complete classification in terms of their mass and spin[15] [Schweber, 1962: 46-53]. In this way, he used the global symmetries of spacetime to define what is meant by an elementary particle.

[15] Or, for massless particles, in terms of their *helicity* which is the component of spin in the direction of motion.

Nowadays it is thought that global symmetries are not fundamental because they require a fixed background outside of the laws of physics. Fundamental symmetries must be local. Gross writes:

> Today we believe that global symmetries are unnatural. They smell of action at a distance. We now suspect that all fundamental symmetries are local gauge symmetries. Global symmetries are either broken, or approximate, or they are the remnants of spontaneously broken local symmetries. [Gross, 2004: 58].

Local symmetry principles express the invariance of physical laws under a transformation of our description of the physical situation. Wigner [1979: 17] calls them “dynamical” because they are formulated in terms of the physical laws, rather than the physical events, and involve only passive transformations (i.e. changes of description), rather than active transformations which yield a different physical situation.

Local symmetry principles are applied by taking a global symmetry of the physical laws and making it depend on local coordinates. For example, Einstein’s Principle of Equivalence (which was his inspiration for the General Theory of Relativity) is a local symmetry principle which says that in any region of spacetime, in an arbitrary gravitational field, one can always choose an unaccelerated coordinate system, without gravitation, in which the laws of physics take the same form. Applying this principle corresponds to taking the global Lorentz symmetry of Special Relativity and making it local.

Local symmetry principles restrict the form of the physical laws [Cheng and Li, 1984: 235-240]. When the local reference frame can change, the significance of comparing two objects (e.g. vectors or tensors) at different spacetime points is lost. To compensate for this, a mathematical connection must be introduced which allows objects to be compared by (notionally) parallel-transporting them along smooth curves in the spacetime manifold until they are at the same point and can be compared directly. In General Relativity, this mathematical connection is called the Christoffel symbol and it plays the role of the gravitational field in Einstein’s equations. By requiring the laws to be invariant under local changes of reference frame (i.e. by imposing *general covariance*), we necessitate that a field be introduced to transport perturbations from point to point and this field is just the gravitational field.

In gauge theories in particle physics an analogous method gives rise to the strong, weak and electromagnetic fields. For example, in electrodynamics the laws for a free-electron field have a global symmetry because they are invariant under global changes in the phase of the field [Cheng and Li, 1984: 229-231]. We can make the symmetry local by requiring that the laws be invariant under local changes of phase (i.e. we make the phase change depend on spacetime position). This necessitates the introduction of a mathematical connection — the gauge field — which, in this case, corresponds to the familiar electromagnetic field. Because changes of phase amount to rotating the unitary wavefunction in the complex plane, the gauge group of electromagnetism is the unitary group U(1).

In modern physics symmetries are more than just classificatory tools — they dictate the kinds of forces and particles that a theory may contain. In General Relativity, a local symmetry principle — Einstein's principle of equivalence— requires the existence of the gravitational force. Similarly, in the Standard Model of particle physics, local gauge symmetries require the existence of the electromagnetic, weak and strong forces and the particles that carry them. There is a sense of everything flowing from an understanding of symmetries, as expressed by Gross:

> ...from a simple principle of symmetry we deduce in an almost unique fashion the nature of the forces of nature and the existence of the carriers of these forces — the graviton that underlies gravitational forces, the photon of light, the gluons that hold nuclei together, and the W and Z mesons that are responsible for their radioactive decay. [Gross, 1988: 8372].

More flamboyantly, Weinberg comments:

> At the deepest level, all we find are symmetries and responses to symmetries. Matter itself dissolves, and the universe itself is revealed as one large reducible representation of the symmetry group of nature. [Weinberg, 1987: 80].

In unified field theory, symmetries predict new forces and new forms of matter. The large symmetries of String Theory offer the promise of a Theory of Everything. Symmetry principles are more fundamental than laws. They are our primary guide in the search for deeper physical theories.

3.2.2 Weyl's world geometry

There is no privileged coordinate frame in General Relativity. The physical content of the theory is a collection of coincidences (or, equivalently, point events; such as a measuring rod being brought into coincidence with the end point of a physical body [Ryckman, 2005: 188]). Such coincidences are intersections of world-lines in spacetime and are preserved under general coordinate transformations. General coordinate transformations just relabel the spacetime coordinates assigned to each coincidence; i.e., they change the point of view without changing the physical content of the theory. The coincidences do not determine a fixed background spacetime[16]. However, Smolin considers General Relativity to be only partly relational because the differential structure, dimension and topology of spacetime are fixed in the theory [Smolin, 2005: 11]. The differential structure determines the system of relationships between the coincidences; such as their causal order and their measure (i.e. the spacetime volume of sets defined by the causal order).

When General Relativity was first introduced Weyl was dissatisfied with its fixed features [Ryckman, 2005: 145-176]. In particular, he objected to the imposed differential structure because it violates the symmetry of local scale invariance. The metric of General Relativity acts like a natural gauge which allows the direct comparison of distant magnitudes (e.g. the magnitude of vectors at distant points can be compared, though not their direction). In effect, Einstein had assumed the existence of rigid rods and perfect clocks. Weyl thought that the geometry of spacetime should allow for arbitrary rescalings of the local unit of length at each spacetime point. He wanted to restore local scale invariance to the theory.

Weyl's motivation was a philosophical one. He was influenced by the transcendental phenomenology of Husserl and "sought to construct physical reality from the material of immediate experience using only what is given in the infinitesimal neighbourhood of the cognising ego" [Ryckmann, 2005: 148]). Thus, in his theory, the world geometry would be entirely local. The imposition of a local coordinate frame would be:

> ... the unavoidable residue of the ego's annihilation in that geometric-physical world which reason sifts from the given under the norm of 'objectivity'. (Weyl as quoted in [Ryckmann, 2005: 149]).

Unpacking this statement, the 'given' is the physical content of the theory — i.e. the collection of coincidences — and a local coordinate frame is unavoidably imposed in order to construct the relationships between the coincidences — i.e. to make sense of what is given — under the norm of 'objectivity' — where Weyl thought that objectivity is achieved by giving objective status only to those relations that are invariant under particular transformations.

According to Weyl, physics is about constructing the underlying relations locally in an objective way. This is what makes it inherently mathematical because it is only by using mathematics that we can

[16] As demonstrated by Einstein's hole argument [Norton, 2003: 114].

abstract away from our individual viewpoint. Reality is not mind independent but it is what would be agreed upon by all cognising egos acting under the norm of objectivity. Poincaré had expressed a similar view:

> But what we call objective reality is, in the last analysis, what is common to many thinking beings, and could be common to all; this common part … can only be the harmony expressed by mathematical laws. [Poincaré, 1906: 14].

The emphasis on objectivity is what makes symmetry so important in Weyl's physics. According to Weyl, "all *a priori* statements in physics have their origin in symmetry" [Weyl, 1952: 126]. Symmetries are what permit reason to sift the objective from the given. They separate what is invariant ('real') from what is ephemeral ('appearance') under a change of perspective. The objects of the theory are characterised by invariance under particular groups of transformations relevant to the context. For example, as previously noted, Wigner had shown that elementary particles are characterised by irreducible representations of the Poincaré group.

Weyl's modification of General Relativity to include local changes of scale (i.e. *gauge* transformations) was the original gauge theory. The requirement of gauge invariance necessitated the introduction of mathematical structures which he identified with the electromagnetic field, so it led to a unified field theory incorporating the equations of General Relativity and electromagnetism. Unfortunately, it did not meet the test of experiment. Einstein pointed out that a consequence of Weyl's theory would be that the spectral lines of atoms would depend on their world-lines since atoms transported along different paths through an electromagnetic field would experience different integral gauge effects [Ryckman, 2005: 86-87]. Experiment shows that the spectral lines of atoms are independent of such effects. In a practical sense, quantum systems such as atoms with definite spectral lines provide us with the standard rods and clocks assumed by Einstein in his original formulation of General Relativity.

Weyl wasn't disturbed by Einstein's objection. He took the natural gauge of the world to be an experimental fact which needed to be explained by a deeper theory, rather than assumed. He thought that the observed absence of local scale invariance in nature was due to dynamical interactions with matter. However, he was unable to incorporate his ideas into an empirically-acceptable unified theory and he was overtaken by other developments in quantum physics. Later, Weyl reframed his gauge theory of electromagnetism as a symmetry of the *phase* of the electromagnetic field, rather than as a geometrical invariance of spacetime. As such, it has become a fundamental component of today's Standard Model of particle physics. The gauge concept suffered many setbacks on its path to acceptance and it is a prime example of a mathematically-elegant idea which some physicists persisted with in the face of (apparent) empirical disconfirmation[17].

3.2.3 The relational strategy

Weyl's thinking has been very influential. Ryckman credits him with "reviving the method of *a priori* mathematical conjecture in fundamental physical theory" [Ryckman, 2003: 85]. Eddington was influenced by his views and created his own geometrical theory of the world which led, in effect, to an alternative derivation of the equations of General Relativity, not depending on the assumption of rigid rods and perfect clocks. Einstein, too, pursued his own (unsuccessful) unified field theory program in which all physical phenomena were to be interpreted as manifestations of spacetime geometry. The early attempts to geometrise physics were spurred on by Kaluza's discovery in 1921 that the extension of General Relativity to a five-dimensional spacetime naturally unifies gravitation and electromagnetism (i.e. a similar result to that achieved by Weyl in his original gauge theory but using a different approach —extra dimensions rather than local scale invariance). Following Kaluza's discovery, Klein suggested that the extra dimension must be *compactified* (i.e. curled up into a circle with a radius which is tiny compared to the size of elementary particles) so that the motion of objects through it is undetectable.

In modern physics, the extra-dimensional theory of Kaluza-Klein and Weyl's gauge theory of electromagnetism are conceptually unified using the mathematics of *fibre bundles*. A fibre bundle is a geometrical object which attaches an abstract internal space (in mathematical terms, the *fibre*) to each point of spacetime (mathematically speaking, the *base*). The internal space has its own local symmetries which generate physical effects in the theory. For example, the fifth dimension of Kaluza-Klein theory can be thought of as a circle attached to each point of a four-dimensional Riemannian spacetime. The circle has a U(1) symmetry which generates the structures of electromagnetism via the introduction of the mathematical connection needed to compare internal directions at different spacetime points. In effect, in Kaluza-Klein theory, the gauge invariance of electromagnetism arises from a larger symmetry

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[17] As discussed in the previous chapter, sometimes it is worth persisting with a beautiful theory which doesn't fit the experimental data because it may point the way to a new principle which resolves the underlying difficulties. Yang-Mills gauge theory provides a historical case in point. It requires the gauge bosons (force carriers) to be massless, whereas the W and Z bosons of the weak force are known to be massive. As a consequence, it was not taken seriously for many years. Then, when augmented by a new understanding of the principle of spontaneous symmetry breaking, it became the cornerstone of the Standard Model.

— that of general coordinate invariance of a higher dimensional theory of gravity [Gross, 1992b: 964]. More generally, the internal space of the fibre bundle can be a multi-dimensional manifold with a complex geometrical form, and the base need not be a four-dimensional Riemannian space. Thus, the Yang-Mills gauge theories of quantum physics can be geometrised using fibre bundles on a base of flat Minkowskian spacetime with an internal space having the appropriate Lie group symmetry. An internal space having SU(3) symmetry would generate the fields and particles of QCD.

String Theory has large, and not completely known, symmetries. It can only be consistently formulated in multiple dimensions — i.e. either 26 spacetime dimensions for the bosonic string or 10 for the superstring. The extra dimensions are compactified into internal spaces. In ten-dimensional String Theory, the six extra dimensions are compactified into a six-dimensional shape called a Calabi-Yau manifold. The precise symmetries of the Calabi-Yau manifold determine the fields and particles which are generated in the theory. Its topological features determine other physical effects. For example, the number of holes in the manifold determines the number of families of particles. A Calabi-Yau manifold with three holes would generate three families of particles, potentially resolving the mystery of why three families of leptons and quarks are observed experimentally. Indeed, the precise geometry of compactified internal spaces has the potential to resolve many mysteries. The catch is that there is no principle we know of which picks out a particular Calabi-Yau manifold from amongst the many possibilities.

The geometrisation of physics has come a long way since the time of Weyl, Eddington, Einstein, Kaluza and Klein. It is now, once again, at the forefront of research in fundamental physics. What all attempts at geometrisation have in common is the pursuit of *a relational strategy*, as defined by Smolin:

> [They] seek to make progress by identifying the background structure in our theories and removing it, replacing it with relations which evolve subject to dynamical law. [Smolin, 2005:10].

This strategy is linked to the desire to achieve an impersonal viewpoint. It leads to wider and wider classes of observers, broader and broader geometries, more and more symmetries. Einstein removed the absolute space and time of Newtonian physics. Weyl sought to remove the fixed differential structure of General Relativity. Kaluza removed the fixed number of dimensions. String Theory removes the fixed topology: it allows the fabric of space to be torn continuously in such a way that solutions can be followed as they evolve from one topology to another. In fact, String Theory totally changes our view of spacetime at distances smaller than the Planck length. Such distances cannot be measured in principle in String Theory because of the finite size of strings (which are the smallest probes in the theory). If we try to squeeze a compactified dimension of space into a radius R less than the Planck length it morphs into an expanding dimension of radius 1/R in a dual theory (an effect called 'T-duality'). String Theory calls into question the conceptual separation of physical events and spacetime which is enforced classically by a background structure in which physical events occur *inside* spacetime. It allows spacetime itself to emerge from a more fundamental dynamics.

A system with perfect symmetry would be invariant under any and all transformations and would have no discernible structure. As the underlying theory of physics becomes more and more symmetric, differentiating structure disappears. In unified theories, the elementary entities are identical structural units (such as strings) or a single, undifferentiated structural unit (such as a quantum field). The texture of the physical world arises dynamically by the spontaneous breaking of the underlying symmetries. In particular, this is how the different forces arise and how distinct entities acquire properties, such as mass and spin. Intrinsic properties are defined relative to the stable ground state which emerges when the underlying symmetry is broken. So, one cannot define the physical observables of the theory without solving the dynamics. Fundamentally, "… all observables are relational" [Smolin, 2005: 13].

3.2.4 Structural realism

Structural realists have incorporated the insights from physics into a philosophy which emphasises the priority of structure and relations over objects and their properties. According to the epistemic version of structural realism, science gives us knowledge of the relations between fundamental entities through the structure of the governing equations, but that is all we can know, we cannot know the nature of the fundamental entities themselves. In fact, we cannot even know that there *are* fundamental entities — it is possible that all entities dissolve into structure and there is no fundamental level. In that case "it's relations all the way down" [Ladyman and Ross, 2007: 152]. According to the metaphysical version of structural realism — ontic structural realism — only structure is held to be ontologically significant (although it may be necessary to think conceptually in terms of objects in order to understand structure in the first place). Ontic structural realists suppose that objects and their properties can be constructed in purely structural terms using mathematical tools such as group theory:

> … the objects of a theory are members of equivalence classes under symmetry transformations and no further individuations of objects is possible. [Ladyman and Ross, 2007: 147].

According to this view, then, it is symmetries and their invariants that determine the content of physical theories.

Eddington characterised our objectification of reality as "the mind's search for permanence" [Ryckmann, 2005: 190]. We are naturally pre-disposed to identifying permanent patterns behind the changing flux of experience so we try to construct invariant objects out of structural relations. However, Eddington did not regard structure as ontologically prior to objects. For him, relations and their relata are equally fundamental:

> The relations unite the relata; the relata are the meeting points of the relations. The one is unthinkable apart from the other. I do not think that a more general starting-point of structure could be conceived. [Eddington, 1928: 230–231].

In the end I will side with Eddington against the ontic structural realists on this very contentious point.

In constructing reality from the background of mathematical possibility we need to develop dimensional concepts such as "mass", "length" and "time" and associated units of measure. It is not clear how this can be done from what is given *a priori*. Eddington, influenced by Weyl, thought that the world has an inherent "natural gauge" that underpins Einstein's assumption of rigid rods and ideal clocks in General Relativity. He reasoned that:

> Since any apparatus used to measure the world is itself part of the world, the natural gauge represents the world as self-gauging. (Eddington as quoted in [Ryckmann, 2005: 231]).

So, the theory of measurement becomes fundamental to our understanding of reality.

In this section I have examined the role of symmetry principles in developing an impersonal viewpoint in physics and the concomitant mathematicisation of our physical theories. The brevity of the presentation has made it all sound too easy, as though the underlying principles were obvious all along. In the next section I will look at some historical case studies to give a sense of the struggle which went into developing our (now) standard theories. As in all such endeavours, it is the difficulties and the struggle which makes the final breakthrough more meaningful.

3.3 It's Not That Easy: Case Studies

3.3.1 Triumphs and failures

Why isn't it easy to construct a unified theory of physics? If symmetry is the answer then why can't we just include more symmetry in the underlying theory until it explains everything? For example, one might try to generalise Kaluza-Klein theory in 5 dimensions to higher dimensions. After all, Kaluza-Klein theory in 5 dimensions served to unify General Relativity and electromagnetism. One might think that by adding extra dimensions and compactifying them we could incorporate the gauge theories of the weak and strong force. As Gross writes:

> ... one can imagine ... a world of ten dimensions, with nine spatial dimensions in which six of these are curled up into little circles so that they are unobservable, except for the remnants of gravity which would appear to us as nuclear, weak and electromagnetic interactions. [Gross, 1992b: 965].

It would be very satisfying to explain all the forces, including the gauge forces, as being consequences of gravity. Well, in 1938 Klein *did* try to construct a theory of the weak and strong force by generalising Kaluza-Klein theory to include more dimensions [Gross, 1992b: 964-965]. He was able to derive the equations of General Relativity and some of the equations of Yang-Mills theory but not to explain the mechanism by which the massless gauge bosons of Yang-Mills theory become the massive mesons of the weak force without destroying gauge symmetry. Since that time, many other physicists have tried to generalise and extend the ideas of Kaluza and Klein to unify all the forces in a gravitational framework. Unfortunately, none of these efforts has been successful.

Adding symmetry is problematic because the resulting theory has to be (i) consistent and (ii) backwards compatible with known theories in the low energy limit. The second criterion implies that we need to find new dynamical mechanisms for breaking or hiding the extra symmetry:

> It is not enough to invent a new symmetry one must also explain how the new symmetry is broken or hidden. After all, if it were not hidden or broken then it would be evident — not a new symmetry. [Gross, 1992b: 960].

In Yang-Mills gauge theory, invariance under the local symmetry requires that the gauge bosons be massless[18]. Also, in any relativistic quantum field theory, the spontaneous breaking of a continuous symmetry produces massless bosons called Goldstone bosons. It took many years to work out a mechanism (the Higgs mechanism) by which the W and Z bosons of the weak force could acquire mass in a gauge invariant theory without generating a host of unwanted (because unobserved) Goldstone bosons. Yang and Mills first presented their theory in 1954 but it wasn't until 1964 that Brout, Englert, Higgs, Guralnik, Hagen and Kibble developed the Higgs mechanism and not until 1967-68 that Weinberg and Salam incorporated it into electroweak theory. The Higgs boson wasn't detected until 2013.

In the Higgs mechanism, a scalar field satisfying the Yang-Mills gauge symmetry is added to the theory. When the Higgs field acquires a non-vanishing vacuum expectation value, thereby spontaneously breaking a continuous symmetry and generating a Goldstone boson, the Goldstone boson mixes with the massless gauge boson in such a way that the former vanishes and the latter acquires mass. In Weinberg-Salam electroweak theory, four Higgs fields are added. Three Goldstone bosons are "eaten" to give masses to the W and Z bosons that mediate the weak interactions.

In applying Yang-Mills theory to the strong interactions more problems arise because quarks and gluons are confined within the nucleus by the strong force so that an isolated quark is never detected in an experiment. The absence of any quark signature in high-energy scattering experiments meant that, for a long time, quarks and gluons were considered to be mere instrumental devices in the mathematical theory rather than real particles. It wasn't until 1973, when Gross and Wilczek discovered asymptotic freedom in SU(3) Yang-Mills gauge theory (thus showing that the gluons of QCD anti-screen the quarks, causing the strong force between two quarks to decrease as they approach one another and, conversely, to increase as they are pulled apart), that confinement could be understood and the mechanism for hiding SU(3) symmetry explained (further details are given in Section 3.3.3).

Nor is it easy to achieve consistency in a relativistic quantum field theory (criterion (i) above). The Lorentz symmetry of Special Relativity is almost incompatible with quantum mechanics. Quantum field theories of mass zero, spin one particles violate Lorentz invariance unless the fields are coupled in a gauge invariant way, but the interactions in a local field theory bring in couplings to arbitrarily high frequency modes, with no suppression, which leads to non-renormalisable (i.e. potentially infinite) terms.

[18] See e.g. [Cheng and Li, 1984]

Quantum electrodynamics (QED) was the first theory to reconcile quantum mechanics and special relativity. In QED, the potential infinities are absorbed into the constants of the theory in a way which renders it finite at any order in a perturbation series (i.e. by the process of renormalisation[19]). QED is a remarkably successful theory, predicting the magnetic moment of the electron to an accuracy of one part in a trillion [Gabrielse, 2013: 64]. However, it is fundamentally a perturbation theory — it is not well-defined as a quantum field theory up to arbitrarily high energy [Gross, 1992a: 6]. The problem is that the renormalised interaction strength increases with increasing energy and there is an energy scale at which it becomes infinite. This problem infects all quantum field theories which are not asymptotically-free. Thus, non-asymptotically-free theories are inconsistent at very high energies. On the other hand, in asymptotically-free theories (such as QCD) the renormalised interaction strength *decreases* with increasing energy so they do not break down at high energies.

Gross [1992a: 3-9] recalls how quantum field theory was in disgrace in the 1950s. Despite the successes of QED, physicists thought that renormalisation was just disguising problems with infinities which would re-emerge when non-perturbative methods were attempted. They thought that the whole structure was potentially unsound. To compound the difficulties, no quantum field theory then developed was adequate to explain the wealth of data which were being generated in high-energy scattering experiments. S-matrix theory seemed a much better bet than field theory. It was tailored to calculating the scattering matrix from quantum mechanical principles "without the unphysical demand of fundamental constituents or equations of motion that was inherent in field theory" [Gross, 1992a: 7]. According to Gross, until QCD was developed in 1973 "it was not thought proper to use field theory without apologies" [Gross, 1992a: 8].

The requirements of consistency and backwards compatibility have spurred the development of unified theories of physics. Consistency considerations led to String Theory being formulated in 10 and 26 dimensions. Supersymmetry was incorporated in order to remove the unwanted tachyons (i.e. imaginary mass particles) which had plagued earlier versions of the theory. This was "the first string revolution" [Greene, 2000: 139-140] in 1983 — a supersymmetric String Theory was developed which had a consistent quantum field theory and no tachyons. Furthermore, its symmetries led to both general coordinate invariance and gauge invariance so it naturally unified gravity and gauge theories. Potentially, it could explain the distinguished role of these theories in our low-energy world [Gross,

[19] See e.g. [Cheng and Li, 1984: 31-56]

2004: 15]. These requirements are so difficult to satisfy in any nontrivial extension of the existing, successful framework of physics that any theory which can do so automatically gains a certain credibility. In the case of String Theory, however, research following the first breakthrough led to more successful superstring theories being found (i.e. five in all), creating the problem of identifying the "right one". Then, in the "second string revolution" in 1995, connections between the five known string theories were discovered and the hypothesis made that they are all subsets of one underlying fundamental theory called M theory. The saga is ongoing …

In the next section I will consider the question of whether the discovery of local symmetry principles in physics in the 20th century depended on the unreasonable effectiveness of mathematics. A case in contention is the introduction of the SU(3) symmetry group, which proved to be an important step on the path to QCD.

3.3.2 The introduction of the SU(3) symmetry group

The introduction of the SU(3) symmetry group into particle physics has been a matter of some contention in the philosophical literature[20]. Steiner suggests that there is something unreasonable about Gell-Mann's successful use of SU(3) in applications to the strong interactions. For one thing, it breaks the link with perceptual symmetries: the SU(3) group is applied to symmetry transformations in an abstract, three-dimensional, complex space which in no way relate to rotations of any real space of any dimension [Steiner, 1998: 92]. French [2000] defuses Steiner's account by showing how SU(3) was arrived at by a series of reasonable approximations and idealisations. Before discussing their dispute more fully, I will endeavour to fill in the requisite technical and historical background.

Group theory is an important tool in physics because the symmetry transformations form a group. If the fundamental symmetry principles of an interaction are known, then their consequences can be deduced using group-theoretical analysis. Symmetries are associated with conserved quantities[21] (e.g. the U(1) symmetry of the phase of the electromagnetic field is associated with electric charge) and particles can be grouped into degenerate multiplets (i.e. representations of the underlying symmetry group having the same energy or, equivalently, the same mass) labelled by the quantum states associated with particular conserved quantities. Thus, Wigner found all the degenerate multiplets

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[20] See e.g. [Steiner, 1998], [French, 2000], and [Pincock, 2012].

[21] Noether's theorem states that to any continuous one-parameter set of invariances of the lagrangian is associated a local conserved current [Itzykson and Zuber, 1985: 23].

corresponding to irreducible representations of the Poincaré group, labelled them according to their mass and spin, and identified them with elementary particles.

If the fundamental symmetry principles of an interaction are not known, group theory can still be used as a classificatory tool. Typically, to investigate the properties of the interaction, scattering experiments are run at relevant energies and new particles are identified in the debris. As data from the experiments is accumulated, rules-of-thumb are developed governing which particles can be produced under which circumstances and these rules give clues as to the underlying symmetries and their associated conserved quantities. Group-theoretical analysis sorts the data into degenerate multiplets associated with the guessed symmetries. Often these multiplets are approximate, rather than exact, in that they link particles which have only approximately the same mass. Then the guessed symmetry is only an approximate symmetry under the ambient experimental conditions. It may, or may not, hold exactly under different conditions (e.g. at higher energies or in the absence of the Higgs mechanism). Of course, only exact symmetries can be truly fundamental. They point the way to the underlying dynamics of the interaction under investigation.

In the early studies of nuclear reactions it was found that protons and neutrons have approximately the same mass and behave similarly under nuclear forces (obviously, they behave differently under electromagnetic forces because they have different electric charge). Ignoring their slightly different masses and different electric charge, Heisenberg postulated that protons and neutrons belong to a degenerate doublet of a new internal symmetry which he called 'isospin'. The name isospin derives from an analogy to the spin of an electron: both isospin and ordinary spin have symmetry which is given by invariance under the group SU(2) (note that SU(2) is a 2 to 1 covering of the rotational group SO(3) and the Lie algebras are isomorphic). However, unlike ordinary spin which has the units of angular momentum, isospin is dimensionless. It is a purely abstract symmetry denoting rotations in an abstract, two-dimensional, complex space. The concept of isospin was given empirical credibility by the discovery in 1947 of the three pions — π^+, π^0, and π^- — with approximately the same mass. They were interpreted as an isospin triplet.

As further particles were discovered in scattering experiments they were assigned to isospin multiplets. But when the Λ and K particles were discovered they showed anomalous decay properties which indicated that they possessed a new property, subsequently called 'strangeness'. Strangeness is associated with a U(1) symmetry and a conserved quantity called 'hypercharge' (by analogy with electric charge). In 1961, Gell-Mann proposed that isospin and hypercharge could be incorporated together in the larger symmetry group SU(3) which contains the product of their individual symmetry groups, SU(2) x U(1), as a subgroup. SU(3) is a group of rotations in a three-dimensional complex space. Each group element is described by eight real parameters, prompting Gell-Mann to give his scheme the (very 1960s) name of 'the Eightfold Way'. Gell-Mann identified degenerate multiplets of the SU(3) symmetry with certain groups of newly discovered particles. This established order in the existing "particle zoo" but there were some gaps in the classification. In a notable success, a gap in an identified decuplet containing nine existing particles was used to predict the properties of the "missing" particle and a particle with the correct properties — called the Ω^- particle — was later discovered. On the other hand, SU(3) was not as good a symmetry as isospin SU(2): there was large mass splitting in some of the multiplets of the Eightfold Way.

Gell-Mann's classification of particles was similar in many respects to the process used by Mendeleev in organising the chemical elements in the periodic table. In both cases symmetry was used as a guide but in neither case was it known at the time what constituted the underlying building blocks of the symmetry. It was hard to believe that all the particles in the zoo were equally fundamental. In 1964, Gell-Mann proposed that the particles of the strong interaction are bound states of fundamental constituents which he called 'quarks'. Quarks were to have spin ½ and come in three 'flavours' — 'up', 'down' and 'strange'. The flavoured quarks transformed as members of the fundamental representation of SU(3). This simple model soon ran into problems, however. For one thing, the quarks had to have fractional electric charge (-1/3 for the down quark and +2/3 for the up quark) and no such particles had ever been seen in experiments. As previously mentioned, the underlying reason for the absence of quarks in the scattering debris of experiments is that they are confined by the dynamics of the strong force, but the mechanism of confinement wasn't known in the 1960s so their absence was a serious problem. Many physicists relegated quarks to the status of useful mathematical fictions and, as late as 1972, Gell-Mann himself described them as "fictitious" [Fritzsch and Gell–Mann, 1972: 243].

A second, more serious, problem for the simple quark model was that it violated the connection between spin and statistics; i.e., it resulted in some spin ½ particles having symmetric wave-functions. This problem was remedied by an elaboration of the model to introduce a 'colour' degree of freedom for the quarks. Each flavour of quark was assumed to come in three different colours, forming a triplet under a colour SU(3) group. It must be emphasised that the colour SU(3) group has nothing whatsoever to do with the flavour SU(3) group. In fact, we now know that there are at least *six* flavours of quark — the three additional quarks being 'charm', 'bottom' and 'top'. The SU(3) symmetry of the Eightfold Way is now considered to be an accidental symmetry which arises because the up, down and strange quarks are relatively light compared to the others. In contrast, the colour SU(3) symmetry is considered to be a fundamental, exact symmetry of the strong interactions as encoded in the equations of QCD.

3.3.3 Philosophical debate

Now let's return to the philosophical dispute between Steiner and French. According to Steiner [1998: 86-90], the introduction of isospin involved the use of a Pythagorean analogy because it assumed the existence of an *abstract* SU(2) symmetry in analogy to the *real* rotational symmetry of ordinary spin. But at least SU(2) is isomorphic to the symmetry of the rotational group SO(3). When discussing the introduction of SU(3) he writes:

> …SU(3), by contrast, is not isomorphic even in the infinitesimal limit, to rotations in any dimension. [It] is just an abstract symmetry — invariance under a transformation of a three-dimensional *complex* space. When we get to SU(3), the link with perception has been snapped. And this makes the SU(3) hypothesis, the analogy to SU(2), grossly Pythagorean. [Steiner, 1998: 92].

Steiner points out that Gell-Mann didn't know about quarks at the time he introduced SU(3) and, later, he didn't believe in them, so he wasn't making a *physical* analogy to the constitution of nuclear particles by nucleons. It was purely an argument from mathematics. It was also "wrong", as it turned out, since the truly important symmetry in the strong interactions is the colour SU(3) symmetry. There *is* no flavour SU(3) symmetry except in a very approximate sense. Nevertheless, Gell-Mann's work led to the discovery of the quarks and gluons of the strong force. Therefore, according to Steiner, SU(3) was unreasonably effective in fundamental physics.

French [2000] aims to show how the introduction of group theory into particle physics proceeded by reasonable steps in all its stages. He focuses on the extension of group-theoretical methods from atomic to nuclear spectra. Through detailed historical analysis, he sheds light on the heuristic moves made by the participants at the time; confronted, as they were, by so many difficult problems in the emerging field of quantum mechanics and needing, as they suspected, new concepts and tools to tackle them. French's work illustrates how the edifice of quantum mechanics was constructed by ratcheting up from known physics using (sometimes dubious) analogies and idealisations. The process involved forging links to unfamiliar structures in mathematics, then importing new ideas from those structures, testing them in their physical theories and, sometimes, re-exporting them to be incorporated in the mathematical structures themselves. According to French:

> Within such a framework, the applicability of mathematics to science comes to be understood in terms of the establishment of a relationship between one kind of structure and another. [French, 2000: 104].

At no point does any individual move seem unreasonable — sometimes moves are bold, sometimes they are wrong but always they lead to new learning. Group theory and quantum mechanics fed off each other in their development, facilitated by the intellectual efforts of physicists-*cum*-mathematicians like Wigner.

French examines Wigner's application of group theory to the analysis of atomic and nuclear structure. He notes that Wigner was well-schooled in the use of matrices and group theory through his early work on symmetry in crystals [French, 2000: 108]. Hence, he was primed for his attack on the problem of atomic spectra which required those very skills. In his analysis of atomic spectra, the important symmetries were (i) the permutation symmetry between indistinguishable electrons in the atoms; and (ii) the rotational symmetry of electron spin. In extending his analysis from atomic to nuclear spectra, the necessary idealisations were (a) the treatment of protons and neutrons as quantum states of indistinguishable nucleons; and (b) the assumption that nucleons obey an isospin symmetry analogous to, and mathematically equivalent to, the symmetry of electron spin. The analogy is incomplete because protons and neutrons are not really indistinguishable and isospin is an abstract, rather than a geometrical, symmetry. However, neither of these idealisations is unreasonable (they are motivated by empirical results) and they enable the application of the SU(2) symmetry group to nuclear interactions.

Having dealt with the introduction of isospin in some detail, French skates through the introduction of SU(3) and further developments on the path to QCD, but he clearly believes that the requisite mathematical analysis proceeds in reasonable steps. His view is summarised as follows:

> In effect, the physics is manipulated in order to allow it to enter into a relationship with the appropriate mathematics, where what is appropriate depends on the underlying analogy. [French, 2000: 114].

According to French, then, there is nothing unreasonable about the effectiveness of mathematics in fundamental physics: we can always understand the introduction of new mathematics into physics by reconstructing the reasoning applied in its historical context.

French's analysis is impressive and undoubtedly right in so far as he explicates the historical progress of science. SU(2) proved useful in analysing the symmetry properties of nuclear spectra so it was not at all unreasonable to introduce SU(3) when empirical hints of a new symmetry arose. The fact that SU(3), considered mathematically, applies to an abstract, three-dimensional, complex space is besides the point. Plenty of abstract mathematical spaces find application in classical areas of physics such as statistical mechanics. Granted, at the time he introduced it, Gell-Mann couldn't give a physical rationale for the underlying symmetry, but he might have hoped that one would be found in the wake of experimental progress. It was not unreasonable that he found a useful, approximate symmetry by making heuristic moves. Nor was it unreasonable that Wigner successfully introduced group theory into physics as a heuristic tool when the underlying mathematics of quantum systems proved intractable.

Nevertheless, there is something unreasonable about the overall success of the group theory strategy in fundamental physics — the heuristic moves led to the uncovering of a beautiful, underlying mathematical theory. This is in spite of all the wrong moves made. We now know that isospin and flavour SU(3) are not deep symmetries at all[22]. Still, they pointed the way towards the discovery of the underlying, exact colour SU(3) symmetry. This is even more amazing because the colour property of quarks is hidden from observation due to confinement. Colour SU(3) symmetry, as implemented in Yang-Mills theory, turned out to be the key to formulating the dynamical equations of QCD. QCD is a beautiful (almost Pythagorean) theory. Steiner does not embrace metaphysical Pythagoreanism but he does finds it remarkable that many of the physicists involved seemed to *expect* to find a beautiful, underlying mathematical theory. Not only that, their Pythagorean beliefs played a large part in their success [Steiner, 1998: 54-60]. Steiner's argument for the unreasonable effectiveness of mathematics in fundamental physics rests on the success of the overall Pythagorean strategy, not on the individual instances which he discusses (and, sometimes, overplays). Further discussion of Steiner's argument is given in Sections 3.4.3 and 3.6.4. Next, I shall explain why I say that QCD is 'almost Pythagorean'.

[22] They are "quasi-true", according to French [2000: 113-114], because they were useful and predictively successful.

3.3.4 Quantum chromodynamics

As a young physicist in the 1960s, Gross was dissatisfied with the phenomenological (bordering on tautological) approach of S-matrix theory and Gell_Mann's Eightfold Way: "[I] longed to find a more powerful dynamical scheme" [Gross, 1992a: 12]. He focused his attention on explaining the results of high-energy, 'deep-inelastic' scattering experiments which had been designed to probe the insides of protons and give information about their possible constituents. The results of these experiments clearly showed that, at short distances, protons can be viewed as composed of (almost free, non-interacting) point-like spin ½ particles. It was natural to identify these particles with quarks. The scaling behaviour of the quark-like constituents of the proton was deeply puzzling because it indicated that, at short distances, the effect of the strong force is small and decreasing whereas for all other forces known at that time effective strength increases with decreasing distance. Thus, deep-inelastic scattering experiments gave the first indications of asymptotic freedom (i.e. the vanishing of the effective interaction strength at short distances).

Gross "felt strongly that scaling was the paramount feature of the strong interactions" [Gross, 1992a: 15]. In a move reminiscent of Weyl's early efforts in General Relativity, he examined the consequences of scale invariance for gauge quantum field theories. He found that only a free, non-interacting theory containing only massless particles could produce exact scaling. Since the strong interactions exhibit scaling behaviour at short distances and they manifestly involve massive particles, this augured poorly for the future of gauge quantum field theory as a framework for strong-interaction dynamics. As Gross [1992a: 16-18] recalls, he decided to *prove* that gauge quantum field theory could not explain the experimental fact of scaling. First he showed that asymptotic freedom was necessary to explain scaling. Then he showed the lack of asymptotic freedom in all renormalisable theories with the exception of non-Abelian gauge theories. Then he and his student Wilczek turned their attention to the difficult case of non-Abelian gauge theories. Contrary to their own expectations, they discovered asymptotic freedom and, along with it, a viable gauge-theoretical framework for the dynamics of the strong force. Thus QCD was born.

There was still the issue of how it is possible to have a theory whose basic fields (i.e. quarks and gluons) do not correspond to particles that can be observed directly in experiments. To explain this I will need to describe the scaling behaviour of QCD in more detail and, for this purpose, I will use a cut-down model of QCD called QCD Lite [Wilczek, 2003: S214-S217]. QCD Lite only contains two massless quarks but it is still an interesting theory because (a) to a good approximation, although six quarks have now been discovered[23], only the up and down quarks are relevant to everyday matter (the other quarks exist for short periods of time inside exotic particles produced in high-energy accelerators); and (b) the up and down quarks have relatively small masses. QCD Lite has features which make it especially interesting in the context of Pythagoreanism:

(i) it has no adjustable parameters;

(ii) it has a fundamental length scale which emerges when the scale-invariance of the classical theory is explicitly broken by quantum fluctuations;

(iii) it accounts for most of the mass of ordinary matter (even though its quarks are massless); and

(iv) it is well-behaved at arbitrarily high energies (like QCD itself and unlike other quantum field theories which break down at high energies and require the imposition of a high-energy cutoff in order to prevent infinities appearing).

I will explain each of these features since they will be of importance in my thesis.

The Lagrangian of QCD Lite is classically scale-invariant and contains a coupling constant g that is a dimensionless number (note that I have previously referred to this coupling constant as 'the interaction strength'). At first glance, it appears as though g parameterises a family of theories which have no fundamental scale of length. However, when we take quantum fluctuations into account the situation is different. The quantum vacuum gives rise to virtual particles which surround the quarks and diminish their coupling strength. The net effect is that as a test probe approaches a quark (i.e. as its distance from the quark decreases) it experiences a decreasing force. The coupling constant g becomes a function of distance or, equivalently, of energy (since a more energetic test probe can approach closer to the quark). As distance decreases the effective coupling becomes weaker — leading to asymptotic freedom. This is responsible for the good high-energy behaviour of QCD. On the other hand, as distance increases the effective coupling becomes stronger — an effect called *infrared slavery*. Infrared slavery is responsible for the fact that quarks cannot be separated from one another and that no isolated quark has ever been detected in experiments.

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[23] Although no quark has been observed as an individual, indirect evidence of their existence can be obtained by observing the decay products of their bound states.

Because g is a function of distance it does not have a fixed value in the theory. Effectively, we could give it any value and it would be equivalent to specifying a distance (i.e. the distance at which g takes that value). In this way the coupling constant is traded for a unit of length — an effect called *dimensional transmutation*. Rather than there being a family of different theories parameterised by g, there is now just one theory for which one can choose different units of length. The coupling constant g is eliminated[24]. QCD Lite is a unique, parameter-free theory.

Typically, the unit of length is chosen to be the distance at which g takes the value 1. This defines a fundamental length scale Λ_{QCD}. It is the length scale at which quark confinement becomes important. There is no corresponding fundamental length scale in classical field theories and, in that case, our attribution of a unit of length is arbitrary. The breaking of scale symmetry in QCD is an inherently quantum effect.

Λ_{QCD} sets the scale against which all the observable quantities in the theory can be calculated uniquely. In particular, the masses of hadrons (which are constituent particles composed of quarks and gluons; notably, protons and neutrons) can be calculated as a function of Λ_{QCD}. Because the quarks of QCD Lite are massless, the masses of its constituent particles arise solely from the energy of the confined fields inside the particles. Mass, as normally understood, is an emergent property of the theory.

The mass spectrum of QCD Lite has been calculated using powerful computers and the results are generally within 10% of the experimental values (which, of course, depend on effects from the full theory of QCD). Since protons and neutrons contribute 99% of the mass of ordinary matter, this means that QCD Lite accounts for most of the mass of ordinary matter [Wilczek, 2003: S217].

I say that QCD Lite is Pythagorean because it has no empirically-tunable input parameters and the physics is isomorphic to the mathematics. The full theory of QCD is *almost* Pythagorean. The (unreasonable) success of QCD gives us the hope of ultimately being able to derive all observable phenomena from first principles in some grand unified theory which is simple, fully-consistent, parameter-free and well-behaved at arbitrarily high energies. If we had such a complete theory then we could use the techniques of renormalisation to construct effective theories at any energy scale. For example, the Standard Model would emerge as an effective theory at energy scales much smaller than

[24] N.B. g cannot be eliminated from the Standard Model because it is required for comparing purely QCD quantities to non-QCD quantities; e.g. the ratio of the diameter of the proton to the Bohr radius [Wilczek, 1999: 305].

the Planck energy and all its dimensionless parameters would be calculated from the underlying theory. This would be the realisation of the Pythagorean dream.

In the next section I explore the relationship between symmetry principles and Pythagoreanism.

3.4 Symmetry Principles and Pythagoreanism

3.4.1 Symmetry principles, mathematics and physics

Symmetry principles and related mathematics guide the search for new physics. Mathematical tools such as groups and Lie algebras, which were developed by pure mathematicians to investigate symmetry principles, have become integral to modern physics. Wigner was one of the first physicists to apply group theory to quantum mechanics and his work in that area straddled physics and mathematics. Recently, there has been an enormous increase in the level of mathematical sophistication required to work at the cutting-edge of theoretical physics. For example, gauge symmetries have been related to the mathematical theory of fibre bundles which describes unified geometrical objects combining multi-dimensional internal spaces and spacetime. Also, the symmetries of String Theory have been explored using esoteric algebra, geometry and topology. Witten, a leading string theorist, won the 1990 Field's medal in mathematics for his work relating topological quantum field theory to the mathematical theory of knots, and for his work relating supersymmetry to Morse theory (part of the mathematical theory of the topology of manifolds). In commemorating Witten's work, Atiyah writes:

> The past decade has seen a remarkable renaissance in the interaction between mathematics and physics. This has been mainly due to the increasingly sophisticated mathematical models employed by elementary particle physicists, and the consequent need to use the appropriate mathematical machinery... many of the ideas emanating from physics have led to significant new insights in purely mathematical problems, and remarkable discoveries have been made [Atiyah, 1990: 31].

As physicists probe deeper and deeper, using symmetry principles as their guide, mathematics becomes ever more effective.

This combined use of symmetry principles and mathematics is the modern continuation of the Pythagorean search for the underlying beauty of nature. Heisenberg, in discussing the application of group theory to particle physics, explicitly recalls Plato's dissolution of matter into mathematical forms (i.e. into the regular solids and the triangles from which they can be constructed). He writes:

> … the particles of present-day physics are representations of symmetry groups… and to that extent they resemble the symmetrical bodies of the Platonic view. [Heisenberg, 1989: 83].

String Theory goes beyond this and beyond anything which could have been foreseen by Heisenberg, Wigner or their contemporaries. To a large extent, it has advanced by discovering mathematical structures and using them to construct physical concepts. It is essentially Pythagorean in its development.

3.4.2 The continuing story

There is still a question mark over whether nature will follow the Pythagorean script to its projected culmination. Will the tremendous advances made using symmetry principles to understand the electromagnetic, weak and strong forces lead on to the unification of all forces in a simple gauge group? Will this, together with the supersymmetric unification of all matter (i.e. bosons, the carriers of force, and fermions, the agents of force), realise the dream of a beautiful symmetrical form determining all matter and the laws which it obeys? Wishing doesn't make it so and there is always the chance that experiment will throw a spanner into the works. The role which symmetry plays in nature is not determined by human aesthetics.

The pursuit of truth through contemplation and the search for beauty has had a checkered record (e.g. astronomers were led down false paths by their obsession with the symmetry of spheres and regular platonic solids). History tells a story of scientific hubris according to which this or that beautiful symmetry was thought to hold the key to nature's mysteries only to be shown by experiment and further theorising to be accidental, or to have a deeper cause. Yet history is a slippery thing and will always be rewritten in the light of the reigning victor. At the end of the 20th century, following the "second superstring revolution" [Greene, 2000: 140], it was tempting to create a narrative telling the story of the triumph of Pythagoreanism. In this narrative, astronomers such as Ptolemy and Kepler would be portrayed as having followed the right ideas, but erring due to their lack of our modern understanding of the way in which the ultimate symmetry of nature's laws is broken in the world of things (e.g. whilst the laws of gravity are indeed spherically symmetrical, this symmetry is not revealed explicitly in the shape of planetary orbits). Opposing narratives — such as the positivistic doctrine that objects which are not directly measurable are unphysical and meaningless and should not form the basis of physical theories — would be deemed to have been proven wrong by the demise of S-matrix theory and by the triumph of field theory and symmetry principles. String Theory would be seen as the ultimate triumph of Pythagoreanism. It would provide a consistent, unified theory of quantum gravity without any adjustable parameters. It would reveal the fragile and perfect mathematical form underlying all physical reality.

Unfortunately, in the 21st century, problems have begun to emerge with the ending of this narrative. String Theory has run into difficulties. In order to connect it to phenomena in our low-energy, lower-dimensional world, it is necessary to detail a process whereby a universe like ours would condense out of the highly-energetic, highly-symmetric string background universe by compactification of extra spatial dimensions and symmetry-breaking of fields as a function of decreasing energy. This task has so far proven to be beyond our capabilities. Initially, string theorists hoped that there would be a unique process with one theoretically-enforced ground state (i.e. our universe). However, rather than finding one phenomenologically adequate solution, many string theorists now believe that there are multiple possible ground states that can condense out of String Theory and that each of these corresponds to a different universe with different dimensionless fundamental constants. It is not known how many physically inequivalent ground state solutions the theory has, but it is believed to be in the order of 10^{500} [Douglas, 2003]. This ties up with speculation from modern inflationary theories of quantum cosmology that the cosmic medium is, in fact, a multiverse and that our universe occupies only a small part of reality [Tegmark, 2008].

In the multiverse view, the values of the dimensionless fundamental constants which specify our universe are completely arbitrary in the sense that they are the result of random processes in the evolution of the cosmic medium. This takes us back to the situation described by Newtonian physics in which arbitrary initial conditions lead to the phenomena we observe today and there can be no conceptual, intrinsic explanation. The detailed structure of the world is contingent and physics becomes an environmental science investigating the properties of our little patch of reality. Of course, the dimensionless fundamental constants are not arbitrary in the sense that any old values would result in the universe we see today. On the contrary, as has often been remarked, they appear to be fine-tuned for the development of life as we know it. Perhaps there is some other principle which picks out our universe uniquely (e.g., a supernatural power, or something like the strong anthropic principle which states that the universe must necessarily have those properties which allow life to develop [Barrow and Tipler, 1986: 21], or some yet-to-be-discovered principle of physics). If conceptual principles could be found which would determine a unique ground-state solution for String Theory then Pythagoreanism would triumph again. Since String Theory is still in its early stages, and has many attractive theoretical features which give it an aura of inevitability, and is still leading to new conceptual insights[25], and has no obviously superior alternative, there are grounds for hope.

[25] For example, the AdS/CFT correspondence [Gross, 2007: 18].

A clear link to phenomenology would help. Pythagoreans sometimes underplay the importance of experiment. Concepts don't come from the contemplation of beautiful mathematical structures in isolation, but from the interplay of mathematics and empirical investigation. The understanding of spontaneous symmetry breaking — leading to the explication of the Higgs mechanism and, hence, of the origin of mass — came from observing behaviour in systems in condensed matter physics. Condensed matter systems also gave clues to the mechanism of quark confinement — leading to important conceptual advances on the path to QCD. It was condensed matter physicists such as Anderson who helped make the deep connection between behaviour observed in these experimentally-accessible systems and unresolved problems in the relatively inaccessible realm of particle physics [Gross, 1992a: 10]. Without such empirical insights, it is unclear how physics would progress. It might degenerate into competing metaphysical theories, all agreeing with known data and meeting common aesthetic standards (e.g. simplicity, unification, backwards compatibility), but differentiated by individual aesthetic biases; as per Wigner's nightmare discussed in the previous chapter. Physics is currently at something of an impasse awaiting further empirical insights but the hopeful lesson of history is that new impetus can come from unexpected sources. Smolin [2005] has suggested that physics look to biology for new constraining principles, such as using an evolutionary principle to determine how the multiverse evolves. Any such optimising principle, though not inherently mathematical, could be incorporated into the mathematical framework of Pythagoreanism.

3.4.3 The case for Pythagoreanism based on symmetry principles

It is as yet unclear where String Theory will lead us. Nevertheless, given the astonishing progress which the search for symmetries has facilitated to date, it does not seem unreasonable for the Pythagoreans in physics to dream of an extraordinarily beautiful and simple Theory of Everything. Even without TOE, we have already learned that the effectiveness of mathematics in fundamental physics is due to a deep fact about the world:

> As we explore physics at higher and higher energy, revealing its structure at shorter and shorter distances, we discover more and more symmetry. [Gross, 1996: 14259].

Extrapolating this trend encourages us to search for new symmetries, using whatever mathematical concepts we can discover.

The deep symmetrical structure of nature explains why mathematics is unreasonably effective in fundamental physics at very high energy levels and it is also explains why mathematics is *reasonably* effective at lower energy scales (with all the caveats explored in the previous chapter). In the everyday world we observe broken, or approximate, or residual, symmetries. For example, Witten showed that Minkowski spacetime is a stable ground state of General Relativity. Thus, Poincaré invariance, which leads to the familiar global symmetries of spacetime, can be regarded as a residual symmetry of the Minkowski vacuum under changes of the coordinates. Similarly, the bare mass of particles can be regarded as arising from a broken symmetry via the Higgs mechanism. Isospin invariance, which postulates invariance under transformations which turn a proton into a neutron (and vice versa), is an important symmetry in nuclear physics, but we now know that it is an approximate symmetry which results from the underlying exact symmetry of quarks and gluons. Also, symmetries observed in condensed matter systems are reflections of underlying fundamental symmetries. Without these reflections in the experimentally-accessible world we would never have been able to get as far as we have in uncovering nature's deep structure.

Our understanding of symmetries has its origins in empirical observations of physical objects and the global invariances of spacetime. Mathematicians, following their aesthetic instincts, have taken it far beyond those empirical origins yet, remarkably, their abstract game playing frequently turns out to be just what is needed in uncovering the next layer of reality in fundamental physics. Thus, in String Theory, the symmetries of abstract internal spaces in multiple dimensions, as represented mathematically by Calabi-Yau manifolds, directly determine the forces and particles operating in our universe! Somehow abstract mathematics represents the deep structure of nature, without approximation or idealisation, and this goes way beyond anything that can be accessed empirically.

Steiner [1998] identifies many historical examples where physicists used mathematical analogies to make discoveries and predictions. He calls such analogies 'Pythagorean' when they are expressible only in the language of pure mathematics. He calls them 'formal' when they are based on the syntax or notation of physical theories, rather than on their meaning [Steiner, 1998: 54]. An example of a Pythagorean analogy is Dirac's prediction of the positron, when he assumed that the purely mathematical negative energy solution of his relativistic quantum mechanical equation for the electron corresponded to a physical particle. An example of a formal analogy is any instance of physicists 'quantising' classical equations using the tricks of correspondence. In Section 3.3.2 we saw Steiner's identification of Pythagorean and formal analogies in the introduction of SU(2) and SU(3) into nuclear physics. According to Steiner:

> In the past, mathematics was used to get quantitative descriptions of phenomena that could also be described qualitatively. Today mathematics gives even the qualitative descriptions, because we often have no deeper language than mathematics. [Steiner, 2005: 641].

Physicists may have tried such analogies for a variety of reasons (e.g. because of Pythagorean beliefs or, simply, because they were desperate to try known tools in a situation where all physical intuition failed). They kept using them because they work. I claim, with the Pythagoreans, that they work because of the underlying mathematical structure of nature.

A detailed historical story of the development of the Standard Model of particle physics would show physicists groping for understanding amidst a plethora of confusing experimental data; applying a series of approximations and idealisations to shoehorn the data into useable mathematical models; pursuing whatever methods they can to get a handle on the problems; biased by their individual inclinations and background knowledge. Thus, any particular example of Steiner's can be deflated — as French [2000] does with Wigner's application of group theory to quantum mechanics, Heisenberg's application of the SU(2) symmetry group to nuclear forces, and Gell-Mann's application of the SU(3) symmetry group to the strong force. But Steiner's point is that, in hindsight, we can see a historical trend:

> What succeeded was the global strategy. [Steiner, 1998: 5].

Not only did the global strategy succeed, it succeeded beyond all reasonable expectations. Wigner's initial application of group theory to quantum mechanics as a classificatory tool may have been perfectly reasonable given his background in investigating the symmetry of crystals, but group theory turned out to be much more important than just a classificatory tool — it has been used to predict new forces and particles. Heisenberg's application of SU(2) to nuclear forces and his introduction of the internal, abstract symmetry of isospin may have followed on from his study of particle statistics in quantum electro-dynamics and may even have been a case of "the physics [being] manipulated in order to allow it to enter into a relationship with the appropriate mathematics" [French, 2000: 114], but internal symmetries have proven to be fundamental in the modern fibre bundle approach to gauge theory in which it is more a case of the mathematics dictating the physics. Gell-Mann's application of SU(3) to the strong force may have been an inspired guess based on approximate symmetries and incomplete patterns in the data, but it pointed the way to the beautiful, exact symmetries of QCD. Attempts (like S matrix theory) to structure the data and make predictions whilst eschewing a fundamental mathematical explanation have proved sterile. History does not show a triumphant, linear march of progress being made by the unreasonable application of Pythagorean strategies. However, it does show that the bulk of the answers in fundamental physics have been arrived at by doggedly pursuing the clues left by observable broken, approximate and residual symmetries in order to uncover the beautiful exact symmetries at their source.

Steiner argues that physicists followed an anthropocentric strategy, using beauty and convenience as criteria, and that the unreasonable success of this strategy indicates some sort of correspondence between our minds and the Cosmos [Steiner, 1998: 176]. He does not endorse Pythagoreanism although he acknowledges it as a respectable position to hold [Steiner, 1998: 5]. Nor does he draw any theistic conclusions. He identifies an epistemic mystery but does not embrace any metaphysical explanation.

In the previous chapter I argued, contra Steiner, that mathematical beauty is not anthropocentric (or species-specific) but is an objective criterion that we learn how to apply as we discover more and more about the mathematical structure of the world and the application of mathematics to itself. In Section 3.2 I argued that physicists have consciously tried to over-ride anthropocentric biases by appealing to general principles to ground their theories. Indeed, it is this very effort which has led to the increasing mathematicisation of physics. In this section I have argued for a metaphysical, essentially Pythagorean explanation of the unreasonable effectiveness of mathematics in fundamental physics, based on the importance of symmetry principles. In the next section I shall explain in more detail what I mean by a Pythagorean theory and expand the analysis to include general principles other than symmetry principles.

3.5 What is a Pythagorean theory?

3.5.1 The criteria

In this section I will be more explicit about what I mean (and don't mean) by a Pythagorean theory. I don't mean that 'everything is number' as claimed by the ancient Pythagoreans, although I find their teachings fascinating. Ancient Pythagoreans thought that numbers are the first and highest form of being and that they possess properties of a metaphysical nature [O'Meara, 1989: 16]. The first ten numbers, the decad, had a special place in their hierarchy of numbers[26]. Their properties were studied extensively and assimilated to particular Gods; e.g., the monad was 'paternal', the dyad, 'generative' and the triad, 'perfective'. Everything — including physics, ethics and theology — was explained in terms of properties which were assimilated to the properties of numbers. This is the rationale behind the typically mysterious Pythagorean saying that 'Justice is five'.

[26] The *Theologoumena* of Nichomachus of Gerasus (writing in the second century as one of the Neopythagoreans) dealt with each of the first ten numbers, their specific mathematical properties and how they related to various non-mathematical subjects [O'Meara, 1989: 20].

Nor do I mean that ‘everything is structure’ (mathematical or otherwise) as is held by ontic structural realists. For one thing, I agree with those philosophers who argue that structure is not sufficient on its own and that there must be something, ultimately inexplicable in terms of structure, which is structured. There is also the question of what makes some structures special. Everything has a structure but some structures are more fundamental than others and are studied as part of mathematics and physics. Not just any old structure will do — that would lead to lazy structuralism of the sort discussed by Wilson [2000] (see Chapter 2). There must be some selection principles which operate over and above structure. It is these principles which make a theory Pythagorean.

One criterion of a Pythagorean theory, then, is that it be a *principled* theory in which the consequences flow logically from a few principles without further empirical input. Empirical input may be needed to develop and test a Pythagorean theory but, once it is in place, it is a logically fragile, beautiful, principled structure whose consequences can be explored by pure thought. Physical theories differ in the extent to which they meet this criterion. QCD Lite meets the criterion. The Standard Model of particle physics does not because it depends on a host of input parameters. General Relativity is a principled theory but is not Pythagorean because its treatment of mass requires empirical input (unlike its elegant intertwining of spacetime and geometry) [27]. String Theory *could* be the ultimate Pythagorean theory, with its applications limited only by the underlying principles (and not by idealisations or approximations), but we don’t know what its underlying principles are yet.

On a cautionary note, Kepler’s original model of the solar system, as set out in his *Mysterium Cosmographicum*, meets the criterion. It explains data such as the ratio of Earth’s distance from the sun to Mar’s distance from the sun in terms of the geometrical properties of inscribed Platonic solids. We reject it because its predictions do not match observations. A Pythagorean theory is especially vulnerable to empirical disproof because there is no wiggle-room in the absence of adjustable input parameters. If it is empirically disproved within its sphere of applicability then we must conclude that one of its underlying principles doesn’t hold. Conversely, this fragility increases our confidence in it when it does receive empirical confirmation.

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[27] Ryckman quotes Einstein as having said that his gravitational field equation is like a building, one wing of which (the left-hand side consisting of the tensor representing the curvature of spacetime for given sources) is built of “fine marble”, the other (the right-hand side consisting of the coupling constant and the energy-momentum tensor) of “low grade wood” [Ryckman, 2005: 219].

Quantum field theories are not as rigidly defined as Kepler's original planetary model — they are not Pythagorean — but they are principled to the extent that their experimental predictions are ultimately derivable from conceptual principles of symmetry, locality and consistency, together with the input parameters. They are not yet at the level of a fully intrinsic explanation of phenomena, but they are a lot further along that path than Newton's theories. According to Wilczek:

> … the fact that the Standard Model is principled in this sense is profoundly significant: it means that its predictions are precise and unambiguous, and generally cannot be modified 'a little bit' except in very limited, specific ways. This feature makes the experimental success especially meaningful, since it becomes hard to imagine that the theory could be approximately right without in some sense being exactly right. [Wilczek, 1998, p. 867].

Deep principles lead to the two outstanding theories of modern physics: the Standard Model of particle physics and General Relativity. Together these two theories explain a wealth of physical phenomena. Unfortunately, they are incompatible with each other at length-scales of the order of the Planck length and below; i.e, $<\approx 10^{-35}$ m. Also, they contain adjustable dimensionless parameters which cannot be derived theoretically. The reason why String Theory has garnered so much support in the theoretical physics community is because it promises to overcome these obstacles. It promises to provide a unified theory of quantum gravity which does not contain any adjustable dimensionless parameters. As such, it is our first candidate for a Theory of Everything — a Pythagorean theory. However, we don't have a fundamental, background-independent definition of the theory yet; just a collection of linked, background-dependent theories which seem to be limiting theories of some unknown, underlying, principled M-theory. Even more problematically, physicists are still struggling with the bridge principles which would connect it to phenomenological laws. Until it has experimental verification, it cannot be accepted as a true fundamental theory.

The examples of String Theory and of Kepler's original model of the solar system illustrate an important difference between mathematical and physical theories. Mathematical theories are subject to criteria of consistency and proof. Physical theories must pass the test of empirical investigation. Mathematicians can explore all possible structures but physicists are only interested in the one which matches the real world.

Whether or not a theory can be classed as Pythagorean depends, not only on its axiomatic form, but on the nature of the underlying principles. The principles on which Pythagorean theories are based should be synthetic *a priori* principles shared by all rational beings. To elaborate on this criterion, let us consider the rational beings with which we are most familiar — human beings. It is my view that human beings have pre-programmed neural algorithms which lead us to structure incoming data in particular ways. This is similar to the Chomskyian view for languages but generalised to all types of input stimuli. One obvious way in which we structure data is by forming them into objects and properties, providing "grammatical" principles to make sense of what we experience. Part of this process involves focusing on what is invariant in our flux of experience under various transformations (e.g. when we turn our heads, walk around, or let time go by). Underlying the pre-programmed algorithms are abstract principles which can be applied to any data. Foremost amongst these are principles of logic, such as the law of non-contradiction. However, there are also more general principles, such as principles of symmetry and the principle of the One and the Many (more on this later). Such principles may extend to space being structured as locally Euclidean, as believed by Weyl, even though we know that it is not globally Euclidean. The underlying principles are metaphysical/mathematical. They are truths which are universal to all rational beings. Without them we could not structure our experience; we could not think. As in the Chomskyian view, the pre-programmed algorithms have parameters which can be set for particular environments. Thus, hypothetical rational beings living in different parts of the (hypothetical) multiverse would set their parameters differently in order to interpret the different laws of physics operating in their environment but they would all be drawn back to the same innate, fundamental principles on which to base their Theory of Everything (and their Standard Interpretation of arithmetic etc.).

The innate principles are structural principles. They determine which structures we find beautiful, interesting and/or useful. We find out about them by using them instinctively in our everyday lives and then abstracting them from our thoughts, actions and experiences. Because they are structural and innate, they can be refined by abstraction and reflection.

3.5.2 Metaphysical principles, mathematics and physics

Let me now elaborate on the principle of the One and the Many (because it will be important in the development of my thesis). It was the first metaphysical principle of Parmenides:

> 'The One, if it is one, may not be many'. For the many must participate in the One. But the One does not participate in the One but is the One itself. [O'Meara, 1989: 202].

As stated, it is a paraphrase of Russell's paradox in set theory — i.e. All sets must be members of 'the set of all sets' but 'the set of all sets' is not a member of 'the set of all sets' but is 'the set of all sets' itself. In ancient times, it was the subject of much discussion in relation to Plato's Theory of Forms. The One and the Many together produce the Forms in a process unfolding from the One (which is transcendent and contains contradictions) by successive applications of unifying and diversifying. Thus, Proclus combined Parmenides' principle with the hypothesis that 'All multitude is unified' to derive the principle that 'All that is unified is other than the One itself'. He then used this principle to unfold truths about the hierarchy of the gods such as:

> Each [class of] god is distinct, forms part of a serial ordering, and is reflected in its effects in corresponding features found on the successive lower levels of reality. [O'Meara, 1989: 205].

The reason why I mention this work is because I see many parallels between it and modern set theory. The underlying metaphysical principles are related and so, therefore, are the derived results. This will become clearer in the following two chapters of my thesis which deal with mathematics. As I will discuss, set theory has the potential to provide a background language for all of modern mathematics (i.e. truths proved in other areas of mathematics can be translated into the language of set theory). This makes it of supreme importance in considering what structures are possible in the world. Furthermore, it is built on principles which are largely innate in that they derive from considerations of how we combine many elements into a unified whole in order to form an object and, conversely, how we decompose a unified object into its parts in a hierarchical fashion.

The extent to which the axioms of set theory are innate is highly contentious. For me, this issue bears on the question of whether or not all mathematical theories are Pythagorean. Certainly, the consequences of mathematical theories flow logically from a few principles without further empirical input, so that criterion is met. The only question is about the nature of their underlying principles. I hold the axioms of mathematical theories to be synthetic *a priori* principles shared by all rational beings. That is what differentiates mathematical theories from purely formal theories and from games like chess.

The fact that mathematical truths can be translated into the language of set theory shows that the essence of mathematics isn't in the mathematical objects themselves but in the underlying principles. Different objects can represent the same principles. This is the case when, for example, different objects are used to represent numbers in standard arithmetic [Benacerraff, 1965]. In physics, different objects represent the same principles when there are different mathematical representations of the same physical theory. We can think about removing the dependence on particular objects in physics in the same way that we think about removing fixed background structure using the relational strategy. Following this trend, we would first remove perceptual biases, then crutches such as dimensions, extension, time, and topology, then, finally, particular objects, until the ultimate physical theory — the Pythagorean theory — is translated into the language of set theory[28]. This would accentuate its conceptual origins but would be highly unintuitive for physicists! For most physicists, geometry is most intuitive branch of mathematics, which is why the geometrisation of physics has been much more successful than axiomatic approaches. Geometry is especially useful as a way of grasping difficult concepts using familiar images. Since ancient times, it has been a privileged mediator between material and immaterial reality [O'Meara, 1989: 169].

My contention, then, will be that theories in mathematics and fundamental physics are grounded in common principles which are synthetic *a priori* in nature. Physicists have gone a long way towards explaining underlying reality using principles of symmetry, locality and consistency. A fourth element of modern physical theories is symmetry breaking and this element is less well-developed (and less well-grounded) than the other three. I speculate that, just as esoteric mathematics (e.g. group theory and algebraic geometry) has guided our growing understanding of symmetry principles in physics, so will it guide our future understanding of symmetry breaking. This might involve mathematical principles not yet familiar to physicists (e.g. from set theory) or maybe even new mathematics. It would provide further cases of mathematics being unreasonably effective in fundamental physics.

I mention set theory because it is a source of profound principles which seem to directly relate to symmetry breaking in that they are concerned with diversity unfolding from unity[29]. It is not known whether or not its principles can be incorporated into physics in a useful way. I note that the subjects of physics and set theory can be put into a closer correspondence if we take the effective field theory view of physics [Lepage, 1989]. The effective field theory view gives us a physical hierarchy which is analogous to the set theoretic hierarchy — at each level there are problems which cannot be solved without going to higher levels of the hierarchy. The higher levels impact the lower levels although, in practice, they are ignored for most purposes. Thus, in physics, if we are only interested in everyday objects, we can treat masses as point-like objects obeying Newtonian mechanics. If we are interested in chemical reactions, we need to consider the structure of atoms in terms of electrons and nuclei (treated as point-like objects) and the electromagnetic interactions between them. At higher energy levels, when

[28] Quine pursued this line of thought in his paper 'Whither Physical Objects?' [1976].
[29] An example is the Reflection Principle (see Chapter 4 Section 6 and note the above quote from [O'Meara, 1989: 169]).

considering nuclear reactions, we need a theory of nucleons (i.e. protons and neutrons inside atoms) whose interactions are mediated by mesons. At still higher energy levels, we need to consider the internal dynamics of nucleons; that is, interactions of quarks mediated by gluons. Each level of the hierarchy can be formulated as an effective field theory with a high-level cut-off such that contributions to the physics from energy levels above the cut-off are averaged over or ignored. Physical phenomena at different scales decouple to a large extent (with gravity being something of an anomaly in that it is a remnant of high-energy physics which is important at low-energy levels because it couples to large objects).

Physicists can investigate the effective field theory hierarchy up to energy-levels in the order of the Planck energy. However, according to String Theory, we cannot, in principle, probe above this energy level (because of T-duality and the limit of the finite length of strings). Similarly, in set theory, according to Woodin[30], there is a level at which the higher large cardinal numbers "fall in" and we can't use the techniques of set theory to probe any further. There is an epistemological veil which limits how far we can see, even in principle. This is a curious feature shared by these hierarchical theories. I shall explore other similarities in the following chapters. For the moment, I shall just say that I suspect that any "missing" principles in physics will turn out to be ones which we are already familiar with under a different guise. They will be based on innate, mathematical principles (rather than biological principles, for example, as suggested by Smolin [2005], since I consider biological principles to be derivative). Exploring mathematical structures to uncover physically-relevant underlying principles is the Pythagorean strategy advocated by Dirac in his quest to understand quantum mechanics. It is the strategy used by string theorists today.

3.5.3 A metaphysical approach to the problem of applicability

Having filled in more details of my version of Pythagoreanism, I would now like to revisit the issue of the applicability of mathematics. The outline of my approach to applications was given in discussing the effective field theory view of physics. It is based on scaling: different theories are applicable at different scales but all are grounded in the ultimate theory which is applicable at the highest energy levels. The ultimate theory is Pythagorean so the physical objects and their relations directly instantiate the mathematical objects and their relations, with both flowing from the underlying physico-mathematico-metaphysical principles of the theory. Symmetry breaking complicates the picture

[30] See Section 4.5.2 in the next chapter.

because it means that the ultimate theory is not directly applicable at lower energy levels. Symmetry breaking leads to complexity. Physical structures at lower energy levels instantiate mathematical structures of some sort but those structures are not always amenable to investigation by principled theories. Nevertheless, there is structure at the lower energy levels which reflects the structure of the ultimate theory and this allows us to (i) use mathematics in applications, and (ii) work our way back to an understanding of the ultimate theory. In this view, the reason why we can apply the laws of fluid dynamics, for example, is a consequence of the microscopic laws of classical dynamics, which in turn depend on the nonrelativistic laws of the quantum mechanics of atoms, which depend on laws governing electrons and nuclei, which depend on laws governing quarks and gluons etc. [Gross, 1988: 8371]. The physics at different scales decouples, allowing us to use effective theories at the appropriate scale whilst ignoring higher-order corrections (which are assumed to be small). This is a reductionist view of science. Nevertheless, it is compatible with the possibility that some problems in science will not be amenable to mathematical analysis despite supervening on underlying mathematical structure[31].

I understand that any individual example of the unreasonable effectiveness of mathematics in physics can be pulled apart. There is always a historiographical story to tell of any discovery. Ideas don't just materialise from nowhere. A detailed history would reveal false leads and wrong ideas: String Theory started out as a theory of hadrons… Gross started out trying to prove that no quantum field theory is fully consistent (he completed the proof for abelian gauge theories and just needed to prove it for non-abelian gauge theories)... the discovery of the omega minus particle relied on a lot of guesswork and Gell-Mann himself didn't believe in the existence of quarks to start with... These are great stories. Nevertheless, somehow we ended up with principled theories based on a few powerful concepts — with symmetry being the key. Somehow this continues a greater narrative, going back to ancient times, on the symmetrical forms underlying reality. The persistence of this thread over the centuries means something. One possible explanation is the Pythagorean one which I have presented in this chapter. Another possible explanation is that human beings always fall for the same story (perhaps for some evolutionary reason to do with religious instinct) and there is really nothing underneath it. We just muddle along explaining phenomena in our world as best we can. I don't think that explains the progress we have made towards an ultimate theory, even if a glorious end to the Pythagorean narrative cannot be guaranteed. It is the remarkable success of mathematicised science in uncovering the

[31] See e.g. Anderson's paper 'More is Different' [Anderson, 1972] which discusses how many-body systems exhibit emergent phenomena.

fundamental structure of the universe that seems unreasonable. The ship of science, navigated by physicists, buffeted on the high seas of empirical investigation, on a course set to naturalistic goals, is drawn into an ever-receding, increasingly-mathematical vortex of ultimate reality in which all familiar distinctions are extinguished.

3.6 Closing Thoughts

In this chapter I have argued that the principles which underlie mathematics and fundamental physics are common and they are common to the mind[32]. They are both in our minds and in the world. That is why we are capable of understanding the laws of nature. It is why it can seem that our minds are attuned to the world. It is why we feel both that we discover mathematics and that we create it. It is why mathematics is unreasonably effective in fundamental physics. It is why ancient metaphysics can seem unreasonably effective, in spite of its disdain for experiment.

My last observation (about ancient metaphysics) may go against trends in philosophy but I can vouch that, for a particle physicist confronted with Plato's *Timaeus*, what leaps out is that parts of it seem like a weird, ancient premonition of a text on modern particle physics! Of course, other parts seem like complete nonsense in the light of modern science. My contention is that the gems and the nonsense correlate with my thesis in the sense that the gems are connected with fundamental concepts (e.g. the creation of time; the seething, chaotic Khora which is reminiscent of the quantum vacuum; the underlying, symmetrical structure of matter) and the nonsense comes when Plato moves into biology and physiology. Because of Plato's methodology, the details of the scientific description are not important to him. They are signposts and as long as we arrive at the right destination, that is all that matters. It is not important that the constituents of the universe are photons, electrons and quarks (rather than fire, earth, air and water). The important thing is the underlying mathematical symmetries that they point us toward. In Plato's philosophy that led to a contemplation of the elements of Euclidean geometry. In modern physics, we are led to the mathematics of non-Abelian gauge groups. In both cases, it is the underlying symmetries that are important. They are the closest things to Forms that human beings know. Contemplating them helps us in our search for the beautiful and the good and

[32] This is in agreement with Frege's fundamental insight that mathematics is "connected very intimately with the laws of thought" [Frege, 1884: §14].

restores the harmony of our soul with the soul of the universe. Mathematics is the paradigm for fundamental physics, just as Forms were the blueprint for the creation of the universe in the Timaeus.

Curiously, physicists in modern times have returned to questions about the origins of the universe, not for religious or metaphysical reasons, but because the science has driven them there. Physicists have been driven to consider origins because of developments in General Relativity and particle physics. According to the equations of General Relativity, the container of the universe (the spacetime geometry itself) evolves together with its contents (mass and energy). This means that cosmologists can model the evolution of the universe as a whole from its beginning. The current Standard Model of cosmology is the Big Bang model according to which the universe began approximately 13.7 billion years ago in a singularity of infinity density and temperature. Particle physicists have been driven to consider origins because their investigations of symmetry and symmetry breaking have led them to a scenario in which, in the beginning of the universe, when temperatures were extremely hot, there was an overarching symmetry principle which linked all the particles which we see today, and all those which we predict but don't see. Then, as the universe cooled, the symmetry was broken in stages, condensates formed, the respective forces peeled off, and particles acquired mass by interaction with the condensates. The holy grail of physics — the resolution of conflicts between General Relativity and particle physics into a theory of quantum gravity — only manifests itself at the very earliest times when energies were at or above the Planck scale.

There is a link between fundamental reality and origins which is important both to physics and to metaphysics. Considering the origins of the universe focuses the mind on contentious distinctions in metaphysics: the difference between the possible and the actual; the abstract and the concrete; the necessary and the contingent; mathematical structure and physical structure. It gets to the question at the heart of metaphysics: What is being? It allows us to focus on the essentials of such issues without the complexity of the later universe. Physicists and metaphysicians alike have reason to contemplate the nature of being and nothingness. Is it possible to get something from nothing? Can the universe materialise out of nothing by way of an energy-neutral quantum fluctuation, without a first cause, or does mathematics and its subset, the Theory of Everything, exist apart from physical reality and somehow engender it? What is the modern creation myth? Perhaps the one consistent mathematical structure which some set theorists seek to define is like Plato's Form of Being with generative parts which give rise to physical structure in the form of an egg of spacetime with its DNA imprinted with the laws of nature which determine how it evolves. Then, physics, metaphysics and mathematics come together at the origin of the universe. Later, the universe diverges from the Form through a process of symmetry breaking. It becomes increasingly diverse and complex. It still reflects its origins, so it still is amenable to mathematical analysis, but only through idealisation and approximation. Metaphysics can't get a grip. Mathematics is unreasonably effective in high-energy physics but only reasonably effective in the other natural sciences and, for some problems, not effective at all.

At the same time as physicists are driven to contemplating nothingness, they are driven to contemplating the other extreme —infinity. If physicists adopt the Big Bang picture of creation then, in order to explain some of the large-scale features of our current universe, such as its homogeneity, they need to introduce a period of rapid inflation in the early universe to stretch and smooth it out (or, at least, this is the best explanation anyone has come up with so far and it has some experimental support from the microwave background data). Furthermore, it seems to some that the nature of that inflation must be such that it is on-going, so that our universe is just one bubble of inflated universe amongst a vast chaotic ocean of bubble universes and that more bubbles of spacetime are inflating all the time. It has been estimated that during the time from the Big Bang to now at least as many as 10^10^77 other universes would have branched off from our own patch of inflating spacetime. If you take into account that this fractal branching process has been occurring since past eternity and will continue into future eternity, then that's a lot of universes to contemplate. So, General Relativity leads to a singularity at the beginning of the universe and to a potential infinity of universes. Similarly, particle physicists have had their own confrontation with singularities and infinities: it is very difficult to construct a renormalisable theory of the electroweak interactions and only QCD is a fully renormalisable theory. These difficulties have contributed to the development of String Theory because strings (being finite in length, though very tiny) don't lead to infinities in the theory in the way that point particles do. In both General Relativity and particle physics, it is the appearance of (and attempt to overcome or explain) technical singularities and infinities in the theory which drives progress forward.

Whichever way you look at it, there is a thought pattern which drives fundamental physicists to contemplate the extremes of infinity and nothingness. Furthermore, thinking about one extreme leads to thinking about the other. This is familiar to metaphysicians who struggle with the paradoxes of thought and to mathematicians who struggle with the paradoxes of set theory. Set theory is the natural milieu for thinking about infinity and nothingness. It is the topic of the next chapter.

Chapter 4. One True Mathematics

4.1 Overview

I have argued that the unreasonable effectiveness of mathematics in fundamental physics leads to a Pythagorean view of the world in which the underlying principles are structural, mathematical principles. This naturally raises one of the key questions which Wigner addressed in his investigation of the relationship between physics and mathematics: 'What is mathematics?' That is the question which I address in the next two chapters. I am particularly interested in exploring the implications of a Pythagorean view for one's philosophy of mathematics. What should Pythagoreans believe about the nature of mathematics? What are the potential problems for this view? It seems clear that, for Pythagoreans, mathematics is necessarily true, eternal and unchanging. Pythagoreans should be realists about mathematics. They should regard the pursuit of mathematics as discovering facts about the world. They should look to mathematics as the ultimate arbiter of questions about existence and truth. It is the last point that I am mainly concerned with in this chapter: can mathematics serve as the ultimate arbiter of questions about existence and truth? More particularly, I investigate whether there is one background theory of mathematics in which any mathematical problem can be interpreted and (potentially) decided. If there is, then the various disciplines of mathematics are part of one, true mathematical universe. If not, then there are alternative mathematical universes and mathematical statements may be true in some universes and false in others, not necessarily true. That would be a problem for Pythagoreanism.

I begin Section 4.2 with historical background describing how mathematics grew from its empirical origins into a freely developed, aesthetic pursuit and how a renewed focus on the rigour of its foundations led to increasing formalisation and the development of set theory as a possible background language for interpreting and comparing different mathematical disciplines. Next I discuss the problems of set theory — the set theoretical paradoxes, incompleteness and independence phenomena. Following Gödel, techniques were developed which led to a plethora of independence results, each one of which creates a potential bifurcation of the underlying set-theoretic universe according to whether it is forced to be true or false. Pluralists see independence results as bringing set theory into the world of modern mathematics, freeing it from the special burden which it had bourn because of its importance for foundational work in the philosophy of mathematics. Universalists see them as challenges to be overcome on the path towards one, true mathematics.

The topic of Section 4.3 is the current debate in set theory between the *pluralists* — i.e., those who believe that there are many distinct concepts of set, each with its own set-theoretic universe — and the *universalists* — i.e., those who believe in an absolute background set concept, with a corresponding absolute set-theoretic universe. This debate is of crucial importance for the Pythagorean view in which mathematics is considered to provide the necessary and unique structural principles of the world. Consequently, I spend some time investigating pluralism and its problems, both from a set-theoretical and a philosophical viewpoint. One interesting problem is the difficulty of articulating the pluralist view. The pluralist must pick a background theory in order to discuss various models then, when pressed, embrace pluralism concerning the background theory. This can lead to conflicts between the meta-theory and the theory itself. Pluralists also have difficulties with reference. If the sets in alternative universes are different objects, satisfying different concepts of set, then they cannot be compared in any useful way. These problems resonate with Quine's work on ontological relativity in the philosophy of language. Some philosophers have taken pluralism to its extreme, embracing the thesis that any mathematical object which possibly could exist does exist (where *possibility* is taken as logical possibility and defined in terms of consistency). This radical form of pluralism so devalues the notions of existence and truth that it becomes on a par with fictionalism as a philosophy of mathematics; i.e., as an account of the objective facts underlying mathematical practice. For Pythagoreans, it is not enough to account for mathematical practice — mathematics has a bigger role to play in the web of knowledge. Regardless of personal philosophy, if set theory is to play a foundational role, then set theorists will have to provide a fixed background language for the purpose, and they will have to find reasons for preferring one set theoretic universe over the others.

In Section 4.4 I take up the argument of the universalists, which depends on the justification of new axioms for set theory. Justifying axioms is a tricky business. Ideally axioms would be self-evident but this is never achievable and what we get in practice is varying degrees of plausibility. Gödel distinguished between 'intrinsic' and 'extrinsic' justification in the search for new axioms, with the former deriving from internal set theoretic concepts and the latter linked to the consequences of set theory for other fields of mathematics and for physics. I list desirable characteristics for new axioms and present the case for *large cardinal axioms* as the preferred continuation of set theory. Large cardinal axioms can be thought of as asserting that certain infinite levels of the universe of sets exist. Collectively, they satisfy the concept that the iterative process of generating sets should not run out and that there should always be another stage extending the class of ordinals. I discuss how large cardinal axioms are generated and explain their central role in set theory. It has been established that any natural candidate for a new axiomatisation of set theory is mutually interpretable with a theory defined by a large cardinal axiom. Then large cardinal axiomatisations can be used to translate back and forth between rival theories from conceptually distinct domains and compare features such as their consistency strength. In many cases this is the only way to compare theories.

In Section 4.5 I investigate the possibility of achieving an effective completion of mathematics in the wake of independence results. There are a plethora of independence results, each one of which creates a potential bifurcation of the underlying set-theoretic universe according to whether it is forced to be true or false. In the view of the universalists, this is just another manifestation of the incompleteness of mathematics and demonstrates the need for new axioms and new techniques. They are actively pursuing a vision of one mathematical universe by searching for an axiomatisation of set theory which is immune to independence results. Their current best hope is the (delineated but not yet constructed) theory of *Ultimate L*. The claim is that Ultimate L would give us a complete picture of the set theoretic universe in the sense that it would be capable of interpreting any extension of set theory and it would decide all set theory propositions.

In Section 4.6 I examine some universalist arguments for the truth of set theory. At one level, the debate between pluralists and universalists turns on potential bifurcation points for mathematics. The pluralists try to show that particular mathematical statements, such as Cantor's continuum hypothesis, are absolutely undecidable and the universalists try to rebuff these challenges. The arguments can get very technical. From a local perspective in the set hierarchy it will always seem that there are many possible branches. The universalists will not be able to argue convincingly for their case until they have developed a theory like Ultimate L which can give a global perspective. Nevertheless, their brilliant recent work keeps the dream of one, true mathematics alive.

The thrust of the universalists' work is to sharpen the language of set theory so as to eliminate indefiniteness. In my closing thoughts in Section 4.7 I suggest that the solution may require a more holistic view, taking account of the place of mathematics in the web of being and knowledge. In the next chapter I will outline a particular Pythagorean metaphysical framework (based on set theoretical concepts) and examine its consequences for the philosophy of mathematics.

On a practical note, set theory is technical subject which builds from relatively simple concepts to very challenging theorems and proofs. This chapter is non-technical but I have needed to define some terminology in order to talk about the philosophical issues. Where possible I have relegated details to an appendix for the interested reader.

4.2 The Development of Set Theory

4.2.1 Historical background

For most of human history, mathematics consisted of two main subject areas: arithmetic and geometry [Boyer, 1991]. Arithmetic was originally developed for applications in accounting, and geometry was developed for applications in building construction and land allotment. A list of early achievements would include: the development of mathematical notation, the axiomatisation of Euclidean geometry, the proofs in number theory of key properties of prime numbers, and the solution of abstract algebraic problems. Very early on, as evidenced by the proof of Pythagoras' Theorem concerning the hypotenuse of a right-angled triangle, mathematicians were inculcated with an aesthetic aspiration for elegance and beauty in their work, and developed a sense of their subject as having importance beyond its applications. Indeed, mathematics acquired a parallel existence through its incorporation into mystical thinking in the Pythagorean cult.

In the 17th and 18th centuries, there was an explosion of new mathematics. Calculus and real analysis were developed, still with a predominant focus on immediate practical applications (e.g., in shipping, warfare and industrial processes). Then, by the 19th century, mathematics began to take on a life of its own, driven by its own standards and values, not tied to applications. Later developments such as non-Euclidean geometry, functional analysis, abstract algebras and topology had their conceptual roots in the physical world but were developed freely, according to the interests of the most innovative mathematicians of the time, and quickly diverged from any obvious physical interpretation. It is true that there had always been a thread of the playful, or mystical, in mathematics, at least from the time of Pythagoras, but this increasingly became the driver of the bulk of new mathematics (presumably, as a beneficial result of economic liberalisation and specialisation).

The development of mathematics for its own sake led to a growing uneasiness with the state of its foundations. The geometry of Euclid had been based on theorem and proof and had set the benchmark for mathematical rigour. It formed the basis of the ideal of mathematics as being necessarily true, eternal and unchanging. However, many of the practical mathematical techniques that were developed in the calculus and in mathematical analysis were not developed rigorously from first principles, but more loosely and creatively, with an eye to what worked and with a justification based on intuition. Towards the end of the 19th century and continuing into the 20th century, there was a renewed focus on rigour and on the underlying assumptions of the various branches of mathematics. The logical tools of formal languages were developed; revolutionising the methods of axiomatisation, deduction and proof.

The possibility began to emerge that set theory could provide a background language for mathematics in which methods and results from different disciplines could be interpreted and compared. For example, after the axioms of arithmetic were formalised by Peano [1889] using a standard interpretation in terms of the natural numbers and their relations, it was found that Peano Arithmetic could be interpreted in set theory by representing the natural numbers as sequences of sets (e.g., '0' by the empty set '{}', '1' by the set containing the empty set '{{}}', '2' by '{ {},{{}} }', and so on). Beyond the natural numbers of arithmetic, it was found that all numbers could be represented in set theory —integers could be constructed from pairs of natural numbers; rational numbers could be constructed from pairs of integers; and real numbers could be constructed from rational numbers using concepts such as 'least upper bound'. Furthermore, mathematical structures as diverse as graphs, manifolds, rings, and vector spaces could be defined as sets satisfying various axiomatic properties. In principle, theorems developed in other branches of mathematics could be derived in set theory, even if this were rarely carried out in practice[33].

It is now accepted that almost all of mathematics can be interpreted in set theory (the exceptions are developments which subsume set theory, such as type theory and category theory). Can set theory serve as the one background theory of mathematics that we are looking for? In order to answer this question, we will need to consider the development of set theory specifically and the current status of the foundational debates which it has spawned.

4.2.2 Cantor and the set theoretical paradoxes

Set theory was founded by Cantor in a paper published in 1874. Intuitively, a set is a collection of objects. These can be objects of any kind, mathematical or otherwise. Objects which can be members of sets, but are not themselves sets and do not have any members, are called *urelements.* For mathematical purposes, it is common to formulate the theory in terms of *pure* sets (i.e., sets whose

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[33] One interesting exception is Harvey Friedman's efforts to 'reverse engineer' textbook mathematics in order to reveal exactly what set-theoretical machinery is used [Friedman and Simpson, 2002].

members are sets, whose members' members are sets, and so on) unless there is some particular application requiring urelements.

A set is *ordered* if, between any two of its elements *a* and *b*, there is a relation '≤' such that either a ≤ b or b ≤ a, or both. A *well-ordered* set is an ordered set such that every subset contains an element which is in the relation ≤ to all the others. Two well-ordered sets that can be brought into one-to-one correspondence under invariance of the relation ≤ are said to have the same *ordinal number*. The finite ordinals correspond to the natural numbers. Hence, 0 can be thought of as the smallest ordinal number, followed by 1, 2, 3 … . The first infinite ordinal number corresponds to the set of natural numbers {0, 1, 2, 3 … } and is denoted ω. After ω comes ω+1, ω+2, ω+3, and so on, proceeding in order to ω+ ω, and further to ω^2, ω^3, ω^ω, and so on indefinitely.

The size of a set is measured by its *cardinality* and denoted by a *cardinal number*. Two sets are said to have the same cardinality if their elements can be brought into one-to-one correspondence. The cardinal number of a set which has the same cardinality as one of its subsets is called *infinite*; other cardinal numbers are called *finite*. Consider sets of natural numbers. The cardinality of a finite set of natural numbers is the number of elements in the set. The cardinality of the infinite set of natural numbers is denoted $\aleph_0$ (read 'Aleph zero') and is identified with the least infinite ordinal ω. Sets that have the same cardinality as ω are called *countably infinite*. The smallest cardinal number greater than $\aleph_0$ is denoted $\aleph_1$. It is the cardinality of the set of all countably infinite ordinal numbers. The smallest cardinal number greater than $\aleph_1$ is denoted $\aleph_2$. It is the cardinality of the set of all ordinal numbers of cardinality $\aleph_1$. In general, the smallest cardinal number greater than $\aleph_\alpha$ is denoted $\aleph_{\alpha+1}$. It is the cardinality of the set of all ordinal numbers of cardinality $\aleph_\alpha$. The sequence of aleph numbers defines an infinity of infinities, sometimes referred to as 'Cantor's paradise'[34].

Cantor showed that there is a hierarchy of infinite sets with increasing cardinality which can be constructed using the *power set operation*. The *power set* of a set A is the set of all possible subsets of A, including the empty set and A itself. Cantor showed that the power set of the set of natural numbers has cardinality greater than $\aleph_0$; i.e., it has cardinality 2^ $\aleph_0$. This is the cardinality of the set of real numbers. If we continue iterating, then the power set of the set of real numbers has cardinality 2^(2^ $\aleph_0$), the power set of the power set of the set of real numbers has cardinality 2^(2^(2^ $\aleph_0$)), and so on.

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[34] Following Hilbert's famous exclamation: "No one shall expel us from the paradise which Cantor has created for us." [Boyer, 1991: 570]

Cantor hypothesised that $\aleph_1 = 2^{\aleph_0}$ (i.e., that there are no cardinals strictly between $\aleph_0$ and the cardinality of the set of real numbers). This is called Cantor's *Continuum Hypothesis*. The *Generalised Continuum Hypothesis* is that $\aleph_{\alpha+1}=2^{\aleph_\alpha}$.

Cantor's work was a great achievement but it was not without its difficulties. In 1899, Cantor himself identified a paradox at the heart of set theory. It was developed from the theorem that there is no greatest cardinal number. This theorem is true by construction since, assuming the existence of a largest cardinal number C, one could always generate a larger cardinal number by taking the power set of a set of size C, leading to a contradiction. But consider the set of all sets. It must contain its own power set so its cardinality must be bigger than itself. This is Cantor's Paradox.

Around 1900, Russell developed a closely related paradox by considering the set of all sets that are not members of themselves. This concept leads to a contradiction because the set must both be a member of itself and not a member of itself. Paradoxes such as these reveal a fundamental problem in the naïve concept of a set. Russell's paradox, in particular, reveals a problem with allowing a set to be defined in terms of a collection of objects sharing a given property. Naively, this is the logical concept of a set (and, indeed, it is the one which Frege [1879] used in trying to develop the foundations of mathematics from logic alone). One wants to be able to form sets arbitrarily according to such prescriptions as 'the set of all prime numbers', 'the set of all red objects', 'the set of all sets containing exactly three objects', etc. However, Russell's paradox shows that this can lead to inconsistency unless restrictions are placed on the property which is used in defining the set. In particular, problems arise if the property is self-referencing.

4.2.3 Modern set theory

Mathematicians got around the paradoxes of set theory by refining the concept of set to include an iterative construction with stages which build in such a way as to make self-referencing impossible. This was achieved through the axiomatisation of set theory by Zermelo [1908] and further refinements by Fraenkel [1922]. Their theory — 'Zermelo-Fraenkel set theory with the Axiom of Choice'(or ZFC for short) — is the standard axiomatisation used today. The axioms of ZFC are interpreted through the cumulative hierarchy of sets V_α , indexed by the ordinal numbers α, which is constructed as follows:

- Let V_0 be the empty set {}:

  V_0 := {}.

- For ordinal number α, let $V_{\alpha+1}$ be the power set of V_α:

$V_{\alpha+1} := P(V_\alpha)$.

- For limit ordinal α, let V_α be the union of all the V-stages so far:

  $V_\alpha := \bigcup \{V_\beta \mid \beta < \alpha\}$.

It is a consequence of the axioms of ZFC that for each ordinal α, V_α exists and, moreover, that for each set X there exists an ordinal α, called the *rank* of X, such that $X \in V_\alpha$. By construction, the sets V_α are *well-founded* in that they are built up inductively from the empty set using operations such as taking unions, subsets and powersets. This precludes any V_α from being a member of itself and so circumvents Russell's paradox.

Consider the sets appearing in V_ω where ω is the ordinal denoting the least completed infinity [Koellner, 2011a: 3-4]. They can be coded by the natural numbers and can be shown to satisfy a theory that is *mutually interpretable*[35] with Peano Arithmetic (PA). In fact, since they do not include ω, they satisfy the theory of ZFC without the *Axiom of Infinity*[36]. At the next level, the sets appearing in $V_{\omega+1}$ are the power sets of the natural numbers (or, equivalently, the real numbers) and satisfy a theory that is mutually interpretable with second-order arithmetic (PA_2). At the third infinite level, the sets appearing in $V_{\omega+2}$ are the power sets of the real numbers (or, equivalently, functions of real numbers) and satisfy a theory that is mutually interpretable with third-order arithmetic (PA_3). Thus, the first three infinite levels interpret, respectively, arithmetic, analysis and functional analysis; i.e., most of normal mathematics. Cantor's continuum hypothesis can be formulated at the third infinite level, $V_{\omega+2}$.

It is not possible to collect all the stages V_α into a set because such a 'set of all sets' would fall foul of Russell's paradox. However, it is useful to be able to talk about the universe of sets, and other collections such as 'all cardinal numbers' and 'all ordinal numbers' (which cannot be sets because of Cantor's paradox), so such inconsistent totalities are given the name *proper classes*[37]. The class of all ordinal numbers is denoted O_N. The class V is defined to be the union of all the stages V_α:

$V := \bigcup \{ V_\alpha \mid \alpha \in O_N\}$

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[35] Let T_1 and T_2 be recursively enumerable axiom systems. We say that T_1 is *interpretable* in T_2 ($T_1 \leq T_2$) when, roughly speaking, there is a translation τ from the language of T_1 to the language of T_2 such that, for each sentence ϕ of the language of T_1, if T_1 proves ϕ then T2 proves $\tau(\phi)$. T_1 and T_2 are said to be *mutually interpretable* when both $T_1 \leq T_2$ and $T_2 \leq T_1$. [Koellner, 2010: 5]

[36] The Axiom of Infinity asserts that there are infinite sets; i.e., completed infinite collections.

[37] Cantor used the term *inconsistent manifold*.

Although V is usually referred to as 'the universe of sets' it is not itself a set and it only contains well-founded sets[38]. Woodin, a modern set theorist, makes the following comment: "The purpose of this notation is to facilitate the (mathematical) discussion of Set Theory—it does not presuppose any meaning to the concept of the universe of sets" [Woodin, 2011b: 89]. Note that the set V_α is called the *rank initial segment* of V determined by the ordinal α.

As far as anyone knows, ZFC is a consistent theory. But its axioms are not as intuitive as the axioms of arithmetic and they are not uniformly accepted by mathematical logicians. Many variants of set theory are still explored and debates over the justification and consequences of particular axioms is ongoing [Maddy, 2011]. Historically, one of the most controversial axioms has been the *Axiom of Choice*. The Axiom of Choice asserts that for any set X of nonempty and pairwise disjoint sets there is a set with just one element in common with each member of X (so it is always possible to form a new set from X by choosing one element from each of its members). Whilst this is intuitive for finite sets, in the infinite case it leads to sets that are not explicitly definable. Intuitions vary on whether or not this is acceptable. Most disturbingly, Banach and Tarski [1924] showed how the axiom could be used to divide a ball into five disjoint sets that could be moved around to form two balls of the same volume as the initial ball. This is the Banach-Tarski theorem. It does not agree with our physical intuition. However, one might simply conclude that the concept of a set is more inclusive than required for representing physical space and that physical regions might better be represented by a restricted subset. Crucially, the disjoint sets of the Banach-Tarski theorem are not Lesbesgue-measurable whereas measurability would seem to be a reasonable restriction to place on physical applications. The Axiom of Choice was used by Zermelo in showing that every set can be well-ordered. It is an underlying assumption of much practically important mathematics and, as a result, it has largely been accepted by the community of modern set theorists [Maddy, 1997: 54-57]. More discussion of the axioms of ZFC is given in Section 4.4.1.

It would be wrong to say that the paradoxes of set theory have been resolved. They are still troubling and resurface whenever an attempt is made to obtain a satisfying closure or completeness for mathematics (e.g., at the level of classes) or, as described in this chapter, when one seeks a unique background theory. Certainly ZFC is not such a theory.

[38] Axiom systems other than ZFC may omit the Axiom of Foundation, and may even include its negation, allowing for non-well-founded sets. Such sets may be "pathological" in some sense and not suitable for some mathematical applications.

4.2.4 Incompleteness and independence phenomena

Continuing the historical exposition, the next major problem for ZFC arose in 1931 as a result of Gödel's Incompleteness Theorems [Gödel, 1931]. Gödel showed that any consistent formal system strong enough to interpret Peano arithmetic contains sentences which are true but unprovable (The First Incompleteness Theorem). Furthermore, he showed that the consistency of such a system cannot be proved from within the system itself (The Second Incompleteness Theorem).

Gödel's First Incompleteness Theorem demonstrates that there are *undecidable* propositions in mathematics. Any such proposition would become decidable in an appropriate stronger system but then that system, in turn, would have its own Gödelian sentence. In his 1931 paper, Gödel wrote:

> …the true reason for the incompleteness inherent in all formal systems of mathematics is that the formulation of ever higher types can be continued into the transfinite… For it can be shown that the undecidable propositions constructed here become decidable whenever appropriate higher types are added… An analogous situation prevails for the axiom system of set theory. [Gödel, 1931: 181]

Gödel thought that incompleteness would inexorably lead to the acceptance of new axioms for set theory. The question for him became 'which new axioms?' and how to justify that decision. At the time, it was anticipated that some of the major unresolved problems of mathematics (e.g., Goldbach's conjecture or the Riemann hypothesis) might turn out to be undecidable, giving urgency to the new axiom project. However, eighty years later, that is yet to be shown. Moreover, the undecidable propositions of Gödel's theorems are remote from the work of normal mathematicians. The undecidable Gödelian sentence from the First Incompleteness Theorem is constructed artificially as a proof of existence and is not of any interest in normal mathematics. The undecidability of the consistency statement in formal systems, from the Second Incompleteness Theorem, is of great interest and importance for logicians but, again, does not impact normal mathematics. A lot of effort by logicians (notably, Harvey Friedman[39]) has gone into finding more concrete examples of undecidable propositions; e.g., in combinatorics. As yet, these are still quite far from the work of normal mathematicians, but it seems inevitable that independence phenomena will permeate normal mathematics at some stage. This could happen in two ways: by logicians showing that some current propositions of interest to normal mathematicians are undecidable in ZFC; or by normal mathematicians

[39] See, for example, Friedman [2010] and [2011].

expanding the boundary of their interest to incorporate areas which require more of the machinery of set theory. The second possibility seems quite likely if we consider a very long time frame.

The question of new axioms became more urgent for set theorists following proof of the undecidability of Cantor's continuum hypothesis in ZFC. The proof was achieved in two stages. Firstly, Gödel [1940] showed that there are models of ZF in which the Axiom of Choice and the continuum hypothesis are both satisfied (assuming ZF is consistent and hence has a model[40]). Therefore, the continuum hypothesis cannot be disproved from within ZFC (i.e., it is consistent with ZF and the Axiom of Choice). Secondly, Cohen [1963] and [1964] showed that there are models of ZFC which violate the continuum hypothesis, so the continuum hypothesis cannot be proved from the axioms of ZFC. In these papers, Gödel and Cohen introduced important new techniques which are essential for understanding current debates in set theory. I will briefly describe these techniques here for future reference. More details are given in Section A.1.1 of Appendix 1.

Gödel's technique involves constructing an *inner model* L of ZF. L is called the *constructible universe of sets*. It is built up iteratively, in a manner analogous to V, by restricting the formation of the set $L_{\alpha+1}$ at the $(\alpha+1)^{th}$ level in the cumulative set hierarchy to include only those subsets of L_α which are *logically definable*[41] from parameters in L_α (rather than including all subsets, as in the power set operation used to construct V). Thus L_α is defined by induction on α as follows:

- Let L_0 be the empty set {}:

  $L_0 := \{\}$.
- For ordinal number α, let $L_{\alpha+1}$ be $P_{Def}(L_\alpha)$, the set of definable subsets of L_α:

  $L_{\alpha+1} := P_{Def}(L_\alpha)$.
- For limit ordinal α, let L_α be the union of all the L-stages so far:

  $L_\alpha := \bigcup \{L_\beta \mid \beta < \alpha\}$.

[40] The statement that ZF has any models at all is equivalent to the statement that ZF is consistent, which is an assumption that cannot be proved within ZF. This follows from Gödel's Second Incompleteness Theorem together with his completeness theorem for first-order logic.

[41] A set X is *logically definable* from parameters in L_α if there exist elements $a_1, \dots, a_n$ of L_α and a logical formula $\Phi(x_1, \dots, x_n)$ in the formal language for set theory such that $X = \{a \in L_\alpha \mid (L_\alpha, \in) \models \Phi(a, a_1, \dots, a_n)]\}$ where the order $(L_\alpha, \in)$ is the set L_α with the membership relation ∈ restricted to it and ⊨ is the symbol for consequence. [Woodin, 2010a: 3].

L is defined as the class of all sets X such that X $\in L_\alpha$ for some ordinal α. L contains all the ordinal numbers of V and it has no extra sets beyond those in V. Conversely, if we were to assume that every set of V is in L, then we would have the *Axiom of Constructibility*, denoted 'V=L'. This axiom is not generally accepted by set theorists but it is important to explore because it settles many open questions in set theory. Gödel showed that L satisfies all the axioms of ZFC and the generalised continuum hypothesis.

Cohen's technique, called *forcing*, involves constructing an *outer model* L[G] of L. The aim of this technique is to extend L by adding a new set G which is forced to have certain desired properties not possessed by the sets in L (e.g., the property of violating the continuum hypothesis). This is done in such a way that L[G] (i.e., the result of adjoining to L the set G and everything constructible from G together with the elements of L) is consistent with the axioms of ZFC. Using forcing, Cohen was able to construct a model which, together with Gödel's earlier work on L, proved the independence of the continuum hypothesis in ZFC.

Since Cohen's original work, the technique of forcing has been improved and generalised[42] and applied to a wide range of problems in set theory. Kanamori [2008] attributes Cohen with a dramatic change in the attitudes and concerns of set theorists. In his view, set theory before Cohen had laboured under a special burden of philosophy because of its importance for foundational work in mathematics. The focus had been on questions about the metaphysical and epistemological basis for sets, and on V as the universe of sets. Then:

> With Cohen there was an infusion of mathematical thinking and of method and a proliferation of models, much as in other modern, sophisticated fields of mathematics. Taking V as the ground model goes against the sense of V as the universe of all sets ... This further drew out that in set theory as well as in mathematics generally, it is a matter of method, not ontology. [Kanamori, 2008: 371]

Here Kanamori alludes to the ongoing debate in set theory between the pluralists and the universalists which I will discuss in detail in the next section.

Kanamori is a little too quick to decide the debate. It is a fact that the forcing technique has led to a plethora of independence results, each one of which creates a potential bifurcation of the

[42] For example, it is not necessary for the ground model to be a countable, standard, transitive model and not all forcing extensions are cardinal preserving.

underlying set-theoretic universe according to whether it is forced to be true or false. Furthermore, it appears that any candidate for V can be extended by forcing (i.e., by taking V as the ground model and adjoining a V-generic set G to create the larger universe V[G]). However, in the view of the universalists, this is just another manifestation of the incompleteness of mathematics and demonstrates the need for new axioms and new techniques. For example, Woodin [2010a] is actively pursuing a vision of one mathematical universe by exploring the possibility that ZFC can be extended to provide an axiomatic foundation for set theory which is immune to independence by forcing. He uses a stronger logic than first-order logic and adds a single formal axiom which, remarkably, can be shown to subsume a defined hierarchy of ever stronger axioms. I will consider Woodin's work later. First I would like to examine the debate between pluralism and universalism.

4.3 The Threat of a Mathematical Multiverse

4.3.1 Pluralism *versus* universalism

Hamkins [2011] comprehensively defends pluralism. Much of his argument emphasises the role of set theory as an autonomous field of mathematics with its own, purely mathematical, goals and values. As such, it should not be subject to imposed metaphysical constraints. He makes the point that, following Cohen, a vast diversity of alternative set theoretical universes has been discovered. Forcing enables set theorists to construct universes with properties tailored to their particular interests; e.g., satisfying particular axioms or their negations; containing certain structures of cardinals or not; or having specific relations to other models. As regards to the continuum hypothesis, it is "something like a light-switch, which can be turned on and off by moving to ever larger forcing extensions" [Hamkins, 2011: 15]. This is how mathematics is done and he thinks that universalists, by insisting on the existence of an absolute set-theoretic universe, are denying the experience of the majority of set theorists and setting up an artificial divide between "real" sets and "imaginary" ones [Hamkins, 2011: 3]. In his view, this is no more valid than the historical rejection of so-called imaginary numbers (which are now an integral part of mathematical applications through the development of complex analysis) or of non-Euclidean geometries (which are fundamental to our current understanding of physical space-time).

Hamkins [2011: 2] asserts the actual existence, in the Platonic sense, of all sets described by set theoretical models. Intuition may lead one to prefer some set universes over others but there is no self-evidently "correct" axiomatisation. Different axiomatisations should be interpreted as describing different set concepts. Similarly, forcing extensions of V are just as real as V itself [Hamkins, 2011: 5]. This is supported by the fact that knowledge of an extension V[G] can be obtained from within V by simulation, analogously to the way in which non-Euclidean spherical geometry can be simulated from within Euclidean geometry by restricting the elementary concepts (e.g., 'line') to lie on the unit sphere. Of course, knowledge so obtained cannot be the complete story because V[G] contains sets which are not available in V. However, it provides us with "a glimpse of what it would be like to live in [the other universe]" [Hamkins, 2011: 11]. Hamkins, at least, having glimpsed other universes, cannot go back to the 'one universe' view. And he suspects that this will be a fundamental problem for the universalists: whatever reasons they come up with for preferring one universe over the others, many set-theorists will think of their own mathematical experience of those other universes and be unable to repudiate them [Hamkins, 2011: 16-19].

To a certain extent one's view on pluralism *versus* universalism depends on ones' underlying philosophy of mathematics. I understand Hamkins to be adopting a *thin realist* position, as described by Maddy:

> Sets just are the sort of thing set theory describes; this is all there is to them; for questions about sets, set theory is the only relevant authority. [Maddy, 2011: 61]

In this view, if set theory doesn't lead to a clear concept of set which assigns values to set theoretic propositions in a determinate way, then we must accept that, rather than seeking to constrain our practice with new axioms of questionable evidential status. In contrast, Gödel [1953/9: 351-353] adopted a more traditional realist position: he believed that the content of mathematics consists in relations between concepts which exist independently of our sensations and outside of space-time reality. Furthermore, he thought that there are no absolutely unsolvable mathematical problems; otherwise "it would mean that human reason is utterly irrational by asking questions it cannot answer, while asserting emphatically that only reason can answer them" [Wang, 1996: 317]. As a consequence, he believed that every mathematical proposition is objectively either true or false. So the continuum hypothesis, for example, cannot be true in one universe and false in another - there is only one mathematical universe. If our current concept of set is inadequate for determining the truth, we must refine and delineate it with new axioms. Gödel is an example of what Maddy terms a *robust realist* [Maddy, 2011: 57]. Woodin appears to be closer to this view and talks in terms of the 'truth' of axioms leading to their consistency and predictive consequences in the world [Woodin, 2011b: 96]. Pythagoreans should be robust realists.

I think that the thin realist view adequately underpins set theoretical practice but understates set theory's foundational role in mathematics. Hamkins makes much of the analogy between set theory and geometry [Hamkins, 2011: 11-12]. Mathematicians used to believe that Euclidean geometry was the only possible geometry and that it correctly described the relations of physical space. Kant even classified geometrical knowledge as being 'synthetic *a priori*' because he thought that we are forced by the way our brains are constituted to understand space as Euclidean. However, this was totally turned on its head by the discovery of non-Euclidean geometries and their role in General Relativity. By Hamkins' analogy, whereas early set theorists like Cantor and Gödel believed in a unique concept of set, mathematicians now have such a wide experience of alternative set theoretic universes that they cannot revert to the absolute universe view. But a robust realist might think that there really is only one correct geometry for describing space-time relations in our universe; that geometers initially misunderstood the range of geometrical possibilities; and that it is an empirical matter which geometry is the right one. By analogy, they might think that there is only one correct set theoretical universe that settles questions of mathematical truth and existence; that modern set theorists have expanded the range of possibilities; and that further evidence will determine which universe is the right one.

What would constitute evidence? Gödel distinguished between 'intrinsic' and 'extrinsic' justification in the search for new axioms, with the former deriving from internal set theoretic concepts and the latter linked to the consequences of set theory for other fields of mathematics and for physics. He suggested that:

> There might exist axioms so abundant in their verifiable consequences, shedding so much light upon a whole field, and yielding such powerful methods for solving problems (and even solving them constructively, as far as that is possible) that, no matter whether or not they are intrinsically necessary, they would have to be accepted at least in the same sense as any well-established physical theory. [Gödel, 1964: 261]

Magidor [2011: 10-12] goes further in suggesting that set theory might eventually impact on physical theories such as quantum mechanics, in which case we might obtain actual empirical evidence (and the analogy with non-Euclidean geometry would be even stronger).

Many of Hamkins' arguments, when considered from the viewpoint of the robust realist, have the opposite import from that intended. For example, he questions why mathematicians are so confident that there is an absolute concept of finite natural number [Hamkins, 2011: 14]. A robust realist might think that the development of our concept of finite natural number came from our experiences in the physical world and preceded our practice of mathematics. Indeed, our mathematics was developed to sharpen and explore concepts such as this. It might be interesting for mathematicians to explore different models of arithmetic, with different arithmetic truths, but there is just one standard model of arithmetic tied to applications in the real world (as explicated by Frege). If incompatible background concepts of set give rise to multiple standard models of arithmetic, then I take that as an argument *against* incompatible background concepts of set and *for* a fixed background concept. As Hamkins [2011: 14] discusses, if we assume the fixed background concept of set embedded in second-order logic, then categoricity arguments determine the structure of the natural numbers up to isomorphism. Without such categoricity arguments it becomes very difficult to say what arithmetic is about. If restricted to first-order logic (i.e., eschewing all set theoretical assumptions), we can't even determine the cardinality of models of the natural numbers[43]. First-order logic is fine for exploring the deductive consequences of theories, but inadequate for exploring questions of structure and reference.

The question of reference is central to the viewpoint of robust realism and it is central to Hamkins' arguments as well. If there are problems with determining reference for the natural numbers, for which many mathematicians feel that we have a full conception, how much more difficult will the problem be for sets, for which our pre-theoretic concepts are manifestly inadequate? Martin [2001] questions the feasibility of a philosophy of mathematics in which the concept of set is not sharp enough to give a truth value to every statement about sets, because it leads to a multiplicity of standard models and statements whose truth value is not the same in all standard models. He argues that multiple universes of sets can always be amalgamated into one universe of sets which the pluralist should regard as the true universe of sets [Martin, 2001: 14]. Then the truths of set theory would be those sentences which are true in the amalgamated model. Martin's argument is based on second-order categoricity arguments and assumes a sharp concept of the natural numbers and of well-ordering. Hamkins presents the argument as follows:

> … the set-theoretic universe is unique, because any two set-theoretic concepts V and V' can be compared level-by-level through the ordinals, and at each stage, if they agree on $V_\alpha = V'_\alpha$ and each is claiming to be all the sets, then they will agree on $V_{\alpha+1} = V'_{\alpha+1}$, and so ultimately V = V'. [Hamkins, 2011: 13]

[43] The Lowenheim-Skolem theorem says that if there exists an infinite model of Peano arithmetic, then there exist models of all cardinality, from countable upwards.

Hamkins claims that the argument begs the question because, in using categoricity arguments, Martin has already assumed a fixed set-theoretic background concept. Pluralists agree that any V and V' agree within a fixed background. However, they assert that there can be different set concepts which lead to different V and V'. Sets falling under different concepts cannot necessarily be compared coherently, even in regards to properties such as well-ordering and the natural numbers.

I accept Hamkins' reply but think that it leads him into his own difficulties with reference. Surely some of the benefits which he touts for the pluralist view intrinsically depend on being able to compare sets in different set-theoretic universes? How to adjudicate between different theories if they refer to different objects [Hamkins, 2011: 14-15]? How to learn about inner models by considering various outer models [Hamkins, 2011: 4]? If the sets in alternative universes are different objects (satisfying different concepts of set) then they cannot be compared in any useful way. For example, the continuum hypothesis might be true in Universe_1 and false in Universe_2 but this only tells us that sets_1 satisfy the hypothesis and sets_2 don't: it doesn't tell us anything about sets *per se* because there are no such things in the pluralist view. The question itself becomes meaningless.

The only statements of set theory which can be said to be true *simpliciter* are those which hold in every universe of the multiverse. If we are to include all consistent theories as defining a set concept (an approach which Koellner [2011c: 20] calls *radical* pluralism) then these will not even include the arithmetic truths [Hamkins, 2011: 14]. Neither will there be a common concept of well-foundedness or well-ordering for sets. In the non- pluralist view, alternative set theories allow us to explore alternative possibilities for the unique concept of set and it makes sense to compare them and see which ones most closely accord with our intuitions. When our intuition fails, we can always hope for future illumination. In the pluralist view, set theory becomes a technical device for creating universes of mathematical objects but it doesn't illuminate the concept of 'set' at all.

Koellner [2011c, 20-29] discusses the difficulty in articulating the pluralist view. The pluralist must pick a background theory in order to discuss various models then, when pressed, embrace pluralism concerning the background theory:

> …each time one attempts to articulate the broad multiverse conception one must employ a background theory but since one is a pluralist about that background theory this pass at using the broad multiverse to articulate the conception does not do the conception full justice. [Koellner, 2011c: 21]

In particular, a problem arises when discussing non-standard models of arithmetic because, in such models, one of the axioms asserts the inconsistency of the other axioms[44]. But the pluralist must adopt a background concept in which they are in fact consistent because consistency is needed in order to apply Gödel's Second Incompleteness Theorem. This leads to a conflict between the background concept (i.e., the meta-theory) and the theory itself.

4.3.2 Radical pluralism in the philosophy of mathematics

Balaguer [2009] expounds a form of radical pluralism for all of mathematics which he calls Full-Blooded Platonism (or FBP, for short). It is the thesis that any mathematical object which possibly could exist does exist. Here, *possibility* is taken as logical possibility and defined in terms of consistency; so another formulation of FBP is that all consistent purely mathematical theories truly describe some part of the mathematical realm. According to FBP, whenever a mathematician describes a consistent mathematical object then it exists. As long as we can account for mathematicians being able to reliably assert the consistency of purely mathematical theories then, assuming FBP is true, we can account for the fact that their mathematical beliefs are reliable, since any consistent mathematical belief will be true of something in the mathematical realm. Their mathematical knowledge will be 'thin' (as in Thin Realism) in the sense that it will not pick out *particular* objects.

The 'full-bloodedness' of FBP relates to the plenitude of mathematical objects which it allows; i.e., anything which is described by a consistent mathematical theory. But it turns out that Balaguer isn't quite so indiscriminate in his embrace of mathematical objects after all: he reserves a special place for objects which satisfy our pre-theoretic notions in the relevant mathematical area [Balaguer, 1998: 77]. Such objects define an 'intended' part of the mathematical realm and a 'standard interpretation' for mathematical sentences. A purely mathematical sentence is said to be true *simpliciter* if, and only if, it is true for all the intended parts of the mathematical realm [Balaguer, 2009: 68]. In arithmetic, our pre-theoretic notions dictate that objects which satisfy our concept of 'numbers' can't be sets or functions or chairs, or have properties such as colour or shape. Our full conception of the natural numbers (FCNN) precludes such objects. It does not lead to a unique sequence of natural numbers, however, because there can be ω-sequences which satisfy our FCNN but differ in properties which no human being has ever thought of. Most likely, there will be an infinite number of such ω-sequences. In summary,

[44] Note that, according to Gödel's Second Incompleteness Theorem, such models must exist if Peano arithmetic is consistent because it is consistent with its own inconsistency statement.

according to Balaguer, there are an infinite number of ω-sequences which satisfy the axioms of Peano arithmetic; some of these also satisfy our FCNN and, hence, constitute different intended parts of the mathematical realm and lead to standard interpretations of arithmetic; others do not satisfy our FCNN but nevertheless truly describe some part of the mathematical realm.

By defining truth *simpliciter* in terms of standard models, Balaguer attempts to evade some of the problems posed by non-standard models (as pointed out by Koellner, above). However, in doing so, he loads himself up with baggage associated with pre-theoretic notions. Balaguer tries to fix reference for mathematical terms like 'number' by drawing on our common conception of numbers — our FCNN — but under his account it is unclear how humans could have developed a concept of numbers which serves to successfully discriminate between them in mathematical talk. We might think of numbers as having properties such as being abstract, non-physical, non-spatiotemporal and acausal, but how do we distinguish '3' from '4'? This would be like "a congenitally sightless population speaking meaningfully about 'red' and 'blue'" [Cheyne, 1999: 171]. Balaguer says that an object occupying the '3 position' in an ω-sequence must have the property of 'being 3' according to our FCNN [Balaguer, 1998: 80]. But how do we refer to objects which possess the property of 'being 3'? This is a semantic problem and Balaguer doesn't seem to have an answer. To provide one he would have to address the issue of how we develop our FCNN in the first place, but that would be difficult for him given his dependence on an *externalist*[45] account of mathematical knowledge.

I find Balaguer's theory of our FCNN to be poorly-defined and problematic. He wants to rule in properties such as 'being 3' which help his argument and rule out properties such as 'uniqueness' which do not (and so get labelled as "un-tutored" conceptions [Balaguer, 1998: 79]). Furthermore, a robust realist cannot accept a definition of truth which depends on standard models as described in FBP. Which models get counted as standard depends on our pre-theoretic notions so mathematical truth becomes dependent on psychological facts about us. It becomes subjective and contingent, rather than objective and necessary.

Balaguer argues that consistency is a matter of syntax, not semantics, so we do not need to know anything about mathematical objects (not even their existence) in order to construct consistent

[45] Externalists think that all that it is needed to justify a belief is for the belief to be acquired by a reliable method (in this case, the construction of consistent mathematical theories); we don't need to know that the method is reliable or be able to explain its reliability [Balaguer, 1995: 309].

mathematical theories. He acknowledges that there are problems with knowing that a system is consistent, but says that these problems are the same for all philosophies of mathematics and create no special problems for FBP. The problem for Balaguer, as I see it, is that the axioms of our mathematical theories follow from our understanding of the relevant mathematical objects of the theory, not the other way around. This is reminiscent of Gödel's argument against the logical positivists' attempt to reduce mathematics to syntax: "… instead of justifying the mathematical axioms by reducing them to syntactical rules, these axioms (or at least some of them) are necessary in order to justify the syntactical rules (as consistent)" [Gödel, 1953/57: 342]. We have to know something about the objects of a story before constructing the story.

Even if thin knowledge were enough to construct consistent mathematical theories and, using the truth of FBP, to account for their being about existing mathematical objects, it would not be enough to account for the *truth* of our theories and, hence, not enough to explain mathematical *knowledge*. This is demonstrated by the fact that Balaguer's account does not seem able to bear a counterfactual formulation. According to his account, if the mathematical truths were different then mathematicians would still have the same beliefs. For example, if there were no mathematical objects (which Balaguer thinks is possible [2009: 85]) mathematicians would still have the same ability to construct consistent theories and, hence, would acquire the same beliefs about arithmetic and set theory. Our beliefs under FBP do not track the truth. Even conceding the truth of FBP, our beliefs about mathematical objects seem more like lucky true beliefs than knowledge.

In practice, mathematicians don't just accept any consistent theory; it has to somehow bear upon their intuitive understanding of what mathematics is about. For example, formal systems in which the consistency statement of Peano arithmetic is derivable are *not* considered to be on a par with ones in which its negation is derivable, even though they are all legitimate subjects of study. Balaguer gets this and it is what drives him to his theory of our FCNN but he cannot explain where that knowledge comes from.

It turns out that Balaguer himself does not subscribe to FBP [Balaguer, 2009: 90-99]. He considers it to be on a par with fictionalism as a philosophy of mathematics (and superior to all other contenders). He argues that the apparent differences between FBP and fictionalism come down to a dispute over the existence of mathematical objects and that that there is *no fact of the matter* to decide the issue. Thus, in his view, mathematics is either fiction or docu-drama. Maddy makes much the same point in relation to her classifications of Thin Realism and Arealism (a position which denies the existence of sets):

> The application of 'true' and 'exists' to the case of pure mathematics isn't forced upon us—as it would be if Thin Realism were right and Arealism wrong—nor is it forbidden—as it would be if Arealism were right and Thin Realism wrong… Thin Realism and Arealism are equally accurate, second-philosophical descriptions of the nature of pure mathematics. They are alternative ways of expressing the very same account of the objective facts that underlie mathematical practice. [Maddy, 2011: 112]

If such is the case then I think we may as well stick with some sort of fictionalism. The only cost seems to be an instantiation of the fictionalist story and, perhaps, some inter-subjective notion of mathematical truth (though this point is not entirely clear). Why bother with troublesome notions like 'truth' and 'existence' at all? As a robust realist, however, this just gives me more reason to reject both positions.

4.3.3 The foundational role of set theory

Regardless of personal philosophy, it seems clear that if set theory is to play a foundational role in making comparisons across the various mathematical disciplines, then set theorists will have to provide a fixed background language for the purpose and they will have to find reasons for preferring one set theoretic universe over the others[46]. At one point of his paper, Hamkins claims that:

> … the multiverse view does not undermine the claim that set theory serves an ontological foundation for mathematics, since one expects to find all the familiar classical mathematical objects and structures inside any one of the universes in the multiverse. [Hamkins, 2011: 2]

Then, at a later point, he calls into question the truths of arithmetic [Hamkins, 2011: 14], thereby muddying the issue. As discussed previously, I think that there is no reason to believe that "the familiar classical mathematical objects and structures inside any one of the universes in the multiverse" are the same as those in any other, so set theory under the multiverse view *cannot* serve as an ontological foundation for mathematics. Nevertheless, it is difficult to make the case that finding new axioms for set theory is an urgent problem (witness the differing views of Friedman [2001], Feferman [2000] and Macintyre [2011]). Hamkins' foundational claim does have some pragmatic validity in the *status quo*.

[46] Magidor [2011: 2] also advocates this view, although he is prepared to contemplate a small number of alternative universes.

The main argument in favour of taking action on new axioms now is that it is to be expected that independence phenomena will impact on normal mathematics at some stage in the future, as mathematics grows and changes, and then the matter will become urgent. It is best not to wait until it becomes urgent. As a practical point, mathematicians would not enjoy having the truths of their particular field depend on which set theoretic background they chose. At some level of interference, as truth became further relativised, the whole endeavour of mathematics would have to be re-thought. It would become more and more like a pointless game, less and less relevant to applications (even potentially). Whilst set theorists should pursue their subject in whatever manner they see fit; in providing guidance to the rest of the mathematical community on foundational issues, some degree of consensus around a preferred background theory will become requisite. Informally, ZFC plays that role now, but its fitness for the task is debatable and it really isn't up to it in the longer term.

Hamkins argues that the multiverse provides a context for universe adjudication: "It is only in a multiverse context that we may sensibly compare competing proposals for the unique absolute background universe" [Hamkins, 2011: 15]. This is undoubtedly correct. In order to delineate a preferred universe, it needs to be distinguished from other, competing concepts and reasons given for its preferential role[47]. However, in the same paragraph he goes on to say that: "In this sense, the multiverse *view* provides a natural forum in which to adjudicate differences of opinion arising within the universe view" (my italics). This is overstating his case. A multiverse context does not entail a multiverse view. One can have a universe view and still wish to explore the possibilities in a multiverse context because one does not yet have good enough reasons for advocating a particular background universe. This is an example in which different aims encourage similar practices.

So much for the problems of the pluralist view. What of the prospects of set theorists pursuing the non- pluralist view? I take it that Woodin and his colleagues (e.g., Koellner, Martin and Steel) are at the forefront of this project and I shall focus my attention on their work. Their case depends on being able to justify new axioms for set theory, as discussed in the next section.

[47] I take this to be Woodin's *modus operandi* in searching for ways to distinguish inconsistent large cardinal axioms from consistent ones [Woodin, 2011b] (see Section 4.6.1 and Section A.6 in the Appendix for clarification of this point).

4.4 New Axioms for Set Theory

4.4.1 Justifying axioms

Justifying axioms is a tricky business. Ideally axioms would be self-evident but this is never achievable and what we get in practice is varying degrees of plausibility. Take the existence of the empty set; i.e., the Empty Set Axiom. This is one of the axioms of ZFC. Is it self-evident that there is a set with no members? Some people's intuition tells them that a set is a collection and "a collection must collect something" [Maddy, 1997: 41] so there is no empty set. Others argue from the logical conception of set (according to which a set is a collection of members defined by a common property) that there is a set corresponding to inconsistent properties such as 'blue and not blue' and this would necessarily have no members. Arguments over the Empty Set Axiom can be clouded by metaphysical issues to do with the 'something from nothing' debate. Given the empty set, the whole edifice of set theory can be constructed and, with it, all the mathematical objects. In Fraenkel's exposition of set theory, the empty set is not really a set [Fraenkel and Bar-Hillel, 1958: 30-31]. It is the 'something' — the one and only memberless object — whose existence is needed to start the unquestionably fruitful process of set building. It is called a set by convention, for ease of exposition. For mathematical purposes, the important thing about the empty set is its usefulness. The Empty Set Axiom is accepted for pragmatic reasons (as is the Axiom of Choice, discussed in Section 4.2.3).

The Axiom of Infinity is another interesting axiom of ZFC which divides the mathematical community. Some people are happy with the idea of *potential* infinity, whereby any finite set can always be extended by adding a new member, but balk at the idea of a *completed* infinity, whereby the totality of infinite members can be collected in a set. Indeed, there is a persistent undercurrent of *finitism* in the philosophy of mathematics, according to which a mathematical object does not exist unless it can be constructed from the natural numbers in a finite number of steps[48]. But completed infinities are needed for defining real numbers, and most mathematicians would be reluctant to jettison the whole of analysis, so we accept the Axiom of Infinity. Once again, this is a pragmatic, extrinsic justification (as emphasised by Maddy [1997: 52]).

The Axiom of the Empty Set, the Axiom of Choice and the Axiom of Infinity are already questionable. When we get to the Power Set Axiom and the Axiom of Replacement things get murkier

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[48] Finitism is often associated with the German mathematician Kronecker, who is attributed with the statement: "God made the integers, all the rest is the work of man". [Boyer, 1991: 570]

still [Maddy, 1997: 52-60]. The idea that such axioms could be justified on the basis of being "self-evident" or "following directly from the concept of set" seems implausible. In the end, although we may require that arguments be mounted in support of individual axioms, it is their consequences as a whole that provide their justification. Thus, the axioms of ZFC are justified on the basis of the success of the theory in providing a foundational basis for accepted mathematics. New axioms for set theory will have to be justified on a similar, extrinsic basis. Magidor [2011: 8-10] lists the following desirable characteristics for a new axiom:

- The new axiom should have intuitive or philosophical appeal. It should conform to some mental image of the basic concepts of Set Theory.
- The new axiom should be strong enough to decide a large class of statements which are undecidable on the basis of the axioms adapted so far.
- The new axiom should produce a coherent elegant theory for some important class of problems.
- To the extent possible, the new axiom should have testable verifiable consequences (e.g., evidence for the axiom could be a result that we intuitively believe is true and that we were not able to derive without the new axiom; or even some consequence for fields outside of mathematics, like physics).
- If possible the new axiom should be resilient under forcing extensions.

Of course, the ultimate test is acceptance by the mathematical community as a whole.

4.4.2 Large cardinal axioms

Let me give an example of how not to create new axioms [Koellner, 2011a: 3-8]. Remember that Gödel showed that formal systems such as ZFC cannot decide their own consistency statement. If this were the only problem then it would be possible to strengthen the axioms by simply adding the consistency statement as a new axiom. Then a stronger formal system would be obtained, its consistency statement would be undecidable and that, in turn, could be added as a new axiom, and so on indefinitely. Such new axioms based on consistency statements have an intuitive and philosophical appeal and they do act to resolve previously undecidable statements. However, it is not clear what new undecidable statements are unleashed at each stage, of what degree of intractability. The theories produced are not ones in which mathematicians would naturally be interested and are not "coherent elegant theor(ies) for some important class of problems". Furthermore, it is known that iterating the consistency statement does not produce theories of sufficient strength to resolve some important undecidable statements arising in the study of sets of real numbers [Koellner, 2011a: 8]. It certainly cannot resolve the continuum hypothesis.

A better argument can be made for extending ZFC using *large cardinal axioms* [Koellner, 2011a: 8-14]. Large cardinal axioms can be thought of as asserting that certain infinite levels of the universe of sets exist. The Axiom of Infinity is the first in this sequence. It asserts that there is a level ω which completes the potential infinity of finite sets. Next comes the Axiom of Replacement. It asserts, equivalently, that there is a level ω + ω. The next axiom in the sequence takes us beyond ZFC. It asserts that there is an *inaccessible* cardinal; i.e., a level which cannot be reached by taking unions of collections of smaller sets, or taking power sets, or taking any combination of operations available in ZFC [Maddy, 1997: 74]. After inaccessible cardinals, the properties defining ever-higher stages in the large cardinal hierarchy become more and more difficult to comprehend, and further and further distanced from the operations underlying the sets of ZFC. There is a *hyperinaccessible* cardinal κ such that there are κ inaccessible cardinals smaller than κ, and a still higher hyperinaccessible cardinal λ such that there are λ cardinals like κ smaller than λ, and so on. [Maddy, 1997: 74-75]. Higher still are *Mahlo* cardinals[49].

Large cardinal axioms at the level of an inaccessible or higher cardinal cannot be shown to be relatively consistent with ZFC[50] and they are hard to justify on an individual basis. However, collectively, they satisfy the concept that the iterative process of generating sets should not run out and that there should always be another stage extending the class of ordinals. This is part of the *maximise* concept of set theory, whereby we should place as few restrictions as possible on set formation so that the criteria of mathematical existence can be as inclusive as possible [Maddy, 1997: 211].

Gödel noted that "only a maximum property would seem to harmonise with the concept of set" [Gödel, 1964: 262-263]. In his 1947 paper on the continuum hypothesis, he specifically mentioned axioms of Mahlo type:

> … [Mahlo] axioms show clearly, not only that the axiomatic system of set theory as known today is incomplete, but also that it can be supplemented without arbitrariness by new axioms which are only the natural continuation of those set up so far. [Gödel, 1947: 182]

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[49] See [Kanamori, 1994: 17]

[50] By Gödel's Second Incompleteness Theorem, it is never possible to prove the relative consistency of a large cardinal axiom with ZFC because, if x is a large cardinal, then it can be shown that V_x is a model of ZFC and, thus, that ZFC is consistent [Maddy, 1997: 74].

This excerpt shows that he thought of such new axioms as being intrinsic (i.e., following naturally from the concept of set[51]).

Although we cannot know exactly why Gödel thought that an axiom asserting the existence of a Mahlo cardinal is intrinsic, Koellner [2009a] has outlined a possible explanation in terms of *reflection principles*. Reflection principles codify the concept that the universe of sets V is absolutely infinite, indefinable and transcends any possibility of description from below, so anything true which we can say of V must already be true of an initial segment V_α [52]. Consistent with Koellner's explanation, Gödel is reported to have said:

> Generally I believe that, in the last analysis, every axiom of infinity should be derivable from the (extremely plausible) principle that V is indefinable, where definability is to be taken in [a] more and more generalised and idealised sense.[Wang, 1996: 285]

To see how reflection principles play out, consider the theory of ZFC without the Axiom of Infinity [Koellner, 2009a: 209]. Since V is closed under the power set operation, reflection principles imply that there is a V_α which is closed under the power set operation. Then α is a completed infinity and, by letting ω be the smallest such completed infinity, we derive the Axiom of Infinity. Similarly, V is closed under the combined operations of replacement and power set, so there is a V_α which is closed under these operations. Then α is inaccessible and, by letting κ be the smallest such α, we derive the smallest inaccessible cardinal. Then by reflecting the statement 'V contains an inaccessible greater than κ' we get an inaccessible greater than κ and, by continuing this reflection process, we get higher and higher inaccessibles until we reflect the statement that 'V contains an inaccessible limit of inaccessibles' which takes us to the next level. This process can be continued to give Mahlo, *weakly compact*, *indescribable*, *ineffable* and *remarkable* cardinals, but generates inconsistencies at a level just below that of an *Erdos* cardinal [Koellner, 2009a: 212-213]. Thus, this interpretation of the reflection principle fails at a finite level and doesn't exhaust the concept of V containing higher and higher levels.

Other versions of the reflection principle have been suggested to further extend the large cardinals, as discussed by Koellner [2009a: 217-218], but these are not as straightforward or, arguably,

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[51] For technical reasons, Gödel thought that the resolution of the continuum hypothesis would require new axioms of an extrinsic kind (i.e. based on hitherto unknown principles). In fact, it is now known that the continuum hypothesis is independent of all remotely plausible large cardinal axioms that have been considered to date.

[52] Here, as always, one must be careful to keep separate the concepts of proper class and set; e.g., V has the property of containing all sets, but no set (and, therefore, no initial segment) can have that property.

intrinsic. For example, by analogy to extending the "height" of V, Koellner suggests extending the "width" of V by reflection. We saw earlier how Gödel defined a universe of constructible sets L which contains all the ordinals, and so, has the same height as V, but does not contain all the power sets, and so, is not as wide as V. L is the smallest inner model. The non-constructible subset of ω called $0^\#$ codes up information on exactly how L differs from V [Maddy, 1997: 76]. By reflecting the statement 'L does not cover V' we derive the existence of $0^\#$. By considering an inner model which is some union of L and $0^\#$ and reflecting the statement that this still does not cover V we get the existence of a further complementary set $0^{\#\#}$, and so on. According to Koellner [2009a: 218], width reflection can be used to generate an inner model with a *measurable* cardinal, taking us beyond the level of large cardinal reachable by height reflection. It is known that the existence of a measurable cardinal is not consistent with the Axiom of Constructibility V=L. This makes it the first of what are deemed *large* large cardinals, in contrast to the large cardinals reachable by height reflection which are deemed *small* by virtue of the fact that their existence is consistent with V=L.

Reflection principles meet Magidor's criterion of having an intuitive and philosophical appeal but they are not able to generate every axiom of infinity, as Gödel may have hoped. The most general method for generating large cardinal axioms uses *elementary embeddings* (as explained in Section A.1.2 in Appendix 1). The elementary embedding procedure allows us to define large cardinals up to an *ω-huge* cardinal [Woodin, 2011c: 456]. The large cardinal axiom asserting the existence of an ω-huge cardinal is the strongest large cardinal axiom which is not known to be inconsistent with ZFC.

The elementary embedding procedure for generating large cardinal axioms does not show us how to build up to the large cardinal from below and so it is not intrinsic. Also, to be very clear, it does not guarantee the existence of the large cardinal or the relative consistency of the associated axiom with ZFC. It simply provides us with a means of pushing the level of cardinals as high as it can go, right up to the point where we bump into proven inconsistency with ZFC. A further sobering fact is that even the largest of large cardinal axioms is not enough to determine the continuum hypothesis. Levy and Solovay [1967] showed that all such large cardinal assumptions are relatively consistent with both the continuum hypothesis and its negation. So something more is needed.

4.4.3 The case for large cardinal axioms

So far I have discussed what large cardinal axioms are and how to generate them but I haven't explained their central role in set theory and why they are considered by many set theorists to be the preferred continuation of ZFC. Part of the reason derives from reflection principles and the richness of the structure of ordinals, but more telling is the accumulated weight of extrinsic and historical evidence. This evidence is reviewed in Section A.1.3 in Appendix 1 and is summarised below.

In the early development of set theory there was a focus on studying the properties of definable sets of real numbers[53] – what is now called *descriptive* set theory [Koellner, 2011b: 16-24]. Problems emerged in the 1960s (following the work of Gödel and Cohen) when independence results showed that many questions about the properties of definable sets of reals are not decidable in ZFC. The situation was greatly clarified in the 1970s and 1980s with the development of new axioms for ZFC. One strand of research studied the implications of large cardinal axioms. The other strand, to do with axioms of *determinacy*, had its origins in a quite different area of mathematics — game theory. The underlying connection between determinacy and the regularity properties of sets of reals is not immediately obvious to outsiders, but to some set theorists "determinacy lies at the heart of the regularity properties and may be considered their true source" [Koellner, 2011b: 26]. There is an intricate web of theorems interweaving large cardinals, determinacy, and regularity properties. These theorems bring the two strands of research on new axioms together in an unexpected way. They show that definable determinacy is necessary and sufficient to prove the regularity properties of certain definable sets of reals; and that large cardinals are necessary and sufficient to prove definable determinacy.

Koellner [2011b: 30-38] has used these results to support the case for definable determinacy and, hence, for large cardinal axioms. Although all set theorists agree that the results are remarkable, they do not all find the case for definable determinacy compelling. They argue that descriptive set theory is only one area and focusing on another area might produce competing, incompatible axioms. Part of Koellner's argument is the claim that this is unlikely but, of course, it depends on what areas one finds interesting and what weight one gives to intrinsically plausible results from different areas. For example, Shelah [2003: 211-213] thinks that Gödel's Axiom of Constructibility V=L is an interesting axiom with some appealing consequences, and it is known to be incompatible with one of the axioms of definable determinacy.

[53] See the previous discussion of Gödel's constructible universe L for a definition of definable sets. By starting with the real numbers R and iterating the definable power set operation along the ordinals we get the hierarchy L(R) of definable sets of reals.

The only response for the universalist is to try to strengthen the case for large cardinal axioms and bring more and more set theorists on board. One thread of evidence which I haven't discussed yet comes from the *Inner Model Program* [Woodin, 2011c: 458-466]. Recall that Gödel's inner model L *does* have many appealing features: it is consistent with ZFC, contains all the ordinals, and resolves many outstanding problems in set theory, including the continuum hypothesis (see Section 4.2.4). Furthermore, there is a detailed and comprehensive understanding of L, including the properties of sets of reals in L, as a result of the fine-structural methods developed by Jensen around 1970 [Martin and Steel, 1994: 3]. However, L only contains large cardinals up to the level of inaccessibles, not a measurable cardinal and certainly not the Woodin cardinals required for the determinacy theorems. Many set theorists think that the axiom V=L is just too restrictive and Woodin goes so far as to declare it "false" [Woodin, 2010a: 1].

The Inner Model Program seeks generalisations for V=L which are compatible with large cardinal axioms [Martin and Steel, 1994: 3-6]. Historically, building on Gödel's L, the next inner model studied was L[U] which satisfies "There is a measurable cardinal". Jensen's fine structure methods were applied to L[U] to develop a detailed description of its properties. Later, Mitchell and Steel [1994] developed a process for extending inner models in a naturally well-ordered way in terms of the increasing strength of the associated large cardinal axiom. This is how Woodin describes the process:

> The inner models which are the goal and focus of the Inner Model Program are defined layer by layer working up through the hierarchy of large cardinal axioms, which in turn is naturally revealed by the construction of these inner models. Each layer provides the foundation for the next, and L is the first layer. Roughly (and in practice) in constructing the inner model for a specific large cardinal axiom, one obtains an exhaustive analysis of all weaker large cardinal axioms. [Woodin, 2011c: 458]

According to Steel [2000: 426], the ability to build a systematic, detailed, fine structure theory at each stage is tantamount to showing the consistency of the associated large cardinal axiom. As one ascends through the hierarchy of large cardinal axioms, the construction generally becomes more and more difficult. It breaks down at the level of a Woodin cardinal which is a limit of Woodin cardinals. Devising techniques to extend the construction to the level of a *supercompact*[54] cardinal is a current focus of the Inner Model Program.

Using the techniques of forcing and inner model theory, it has been established that any natural candidate for a new axiomatisation of ZFC is mutually interpretable with a theory defined by a large cardinal axiom. Thus, if Θ is a candidate for a new axiom then one can generally find a large cardinal axiom Φ such that ZFC + Θ and ZFC + Φ are mutually interpretable [Koellner, 2011a: 12]. Then large cardinal axiomatisations can be used to translate back and forth between rival theories from conceptually distinct domains and compare features such as their consistency strength. In this way, large cardinal axioms are "crucial to organising and understanding the family of possible extensions of ZFC…it seems that the consistency strengths of all natural extensions of ZFC are well-ordered, and the large cardinal hierarchy provides a sort of yardstick which enables us to compare these consistency strengths" [Steel, 2000: 426-428]. There are many examples of set theoretical statements whose consistency strength can be precisely calibrated in terms of the large cardinal hierarchy. In many cases this is the only way to compare theories; there is no direct interpretation and one must proceed by translating into a background theory mediated by large cardinal axioms [Koellner, 2011a: 14]. This provides a strong plausibility argument for large cardinal axioms as the preferred continuation of ZFC.

4.5 Effectively Complete Theories

4.5.1 Overcoming independence results

The main threat to the completion of mathematics is the technique of forcing. Of course, Gödel's Incompleteness Theorems tell us that there will always be a sense in which mathematics is incompletable: if we strengthen a given formal system to resolve its independent statements, new independent statements will arise in the form of a new Gödelian sentence, a new consistency statement and new statements of first-order arithmetic (i.e., Diophantine equations), which can then be resolved by moving to the next level, and so on. However, Gödel initially envisaged that mathematics might be made effectively complete via large cardinal axioms:

> It is not impossible that for such a concept of demonstrability [namely, provability from true large cardinal axioms] some completeness theorem would hold which would say that every

[54] See [Kanamori, 1994: 298]

proposition expressible in set theory is decidable from the present axioms plus some true assertion about the largeness of the universe of all sets. [Gödel, 1946: 151]

This hope was seemingly dashed by the development of forcing techniques. The insidious thing about forcing is the way in which it can be used to switch the value of a statement from true to false in some generic extension of V whilst having a relatively benign impact on other properties of V (e.g., not collapsing any cardinals). So it is for the continuum hypothesis, according to the pluralist view: whatever universe one is in, one is always very close to a universe in which the continuum hypothesis has the opposite truth value [Hamkins, 2011: 15].

Are there supplementary axioms, or other techniques, which could be used to resolve the continuum hypothesis? The status of this question is reviewed in Sections A.1.4 and A.1.5 in Appendix 1. A summary is given below. I warn that this work is at the cutting edge of research and, hence, quite fluid and messy.

We know that in order to resolve the continuum hypothesis a new notion is needed. Part of the answer might be to introduce a logic which is stronger than first order logic and is well-behaved in the sense that its consequences cannot be altered by forcing in the presence of large cardinal axioms. Ω-logic is such a logic[55]. It has a notion of consequence which is robust to forcing under the assumption of an appropriately strong large cardinal axiom (i.e. under the axiom "there is a proper class of measurable Woodin cardinals"). It also has a quasi-syntactic proof relation. Ω-logic is known to be sound (i.e., if a sentence φ is Ω-provable in a set theory T then it is necessarily an Ω-consequence of T). However, it is not known whether Ω-logic is complete (i.e., it is not known whether every sentence φ which is an Ω-consequence of a set theory T is necessarily Ω-provable in T). The conjecture that Ω-logic is both sound and complete is called *the Ω conjecture*. Ω-completeness provides a notion of effective completeness at higher levels of the set hierarchy.

It is known that, if the Strong Ω Conjecture holds[56], one cannot have an Ω-complete theory of the whole of the structure $V_{\omega+2}$ (note that the continuum hypothesis is a statement in $V_{\omega+2}$). If the Strong Ω Conjecture does not hold, then there is the possibility that we could have an Ω-complete theory for $V_{\omega+2}$. In fact, it might be possible to achieve an Ω-complete theory for any specified level V_α and to piece

[55] For more information on Ω-logic and its relation to full first- and second-order logic see Koellner[2010].
[56] The Strong Ω Conjecture is the Ω Conjecture plus a conjecture about an extended form of the Axiom of Determinacy. See [Koellner, 2011: 16-17].

these together into a coherent theory of the entire universe of sets. If this were achieved, then the theory would be a candidate for the effective completion of set theory along the lines envisaged by Gödel [1946: 151] (quoted previously). However, Koellner and Woodin [2009] have shown that such a theory could not be unique. If there is an Ω-complete theory of V_α for $\alpha \geq (\omega+2)$ then there is another, equally good but incompatible Ω-complete theory which differs on the truth value of the continuum hypothesis[57]. This seriously raises the spectre of pluralism. The existence of two, or more, coherent but incompatible Ω-complete theories of V would undermine the universalist's concept of truth. It would not be possible to combine such theories in the manner suggested by Martin [2001]. Such a scenario is considered by Steel [2004] and discussed at length by Maddy [2005: 369-373]. The theories would all be universal and equivalent in terms of the mathematics that would result (i.e., choosing one theory over another would not result in any behavioural or methodological differences). In choosing one theory, the others would be accessible as generic extensions. The continuum hypothesis (and other undecidable statements) would be meaningless in an absolute sense, but acquire a relative value as a matter of convention.

This scenario is highly speculative but does serve to crystallise a plausible pluralist case. Note that a multiverse of this kind does not conform to the radical pluralist view of what a multiverse might consist of (i.e., all consistent theories). It locks in a universalist view at least to the level of ZFC + "there is a proper class of measurable Woodin cardinals" which is essential for the application of the notion of Ω-completeness. In this way, it avoids the problems which plague the radical pluralist view. It is a challenge for the universalist view (and, hence, for Pythgoreanism).

In responding to the challenge of the plausible pluralist case, Woodin [2011d] uses a global perspective to define and investigate a multiverse which he calls 'the generic multiverse'. He sets up the generic multiverse as a strawman for the purposes of showing that it is untenable under fundamental principles of set theory, *provided that the Ω Conjecture is true*. Here, once again, we see the importance of the Ω Conjecture. The pluralists' best response is to deny the truth of the Ω Conjecture. Woodin ([2011d: 28] and [2011b: 111-112]) argues that there is evidence for the Ω Conjecture, albeit nothing conclusive. Firstly, the Ω Conjecture can be shown to be consistent with the theory ZFC + "There is a proper class of Woodin cardinals". More importantly, "… recent results indicate that if [the Inner Model] program can succeed at the level of supercompact cardinals then no large cardinal hypothesis

[57] Note that the continuum hypothesis is just a particular case; there are other suitable sentences which could be forced to differ between Ω-complete theories of V_α.

whatsoever can refute the Ω Conjecture" [Woodin, 2011d: 28]. These recent results are what I shall address in the next section. They are the current best hope for completing mathematics in such a way as to decide every proposition expressible in set theory, including the continuum hypothesis.

4.5.2 The theory of Ultimate L

Recall that the Mitchell-Steel inner model method enabled Gödel's L to be generalised and made compatible with large cardinals up to the level of a Woodin cardinal which is a limit of Woodin cardinals (see Section 4.4.3). The method is incremental. At each stage, an inner model is constructed for a given large cardinal axiom, together with its detailed fine-structure, including an exhaustive analysis of all weaker large cardinal axioms. The inner model for a given large cardinal axiom is inconsistent with the existence of stronger large cardinal axioms. Just as the axiom V=L is inconsistent with the existence of a measurable cardinal, so the inner model at the level of one measurable cardinal is inconsistent even with the existence of a second measurable cardinal. By virtue of its incremental nature, together with the expectation that there is no strongest large cardinal axiom, this method cannot provide any sort of completion of mathematics. However, Woodin [2010b: 104] has shown that *if* an inner model could be constructed at the level of a supercompact cardinal *then* that model would necessarily be close[58] to V and inherit all large cardinals from V. It would yield the ultimate enlargement of L, called *Ultimate L.* It is not yet known how to construct Ultimate L or, indeed, whether it can be constructed at all, but Woodin has shown that these results would hold no matter how it was constructed.

Woodin's work takes a global approach and investigates the coarse structure of an inner model with a supercompact cardinal. The conjectured theory would be axiomatised by ZFC + $V=L_{\text{ultimate}}$, where $V=L_{\text{ultimate}}$ is a single axiom which would imply the existence of a supercompact cardinal and have, as a corollary, the truth of the Ω Conjecture and the continuum hypothesis. $V=L_{\text{ultimate}}$ would yield an Ω-complete theory, immune from the effects of forcing. Unlike V=L, it would be compatible with all large cardinal axioms, so any instances of independent statements from Gödel's incompleteness theorems at a given level would be determined by ascending to the next level of the large cardinal hierarchy. This would put set theory on the same footing as number theory as far as independence goes [Woodin, 2011d: 30]. The complete theory of Ultimate L would have the form ZFC + $V=L_{\text{ultimate}}$ + LCS, where LCS is

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[58] For example, it would be correct about singular cardinals and compute their successors correctly [Woodin, 2011c: 465].

an axiom schema standing for that part of the large cardinal hierarchy stronger than a supercompact [Koellner, 2011c: 34]. The claim is that Ultimate L would give us a complete picture of V in the sense that it would be capable of interpreting any extension of ZFC attainable from supplementary axioms or forcing and it would decide all set theory propositions.

If $L_{ultimate}$ were a unique axiom with the properties outlined above, then Ultimate L might be the unique background theory of mathematics which would serve as the ultimate arbiter of mathematical existence and truth. The catch is that there are (at least) two competing candidates for $L_{ultimate}$ — called L_{Ω} and $L_{\Omega s}$ — which are Ω-complete and compatible with all large cardinal axioms but have different structural properties. Woodin [2011a] makes a case for preferring $L_{\Omega s}$ over L_{Ω} but it is very much an extrinsic case. This research is at such a fluid stage that no particular scenario should be taken too seriously. The main point is that now there is an argument for a theory which would be an ultimate completion of mathematics. Woodin writes:

> There is in my view no reason at all, beyond a lack of faith, for believing that there is no extension of the axioms ZFC, by one axiom, a posteriori true, which settles all instances of the Generalised Continuum Hypothesis and more generally which yields a theory of the universe of sets which is as "complete" as the theory of Gödel's constructible universe, L, which is given by the axioms of ZFC.…Until recently I have always viewed such a possibility as very implausible at best. [Woodin, 2011d: 30]

4.6 Large Cardinal Axioms and Mathematical Truth

4.6.1 Woodin's arguments for the truth of set theory

The relative consistency of large cardinal axioms with ZFC cannot be proven but Woodin (like Steel) believes that "large cardinal axioms for which there is an inner model theory are consistent" [Woodin, 2011c: 458]. He goes so far as to predict that there will be "no discovery ever" of an inconsistency in the theory ZFC + "there exist infinitely many Woodin cardinals" [Woodin, 2011c: 453]. I would say that his confidence comes from his overall experience with large cardinals; e.g., their centrality in the hierarchy of interpretability; their role in developing a coherent theory of the definable real sets; their explication through the detailed modeling of fine structure theory; their pointing the way to an effective completion of mathematics. It seems that large cardinal axioms are so fundamental for questions of mathematical truth and existence that if one can build an inner model for a large cardinal axiom, then the axiom must be consistent and the large cardinal must exist. This is Woodin's Set Theorist's Cosmological Principle:

> The large cardinal axioms for which there is an inner model theory are consistent; the corresponding predictions of unsolvability are true *because the axioms are true*. [Woodin, 2011c: 458]

Needless to say, this is a very controversial view. Shelah writes:

> Generally I do not think that the fact that a statement solves everything really nicely, even deeply, even being the best semi-axiom (if there is such a thing, which I doubt) is a sufficient reason to say it is a "true axiom". In particular I do not find it compelling at all to see it as true. [Shelah, 2003: 212]

Feferman also expresses his skepticism:

> …the usual idea of mathematical truth in its ordinary sense is no longer operative in the research programs of Martin, Steel, Woodin, et al. which, rather, are proceeding on the basis of what seem to be highly unusual (one might even say, metaphysical) assumptions. [Feferman, 2011: 8]

In "The Realm of the Infinite" [2011b] Woodin mounts an argument for the Set Theorist's Cosmological Principle and its extension to the truth of all consistent large cardinal axioms. It is an argument in many parts. He begins by considering the view that only assertions about small finite numbers (e.g., physically meaningful magnitudes) are meaningful. For the purposes of this argument, an example of a small finite number would be 10^{24}. He points out that V_n is relatively large, even for small values of finite n, and that for $n=|V_{1000}|$ it is way beyond any boundary which might be set by the small finitist. He then uses a Gödelian sentence to exhibit a proof of length less than 10^{24} bits which, if it exists, entails the non-existence of V_n for $n=|V_{1000}|$. He argues that (conceivably) a quantum computer could be programmed to make a truly random search for a proof of this size. The prediction that the computer would never find such a proof is a prediction about the physical world. Thus, the non-physical realm (i.e., the large finite as represented by V_n for $n=|V_{1000}|$) has consequences for the physical realm. Of course, if the computer *did* find the proof then it would show that the conception of finite numbers is not meaningful beyond a certain limit and then much of modern mathematics would collapse. On the other hand, if it went through all the finite combinations and *didn't* find the proof, it still wouldn't demonstrate the existence of the large finite. The point seems to be that our intuition strongly favours the prediction that the proof would never be found and this shows that we implicitly accept the meaningfulness of the large finite.

Woodin [2011b: 91] argues that, if this is persuasive, then there seems little to prevent us from accepting the universe of sets. This part of his argument is based on the question of 'where do you draw the line?' It is one of the strongest arguments used by advocates of new axioms for mathematics, including Gödel and Friedman. Our mathematical intuition leads us from small finite numbers to large finite numbers to infinity and (via Cantor's process) to infinite cardinals. Gödel thought that large cardinals were a natural continuation of this sequence. In fact, like Woodin, he thought that "if set theory is inconsistent then elementary number theory is already inconsistent" [Wang, 1996: 216]. Extensions beyond the smallest infinity are so closely linked from one stage to the next that a serious weakness at one point would bring down the whole edifice. Friedman [2002: 9] uses the language of 'blessed objects' and 'excluded objects': it is difficult to hold the exclusionary view when axioms associated with excluded objects are needed to prove statements involving blessed objects. Woodin makes essentially the same point: "... assuming the large cardinal axioms to be true one can infer as true specific statements of Number Theory *which arguably cannot otherwise be proved*" [Woodin, 2011c: 470].

In the next part of his paper Woodin gives further arguments for the truth of set theory, but they are of such a technical nature that I have relegated them to Section A.1.6 in Appendix 1. To summarise, Woodin argues that Ultimate L could provide us with the means to explore the boundary between possible and impossible large cardinal axioms. This could never be done using Inner Model Theory because of its incremental nature. His hunch is that Ultimate L will enable us to eliminate essentially all the large cardinal axioms known to contradict the Axiom of Choice (which would be tantamount to a proof of the Axiom of Choice [Woodin, 2011c: 470]). Then, in his view, any suggested set theory would need to be justified by first establishing its equiconsistency with a specific level of Ultimate L. Ulimate L, in its turn, would be justified by our understanding of the hierarchy of large cardinal axioms as "true axioms about the universe of sets" [Woodin, 2011b: 96].

4.6.2 Comments on highlighted views

To summarise Woodin's view:

- Set theory at all levels makes new predictions about the physical realm, which can be investigated experimentally to some extent, and these predictions need to be accounted for.
- The best account of the predictions of set theory justifies them through their "calibration by a large cardinal axiom in **conjunction** with our understanding of the hierarchy of such axioms as **true axioms about the universe of sets**". [Woodin, 2011b: 96]

The axioms themselves are justified by a combination of intrinsic and extrinsic reasoning (as outlined in this chapter and Appendix 1) with the extrinsic reasons becoming less compelling for the higher levels. Most importantly, there is still the very strong possibility that we will be able to find an effective completion of mathematics via a single formal axiom $L_{ultimate}$. Woodin states:

> A far stronger view … which I also currently hold … is that there *must* be such an axiom and in understanding it we will understand why it is essentially unique and therefore true. [Woodin, 2011d: 31]

Gödel held similar views. He believed that there could be an effective completion of mathematics via some true assertion about the largeness of the universe of all sets, in spite of his Incompleteness Theorems. He thought that the predictions of mathematics are grounded in the truth of the fundamental axioms [Gödel, 1953/9: 340]. Of course, Gödel also thought that we can understand the truth of the axioms by using an additional sense which enables us to perceive mathematical reality [Gödel, 1953/9: 351-353]. He has often been pilloried in the philosophical literature for this view. Personally, I think it is hard to deny a role for mathematical intuition when you consider the remarkable abilities of mathematicians like Ramanujan [Kanigel, 1991]. Regardless, Gödel's discussion of the intrinsic and extrinsic justification for axioms indicates a much more sophisticated understanding of the philosophical issues than he is often given credit for; not dissimilar to modern views. For example, in his paper 'Is Mathematics Syntax of Language?', Gödel [1953/9: 338] argues that if one needs to posit certain concepts in order to use a theory rationally then one is committed to some sort of existence for them, just as when one posits concepts of physical objects (e.g., quarks) in order to best explain empirical data. He also argues that only the laws of nature *together* with mathematics have consequences verifiable by sense experience, and so it is arbitrary to separate out scientific statements from mathematical ones and claim that evidence only applies to them [Gödel, 1953/9: 355-356].

A not-inconsistent-but-diametrically-opposed view is held by Nelson [2011]. He sees Gödel's Second Incompleteness Theorem as indicating the possible collapse of contemporary mathematics.

However, his solution is to stick to finite mathematics. He thinks that mathematics went wrong with the introduction of mathematical induction. He interprets Gödel's theorem as indicating the inconsistency of Peano arithmetic.

There is no doubt that large cardinals are outside the ken of normal mathematics. Normal mathematicians do not work with formal axiomatic systems like ZFC. They work with integers, real and complex numbers, elements of geometry, and relations between objects such as these. Normal mathematics includes geometry, number theory, differential equations, algebra, and functional analysis. For most purposes, only subsystems of second-order arithmetic are required. Feferman [2006b: 446] has argued that the mathematics that is applicable in science can be formulated in a basically predicative theory[59]. Feferman [2000] sees the logical need for new axioms for set theory but questions 'which ones?' and 'why those?' He sees no practical need for new axioms to settle open arithmetical and finite combinatorial problems (despite Friedman's efforts to bring such problems closer to normal mathematics). He thinks that the example of Fermat's Last Theorem shows that the basic axioms will suffice if we work them hard enough. In that case, Andrew Wiles' proof of Fermat's Last Theorem made an assumption about the existence of an inaccessible cardinal but mathematicians think that this was just a removable short cut and that a proof will be found in Peano arithmetic or some slight extension of it.

Feferman's philosophy of mathematics is *conceptual structuralism* [Feferman, 2011]. Emphasis is not on mathematical objects but on how they relate to one another; i.e., their structure. Mathematics is taken to be a construction of human minds. It achieves a form of intersubjectivity (not full objectivity) through the combined judgement of the community of mathematicians over history. According to Feferman [2011: 11], mathematical truth in full is applicable only to completely clear conceptions. He thinks that the concept of set is not completely clear; in fact, he thinks that it is inherently vague. This is because he sees the property of arbitrariness as being essential to the concept of set. The naive concept of set is of a definite totality determined solely by the objects which it contains. However, the paradoxes of set theory show that it is not possible to create a definite totality of *arbitrary* objects. The iterative hierarchy of sets is a new concept, designed to exclude the paradoxes. It contains the concept of a definite totality of arbitrary subsets of a set (i.e., the power set). In Feferman's view, the lack of clarity of this concept is what leads to the indeterminateness of the continuum hypothesis [Feferman, 2011: 2]. In

[59] According to predicativists, the natural numbers form a definite totality but the collection of all sets of natural numbers (i.e. the power set) does not. For more on predicativity see Feferman [2005].

order to sharpen it, we would need impose restrictions on the allowed subsets; e.g., by making them definable or using a stronger logic. Then we would no longer be talking about *arbitrary* subsets. We would be violating what set theory is supposed to be about by assuming definiteness.

Arguably, mathematicians have already imposed restrictions on the arbitrariness of sets by adopting the iterative concept. This was done in order to avoid the set paradoxes and their concomitant inconsistencies. But why should set theory be restricted to consistent theories? Beall [1999] wonders why mathematicians don't fully embrace the concept of arbitrariness by incorporating theories that are inconsistent but non-trivial (i.e., using paraconsistent logic). This would involve expanding our idea of what is logically possible but follows legitimately from the overall strategy of forming sets of arbitrary objects. There seems to be no good reason to reject Beall's suggestion other than this is not what mathematics is about, which threatens to beg the question. Maybe we are violating what set theory is supposed to be about by assuming consistency? I don't think so; but then, as a robust realist, I don't see any problem in further restricting our concept of set in order to ensure definiteness.

Feferman's argument has no force for robust realists. Robust realists assume that sets exist and so assume definiteness. Steel writes: "If the language permits vague or ambiguous sentences, then it is important to trim or sharpen it so as to eliminate these" [Steel, 2000: 435]. I take it that this is the thrust of Woodin's work; sharpening the language of set theory so as to eliminate indefiniteness. Historically, talk about the existence of mathematical objects in the structure of set theory was facilitated by the categoricity results developed by Zermelo. That required second-order logic since the Lowenheim-Skolem theorem stymies any such discussion in the language of first-order logic. But we know that second-order logic is not strong enough to determine the continuum hypothesis, so Woodin's work uses a stronger logic — Ω-logic — which may enable the resolution of that problem and all other outstanding candidates for absolutely undecidable propositions. I see this as searching for a characteristic language that will enable human beings to make sense of set theory.

4.7 Closing Thoughts

Let me explain my thoughts with an analogy. Human beings use natural languages which have evolved to enable us to make sense of the world around us; at least, those parts of it which impact our survival as a species. Natural language is fundamentally of a subject-predicate form because of the adaptiveness of our object-based view of the world. Philosophers recognise that natural language makes many assumptions about the nature of reality which obscure the underlying phenomena. In the 20th century, logical positivists tried to develop a language whose non-logical atoms were basic phenomena (e.g., "the triggerings of our sensory receptors" [Quine, 1981: 1]). It would have had the form of a first-order set theoretical language with ur-elements and been used to reconstruct our experience of the world from sense data. But the project proved to be too difficult. I take it that one reason for this was that human beings can't make sense of the world in this way; our sense data are themselves too nebulous. It is a limitation of ours. Reality doesn't depend on how our language packages it (i.e., in Quinean terms, it doesn't depend on how we divide it up into ontology and ideology) but there are some forms which we are naturally adapted to make sense of. In set theory, the historical development of the language has been in the opposite direction. It has proceeded from the first-order language, which we know is inadequate even for defining mathematical objects. We need to find a characteristic language for set theory which will enable us to make sense of mathematical reality. It won't change reality, but give us better access to it, given the limitations of human intelligence. It will enable us to discriminate between what is consistent and what is not consistent, what exists and what does not. Any such characteristic language would necessarily jettison some lower level detail for the sake of understanding the bigger picture.

Is Ω-logic the characteristic logic for set theory? I don't know. I'll wait for Woodin and his colleagues to work this out and persuade other set theorists. I think they are on the right track, but that is because I am a robust realist and believe, with Gödel, that a solution exists which is accessible to human rationality. When the solution is found, objections will melt away. Until then, the weight of opinion will probably be on the side of the pluralists. Hamkins asks us to imagine that set theory had followed an alternative history:

> … that as the theory developed, theorems were increasingly settled in the base theory; that the independence phenomenon was limited to paradoxical-seeming meta-logic statements; that the few true independence results occurring were settled by missing natural self-evident set principles; and that the basic structure of the set-theoretic universe became increasingly stable and agreed-upon. [Hamkins, 2011: 3]

As he says, such developments would have constituted evidence for the universalist view. But, contrary to his implication, the fact that actual history is not like this does not weigh *against* the universalist view. It is the nature of the problem. There are many points at which set theory could bifurcate. Without the complete picture, it will always seem that there are many possible branches. The pluralists cannot clinch their case until they have demonstrated an absolutely undecidable sentence. So far, Woodin and his colleagues have been able to combat such attempts with great effort. On the other hand, if there is one, consistent, effectively-completable mathematics then the universalists might not be able to clinch their case until the final piece is put in place. Woodin's achievement is to have kept the dream alive. His work is very finely-balanced at the cutting edge of research. One argument supports the continuum hypothesis; another refutes it. New developments could finally derail the path to Ultimate L. Without a unique axiom for $L_{ultimate}$ it will difficult to justify one effectively-complete theory over another.

Maybe the solution will require a more holistic view, taking account of the place of mathematics in the web of being and knowledge. In this thesis, I argue that mathematical reality is all there is: physical reality is the mental construction of self-aware systems in a tiny subset of the vast mathematical web of being. Following this approach, mathematics at its foundations will be drawn more towards the outer edges of the Quine's web of knowledge [Quine, 1953: 45-46]; its justification more connected to science and empirical reasoning; away from the faux-analytic core in which mathematical truths are deemed to be derivable in first-order logic from self-evident truths, or from concepts inherent in the nature of mathematical objects.

I see the multiverse approach as collapsing into incoherence or formalism. I see intermediate approaches as trying to draw an artificial line between those levels of infinity which we accept and those we don't, with opinions varying widely as to where to draw that line. I say: don't draw a line. Find the language which we can use to unlock the secrets of existence of set theory and, thereby, all of mathematics and, according to the view of this thesis, all of reality.

Chapter 5. What Mathematics Is About

5.1 Overview

The focus of the previous chapter was foundational issues for mathematics; in particular, whether there is one background theory which could serve as the ultimate arbiter of mathematical existence and truth. The outcome was inconclusive. The current situation is that there are many background theories which agree on commonly held intuitions about mathematics but which differ on the status of undecidable propositions such as the continuum hypothesis. Pluralists have arguments to support the view that this will always be the case and that questions of mathematical truth and existence are relative to a chosen background. Universalists believe that, with further work, a preferred set of axioms will emerge which will effectively determine the truth value of all mathematical propositions. Profoundly important steps have been taken by the universalists in their quest for the ultimate background theory and there are signs that a solution is emerging. However, the work is very preliminary and the weight of opinion in the debate is still on the pluralist side.

In fact, we saw that debate is not restricted to extensions of ZFC but that potentially all the axioms of set theory are debatable. Justification often comes in a pragmatic form; i.e. certain axioms are required for the formulation of practical, or interesting, or fruitful mathematics and so become accepted by the community. There seems to be no prospect of convergence if justification is restricted to purely mathematical criteria. Whereas most mathematicians can agree on a basic requirement for consistency there is no general agreement on issues as fundamental as well-foundedness, let alone criteria such as Lesbegue measurability for sets of real numbers. Whilst there is much to be said for just letting mathematicians get on with it and doing whatever mathematics they find interesting, without artificial philosophical constraints, I see no reason why philosophers should abandon the quest of integrating mathematics into a coherent world view. Indeed, I think that it is only by the pursuit of some such holistic view that the underlying issues of mathematics and science will ever be resolved. So, in this chapter, in discussing what mathematics is about, I will venture to embed it in a broader metaphysics in which the issues of the previous chapter can be clarified. Of course, any such metaphysics lives or dies by the coherence which it brings to the total web of knowledge, but I do not think that philosophy should always follow in the wake of science.

Early philosophers considered themselves to be searching for underlying principles behind empirical phenomena and, later, when science had largely usurped that role, to be searching for explanations behind the laws of science, often couched in a theological context. Kant took himself to be uncovering the framework of empirical knowledge itself, to which our perceptions must conform. In contrast, Quine opposed the view that there is any framework beyond science which philosophy can access. The main premise of Quinean naturalism is that it is within science itself that reality is to be identified and described. Whilst sympathetic to the motivation behind this view, and cognisant of the damage which an ideological, philosophy-first approach has wreaked in the past; I think that time has revealed the weaknesses of a naturalistic approach. Modern science consists of silos of expertise which can explain and predict a lot of empirical data but which give no overlying coherent picture. Each silo has its own set of unsolved, fundamental puzzles. Physics is reaching practical limits in terms of the cost and size of equipment which is needed to explore higher energy levels, or unknown regions of the universe, and it has yet to find the evidence which would point the way beyond the (manifestly inadequate) standard models of particle physics and cosmology.

Theoretical physicists have taken on the role of explaining the origins of the universe but their fantastical schemes have little connection to hard science beyond asymptotically reducing to known laws. Are they doing science or philosophy? Are their explanations ultimately more satisfying than those which religion provides? Many people don't think so and this is leading to a split in society between those who seek ultimate answers in science and those who seek ultimate answers in religion. I think that the increasing polarisation of science and religion is unnecessary. There is truth to be found in both approaches and philosophy should be able to build a bridge. It cannot do this if it restricts itself to following behind science and forensically analysing the arguments of religion. Science has a special place in the search for knowledge, but it cannot provide the ultimate answers which human beings crave. Religion purports to give the ultimate answers, but its arguments crumble under analysis. Surely there is a task for philosophy here.

Of course, there may be no overlying coherent picture. It may be that that there are multitudes of multiverses, subject to different laws of physics and different mathematical truths, containing all possible objects (sentient or otherwise) related in all possible ways, with no explanation for our little patch of reality other than that's how it had to be in order for us to be the way we are. But I don't think so, and I have no doubt that the weight of commonsensical opinion, both now and historically, is on my side. In this chapter, I shall endeavour to put an alternative view. Central to this view is what mathematics is about.

If taken at face value, it is clear what mathematics is about: arithmetic is about numbers; group theory is about groups; Euclidean geometry is about points, lines and shapes; etc. Does this mean that all these objects exist in the platonic sense? Or, since mathematical propositions can be expressed in terms of set theory, is all of mathematics about a universe (or universes) of existing sets? In previous chapters, I have discussed some alternative philosophies of mathematics. I lean towards Realist Structuralism: the thesis that mathematics is about abstract structures and that mathematical objects are just place-markers in structures. It is a philosophy which is consistent with the historical development of mathematics away from a view which focuses on the objects themselves and towards a view which emphasises structural properties and relations. It is consistent with the trend towards further abstraction, formalisation and axiomatisation in mathematics. It is also consistent with the structuralist approach to physics which emphasises the continuity of structure (i.e., mathematical equations in physical theories) over ontology (i.e., objects that the equations are deemed to be about). But it retains the platonistic insight that mathematics is about something which exists independently of the human mind.

Can structuralism be the whole story? That is the main question that is addressed in Section 5.2. It is concluded that structuralism cannot be the whole story because of the infinite regress of structure *versus* structured and the fact that mathematics needs non-mathematical objects for its definition. In a historical perspective, the link is made to fundamental problems concerning being and existence in philosophy.

Section 5.3 is the core of this chapter. It introduces a suggested framework to address the problems of existence and associated problems in modern mathematics and physics. I won't discuss the details here. Suffice it to say that the framework is based on the ancient dichotomy of the One and the Many. The framework provides an ontology, an account of meaning and reference, and has implications for epistemology. Reflection principles form one of the key concepts, and are suggested as a potential practical tool. The importance of reflection principles follows from the set-theoretical structure used in the framework.

In Section 5.4 I discuss the metaphysical overtones which historically have been associated with the mathematical concepts of the empty set and the universe of sets V. I demonstrate the appropriateness of the role which they play in the suggested metaphysical framework.

In Section 5.5, I draw out the consequences for mathematics of the suggested metaphysical framework. I conclude that mathematics is about the structure of Being; that it is unchanging, necessary and true; and that we come to know about it by abstracting structure from the world around us and by self-reflection. I make a distinction between mathematics and *human* mathematics which is a cultural product of our society. There is one, true mathematics and our current best description of it is Woodin's Ultimate L. Consequently, some human mathematics is fiction, but that does not mean that it is uninteresting or useless. On the contrary, it is only by pursuing all possible avenues that we will learn how to make the distinctions that will lead us towards true knowledge.

Up to this point, my thesis has explored the nature of mathematics and physics and their connection via applications. It seems that whatever subject one explores, one arrives at deep puzzles which resist efforts to resolve them using the tools of the subject at hand. Sometimes one is led to paradoxes — as in quantum theory and set theory. Sometimes one is led to a mind-boggling explosion of possibilities which one struggles to make any sense of — as in physical and mathematical multiverses; or the proliferation of unintended, non-isomorphic models satisfying first-order formal theories; or the over-arching indeterminacy of meaning and reference in languages. For many good reasons, modern philosophers balk at following the traditional path of attempting a resolution via transcendental metaphysics, but I think that it is counter-productive to close off this path which has led to so many insights historically and which expresses a natural human instinct of a Humean kind. I think that the puzzles raised by the applicability of mathematics hint at a transcendental reality beyond the limits of our understanding (though still accessible to some extent) and, potentially, a keystone to resolve philosophical issues across a broad spectrum of areas. I cannot give up this belief in the rationality of the universe and our ability to investigate it. Accordingly, I am going to put forward a metaphysical framework which represents an initial guess. There is no pretence of firm knowledge here; rather, it is an attempt to provide a consistent and coherent account which is flexible enough to address a broad spectrum of issues and specific enough to determine a position on those issues. It draws on the work of many philosophers; in particular, Plato, Descartes, Leibniz, Kant, Hegel and Gödel.

This chapter is very speculative. In the next chapter I develop a more detailed model of how actuality emerges from potentiality, based on existing paradigms in philosophy and physics. The model introduced in Chapter 6 has a different focus to the one developed here, but it is built using common elements (i.e., it is a monadology which incorporates the structure of set theory). The two models should be viewed as stages on the way to a final formulation.

5.2 Structuralism and the Problems of Existence

5.2.1 Historical background

Any metaphysical framework should begin with a discussion of the problems of existence: Does anything exist? What exists? What does it mean to exist? As far as the first question goes, I am happy to follow Cartesian reasoning and conclude that my consciousness of thinking indicates the existence of something. That something may be a bare instant of self-consciousness, rather than a persistent 'I', but it is something and, furthermore, it has structure and meaning.

As for what exists, Descartes famously concluded that there are two substances — mind and matter. Mind has the essential property of thinking and is a distinct substance, separable from matter (though, in normal life, inextricably interwoven with it). Matter has the essential property of extension. Descartes' substance dualism foundered on the issue of how mind and matter can interact when they are essentially different. Much more common these days is the view that everything supervenes on the physical and that science will ultimately be able to explain how mind emerges from the neural activity of the brain. However, this view has its own problems. For one thing, it has difficulty explaining the nature of our conscious experience; i.e., so-called *qualia*, such as our subjective experience of colours and sounds.

An alternative to dualism and physicalism is the monist position that everything supervenes on mind, as advocated by Berkeley. According to this view, we perceive things in our minds but it is the perceptions themselves that exist, rather than any external matter causing them. In the final analysis, everything in the universe is perceived continuously in the mind of God. Berkeley's idealism struggles to account for the complex structure of our experiences; e.g., their temporal structure, as targeted by Kant in his refutation of idealism [Dicker, 2008]. What is it that allows us to determine the temporal order of our experiences? Kant argues that this is achieved by making reference to persisting objects outside the mind.

Leibniz developed a metaphysics which combines aspects of idealism with the existence of an independent reality. His model is one in which reality is grounded in the perceptions and appetites of mind-like simple substances called *monads*. He writes:

> I don't really eliminate body, but reduce it to what it is. For I show that corporeal mass, which is thought to have something over and above simple substances, is not a substance, but a phenomenon resulting from simple substances, which alone have unity and absolute reality. [Leibniz, 1989: 181]

In Leibniz's monadology, each monad is distinct from every other monad but, as part of a continuous plenum of substance, feels the effects of all the others and, by acting according to a pre-established harmony, becomes a "perpetual living mirror of the universe"[60] [*Monadology* §56-59]. At the same time, each monad provides its own perspective on the universe and, taken together, the infinity of perspectives constitute God's omniscience. Thus, a coherent view of the universe is constructed from many individual perspectives. Leibniz establishes a hierarchy in which all monads are *entelechies* (beings endowed with perception and desire) but some are *souls* (beings which, in addition, have memory) and some are *spirits* (beings which, in addition, have the ability to reason). Leibniz thought of space as a certain order in the community of monads and of time as the dynamic sequence of their states [Kant, 1781: 374]. So, space and time depend on the structure of the monads and their perceptions.

I agree with Leibniz that mind and some sort of independently existing structure are the essential grounds of existence, as I will endeavour to make clear in what follows.

5.2.2 Structuralism in physics

We saw in Chapter 3 that the scientific search for an ultimate ground of matter has proved elusive. In quantum field theory, mass and energy are interchangeable. Every particle has an antiparticle and the creation and annihilation of particles is ubiquitous. Ninety percent of the mass of protons and neutrons (which account for over ninety-nine percent of the mass of ordinary matter) is due to energy stored in the quark and gluon fields. The remaining small component due to the mass of constituent fermions is explained in terms of an omnipresent Higgs condensate which formed shortly after the Big Bang (remember that particles produced by the symmetry breaking of quantum fields are, by default,

[60] "Now this connexion or adaptation of all created things to each and of each to all, means that each simple substance has relations which express all the others, and, consequently, that it is a perpetual living mirror of the universe" [Leibniz, *Monadology* §56].

massless). In applied physics, the proton is sometimes modelled as consisting of an infinite number of point-like, structure-less quarks and anti-quarks [Sekine, 1985]. It is viewed as a cloud of virtual particles which continually pop into and out of existence.

The identity and individuality of quantum particles is further brought into question by the dual properties of quantum mechanics (e.g.; momentum and position are dual properties so, if the momentum of a particle is known, then its position is completely undetermined and there is a non-zero probability of its being detected anywhere in the universe) and by the entanglement of quantum states (e.g.; if the spin states of two protons are entangled then the measurement of the spin state of one of the protons at a later time instantaneously determines the value of the other). These are examples of the counter-intuitive causal effects of the inter-relationality of quantum fields. Even weirder examples can be found in delayed-choice experiments in which the type of measurement which an observer chooses to make seems to determine the wave or particle behaviour of a photon long after the behaviour of that photon had been forced by passing through a double-slit [Jacques et al., 2007].

One lesson to be drawn from these examples is that, in quantum field theory, objects should be understood as properties of a field in that they are ways in which a field is manifested at a particular place and time. The fields themselves become the ontologically basic features of the theory. Mermin goes even further in saying that:

> The proper subject of physics is correlation and only correlation... Correlations have physical reality; that which they correlate does not… Quantum mechanics is a theory of correlation without correlata. [Mermin, 1998; 753]

Ontic structural realists have incorporated these lessons into a philosophy of physics which emphasises the ontological priority of structure and relations over objects and their properties[61]. Indeed, Saunders believes that objects *are* structures and that there is no reason to suppose that there are ultimate constituents of the world which cannot themselves be understood as structures [Saunders, 2003: 129]. There is a hierarchy of structures. This position is buttressed by the tendency of physicists to think of their theories as a hierarchy of effective field theories.

Interestingly, string theory presents arguments that the Planck scale is a fundamental level below which no further structure can be resolved. However, this is still very speculative and doesn't

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[61] See e.g. [Ladyman and Ross, 2007]

really affect arguments about the ontology of structure — it is more in the nature of an epistemic limit. If anything, string theory supports a move towards structural realism because it is a mathematical theory which can be expressed in terms of empirically-equivalent, but logically different, elementary ontologies. In practice, thinking about particles as "strings" was useful in developing the equations in the first place but that has now been totally superseded by thinking about the mathematics itself and its topological structure. In string theory, mathematics is all. The theory is completely divorced from our everyday experience of the world – even space and time do not appear in the fundamental equations of the theory but emerge as derived quantities. Nevertheless, string theorists tend not to be instrumentalists but to imbue the mathematics with ontological significance. Weinberg writes:

> We are discovering laws that are becoming increasingly coherent and universal, and beginning to suspect that this is not merely an accident, that there is a beauty in these laws that mirrors something that is built into the structures of the universe at a very deep level. [Weinberg, 1992: 243].

In the extreme case, physicist Max Tegmark has suggested that "our external physical reality *is* a mathematical structure" [Tegmark, 2008: 101]. He calls this the Mathematical Universe Hypothesis. On the face of it, he is committing a category mistake. Mathematical structure is formal. It is causally inert. It cannot have any physical meaning without being interpreted and the interpretation cannot be purely structural in nature. Thus, for example, whilst the rotation group can be used to describe the symmetry properties of elementary particles, it can also be used to describe the symmetry properties of oranges, and in neither case would we say that it is *about* elementary particles or *about* oranges. Nor is it just a question of the underlying ontology – the same mathematical structure can have quite different physical interpretations and, then, different empirical consequences. Nevertheless, I take Tegmark's views very seriously (and the views of like-minded physicists working at the coal-face of fundamental theory). Tegmark says that a mathematical structure has physical existence if it contains a self-aware structure that subjectively perceives itself as existing [Tegmark, 2008: 125]. He holds that consciousness supervenes on mathematical structure. Whilst not agreeing with this, I think that it is the key to formulating his views in a more philosophically acceptable way: mind and mathematical structure together may be sufficient to ground physical reality.

How do the ontic structural realists interpret "structure"? They say that objects are mere place-holders which play a heuristic role in enabling us to construct the correct theoretical structure (e.g. the group structure of quantum chromodynamics). But surely it is important that the objects are quarks and gluons, for example, with their concomitant causal powers, rather than objects at a different scale with different causal properties? Part of the answer is that ontic structural realists transfer causal powers to the structures themselves [French and Ladyman, 2003: 75]. The structure is "physical" and "causal" and so not mathematical, even though it is described using mathematics. This seems uncontroversial. However, the structure cannot then be explained in purely structural terms – it needs to be interpreted in terms of physical phenomena. It is in danger of itself becoming a physical object (i.e., in other people's terminology [Cao, 2003: 59]). The arguments degenerate into equivocation.

It is important to keep in mind the difference between mathematical (uninstantiated/abstract) structure and physical (instantiated/concrete) structure. It may be true, as Resnik says, that "modern physics blurs the mathematical/physical distinction as drawn in spatiotemporal or causal-informational terms" [Resnik, 1990: 369-370] but there is still a distinction. In this respect, it is interesting to think about how scale and dimensional parameters could arise in a mathematical scheme such as Tegmark's. One option would be to assume a fractal structure for the universe; i.e., no fundamental scale. This seems to be what Leibniz had in mind in his *Monadology* when he explained how matter can be sub-divided without end and have the properties of its infinitesimal parts:

> ...each portion of matter can be conceived as like a garden full of plants, or like a pond full of fish. But each branch of a plant, each organ of an animal, each drop of its bodily fluid is also a similar garden or a similar pond. [*Monadology* §67]

It is also what Weyl envisaged for the geometry of physics (see Section 3.3.2). He wanted to base physics on a purely infinitesimal geometry which allows arbitrary rescalings of the local unit of length at each point of spacetime. I will return to this issue again in the next chapter when I discuss how physical structure emerges from mathematical structure.

5.2.3 Structuralism in mathematics

The arguments surrounding ontic structural realism echo those discussed in structuralism in the philosophy of mathematics. Mathematics, more clearly than physics, seems to be about the structures themselves rather than the objects structured. According to Resnik:

> The objects of mathematics… are themselves atoms, structureless points, or positions in structures. As such they have no identity or distinguishing features outside a structure. [Resnik, 1997: 201].

Shapiro has a similar view:

> The structure is prior to the mathematical objects it contains, just as any organisation is prior to the offices that constitute it. [Shapiro, 1997: 78].

However, we will see that it is not so easy to separate structure from structured.

In Shapiro's *ante rem* version of structuralism, a mathematical structure is a universal: a one-over-many with many possible realisations [Shapiro, 1997: 84-85]. It exists even if it has no instantiation. In eliminative structuralism, by contrast, structures are assumed to instantiate some background ontology (usually, the domain of set theory). Eliminative structuralism has the advantage that it can use all the machinery of set theory; i.e.:

(i) a model-theoretic notion of structure in terms of a domain of objects and relations between those objects;

(ii) the modeling of any given mathematical structure (except that of set theory itself) in the set-theoretic structure; and

(iii) truth and reference in terms of Tarskian truth theory.

The problem is that it requires either a separate treatment of set theory (so structuralism would apply to all of mathematics except set theory – an unpalatable option) or the assumption of a suitable background ontology of objects for set theory (and it is not clear what could serve this purpose). Shapiro comments:

> Perhaps from a different point of view, set theory can be thought of as the study of a particular structure U, but this would require another background ontology to fill the places of U. This new background structure is not to be understood as the places of another structure or, if it is, we need yet another background ontology for its place… we have to stop the regress of system and structure somewhere. [Shapiro, 1997: 87].

However, Shapiro's own alternative of *ante rem* structuralism has similar problems. It cannot use the machinery of set theory in a definitional sense because of the threat of circularity. Model theory assumes set theoretic constructions so it cannot be used to define structure if we want sets to be structures. According to Shapiro [1997: 90-96], a "theory of structures" is required to stop the regress of system and structure at a universe of structures. But any such theory must itself fall prey either to circularity or to an infinite regress [Isaacson, 2011: 22-23]. Shapiro's theory of structures aims to axiomatise the notion of structure directly but it, in turn, must be accounted for as a mathematical theory since it is part of mathematics. It has no obvious advantages over using a background theory of sets and it does not have the status of set theory[62].

To circumvent these difficulties, when Resnik endorses structuralism he simply denies that a theory of structures is needed: "It is not clear that structuralism… need be formulated as a theory, much less as a regimental mathematical theory" [Resnik, 1997: 253-254]. However, this leaves some sort of explanatory gap.

In epistemological accounts of structuralism, mathematical structures are arrived at by abstracting from physical structures. This leads to a blurring of the distinction between mathematical and physical objects. Often there is an in-between stage involving what Parsons calls "quasi-concrete objects" [Parsons, 1990: 304]. These are objects which cannot be thought of as bare positions in a structure because they are directly instantiated in the concrete. For example, geometry was initially formulated in terms of points and lines in physical space and only much later was it formulated in abstract axiomatic terms. As Parsons says: "a purely structuralist account does not seem appropriate for quasi-concrete objects, because the representation relation is something additional to intrastructural relations" [Parsons, 1990: 304]. Sets are an interesting case. Gödel describes them as "quasi-physical" because they are constituted by their elements [Wang, 1996: 270]. Their existence presupposes the existence of individuals which are not sets to act as their constituents (N.B. the empty set is taken to be a sort of individual, rather than a true set because, like individuals, it has no members). In this sense, structuralism is not the whole truth for sets. Shapiro agrees that quasi-concrete objects cannot be eliminated from our thinking:

> The best way to show that a structure exists is to find a system that exemplifies it. At some point, we have to appeal to items that are not completely structural (unless somehow everything is completely structural). And at some point, we have to appeal to items that are not completely concrete, given the size of most mathematical structures. So we appeal to the quasi-concrete. [Shapiro, 1997: 105].

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[62] Shapiro himself says that talk of structures can easily be rendered as talk of sets [Shapiro, 1997: 96].

5.2.4 Summary of the discussion so far

For convenience, I provide an interim summary of the discussion on metaphysics in this section:

(i) Descartes advocated a form of substance dualism based on mind and matter.

(ii) Berkeley and, separately, Leibniz (and, then, Husserl and others following in the continental tradition) developed philosophies in which mind is the dominant principle[63]. Leibniz's monadology depended to some extent on the structure and relations of the monads.

(iii) Modern analytic philosophers tend to favour physicalism, according to which everything supervenes on matter.

(iv) Fundamental physics has become increasingly dependent on mathematics and divorced from an ordinary understanding of things like space, time and matter.

(v) Leveraging this, ontic structural realists in the philosophy of physics have advocated the priority of structure over objects — with extreme versions morphing into Pythagoreanism (i.e. "everything is mathematics").

(vi) In a parallel way, contemporary mathematics has become increasingly formal and abstract, leading to the development of structuralism in the philosophy of mathematics.

(vii) The triumph of structure as the key principle of the universe has been foiled by issues to do with the infinite regress of structure *versus* structured and the fact that mathematics needs non-mathematical objects for its definition.

According to Gödel, "'The study of structure' is a confession that we don't know what the things are" [Wang, 1996: 292] and I agree with him.

5.2.5 Some further considerations

To provide more pieces of the puzzle, it will be necessary to review some of the oldest and deepest arguments in the history of philosophy on the nature of existence. Traditionally, everything that exists is said to share in a property called *being* which is the fundamental and ultimate element of reality. Heraclitus argued that things must also contain an element of non-being, so that they can come

[63] Although, only Berkeley denied the reality of matter *per se*.

into and go out of existence. Clearly things change; therefore, they must contain what they are not. According to Heraclitus , contradictions exist and are an essential component of reality.

In opposition to Heraclitus, Parmenides argued that non-being is nonsense: “For in no way may this prevail, that things that are not, are” (as quoted in [Cohen et al., 2005: 38]). Consequently, change is an illusion and not the truth about being. Being is eternal and unchanging. Changing appearances are not a proper object of thought; they cannot lead to true knowledge. According to Parmenides, we should try to fix our thoughts on the unchanging reality behind appearances in order to learn the way of truth.

Plato followed Parmenides in seeking true knowledge in an unchanging reality — the world of Being — containing the perfect, eternal and unchanging Forms. Our everyday world — the world of Becoming — contains objects which are imperfect, ephemeral and changing. Plato thought that these earthly Sensibles derive their characteristics from a participation in the Forms, although the nature of that participation is difficult to pin down. In some ill-defined sense, the Forms are universals and the Sensibles are their corresponding particulars. Sometimes Plato identifies mathematical objects with the Forms. For example, in *Republic*, he complains that geometers wrongly refer to geometrical objects using dynamical language when they should be using tense-less language:

> [Geometers talk] of “squaring”, “applying”, “adding”, and the like, whereas the entire subject is pursued for the sake of knowledge… of what always is, not what comes into being and passes away. [*Republic*, 527a-6].

In *Republic* 508e2–3, Plato talks about the Form of the Good which gives the other Forms their meaning and existence [Cohen et al., 2005: 500]. In more mathematical contexts, it is referred to as the One. It is the first of his two fundamental principles. The second principle is variously referred to as the Many, the Indefinite Dyad, and the Unlimited. The One and the Many, being basic principles, are not rationally accessible but, together, they give rise to all rationally accessible being. In *Philebus*, Plato says: “Whatever things are said to be are composed of one and many, and have the finite, and infinite implanted in them” [Jowett, 2006: 10-11].

Leibniz, also siding with Parmenides, argues in the Monadology that “the final reason of things must be in a necessary substance, in which the variety of particular changes exists only eminently, as in its source” [*Monadology* §38]. This ultimate substance is unique, universal and illimitable [*Monadology* §40]. It is the source of what exists and of what is real in the possible [*Monadology* §43].

Hegel's philosophy is a mixture of the views of Parmenides and Heraclitus. It has perfect Absolute Being at its core. On the other hand, it portrays our experience of the world as being inherently contradictory:

> "What is" is always becoming "What is not"; "Here and Now" become "There and Then". (Hegel's *Phenomenology of Spirit* as translated by [Miller, 1977: 60-66]).

The reality of our experience is multi-relational and multi-temporal. We use boundaries to define things but what they *exclude* (i.e. "the Other") is just as important in determining a thing's identity as what they *include* (i.e. "essential properties"). For Hegel, being is only meaningful in opposition to non-being. To think about being is to think about non-being, and vice-versa. The only way to differentiate them in thought is via a third category, "becoming", which keeps the poles apart whilst passing from one to the other. Becoming is said to *sublate* the two moments of being and non-being in this Hegelian triad. Hegel's *Phenomenology of Spirit* unfolds through a process of contradiction and resolution, followed by a new contradiction, a new resolution, and so on. Contradiction is the dynamic force that leads to change, development and learning in the world.

In a modern context, Tegmark [2008: 3-4] discusses a resolution of the static and dynamic views of physical reality in terms of the *bird perspective* and the *frog perspective*. For Tegmark, physical reality is a mathematical structure and, from his bird perspective, it is an eternal, unchanging object existing outside of space and time. From the other perspective, that of a frog living inside the structure, it is experienced as the familiar, continuously changing kaleidoscope of existence. An extra twist is added by the fact that, in Tegmark's Mathematical Universe, the frog itself is a mathematical structure. Only the bird perspective gives us the full picture.

5.3 Suggested Metaphysical Framework

5.3.1 Outline

The purpose of the discussion of the problems of existence in Section 5.2 has been to focus on debates which resonate from the beginning of philosophy to current mathematics and physics. I claim that the pieces of the puzzle which have been presented can be reconciled in a surprising way if we consider Being to be the ultimate structured substance. Structure is important. It introduces boundaries, it makes distinctions, and it contains. But it is not the whole story.

In the suggested metaphysical framework, Being has two poles, which I shall call the One and the Many. These generate, and are held distinct by, the structured world of rationally-accessible beings of which we are a subset. A simple analogy is to North and South magnetic poles which generate a magnetic field permeating all space. This picture is at once static and dynamic; there is implied movement from one pole to the other. However, the analogy is too limited for my purposes.

The picture which I shall actually work with, as an aid to understanding, is a set-theoretical model of the structure of Being. Let us interpret the empty set as the One, and the universe of sets V as the Many. The Von Neumann hierarchy of sets is interpreted as the world of Becoming. The One and the Many are not themselves rationally-accessible. They are two poles of Being, which is a falling together of contradictions at the limits of thought and expression: it is and is not; it is one and it is infinite; it is the beginning and the end; it is everything and nothing. The One is the ultimate monad. Monads can be thought of as atoms of Being. The One and the Many generate the world of Becoming in which beings (modelled by sets of monads) are consistent, rationally-accessible objects. Monads themselves have properties of mind, so that they have perception and desire and perceptual states. From another point of view, then, this is a model of Being as a gigantic mind: a mind constituted by structured monads and their states, not a mind supervening on a physical brain with hard-wired connections.

What does this mind think about? It thinks about itself, its own structure. In thinking about itself it creates an interpretation of itself and that is what I say physical reality is. Human beings are part of a bigger story which is the actualisation of Being. Being contains all possible forms, all possibilities, but it chooses a pathway through that, an interpretation, which is physical reality.

I contend that my view is an extrapolation of what physics is telling us. The standard model of particle physics is a group — SU(3) x SU(2) x U(1) — which describes the strong and weak nuclear forces and electromagnetism and the generators and carriers of these forces. Grand Unified Theories attempt to unify these forces by embedding them in a bigger group, SO(10) say. Then the story is that at the time of the Big Bang temperatures were so hot that all the non-gravitational forces were unified and, as the universe cooled, through a process of spontaneous symmetry breaking, the known forces and particles condensed out. String theory (or M-theory, as it has become) takes this back a step further. Now we try to unify all the forces including gravity. We start with a mathematical structure (M-theory, nobody knows what it is yet) and space and time condense out of that, followed by the forces and particles. This can happen in 10^{500} different ways and each of those is interpreted as a different universe with, maybe, different numbers of spatial dimensions and different forces and numbers and types of particles. I am saying: take the process a step even further back. Start with a mathematical structure which describes all of mathematics, all possible forms. Physical reality condenses out of that, not by “compactification” or “spontaneous symmetry breaking”, but by Being making choices and actualising itself. Nor does it happen in $10^{\text{gazillion}}$ different ways: there is one actualisation, one pathway chosen through all the possibilities. This particular actualisation is the one which gives rise to rational beings which can understand and interpret it. Mind and rationality are essential to my viewpoint, and they are essential to modern physics in the form of the observer in general relativity and quantum mechanics. According to this view, mind and mathematical structure are the principles of Being.

5.3.2 Details of the framework

Being is eternal and unchanging. It is all that really exists; everything else has a derivative existence. To exist is to be part of Being. Being contains all beings but is not constituted by them. It is not itself a being (i.e. not a “one which is, at the same time, many”) but an inconsistent manifold. Particulars are either beings or the thoughts of beings (which supervene on their states). Physical reality is constituted in the thoughts of beings. Universals are structures abstracted from particulars. Mathematical structure is abstracted from the structure of Being. Particulars and universals which exist in the world of Becoming are consistent and, in this sense, their type of existence can be contrasted with that of Being itself, which is inconsistent.

Being contains the ordinal progression of sets which is a model of time, like freeze-frames in a movie. This corresponds to subjective time as experienced by beings. At each time step, the monads are in a particular state. Each monad perceives and is influenced by all other monads, with a strength determined by pair-wise correlations. Monads experience this influence as a desire to move to a new state, and so the whole system is updated at each time step. Monads have local objectives, to satisfy their desires. Their local objectives are determined by the global objectives of the system and are communicated via laws of nature in the form of correlations. The global objectives of the system are determined by the desires of the two poles: the desire of the One to become Many (i.e. to share the joy of existence), the desire of the Many to become One (i.e. to achieve self-identity, to think itself as a unity, which its status as an inconsistent manifold prevents), and the desire for self-knowledge. Of course, no dynamic optimisation process takes place because Being is eternal and unchanging, but it is *as if* the universe has been optimised and is, as Leibniz surmised, the best of all possible worlds.

The desire of the One to become Many drives the universe to multiplicity, complexity and differentiation. It is reflected in the Second Law of Thermodynamics (i.e. increasing entropy) and in the Big Bang model of cosmology. It is also reflected in the laws of Darwinian evolution and, so, in Life, in the desires of beings, and in the ethics of rational beings. In an expanding universe, it is the dominant ethic.

The desire of the Many to become One leads to gathering together, eliminating differences, unifying through simplification and merging, focusing on the singularity of existence. It is reflected in the Big Crunch model of cosmology and in the collapse of the wave function in quantum mechanics. It is reflected in the need that minds have to think of many objects as one object; to comprehend multitudes as unities. It is reflected in Death, which is a way of eliminating differences and returning to the One. In a contracting universe, it would be the dominant ethic[64].

The desire for self-knowledge drives Being towards a unique, intended, "real" interpretation, as reflected in our experience of physical reality. It is satisfied by beings interpreting structure and giving it meaning, in a way enabled by, and enabling, communication between minds at all scales of the universe. Meaning is built up from very simple components. In the "beginning" (in a dynamic view of the universe) is The One; which is a perfect simple, without parts. The One is the transcendental "I"; a subject with no object, no properties, and no thoughts. It is unconscious being and desire. At the next level is the unit set. It has consciousness and thinks "I am" and "I am desiring". It has the notions of membership and unity, of everything and of nothing. At the next level, there is perception of an Other. This leads to self-consciousness and the logical notions of sameness (i.e. "I am identical to myself") and difference (i.e. "I am different from the Other"). The self-conscious structure perceives the ordinal progression as time and change, and interprets the Other as existing in space. It comprehends the finiteness of the Other as

[64] It is hard to imagine what consequences this would have for the drives and ethics of rational beings. One might conjecture that different values would be attached to outcomes; i.e., that "good" and "bad" would label different things, and that we would want different things. Such considerations shed light on some aspects of the problem of evil in our world. Good and bad are not objective labels and certain natural disasters, which we might loosely consider "evil", might be contributing to good from another perspective. However, this does not solve the problem of evil. Evil is a matter of intention. It involves intentionally doing what one considers bad to another. What possible purpose could evil serve in achieving the objectives of the system? Here I will conjecture that evil could play a role in achieving the *global* optimization of good. The analogy is to a mathematical system undergoing an optimization process. Such systems often get stuck in a local maximum. Optimisation programs contain procedures to jolt systems out of local maxima by introducing an element which moves against the direction of the objective function. So, if the system is optimising "good" then a jolt of "bad" would get it out of a local maximum. In human terms, evil shakes up the system as a whole, helps us better understand the nature of good, and prevents us lapsing into complacency, thereby facilitating progress towards good overall.

extension. For the two-set itself, there is the notion of containing parts. At higher levels, more complex logical notions arise; such as the notions of overlapping with another set, of disjointness, and of inclusion. Associated with these are notions such as number, greater than and less than. Structured sets of monads are interpreted as objects. The notions of near-ness and far-ness are developed in response to the perceived strength of correlations between objects.

As the hierarchy is ascended, a great complexity of relationships and notions can be formed. At each stage, Being must make choices about how to interpret its structure, but these are constrained by the initial interpretation. So meaning is given to physical reality through a process of developing and interpreting meaning units and their relations. Shared meaning is already required for the understanding of asymmetric relations such as greater than and less than. As the universe becomes more complicated, shared meaning is what enables the encompassing mind to approach self-knowledge.

5.3.3 Meaning and reference

At some stage of the hierarchy, beings emerge which have thoughts such as “I am feeling” and, later, “I am thinking”. The “I” of “I am thinking” is a rational being with powers of memory, self-reflection and language. Language is the vehicle for shared meaning in our everyday world. The creation of language is a reflection of the process of creating meaning in the universe. In that process, meaning units are interpreted from independently existing background structure in an incremental way. New meaning units are built up from previously existing meaning units, just as sets are built up from previously existing sets. What is added is constrained to be consistent with what already exists. In the development of language, language units are associated with simple meaning units and, incrementally, a shared, intersubjective interpretation of language is built up within a community of rational beings. The language process cannot get started without a background of shared meaning.

From our current perspective in the everyday world, it is difficult to find any grounds for meaning and reference. In language, the meaning of particular words can, and does, change over time. There is no way to know whether one person’s private meaning attached to language units is the same as another’s. All we can do is observe their behavior in using and responding to language in different situations, and judge its appropriateness. In mathematics, it is difficult to understand how we achieve an intended interpretation of structures such as the natural number sequence, when we know that first-order axiomatised structures have non-standard models which differ even in their cardinality. In physics, it is difficult to understand how we observe one outcome from the infinite possibilities encoded in the quantum wave function of the universe. These problems cannot be resolved by working our way backwards in logical steps. The only possible resolution is by building up from clear and distinct perceptions in a Cartesian analysis, assuming a coherent metaphysical framework. However, I also suspect that, at least in mathematics, a unique solution cannot be worked out from a perspective limited to an initial segment of the universe. This was one of the lessons from Chapter 4. Uniqueness may not be a property from any perspective lower than that of the universe of sets V. Our only hope for a unique solution, therefore, relies on mathematicians somehow being able to bootstrap a global perspective from a local perspective, as Woodin has attempted to do with Ultimate L. I shall have more to say about these matters in the next chapter when I discuss the creation of physical reality from mind and mathematical structure.

5.3.4 Reflection principles

From my viewpoint, the evolution of physical reality is less about *things* and more about mind acquiring self-knowledge by giving *meaning* to its internal structure of relations. The emergence of rational beings with language is a critical point in this process. As human beings, we can reflect on ourselves and our experiences. Remarkably, when we do so, we find the underlying structure of reality reflected there. I have already pointed out some of these reflected properties in the course of my exposition, but now I would like to isolate and add to them. The contradictory nature of Being, with its separated poles mediated by Becoming, is reflected in physics in:

(i) The vacuum of quantum field theory, which is a conflation of everything and nothing in which particles are continuously being created and annihilated;

(ii) The structure of all field theories (e.g. the simple example of the magnetic field, mentioned earlier);

(iii) Forces (interpreted in terms of the objectives of the system and the desires of its components);

(iv) The law of entropy, according to which information in the universe increases;

(v) Models of cosmology (whether it be the Big Bang, the Big Crunch, or perpetual inflation);

(vi) The collapse of the wave function in quantum mechanics, which reflects the way in which the Many collapses to the One when we try to transcend or comprehend it;

(vii) The dual properties of quantum mechanics (e.g. if the momentum of an electron is known then its position is completely undetermined) which reflect the relationship of the One and the Many in that they cannot be comprehended together.

More directly, the nature of Being is reflected in the paradoxes which we encounter at the limits of thought, cognition, iteration and expression (see [Priest, 1995]). It is reflected in the hierarchies which we see all around us, and in our inherent tendency to group and categorise things. It is reflected in the structure of time, in Darwinian evolution, in Life and Death, and in language development. The property of extension reflects our understanding of our status as finite beings. Space reflects our understanding of our status as distinct beings. Then there is our notion of infinity itself, which mathematicians have refined to give us all the wonders of the set theoretic hierarchy. I contend that our understanding of set theory is grounded in a reflection of the structure of Being within us.

Furthermore, if Being has a set theoretical structure, and physical reality is an interpretation of that, then I contend that we should *expect* to see the properties of Being reflected in physical reality. This follows from the fact that reflection principles are fundamental to set theory; i.e.:

(1) The universe of sets V is absolutely infinite, indefinable, and transcends any possibility of description from below, so anything true which we can say of V must already be true of an initial segment[65] ;

(2) "The universe of sets does not change its character substantially as one goes over from smaller to larger sets or cardinals… the same or analogous states of affairs reappear again and again (perhaps in more complicated versions)". [Wang, 1974: 189-190]

Could we use the reflection principle, in a manner analogous to its use in set theory, to intuit unsubstantiated features of the universe? Perhaps we could argue as follows:

[65] To make the logic of this more explicit: if V were definable then we could define a property that it alone possesses and that allows us to single it out from all the initial segments that do not share this property. However, it is not definable, so any of its definable properties must be shared by an initial segment.

> Human beings have the essential property of rational thought. Reflect that property up to Being which, we conclude, has infinitely greater powers of thought. If Being were rationally accessible, we might try to define it as the being whose powers of thought are greater than any other, or as some sort of totalisation of rational beings. But it is not rationally accessible, so its powers are beyond anything which we can think or define. Its property of thought is inaccessible. We can also reflect the property of thought down to an initial segment in which that property emerges (perhaps, in human beings, or perhaps, before); and to later segments (perhaps, later stages of Darwinian evolution) in which higher forms of rational being exist with more powerful minds than humans.

Whether or not we could detect such higher forms of rational being in our everyday world is a moot point but, doubtless, many people throughout the course of history have made the claim.

5.3.5 Epistemology

The metaphysical framework which I have presented suggests a pursuit of knowledge via self-reflection, as in Husserlian phenomenology, combined with the objective probing of the structure of nature via empirical science. The aim would be to focus on structure abstracted from the world, and from concepts in our mind, and to understand reality in terms of mathematical structure, rather than sense data. This is the agenda of structuralism. What are "sense data" after all? Sense data are already interpreted; they are not immediate. Light and sound are reduced by analogue methods in the brain to electrical signals which it can interpret. Really, information is sensed by the brain in the form of mathematical structure, albeit, embedded in a physical signal. Science, similarly, reduces various forms of information, not to phenomena, but to mathematical structure. Physics is most advanced in this respect.

Perhaps we don't need a new science of transcendental phenomenology (as advocated by Husserl) in order to understand reality; perhaps we just need to proceed with science as usual? Physics will converge towards mathematics (by a process of mutual interaction) and other sciences will become more quantitative as they develop. But this judgement is too hasty. We will surely need a better understanding of the way mind works and this will not be achieved by physics alone. Phenomenology and other mind-related studies have already provided valuable insights into our psychology and the transcendental conditions required for us to experience the world in the way that we do. A functional model of the mind (e.g. one input signal processed into one output signal — which would allow mind to be connected to structure directly rather than through "body" as such) does not capture our internal beliefs such as our feeling of 'I'. Searching for the 'I' is like searching for the ultimate grounds of matter or of mathematical structure. I have argued that these things are grounded in the One and that rational beings can reach back to this concept through self-reflection. Understanding via self-reflection is the agenda of phenomenology.

The suggested metaphysical framework is compatible with extant interpretations of quantum mechanics. It favours some interpretations over others, as will be discussed in the next chapter. It also has consequences for many outstanding issues in the philosophy of mathematics, as will be discussed in Section 5.5.

In the next section, I will discuss how, in the history of mathematics and theology, the empty set and the universe of sets V have had metaphysical overtones consistent with their role in the suggested framework.

5.4 The Role of the Empty Set and the Universe of Sets

Famously, Cantor identified Absolute Infinity of his mathematics with God:

> The *transfinite* with its plenitude of formations and forms necessarily indicates an Absolute, a 'true infinite' whose magnitude is capable of no increase or diminution, and is therefore to be looked upon quantitatively as an absolute maximum…
>
> What surpasses all that is finite and transfinite… is the single completely individual unity in which everything is included, which includes the 'Absolute' incomprehensible to the human understanding. This is the 'Actus Purissimus' which by many is called 'God'. [Priest, 1995: 128]

The qualities of Cantor's Absolute (or, in our terminology, the universe of sets V) which make its identification with God so irresistible are its incomprehensibility, its transcendence (it is beyond mathematics and the limits of thought), its paradox, its inclusivity and its reflection properties. God is often described in these terms.

In the eleventh-century, the theologian Anselm described God as "that than which a greater cannot be thought but also something greater than can be thought" [Priest, 1995: 62]. Nicholas of Cusa, in the fifteenth-century, described God as the Maximum:

> There can be nothing greater in existence than the simple, absolute maximum; and since it is greater than our powers of comprehension – for it is infinite truth – our knowledge of it can never mean that we comprehend it. [Priest, 1995: 23]

Correspondingly, in Greek philosophy, there is a metaphysical infinite:

> For Anaximander the [infinite] is a kind of eternal and limitless… elemental plenum from which the finite and bounded cosmos has been extracted and against which the cosmic order is continuously preserved… For the Pythagoreans it is born of number that subdues the limitless and gives dimensions (and thereby existence) to finite things. [Hart, 2011: 258].

In Plato's *Philebus*, as was discussed in Section 5.2, the infinite is mixed with the finite to produce the Forms and Sensible things.

God is also described in terms of the falling together of contradictions. Nicholas of Cusa used mathematics to explore this idea [Infinity: 35-38]. He argued that the infinite line, the circle and the triangle coincide in infinity, and this insight led him to a new methodological approach for calculating the circumference of a circle. He said:

> The absolute maximum… is all things and, whilst being all, is none of them; in other words, it is at once the maximum and minimum of being. [Priest: 24]

In this quote he is describing two contradictory aspects of God: His aspect as the Many, and His aspect as the One.

The third century philosopher Plotinus emphasised the aspect of God as the One [Achtner, 2011: 23-27]. The One has no division, multiplicity or distinction, and is beyond all categories of being and non-being. It has no sentience, self-awareness or activity. Owing to its simplicity, it is indescribable directly. It has infinite power, in the sense that it is the cause of all beings. However, it does not create beings, as such, since it has no activity. Rather, by being perfect simple, it is the "not what exists" which allows the "what exists" to be.

In mathematics, the empty set plays a role analogous to the One. Historically, Peano defined the empty set as the intersection of all sets [Kanamori, 2003: 275]. He used the notation "There Exists" to indicate that a set is not equal to the empty set. Hence, his definition of mathematical existence was by opposition to the empty set, analogous to Plotinus' definition of metaphysical existence by opposition to the One. Frege defined the number 0 (which is interpreted as the empty set in set theoretical models) as the extension of the concept “not being self-identical” or, equivalently, the extension of any contradictory concept [Kanamori, 2003: 274]. Again, there is a clear analogy to Plotinus’ attempts to describe the One. Russell had philosophical problems with the empty set. He said: “A class which has no terms fails to be anything at all: what is merely and solely a collection of terms cannot subsist when all the terms are removed” [Russell, 1903: § 73]. In Fraenkel’s exposition of set theory, the empty set is really a non-set. It is the one and only memberless object whose existence is needed to “start” the iterative set hierarchy [Fraenkel and Bar-Hillel, 1958: 30-31]. It is called a set by convention, for ease of exposition. Modern set theorists attach little significance to it beyond its usefulness; e.g., it allows the intersection of any two sets to belong to the universe even when these sets have no member in common.

In summary, the universe of sets V is an inconsistent manifold, the empty set is an indefinable unity. Just as V transcends all sets and is their union; so Being, in its aspect as the Many, is transcendent beyond all finite beings and contains them. Just as the empty set is contained in all sets and is their intersection; so Being, in its aspect as the One, is transcendentally present in all beings and is their source.

5.5 Consequences for mathematics

In this section, I will draw out some initial consequences for mathematics; then, discuss potential problems, suggest solutions, and complete the list of consequences.

5.5.1 Initial discussion of consequences

1. Mathematics is about the structure of Being.

Mathematical objects are not important. We describe mathematics in terms of objects and relations but really only the relations are important. Different areas of mathematics can be considered as different ways of describing fundamental mathematical structure. Sometimes, having a different perspective can give us more insight. Similarly, some perspectives are more directly applicable to particular practical problems than others. However, all of mathematics can be translated into a single background language. Set theory is our best attempt at that background language even if we do not fully understand it yet. In set theory, it is not possible to transcend the universe of sets V because that is the nature of the structure of Being which it describes. V is a pole and, in the iterative set process, rather than being able to transcend it, the process just collapses back to the beginning again and the set theoretic hierarchy is recreated. In this sense, the process is circular.

2. Mathematics is an unchanging, necessary and true derivative of Being.

In the mathematical realm, the laws of thought apply. Mathematics does not contain any contradictions. However, its domain is defined relative to inconsistent, non-mathematical objects.

3. Mathematical objects do not exist in a mysterious Platonic heaven totally independent of the physical world.

Rather, physical reality is a mental interpretation of a subset of mathematical structure.

4. Human mathematics is the cultural product of a community of rational beings.

It is a language which we use to describe structures which we intuit from observing the physical world and then further refine through reflection. Human mathematics is subject to error but, by a public process of being scrutinised by many different minds, and by the efforts of a few truly inspired minds which seem to have special insight into the structure of Being, we evolve towards a true understanding of mathematics.

There are features of mathematical practice which give us insight and pleasure but don't reveal anything more about what mathematics is. Mathematics can have aesthetic appeal. It can be like a game. Human mathematics is pursued both for aesthetic pleasure and for practical applications. It is part of human culture, like natural language and art. Human mathematics is a language for describing the structure of Being. Natural language is a language for communicating our experience of Being. Physics is a hybrid language connecting our experience of Being with our understanding of the structure of Being. Mathematics, science, art, poetry, and religion are all part of a single endeavour to understand Being and our place in it.

5. We come to know about mathematics by abstracting structure from the world around us and focusing on the structure of Being mirrored in us.

According to Leibniz:

> It is not as regards their object, but as regards the different ways in which they have knowledge of their object, that the Monads are limited. In a confused way they all strive after the infinite,

> the whole; but they are limited and differentiated through the degrees of their distinct perceptions.[*Monadology* §60].

It is debatable how we come to recognise structure in the first place. There is more to it than perceiving immediate sense data. Our minds seem to have an a priori ability to interpret sense data in terms of objects and relations. This ability involves comparing, contrasting, collecting and classifying data. Despite receiving quite different sense data, we all tend to interpret the same physical reality. As Kant pointed out, this means that what we interpret may not correspond to any aspect of the things in themselves, although they may still objectively cause our interpretation. Gödel wrote:

> That something besides the sensations … is immediately given follows (independently of mathematics) from the fact that even our ideas referring to physical objects contain constituents qualitatively different from sensations or mere combinations of sensations, e.g. the idea of object itself, whereas, on the other hand, by our thinking we cannot create any qualitatively new elements, but only reproduce and combine those that are given. [Gödel, 1953/9: 351-353].

Gödel thought that there is a "given" underlying mathematics and that we can perceive it by mathematical intuition. This would correspond to the structure of Being in the suggested framework. I am suggesting that it is all there is and that it is what we detect, also, with our physical senses. Our minds are trained to detect and interpret it.

In this framework, mathematical and physical structure is blurred. Mathematical structure describes certain patterns which we abstract from physical reality whilst physical structure instantiates these patterns and helps us recognise the properties of the associated mathematical structure [Resnik, 1997: 102-107]. At higher levels, there is a thorough relativity of ontology because any given structure can be interpreted in terms of objects and relations in different ways. However, any given community of beings will interpret structure in a unique way (i.e. develop a "public" model with shared meaning which, if the beings are rational, can be communicated in an associated language). Also, at a deeper level, there is a unique interpretation because of the structure of Being and because Being is an encompassing mind which provides uniqueness.

The development of meaning in the community of human beings has a historical and cultural context, as emphasised by Hegel in *The Phenomenology of Spirit* [1807]. According to Hegel, Spirit is that which constitutes the process of reality [Kelly, 1988: 120]. Our knowledge of Absolute Being evolves as

Spirit unfolds over the course of human history, through a process of encountering and overcoming contradictions. In mathematics (one of the many cultural activities which articulates Spirit) knowledge builds in the community over time. The Ancient Greeks didn't have the knowledge which we have now but they were in tune with Spirit and made a number of significant discoveries at a deep, fundamental level. Arguably, we are now at a time of great polarisation; with religion set in opposition to science and dangerous rifts developing in religious *versus* secular society. Hopefully, this will be overcome in a new sublation. In physics and mathematics, this could be through a coming together at the most fundamental level and the development of an understanding of the one, true mathematics.

However we do come to recognise structure in the first place, it is clear that pattern recognition is very important for our heuristic development. Human beings are learning machines. We abstract patterns from physical reality and search for invariants which can guide us in making predictions about future events. Some of the first patterns we recognise are instantiations of the "one *versus* many" dichotomy (i.e. we learn to recognise objects as a unity, even though they are presented to us as a manifold of contrasting and changing properties). Mathematics is an integral part of pattern recognition; hence, its importance in applications. We find patterns in nature and intuit the underlying mathematical rules. This is mathematics in practice. Most mathematicians gain understanding of their subject and make discoveries through an intuitive process. Theorem and proof, and axiomatisation, come later. Some particularly intuitive mathematicians, such as Ramanujan, don't even bother with them.

We might worry that an understanding of our approach to the world in terms of heuristics could lead us to a skeptical view. If all we do is learn and follow rules, then we don't really need to believe in the underlying story which we tell ourselves to explain those rules. However, if there were no independent, objective reality (or, if it didn't have invariant features) then we wouldn't be able to formulate rules in the first place; indeed, we wouldn't be able to think or formulate rules at all. Stable features in an objective reality are a transcendental condition for our subjective experience.

Once mathematics gets started, it is led inexorably towards more abstraction, formalisation and axiomatisation. Axiomatisation is a way of organising and systematising mathematical rules. It involves focusing on the structures themselves, as revealed to us through reflection and thought. By the method of implicit definition, we can reach levels inaccessible by constructive means. This knowledge appears to be a priori:

> …if sensory experience is not involved in the ability to understand an implicit definition, nor in the justification that an implicit definition is successful, nor in our grasp of logical consequence, then the knowledge about the defined structure(s) obtained by deduction from implicit definition is a priori. [Shapiro, 1997: 132].

We can go beyond intuition to the very limits of thought.

6. Mathematics is applicable because it truly describes the fundamental structure of reality.

Human mathematics progresses from quasi-concrete objects to free-standing structures. Fundamental physics progresses from concrete objects to quasi-concrete objects. At a fundamental level, as was discussed in Chapter 3, physics directly instantiates mathematical structure. At higher levels, this behavior is scaled-up to effective field theories. At still higher levels, it is still responsible for the underlying patterns that occur, but the effects are masked by complexity.

Superficially it may appear that mathematics is applicable because it is a science of patterns and it can provide a pattern to fit any given set of empirical data. There is some truth in this and it correctly describes the way in which mathematics is used in some applications. However, it does not explain the law-like properties of the mathematical structure which describes fundamental physics. Not just any pattern which fits the data will do in this case.

7. There are rational beings with mental powers surpassing those of humans who have a deeper insight into mathematics than we do.

Sets are often described in terms of collections which can be overviewed in thought (i.e. thinking the many as a one). Human beings have limited capacity for overviewing collections. We struggle even with the power set axiom; i.e., we cannot overview all the subsets of an infinite set. But, as Gödel said, "for every set, there is some mind which can overview it in the strictest sense" [Wang, 1996: 260]. This is the idealised subjective view of sets. However, even by using the idealised subjective view, we cannot get to the indefinability of the universe of sets V.

Intuitionism considers mathematics to be a construction of the human mind. Idealised intuitionism considers it to be the construction of rational minds more powerful than human minds. However, even the most idealised form of intuitionism cannot encompass all of mathematics. Mathematics is incompletable by rational beings. The whole truth can only be overviewed by Being. For initial segments, there will always be true statements which cannot be proved.

8. Model theory provides the correct picture of how mathematical languages describe mathematical reality.

The domain of discourse is a static collection which can be thought of as the collection of everything which Being can mentally construct. Thus, the dynamic, constructionist language of intuitionism is incorporated into the static view of Platonism.

9. No mathematics is surplus.

Quinean naturalism accepts the objective existence of things which are part of our best explanation of the world. Hence, any mathematics which plays an indispensable explanatory role in the web of human knowledge is accepted into the naturalist's ontology. Within the suggested framework, all of mathematics is indispensable. This absolves us from the need to figure out where to draw the line; something which Quine himself found difficult[66]. As was discussed in Chapter 2, seemingly "surplus" mathematical structure has often given rise to new physics. Naturalism can accommodate this after the fact, by including the extra structure in its ontology, but it doesn't give a satisfactory explanation of the deep relationship between mathematics and fundamental physics. Furthermore, as argued in Chapter 4, every additional level of the set theoretic hierarchy leads to the solution of some previously undecidable propositions of finitary number theory, where the meaningfulness and unambiguity of the concepts is generally accepted; so, in this sense also, no mathematics is surplus.

5.5.2 Potential problems

Let me address some potential problems for my central argument before drawing out the final consequences. I start from the realist position that there is something which mathematics is describing. A natural question to ask is then: How are we to determine which of our alternative set theories is true? This, in turn, is linked to the question: What is it that any given set theory is describing? Here we immediately run into problems because, by the Lowenheim-Skolem theorem, any consistent first-order formal theory with an infinite model has models of every infinite cardinality. For example, there are models of first-order set theory that are countable. Such models are called "non-standard" because they

[66] At one point, Quine advocated accepting all of Gödel's constructible universe of sets L on the grounds that it would cover all imaginable scientific needs.

satisfy the axioms of set theory but do not have the property of "uncountability" that is intended. The reason is that notions like "uncountability" amount to different things in different models. A given model can satisfy the formula that says that a certain set is uncountable even if the set (considered from a different structure) is really countable. From this perspective, the implied paradox turns on an equivocation between the semantics of ordinary English set theory and the model-theoretic semantics of formulas in particular models [Bays, 2007: 642-643]. Nevertheless, it creates a very real problem for reference in first-order theories.

How can we constrain the class of permissible models of the formalised set theory so as to retain the connection with "intuitive" set theory? It cannot be done if the sole determinant of the meaning of the axioms is their first order structure. There must be some independent constraints on meaning. A solution which is often mooted is to go to a second-order logic[67]. In second-order set theory the models are determined up to isomorphism; i.e., it is *categorical*. The problem with this solution is that it involves making our choice of logic determine the range of models which we are prepared to consider. From the realist point of view, language is used to *describe* what exists, not to *determine* what exists.

It is often held that full second-order quantification depends on set theory and, so, second-order logic cannot be used to investigate the foundations of set theory without circularity [Isaacson, 2011: 3-4]. Koellner [2010] has undertaken a thorough investigation of strong first and second-order logics which endorses this view.

Firstly, he defines a notion of "strength" for a logic according to which, to generate a stronger logic, one narrows the class of test structures that are consulted in certifying logical implication and validity. For example, standard first-order logic is the limiting case where no constraint is placed on the test structures. It is the weakest first-order logic. At the other extreme, Woodin's Ω logic (which consults universally Baire sets of reals as test structures) is the strongest first-order logic.

Next, Koellner defines a notion of "absoluteness" for a logic according to which, provided disputants agree on the background assumptions of the logic, they will agree on what implies what in the logic. Clearly, standard first-order logic has a high degree of absoluteness.

One component of absoluteness is invariance under forcing. We have seen in Chapter 4 that Woodin's Ω logic has this property. It is secured relative to the background theory of ZFC + "There exists

[67] See e.g. [Shapiro, 1997: 54-57]

a proper class of Woodin cardinals", which also secures that sets of reals have the property of being universally Baire. Koellner [2010: 14-17] completes the further steps required to show that Ω logic has a high degree of absoluteness. In contrast, he shows that:

> In the case of full second-order logic there is a radical failure of absoluteness due to a direct "coupling" with truth in the universe of sets. Instead of being an invariant constant against which differences about Π_2 truth in the universe of sets can be articulated (as is the case with all of the strong logics we have considered) full second-order logic co-varies with Π_2-truth in the universe of sets and so to agree on the logic is to agree on Π_2-truth in the universe of sets. [Koellner, 2010: 30-31]

Furthermore, he argues that full second-order logic cannot secure that set theory be categorical in some absolute sense that is not available without full second-order logic. He concludes that "Quine was right to think that full second-order logic is set theory in sheep's clothing" [2010: 34]. Thus, second-order logic cannot rescue us from the problems of reference introduced by the Lowenheim-Skolem theorem. Using second-order logic might still be appropriate for descriptive purposes, but some other justification is needed.

In a more general context, Putnam [1980] argues that we cannot separate a model from the language which describes it. The two are created in tandem; i.e., the conceptual scheme which determines how we divide up the world into objects and properties at the same time determines how we describe it in terms of subjects and predicates. There isn't a "correct" model independent of language; there are just different conceptual schemes. According to Putnam, this is also true in mathematics and physics. The objects and properties in mathematics and physics are relative to a conceptual scheme (e.g. the Von Neumann universe of sets and set theory, or electromagnetic particles and fields and Maxwell's equations).

Theory alone cannot determine its own reference. Putnam [1980: 473] imagines that we have an ideal theory of the world which correctly incorporates all possible empirical and structural knowledge in the form of theoretical and operational constraints. Using the Lowenheim-Skolem theorem, he shows that there are an infinite number of models that can be made to satisfy this theory. He argues that since each of these models satisfies the theory there is nothing to determine which one is the "intended" model. All are equally good interpretations of reality. Furthermore, since the theory already incorporates all theoretical and operational constraints, there is nothing more that we can do to constrain it. Any attempts would be “just adding more *theory*” [1980: 477] which wouldn’t affect the underlying argument. So there can be no independent, objective, true model of reality (as envisioned by metaphysical realists); just many equally good interpretations.

Putnam’s answer is to join Quine [1969] in advocating a thoroughgoing relativity of ontology and reference. Objects do not exist independently of a conceptual scheme. Within a conceptual scheme, we can match objects and language (i.e. we can use the language and attach meaning to it), but there is no privileged conceptual scheme. Putnam says:

> To speak as if this were my problem, "I know how to use my language, but, now, how shall I single out an interpretation?" is to speak nonsense. Either the use already fixes the "interpretation" or nothing can. [Putnam, 1980: 482].

According to Putnam, then, meaning is exhausted by use. Meaning is built up in communication between individuals. There is no private meaning. Nothing grounds meaning. Words can be explained in terms of other words, in a never-ending regression, but we can never break out of language. This is a point which Quine repeatedly emphasises in his own work. For example:

> It is meaningless to ask whether, in general, our terms 'rabbit', 'rabbit part', 'number', etc., really refer respectively to rabbits, rabbit parts, numbers, etc., rather than to some ingeniously permuted denotations. It is meaningless to ask this absolutely; we can meaningfully ask it only relative to some background language… And in practice we end the regress of background languages, in discussions of reference, by acquiescing in our mother tongue and taking its words at face value. [Quine, 1969: 48-49]

This is a key point. In any realist theory, truth is central to meaning, and truth is based on a correspondence between language and reality. In model theory, for example, statements of a language are determinately true or false depending on an interpretation which is set up between terms in the language and objects in the model. But if language and truth cannot determine reference, as Quine and Putnam argue, then truth cannot be central to meaning. It was just these considerations which led the intuitionists to take proof, instead of truth, as evidence of meaning and understanding in mathematics [Dummett, 1973: 101-102].

5.5.3 Talking to extra-terrestrials

Let me illustrate the problems discussed in section 5.5.2 using an example originally due to Planck [1899]. Suppose we wanted to communicate with other rational beings in the universe. How would we do it? Presumably, they wouldn't recognise any human languages or have the same conceptual scheme that we have. Apart from all the difficulties of formatting, sending and receiving the message, what could it possibly be about that they would understand? Not about cats and dogs, or galaxies; since these things might not have a correlate in their conceptual scheme. Planck supposed that they would understand numbers and have the same physical laws, so they would understand information sent in dimensionless units constructed from the universal constants of physics (i.e. the speed of light in a vacuum, Newton's gravitational constant, and Planck's constant). But what if they didn't have the same physical laws or the same reference for the terms in those laws? Then they wouldn't understand the reference of the numbers. Would they understand what numbers are? This is the minimal requirement typically assumed by SETI programs (i.e., intelligent extra-terrestrials will understand the properties of numbers and be able to interpret information such as a sequence of prime numbers). But maybe we can't even assume this, according to Putnam?

Putnam's view on this point is not entirely clear because he does not make clear where his bedrock lies. He imagines an extra-terrestrial species of intelligent beings who have developed a high level of mathematics but reject the Axiom of Choice for set theory. That is quite rational, he says [Putnam, 1980: 470]. However, he also says there are some mathematical axioms that it might *not* be rational for them to reject; e.g. "every number has a successor" [Putnam, 1980: 471]. So, perhaps Putnam thinks that his extra-terrestrials should rationally accept the same axioms of number theory as we do. Even then, mightn't they interpret 'numbers' in terms of a non-standard model; i.e., an uncountable model? Presumably, even Putnam would be uncomfortable with the scenario of extra-terrestrials who could build sophisticated equipment to receive signals from outer-space but couldn't count to three! Nevertheless, he does seem prepared to countenance the existence of rational beings who are so different from us that all communication with them is impossible, even in theory. I take it that this is a corollary of his preparedness to accept that the set of truths that any species of rational inquirers would eventually acknowledge may be empty [Putnam, 1980: 478].

Is there no way that we can send an unambiguous message to other rational beings in the universe, even theoretically? I think there is. I think that all rational beings share an a priori knowledge of the structure of Being and that this determines the intended model of mathematics. It singles out the full universe of sets and the membership relation on it. So, mathematics can be the basis of a *lingua franca* for the universe.

In practice, our ability to communicate with other societies will depend on the respective maturity of our cultures. It can be expected that they, like us, will have some intuitive knowledge of terms like 'set' and 'membership' from observing the structure of their world. There is an interplay between intuiting some aspects of the intended mathematical model, then formalising and axiomatising those intuitions, gaining deeper insight into the intended model, further axiomatisation etc. That is the process which human mathematics is going through at the moment. We haven't arrived at a full understanding of the intended model yet, but I share Gödel's rationalist belief that we will. In this endeavour, the development of an appropriate descriptive logic (e.g. Woodin's Ω logic) which enables humans to understand salient features of structure will be very important. In other words: we have the vague concept of 'set'; the sharp concept exists; and we need to develop the tools to make our understanding of it precise.

5.5.4 Final discussion of consequences

Following the discussion of Section 5.5.3, my first claim is that rational beings share a common conceptual scheme based on the structure of Being. This has the following consequences:

10. There is one, true mathematics.

There may be different communities of rational beings who interpret physical reality in different ways; e.g., by having greater perception of parts of structure which we have lesser perception of, or which we cannot perceive at all; or, alternatively, by their community developing a different physical interpretation of the same structure; which would then be like a different perspective for an encompassing mind. However, all communities develop mathematics which is mutually interpretable in a background language. Furthermore, all have the same standard interpretation of mathematics.

11. There are no absolutely undecidable propositions in mathematics.

12. Some human mathematics is fiction.

Consistency is not enough to determine existence. Every mathematical proposition has a bivalent truth value and theories which give the wrong value are wrong. This doesn't mean that they are uninteresting or useless. The only way that mathematicians will make any progress in uncovering the true theory will be by investigating false theories and learning how to discriminate false from true. Similarly, in model theory, the whole point is to investigate the interaction between models and formulas. In this context, it makes no sense to design semantics which restricts investigation to a standard model. We would be missing out on potentially useful mathematics.

Now, back to the original question posed in Section 5.5.2: How are we going to tell which human mathematics is fiction and which is true? My discussion entails that we should use the criteria of consistency, categoricity, well-foundedness and decidability. However, this doesn't narrow things down enough. What about something like Gödel's universe of constructible sets L? Gödel's L meets these criteria and it is consistent with ZFC and contains all the ordinals. Yet it is rejected by most set theorists as a candidate for the true universe of sets. Why? Interestingly, the problem with L is linked to the sense in which it "contains all the ordinals" and this, in turn, is linked to a problem with Putnam's argument in *Models and Reality* (see e.g. [Lewis, 1984] and [Resnik, 1986]). Let's investigate further.

It is known that L is inconsistent with the assertion that it contains a measurable cardinal so, in this sense, it cannot contain all the ordinals. However, this is simply because L cannot "recognise" that it contains a measurable cardinal because it does not have the semantic resources to witness measurability [Koellner, 2011: 10-11]. This is the same semantic equivocation that we encountered with non-standard models. In the same way that a theory can "assert" that its domain is uncountable, even when it is countable from another viewpoint, so a theory can "assert" that it doesn't contain a measurable cardinal, even when it does contain one from another viewpoint. In regards to Putnam's argument, Resnik makes the rebuttal that "we cannot infer that the interpretations of [the ideal theory] T meet all the theoretical constraints simply because they verify T… there is a difference between imposing a constraint C on the interpretation of T and requiring that T assert that C is satisfied" [Resnik, 1986: 155-156].

My second claim follows on from these considerations: Being cannot be wrong about itself or lack the resources to describe itself. Being knows all truth. It has the ultimate viewpoint which transcends all others. Furthermore, we do not have the ability to think of cardinal numbers which transcend the cardinality of Being. Therefore, if we can define a large cardinal then it exists, unless it can be ruled out by consistency arguments (e.g., as Woodin does for Reinhardt cardinals[68]). This leads to the final consequence:

13. The universe of sets V cannot be Gödel's L; it must be something like Woodin's Ultimate L which contains all possible large cardinals and has the semantic resources to witness them.

These additional criteria are still not sufficient for mathematicians to determine a unique set theory at this point in time. This does not discourage me. Great progress has been made in the last decade and I think that eventually we will find the criteria which lead to the one, true mathematics.

5.6 Summing Up

In this chapter I have the reviewed the many puzzles presented in previous chapters and suggested a potential resolution through the introduction of a coherent metaphysical framework. In the next chapter I will use the elements of the framework to explore the relationship between actuality and potentiality. Two paradigms are considered: (1) Leibnizian possible worlds, which is rooted in classical physics; and (2) the consistent histories quantum theory of Griffiths, Gell-Mann, Hartle, and Omnès. By comparing and contrasting these two paradigms I seek to highlight the disconnect between the classical and quantum world views and to show how it can be transcended in a monadology.

[68] See Chapter 4

Chapter 6. Actuality from Potentiality

6.1 Overview

In this chapter I am going to explore how actuality emerges from potentiality. In the last chapter I engaged in some speculative thinking about the general process but in this chapter I want to be more specific and that will entail a retreat to a more manageable problem. I will be considering two paradigms: (1) Leibnizian possible worlds, which is rooted in classical physics; and (2) the consistent histories quantum theory of Griffiths, Gell-Mann, Hartle, and Omnès (I will refer to this as CHQT). Hence, at least initially, I will not be considering all of mathematics but merely that part which is required for these paradigms. It is a moot point how much mathematics that encompasses and I will have more to say about that at the conclusion.

There are interesting parallels between these two paradigms which, to my knowledge, have not been explored previously and which I believe will shed new light on some important philosophical issues; primarily, how actuality emerges from potentiality but, also, the problem of the One and the Many, the role of mind in interpreting reality, and the role of chance versus determinism. The analysis goes beyond a comparison of classical and quantum physics to consider how modern physics might be integrated into a more comprehensive view of the world, in the spirit of Leibniz's own philosophy.

Leibniz was one of the last philosophers to develop an encompassing, coherent philosophy based on the most advanced physics and mathematics of his day. His efforts were directed towards answering the big philosophical questions, rather than towards achieving a scientific understanding of part of reality. In contrast, the consistent histories approach to quantum mechanics has been developed by contemporary physicists focused on scientific questions, with only a tangential interest in philosophical issues. Nevertheless, because it is the compendium of three (more or less independent) groups of physicists whose particular expertise includes quantum information theory, logic, advanced semiclassical physics, cosmology and linguistics, the net result shows considerable overlap with the issues addressed by Leibniz. For example, CHQT addresses the issue of how the logical description of reality is linked to ordinary language. It also offers a cosmological view of how the macroscopic world of people and everyday objects emerges from quantum histories traceable to the Big Bang.

Leibniz's philosophy is not consistent with facts which we now know about the quantum world. Classical science is based on the intuitive representation of reality which we have formed as part of our evolutionary struggle to understand and predict physical phenomena relevant to our survival. According to our intuitions, reality is unique, continuous, and separated into objects which are localised in space and time and have causal relations. There is a clear-cut distinction between actuality and potentiality. All these characteristics break down in the quantum world.

By comparing Leibniz's philosophy to CHQT, I hope to make the contrast particularly stark. I choose Leibniz for the comparison because I think that his comprehensive metaphysical/mathematical approach gets closer to the mysteries of ultimate reality than the more piece-meal empirical/mathematical approach of great classical scientists in the Newtonian tradition. I choose the consistent histories approach because it parallels Leibniz's representation of reality in terms of propositions constructed from fundamental properties in accordance with logical rules; and because it offers a corresponding explanation of the way in which our experience of the everyday world emerges from ultimate reality. It also provides an interesting test case for considering the role of surplus mathematics in physical theory.

One conclusion will be that our minds are not matched to reality in the way that Leibniz supposed. Ultimate reality obeys a form of quantum logic, not classical logic. Experimental results in quantum physics are shockingly counter-intuitive. They are not just a refinement of classical concepts. This makes it difficult to deny the role of an external world — independent of the human mind — in our discovery of science. It also makes the effectiveness of mathematics in science more mysterious… more in line with Wigner's views than with those of naturalists who claim that "our mathematical ideas fit the world for the same reason that our lungs are suited to the atmosphere of this planet" [Hersch, 1979: 45]. We cannot simply extrapolate concepts learned from the everyday world to the quantum world. The quantum world obeys fundamentally different laws. There is a bridge (as discussed in previous chapters) and it can be crossed (by a painstaking process of conjecture and experiment, as history shows) but it is still remarkable how mathematicians seem so often to have parachuted down from above and cleared the territory before the ground forces arrive.

In the final section I explore the possibility of combining a Leibnizian phenomenology with concepts drawn from CHQT using the elements of the metaphysical framework introduced in the previous chapter. Leibniz's philosophy is consistent and comprehensive, and prescient in many respects, but it does not incorporate the insights of quantum mechanics. On the other hand, quantum mechanics is a very successful formalism which has no accepted interpretation. My basic idea is that CHQT brings the formalism of quantum mechanics into the logical realm of possible actuals where it can be interpreted in an updated version of monadology. A translation of CHQT into the language of monadology brings together many disparate threads of physics and philosophy.

The format of this chapter is as follows. Section 6.2 focuses on Leibniz's philosophy. Section 6.3 is an overview of the consistent histories approach to quantum mechanics. Section 6.4 is the comparison and critique. Section 6.5 is my attempt to create a blended philosophy in the form of a quantum monadology.

6.2 Leibniz's Philosophy

The purpose of this section is not to give a comprehensive critique of Leibniz's philosophy, but to emphasise those parts that are relevant to my thesis and, especially, to a comparison with CHQT.

6.2.1 Monads

Leibniz's philosophy is reductionist. Everything that exists can be resolved into its parts and, at the most fundamental level, reality consists of simple substances called *monads*. Monads do not have parts but they do have properties. Each monad has an infinity of properties which identify it as a particular individual, distinct from all other individuals [Rescher, 1967: 14-16]. This collection of properties can be termed its Complete Individual Concept (or CIC, for short). The CIC fully describes the monad but the monad is something more than its CIC — it is a mind with perception and will. The monad is Leibniz's solution to the problem of the One and the Many. Mind is the means through which many disparate properties are unified into a single perception. If perception is combined with the conscious activity of the percipient then it is thought.

A monad's existence consists of a continuous sequence of perceptions played out like a movie according to the program which is encoded in its CIC. The tendency of one state to follow another is inherent in the monad and is called *appetition*. Appetition is an expression of the will of the monad in striving towards its goal. It is important to Leibniz that a monad has will and agency and is not just a passive palette on which the diverse properties of the world are exhibited. According to Leibniz's version of compatibilism, each monad plays an active role in the construction of the universe whilst being subject to deterministic laws. Physical forces in the world are derivative on the appetition of monads and the natural laws which govern the world have their ultimate explanation in the teleological laws which govern appetition.

Monads can have no causal interaction with one another since they are mental, and not physical, substances. This is Leibniz's solution to the Cartesian mind-body problem. His attitude towards the existence of physical substance is not clear and probably changed over his lifetime as his philosophy developed [Garber, 2009: 382-384]. In some passages he writes as though there is both a physical realm and a monadic realm and the two operate in perfect harmony in parallel. In other passages he seems to endorse a form of phenomenalism, verging on idealism, in which bodies are nothing beyond the organised perceptions of monads, grounded in the coherence of those perceptions within individual monads at different times and across all monads at particular times. For example, in discussing the nature of matter and motion he writes:

> Indeed, considering the matter carefully it should be said that there is nothing in things except simple substances and in them perception and appetite. Moreover, matter and motion are not so much substances or things as the phenomena of perceivers, the reality of which is located in the harmony of perceivers with themselves (at difference times) and with other perceivers. (Leibniz as quoted in [Garber, 2009: 363]).

In discussing space, he writes:

> For monads in themselves do not even have situation with respect to each other—at least one that is real, which extends beyond the order of phenomena. Each is, as it were, a certain world apart, and they harmonise with each other through their phenomena, and not through any other intrinsic intercourse and connection. (Leibniz as quoted in [Garber, 2009: 360]).

Space, time and motion are purely phenomenal and appear only in the contents of monadic perceptions. Physical phenomena are well-founded in monadic perceptions but are not constituted by the monads themselves (in contrast to a materialistic, atomic view of the world). I will adopt a phenomenological interpretation of Leibniz's philosophy, without any assumption that this was his settled view.

6.2.2 Properties

In the phenomenological interpretation, monads do not actually perceive other monads in the sense of "seeing" them (since they have no causal interactions). Rather, the content of their perceptions is a form of expression of the external world which is built into their CIC. According to Leibniz, the complete description of any individual must include its relation to all other existents. He argues that “there is no term so absolute or detailed as not to include relation, and the perfect analysis of which does not lead to other things and even to all others” (as quoted in [Ishiguro, 1972: 77]). I see this as a form of holism in which a term cannot be defined in isolation — its meaning is derived from the whole network of relations between terms. Thus, the CIC of any particular monad includes its relations to all other monads for all times. From any of its states, it is possible to infer to its past and future states and, ultimately, to the complete state of the universe at any time over the course of its history. In this way, “each monad is a perpetual living mirror of the universe” [Monadology, §56].

A monad ‘expresses’ something by containing relations which correspond to the relations of the thing expressed [Garber, 2009: 216]. This is analogous to the way in which a blueprint expresses the structure of a building, an algebraic formula expresses a sphere or, more pertinently, DNA expresses the structure of a living being. A monad’s perceptions express all the relations of the universe. However, not all monads have consciousness and are aware of their perceptions. Furthermore, even for conscious monads, not all perceptions are distinct — some are confused, and some are so tiny as to fall below the threshold required for awareness. A monad is said to be more perfect in so far as it has distinct perceptions and less perfect in so far as it has confused perceptions.

In a strange twist, the interconnectedness of everything (i.e. the adaptation of each monad to all the others and its expression of all their relations) when combined with the thesis of causal isolation leads to the possibility of solipsism. The actions of a monad derive from its own nature, as determined by God, and so it behaves as though there were nothing but God and itself in the world. One might think that a monad would be compelled to transition from state to state according to its CIC, with all the concomitant perceptions of an external universe, whether or not an external universe existed. However, this is not possible in Leibniz’s philosophy because the perceptions given to a monad by God are an expression of the truth. The world may not be exactly as the monad perceives it (because expression is an isomorphism of relations, rather than a direct representation, and because the perceptions of monads are confused) but something external exists which corresponds to its perceptions and is the cause of their variety. Each monad has its own point of view, in perfect harmony with all other monads, and God can combine this infinity of points of view into a complete picture of the universe.

The CIC of a monad contains everything that can be predicated of it. Any proposition of the form ‘Monad #75 has the property P’ is true just in case that property is included in the list of properties which comprise the CIC of that monad. Furthermore, any proposition about the things of this world can be constructed from elementary propositions about monads, since they are the only existents. Hence, reality is expressed in true and false monadic propositions and, conversely, true and false monadic propositions represent reality [Martin, 1964: 5]. In principle, there is a basic set of independent properties which distinguishes each monad via a sequence of simple questions of the form ‘Does Monad X have property Y?’ A monad’s CIC can be represented as a string of bits in this basis. Distinct monads must differ in at least one of their basic properties.

6.2.3 Concepts and possible worlds

For Leibniz, the logical construction of thought and language from simple concepts and their associated signs reflects the way in which the structure of the world is logically constructed from the basic properties of monads. In principle, any concept can be analysed into a combination of absolutely simple irreducible concepts. Conversely, by combining simple concepts according to the laws of thought, any possible concept can be constructed. If we were able to express every simple concept by an arbitrarily chosen sign and encode the laws of thought in syntactical rules then we would have a formal language which we could use to analyse any mathematical, scientific or metaphysical proposition whatsoever. This is Leibniz’s dream of a universal science [Rescher, 1967: 131]. It combines a universal notation — an alphabet of human thought — with rules of computation. Mathematics is its paradigm but it goes beyond mathematics to encompass all of reasoning[69]. Every well-formed formula would express a possible concept and every provable formula would express a true concept. Impossible concepts (such as a round square) would be reduced to a contradiction when analysed in terms of their simple components.

Of course, human beings would not always be able to carry out a complete analysis of a given concept — that might involve an infinite chain of reasoning or rely on simple concepts of which we were unaware. The human mind is incapable of forming complete, or exhaustively individuating, concepts. Nevertheless, God is capable of this and all true concepts exist in divine thought:

> There it is that I find the original of the ideas and the truths which are engraved upon our souls, not in the form of propositions but as the sources whose application and occasions give birth to actual statements. (Leibniz as quoted in [Martin, 1964: 99-100]).

[69] Drawing on Leibniz’s dream of a universal science, Gödel dreamed of a theory of concepts which would go beyond the theory of sets (i.e. beyond mathematics). Whereas set theory deals with extensions, concept theory would deal with intensions. Note that different concepts can have the same extension whereas, by definition, different sets cannot since they are determined by their extension. For more on this topic, see [Wang, 1996: Chapter 8].

Furthermore, all possible concepts have their source in God's understanding insofar as they are real:

> Without God's understanding there would be nothing real in possibilities and not only would nothing exist but nothing would be possible. [Monadology, §43].

Leibniz is one philosopher who gives great consideration to God's act of creation of the universe [Rescher, 1967: 11]. He says that before creation all possible monads subsist as concepts in the mind of God. They all strive for existence, competing against one other in a pre-Darwinian vision of the survival of the fittest. Possible monads can conflict with each other in that they can have incompatible properties (i.e. analysis of their conjoint CICs leads to a contradiction). Monads with incompatible properties cannot be realised together and they are said to be incompossible. Only monads with completely compatible histories are compossible with one another and this is encoded in their CIC. Therefore, the manifold of possible monads splits into mutually exclusive possible worlds in which all the monads are reciprocally adjusted to one another. Each possible world has a history which is defined by its smallest part and the smallest part cannot exist without the whole. Any change in any property of a constituent monad throughout its history would put it into a different possible world.

The act of creation amounts to God's choosing the best possible world by maximising perfection. Leibniz says:

> Now, since in the ideas of God there is an infinity of possible universes, and since only one can exist, there must be a sufficient reason for God's choice of that one — a reason that leads him to choose one rather than some other of the possible universes. And this reason can only be found in the *suitability* or *degrees of perfection* that these worlds contain, with each possible world's right to claim existence being proportional to the perfection it contains. [Monadology, §53-54].

This is a statement of the Principle of Perfection. Since Leibniz considers existence to be a mode of perfection, one consequence is that God maximises existence by actualising an entire possible world containing an infinity of monads. Leibniz says that "one can define an existent as that which is compatible with more things than anything incompatible with it" [Rescher, 1967: 28]. Furthermore, since harmony is also a mode of perfection, God maximises the harmony of the monads far beyond the requirements of mere compossibility. This means that the relations between monads are governed by simple laws; their changes of state are continuous and orderly; and their perceptions give foundation to coherent, well-founded phenomena (such as the spatial and temporal structure of the world). In summary, the best possible world is the one which exhibits the greatest variety of its contents together with the greatest order and harmony (and, hence, the greatest simplicity of its laws). By choosing the best possible world God shows his goodness and benevolence.

Empirical facts about the world (such as those established by physics) are contingent on God's choice of the best possible world. Leibniz calls them *truths of fact* and contrasts them to *truths of reasoning* which are true in all possible worlds. Mathematical truths are truths of reasoning. The laws of nature are truths of fact. There are possible worlds in which our laws of nature do not hold and the monads exist in a chaos of kaleidoscopic changing perceptions, leaving them in a perpetual daze [Monadology, §24]. These are the least perfect possible worlds. According to Leibniz, the degree of perfection of an individual monad is proportional to the clarity with which it perceives other monads and the perfection of the world is the sum total of the perfection of its individual constituents. The most perfect world not only satisfies the global constraints on the variety of its contents and the simplicity of its laws, discussed in the previous paragraph, it is also locally optimised so that each monad is as perfect as possible. The perfection of individual monads is linked to the laws because, in relations between monads, the more perfect one is seen to be the active agent in phenomenal causation (even though monads have no real causal influence over one another). The laws of nature are ultimately explainable in terms of the relative perfection of monads [Monadology, §49-52].

6.2.4 The construction of physical structure from mathematical structure

The Principle of Perfection is needed to get from potentiality to actuality — and from mathematics to physics. But, whereas the transition from potentiality to actuality takes place in the mind of God, the construction of physical structure from mathematical structure takes place in human minds and is subject to their limitations. All of God's concepts are complete and have a degree of reality, but the concepts of the human mind are incomplete and abstract since finite minds are incapable of forming complete concepts. According to Leibniz, mathematical objects are abstract constructs of the human mind. They form the imaginary realm of possibilities which human minds are capable of comprehending. God doesn't need mathematics because he sees all possibilities, altogether at once, as they really are, without need of idealisation or derivation. Nevertheless, He is the ultimate source of mathematical truth, as of all eternal truths.

To understand the construction of physical structure from mathematical structure in Leibniz's philosophy of mathematics we must first review some aspects of his epistemology. According to Leibniz, our monadic perceptions are the source of all our knowledge. The extent to which our knowledge is complete depends on the extent to which we can consciously access our perceptions in clear and distinct form. Our senses give us clear but confused notions. From them, our reason constructs the complex of relations which are well-founded phenomena. God is the reason why we all experience the same physical reality:

> … the reason why different minds sense things in the same way is because they all communicate with the same Entity, namely God. (Leibniz as quoted in [Garber, 2009: 287]).

In addition to our single senses, there is a 'common sense' which allows us to describe qualities accessible to more than one external sense. Common sense gives us clear and distinct notions, such as number, extension, shape and space. They are distinct notions because they can be defined. From them, our imagination constructs the objects and relations which are used in mathematics. According to Leibniz:

> There must be an internal sense where the perceptions of these different external senses are found united. This is called the *imagination*, which comprises at once the concepts of particular senses, which are clear but confused, and the concepts of the common sense, which are clear and distinct. And these clear and distinct ideas which are subject to the imagination are the objects of the mathematical sciences, namely arithmetic and geometry, which are the pure mathematical sciences, and their applications to nature, which make up mixed mathematics (such as optics and astronomy). [Leibniz, 1989: 187-188].

The constructions of the imagination are things like mathematical lines, triangles and circles. They enable us to discover the truths of mathematics but they are not necessary for mathematical truth. Mathematical truths are known by the understanding and "not at all by the experience of the senses" [Leibniz, 1989: 189]. Indeed, according to Leibniz, imagination only serves to provide material for the mathematical understanding to work with. It is the understanding itself which is responsible for the truth and necessity of mathematics.

Mathematical knowledge is formal knowledge of the logic of mathematical relations and it does not depend on the nature of the relata or their source. It only requires "*intelligible notions and truths*, which alone are capable of allowing us to judge what is necessary" [Leibniz, 1989: 189]. Leibniz considers intelligible notions and truths to be those which are discoverable by introspection alone, independent of sensations. Metaphysical concepts such as being, identity and self are intelligible notions. Without them we would not be able to think at all — they are the preconditions for thought. They feed into the understanding and enable us to structure data and abstract meaning from it.

As an illustration of the process, consider how our imagination takes the concept of a line, abstracted from the common sense, and uses it to generate an entire realm of abstract possibles (i.e. numbers) which the mathematical understanding then uses to discover the laws and relations of arithmetic. It is the laws and relations which constitute mathematical knowledge, not the numbers themselves. Similarly, the mathematical understanding acts on triangles and circles — and such-like constructions of the imagination — to discover the science of geometry. The constructions of the imagination are instantiations of certain mathematical relations but once we have knowledge of the relations themselves (and the axiomatised theory of which they are a part) then it can be formalised in ways which do not depend on particular objects.

According to Leibniz, mathematical reasoning largely consists of the manipulation of symbols. He writes that:

> [Certain thoughts] are empty of perception and sensibility, and consist in the wholly unaided use of symbols, as happens with those who calculate algebraically with only intermittent attention to the geometrical figures which are being dealt with. [Leibniz, 1996: II.xxi.186].

His understanding of notation and its role in facilitating thought enables him to develop a formalist attitude to mathematics in advance of his contemporaries. For Leibniz, the mathematical understanding is not constrained to work only with the outputs of imagination; it can work with any symbols suitably construed. For example, its content can come from mathematics itself via a limiting process — like the mathematical point, which Leibniz considers to be a fictional object posited by the understanding as the limit of the process of indefinitely dividing the line. Leibniz says of infinitesimals and infinitely large magnitudes:

> … I consider both to be fictions of the mind, due to abbreviated ways of speaking, which are suitable for calculation, in the way that imaginary roots in algebra are. Moreover, I have demonstrated that these expressions have great usefulness for shortening thinking, and thus for discovery, and that they cannot lead to error, since it would suffice to substitute for the infinitely small as small a magnitude as one wishes, so that the error would be less than any given; whence it follows that there can be no error. [Leibniz, 2007: 33].

Such objects are useful fictions which shorten mathematical demonstrations whilst conserving the truth of statements about imaginable mathematical objects.

Leibniz defines mathematics as “the science of imaginable things” [Jolley, 1995: 184]. This is how he reconciles its *a priori* nature with its applicability. It is *a priori* because mathematical demonstration only depends on intelligible notions and truths. It is applicable because it sources content from imagination, which operates with the outputs of common sense. There are objects of mathematical reasoning which are not sourced in imagination, but they are fictions which are useful for establishing truths about imaginable things. Leibniz doesn’t concern himself with fictional mathematical objects as a source of aesthetic pleasure. He seems to think that, one way or another, all mathematics is applicable.

Some of Leibniz’s views are reminiscent of those of modern day fictionalists but, for him, imaginable things are just as real as physical things. Mathematics is needed for the construction of objects from our confused perceptions; e.g., we need mathematical concepts such as extension and continuity in order to form boundaries around and between phenomena. All our knowledge is built on relations which we abstract from our perceptions using innate concepts and truths. The relations which we perceive as physical phenomena (i.e. as nature) are truths of fact. The relations of mathematics are truths of reason. For Leibniz, all relations are abstract mental constructions.

The foundations of physical phenomena are infinitely complex aggregations of monads. According to Leibniz:

> Every portion of matter can be thought of as a garden full of plants or a pond full of fish. But every branch of the plant, every part of the animal (every drop of its vital fluids, even) is another such garden or pond. [Monadology, §67].

Hence, reality has a fractal structure. Continuous geometrical shapes are not instantiated in the real world but are an idealisation.

As previously remarked, Leibniz denied that there is an infinite number in mathematics. He thought that mathematical definitions must involve a proof of the possibility of the thing defined; e.g., by construction [Rescher, 1967: 105]. According to Leibniz, there are arbitrarily large numbers, but no infinite number as such. Nor are there infinitely small numbers, infinitesimals or mathematical points; only finite fractions of arbitrary smallness. The mathematical line is a whole prior to its parts and it can be indefinitely divided, but it is not constituted by points. A mathematical point is the fictional limit of an infinite subdivision.

In metaphysics, however, Leibniz's views are quite the reverse. He considers that real objects are constituted by monads (i.e. by metaphysical points) and that any continuous features which we might impute to them are mere phenomena. Furthermore, there *is* an actual infinity of monads even though it is not denotable by a mathematical number: "It surpasses all numerical bounds in content and is immeasurable by an infinite number" [Rescher, 1967: 107]. So, for Leibniz, reality has actual infinity but mathematics does not. Actual infinity is God's signature.

In the human endeavours of mathematics and physics, the real is governed by the ideal and vice versa:

> It is found that the rules of the finite succeed in the infinite, as if there were atoms …, and that vice versa the rules of the infinite succeed in the finite, as if there were infinitely small metaphysical [points]. (Leibniz as quoted in [Rescher, 1967: 107]).

Continuously extended structures are abstractions which can be indefinitely divided whereas real structures are composed of parts.

Well-founded phenomena contain something of both the ideal (contributed by the human mind) and the real (contributed by monads). They can be indefinitely divided like the mathematical line but they obey ordering relations which derive from their monadic foundations. Therefore, both mathematics and metaphysics are needed to understand them (and to understand physics). Leibniz says:

> I concluded that besides purely mathematical principles subject to the imagination, there must be admitted certain metaphysical principles perceptible only by the mind… since all the truths about corporeal things cannot be derived from logical and geometrical axioms alone, but there must be added those of cause and effect, action and passion in order to give a reasonable account of the order of things. (Leibniz as quoted in [McRae, 1976: 125]).

According to Leibniz, we mentally construct the physical world from sense data using mathematical and metaphysical understanding. So, although he is a nominalist in the sense that he thinks everything can be explained in terms of individual substances and their properties, he is not a nominalist in the same sense as someone like Hartry Field. Field thinks that the objects of our physical theories are real entities which causally interact with one another and that mathematical entities, being abstract, cannot be essential for their description. Leibniz thinks that the only real entities are monads and they are in complete causal isolation from one another. The only action is their internal striving which causes them to transition from state to state. Causal relations and physical forces in nature are phenomena which derive from the harmony of monads.

6.2.5 Metaphysics

Leibniz encapsulates his metaphysical views in a letter to Arnauld in 1687:

> … I hold that philosophy cannot be better reestablished and reduced to something precise, than by recognising only substances or complete beings endowed with a true unity, together with the different states that succeed one another; everything else is only phenomena, abstractions, or relations. (Leibniz as quoted in [Garber, 2009: 294]).

He arrives at this position after many years of contemplating the problem of the One and the Many[70]. He focuses on two competing requirements:

(1) A true being must be a unity and not composed of parts; and

(2) Out of that unity, all the variety of the world must emerge.

From the first requirement, Leibniz comes to the view that the fundamental entities must be spiritual beings, or minds. However, as minds, they cannot causally interact with one another, so each one taken individually must contain within it the means to express all the variety of the world. Hence, Leibniz proposes his mirror thesis whereby every monad has perceptions which mirror the entire universe. The monad and its perceptions are the expression of the many in the one. Unity comes from mind. Considering the unity of physical objects, Leibniz says:

> … a thing that is aggregated from many is not one except from a mind, and has no reality except that which is borrowed from what it contains. (Leibniz as quoted in [Garber, 2009: 363]).

There is no reality in anything except the reality of unities and physical objects are not composed of constitutive unities; rather, they are founded in the coherence of monadic perceptions. "Unlike unities

[70] Heinrich Schepers said that "one can view Leibniz's metaphysics as the work of rationally explicating the different expressions of the polarity of the One and the Many" (as quoted in [Kulstad, 2005: 21]).

per se, which are one in the fullest sense, aggregates are unified only with respect to the mind that conceives them as unified" [Garber, 2009: 295].

The Principle of Perfection leads to further expressions of the One and the Many. The actual world is optimised so that the simplest possible laws lead to the greatest possible variety. This is achieved through harmony, which is a unity in multiplicity. Each monad acts according to its will in transitioning from state to state, but harmony ensures the overall coherence of the individual and combined monadic perceptions, in accordance with God's will.

The disconnect in his philosophy between underlying reality and everyday phenomena led Leibniz to address the question of how the familiar world is grounded in the world of monads. This is not dissimilar to the challenge which we face today in trying to explain how the classical world of macroscopic objects emerges from the quantum world. Leibniz explains the variety of the world as emerging from correlations between monadic properties, as governed by harmony. Nowadays we explain the variety of the world as emerging from correlations between the properties of quantum particles, as governed by natural laws. In both cases what we call causality consists in the regular correspondence between certain changes in the properties of one thing with those of another, governed by structural laws. According to Omnès, there is no action in quantum mechanics:

> … there are [quantum] histories, they interfere, and the observed data results; nothing acts. Action and causality appear only beyond decoherence, which is an interference effect… *the fundamental laws of nature are pure mathematical forms accounting for the phenomena though providing no cause for them and showing no action.* [Omnès, 2005: 157]. Original italics.

One could say that in Leibniz's philosophy there are monadic histories, they accord with one another, and the observed data results; nothing acts. Action and causality are appearances resulting from the coherence of monadic perceptions. The fundamental laws of nature account for the phenomena though providing no cause for them and showing no action.

The only caveat to this is that Leibniz attributes action to the monads themselves in their transitions from state to state. This is his way of retaining some notion of free will in his philosophy. Monads are not passive particles but are beings with minds of their own. Leibniz introduces mind as a basic property of all true existents. Quantum mechanics does not associate mental properties with particles but, then, there is the question of how mind and consciousness emerge in the world. There is also the issue of the role of the conscious observer in the standard interpretation of quantum mechanics. According to Wigner:

> When the province of physical theory was extended to encompass microscopic phenomena, through the creation of quantum mechanics, the concept of consciousness came to the fore again: it was not possible to formulate the laws of quantum mechanics in a fully consistent way without reference to the consciousness. All that quantum mechanics purports to provide are probability connections between subsequent impressions (also called "apperceptions") of the consciousness, and even though the dividing line between the observer, whose consciousness is being affected, and the observed physical object can be shifted towards the one or the other to a considerable degree, it cannot be eliminated. [Wigner, 1979: 172].

This quote from Wigner, with its reference to the fundamental importance of the apperceptions of conscious minds, is very much in the spirit of Leibniz's monadology. It expresses the view that that physics needs something external (i.e. unphysical observing minds) for its formulation. In opposition to this view, the consistent histories approach is an attempt to formulate quantum mechanics without the need for observers. Its underlying philosophy is that everything can be explained by the laws of physics and their initial conditions. The next section is an exposition of the consistent histories philosophy. Critical comments will be deferred to Section 6.4.

6.3 A Consistent Quantum Philosophy

6.3.1 The standard interpretation of quantum mechanics

Quantum mechanics is a very successful, thoroughly tested theory of physical phenomena at the micro-scale but, after almost one hundred years of operation, it has no commonly accepted interpretation. This statement is at the same time both extraordinary and a cliché. The theory mathematically represents physical phenomena and accurately predicts the results of experiments but it doesn't tell us what the real world is like. Different interpretations of quantum mechanics, all consistent with the mathematical theory, describe reality in radically different ways. For example, the Many Worlds interpretation tells us that reality is fully deterministic whilst the consistent histories interpretation says that it is fully stochastic.

In the standard exposition of quantum mechanics by von Neumann [1955] the physical state S of a quantum system is fully-describable by a non-zero vector in an associated Hilbert space H and every normalised vector in H represents a physically possible state. As long as the system remains isolated its state evolves deterministically according to Schroedinger's equation. Observables of the system (e.g., position, momentum, spin) are represented by Hermitian operators[71] in H whose eigenvalues correspond to the possible values of the observable. For example, if the system is in a state corresponding to an eigenstate of the observable Q with eigenvalue q then a measurement of Q will register the value q. On the other hand, if the system is in a superposition (say, $|S> = \Sigma c_j |A_j>$ where the $|A_j>$ are a complete basis of eigenvectors of Q) then a measurement of Q will give one of its eigenvalues and the probability of measuring a given eigenvalue q_k is $|ck|^2$. This is Born's rule. Immediately following a measurement which yields q_k, the system is in the pure eigenstate A_k. The act of measurement is said to "collapse" the system into an eigenstate of the measured observable.

In von Neumann's exposition there are two processes: (1) a continuous deterministic process in which the state evolves according to Schroedinger's equation; and (2) a stochastic process in which the state collapses to one of its eigenstates. The second process is associated with a measurement of some sort. The state vector can be represented by a wave function in position or momentum space. In the Copenhagen interpretation, the wave function is not considered to be an objectively real entity; it is simply a mathematical device for calculating the probabilities of measurement outcomes. It links macroscopic input from the prepared system to macroscopic output from future experiments performed on the system. What really happens between measurements is not known. In particular, wave function collapse is not thought of as a real physical process.

In the Copenhagen interpretation there is a divide between the quantum world and the classical world; i.e., between the world of potentiality, which contains superpositions, and the real world as we perceive it, which only contains definite states recorded by our measuring devices. Von Neumann thought that the quantum/classical divide could not be drawn before the outcome of an experiment was registered in consciousness. Before that point, it would always be possible to represent the measured quantum system and the measuring device as a quantum system in a coherent superposition. Since macroscopic superpositions (such as, notoriously, Schroedinger's superposition of a live and dead cat) are never observed, a conscious observer was seen as being necessary to break the chain of superposed

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[71] A Hermitian operator is an operator which is equal to its adjoint, so its eigenvalues are real numbers.

quantum state and measuring device. That is why Wigner said that it was not possible to formulate the laws of quantum mechanics in a fully consistent way without reference to the observer (see his quote in Section 6.2.5).

The role of the conscious observer in quantum mechanics is controversial. Sophisticated modern versions of the double-slit experiment suggest that an interference pattern can only be manifested if there is no way to know, even in principle, which path the particle took through the apparatus[72]. It is irrelevant whether or not the information is registered in the consciousness of an observer. Although it is the experimentalist who chooses which information manifests itself in an experiment (i.e., by choosing the setup), the objective randomness of quantum measurement provides a limit to the control any experimentalist has. There is an external reality outside the control of the observer.

The quantum/classical divide imposed by the Copenhagen interpretation precludes a more fundamental explanation of why we don't observe superpositions of macroscopic states and why classical determinism holds at the macroscopic scale. It cannot lead to a unified description of physical phenomena at all scales. It cannot serve the purposes of quantum cosmology, which seeks to treat the whole universe as an isolated system whose evolution is independent of the consciousness of any observer [Hartle, 2004]. Many alternative interpretations of quantum mechanics have been suggested over the years. I will not be reviewing them all. Rather, as discussed in the overview, I will be focusing on one — Consistent Histories Quantum Theory (CHQT) — which gives a consistent interpretation of quantum mechanics over a vast range of physical phenomena.

6.3.2 Introducing Consistent Histories Quantum Theory (CHQT)

CHQT combines the interpretations developed in the work of Griffiths, Omnès, Gell-Mann and Hartle[73]. It is built on the concept of 'histories', where a history is a sequence of properties of a system holding at a sequence of times[74]. This is a very general concept which can be used to describe the behaviour of diverse systems. For example, the result of a laboratory experiment might be described by a history which states: "at time t1 the z-component of electron spin was measured to be +1/2 and at time t2 the x-component of electron spin was measured to be -1/2". Equally, a history could be used to describe the motion of the centre of mass of the moon as it orbits around the Earth. Griffiths' work

[72] See e.g. [Zeilinger, 1999] and [Yu and Nikolic, 2011].
[73] See e.g. Griffiths [2002], Omnès [1994], and Gell-Mann and Hartle [1990].
[74] A property of a system is given when one can assert that the value of an observable A is in some given set D of real numbers [Omnès, 1992: 342].

focuses on histories definable for systems of quantum particles in laboratory experiments. Gell-Mann and Hartle focus on histories definable for the universe as a whole (treated as an expanding box of quantum particles and fields in a fixed background spacetime). Omnès links the two together using the tools of logic and advanced semiclassical physics to show how the micro-world of quantum physics gives rise to the macro-world of Newtonian dynamics.

To avoid any possible confusion, let me emphasise that the histories of CHQT are not the same as the histories used in the Feynman path integral formulation of quantum mechanics. Feynman's histories include all possible histories for a system, including ones that interfere, to which probabilities cannot be applied. In contrast, the histories of CHQT form mutually exclusive sets obeying a consistency rule which ensures that they may be assigned probabilities in the usual way (i.e. the probabilities sum to one and can be combined according to the laws of standard probability theory). For exact consistency, the rule requires that the interference effects between histories as measured by the decoherence functional vanish [Griffiths, 2002: 140-142]. In practice, it is usually sufficient to apply an approximate consistency condition whereby the off-diagonal elements of the decoherence functional are small enough to result in negligible interference. Dowker and Kent [1996: 1578] suggest that only exactly consistent sets need be considered in any fundamental formulation of the theory because an exactly consistent set can always be found in the neighbourhood of generic approximately consistent sets of histories. Consistent sets of mutually exclusive histories are called 'frameworks'. Fundamentally, CHQT says that the basic task of quantum theory is to use the time-dependent Schroedinger equation to assign probabilities to histories in frameworks. The predictions of the theory are intrinsically probabilitistic.

The notion of 'decoherence' is very important. In the context of CHQT it means the absence of quantum mechanical interference between individual histories. More generally, it is the loss of local phase correlation between the components of a system in a quantum superposition. It is something that happens naturally in quantum systems as information about selected observables leaks into the environment. For example, calculations show that a superposition of two positions of a grain of dust, 1mm apart, is decohered simply by the scattering of the cosmic background radiation on the timescale of a nanosecond [Hartle, 2004: 9].

Decoherence is the key notion in Gell-Mann and Hartle's formulation of CHQT because they are interested in the prediction of probabilities for alternative histories of the universe considered as a single, closed quantum mechanical system. Probabilities can only be assigned to such histories if they decohere (or, at least, approximately decohere). In principle, decoherent histories constructed from the initial quantum state of the universe and the laws of quantum dynamics contain all the information we can use to extract knowledge of the features of the universe. It should be possible to use them to predict the emergence of features of our everyday world (e.g., the existence of macroscopic objects obeying classical laws of motion).

6.3.3 The emergence of the classical world from the quantum world

Much of Gell-Mann and Hartle's work is directed to the question of how the classical world emerges from the quantum world. They even consider such important developments as the evolution of Information Gathering and Utilising Systems (i.e. IGUSs), of which human beings are one example[75]. Less ambitiously, they consider how a statement such as "the moon moves on a classical orbit around the Earth" can be framed in the language of decoherent histories. Hartle says:

> In quantum mechanics, the statement that the moon moves on a classical orbit is properly the statement that, among a set of alternative histories of its position as a function of time, the probability is high for those histories exhibiting the correlations in time implied by Newton's law of motion and near zero for all others. [Hartle, 2007a: 2].

Gell-Mann and Hartle call quantum systems which exhibit classical determinism over a wide range of time, place and scale 'quasiclassical realms'. It is a fact of our everyday experience that we live in a quasiclassical realm. We are mainly interested in predicting the behaviour of macroscopic objects and the laws of Newtonian dynamics are adequate for that purpose.

In Newtonian dynamics, a system is described using collective variables such as averages over suitable volumes of mass density. Analogous quasiclassical variables can be defined in quantum mechanics. Quasiclassical variables are those which obey conservation laws arising from the symmetries of spacetime; e.g., densities of energy, momentum, and baryon number [Hartle, 2011: 990]. Because they are conserved quantities (or, nearly-conserved quantities), they resist degradation due to interaction with the environment. Technically, quasiclassical variables are defined by 'coarse-graining' the underlying Hilbert space in an appropriate way. The analogy of a classical property such as "the mass

[75] More generally, an IGUS is a complex adaptive system which is coupled to its environment in some way, able to model its local environment by some form of logical processing, and able to act on the results of its computations. It could be biological or it could be something like a thermostat or a robot [Hartle, 2007b: 3111].

density in region X has the value Y" is a quasiclassical projector onto a multi-dimensional subspace of a suitable Hilbert space [Omnès, 1992: 352].

Quasiclassical realms are consistent sets of decohering histories constructed from quasiclassical properties. The process of coarse-graining the underlying Hilbert space is not unique. There will be many different ways in which it can be partitioned to give histories of quasiclassical properties which approximate the deterministic behaviour of the classical system. The important points are: (i) that the set of coarse-grained histories decoheres; and (ii) that the correlations in time that define classical determinism occur with very high probability [Hartle, 2007a: 2].

When demonstrating the emergence of approximately-valid classical equations from the underlying quantum mechanical laws applied to decoherent histories, as discussed in [Omnès, 1992: 348:353], it is legitimate to posit the quasiclassical variables of interest and proceed from there. However, there is nothing in the laws of quantum mechanics to identify the quasiclassical variables of interest. This points to an important difference between CHQT as applied to quantum cosmology and CHQT as applied to laboratory situations. In laboratory experiments, we have the luxury of knowing what the observables of interest are and how the system has been prepared. In quantum cosmology, the aim is to derive the features of the universe from the initial conditions and dynamical laws. Therefore, objective criteria must be sought for identifying the appropriate quasiclassical variables.

One criterion is arrived at by considering the limits of coarse-graining and decoherence. These are the fundamental processes which lead to the emergence of quasiclassical realms. Coarse-graining is inevitable because there are no non-trivial completely fine-grained sets of histories that decohere [Gell-Mann and Hartle, 2007: 022105][76]. Given any coarse-grained decoherent set of histories we can imagine refining it by increasing the number of time steps of each history (at least, down to the scale of Planck time) or sub-dividing their constituent properties. Maximally-refined decoherent sets of histories are those for which there are no finer-grained sets that are decoherent.

A lower limit on the size of the volumes used to define quasiclassical variables is imposed by the requirement that the volumes retain their integrity in interactions with the environment. This criterion sifts through the Hilbert space of a system interacting with its environment and selects the states that

[76] Intuitively, if the components of a quantum system are in a superposition the only way they can decohere is through interaction with an environment and an environment can only be defined by coarse-graining.

are the most predictable. The most predictable states are the ones that approximately obey the equations of classical physics [Zurek, 2002: 17]. So, predictability and classicity go together.

This is notionally how maximally-refined quasiclassical frameworks can be understood as objective features of the underlying Hilbert space. It is not a rigorous demonstration since there are not yet measures of maximality and classicity adequate for the task of consistently reconstructing these features from first principles. Currently, all attempts to derive the emergence of the classical world from the quantum world involve an element of knowing the desired outcome and imposing external constraints to achieve it (e.g. in selecting the collection of variables to follow in a history).

6.3.4 CHQT in the laboratory

Whereas Gell-Mann and Hartle have the ambitious project of extending the interpretation of quantum mechanics to give a unified account of the evolution of the universe from the Big Bang to the emergence of consciousness and the classical world of objects, Griffiths [2002] is more focused on giving an interpretation that will allow us to reason intuitively about quantum systems in technology and in the laboratory. It is a challenge to understand the operation of modern devices such as quantum computers and SQUIDS, and to explain the outcome of sophisticated experiments designed to test the principles of quantum mechanics (e.g. delayed-choice experiments). By following textbook rules one can compute the behaviour of quantum systems but the lack of an intuitive interpretation hampers understanding and innovation.

Using the histories approach which he introduced in 1984, Griffiths constructed a language for quantum systems which allows one to talk about time-sequences of quantum properties using the usual rules of logic and probability theory, provided certain consistency conditions are satisfied. He uses the same consistency conditions as Gell-Mann and Hartle; however, he does not rely on decoherence to achieve consistency since the systems he wishes to describe exhibit the full range of quantum weirdness due to superposition. He finds sets of consistent histories by appropriately partitioning the mathematical and logical structure of Hilbert space. The underlying principle is that the structure of quantum theory reflects the structure of the world and so anything one can meaningfully say about the physical world (e.g., "there are two neutrons and there are two electrons in this region of space") can be expressed in terms of subspaces of a suitable Hilbert space.

Certain syntactical rules are applied to determine when and how quantum properties can be combined into meaningful propositions. In particular, rules are given for negation, conjunction, disjunction and implication. Once a consistent logical framework has been defined, one can use it to discuss quantum mechanical systems in a common sense way, using intuition and plain language. For example, in an experiment set up to measure the z-component of electron spin, one can talk about the spin of the electron as it moves through the apparatus between preparation and measurement. So, one can relate the outcome of the experiment to a property which the particle had *before* measurement whereas, in the standard interpretation, it is the act of measurement itself which somehow causes the wave function of electron spin to collapse into a definite state.

The consistent histories approach is fully stochastic. The wave function becomes an instrumental tool used to calculate probabilities for certain events to occur in a stochastic dynamical process. Talk is not about the behaviour of the wave function, but about the probabilistic time-development of quantum properties.

A greatly expanded vocabulary of quantum properties and events can be discussed in CHQT compared to the standard interpretation (and it does not make use of hidden variables). According to Griffiths, it makes sense of the way experimental physicists normally discuss their work. Furthermore:

> … what is allowed covers what physicists need to talk about, and what is excluded includes things like the mysterious instantaneous nonlocal influences that have long been an embarrassment to quantum foundations and are often thought to be an impediment to connecting quantum theory with special relativity.[Griffiths, 2013: 102].

Griffiths [2013: 93] considers that CHQT provides a consistent realistic ontology for quantum mechanics.

6.3.5 The logical construction of frameworks

Griffiths' histories approach was given a rigorous logical interpretation by Omnès [1988]. The exposition in this section is based on [Griffiths, 2013].

Any quantum property of a system can be associated with the closed subspace ϕ of Hilbert space defined by all the state vectors which represent states having that property. For example, consider a system consisting of one electron and the property that "the spin in the z-direction, S_z, is equal to + ½ ". All the state vectors which represent states having that property form a closed subspace (in this case, a ray) in the two-dimensional Hilbert space of electron spin. So, in mathematical terms, when we talk about quantum properties we can translate it into talk about closed subspaces of a suitable Hilbert space.

Similarly, a quantum property can be associated with the operator P, called a 'projector'[77], which acts on the Hilbert space in such a way as to pick out the closed subspace ϕ of state vectors which represent states having that property; i.e.,

For any state vector |x> in the Hilbert space, P|x> = |x> if and only if |x> is in ϕ.

So, talk about the quantum property can be expressed, equivalently, in terms of the projector P.

In CHQT, the negation of a quantum property corresponding to projector P is a property corresponding to projector I – P, where I is the identity operator on the Hilbert space. The projector I – P picks out the orthogonal complement of the subspace ϕ. The union of ϕ and its orthogonal complement does not constitute the whole of Hilbert space and so there are state vectors which cannot be assigned either the property or its negation. In the electron spin example given above, there are state vectors for which the property "S_z = + ½" is neither true nor false. The property "S_z = + ½" is true for state vectors in the subspace ϕ given by a ray in the associated two-dimensional Hilbert space. Its negation is represented by the ray which is perpendicular to ϕ and corresponds to the property "S_z = - ½". For all other state vectors, the property is undefined. *It is at this point in the logical scheme that quantum physics comes apart from classical physics*. A classical property must either be true or false, it cannot be undefined.

The other logical tools needed for constructing sentences in the language of CHQT are conjunction, disjunction and implication. The conjunction P.AND.Q of two quantum properties corresponding to projectors P and Q is represented by the product PQ *provided that PQ = QP* (i.e., provided that P and Q commute). If PQ = QP, then PQ is itself a projector and it projects onto the intersection of the subspaces associated with P and Q. If PQ ≠ QP, then PQ is not a projector and P.AND.Q is undefined. In the electron spin example, if the property "S_z = + ½" (where S_z is the spin the z-direction) is represented by projector P and the property "S_x = + ½" (where S_x is the spin the x-direction) is represented by projector Q, then P.AND.Q is undefined. This can be justified by considering that the spin components in the z- and x-directions do not commute and, so, cannot be measured simultaneously. That is Bohr's complementarity principle. In a similar way, the disjunction P.OR.Q is represented by the projector P + Q – PQ, provided that PQ = QP, and is undefined otherwise.

[77] A projector is a Hermitian operator on the Hilbert space which is equal to its square, so its eigenvalues are 0 and 1.

The definition of implication will require a little more work as we will need to understand how a sample space and its associated probabilities can be constructed in quantum mechanics. A sample space S is a set of mutually exclusive properties whose probabilities p_j sum to 1. It is represented by a set of projectors $\{P_j\}$ which are mutually orthogonal ($P_j P_k = 0$ for $j \neq k$) and sum to the identity. S can be used to define probabilities for the event algebra represented by sums of the elementary projectors P_j (e.g., P_1.OR.P_2 is represented by $P_1 + P_2$ and has probability $p_1 + p_2$). Given two properties *a* and *b* in the event algebra, *a* implies *b* when the conditional probability for *b* and *a* is 1; i.e.,

Prob($b|a$) = Prob(*a*.AND.*b*) / Prob(*a*) = 1.

Note that *a*.AND.*b* is well-defined because *a* and *b* are represented by sums of mutually-orthogonal projectors P_j.

A sample space for a single time in a quantum stochastic process consists of a collection $\{P_j\}$ of mutually-orthogonal projectors which sum to the identity. Two frameworks given by sample spaces $\{P_j\}$ and $\{Q_k\}$ are said to be *compatible* if every P_j commutes with every Q_k. Otherwise they are *incompatible*. Compatible frameworks can be combined using a common refinement.

To construct a sample space for a system at successive times $t_1, t_2, \ldots t_k$ we take the tensor product of copies of its Hilbert space:

$H^h = H_1$ ⓟ H_2 ⓟ … H_k

where each H_j is a copy at time t_j of the Hilbert space H associated with the quantum system and the symbol ⓟ is used to denote the tensor product. H^h is called the "history Hilbert space". A *history* is then defined by a projector:

$P^{\alpha} = P^{\alpha 1}{}_1$ ⓟ $P^{\alpha 2}{}_2$ ⓟ … $P^{\alpha k}{}_k$

onto H^h where each $P^{\alpha i}{}_i$ is a projector from the sample space $\{P^{\alpha i}{}_i\}$ taken at time t_i. A sample space of history projectors can be used to define a framework of histories. Histories in a framework must obey certain consistency conditions, as discussed previously. They are mutually exclusive and have probabilities summing to 1. Analogously to the case for a single time step, two frameworks are compatible if and only if all their history projectors commute with one another.

6.3.6 The single framework rule

The single framework rule (the central tenet of CHQT) says that any description of a physical system should consist of properties belonging to a single framework and any reasoning about it should consist of valid implications. Descriptions from incompatible frameworks cannot be combined. There are always many possible frameworks to choose from and there is no common refinement which could be used to construct a unified description.

Griffiths [2013: 99] gives a simple example which is helpful in understanding this view of reality and its relationship to Bohr's complementarity principle. Consider an experiment in which an electron is prepared in a state with $S_x = + ½$ at time t_1 and then detected to be in the state $S_z = + ½$ at time t_2. What can one say about the spin state of the electron at the intermediate time? There is a frame containing the properties "$S_x = + ½$" and "$S_x = - ½$" and in this frame one can say that the electron is in the state $S_x = + ½$ at the intermediate time with probability 1. There is an alternative frame containing the properties "$S_z = + ½$" and "$S_z = - ½$" and in that frame one can say that the electron is in the state $S_z = + ½$ at the intermediate time with probability 1. The two frames cannot be combined because the projectors do not commute. The first frame is useful for discussing the preparation of the electron spin state and the second one is useful for discussing the outcome of the measurement. Compare this with the standard interpretation of quantum mechanics in which nothing can be said about the electron spin state at the intermediate time. In the standard interpretation, there is one unified reality but our knowledge of it is confined to the measurement outcomes.

The analysis gets more challenging when considering the behaviour of particles in complicated setups such as those used in quantum interference devices or delayed choice experiments [Griffiths, 2002: 273-282]. Then the consistent histories approach allows one to sort out which statements can be made meaningful in a framework and which statements are always meaningless (with the latter type of statement leading to paradoxes in the standard interpretation[78]).

It is important to note that CHQT is internally consistent. Frameworks may be incompatible but they are never contradictory:

> … as long as both frameworks contain all the events needed to express the initial data and also those needed for drawing conclusions, the probabilities linking data and conclusions will be

[78] For example, it is meaningless to ask which slit a photon passed through in Young's double-slit experiment when an interference pattern is observed.

identical even if the two frameworks in question are mutually incompatible. [Griffiths, 2014: §5.1].

It is also important to realise that all frameworks are equally valid. None can be picked out as giving the single true history of what happened. Alternative frameworks contain different information and, so, to pick out one of them and reject the others would be to throw away some information about the system. Nevertheless, a specific choice of framework must be made for problem solving and this is done based on considerations of utility. In Griffith's work, the choice of framework depends on what the experimenter wishes to investigate, or on what the theoretician wishes to analyse in a gedanken experiment. Different frameworks are applied as the focus of investigation changes. For Gell-Mann and Hartle, the aim is to explain the emergence of familiar features of our universe based on its initial quantum state and the laws of quantum physics, so they focus on quasiclassical realms. Still, Hartle cautions that:

> All of the totality of incompatible realms are necessary to give a complete account of the universe because they are, in principle, equally available for exhibiting regularities and constructing explanations. [Hartle, 2007: 3119].

6.3.7 Useful frameworks

In practice, some frameworks will be of little interest to human investigators. Every subspace of Hilbert space corresponds to some property of the system and it would be possible to construct histories based on ordered strings of arbitrary projectors which decohere but carry no meaning for us. Interest focuses on operators corresponding to the fundamental fields, with evolution given by fundamental laws [Gell-Mann and Hartle, 1990: 330]. To a certain extent, these naturally form into decoherent frameworks of quasiclassical variables due to objective processes. Because quasiclassical variables are conserved quantities (or, nearly-conserved quantities), they resist degradation due to interaction with the environment. Also, because the universe started in a highly ordered state and obeys the second law of thermodynamics, quasiclassical variables leak information into the environment and their histories decohere.

Not all decohering sets of histories are quasiclassical, but we focus on quasiclassical realms because they have the most predictability [Hartle, 2007b: 3101]. Human IGUSs have evolved to exploit predictability. We use internal mental models based on grossly coarse-grained input data to predict the future with the aim of maximising the probability of achieving our goals (principally, survival). We have developed sophisticated tools to improve our inbuilt prediction model. Language and communication are central to this. Our models are coordinated (both by evolution and by communication) to agree on features such as what variables and properties are to be tracked; what data should be labeled 'past', 'present' and 'future'; what consequences are to be drawn from data; and how all these features are to be represented in words.

Retrodicting past events helps us to detect patterns which improve our ability to predict future events, so reconstructing the past is an important activity for us. We construct a past for ourselves by using classical reasoning applied to a particular coarse-grained quasicclassical history which we have adapted to (a tiny subset of quantum reality). More rigorously, in CHQT, an event is said to have 'occurred' if its conditional probability given the present data is close to one. Probabilities are calculated by working forward in time from the initial state, constructing alternative histories which might contain the event of interest, and conditioning on present data. If an event has occurred in one framework, then it will have occurred in all frameworks which can describe it, because of the internal consistency of CHQT. Typically, in reconstructing the past we choose events of a type which are compatible with present data and which will help us in predicting the future. This is how we choose an appropriate framework.

The actual history which is realised in any framework depends on a sequence of random events. Sometimes these events leave explicit marks, as when the decay of a uranium nucleus leads to the production of fission tracks in mica deep in the earth [Gell-Mann and Hartle, 1990: 338]. Such marks help us in reconstructing the past. More generally, the specific structures which form in the universe are contingent (e.g. the Earth can only be formed in a system in which the sun has already formed) so conditioning on present data serves to prune the branches of the framework of alternative histories which can be constructed.

The probability of an event to have occurred is highly branch dependent. However, in general, this dependence is not the same as causality. Causality and determinism are features of quasiclassical histories associated with the emergence of quasiclassical variables approximately obeying the laws of classical physics. In principle, in classical physics one can use causality and determinism to retrodict a unique past from present data (and to predict a unique future).

It is only because the initial state of the universe was so highly ordered that we can distinguish the past from the future. This fundamental asymmetry leads to the Second Law of Thermodynamics, decoherence, and the direction of time and event dependence. These features are probabilistic and are not the same as causation. Typically, later events seem to depend on earlier events. However, in particular cases it is possible for earlier events to seem to depend on later events (e.g. in laboratory experiments set up to demonstrate this effect using beam-splitters and interferometers) but this should not be interpreted as later events *causing* earlier events [Griffiths, 2002: 196-201]. It is simply the result of time-dependent correlations.

In practice, because of our ignorance of present data[79], probabilities and coarse-graining are also needed to reconstruct the past in classical physics. The practical process of retrodiction in classical physics is much the same as described above for CHQT. When we say that Socrates died in 399 BCE it simply means that the conditional probability of this event having happened, given present data, is near unity. We cannot be certain of the accuracy of our historical records (or even of our present memories of events in our own lifetimes). The difference in CHQT is that probabilities and coarse-graining are fundamental and lead to incompatible frameworks. Then, in principle, the past is not unique. To retrodict past events in CHQT one needs present data, the initial state of the universe *and a chosen framework*. There is no underlying, fine-grained framework which can be used to reconstruct a unique past.

6.3.8 The world according to CHQT

Let us step back to consider the picture of reality given by CHQT. According to CHQT, quantum mechanics is universal — the same laws apply at the micro-scale and the macro-scale. Observers and measurement have no special role. Observers are IGUSs whose information-processing functions are carried out by physical systems subject to the laws of quantum mechanics. They present a puzzle only in so far as it is not yet understood how their consciousness emerges. Measurements can be fully explained in terms of the principles of CHQT[80]. Classical measurement is a special kind of quantum measurement. Similarly, the perceptions of IGUSs are a special case of quantum measurement. All phenomena can be described in the formal language of quantum mechanics without making reference to observers, consciousness or measurement.

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[79] And because of complications such as thermodynamic irreversibility and chaos.

[80] Omnès [1994] has applied the methods of quantum information theory to rigorously reconstruct measurement theory from the principles of CHQT — from quantum observables to macroscopic observables.

There is a reality independent of observers. It can be described in terms of frameworks which are sets of mutually-exclusive histories satisfying certain consistency conditions. Within a framework, normal reasoning (i.e. ordinary propositional logic) can be applied to combine statements about properties of the system into a meaningful narrative history subject to criteria of truth. In general, the Hilbert space of a quantum system contains sets of mutually-exclusive histories which do not satisfy the consistency conditions, cannot be assigned probabilities and, so, do not form frameworks. In CHQT these are considered to be meaningless. CHQT restricts which constructions of a Hilbert space can be talked about meaningfully — i.e. frameworks. The claim is that frameworks are adequate to describe reality (see e.g. [Gell-Mann and Hartle, 1990: 323-324] or [Griffiths, 2013: 112]). Sometimes the further claim is made that frameworks are the real content of quantum mechanics, the ontology [Griffiths, 2013: 93].

Time development in CHQT is fully stochastic. Only one of the possible histories in a framework actually occurs. Measurement is not needed to collapse the wave function into an actual state. It can be shown that macroscopic systems approximately follow classical determinism as a direct consequence of the underlying quantum mechanical laws. Decoherence, not measurement, is the key process in the emergence of classical reality from quantum reality.

For everyday purposes, we describe reality in terms of quasiclassical realms of quasiclassical variables. Such frameworks can be refined to get closer to ultimate reality; e.g., by increasing the number of the time steps or by refining the properties to describe more fundamental objects. If two frameworks have a common refinement of which they are coarse-grainings they are said to be compatible. Then, a description in one of the frameworks can be translated into a description in the other framework by using the linguistic resources of the common refinement. It is very important to note that there is no ultimate framework which underlies all of reality:

> *There is no (non-trivial) fine-grained, probabilistic description of the histories of a closed quantum-mechanical system.* [Gell-Mann and Hartle, 2007: 2]. Original italics.

In the context of quantum cosmology, coarse-graining is necessary for the construction of frameworks because it is decoherence which leads to consistency and decoherence requires coarse-graining.

There are many different coarse-grainings which result in maximally-refined frameworks and not all of them are compatible. When frameworks are incompatible their descriptions cannot be combined into a single viewpoint. Statements which are true in one framework may be meaningless in another (although they cannot be false because that would imply an inconsistency and CHQT is fully consistent). Events describable in one framework may not be describable in another. There is no unique past conditioned on given present data — incompatible frameworks can provide different stories of what happened. Hartle suggests that words such as "true" and "happened" should be used relative to a framework, just as "motion" is relative to a framework in special relativity [Hartle, 2007b: 3101].

The existence of multiple incompatible frameworks does not lead to any problems for predicting the outcome of experiments in CHQT, provided that reasoning is confined to a single framework. This is known as the 'Single Framework Rule'. Any framework can be chosen to describe a system but descriptions cannot be combined across frameworks. Griffiths [2002: 261-344] has shown that many of the paradoxes of quantum mechanics can be traced to violations of the single framework rule. In practice, the choice of framework for a system will be determined by what features we wish to investigate, at what resolution. What can be discussed will depend on the choice but, clearly, reality does not depend upon it. In effect, the single framework rule is a reconceptualisation of Bohr's complementarity principle for quantum mechanics.

It is an essential feature of CHQT that there is no unique description of reality and, therefore, no universal truth functional. This is very different from classical physics in which there is always a unique exhaustive description which underlies any possible coarse-graining of a system. Griffiths talks about the principle of *unicity* in classical physics:

> [Unicity] implies that every conceivable property of a particular physical system will be either true or false, since it either is or is not contained in, or implied by, the unique exhaustive description. [Griffiths, 2002: 363-364].

According to CHQT, unicity is not part of quantum reality.

The absence of unicity in CHQT — the fracturing of reality into multiple incompatible frameworks — is its single most challenging feature. It might lead one to search for an alternative, more palatable interpretation of quantum mechanics. However, one would find that all current interpretations challenge our intuitions in some major way. Either they incorporate the collapse of the wavefunction, which is problematic in a realist interpretation, or they make truth indexical (e.g.; in CHQT, truth is indexed by a framework; in Many Worlds, by a world; in Many Minds, by a mind). Whichever way you cut it, quantum mechanics is telling us something non-intuitive about the nature of reality. One approach might be to say that quantum mechanics is incomplete and that its interpretational problems will be resolved when constraints of some kind are added (e.g. hidden variables). This might be palatable except that there are no empirical drivers for adding constraints and, currently, no hidden variables theory can compete with standard quantum mechanics in describing the full range of physical phenomena (e.g. there is no relativistic version of Bohmian mechanics).

6.3.9 Combining into a consistent view

Together the work of Griffiths, Omnès, Gell-Mann and Hartle forms a consistent histories interpretation of quantum mechanics which applies over a vast range of physical phenomena. In Griffith’s work, meaningful propositions about a quantum system are extracted from the logical structure of the associated Hilbert space and combined time-sequentially using the dynamical laws to construct a complete set of mutually-exclusive narrative histories describing the evolution of properties of interest. Omnès uses the tools of logic and advanced semiclassical physics to show how this detailed approach at the microscopic level can be scaled up to apply to quasiclassical variables and their environment, thus demonstrating the emergence of decohered histories at the macroscopic level and providing a plausible explanation for our experience of classical reality. The reconstruction of the classical world from the quantum world is far from rigorous because the collections of variables that will emerge as quasiclassical cannot be identified fundamentally in the Hilbert space and one has to make a lot of assumptions about what the right quasiclassical variables are. Nevertheless, the combination of histories, logic, decoherence and advanced semiclassical physics provides links all the way from systems of elementary particles in the laboratory to the closed system of the universe itself, which is the focus of Gell-Mann and Hartle’s work.

One of the great strengths of CHQT is that the content of the histories is very flexible — degrees of freedom can be integrated over or inserted as required. In fact, to a large extent the formalism floats free of any underlying ontology because it is expressed as a logical framework of propositions linked by conditional probabilities. Hartle [2007c] suggests how the generalised framework of CHQT could be applied to gauge field theory, general relativity and quantum gravity. He envisages pushing the envelope of explanation back to scales below the Planck length where even the geometry of spacetime is in a quantum superposition and the operators defining the fixed background spacetime of quantum mechanics emerge as quasiclassical.

In the next section I compare the whole world picture provided by CHQT with the Leibnizian view presented in Section 6.2. I attempt to show that Leibniz (in spite of his ignorance of the quantum nature of reality) had fundamental insights which are still relevant and which can be combined with CHQT in a useful way.

6.4 Leibniz's Philosophy in the Light of Modern Science

6.4.1 General comparison

6.4.1.1 Physics

Leibniz's metaphysical approach, informed by his insight into mathematics, led him to focus on questions which are still the subject of important debate in science. More surprisingly, his answers are still relevant in many cases. I take this as evidence of the unreasonable effectiveness of metaphysics and mathematics. I will highlight some examples in this section before moving to a more detailed comparison of his philosophy with CHQT in the Section 6.4.2.

Leibniz shows prescience in his explanations of familiar phenomena in terms of the relationality of observers. For example, regarding space, he says:

> … we call a "body" whatever is perceived in a consistent way, and say that "space" is that which brings it about that several perceptions cohere with each other at the same time. (Leibniz as quoted in [Garber, 2009: 277-278]).

According to Leibniz, space is an ordering relationship among monadic perceptions. So is time. So is motion. Their reality is only in the agreement of the appearances of monads. This approach foreshadows the observer-oriented physics of our current theories of relativity. It led Leibniz to many keen insights:

(i) the relativity of space, time and motion;

(ii) the non-substantiality of spatial extension;

(iii) the notion of topological space — i.e. a space of real points without a metric but with a relation of closeness (such as that determined by the closeness relation between monadic points of view); and

(iv) the absence of void.

Leibniz argues for the absence of void from the way in which relations in monadic reality are expressed in the world of phenomena. In monadic reality, each monad mirrors the whole universe so, in the world of matter, each material body "feels the effects of everything that happens in the universe, so that he who sees everything could read off from each body what is happening everywhere" [Monadology, §61]. This is achieved by the world of matter being a plenum in which the effects of motion of any part are propagated to all parts to some extent. Leibniz's description of the plenum foreshadows the concept of space-filling fields in physics. His plenum is like a classical field because every part can be divided indefinitely [Monadology, §65]. Nevertheless, since monadic reality is grounded in individual substances, it is clear that Leibniz is acutely aware of the dichotomy between the discrete and the continuous — the particle-and-field-like nature of reality — which is so central to quantum field theories. This is not to say that Leibniz has any inkling of the quantum world. On the contrary, he adheres to the classical view that "nature doesn't make jumps". A principle of continuity governs his monadic world — from the continuous changes of state of the monads to their continuous ordering in terms of perfection or points of view — and this is expressed in the continuous nature of phenomena such as space and time. However, a principle of discreteness is also central to his philosophy because he considers that individual substances — genuine unities — are the only true existents. Furthermore, this is a truth of reason whereas continuity is a truth of fact.

One reason why Leibniz believes that nature doesn't make jumps is because otherwise it would behave probabilistically rather than following a deterministic program. Of course, that is just what quantum mechanics tells us about nature, but it is a truth of fact which came as a complete surprise to the 20th century scientific community and was not even contemplated in Leibniz's day. Leibniz did think probabilistically about the world but, in his philosophy, all the probabilistic action takes place in God's mind before creation. Once God has chosen the laws of nature according to the Principle of Perfection, everything unfolds deterministically.

According to Rescher:

> Leibniz, more than any other modern philosopher, took seriously the idea of a *creation* of the universe, giving it a centrally important place in his system. [Rescher, 1967: 11].

Modern philosophers may not pay much attention to creation, and Leibniz's focus on God as the author of the universe goes beyond any scientific remit, but modern physicists have been driven to considering the origins of the universe in their attempts to unify the forces and to understand the implications of

General Relativity. The questions asked by physicists are the same as those addressed by Leibniz. How does actuality emerge from potentiality? How does variety emerge from simplicity? Which laws are universal and which are particular to our world? Is there some optimisation principle which acts to choose our world out of all the possible worlds? Some of these questions were discussed in previous chapters. In the next section I will compare the answers given by Leibniz with those given by consistent histories quantum physicists.

Leibniz's sympathy with modern physics is further evidenced by his clever use of symmetry arguments. In his correspondence with Clark, he uses symmetry arguments to support his thesis that space and time are relative, rather than absolute as in Newtonian physics (N.B. this argument was discussed in Section 3.2.1). In the Monadology, he uses symmetry arguments to support his thesis that monads have some properties, although they have no parts. He says:

> If monads all had the same qualities, they would be indistinguishable from one another (given that they don't differ in any quantitative way, e.g. in size). That would make all composite things •such as portions of matter• indistinguishable from one another also, because whatever is the case about a composite thing has to come from its simple ingredients… So, assuming a plenum and no qualitative variety, any moving around of matter would only result in each place containing something exactly like what it had contained previously, so that one state of things would be indistinguishable from another. [Monadology, §8].

This is similar to the argument given at the end of Section 3.2.3 when discussing symmetry in quantum field theory: "A system with perfect symmetry would be invariant under any and all transformations and would have no discernible structure". In quantum field theory, the properties of particles arise dynamically as a result of symmetry breaking as the universe cools after the Big Bang. They are defined relative to the stable ground state which emerges when the underlying symmetry is broken. Intrinsic properties follow from the underlying symmetry of the laws. For example, the intrinsic properties of fundamental particles that would result from a single representation of the SO(10) symmetry group is shown in Table 6.1 below:

Quantum number/ Particle	R	W	B	G	P
u	+	-	-	+	-
u	-	+	-	+	-
u	-	-	+	+	-
d	+	-	-	-	+
d	-	+	-	-	+
d	-	-	+	-	+
u^c	-	+	+	-	-
u^c	+	-	+	-	-
u^c	+	+	-	-	-
d^c	-	+	+	+	+
d^c	+	-	+	+	+
d^c	+	+	-	+	+
ν	+	+	+	+	-
e	+	+	+	-	+
e^c	-	-	-	+	+
N	-	-	-	-	-

Table 6.4.1: Intrinsic properties of particles associated with SO(10). Taken from [Wilczek,2005: 245].

The properties in the table are labeled R, W, B, G, and P for red, white, blue, green and purple, but the colour terminology has no purpose other than differentiation. We could envisage a similar table of monads, with the columns labeled by a basis of independent properties. However, we would need to be mindful of some notable disanalogies.

As already noted, monadic properties are determined at creation whereas quantum properties arise dynamically. More importantly, there are an infinity of monads and monadic properties whereas only a finite number of particles and properties are associated with any group representation. Unlike monads, quantum particles are not individuals. There are identical quantum particles which can be interchanged with no change to the world. Of course, Leibniz would consider quantum particles to be phenomena so this does not contradict his assertion of the individuality of monads. Quantum physics has nothing which directly corresponds to monads. It is a theory of the relations between physical phenomena.

The existence of an infinity of monads follows from the Principle of Plenitude which in turn is a consequence of Perfection. Plenitude implies that everything possible consistent with the laws is actualised. It is a principle which does seem to be shared by quantum physics. In quantum physics, it means that when certain symmetries are satisfied by the laws, all the particles implied by those symmetries exist (e.g. the Omega minus particle as hypothesised by Gell-Mann following SU(3) symmetry, and subsequently discovered; see Section 3.3.2).

6.4.1.2 Mathematics, logic and biology

According to Leibniz, the infinity exemplified by monads is the actual infinity which surpasses all numerical bounds. It is a sign of God's authorship. It is not something that can be comprehended by finite minds or defined in human mathematics. Because Leibniz understood the formal manipulation of symbols in mathematics, he would have been able to countenance Cantor's hierarchy of infinities (though he would have considered them to be fictions). Cantor's work does not contradict the central insight of Leibniz's actual infinity which is that there is something beyond mathematics which is needed for its complete definition. Cantor called it 'absolute infinity' and, like Leibniz, associated it with God.

Leibniz's appreciation of symbols and their manipulation is another prescient aspect of his philosophy. His separation of mathematical understanding from mathematical content foreshadows the formalisation of modern mathematics, logic and computing. He was able to envisage a universal language of thought in which signs representing our most basic concepts could be manipulated logically so as to create all possible propositions comprehensible by the human mind and to determine their truth or falsehood. In this way our finite minds would achieve the greatest possible understanding of the mind of God which is the source of all truths and all possibilities insofar as they are real.

For Leibniz, propositions represent reality, or reality instantiates propositions… it is not always clear which. Garber thinks that Leibniz never came to a final position on realism versus idealism. He writes:

> Certain metaphysical arguments convinced Leibniz that, at root, simple substances, monads, had to be at the bottom of everything. What he hadn't fully figured out, though, is how exactly bodies are to be grounded in the world of monads… I don't think that he ever arrived at an answer that fully satisfied him. [Garber, 2009: 384].

Leibniz is not entirely convincing in his description of a monadic hierarchy in nature. In his defense, there are many facts about biology which were not known at that time — including things as fundamental as the theory of evolution and the existence of DNA — and this hampers his efforts.

According to Leibniz, every living body has a dominant monad (which in an animal is its *soul*) and the parts of that living body are full of other living things, each of which has its own dominant monad [Monadology, §70]. The souls of spermatic animals are raised, through an act of conception, to the level of *reason* and to the privileges of *spirits* [Monadology, §82]. Souls in general are living mirrors of the universe of created things, but spirits are also images of God himself; in that they are capable of knowing the system of the universe, and of imitating aspects of it through sketchy constructions of their own [Monadology, §83].

Leibniz's main concern here is to explain how aggregations of monads can form a single body — what does the unity of that body consist in? His answer is that the dominant monad provides unity by having clearer perceptions of the other monads in the aggregate than they have of it. This means that their functions are subordinate; e.g., the essential organs serve the needs of the animal, the kidney cells serve the needs of the kidney, and so on down the line in the never-ending fractal hierarchy of existence. Another concern is to explain the role of animals and, in particular, the very special role of human beings, in the hierarchy. This is done by classifying monads as bare monads, souls or spirits according to the degree of clarity and awareness they have of their own perceptions [Rescher, 1967: 118-119]. Bare monads might form an aggregate such as a rock, which would derive its unity from certain correlations between the perceptions of the bare monads themselves and from its being perceived as one unified object by a spirit. Animals derive their unity from their dominant monad which is a soul having awareness. Human beings derive their unity, and their special role at the apex of the hierarchy, from their dominant monad which is a spirit having awareness and reason. Humans have finite minds which mimic in some much lesser way the constructions of the infinite mind of God. For example, they might confusedly strive for understanding of the way in which physical structure emerges from mathematical structure; but there is a deeper reality in which God chooses the actual world from all possibilities so as to maximise perfection.

Things get complicated when considering the great variety of life forms. Nowadays one might say that animals derive their unity from a program, encoded in DNA, which is in every cell of their bodies. Being generous to Leibniz, this is not too dissimilar from his main concept. One might even see embryonic notions of "the survival of the fittest" in Leibniz's depiction of the struggle of possible entities for existence. These examples illustrate the power of his metaphysical reasoning in discerning essential features of the world, even when deprived of detailed scientific facts. However, even being generous, one cannot paper over the cracks revealed in Leibniz's philosophy by the facts which we now know about the quantum world.

6.4.2 Comparison with CHQT

6.4.2.1 Properties

In Leibniz, the structure of the world is built up from the perceptions of monads so, in principle, anything one can sensibly say about the world can be expressed in terms of monadic properties. One could envisage a table like Table 6.4.1 listing the Complete Individual Concept of each monad. The columns would be labeled by a basis of independent simple properties and the description of a monad would be given by a sequence of yes/no questions of the form "Does Monad X have property Y at time t?". For a complete description, we would have to add columns for the relations of the monad to each of the other monads (i.e. its perceptions of other monads) and, also, for its properties and relations at all other times. The CIC of any monad ends up being a complete description of the whole universe from a particular point of view. The monad itself has only imperfect knowledge of its CIC. God has perfect knowledge and can combine the CICs of all monads to get a complete picture of the whole universe from an infinity of viewpoints.

Any proposition about the world can be reduced to simple propositions about monadic properties and is true just in case the monads have the purported properties. For example, "Monad X has property Y at time t" is true just in case Monad X has property Y at time t. Otherwise it is false. There are no undecided truth values in Leibniz's philosophy.

Quantum theory is a mathematical and logical structure which represents the structure of the world and so, in principle, anything one can sensibly say about the world can be expressed in terms of quantum properties (where a quantum property is represented by a projector onto a subspace of a suitable Hilbert space). In contrast to the classical world, if a quantum proposition is false it does not mean that its negation is true. A quantum proposition can be undefined. For example, in the two-dimensional Hilbert space associated with the spin of an electron, if the proposition "$S_z = + ½$" is false we cannot conclude "$S_z = - ½$" because the system may be in a superposition for which "$S_z = - ½$" is undefined. However, CHQT defines frameworks within which the usual rules of logic can be applied; e.g., within a framework, if a proposition is false then its negation is true. In CHQT, alternatives at any time can always be reduced to a set of yes/no questions (represented by the two eigenvalues, one and zero, of the associated projection operators). CHQT is fully stochastic and only gives the probability of mutually exclusive alternatives, although sometimes a conditional probability can have value 1, in which case the underlying proposition constitutes a reliable fact. In general, propositions can be thought of as "possibly actual" properties and the theory gives their probability of being instantiated (independently of whether they can be measured or observed).

In CHQT, the ability to describe a system in a particular framework by combining quantum properties according to the usual rules of logic is acquired at the expense of being able to combine the frameworks into a single, unified picture of the world. Reality splinters into complementary frameworks. Criteria of truth can be applied in single framework but there is no universal truth functional. A proposition which is true in one framework cannot be false in another framework but it may be undefinable.

6.4.2.2 Laws

According to Leibniz, God chooses the states of the world so as to maximise perfection. Also, from a relative point of view, each individual strives to maximise the perfection of its own state so as to achieve the highest degree of reality. The laws of nature act in harmony with the strivings of individuals to give the best possible world. The states are prior to the laws. The chosen states and the form of the laws are both contingent on the Principle of Perfection.

In modern physics it is more usual to think of the laws as having priority. They are seen as being the transcendent, eternal and immutable mathematical forms expressed in the equations of the Theory of Everything. They act on any given state of the world to determine its evolution. All the variety of the world comes from the laws and the initial state. Even in the multiverse view, there is a transcendental Theory of Everything which spawns universes; then each universe has its own effective laws generated by spontaneous symmetry breaking as it cools. In the multiverse view, the states of our world co-exist with the states of an infinite number of other worlds and ours just happen to include intelligent life.

Wheeler questions the paradigm of a Theory of Everything. He argues:

> In the domain in which each law is applicable that law states itself most completely as the consequence of a symmetry; but that symmetry hides any view of the next layer of structure. [Wheeler, 1996: 45].

So, by their very nature, dynamical laws cannot be fundamental. As we saw in Chapter 3, physics in the 20^{th} century advanced by progressively removing fixed structure, revealing deeper layers of symmetry and new laws. According to effective field theory, the laws applicable at any given scale are approximate and their deficiencies can only be understood by going to the next level. Wheeler thought that laws might emerge with the states of the world, as the result of random processes, much like the Second Law of Thermodynamics. He called this “law without law” [Wheeler, 1980]. After all, many physicists believe that space and time emerge, together with their symmetries and conservation laws. Perhaps the laws and the states emerge together and act on one another, with neither having priority.

CHQT is compatible with all these views but it is naturally described in terms of states. Systems are described by networks of alternative states linked by conditional probabilities. Laws are used to calculate conditional probabilities for a given structure. Schroedinger’s equation can be used to generate unitary states but, according to Griffiths, these should be treated (in general) as mathematical objects for calculating probabilities, rather than as physical properties of the system. In Gell-Mann and Hartle, useful histories are constructed by conditioning on current known structure (which includes human IGUSs). CHQT is not inconsistent with a view in which the probabilities/laws adjust themselves to produce an optimal structure (although a new optimising principle would be required, along the lines of Leibniz’s ‘perfection’). The particular form of the laws is not crucial to a generalised CHQT: what are needed are fine-grained histories and mechanisms for coarse-graining and decoherence. Space, time, laws of quantum dynamics, causality and Newtonian laws can all emerge from structure at a deeper scale.

6.4.2.3 Possibility

In both Leibniz’s philosophy and CHQT the actual world emerges from a realm of possibilities. In Leibniz, God is the ground of meaning and of what exists and what is possible. All possible monads subsist as concepts in his mind. They correspond to all possible combinations of a basis of simple concepts. The rules of combination are the laws of thought — the laws of classical logic. God is subject to these so he cannot think a contradiction. Possible monads cannot have contradictory properties and two monads cannot co-exist if analysis of their complete individual concepts leads to a contradiction. The manifold of possible monads splits into sets of compossible monads; i.e., possible worlds. Furthermore, in order to be compossible, monads must be adjusted to one another to the extent that each monad in a possible world mirrors all the others. This is the end point of the process of differentiating an individual: the complete description becomes a list of the properties of the individual and its relations to all other monads and, ultimately, of everything. Each monad represents the whole world from its own point of view and its complete individual concept is, in effect, a consistent history of the state of the world for all times.

Possible monads vie for existence as individuals but God's choice of which ones to actualise reduces to a choice between mutually exclusive possible worlds. He chooses the best one, according to the criterion of perfection. All choice is restricted to the act of creation. Subsequently, the states of the world play out like a program — appearing, in the world of phenomena, to be subject to laws.

In Leibniz, the principle of contradiction determines what can exist. Compossible monads cannot have contradictory properties. In quantum mechanics, contradictory properties can co-exist as superpositions. There is a realm of quantum potentiality in which Schroedinger's cat can be simultaneously dead and alive. However, there is no actual state in which a cat can be both dead and alive.

If one thinks of quantum mechanics as way for incompatible possibilities to coexist, then CHQT is a method for sifting out the 'possible actuals' (i.e. the set of possibilities which can be actualised) from the realm of quantum potentiality. Consistent histories are mutually exclusive, ordered sequences of quantum properties which can be assigned a probability for actualisation.

The price paid for being able to split out the possible actuals is that reality itself is split into incompatible frameworks which cannot be combined to give a single, true view. Incompatible properties, represented mathematically by non-commuting projectors, exist in alternative frameworks and cannot be measured simultaneously. Trying to combine quantum properties across alternative frameworks can lead to inconsistencies. However, as long as one remains within a single framework, quantum properties can be combined according to the usual rules of logic to create possible actual histories.

Which history actually occurs is completely random. Thus, each actualisation eliminates all histories in the framework save one. The actual world evolves as a stochastic process — it is not optimised and its evolution cannot be predicted.

6.4.2.4 Conceptual frameworks

We can only describe a possible world by using concepts that are available to us. In CHQT, there is a nested hierarchy of conceptual schemes which defines what is possible. The most general scheme is a notional one in which laws and properties evolve together. I will address this in a later section (see

Section 6.5). The issue is that, without God to give a ground to meaning, properties cannot be defined independently of laws and we must find some other way to avoid the “chicken and egg” question of the priority of states *versus* laws. For the moment, imagine a very general conceptual scheme which applies below the Planck scale and from which relevant concepts such as space and time emerge to set the fixed background for quantum mechanics. The conceptual scheme used by Gell-Mann and Hartle is derived from this. It assumes a closed quantum system in which gross quantum fluctuations in the geometry of spacetime can be neglected:

> The closed system can then be thought of as a large (say >~20,000 Mpc), perhaps expanding box of particles and fields in a fixed background spacetime. [Gell-Mann and Hartle, 2014: 052125].

Within this conceptual scheme, there are infinitely many completely fine-grained sets of histories, specified by giving the values of a complete set of operators at all times. One such set is given by specifying all field variables at all points of space at every time, as in Feynman’s sum over histories approach. Another set is given by specifying all field momenta. There are also degenerate sets in which the same complete set of operators occurs at every time [Gell-Mann and Hartle, 1989: 307]. A complete set of operators defines the orthonormal basis of properties from which states can be constructed: it defines a Hilbert space at a given time. A completely fine-grained set of histories defines a History Hilbert space (see Section 6.3.5).

To do calculations we must choose a particular History Hilbert space; say, one applicable to Feynman’s sum over histories. Then, in general, the histories will interfere and the quantum properties will not be combinable using classical logic. In order to extract possible actuals we have to further restrict the scheme to consistent sets defined by a decoherence condition. Non-trivial consistent sets are always a coarse-graining of the underlying completely fine-grained set. In principle, given the initial state, it is an algebraic exercise to find all the consistent sets and isolate the maximal sets (i.e. those for which there is no refinement which decoheres). The maximal sets define the most general conceptual scheme for possible actuals in CHQT. Only a tiny subset of these consistent histories will be of interest. The majority will be composed of quantum properties which correspond to complicated combinations of underlying field variables from disconnected regions of space. This is loosely analogous, in the classical case, to considering the properties of arbitrary mereological sums such as my right thumb and the Statue of Liberty. We really want to focus on consistent histories which tell a meaningful story about properties of interest — we want to coarse-grain maximal sets into useful ‘narrative’ frameworks.

According to Gell-Mann and Hartle, it may be possible to objectively define quasi-classical frameworks using some sort of classicity measure; however, this is yet to be conclusively demonstrated. They speculate that:

> It would be a striking and deeply important fact of the universe if, among its maximal sets of decohering histories, there were one roughly equivalent group with much higher classicities than all the others. That would then be *the* quasiclassical domain, completely independent of any subjective criterion, and realised within quantum mechanics by utilising only the initial condition of the universe and the Hamiltonian of the elementary particles. [Gell-Mann and Hartle, 1989: 321].

The consistent histories in such a quasiclassical framework would contain all possible choices that might be made by all possible observers that might exist, now, in the past, or in the future for that framework.

As it is, useful narrative frameworks are defined by observers rather than by an objective mathematical procedure. The most important quasiclassical framework for human beings is the one which we call “the real world” and describe in natural language. Other useful narrative frameworks (not necessarily quasi-classical) have been defined by physicists to describe particular experimental setups or thought experiments. In such cases, the properties of interest are determined by considering the whole experimental arrangement in context.

Griffiths’ work is primarily directed towards constructing narrative frameworks for quantum information theory. In answer to the question “Information about *what*?”, he says:

> … the information is about *quantum properties*, represented mathematically as subspaces of, or the corresponding projectors on ,the quantum Hilbert space. More generally, information can be about a time sequence of such properties, a quantum history. [Griffiths, 2013: 109].

So, for Griffiths, to exist is to be describable by a consistent history which is actualised. The ontology is identified with elements of the conceptual scheme used to set up the Hilbert space in the first place; i.e., a single electron, in the simple example given in Section 6.3.6 (or, an expanding box of quantum particles, in Gell-Mann and Hartle’s work). CHQT is concerned with what can be known about an electron, what can be said about it, and how its properties and behaviour can be described. It provides a linguistic framework giving syntactical rules governing how logical expressions combining quantum properties can be formed in a meaningful way. The assumption is that meaningful expressions must refer to possible actuals and obey the standard rules of reasoning. In this way, a link with natural language is preserved and intuition is facilitated.

According to Carnap:

> If someone wishes to speak in his language about a new kind of entities, he has to introduce a system of new ways of speaking, subject to new rules; we shall call this procedure the construction of a linguistic framework for the new entities in question. [Carnap, 1956: 206].

Griffiths has introduced a linguistic framework for discussing possible actuals in quantum mechanics. He emphasises that, when analysing and describing a quantum system, there are always many alternative frameworks which are equally good from a fundamental point of view. He advocates 'liberty' for the physicist in constructing different, perhaps incompatible, frameworks. 'Utility' is the criterion suggested for choosing frameworks: "… it is important to avoid thinking that the physicist's choice of framework somehow influences reality… quantum reality allows a variety of alternative descriptions, useful for different purposes" [Griffiths, 2014a: 618]. This is all very much in the spirit of Carnap. The unusual feature of linguistic frameworks in CHQT is that they can be incompatible; in which case, they cannot be combined to give a single view.

According to my understanding, the question of ontology is an external question. Any framework which we choose to describe the world could be translated into a quantum framework in generalised CHQT. A quantum framework could be introduced to describe Newtonian physics, gauge field theory, general relativity, or (hypothetically) quantum gravity. The entities describable in such a framework would depend on the completely fine-grained conceptual scheme and on the level of refinement of the consistent histories. Since the completely fine-grained scheme of the world does not decohere, information is sacrificed in constructing consistent frameworks. In practice, information leaks into the environment and this is what makes quantum systems decohere. The implication is that there is more to quantum reality than can be described in a language of possible actuals. We need the fine-grained scheme in order to extract consistent histories. It may nevertheless be true, as Hartle claims, that:

> The probabilities of the histories of the possible *decoherent* sets of coarse-grained histories and the conditional probabilities constructed from them are the totality of predictions of the quantum mechanics of a closed system given the Hamiltonian and initial state. [Hartle, 2007b: 3118].

This is an epistemological claim: consistent frameworks are what we can reason about.

6.4.2.5 Language, thought and reality

In Leibniz's philosophy, thought and reality are matched. The logical construction of thought (and language) from simple concepts reflects the way in which the structure of the world is logically constructed from simple entities. In quantum mechanics, there is a fundamental layer of reality which is not matched to thought; it obeys quantum logic. So the logical way we combine concepts in our thinking does not reflect the logical structure of ultimate reality but, rather, the structure of some quasiclassical realm which we inhabit and have adapted to. Hartle [2007b: 3112-3113] has hypothesised beings which think in a different way and latch on to different aspects of reality.

Quantum reality can be described using mathematical language — e.g., mathematics can represent incompatible properties using non-commuting operators — so it is not inaccessible to human thought. It is merely unintuitive. It cannot be mapped directly onto our experience of the world and we struggle to assign meaning to some aspects of it. A lot of the mathematical structure of quantum mechanics is uninterpreted, and referred to as "surplus", but it seems to be essential for describing that unknown out of which meaningful reality emerges.

Leibniz was able to envisage a language of thought which would form the basis of all human science. To a certain extent, his dream has been realised with the invention of computers and computing language. Computers don't need to understand the meaning of language; they just follow rules of syntax to construct well-formed formulas and use rules of inference to derive consequences. They are quantum systems but they use classical logic. They are modeled on the human brain, which is quantum mechanical but reasons classically. Like us, they are adapted to operating in a single quasiclassical framework.

Nowadays, there is the exciting prospect of quantum computers. Quantum computers will employ histories in a framework which is not quasiclassical in order to speed up computation. Nevertheless, their outputs will still be recorded in the single quasiclassical framework which we inhabit, so any information which we extract from them could have been obtained using a classical computer. In fact, as far as we know, any physical process can be simulated by a universal Turing machine. According to Gerard 't Hooft [2012], there is even a cellular automaton interpretation of superstring theory via a discretisation on a space-time lattice. Of course, if our universe were actually a cellular automaton (rather than merely being approximated by one) it would mean that there are no truly random processes because random processes can be used to generate sequences which are not Turing-computable. This would contravene our current understanding of quantum mechanics. Nevertheless, it cannot be ruled out based on available experimental evidence.

There is a reality independent of observers but what we can know about it, and how we think about it, depends on our biological, intellectual and cultural context. According to Hartle:

> The general point is that notions of reality reside in the models (schemata) that IGUSes construct of the world around them, both individually and collectively. Different models have different notions of reality. Therefore, when using the words 'real' or 'reality' it is important for clarity to specify which model is being referred to... Everyday notions of physical reality arise from the agreement among human IGUSes, both individually and collectively, on their observations and on the models of the world (schemata) that they infer from them. These models are formed from the gathered data by processes of selection, communication and schematisation, consistent with built-in biases. The models are constantly updated as the IGUSes acquire new information, integrate it with previous experience, infer new useful regularities, and check the model against other schemata. The everyday notions of physical reality reside in these models. [Hartle, 2007b: 3110].

We interpret evidence, retrodict the past, and predict the future, based on our models of reality. There are many alternative narratives of the past which are consistent with current evidence and our reconstructions can only ever be assigned a conditional probability, not certainty. This is true even for classical models of reality because we don't have all the facts. In CHQT, there is the added complication of incompatible frameworks. Yet, our reconstructions are not arbitrary; they are conditioned on an external reality. If a fact is recorded in one framework, and it is describable in another framework, then it must still be true that framework. There is one world which can be interpreted in many frameworks; not many worlds. It is the constraint of external reality which makes some models — some concepts — more useful than others. We have evolved to think in terms of these useful concepts.

According to Putnam [1980], we cannot separate a model from the language which describes it: the two are created in tandem. The conceptual scheme which determines how we divide up the world into objects and properties at the same time determines how we describe it in terms of subjects and predicates. There isn't a "correct" model independent of language; there are just different conceptual schemes. I am going to suggest that concepts, language, meaning and laws emerge together from quantum reality; with mind as the fundamental principle which operates on potentiality to create possible actuals. Leibniz's philosophy has a phenomenological interpretation in which only minds exist — bodies are the artefacts of coherent monadic perceptions. I adopt this approach with the added assumption that minds have quantum properties. I hypothesise that minds are subject to constraints imposed by the nature of their perceptions and their operations which restrict the way in which they can construct a meaningful narrative framework. Then, our models of reality are the way they are because of the way that minds think. Before attempting this, however, I will consider the view of Gell-Mann and Hartle: that we think the way we do because we are adapted to a single quasiclassical realm.

6.4.2.6 The role of observers

The formulation of the standard interpretation of quantum mechanics requires an observer outside the system. In CHQT, the claim is that there is no fundamental division into observer and observed. An observer is just another physical system — an IGUS — and thinking is an information-processing function which supervenes on its physical states. The fundamental frameworks of CHQT — the maximal sets — are objective so that all possible observers would agree on the nature of fundamental reality. Observers are not needed to determine the possible actuals. A measurement situation occurs when there is correlation between the ranges of values of operators of a quasiclassical domain. It happens as the result of decoherence and does not need an observer.

Yet, observers are very important in CHQT. They assign meaning to possible actuals. A measurement situation needs an observer in order to become a measurement of something meaningful. Consider the identification of quasiclassical frameworks. Quasiclassical variables are collections of quantum properties which persist in time and show a certain predictability. If this were all, then there would be some chance of identifying them objectively; i.e., by looking for aggregates of conserved quantities (as outlined by Hartle, see Section 6.3.3). We could imagine some measure of classicity which would allow us to find all the candidate quasiclassical frameworks from first principles. But most of these would not correspond to standard quasiclassical descriptions of the world. Quasiclassical variables are highly contingent; they depend on the epoch, the spatial region and on frozen accidents of history. Specific information about our history is needed to select a standard quasiclassical framework. More than this, what is needed is a subjective criterion of utility to us. A standard quasiclassical framework is one which has explanatory power and predictive reliability for us. The quasiclassical variables must have meaning for us. Even the choice of focusing on decoherent histories seems to be skewed towards our notions of what existents should be like: we choose decoherence as a criterion so that we end up with properties that we can reason about intuitively. Information is lost to us through choosing a system/environment split.

If we consider observers other than human observers then a framework will be chosen by an IGUS of some sort according to subjective criteria. The selection of a framework doesn't change reality – it is a particular description of part of reality and an observer is needed to decide what is useful to describe. Quantum reality without observers is meaningless; i.e., empty of meaning.

Can IGUSs perform the role of observer as required by CHQT? To answer this question it will first be necessary to consider what an IGUS is and how it is to be modeled in the formalism. This will open up all sorts of fundamental philosophical problems … What is consciousness and what exists before consciousness emerges? How is an IGUS to be identified? What accounts for its identity and persistence over time? How do IGUSs communicate? These problems are not peculiar to CHQT but it has added complications due to multiple frameworks. Difficult cases of personal identity can be imagined involving different branchings and recombinations of the IGUS operators. For a discussion of these problems see [Dowker and Kent, 1996].

An IGUS is a physical system. Its perceptions and thoughts supervene on its physical states. Its behaviour is determined by boundary conditions and probability rules — it does not have free will. In the formalism, it is identified with a collection of properties. There is a question over where to draw the boundary; where to make the cut between the IGUS and its environment. There is the possibility of multiple copies being projected into the future. What makes an IGUS an individual and distinguishes it from nearby collections of properties in Hilbert space? According to Dowker and Kent [1996: 1625] we need a theory of experience to show that an IGUS and its experiences exist in the same consistent set. Operators that allow us to predict our own future experiences must be quasiclassical descriptions of the brain in operation. The link between our experiences and the subset of quasiclassical operators which are most convenient for describing the brain's logical functioning is not obvious.

How do we understand the agreement among human IGUSs on the facts of their physical reality in a quantum universe characterised fundamentally by the distributed probabilities of the alternative histories of a vast number of incompatible realms? According to Hartle:

> The simplest explanation is that human IGUSes are all making observations utilising coarse-grainings of the quasiclassical realms in order to exploit the quasiclassical regularities that these

> realms exhibit. They thus are adapted to develop schemata in more or less the same way. [Hartle, 2007b: 3110].

In other words, evolutionary pressures have led us to develop shared, coherent models of reality which give us the illusion of a unique past and alternative futures projecting its regularities (an approximation good enough for our purposes).

Dowker and Kent [1996: 1626-1633] question this explanation. They point out that there is no clear agreement on how the actual facts needed for reconstructing the past should be selected. If we only accept our own personal perceptions at a given moment of time then we are led to “solipsism of the moment”. If we accept other evidence (e.g. fossils, historical records, other people’s testimony) then we are in the position of building our model of reality based on those facts. If history is constructed based on the experiences of individual IGUSs then we end up with parallel histories of multiple IGUSs and no correlation between their experiences:

> … each may well believe itself in communication with others, but the others may be experiencing a quite different history, or nothing at all. [Dowker and Kent, 1996: 1626].

Dowker and Kent caution that it is important to distinguish between genuine communication and the IGUSs’ beliefs about the matter. IGUSs can believe that they are communicating with one another whilst not agreeing on the content of the communication.

In order for genuine communication to take place, a history must be chosen which can describe all the experiences of all the IGUSs simultaneously; e.g. a single, common quasiclassical framework. It is assumed that if a pair of IGUSs are in communication, there must be a joint probability distribution for their experiences. However, Dowker and Kent [1996: 1629] claim that when there are three IGUSes, pairwise in communication, there exists no joint probability distribution on all three which respects the joint distributions for the pairs. Then there is no genuine communication and CHQT falls into solipsism. IGUSs have their own experiences. They can also describe the experiences of other IGUSs, but they can’t communicate [Dowker and Kent, 1996: 1631]. IGUSs seem to be in a similar situation to Leibniz’s monads!

CHQT does not resolve Wigner’s dilemma of the two kinds of reality:

> The reality of my perceptions, sensations, and consciousness is immediate and absolute. The reality of everything else consists in the usefulness of thinking in terms of it; this reality is relative and changes from object to object, from concept to concept. [Wigner, 1964: 261-262].

In CHQT, subjective reality emerges when IGUSs develop consciousness. In the best case scenario (i.e. according to Hartle, but disputed by Dowker and Kent), communities of IGUSs can develop inter-subjective models of reality which they all, more or less, agree on. Even using these models, they cannot assert statements about definite, observer-independent physical events: all events need to be qualified by a probability and by reference to a framework[81]. For example, "Caesar invaded Britain in 55 BCE" is a statement whose truth value is conditional on choosing a particular quasiclassical framework to describe that event, together with a sequence of historical links to current evidence, all of which have a certain probability of having "actually happened" within that framework [Hartle, 2007b: 3108-3109]. There are many past histories consistent with current evidence within which Caesar did not invade Britain in 55 BCE. Worse than this, "different events can have happened in different incompatible pasts, even seemingly contradictory events" [Hartle, 2007b: 3110][82]. Then we must also consider the possibility of fundamentally different IGUSs which do not exploit the regularities of quasiclassical realms, as we do, but utilise non-quasiclassical realms to develop a totally different (for us, unimaginable) concept of reality. This is the relativity of reality in CHQT.

Furthermore, it is not known how IGUSs develop consciousness and it is not clear what reality consists of before that happens. There may well be histories of the universe which do not contain IGUSs but, without IGUSs, there is no interpretation of reality. Even particles and fields are constructs which we impose on reality in order to extract information in a form which is interesting and useful to us. Without IGUSs, there is the reality of the fundamental theoretical framework but it dissolves into the mathematical formalism — i.e. bits without its. It is mathematical reality, independent of observers and purely formal.

The problem of subjective *versus* objective reality is front and centre of Leibniz's philosophy. For him, reality consists of individual minds which persist through change. The minds create unity in

[81] There are some events which have probability one in a particular framework but it is difficult to identify events which occur with probability one in any fundamental consistent set fine-graining the actual history [Dowker and Kent, 1996: 1611-1616].

[82] Note that "seemingly contradictory events" contradict our intuitions but do not lead to inconsistencies in the formalism.

multiplicity. They objectify reality. They do not communicate but their harmonious perceptions give rise to the phenomena of experience. They are simple unities but, at the same time, they contain all the variety of the universe. They are individuated by their properties, but their individuality is something beyond a bundle of properties.

Leibniz talks in terms of "two kingdoms": the world of physics and the world of mind, each governed by its own laws, with no confusion between them, but perfectly harmonised with each other [Garber, 2009: 263-264]. An explanation of events in one world could be translated into the language and concepts of the other world. He believed that, for scientific purposes, phenomena in the world of physics should be explained in terms of the language of physics. Anything else would be counterproductive. However, for some purposes (e.g. explaining the conditions of possibility for natural science), the language of the world of mind is more appropriate.

History has shown that the language of the world of mind can be motivational for strategic thinking in physics. For example, Weyl was motivated to develop gauge theory by his interest in Husserlian phenomenology (see Section 3.2.2). His quest was to construct physical reality from the material of immediate experience using only what is given in the infinitesimal neighbourhood of the cognising ego. It led to valuable insights for modern physics (even though phenomenology itself has not had the scientific impact that Husserl intended). In the next section I explore the possibility of translating the language of CHQT into the language of monadology.

6.5 Quantum Monadology

6.5.1 Overview of the process

As a premise, I will assume that only monads and their perceptions and appetitions are real; bodies and physical phenomena are the coherent perceptions of sets of monads. Each monad is an individual and there are as many of them as can possibly be conceived (i.e., an absolute infinity beyond number). The monads are embedded in a deeper reality which provides the raw material for their perceptions.

In the language of CHQT, a monad is described by a consistent history of the universe. Each monad has a subjective time according to which it updates its perception of the world. At each time step, it projects a consistent framework of mutually exclusive possibilities onto the space of potentiality, according to the conceptual resources available to it at that time. To start with, it only has knowledge of one truth — "I exist" (which is the initial state) — and it uses introspection and projection to sequentially build up an alphabet of concepts with which to think and interpret the world. A monad projects, records and learns. It projects possible actuals and records what happens from its perspective. It does not choose what happens or cause it in any way. It generates a recording which is the history of the universe from its point of view, sequentially updated, using its conceptual framework. The future, for it, is a sea of potentiality onto which it projects a framework of possible actuals derived from its history using learned rules. It is important that the monad be represented by a *consistent* history so that it does not interfere with other monads when it makes a record (i.e. it does not "collapse the wave function"). All possible monads — all possible consistent histories — exist. Each can be described by a sequence of projection operators acting on the initial state, or, equivalently, by a sequence of properties.

In metaphysical language, Being always exists. It contains all possibilities. It contains all monads, but is not constituted by them. It can be represented by set theory, as in Chapter 5, or it can be represented by a superposition of monadic properties, in analogy to a quantum computer. Potential contradictions coexist as superpositions in the structure of Being. Monads act on potentiality to create possible actuals in a process of resolving contradictions. Reality appears the way it does because of the way that minds function: combining concepts sequentially using classical logic. All reality starts with the initial thought — "I exist" — which is a universal truth revealed to all minds. Monads bootstrap from this by sequentially projecting and learning concepts to build up a meaningful narrative. At each stage, more contradictions are resolved, more concepts are learned and more meaning is revealed. Monads are in competition with one other to achieve actuality through maximising their degree of perfection. They are more perfect if they have greater conceptual resources, richer introspection, and a superior ability to perceive and interpret reality.

Concepts, language, meaning and laws emerge together; with mind as the fundamental agent. The quantum nature of the process dictates that the details of the future are intrinsically unpredictable. The world unfolds, but not in a deterministic way. Probability and choice are not restricted to the beginning of the process.

Monads are Leibniz's answer to the question of how all the variety of the world emerges from simplicity. In modern physics, structure comes from underlying symmetry. In CHQT, without the specification of some set of preferred operators, the state is structureless, and that specification depends on the generators of the relevant symmetry group (e.g. the Poincaré group for quantum mechanics). At higher energies, there is more symmetry, a bigger symmetry group, and more fundamental particles distinguished by more basic properties (essentially, formal properties, often denoted by "colours"). The fundamental particles are physico-mathematical objects corresponding to fundamental representations of the symmetry group. They transform into one another through the operations of the group's generators. The process of symmetry breaking leads to the emergence of particles with physical properties such as mass, spin and charge.

In the suggested metaphysical framework, monads can be thought of as the limit of the process of symmetry unification. They are aspects of total symmetry, distinguished by an infinity of basic, formal properties which they use to record the state of the universe from their particular perspective. The process of conceptualisation leads to the emergence of meaningful properties which are used to construct narrative histories. Actuality is like a fractal structure condensing from total symmetry.

6.5.2 The local view *versus* the global view

One can take the local view or the global view. The local view is the experience of a monad running along a consistent history of the universe. The global view is the structure of all consistent histories drawn from the History space (otherwise referred to as 'quantum reality' or 'the sea of potentiality'). There is no wave function collapse, no reduction of state. Quantum reality always exists and is acted upon by monads projecting possible actuals. Monads have a local conceptual framework which dictates their interpretation of actual outcomes. They do not determine what happens. They do not create reality by observation. They are registers of a particular perspective.

For the moment, I will assume that the local process is a random stochastic process. Later, in Section 6.5.6, I will consider the possibility that actual outcomes are determined by a global choice (as in Leibniz, where God chooses the best possible world). To recapitulate, the working model is that minds convert potentiality (mathematics) into possible actuals (physics) and that actuality is determined by a currently mysterious process, which might be random or might be the global choice of an encompassing mind.

Minds act to create meaning through the construction of narrative histories. Narrative histories are the objects of understanding and they are constructed by combining properties according to rules of syntax. This is assumed to be a fundamental principle of the way that minds function and it determines the way that reality is interpreted: reality is always classical because that is what mind can understand.

Minds cannot interpret superpositions so they convert them into possible actuals. When minds encounter incompatible properties which they cannot conjoin, they choose one or the other and reality is split into incompatible frameworks. As a result, in the physical world, those properties cannot be measured simultaneously.

The laws determining how properties are correlated are the laws of nature. They are learned by monads and depend on the scale at which they are projecting. Minds are not passive spectators; their properties influence other minds via correlations, and they are influenced in turn. There is no fundamental division into observer and observed. According to Zurek:

> While abolishing this boundary, quantum theory has simultaneously deprived the "conscious observer" of a monopoly on acquiring and storing information: Any correlation is a registration, any quantum state is a record of some other quantum state… Moreover, even a minute interaction with the environment, practically inevitable for any macroscopic object, will establish such a correlation: The environment will, in effect, measure the state of the object, and this suffices to destroy quantum coherence. [Zurek, 2002: 21].

In monadology, monads are not aware of all their perceptions; nevertheless, they register them. This leads to decoherence and the creation of quasi-classical frameworks. Mind makes phenomena mirror its workings. The registrations of monads correlate with events in the world. Quantum reality doesn't change: monads record a pathway (narrative history) through it.

6.5.3 The story of the universe

The following is a notional story about how the universe unfolds, in the spirit of the *Timaeus*. Before the beginning of time, monads exist in a superposition of being and non-being. Quantum reality is a superposition of everything/something/nothing. Monads have no thoughts and there are no phenomena. At the beginning of subjective time, all monads are in the same state; they are like one thing with one thought: "I exist". This is the transition from the Many to the One. The "I" in the initial thought has ambiguous reference; it refers both to the monad itself and to Being. Knowledge of the initial thought is knowledge of the One and the Many — of the inexhaustible multitude of existents and their complete and perfect unity. Subsequently, the "I" splits into an infinity of different perspectives, like white light being split into its colours. Each monad develops its own history by learning, projecting and recording. At each step of subjective time, it forms a thought. All histories contain the initial thought which is the primary truth of every monad. Knowledge of the One and the Many is projected at every time step, so that monads are always seeking unity in diversity through the creation of a thought.

The monad's desire for unity is in creative tension with its desire to learn more and differentiate itself from others. It creates an identity for itself through its construction of a narrative history. All possible monads exist simultaneously. Some have only limited concepts; perhaps, projecting the same concept over and over again without learning. Some form into communities, developing diversity through the harmonisation of rules for projecting concepts. Some rules lead to more and more concepts, more and more diversity. Monads in these communities are more perfect because they have richer experiences and greater capacity for introspection. Actuality results from the competition of monads for greater perfection.

Monads self-organise into consistent frameworks with compatible histories. Within such a framework, they can form a consistent view of reality. They don't communicate directly with one another; however, every monad is in an ongoing conversation with Being through its projection of possible actuals and recording of actual outcomes, so Being is the mediator of meaning creation. Monads respond to each other's properties according to "laws" which they develop as they go along, through learning and coordinating. All laws are the result of random processes, like the laws of thermodynamics[83]. Monads within a framework project at different scales (i.e. using different coarse-graining). Meaning is scale-dependent, as are laws, so there are scale-dependent families within a consistent framework. Laws at different scales in a consistent framework have to be compatible with one another.

If alternative frameworks have the same laws and concepts, they can be grouped into a possible world, even though they have incompatible histories and can't be combined to give one view of that world. This is an important point. In quantum monadology, monads can be incompatible but still compossible. They may be incompatible because their histories contain dual quantum properties which are incompatible, yet still be compossible because they are describing the same possible world. In contrast, in Leibnizian monadology, if monads are incompatible then they are not compossible. An analysis of their histories would lead to a contradiction. Quantum monadology provides a way for some incompatible monads to coexist, in accordance with the principle of existence maximisation.

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[83] A force that originates in a system with many degrees of freedom by the statistical tendency to increase its entropy is called an *entropic force*. It has been suggested that gravity is an entropic force [Verlinde, 2011].

There are different stories being played out all the time, at different scales: "gardens within gardens", in the terminology of Leibniz [*Monadology* §67]. There is a scale below which monads cannot project because they need coarse-graining in order to construct possible actuals. Hence, reality cannot be divided indefinitely; there is a minimum scale.

6.5.4 Coherent phenomena

Some monads have the ability to introspect based on the conceptual framework they have built up. They have private thoughts, as well as a shared model of reality. There is a hierarchy of consciousness which is related to the ability of a monad to understand other points of view and to use models other than the default, shared model to get a more objective picture of reality. Monads with higher powers of reasoning can create new concepts by introspection and test their consistency with the shared model by doing experiments. If they are smart enough and have clear enough perceptions, they don't need to do experiments: they can test the validity of new theorems by deduction based on their knowledge of the concepts used to build the shared model. There is an interesting dynamic whereby monads become more and more complex, and develop greater and greater powers of intelligence, and use this complexity and intelligence to discover and test the underlying simple laws of their shared model of reality.

The method of CHQT is to reason from actual facts via conditional probabilities. If the only actual facts which are conceded are the material of immediate experience then this reduces to the method of phenomenology which is the local view of a monad. The question becomes "what are the conditions for this state of affairs to be"? The monad, situated in the now, uses its conceptual abilities to model the past and predict the future. The material which it works with are its perceptions of the present state of the universe and its recorded past, all of which may be confused. There is one, unconditional true fact which all monads are aware of: the initial thought which is true in all histories. From this initial thought, the monad reasons to the existence of others, like itself, with perceptions like it has. It begins to distinguish others in the sea of changing properties which it perceives. Its perceptions are not updated continuously, but according to subjective time, which is discrete and not necessarily synchronised with the subjective time of other monads. As in Leibniz, space and time owe their existence to the ordering relations that obtain among monads. Monads are ordered by the closeness of their histories. Objective time emerges as a coherent phenomenon arising from the local ordering of monadic perceptions. A monad coordinates with others that are close to it and, hence, have a compatible conceptual framework. It develops the concept of space, in which the others are situated, and of time, related to the order of their changing properties. All the while it is recording more and more concepts; thereby individuating itself and, at the same time, differentiating itself from others. It does not become a complete individual until the "end" of the process.

Phenomena are constructed through agreement between monads which are sufficiently close to each other. Once space and time emerge, monads can use local symmetries to sift objects from the sea of perceptions. For example, important structure is invariant under translations of space and time. Decoherence plays a role in determining what structures persist and what observables are selected as preferred concepts (e.g. quasi-classical hydrodynamic observables as discussed in Section 6.3.3). Monads pick out local densities as the source of habitually decohering quantities.

Different families of monads construct different models of reality. At a sophisticated stage of conceptual development, the contemplation of what is invariant under a change of perspective (i.e. under a change of framework) gives a deeper understanding of what is 'real' and what is 'appearance' in a possible world. Using the tools of mathematics, sophisticated families of monads can investigate invariances under local symmetry transformations to discover the conservation laws of physics and their associated forces and elementary particles.

6.5.5 Differentiation

Now I am going to speculate on how some monads develop richer conceptual resources than others. This does not happen by direction but by an accumulation of stochastic accidents, in a process parallel to that of decoherence and structure formation. It is an example of self-organised criticality in which self-similar structure develops at all scales.

In the beginning, a monad has the experience of being one entity which exists. By introspection, it learns the concepts of "I" and "not-I" and projects this onto quantum reality, as in asking the question "Is there an Other?" Depending on the random outcome of this projection, some monads do not develop any new concepts at this time step — they still have the experience of being the one and only entity which exists — while some develop new concepts such as "difference" and "change". Monads can combine concepts into new yes-no questions to project onto quantum reality and then learn still more concepts. Following the outcome of each projection, they will find themselves on different branches of adaptive histories. Some will be in rich environments which facilitate the learning of new concepts. Some will develop "on the edge" of criticality between order and disorder. Too much order leads to more similarity, more agreement, fewer concepts and less learning. Too much disorder leads to a chaos of perceptions, no narrative, no harmony, the inability to construct shared models and, again, fewer concepts and less learning.

Criticality maximises the signal-to-noise ratio in the information embedded in monadic perceptions. Monads near criticality learn and adapt more easily. They develop symbiotic relationships with other monads. Criticality guides the growth and development of symbiotic groupings of monads. Such groupings have patterns of collective activity which facilitate the formation of distributed structures enhancing cognitive ability. They are living beings which form and disform; storing information and replicating it; competing and evolving. Fluctuations near criticality act like mutations to prevent natural selection from becoming mired in non-optimal configurations. Typically, living beings have a dominant monad which is the one with the most developed cognitive resources. All benefit from the symbiotic grouping but the dominant monad benefits most because it can exploit the enhanced cognitive machinery to generate more advanced concepts.

Monads strive for existence and their competition leads to differentiation in terms of their amount of perfection. In Leibniz, the amount of perfection of a possible monad is identified with its "quantity of essence" and determines its likelihood to be actualised:

> "Everything possible… tends with equal right towards existence according to its quantity of essence or reality, or to the degree of perfection which it involves, perfection being nothing but the quantity of essence" (*Phil*., VII, p. 303). Since the more clearly a substance perceives another, the greater the compatibility of its own state with that perceived, perfection increases with compossibility. Thus, "just as possibility is the principle of essence, so perfection or the quantity of essence (which means the number of compossibles of a thing) is the principle of existence" (*Phil*., VII, p. 304). Hence the Principle of Perfection makes it possible for Leibniz to state that "one can define as existent that which is compatible with more things than anything incompatible with it" (Couturat, *Opuscules*, p. 360). [Rescher, 1967: 28].

In Leibniz, the world is both locally and globally optimised for perfection. In quantum monadology, a monad becomes more perfect by learning more distinct concepts and understanding more about the world. It does this by projecting, introspecting and forming advantageous groupings which enhance learning (e.g. possible worlds, consistent frameworks, scale-dependent families, close relatives and symbiotic life forms). The more a monad learns about the world, the more it can share. Its subjective feelings are concepts that it can share with close relatives. Its perception of objects can be shared with scale-dependent families. Physical laws are concepts which can be shared within possible worlds. Mathematical truths are concepts which can be shared between all worlds. If an existent is that which is compatible with more things than anything incompatible with it, then it helps to have more shareable concepts. The most perfect monads develop at the edge of criticality so actuality has a self-similar, fractal structure.

6.5.6 Global optimisation

This brings me back to a question which I put aside earlier. Is there any sort of global optimisation which chooses actuality from potentiality? Despite all the discussion so far, actuality is still mysterious. I have said that there are degrees of reality, that the most real monads are the most perfect and that they develop at the edge of criticality. But what distinguishes actuality? I have two suggestions to make here. One is that actuality is the construction of higher consciousness and that higher consciousness only develops at criticality. Then, actuality would emerge in the possible world which contains the most perfect monads. The construction of actuality by higher consciousness would be consistent with the views of some anti-realist philosophers. My other suggestion, more consistent with Leibniz and with the metaphysical framework given in Chapter 5, is that there *is* a global optimisation and the actual world is chosen as the best possible world by a transcendent entity.

Let's first consider the implications of taking higher consciousness as a condition for actuality. Tegmark conjectures that it severely constrains physical theory. Considering the number of dimensions, he writes:

> …all but the combination 3+1 appear to be "dead worlds", devoid of [IGUSs]. If there were more or less than one time-dimension, the partial differential equations of nature would lack the hyperbolicity property that enables [IGUSs] to make predictions. If space has a dimensionality exceeding three, there are no atoms or other stable structures. If space has a dimensionality of less than three, it is doubtful whether the world offers sufficient complexity to support [IGUSs] (for instance, there is no gravitational force). [Tegmark, 1998: 40].

In support of Tegmark's conjecture, Bernal et al. [2008] show that if we assume (i) that physical theory has the structure of a differentiable 4-manifold, (ii) that it contains IGUSs, and (iii) that no IGUS has a privileged perspective in space or in time, then only four possible models of spacetime emerge. One of these models corresponds to Newtonian spacetime and another to General Relativity. Their work suggests the existence of a strong Pythagorean component in the universe, with monads deriving physical meaning from mathematical symmetries in the limited range of possible spacetimes [Bernal et al., 2008: 7].

Taking a different approach, Benioff [2005] has arguments to suggest that higher intelligence would necessarily emerge in a coherent theory of physics and mathematics that was maximally self-validating. By a coherent theory he means a Theory of Everything, containing all physical and mathematical systems described together in a single theory (rather than as two separate types of systems with different theoretical descriptions). In such a theory, mathematical reasoning and experimental validation would be carried out by intelligent subsystems subject to its structural laws. In turn, its structural laws would be constrained by the ability of its intelligent subsystems to provide validation. The self-validating property implies a strong anthropic principle: the universe is such that intelligence will emerge, become more and more powerful, and never die out.

The uniqueness of Benioff's coherent theory is an interesting possibility to consider. Uniqueness could only ever be shown in an analogous sense to Woodin's Ultimate L: we will never be in possession of the full theory, just a hierarchy of ever stronger theories, and the demonstration of uniqueness would hinge on finding a level of the hierarchy at which all higher levels are subsumed. Knowledge cannot exceed this level even though there are levels of the hierarchy which exist beyond it. There will always be something transcendent to the theory which is needed to define it. In set theory that something is the universe of all sets V which, by Russell's Paradox, cannot itself be a set. A maximally self-validating theory would be one in which intelligent beings develop to the highest epistemological level of the hierarchy. At that point, they would have the ability to test all the testable predictions of the theory and would know why our universe has the properties it does:

> The physical universe could not be otherwise as it is the only one whose properties emerge from or are determined by and determine a coherent theory. [Benioff, 2002: 1006].

It would still be necessary to carry out experiments to test the theory.

Benioff's coherent theory illustrates a problem for all physical and mathematical theories: they are subject to the consequences of Gödel's Incompleteness Theorems. We know that they contain statements which are true but unprovable and, hence, predictions that cannot be tested. They can never be fully self-validating.

Omnès stresses that actuality is the only point where theory and reality come into contact with each other.

> All the rest is a matter of relations between phenomena and observations of their frequencies, which are obtained entirely within the framework of theory if one includes in theory its account of common sense. This is also the only point for which theory does not provide an explanation, nor a mechanism, nor a cause for what is observed. [Omnès, 1992: 379-380].

The attribution of actuality is something that comes from outside the theory, beyond physics and mathematics. This brings me back to my other suggestion about the creation of actuality from potentiality — that it is the result of global optimisation by a transcendent entity. I have said that in quantum monadology there is an absolute infinity of monads and that they self-organise to achieve perfection. Yet I have described their individuation as a process, and a process can only ever describe a potential infinity. A monad does not become a complete individual until the "end" of the process. What I am suggesting here is that the end of the process is the Perfect Monad which is the Many in the One as represented by the universe of sets V. It contains all possibilities, all thoughts and all monads, but is not constituted by them. It is that which is outside the process and which makes the process possible. It chooses the best possible world to be the actual world.

According to this suggestion, whilst the process of differentiation appears to be random and built from below in the local view, it is actually necessary and determined from above in the global view. Nevertheless, it is not pre-determined. There is no more efficient way of programming it than to let it unfold. As time goes on, the attribution of meaning is constrained by what already exists; fewer choices will give consistent meanings. In the end, there is one choice, one will, one actuality. From the global view, there just is actuality and it determines what is true. It is as though possible monads carve out the truth but, really, the truth is always there. In quantum monadology, the truth consists of many, incompatible (though consistent) perspectives; rather than a single perspective.

6.5.7 Consequences

There is a moral dimension to choice which Leibniz explored in his philosophy. He wrote:

> God has… created the soul or any other unity in such a way that everything arises in it from its own internal nature through a perfect spontaneity relative to itself, and yet with perfect conformity to external things. (Leibniz as quoted in [Ishiguro, 1972 : 114]).

He was very concerned to allow room for free will in his pre-determined universe. It is doubtful whether he succeeded, or whether free will was ever possible in a classical view of the world. The effective randomness of the quantum world offers more hope. In the local view, monads feel that they are presented with alternatives and make choices. There is a sense in which they create meaning in the world.

What is the role of quantum mechanics in this view? Firstly, as discussed in the previous paragraph, it injects randomness into the process and prevents it from being pre-determined. Secondly, it provides a way for incompatible possibilities to coexist as superpositions. Thirdly, it enables a principle of plenitude — anything that can be described by a consistent history is a possible actual and can be assigned a probability. Everything has a way of coming into being.

Finally, let me return to the question raised in the opening paragraph of this chapter: how much mathematics is needed for physics? In CHQT, it appears that there is surplus mathematics corresponding to that part of quantum reality that minds cannot interpret. However, the whole of History Hilbert space is needed for the formulation of the theory. According to Omnès [2002], mathematics is a “rounding out” of physics. All parts of mathematics can be reached from physics and are needed for physics, given the requirement of consistency. Also, physics puts some constraints on mathematics (e.g. well-ordering and the Axiom of Choice). For quantum monadology, all of mathematics is needed: the hierarchy of sets can be extracted from the superposition of monadic states.

Chapter 7. Conclusion

On the face of it, physics and mathematics are about different things. The objects of physics are those which we causally interact with every day and which give us our sensations of heat, light, texture, smell and sound. Physicists assume their existence and try to discover their properties and how they cause the experiences that they do. In contrast, the objects of mathematics are usually deemed to be irrelevant to the causal nexus. If mathematical objects exist at all then they exist outside of space and time in an eternal, unchanging state of pure being. We can't interact with them in a sensual way, only by pure intellection, and any such interaction defies the laws of physical objects because it does not involve any interchange of energy, or any change at all in the grasped object, merely a change in our own mental state.

None of this would matter if mathematical objects were content to stay in their Platonic heaven as objects of aesthetic pleasure for pure mathematicians. However, they insist on finding their way into our scientific theories and, indeed, they seem to be indispensable to scientific theorising. Mathematics refuses to be just a meaningless game (as proposed by formalists) or an instrumental deductive tool (as proposed by nominalists). It has content and meaning and it seems to be telling us some important things about the nature of ultimate reality. As our physical theories probe deeper and deeper into the fundamental nature of reality and reach realms where the intuitions developed in our everyday world no longer apply, they require deeper and deeper mathematical structures. Their objects become less intuitive; less like familiar macro-objects, more like physico-mathematical signs. Physical explanations dissolve into mathematical explanations.

On the other hand, some mathematical objects don't seem to have relevance to the physical world at all. For example, it is hard to envisage any role for the inaccessible numbers of set theory in any physical process. We would not reach that sort of cardinality even in a multiverse… even if an infinite number of universes had been continually generating baby universes since past infinity. An inaccessible number is just that: it cannot be reached by the iteration of constructive processes such as taking unions of collections of smaller sets, or taking power sets, or taking any combination of operations available in set theory. If one imagines translating all mathematical objects into the background language of set theory then the subset which is relevant to known physical processes is tiny.

These considerations seem to stymie the strategy of bridging the Platonic divide by bringing the content of physical and mathematical theories together. Whilst mathematics is needed for physics, it goes way beyond it. There is no possibility of physicalising all of mathematics. However, there are still options available for those wishing to adopt this strategy.

One option is to abandon the need for the physicalisation of mathematics by embracing all objects as abstract objects. That is the approach taken in this thesis. I develop a metaphysical framework which is a version of idealism. It is a view of the universe as an abstract mind: a mind constituted by monads and their states, not a mind supervening on a physical brain with hard-wired connections. The structure contains all of mathematics, all possible forms, and physical reality condenses out by a process of self- actualization in thought.

Let us review the argument of the thesis. I began in Chapter 2 by examining Wigner's paper on the unreasonable effectiveness of mathematics in the natural sciences. Whilst many issues surrounding the applicability of mathematics disappear upon examination (depending, to some extent, on one's philosophy of mathematics), I argued that there are some core issues to do with the effectiveness of mathematics in fundamental physics which remain (and these are, to a large extent, independent of one's philosophy of mathematics). The core issues are the existence of the laws of nature and our minds ability to fathom them; the use of formal mathematics in discovering things about the quantum world; the fact that deep mathematics is needed to describe fundamental physics; and the asymptotic nature of the quest for knowledge. These issues were the focus of the thesis. I sought to explain them within a metaphysical framework that takes mind and mathematics as its fundamental principles. The framework is an extension of the Pythagorean metaphysics which inspired many great physicists.

In Chapter 3 I argued that the unreasonable effectiveness of mathematics in fundamental physics has become evident through our growing understanding of the significance of symmetry principles in nature and the ever-expanding repertoire of mathematical concepts and tools developed to extend that understanding. This is a story of the success of Pythagorean metaphysics. It leads to dreams of a final theory — a mathematical form of great beauty, simplicity, logical inevitability and explanatory power. Wigner's work in quantum mechanics features at the beginning of the story. Then there is the development of QCD. The culmination is the development of String Theory — a candidate Theory of Everything which promises to provide a consistent, unified theory of quantum gravity without any adjustable parameters.

It is as yet unclear where String Theory will lead us. Whatever the outcome, we have already learned that the effectiveness of mathematics in fundamental physics is due to a deep fact about the world: as we explore physics at higher and higher energy, revealing its structure at shorter and shorter distances, we discover more and more symmetry. Extrapolating this trend encourages us to search for new symmetries, using whatever mathematical concepts we can discover.

The deep symmetrical structure of nature explains why mathematics is unreasonably effective in fundamental physics at very high energy levels and it is also explains why mathematics is reasonably effective at lower energy scales (with all the caveats explored in Chapter 2). In the everyday world we observe broken, or approximate, or residual, symmetries. Without these reflections in the experimentally-accessible world we would never have been able to get as far as we have in uncovering nature's deep structure.

My explanation of the applicability of mathematics is based on scaling: different theories are applicable at different scales but all are grounded in the ultimate theory which is applicable at the highest energy levels. The ultimate theory is Pythagorean so the physical objects and their relations directly instantiate the mathematical objects and their relations, with both flowing from the underlying principles of the theory. Symmetry breaking complicates the picture because it means that the ultimate theory is not directly applicable at lower energy levels. Symmetry breaking leads to complexity. Physical structures at lower energy levels instantiate mathematical structures of some sort but those structures are not always amenable to investigation by principled theories. Nevertheless, there is structure at the lower energy levels which reflects the structure of the ultimate theory and this allows us to (i) use mathematics in applications, and (ii) work our way back to an understanding of the ultimate theory.

I explored connections between set theory and physics in the context of the effective field theory view of physics. I speculated that any "missing" principle in physics would turn out to be an innate, mathematical principle with which we are already familiar under a different guise. Such a principle could have its origins in ancient metaphysics, just as ancient philosophers introduced symmetry principles. I discussed the principle of the One and the Many and showed that its formulation by Parmenides foreshadowed Russell's paradox in set theory.

Referring to the creation story in Plato's *Timaeus*, I argued for the unreasonable effectiveness of ancient metaphysics in dealing with fundamental concepts such as time and the ultimate structure of reality. Reviewing recent work in cosmology and particle physics, I identified a thought pattern which drives physicists to contemplate the extremes of infinity and nothingness, and to oscillate from one extreme to the other. This is familiar to metaphysicians who struggle with the paradoxes of thought and to mathematicians who struggle with the paradoxes of set theory.

Chapters 4 and 5 addressed one of the key questions of the thesis: 'What is mathematics?' Chapter 4 focused on set theory. It investigated whether it is feasible to hold a Pythagorean view in the wake of Gödel's Incompleteness Theorems and the threat of a mathematical multiverse. Does set theory provide a background language in which any mathematical problem can be interpreted and (potentially) decided? If so, then the various disciplines of mathematics are part of one, true mathematical universe and mathematics can serve as the ultimate arbiter of questions about existence and truth. If not, then there are alternative mathematical universes and mathematical statements may be true in some universes and false in others. That would be a problem for Pythagoreanism.

I discussed the current debate in set theory between the pluralists — who believe that there are many distinct concepts of set, each with its own set-theoretic universe — and the universalists — who believe in an absolute background set concept, with a corresponding absolute set-theoretic universe. After Gödel, techniques were developed which led to a plethora of alternative set theoretical universes. Pluralists see this work as bringing set theory into the world of modern mathematics, freeing it from the special burden which it had bourn because of its importance for foundational work in the philosophy of mathematics. Universalists see it as a challenge to be overcome on the path towards one, true mathematics.

I took up the argument of the universalists, which depends on the justification of new axioms for set theory. The current best hope is the (delineated but not yet constructed) theory of Ultimate L. The claim is that Ultimate L would give us a complete picture of the set theoretic universe in the sense that it would be capable of interpreting any extension of set theory and it would decide all set theory propositions.

At one level, the debate between pluralists and universalists turns on potential bifurcation points for mathematics. The pluralists try to show that particular mathematical statements, such as Cantor's continuum hypothesis, are absolutely undecidable and the universalists try to rebuff these challenges. The thrust of the universalists' work is to sharpen the language of set theory so as to eliminate indefiniteness. The arguments get very technical. From a local perspective in the set hierarchy it will always seem that there are many possible branches. The universalists will not be able to argue convincingly for their case until they have developed a theory like Ultimate L which can give a global perspective. Recent results in set theory keep the dream of one true mathematics alive and offer hope for Pythagoreans. My suggestion is that the ultimate solution will require a holistic view, taking account of the place of mathematics in the web of being and knowledge.

In Chapter 5 I argued that philosophers should not abandon the quest of integrating mathematics and science into a coherent world view. It is only by the pursuit of some such holistic view that the underlying issues of mathematics and science will ever be resolved.

I considered modern structuralist philosophies of physics and mathematics which seem to offer the possibility of a coherent approach through the prioritisation of structure over objects — with extreme versions, such as Tegmark's Mathematical Universe, morphing into Pythagoreanism. However, it was concluded that structuralism cannot be the whole story because of the infinite regress of structure versus structured and the fact that mathematics needs non-mathematical objects for its definition. In a historical perspective, the link was made to fundamental problems concerning being and existence in philosophy.

I ventured to introduce a broad metaphysical framework based on the principles of mind and mathematics. In order to explain how physical reality condenses out of mathematical structure, I used Leibniz's ploy of identifying the fundamental constituents of the universe with simple minds — what he called monads. I considered an interpretation of the set theoretic hierarchy in which the individuals are monads. This interpretation makes an analogy between all of reality — Being — and an infinite mind. Being has structure which is mathematical, due to the arrangement of parts, and it has thoughts. Its thoughts are interpreted as correlated states of monads. Being thinks about itself, its own structure, and creates physical reality through a process of self-actualisation.

I used the suggested metaphysical framework to draw out consequences for the philosophy of mathematics. I concluded that mathematics is about the structure of Being; that it is unchanging, necessary and true; and that we come to know about it by abstracting structure from the world around us and by self-reflection. Mathematical objects do not exist in a mysterious Platonic heaven totally independent of the physical world. Rather, physical reality is a mental interpretation of a subset of mathematical structure. Mathematics is applicable because it truly describes the fundamental structure of reality. No mathematics is surplus. I made a distinction between mathematics and human mathematics which is a cultural product of our society. There is one, true mathematics and our current best description of it is Woodin's Ultimate L. Consequently, some human mathematics is fiction, but that does not mean that it is uninteresting or useless. On the contrary, it is only by pursuing all possible avenues that we will learn how to make the distinctions that will lead us towards true knowledge.

Chapter 5 was very speculative. A more detailed model of the relationship between physical structure (actuality) and mathematical structure (potentiality) was presented in Chapter 6. I considered two paradigms: (1) Leibnizian possible worlds, which is rooted in classical physics; and (2) the consistent histories quantum theory of Griffiths, Gell-Mann, Hartle, and Omnès (called CHQT). The analysis went beyond a comparison of classical and quantum physics to consider how modern physics might be integrated into a more comprehensive view of the world, in the spirit of Leibniz's own philosophy and following the approach of Chapter 5. My basic idea is that CHQT brings the formalism of quantum mechanics into the logical realm of possible actuals where it can be interpreted in a quantum monadology. Quantum monadology brings together many disparate threads of physics and philosophy.

The model introduced in Chapter 6 has a different focus to that in Chapter 5, but it is built using common elements (i.e., it is a monadology which incorporates the structure of set theory). The two models should be viewed as stages on the way to a final formulation.

Leibniz talked in terms of "two kingdoms": the world of physics and the world of mind, each governed by its own laws, with no confusion between them, but perfectly harmonized with each other. An explanation of events in one world could be translated into the language and concepts of the other world. He believed that, for scientific purposes, phenomena in the world of physics should be explained in terms of the language of physics. Anything else would be counterproductive. However, for some purposes (e.g. explaining the conditions of possibility for natural science), the language of the world of mind is more appropriate. I explored the possibility of translating the language of CHQT into the language of monadology.

In quantum monadology, a monad is described by a consistent history of the universe which is a sequence of properties holding at a sequence of times. Monads are embedded in Being. They build up histories over time by sequentially projecting yes-no questions onto Being and recording the answers. In doing so, they act to resolve the potential contradictions which exist as quantum superpositions in the structure of Being. Reality appears the way it does because of the way that monads function: combining concepts sequentially using classical logic. All reality starts with the initial thought — "I exist" — which is a universal truth revealed to all monads. Monads bootstrap from this by sequentially projecting and learning concepts to build up a meaningful narrative history. At each stage, more contradictions are resolved, more concepts are learned and more meaning is revealed. Concepts, language, meaning and laws emerge together; with mind as the fundamental agent. Reality is constructed from different stories being played out at different scales.

Although the process appears to be totally random from the local point of view, it is necessary and determined from the global point of view. The attribution of actuality is something that comes from outside the theory, beyond physics and mathematics. The actual world is chosen as the best possible world by a transcendent entity. It is the world which exhibits the greatest variety of its contents (richness of phenomena) consonant with the greatest simplicity of its laws. It is a world in which physicists are right to search for an underlying Pythagorean theory.

I will conclude with a quote from the *Timaeus*, since that was the inspiration for my thesis. Plato made this argument for the uniqueness of our universe:

> "There is but one universe, if it is to have been crafted after its model. For that which contains all of the intelligible living beings couldn't ever be one of a pair, since that would require there to be yet another Living Being, the one that contained those two, of which they then would be parts, and then it would be more correct to speak of our universe as made in the likeness, now not of those two, but of that other, the one that contains them. So, in order that this living being should be like the complete Living Being in respect of uniqueness, the Maker made neither two, nor yet an infinite number of worlds. On the contrary, our universe came into being as the one and only thing of its kind." [Plato, *Timaeus*]

An analogous argument is used by set theorists today. Universalists argue that multiple universes of sets can always be amalgamated into one universe of sets which should be regarded as the true universe of sets. Pluralists reply that that would be true if we all agreed on a fixed set-theoretic background concept… but we don't. There can be different set concepts which lead to different universes of sets. Theory doesn't pin down the concept of a set.

This is the nub of the problem: at any stage of the set theoretic hierarchy there will always be things that can't be defined, truths that can't be proved without going to the next level of the hierarchy. For related reasons, there isn't a unique interpretation of mathematical structures; physical theories can be satisfied by multiple models; there is radical indeterminacy of meaning and reference. Theories can be wrong about themselves; for example, the theory of constructible sets L can "assert" that it doesn't contain a measurable cardinal, even though it does contain one from a higher viewpoint. This is because it cannot "recognise" that it contains a measurable cardinal because it doesn't have the semantic resources to witness measurability.

Uniqueness is not a property from any perspective lower than that of the universe of sets V. If there is to be a fully-interpreted theory — a universal framework — then meaning has to be transcendental. Being cannot be wrong about itself or lack the resources to describe itself. It knows all truth. It has the ultimate viewpoint which transcends all others.

Here is the dilemma: either we drop the idea that there is determinate meaning (which is the standard response of philosophers these days) or we accept that there is transcendental meaning (which was the standard response prior to the 20th century and is perhaps coming back into vogue with the talk of superminds and the omniscient, omnipresent data cloud). My thesis opts for transcendental meaning which is revealed to separate individual minds by an encompassing mind.

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Appendix 1: Technical Notes on Set Theory

A.1.1 Cohen's Techinique of Forcing

Cohen's technique of *forcing* involves constructing an *outer model* L[G] of L. The aim of this technique is to extend L by adding a new set G which is forced to have certain desired properties not possessed by the sets in L (e.g., the property of violating the continuum hypothesis). This is done in such a way that L[G] (i.e., the result of adjoining to L the set G and everything constructible from G together with the elements of L) is consistent with the axioms of ZFC.

Using the work of Gödel, Cohen [1963: 1143] assumed the existence a model M which satisfies the axioms of ZF + V=L (i.e., ZF with the Axiom of Constructibility). He took M to be a *countable standard transitive*[84] model. Countability is convenient and, anyway, it follows from the Lowenheim-Skolem theorem that if a first-order set theory has a model at all, then it has a countable model. Cohen's focus on M being standard and transitive also made his proof easier, but these are not essential criteria for the technique of forcing in general.

For every object X which can be proved to exist in ZF + V=L, M contains an analog object Y, which is often equal to X [Chow, 2008: 6]. For example, M contains all the ordinals and $\aleph_0$. However, the analog Y is not *always* equal to X. For example, since M is countable, the set y which is the analog of the powerset of $\aleph_0$ in M will also be countable, so it cannot be the same as the powerset of $\aleph_0$ in ZF + V=L (which Cantor showed to be uncountable). Many subsets are missing from y. We say that y is 'uncountable in M' even though it is countable in ZF + V=L. The notions of 'countability' and 'powerset' are relative to the model. Notions, such as 'the empty set', which do not depend on the model are said to be *absolute*.

Cohen's aim [1963: 1144] was to extend M by adjoining a subset U of $\aleph_0$ that is missing from M and that violates the continuum hypothesis (e.g., contains $\aleph_2$ or more reals). The resulting model M[U] would be a model for ZF and the Axiom of Choice and the negation of the continuum hypothesis. He was concerned that, if U were chosen indiscriminately, then the set that plays the role of a particular cardinal

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[84] A set X is *countable* if its elements can be put in a one-to-one relation with the elements of ω.
A set X is *transitive* if every element of X is a subset of X.
A model M of a set theory is *standard* if the elements of M are well-founded sets ordered with the set membership relation.

number in M might not play the same role in M[U]. For example, the set $\aleph_2$ in M might become countable in M[U] [85]. However he argued that, if U were chosen in a generic manner from M[86], then no new information would be extracted from it in M[U] which was not already contained in M, so sets such as the cardinals would be preserved. Cohen wrote:

> … we do not wish U to contain "special" information about M, which can only be seen from the outside… . The U which we construct will be referred to as a "generic" set relative to M. The idea is that all the properties of U must be "forced" to hold merely on the basis that U behaves like a "generic" set in M. This concept of deciding when a statement about U is "forced" to hold is the key point of the construction. [Cohen, 1966: 111]

Cohen formalised the notion of a set U being M-generic and of the forcing relation, through which satisfaction for the extension of M could be approached from inside M. He showed how to build the new set U one step at a time, tracking what new properties of M[U] would be "forced" to hold at each step [Chow, 2008: 8]. The result was a model which, together with Gödel's earlier work on L, proved the independence of the continuum hypothesis in ZFC.

A.1.2 Generating Large Cardinal Axioms Using Elementary Embeddings

The most general method for generating large cardinal axioms uses *elementary embeddings* [Woodin, 2011b: 97-99]. An elementary embedding of V into a transitive class M is a mapping j: V → M which preserves logical truth (i.e., if arbitrary variables $a_0,\dots,a_n$ in V satisfy an arbitrary first-order formula Φ in the formal language for set theory, then their mappings $j(a_0),\dots,j(a_n)$ satisfy Φ in M, and conversely). A basic template for large cardinal axioms asserts that there exists such an embedding j which is not the identity. If j is not the identity then there will be a least ordinal which is not mapped to itself (i.e., such that $j(\alpha) \neq \alpha$) and this is called the *critical point* of j. Then the critical point of j is a measurable cardinal and the existence of the transitive class M and the elementary embedding j are witnesses for this.

By placing restrictions on M, by requiring it to be closer and closer to V (in a manner analogous to width reflection for inner models, but precisely defined in terms of the closure properties of M

85 This behaviour is called *cardinal collapse*.

86 For example, to be generic, U should contain infinitely many primes.

[Koellner, 2011a: 10-12]), we can generate stronger and stronger large cardinal axioms. By continuing this process to its natural limit; i.e., by taking M = V, the axiom generated asserts the existence of a *Reinhardt* cardinal which is the critical point of a non-trivial elementary embedding j: V → V. However, this axiom has been shown to be inconsistent with ZFC by a theorem of Kunen [Kanamori, 1994: 319]. The strongest large cardinal axiom which evades Kunen's proof, and so is not known to be inconsistent with ZFC, is one asserting the existence of an *ω-huge* cardinal:

> A cardinal κ is an *ω-huge* cardinal if there exists λ > κ and an elementary embedding j: $V_{\lambda+1} \rightarrow V_{\lambda+1}$ such that κ is the critical point of j. [Woodin, 2011c: 456]

The outlined procedure for generating large cardinal axioms does not show us how to build up to the large cardinal from below and so it is not intrinsic. Also, to be very clear, it does not guarantee the existence of the large cardinal or the relative consistency of the associated axiom with ZFC. It simply provides us with a means of pushing the level of cardinals as high as it can go, right up to the point where we bump into Kunen's theorem. A further sobering fact is that even the largest of large cardinal axioms is not enough to determine the continuum hypothesis. Levy and Solovay [1967] showed that all such large cardinal assumptions are relatively consistent with both the continuum hypothesis and its negation.

A.1.3 Large Cardinal Axioms and Definable Determinacy

In the early development of set theory there was a focus on studying the properties of definable sets of real numbers[87] – what is now called *descriptive* set theory [Koellner, 2011b: 16-24]. Many important results were proved. Bendixson [1883] and Cantor [1884] showed that all *closed* sets of reals have the *perfect set property*[88] and so satisfy the continuum hypothesis. This result was later extended to the first-order in a hierarchy of certain definable *open* sets[89]. Another property of interest is Lesbesgue measurability. It was known that there are sets of reals which are not Lesbesgue measurable

[87] See the previous discussion of Gödel's constructible universe L for a definition of definable sets. By starting with the real numbers R and iterating the definable power set operation along the ordinals we get the hierarchy L(R) of definable sets of reals.
[88] A set of reals is *perfect* iff it is nonempty, closed and contains no isolated points. A set of reals has the *perfect set property* iff it is either countable or contains a perfect subset. If a set of reals contains a perfect subset it must have the same size as the continuum. Therefore, if a set of reals has the perfect set property it must satisfy the continuum hypothesis. [Koellner, 2011, 20].
[89] See [Koellner, 2011: 17-18] for a definition of the hierarchy of definable open and closed sets of reals.

(giving rise to the Banach-Tarski paradox discussed earlier) but it was not known whether or not any *definable* sets of reals lacked this property. Again, proofs of Lesbesgue measurability were limited to the first-order in a hierarchy of certain definable open sets. The story was much the same for other regularity and structural properties of interest; e.g., the *property of Baire* and the *uniformisation property*[90]. Beyond this, independence results developed in the 1960s (following the work of Gödel and Cohen) showed that many questions about the properties of definable sets of reals are not decidable in ZFC.

The situation was greatly clarified in the 1970s and 1980s with the development of new axioms for ZFC [Koellner, 2011b: 24-30]. One strand of research studied the implications of Large cardinal axioms. The other strand, to do with axioms of *determinacy*, had its origins in a quite different area of mathematics — game theory. *The Axiom of Determinacy*, AD, is an assertion about a two-person infinite game. It says that for every set of reals A there is a winning strategy for one of the players in the game associated with A[91]. AD is known to be inconsistent with the Axiom of Choice so, in the search for new axioms for ZFC, it is usual to consider restrictions of AD to certain definable sets of reals, giving rise to axioms of *definable determinacy* which are consistent with Choice. For example, by restricting AD to the hierarchy of definable sets of real numbers L(R)[92] one gets the new axiom candidate $AD^{L(R)}$.

The underlying connection between determinacy and the regularity properties of sets of reals is not immediately obvious to outsiders, but to some set theorists "determinacy lies at the heart of the regularity properties and may be considered their true source" [Koellner, 2011b: 26]. What *is* obviously remarkable is the intricate web of results interweaving large cardinals, determinacy, and regularity properties; as outlined by the examples below:

Theorem A.1.3.1. Assume ZFC + $AD^{L(R)}$. Then all the sets of reals in L(R) have the perfect set property, are Lesbesgue measurable and have the property of Baire.

Theorem A.1.3.2. Assume ZFC + "there exist infinitely many Woodin cardinals with a measurable cardinal above them all". Then $AD^{L(R)}$.

[90] Note that there are many such properties other than the ones emphasised in the text; see e.g. [Koellner, 2011].
[91] See [Woodin, 2010: 4] for details.
[92] See footnote 22 for a definition of L(R).

Theorem A.1.3.3. Assume $AD^{L(R)}$. Then there is an inner model N of the theory ZFC + "there exist infinitely many Woodin cardinals".

Theorem A.1.3.4. Assume ZFC + "there is a proper class of Woodin cardinals". Then the truths of L(R) cannot be shown to be independent by forcing from L(R).

Many more such linked theorems are discussed by Koellner [2011b: 26-35]. These theorems bring the two strands of research on new axioms together in an unexpected way. They show that definable determinacy is necessary and sufficient to prove the regularity properties of certain definable sets of reals; and that large cardinals are necessary and sufficient to prove definable determinacy.

Koellner [2011b: 30-38] has used these results to support the case for definable determinacy and, hence, for large cardinal axioms. Some of his main points are:

- The regularity properties of structure theory are desirable properties for definable sets of reals (e.g., they circumvent the Banach-Tarski paradox).
- $AD^{L(R)}$ lifts the regularity properties of structure theory to the level of L(R). Conversely, all theories which have this desirable consequence imply $AD^{L(R)}$.
- For sufficiently strong large cardinal axioms, which themselves imply $AD^{L(R)}$, the results regarding regularity properties cannot be undone by forcing from L(R).
- $AD^{L(R)}$ is implied by many theories which have proved useful in strengthening ZFC and "there is reason to believe that $AD^{L(R)}$ lies in the *overlapping consensus* of all sufficiently strong, natural theories". [Koellner, 2011b: 36]
- There is promising research to indicate that the case can be strengthened beyond L(R).
- Definable determinacy and large cardinal axioms are at root equivalent: "We have here a case where intrinsically plausible principles from completely different domains reinforce each other". [Koellner, 2009b: 111]

Although the results are remarkable, not all set theorists find the case for definable determinacy compelling. For example, Shelah [2003: 211-213] thinks that $AD^{L(R)}$ may be an interesting axiom for descriptive set theory but that descriptive set theory is only one area and focusing on another area might produce a competing, incompatible axiom. Part of Koellner's argument is the claim that this is unlikely but, of course, it depends on what areas one finds interesting and what weight one gives to intrinsically plausible results from different areas. Shelah thinks that V=L is an interesting axiom with some appealing consequences, and it is known to be incompatible with $AD^{L(R)}$.

A.1.4 Defining Effective Completeness

Theorem A.1.3.4 offers the hope of overcoming forcing. It says that, assuming sufficiently strong large cardinal assumptions, the truth value of statements in L(R) cannot be changed by forcing. Furthermore, the results of descriptive set theory show that the same large cardinal axioms effectively resolve all of the open problems relating to the regularity and structural properties of definable sets of reals. So, in this sense, we can achieve effective completeness at the level of L(R). Recalling that L(R) is the same as $L(V_{\omega+1})$, this means that we can achieve effective completeness at a level beyond, and inclusive of, the level of second-order arithmetic, $V_{\omega+1}$, but below that of third-order arithmetic, $V_{\omega+2}$. It would be desirable to extend this result to $V_{\omega+2}$ and higher levels. However, the continuum hypothesis is a statement in $V_{\omega+2}$ and we know that it cannot be resolved by any standard large cardinal axioms, so something more will be needed. To define effective completeness at higher levels of the set hierarchy it is necessary to introduce some technical notions.

Firstly, there is the notion of *the Levy hierarchy of formulas*. A formula in a language that contains the language of set theory is of type Σ_0 (or, equivalently, Π_0) if it is built from atomic formulas through the use of logical connectives and bounded quantifiers $\forall x \in y$ and $\exists x \in y$. A formula is of type Σ_1 if it is of the form $\exists \mathbf{x}\, \varphi$ where **x** is a list of variables and φ is of type Π_0. A formula is of type Π_1 if it is of the form $\forall \mathbf{x}\, \varphi$ where φ is of type Σ_0. In general, a formula is of type Σ_n, n>1, if it is of the form $\exists \mathbf{x}\, \varphi$ where φ is of type Π_{n-1}. And a formula is of type Π_n, n>1, if it is of the form $\forall \mathbf{x}\, \varphi$ where φ is of type Σ_{n-1}. In set theory, an assertion is called Π_2 if it is of the form: "For every infinite ordinal α, $V_\alpha \models \varphi$", for some Σ_1-sentence φ, where $\models$ is the symbol for consequence in the language. Similarly, an assertion is called Σ_2 if it is of the form: "There exists an infinite ordinal α such that $V_\alpha \models \varphi$", for some Π_1-sentence φ. A Σ_2-sentence is expressible as the negation of a Π_2-sentence.

Secondly, there is the notion of *a Σ^2_n sentence*. A Σ^2_n sentence is one of the form "$V_{\omega+2} \models \varphi$", where φ is a formula which is Σ_n in the Levy hierarchy. The continuum hypothesis is equivalent to a Σ^2_1 sentence. Hence, it is a sentence at the level of third-order arithmetic, $V_{\omega+2}$, and at the first level of the Levy hierarchy.

Thirdly, there is the notion of stratifying the universe of sets in terms of the cardinal partition, *H(κ)*, rather than the familiar ordinal partition, V_α. Here, κ is an infinite cardinal and H(κ) is the set of all sets whose transitive closure has cardinality less than κ (i.e., the set of all sets which have cardinality less than κ, whose members have cardinality less than κ, whose members' members have cardinality less than κ, and so on) [Koellner, 2011c: 13]. For example, $H(\omega) = V_\omega$ and the theories of the structures $H(\omega_1)$ and $V_{\omega+1}$ are mutually interpretable. If the continuum hypothesis is true then $H(\omega_2)$ and $V_{\omega+2}$ are mutually interpretable, otherwise $H(\omega_2)$ is less rich than $V_{\omega+2}$. So, in investigating the continuum hypothesis it makes sense to consider $H(\omega_2)$ first.

Fourthly, and lastly, there is the notion of *Ω-completeness*. This will take a bit more explanation. We know that in order to resolve the continuum hypothesis a new notion is needed. Part of the answer might be to introduce a logic which is stronger than first order logic and is well-behaved in the sense that its consequences cannot be altered by forcing in the presence of large cardinal axioms [Koellner, 2011c: 15-17]. Ω-logic is such a logic[93]. It has a notion of consequence, denoted $\models_\Omega$, which is robust to forcing under the assumption of a proper class of Woodin cardinals. It also has a quasi-syntactic proof relation, denoted $\vdash_\Omega$, defined in terms of universally Baire sets of reals which act as witnesses for the "proofs" in the logic [Woodin, 2001b: 684]. Ω-logic is known to be sound. Hence, if a sentence φ is Ω-provable in a set theory T then it is necessarily an Ω-consequence of T. However, it is not known whether Ω-logic is complete; i.e., whether every sentence φ which is an Ω-consequence of a set theory T is necessarily Ω-provable in T. The conjecture that Ω-logic is both sound and complete is called *the Ω conjecture*. This is so important that I will highlight it:

The Ω conjecture says that Ω-logic is both sound and complete.

A theory T is said to be *Ω-complete* for a collection of sentences Γ if for each $\varphi \in \Gamma$, either T $\models_\Omega \varphi$ or T $\models_\Omega \sim\varphi$ [Koellner, 2011c: 17]. Hence, the Ω-completeness of a theory for some collection of sentences means that those sentences are decidable in Ω-logic.

A.1.5 Exploring the Effective Completability of Set Theory

Using the notions introduced in Section A.1.4, Theorem A.1.3.4 can be expressed as:

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[93] For more information on Ω-logic and its relation to full first- and second-order logic see Koellner[2010].

Theorem A.1.5.1. Assume ZFC + "there is a proper class of Woodin cardinals". Then ZFC is Ω-complete for the collection of sentences of the form "L(R) $\models \varphi$ ".

Because of the nature of the continuum hypothesis, we know that there is no standard large cardinal axiom LCA such that ZFC + LCA is Ω-complete for Σ^2_1 sentences. However, there is the possibility that a supplementary axiom could achieve this outcome and, surprisingly, it turns out that simply assuming the continuum hypothesis as an additional axiom *does* achieve it. Thus, Woodin (1985) proved:

Theorem A.1.5.2. Assume ZFC + "there is a proper class of measurable Woodin cardinals". Then ZFC + CH is Ω-complete for Σ^2_1. [Koellner, 2011c: 29]

Moreover, up to Ω-equivalence, CH is the unique Σ^2_1-statement that is Ω-complete for Σ^2_1 [Koellner and Woodin, 2009: 1164].

Of course, assuming the continuum hypothesis as an axiom doesn't count as resolving it. On the contrary, for many years Woodin promoted a parallel case in which the continuum hypothesis is *negated* [Woodin, 2001b]. Thus, assuming that there is a proper class of Woodin cardinals and that the Strong Ω Conjecture[94] holds, Woodin [1999] proved that there is an axiom A such that ZFC + A is Ω-complete for the structure $H(\omega_2)$ and that any such axiom has the feature that the continuum hypothesis is *false* in the associated theory. The axiom A is not unique, there are many such Ω-complete theories [Koellner and Woodin, 2009]. However, Woodin found a particular axiom, called *Axiom (*)*, which gives a "maximal" theory in the sense that it satisfies all Π_2-sentences that can possibly hold [Koellner, 2011c: 14]. Assuming Axiom (*), the continuum hypothesis is false and $2^{\wedge} \aleph_0 = \aleph_2$. Woodin [2001b: 687] saw Axiom (*) as being a candidate for the generalisation of $AD^{L(R)}$ to the structure $H(\omega_2)$; i.e., settling in Ω-logic the full theory of $H(\omega_2)$ just as $AD^{L(R)}$ had settled the open problems of descriptive set theory in L(R).

So now we have two cases for extending our Ω-complete theories: Case 1, in which the continuum hypothesis is assumed as an axiom, and Case 2, which assumes Axiom (*) and has the consequence that the continuum hypothesis is false. Case 1 is only Ω-complete for Σ^2_1 sentences, whereas Case 2 is Ω-complete for the whole of the structure $H(\omega_2)$. Case 2 uses the additional assumption that the Strong Ω Conjecture holds.

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[94] The Strong Ω Conjecture is the Ω Conjecture plus a conjecture about an extended form of the Axiom of Determinacy. See [Koellner, 2011: 16-17].

Does this mean that there is a bifurcation of the universe of sets at the level of the continuum hypothesis? This question can't really be answered at the local level since there is always the possibility that further extensions of the theory will favour one path over the other. It is known that if the Strong Ω Conjecture holds then one cannot have an Ω-complete theory of the whole of the structure $V_{\omega+2}$:

Theorem A.1.5.3. Assume ZFC + "there is a proper class of Woodin cardinals". Assume the Strong Ω Conjecture. Then there is no recursively enumerable theory A such that ZFC + A is Ω-complete for Σ^2_3. [Koellner, 2009b: 117]

So this path is effectively closed off at the Σ^2_3 level. On the other hand, the following theorem shows that the continuum hypothesis is not sufficient to provide an Ω-complete theory at the Σ^2_2 level:

Theorem A.1.5.4. ZFC + CH is not Ω-complete for Σ^2_2. [Koellner and Woodin, 2009: 1166]

So this path is also closed.

One can consider further local scenarios at the Σ^2_2 level[95]. I take it that the upshot of these is that, if the Strong Ω Conjecture holds, then it is likely that there is a "best" theory in which the continuum hypothesis has a definite answer, even though there is not yet a compelling case to tell us which way that answer will fall [Koellner, 2011c: 33]. My impression is that the local approach is nearing the end of its usefulness. There were remarkable successes at the level of $V_{\omega+1}$ which were extended with great effort to fragments of $V_{\omega+2}$ but without any clear result for the continuum hypothesis. Cases multiply and it becomes hard to keep track of the arguments. The justification for the "best" cases becomes less compelling. Perhaps this situation will improve with further research but, regardless, the local approach has inherent limitations.

If the Strong Ω Conjecture does not hold, then there is the possibility that we could have an Ω-complete theory for $V_{\omega+2}$. In fact, it might be possible to achieve an Ω-complete theory for any specified level V_α and to piece these together into a coherent theory of the entire universe of sets. If this were achieved, then the theory would be a candidate for the effective completion of set theory along the lines envisaged by Gödel [1946: 151] (quoted previously). However, Koellner and Woodin [2009] have shown that such a theory could not be unique. If there is an Ω-complete theory of V_α for $\alpha \geq (\omega+2)$ then there is another, equally good but incompatible Ω-complete theory which differs on the truth value of

[95] See, e.g., Koellner [2011: 27-33].

the continuum hypothesis[96]. This seriously raises the spectre of pluralism. The existence of two, or more, coherent but incompatible Ω-complete theories of V would undermine the universalist's concept of truth. It would not be possible to combine such theories in the manner suggested by Martin [2001]. Such a scenario is considered by Steel [2004] and discussed at length by Maddy [2005: 369-373]. The theories would all be universal and equivalent in terms of the mathematics that would result (i.e., choosing one theory over another would not result in any behavioural or methodological differences). In choosing one theory, the others would be accessible as generic extensions. The continuum hypothesis (and other undecidable statements) would be meaningless in an absolute sense, but acquire a relative value as a matter of convention.

This scenario is highly speculative but does serve to crystallise a plausible pluralist case. Note that a multiverse of this kind does not conform to the radical pluralist view of what a multiverse might consist of (i.e., all consistent theories). It locks in a universalist view at least to the level of ZFC + "there is a proper class of measurable Woodin cardinals" which is essential for the application of the notion of Ω-completeness. In this way, it avoids the problems which plague the radical pluralist view.

Woodin [2011d] has used a global perspective to define and investigate a multiverse of a related kind, which he calls *the generic multiverse*. Ultimately, he sets up the generic multiverse position as a strawman, only to show that it is untenable under fundamental principles of set theory, *provided that the Ω Conjecture is true*. Here, once again, we see the importance of the Ω Conjecture.

Woodin's generic multiverse is generated from V by closing under generic extensions (forcing) and under generic refinements (inner models of a universe which the given universe is a generic extension of). The associated generic multiverse view of truth is that a sentence is true if and only if it holds in each universe of the generic multiverse. Letting the background theory be ZFC + "there is a proper class of measurable Woodin cardinals", this view entails that the continuum hypothesis is indeterminate whilst $AD^{L(R)}$ is true (and, hence, all the results of descriptive set theory, described previously, are locked in).

Woodin [2011d: 19-21] proves that for each Π_2-sentence φ; the following are equivalent:

1. φ holds across the generic multiverse;

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[96] Note that the continuum hypothesis is just a particular case; there are other suitable sentences which could be forced to differ between Ω-complete theories of V_α.

2. "φ is true in Ω-logic" holds across the generic multiverse;
3. "φ is true in Ω-logic" holds in at least one universe of the generic multiverse.

This means that if φ is Ω-valid in one universe then it holds in all the universes of the generic multiverse. Notably, a sentence asserting the existence of a proper class of Woodin cardinals (and its consequences) and a sentence asserting the truth of the Ω conjecture would be invariant across the generic multiverse. Woodin says: "...the notion of truth is the same as defined relative to each universe of the multiverse, so from the perspective of evaluating truth all the universes of the multiverse are equivalent" [Woodin, 2011d: 15]. He sees this as an important point in favour of the generic multiverse view. For one thing it implies that a statement (such as a consistency statement) can't be indeterminate in one universe and determinate in another, as might happen under the radical pluralist view. Thus, truth is not so sensitive to the meta-universe in which the generic multiverse is being defined and the pluralist isn't forced to retreat to an infinite regression of meta-universes in order to define his position.

Next, Woodin [2011d: 17-18] formulates two related multiverse laws:

1. The set of Π_2-multiverse truths cannot be recursive in the set of multiverse truths of V_α for any specifiable α; and
2. The set of Π_2-multiverse truths is not definable in V_α across the multiverse for any specifiable α.

These laws capture the idea that any reasonable concept of multiverse truth cannot be reduced to, or defined in, an initial fragment V_α of the universe V. That would lead to "a brand of formalism that denies the transfinite by reducing truth about the universe of sets to truth about a simple fragment" [Woodin, 2011d: 17]. Woodin's multiverse laws are in the spirit of the reflection principles of set theory.

Woodin [2011d: 23-25] goes on to show that, assuming both the Ω Conjecture and the existence of a proper class of Woodin cardinals, both multiverse laws are violated by the generic multiverse position. Specifically, he proves that both laws are violated at the level of V_{δ_0+1} where δ_0 is the least Woodin cardinal. This result can be sharpened to $V_{\omega+2}$ for the first multiverse law, but the import is the same in any case: the generic multiverse position is untenable because the set of all its truths is reducible to the set of truths of some initial fragment of V.

The pluralists' best response, short of placing restrictions on the generic multiverse, is to deny the truth of the Ω Conjecture. Woodin ([2011d: 28] and [2011b: 111-112]) argues that there is evidence for the Ω Conjecture, albeit nothing conclusive. Firstly, the Ω Conjecture can be shown to be consistent with the theory ZFC + "There is a proper class of Woodin cardinals". Secondly, the assertion of its Ω-satisfiability is a Σ_2-assertion which is invariant across the generic multiverse and there are no known examples of Σ_2-statements that are provably absolute and not settled by large cardinal axioms. Thirdly, "… recent results indicate that if [the Inner Model] program can succeed at the level of supercompact cardinals then no large cardinal hypothesis whatsoever can refute the Ω Conjecture" [Woodin, 2011d: 28]. These are the results from the theory of Ultimate L (as discussed in Section 4.5.3 in the main text).

A.1.6 Fending Off a Potential Threat to the Universalist View

In "The Realm of the Infinite" [2011b] Woodin begins by arguing that we implicitly accept the meaningfulness of the large finite and that there is nothing to prevent us from accepting the meaningfulness of the universe of sets (as discussed in Section 4.6 in the main text). In the next part of his paper he gives further arguments for accepting the truth of set theory, this time in opposition to a hypothetical Skeptic who denies any genuine meaning to a conception of uncountable sets (i.e., he accepts the meaningfulness of the large finite and the completed infinity of the natural numbers, but not of uncountable sets, and certainly not of large cardinals).

Woodin lays out the evidence for large cardinals and makes his prediction that the theory ZFC + "there exist infinitely many Woodin cardinals" will never be shown to be inconsistent (even though, analogously to the large finite case, an inconsistency *could* conceivably be found in the physical realm). The Skeptic accepts the evidence for the consistency of the theory, and agrees with the prediction, but doesn't accept that it is the result of the truth of the axioms (or, equivalently, the existence of the large cardinals). He suggests that the infinite realm is so inclusive that any reasonably-defined large cardinal axiom would turn out to be consistent, provided that there isn't a simple proof of its inconsistency. In particular, he points to the Reinhardt cardinal axiom, which is the strongest possible large cardinal axiom generated by the elementary embedding template (as discussed in Section A.1.2). Kunen produced a simple proof of its inconsistency with ZFC, but that proof depended on using the Axiom of Choice, and it is still possible that it is consistent with ZF. Furthermore, if the theory ZF + "there is a weak Reinhardt cardinal" *is* consistent then it implies the consistency of ZFC + "there is a proper class of ω-huge cardinals" and thereby accounts for all the consistency predictions of Woodin's accepted large cardinal hierarchy. So this one claim — the formal consistency of the Reinhardt axiom with ZF — would account for all the claims of set theory.

The Skeptic's claim creates a serious problem for Woodin. The consistency of the Reinhardt cardinal axiom could not be grounded in the *existence* of a Reinhardt cardinal because then we would have an immediate bifurcation of set theory into a universe in which the Axiom of Choice holds and one in which it does not hold. Anyway, there is no fine structure theory for a Reinhardt cardinal, so no witnesses to its existence. Rather, the set theorist might try to account for its formal consistency by showing that the theory ZF + "there is a weak Reinhardt cardinal" is equiconsistent with a theory of the form ZFC + LCA, for some large cardinal axiom LCA; in the same way that ZF + AD can be shown to be equiconsistent with ZFC + "there exist infinitely many Woodin cardinals", even though AD (the Axiom of Determinacy) itself is incompatible with the Axiom of Choice [Woodin, 2011b: 100-101]. However, this approach will not be possible because the Reinhardt cardinal axiom is stronger than any large cardinal axiom not known to refute the Axiom of Choice and, so, will not be equiconsistent with any such theory. Next Woodin explores the idea that the generic multiverse view of truth might be able to account for the prediction that weak Reinhardt cardinals are consistent with ZF [Woodin, 2011b: 107-108]. Again, this approach will not be possible because, as we saw previously, the generic multiverse view of truth is untenable under the assumption of the Ω Conjecture because it violates the multiverse laws.

Next Woodin plays his trump card — Ultimate L. If a solution to Ultimate L could be found then one corollary would be that the Reinhardt cardinal axiom is inconsistent with ZF [Woodin, 2011b: 115-116]. So, rather than contorting ZFC to account for the consistency of weak Reinhardt cardinals with ZF, it could be proved that they are, in fact, inconsistent. This would comprehensively refute the Skeptic's claim. More than that, it would provide "for the *first time* an example of a natural large cardinal axiom proved to be inconsistent as a result of a deep structural analysis" [Woodin, 2011b: 116]. Such an inconsistency result could never be achieved by Inner Model Theory because of its incremental nature. But Ultimate L would provide us with the means to explore the boundary between possible and impossible large cardinal axioms, strengthening its claim to being the ultimate arbiter of mathematical truth and existence. Woodin's hunch is that it would enable us to eliminate essentially all the large cardinal axioms known to contradict the Axiom of Choice, tantamount to a proof of the Axiom of Choice [Woodin, 2011c: 470]. Then, according to Woodin's view, any suggested set theory would need to be justified by first establishing its equiconsistency with a specific level of Ultimate L. Ulimate L, in its turn, would be justified by our understanding of the hierarchy of large cardinal axioms as "true axioms about the universe of sets" [Woodin, 2011b: 96].

Woodin thinks that one of the main pieces of evidence supporting the truth of large cardinal axioms is our ability to construct their fine structure theory. Using Inner Model Theory, evidence for each large cardinal axiom is built up from below. Starting from Gödel's L, minimal extenders are added at each stage to witness the existence of the next large cardinal in the hierarchy. However, we know that this process cannot be extended beyond the level of an inner model with one supercompact cardinal. The reason is that Woodin's investigation of this case has shown that such a model would automatically inherit all stronger large cardinals. Then these stronger large cardinal axioms would require some further extrinsic justification. Woodin suggests that the answer to this dilemma is to use the structural analogy of the theory of L(R) under $AD^{L(R)}$ to the theory of $L(V_{\lambda+1})$ under the assumption of an ω-huge cardinal to reveal the structure theory of ω-huge cardinals without constructing a full inner model theory [Woodin, 2011c: 469-470].

Appendix 2: Problems with Plenitude

A.2.1 Overview

This appendix considers the problems that arise if too much plenitude is postulated. It refers to David Lewis' *On the Plurality of Worlds* [1986] for a fair statement of the problems and a defence of plenitude. Lewis' main reason for embracing plenitude is that it allows a great benefit in terms of unity and economy of theory. He thinks that logical space for philosophers can be developed in an analogous way to set theory for mathematicians, thus enabling philosophers to use a vast realm of possibilia to reduce the required number of primitives and premises [1986: 4]. To pave the way he has shown in *Parts of Classes* [1991] how the constructions of set theory can be reproduced in mereology (i.e., the study of parts and wholes).

A.2.2 How Much Plenitude?

Lewis defends the thesis of modal realism which he states as:

> … the thesis that the world we are part of is but one of a plurality of worlds, and that we who inhabit this world are only a few out of all the inhabitants of all the worlds. [1986: vii].

According to this thesis, there are many possible worlds, some just like ours and some completely alien to our understanding. A possible world has parts which are possible individuals. A world is the mereological sum of all the possible individuals that are its parts[97]. Spatiotemporal relations make individuals cohere in a world.

Lewis advocates a principle of recombination. Parts of any world can be duplicated, multiplied and combined into a new world. There is just one proviso:

> The only limit on the extent to which a world can be filled with duplicates of possible individuals is that the parts of a world must be able to fit together with some possible size and shape of spacetime. Apart from that, anything can coexist with anything, and anything can fail to coexist with anything. [Lewis, 1986: 89-90].

[97] The mereological sum of several things is the least inclusive thing that includes all of them as parts.

The proviso limits the ways in which spacetime may be occupied (cf. set theory, in which there is no limit to the ways in which sets can be construed and the result is Cantor's hierarchy of infinities). If spacetime is a continuum, then it cannot accommodate more individuals than the cardinality of the real numbers. Lewis is prepared to contemplate different possible spacetimes but he is not prepared to let the cardinality of possible individuals yield consequences about the possible size of spacetime. Therefore, the cardinality of possible individuals is limited.

Forrest and Armstrong [1984] show that, without the proviso, the principle of recombination leads to paradoxes for Lewis' possible worlds analogous to the paradoxes of set theory. For example, consider the world which contains all the possible worlds as parts. This "biggest world" exists according to Lewis' mereology of possible worlds. Suppose that it has cardinality K. Forrest and Armstrong show how to construct a world with cardinality greater than K by extracting parts of the biggest world, duplicating and recombining them. However, by hypothesis, no world can have cardinality greater than K. Fortunately, the proviso prevents this argument from going through: the "bigger than biggest" world is too big to fit into the biggest world and, so, too big to fit into any possible spacetime.

Lewis relies on the proviso to prevent the construction of "bigger than biggest" worlds. This forces him to contemplate a cleavage between the structure of mathematics and the structure of possible worlds. There will be some set theoretical objects that are bigger than any possible world. Nevertheless (at some cost in terms of unity and economy of theory) Lewis sticks to the principle of plenitude in listing his conception of what there is altogether: "the possible worlds, the possible individuals that are their parts, and the mathematical objects, even if those should turn out to be pure sets not made out of the parts of the worlds" [1986: 111]. Lewis thinks that we should accept mathematics and he rejects the suggestion that any mathematical objects could be fictional. He is less concerned by Benacerraf's dilemma than some:

> If we are prepared to expand our existential beliefs for the sake of theoretical unity, and if thereby we come to believe the truth, then we attain knowledge. We can know that there exist countless objects causally isolated from us and unavailable to our inspection. [1986: 109].

According to Lewis, mathematics and possible worlds give us knowledge beyond the reach of our causal acquaintance. We arrive at this knowledge by reasoning from general principles (such as recombination). Indeed, the same principles are involved in constructing sets and possible worlds, it's just that possible worlds have an additional constraint.

The constraint on possible worlds derives from the intuition that they are just like our world in many relevant ways. They can be described as "concrete", just like our world, and so can be distinguished from the "abstract" world of mathematics, even though Lewis cannot find a principled distinction between concrete and abstract [1986: 81-86]. The question arises as to where to draw the line. Do concrete individuals have to fit into finite-dimensional spacetimes? Lewis thinks that that is too restrictive and he hopes that there is a natural break higher up: "high enough to make room for all the possibilities we really need to believe in" [1986: 103]. The break must be *natural* because, if it were arbitrary, then we could always imagine a possible world in which the boundary is moved. It is hard to see how this natural break would be defined and Lewis does not offer an answer. The constraint on possible worlds leads to an inevitable clash of intuitions.

A.2.3 The Convergence of Physical and Mathematical Objects

I think that Lewis' goals of unity and simplicity would be much better served if he abandoned the concrete/abstract distinction altogether or, at least, removed the structural distinction between concrete and abstract possibilia (in other words, if he allowed all possible worlds to be isomorphic to some mathematical structure and vice versa). Of course, this is a difficult option for those who see possible worlds as a combination of concrete parts because it means either abandoning the higher reaches of set theory (which Lewis will not do) or finding some physical interpretation for mathematical objects like large cardinals (impossible for Lewis).

A similar dilemma is faced by philosophers of mathematics who favour Aristotelian realism (see e.g. [Franklin, 2014]). Aristotelian realists take the view that the world consists of particulars and universals and that mathematical objects are universals which are *in re* (i.e., in the things themselves). In their view, mathematics studies structures which are instantiated in real objects such as bathroom tiles (groups) and coastlines (fractals). This leaves them with the problem of what to say about uninstantiated mathematical objects. In the modal version of Aristotelian realism, the response is that some mathematical objects can be constructed combinatorially from concepts sourced in the real world — they are possibilia similar to Lewis' possible worlds. Still, the assumption is that they *could* be instantiated in concrete things and this puts a constraint on what mathematical objects are supported by this view.

Some mathematical objects don't seem to have relevance to the physical world at all. For example, it is hard to envisage any role for the inaccessible numbers of set theory in any physical process. We would not reach that sort of cardinality even in a multiverse… even if an infinite number of universes had been continually generating baby universes since past infinity. An inaccessible number is just that: it cannot be reached by the iteration of constructive processes such as taking unions of collections of smaller sets, or taking power sets, or taking any combination of operations available in set theory. If one imagines translating all mathematical objects into the background language of set theory then the subset which is relevant to known physical processes is tiny. Feferman [2006b: 446] has suggested that it can be described by the predicative part[98].

One option is to restrict mathematics to the point where it can be physicalised in some sense, by claiming that some part of it is inconsistent or fictional. For example, one might hold (with Nelson [2011]) that only the finite part of mathematics is consistent. This could be combined with a view of the universe as a digital computer operating on a discrete system of binary information (as held by Fredkin [1992] and Wolfram [2002]). In this view, there would be a core part of mathematics — the finite part — which is consistent, firmly grounded in physical intuition, and real, and physics would only be concerned with that part. Physics and mathematics would both be translatable into a language of finite information processing. Presumably, Aristotelian realists could adopt this ultrafinitist position and, indeed, Franklin [2014: 133] questions whether infinities are really needed in applied mathematics. However, he seems to prefer to adopt a version of modal realism in which some of the truths of mathematics refer to actual physical instantiations whilst others refer to possible instantiations, and the possible instantiations can be infinite. This reinstates the problem of where to draw the line. Beyond the finite, there is no principled place to draw the line between physics and mathematics. Aristotelian realism supports an epistemology in which our knowledge of mathematical objects is explained in terms of our knowledge of everyday objects but it does not account for our knowledge of infinity.

Hartry Field is a philosopher who abolishes the distinction between concrete and abstract objects by denying the existence of abstract entities. He champions the view that mathematics is fictional and of only instrumental use in physical theories. As a test case, Field [1980] tries to identify the 'physical' part of Newtonian gravitational theory (i.e. the part which has empirical support and/or corresponds to physical structure) and separate it from the 'surplus' part (i.e. the part which provides logical and instrumental support). He develops a nominalised version of the theory which refers only to space-time points and regions, not to abstract mathematical objects. He defines bridge laws which map statements from the nominalised theory to its mathematical counterpart (and vice versa). According to

[98] According to predicativists, the natural numbers form a definite totality but the collection of all sets of natural numbers (i.e. the power set) does not.

Field, the advantage of using the mathematical theory is that it allows us to use all the deductive power of mathematics to demonstrate consequences which might have been obscure or difficult to prove in the nominalised theory. The mathematics doesn't lead to any new assertions about the physical world — all the physical consequences of the theory can be derived from the underlying nominalised theory. However, in order for the nominalised theory to retain the full power of the mathematical theory, Field is forced to build almost all the structure and complexity of the real number system into space-time itself. Effectively, he adds pseudo-mathematical structure to the nominalised theory until it converges towards the structure of the mathematical theory. Thus, his attempt to nominalise Newtonian gravitational theory leads to a counter-productive blurring of the concrete/abstract boundary.

I argue that the concrete/abstract boundary should be abolished by embracing all objects as abstract objects. What exists are minds (abstract individuals) and thoughts (abstract classes[99]). In this view, physical objects are a subset of mathematical objects but are not essentially different. Both physical and mathematical objects are the objects of intention of some mind (i.e. they are thoughts). The structure of the world instantiates all of mathematics, all possible forms, and physical reality condenses out by a process of self-actualisation in thought. I assume that the world is fully rational and that there are minds which are able to understand, test and validate their theories about its structure at all levels. Thus, any set is oversee-able by some mind. Gödel's incompleteness theorems show that every level of the set-theoretical hierarchy contains true statements which can only be proved using the resources of higher levels. The hierarchy of effective field theories in physics is similarly incomplete. A coherent and fully rational theory of physics and mathematics together requires a mind which embraces all of mathematics but transcends it. That mind is the encompassing mind which is the mereological sum of all individual minds and their thoughts. It is a proper class.

A.2.4 Comments on Lewis' *Parts of Classes*

In Lewis' metaphysical framework, reality divides exhaustively into individuals and classes [1991: 7]. An individual is anything that has no members but is a member of some class. A class is either a set or a proper class. A singleton is a class that has no parts except itself — it is a mereological atom. Every individual has a singleton and so does every set. The only things that lack singletons are the proper classes. A class is any mereological sum of singletons. A proper class does not have a singleton but it is the mereological sum of singletons.

[99] Not all classes are objects, some are proper classes.

The notion of a singleton is a primitive. Lewis finds it mysterious [1991: 29-35]. Singletons have only a single member so they do not fall under the usual description of a set as being a collection of many into one. He says: "Our utter ignorance about the nature of the singletons amounts to utter ignorance about the nature of classes generally" [1991: 31]. I say that a singleton is a thought: it is the result of a mind taking some individual or class as an object of intention. Classes are not in space-time and they are not located where their members are, but they are not nowhere; rather, they are associated with minds.

According to Lewis, any class includes the null set, but the null set is not a subclass or a part [1991: 10-11]. Classes do not have the null set as part because the null set is not a class, it is an individual. Any individual will do for the null set. Lewis redefines the null set as the mereological sum of all individuals [1991: 14]. It is everywhere rather than nowhere. This fits well with my framework. The null set is the mereological sum of all monads and so is the encompassing mind bare of its thoughts. It is the One which is transcendentally present in all beings and is their source [Section 5.4: 159].

A.2.5 Paradoxes of Plenitude

What about Forrest and Armstrong's paradox which arises when we allow unrestricted plenitude for possible worlds? Their argument does not get off the ground in my framework because it assumes that there is a big world which contains all the possible worlds as parts. In my framework, there is no world of all worlds, just as there is no set of all sets, because each individual monad is a possible world and the class of monads is a proper class.

Lewis [1986: 104-108] considers a second paradox of plenitude concerning the set of worlds that characterises the content of somebody's thought[100]. There is no natural bound on the cardinality of the set of somebody's thoughts and the argument uses this fact to construct a possible world whose cardinality exceeds the cardinality of the set of all possible worlds. As before, this argument fails in my framework because there is no set of all possible worlds. However, some interesting points arise in Lewis' discussion. For example, Lewis claims that:

> Not just any set of worlds is a set that might possibly give the content of someone's thought. Most sets of world, in fact all but an infinitesimal minority of them, are not eligible contents of thought. [1986: 105].

[100] Here, "somebody" is not considered to have human limitations (for example, it might refer to a god) and so it should be read as "some mind", maybe an infinite mind.

I do not agree with this. Any set defines a possible thought although not all possible thoughts define a set. Lewis is committed to the view that there are unthinkable contents. I hold that all content is thinkable by some mind. Lewis' discussion of unthinkable content seems only to show that there is content which it is perverse for a mind to think about its world, given its beliefs and desires. He does not show that such content is unthinkable. I can think "I want a saucer of mud" even if I cannot put myself into a state in which I would want to turn that thought into action. Another mind in another possible world might actually want a saucer of mud!

A.2.6 Actuality

Lewis' indexical analysis of actuality enables him to dodge some potential problems of plenitude, including ethical problems. According to Lewis:

> … every world is *actual* at itself, and thereby all worlds are on a par. This is *not* to say that all worlds are actual – there's no world at which that is true… [1986: 93].

For us, our world is actual and other possible worlds are not actual. Therefore, we should concern ourselves with evils in this world, even if there would be the same sum total of good and evil throughout the worlds no matter what we do in this one [1986: 123-128]. Reasonably, it is the good and evil in our actual world which we care about. My framework also denies that other worlds are actual so I can refer to Lewis' arguments in defense of these problems.

However, my analysis of actuality differs significantly from Lewis'. In quantum monadology, the actual world is the one and only, necessary, best possible world. In Lewis' modal realism, actuality is a contingent matter — it is not the case that one world alone is absolutely actual. First, Lewis asks how we would know that we are absolutely actual. Then, he writes:

> And yet we *do* know for certain that the world we are part of is the actual world – just as certainly as we know that the world we are part of is the very world we are part of. [1986: 93].

I take this knowledge that each of us has — of our own existence and actuality — as the most certain knowledge which we can have. It poses a problem for monadology in that it is not clear how an actual monad can know that it is actual rather than merely possible. One response is to say that it is a special property which actual monads have — to be immediately acquainted with their own actuality — but this is not satisfactory as an explanation. Another response is to invoke the concrete/abstract divide — the actual world is concrete and possible worlds are abstract – but this distinction is not available in the idealistic interpretation of monadology which I have adopted.

I propose to make a distinction between actual and possible monads in terms of consciousness — actual monads are conscious whereas potential monads are like philosophical zombies. Consciousness is a property of actual monads because they follow the critical path from beginning to end and complete the neural circuit in which meaning is communicated to each individual mind by the encompassing mind. Actual monads are individuated by a proper class of thoughts; they mirror the structure of the encompassing mind. The content of their thoughts is private to them and is revealed in ongoing communication with the encompassing mind. The way in which actual monads project possibilities and perceive actuality is like a question-and-answer session with the encompassing mind. In the process, their conceptual framework is expanded and the truth is unconcealed. Monads are essentially solipsistic and communication between them is mediated by the encompassing mind. The subjective aspect of a monad is determined by the content of its thoughts but the objective aspect is determined by the structure of its thoughts and is perceived by other monads as mathematical structure. Objectively, monads can be read like a mathematical string. Actual monads are a never-ending mathematical string which contains all the hierarchies of set theory and is transcendentally completed by the encompassing mind. Possible monads do not contain all the hierarchies of set theory; they deviate from the path of criticality at some point and never complete the circuit of consciousness. Actuality is carved out of potentiality by the process of quantum monadology — like Michelangelo's statue of David in the marble block, it is always there, but it is unconcealed by the process.

A.2.7 The Construction of Possible Worlds

Lewis' view of the construction of possible worlds is a static, mereological view involving the combination of parts into wholes. I don't find this way of constructing possible worlds satisfactory. It misses the dynamic, process component of the construction of possible worlds which imposes a hierarchy of priority on thoughts; the ordinal hierarchy as opposed to the cardinal hierarchy which Lewis draws from in his imaginative experiments. Spatiotemporal relations are not what makes a possible world cohere. It is the laws which enable monads to have harmonious relations. Possible worlds have to be built up coherently according to local laws.

Lewis seems to think that possible worlds help us to say what it means for a false theory of nature to be close to the truth: "A theory is close to the truth to the extent that our world resembles some world where that theory is exactly true" [1986: 24]. This doesn't seem right. Our world obeys a hierarchy of laws at different scales which result in harmony. A world in which (for example) Newton's laws were exactly true would not be like our world at all. For one thing, quantum mechanics is responsible for the stability of fundamental matter in our world. Lewis' worlds seem to be more like the dreams (imaginings) of monads at a particular scale, more wish than possible fact, not taking total harmony into account. As he says:

> … imaginability is a poor criteria of possibility. We can imagine the impossible, provided we do not imagine it in perfect detail and all at once. We cannot imagine the possible in perfect detail and all at once, not if it is at all complicated. [1986: 90].

I can imagine a hundred foot elephant walking around the earth but I know that it is not possible according to our physical laws and I doubt whether it is possible at all. It is not enough for possible worlds to be imaginable, they have to be actualisable.

I think that to properly define the construction of possible worlds it will be necessary to go beyond the mereological approach of *Parts of Classes* to embed mathematics as a bare structure in a bigger theory of concepts which has static (cardinal, to do with size) and dynamic (ordinal, to do with priority) components.