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ESTIMATING DEMAND WITH VARIED LEVELS OF AGGREGATION

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ABSTRACT

The response of consumer demand to prices, income, and other characteristics is important for a range of policy issues. Naturally, the level of detail for which consumer behaviour can be estimated depends on the level of disaggregation of the available data. However, it is often the case that the available data is differently aggregated in different time periods, with the information available in later time periods usually being more detailed. The applied researcher is thus faced with choosing between detail, in which case the more highly aggregated data is ignored; or duration, in which case the data must be aggregated up to the "lowest common denominator". Furthermore, since parametric demand systems invariably involve a large number of parameters, with the number increasing at least linearly with the number of expenditure categories, it may well be that only the second option is feasible. That is, there is simply not enough data available at the finer aggregation level for the chosen model to be estimated.

This paper develops a specification/estimation technique that exploits the entire information content of a variably-aggregated data set. The technique is based on the observation that the more highly aggregated data does in fact contain information on the finer subcategories: viz, the *sum* of certain subcategory expenditures is observed. It is thus possible, under certain simplifying assumptions, to write down, and maximize, the likelihood of the observed data as a function of the parameters of the chosen model written for the finest available level of disaggregation. The technique is applied to an ABS dataset containing historical information relating to private final consumption expenditures on up to 18 commodities, and found to be feasible for both the LES and AIDS.

KEYWORDS: Singular demand systems, Linear expenditure system, Almost ideal demand system, Missing data.

JEL classification: C32, C51, D12, E21

1. Introduction.

The response of consumer demand to prices, income, and demographic and other characteristics is important for a range of policy issues, such as the effects of a change in the tax mix, and welfare calculations. Estimation of such response depends on economic theory, a statistical model, and a data source. Naturally, the precision and reliability of parameter estimates relies critically on the accuracy and time span of the available data; and the level of detail to which consumer behaviour can be estimated depends on the level of disaggregation of the available data.

Typically, however, the available data is differently aggregated in different time periods, with the information available in later time periods generally being more detailed. The applied researcher is thus faced with choosing between detail, in which case the more highly aggregated data is ignored; or duration, in which case the data must be aggregated up to the “lowest common denominator”. Furthermore, since parametric demand systems invariably involve a large number of parameters, with the number increasing at least linearly with the number of expenditure categories, it may well be that only the second option is feasible. That is, there is simply not enough data available at the finer aggregation level for the chosen model to be estimated.

The aim of this paper is the development of a specification/estimation technique that exploits the entire information content of a variably-aggregated data set. The technique is based on the observation that the more highly aggregated data does in fact contain information on the finer subcategories, in that the *sum* of the missing subcategory expenditures is observed. It is therefore possible to construct the likelihood of the observed expenditure data as a function of the parameters of the chosen model written for the finest available level of disaggregation. The precise form of the resulting likelihood function is indicated for the Linear Expenditure System (LES) and the Almost Ideal Demand System (AIDS), chosen as illustrative examples.

The technique is then applied to an ABS dataset containing detailed historical information relating to private final consumption expenditures on a wide range of commodities[1], resulting in more detailed and more precise parameter estimates than would normally be available. Implications for the detailed analysis of policy questions of current interest, such as the effect on behaviour of a change in the tax mix, should be obvious.

2. Model and notation.

Consider a system of demand equations $\boldsymbol{q} = \mathcal{Q}(\boldsymbol{p}, m, \theta)$, where $\boldsymbol{q}$ is an N-vector of goods with price vector $\boldsymbol{p}$, m is income (assumed equal to total expenditure), the vector θ contains the parameters of the utility function, and the functions $\mathcal{Q}_i(\cdot)$, $i = 1,\ldots,N$, satisfy the restrictions implied by the theory of consumer demand.

For the purposes of estimation the endogenous variables q_i are generally transformed to expenditures $x_i = p_i q_i$, or, further, to expenditure shares $w_i = p_i q_i / m$ (to be more consistent with an assumption of homoscedasticity and to remove dependence on the numeraire). This leads to the standard specification in demand analysis: the estimation of the parameters of the system of share equations

$$w_i = \mathcal{W}_i(p_1, \ldots, p_N, m; \theta) + u_i \; ; \qquad i = 1, \ldots, N \; .$$

More precisely, the $1 \times N$ vector comprising the t^{th} observation on the N expenditure shares[2] $\tilde{\mathbf{w}}_t' = \tilde{\mathbf{x}}_t' / m_t$, is modelled as a function of the N-vector of prices $\tilde{\mathbf{p}}_t'$, income in the t^{th} period $m_t = \sum_{i=1}^{N} x_{it}$, the parameter vector θ, and an additive, serially independent, zero mean disturbance, with constant variance-covariance matrix $\tilde{\Sigma}$; ie,

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[1] The dataset was compiled in previous joint work (McLaren, Rossiter and Powell (2000)) with the Australian Bureau of Statistics (ABS); and in effect extends the publicly available expenditure data back to 1969/70 for certain subcategories of Other Goods and Services.

[2] Quantities pertaining to the complete N-commodity system are henceforth indicated by a "~" over the symbol for the corresponding "full rank" quantity – *cf.* equation (2.2).

$$\underset{1\times N}{\tilde{\mathbf{w}}'_t} = \mathcal{W}(\tilde{\mathbf{p}}'_t, m_t, \theta) + \tilde{\mathbf{u}}'_t\ ; \quad \tilde{\mathbf{u}}_t \sim (0, \underset{N\times N}{\tilde{\boldsymbol{\Sigma}}})\ . \tag{2.1}$$

The demand system "adding-up" condition, making $\tilde{\boldsymbol{\Sigma}}$ singular, of rank $n = N - 1$, is then as usual avoided by "dropping" one of the expenditure categories, so yielding a full rank system involving T observations on n categories:

$$\underset{1\times n}{\mathbf{w}'_t} = \mathcal{W}(\tilde{\mathbf{p}}'_t, m_t, \theta) + \mathbf{u}'_t\ ; \quad \mathbf{u}_t \sim (0, \underset{n\times n}{\boldsymbol{\Sigma}})\ . \tag{2.2}$$

The model is completed by the conventional assumption that $\mathbf{u}_t$ is distributed n-variate normal[3], and the standard Gaussian log-likelihood[4] follows; ie,

$$\ell(\theta, \boldsymbol{\Sigma}) = -\frac{T}{2}\ln|\boldsymbol{\Sigma}| - \frac{1}{2}\operatorname{tr}(\boldsymbol{\Sigma}^{-1}\mathbf{U}'\mathbf{U})\ , \tag{2.3}$$

where

$$\underset{T\times n}{\mathbf{U}} = \underset{T\times n}{\mathbf{W}} - \mathcal{W}(\underset{T\times N}{\tilde{\mathbf{P}}}, \underset{T\times 1}{\mathbf{m}}, \theta) \tag{2.4}$$

is the $T \times n$ matrix of disturbances, $\mathbf{W}$ is the $T \times n$ matrix of observed expenditure shares, and $\mathcal{W}$ is the $T \times n$ matrix of expected expenditure shares, conditional on the $T \times N$ matrix of prices $\tilde{\mathbf{P}}$, the $T \times 1$ vector of total expenditures $\mathbf{m}$, and the vector of "mean" parameters, θ.

3. The "aggregated" likelihood.

Now consider the situation in which the expenditure data is available at differing levels of disaggregation in different subperiods. For example: suppose that expenditure data is initially collected for categories "Food", "Durables" and "Other"; where "Other" is later split into "Other goods" and "Other services". That is, data is available for shares of $N = 3$ commodities (A, B, C) in the total budget for the first

[3] Although, as the dependent variable is now by definition constrained to be both non-negative, and to sum to unity, it can be argued that the disturbance distribution should be specified so as to avoid violating this constraint. See, for instance, Fry, Fry and McLaren (1996).

[4] For clarity, all likelihoods will be written without their density function constants.

time period, and for N = 4 commodities (1, 2, 3, 4) for a later time period. Thus we observe only

$$w_{At}, w_{Bt}, w_{Ct}; \quad t = 1,\ldots,T_1,$$

and
$$w_{1t}, w_{2t}, w_{3t}, w_{4t}; \quad t = T_1 + 1,\ldots,T;$$

where, by the nature of the problem (and for later convenience setting "*A*" equal to "Other goods and services", "*B*" equal to "Durables" and "*C*" equal to "Food"),

$$\begin{aligned} & w_{At} \equiv w_{1t} + w_{2t}, \\ & w_{Bt} \equiv w_{3t}, \\ \text{and} \quad & w_{Ct} \equiv w_{4t}, \qquad t = 1,\ldots,T_1. \end{aligned}$$

Standard estimation strategies in such a situation would be to:

(a) aggregate the data for the period $t = T_1 + 1,\ldots,T$ and apply the theory to the case of N = 3, for the entire period $t = 1,\ldots,T$; or

(b) use a statistical method to interpolate the data on w_{At} for the period $t = 1,\ldots,T_1$ to construct an approximate statistical series for w_{1t} and w_{2t} for the period $t = 1,\ldots,T_1$, and then carry out estimation for the case of N = 4 for the period $t = 1,\ldots,T$; or

(c) estimate separate models for the subperiods $t = 1,\ldots,T_1$ and $t = T_1 + 1,\ldots,T$.

However, it must be the case that the *expected* expenditure on commodity A is just the sum of the expected expenditures on the component commodities 1 and 2, and hence the stochastic part of w_A is also the sum over the sub-commodities; that is,

$$\mathcal{W}_{At}(p, m; \theta) \equiv \mathcal{W}_{1t}(p, m; \theta) + \mathcal{W}_{2t}(p, m; \theta),$$

and
$$u_{At} \equiv u_{1t} + u_{2t}.$$

In other words, an economic model specified for the most disaggregated data necessarily implies a corresponding economic model applying to the data at any level of aggregation. The same statement applies to the accompanying statistical model.

To generalise this, let us assume $S > 1$ subperiods $\mathcal{T}_1, \ldots, \mathcal{T}_S$ with differing degrees of expenditure category aggregation; and note that the *observed* expenditure shares in each subperiod are necessarily a linear combination of the underlying (partially unobserved) expenditure set $\tilde{\mathbf{w}}$. That is, for $t \in \mathcal{T}_r$ we observe only the linear combination $\tilde{\mathbf{y}}_t = \tilde{\mathbf{A}}_\mathbf{r}\tilde{\mathbf{w}}_t$, where $\tilde{\mathbf{A}}_\mathbf{r}$ is a $N_r \times N$ "aggregation matrix", of rank $N_r \leq N$, taking the N-vector $\tilde{\mathbf{w}}$ into the N_r-vector of observed, but more aggregated, expenditures $\tilde{\mathbf{y}}$. We also assume that, for at least one of our subperiods (usually the last), expenditures on all N commodities are observed, in which case the implicit aggregation matrix for that subperiod is the $N \times N$ identity (ie, $\tilde{\mathbf{y}}_t = \tilde{\mathbf{w}}_t$ for $t \in \mathcal{T}_S$). With $\tilde{\mathbf{w}}_t$ generated as per (2.1) the model for $\tilde{\mathbf{y}}_t$, $t \in \mathcal{T}_r$, is then just

$$\tilde{\mathbf{y}}_t' = \tilde{\mathbf{w}}_t'\tilde{\mathbf{A}}_\mathbf{r}' = \mathcal{W}(\tilde{\mathbf{p}}_t', \mathbf{m}_t, \theta)\tilde{\mathbf{A}}_\mathbf{r}' + \tilde{\mathbf{u}}_t'\tilde{\mathbf{A}}_\mathbf{r}' \quad ,$$

with[5]

$$\tilde{\mathbf{A}}_\mathbf{r}\tilde{\mathbf{u}}_t \sim \mathrm{N}(\mathbf{0}, \tilde{\mathbf{A}}_\mathbf{r}\tilde{\Sigma}\tilde{\mathbf{A}}_\mathbf{r}') \,.$$

Thus, in the context of our introductory example, with $\{w_1, w_2, w_3, w_4\}$ denoting the (partially unobserved) expenditure shares for the "disaggregated" set "Other goods", "Other services", "Durables", "Food"; and $\{w_A, w_B, w_C\}$ denoting expenditure shares for "Other", "Durables", and "Food" respectively, the additional information that $w_A \equiv w_1 + w_2$ instantly implies

$$\begin{pmatrix} w_A \\ w_B \\ w_C \end{pmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix} = \tilde{\mathbf{A}}\tilde{\mathbf{w}} \,.$$

[5] The assumption of an additive multivariate normal disturbance is clearly advantageous in this setting.

$\tilde{\mathbf{A}}$ thus aggregates $\tilde{\mathbf{w}}$ = {w_1, w_2, w_3, w_4} into $\tilde{\mathbf{y}}$ = {w_A, w_B, w_C}, and for at least some subset of the sample period (specifically, t = 1,…,T_1) only the linear combination $\tilde{\mathbf{y}}_t = \tilde{\mathbf{A}}\tilde{\mathbf{w}}_t$ is observed. The model for $\tilde{\mathbf{y}}_t$ then follows from that assumed for $\tilde{\mathbf{w}}_t$.

As before, converting the model for $\tilde{\mathbf{y}}$ into a "full rank" equivalent is most simply accomplished by omission of one of the (possibly aggregated) expenditure categories, corresponding to deletion of the matching row from the aggregation matrix. More formally, note that elimination of the last equation/commodity from the N-vector $\tilde{\mathbf{w}}$ corresponds to pre-multiplication by the $n \times N$ matrix $\mathbf{J} = \left[\mathbf{I_n} \;\vdots\; \mathbf{0}\right]$; ie:

$$\underset{n\times 1}{\mathbf{w}} = \mathbf{J} \underset{N\times 1}{\tilde{\mathbf{w}}} .$$

Accordingly, for the r^{th} subperiod, with N_r observed expenditure shares $\tilde{\mathbf{y}} = \tilde{\mathbf{A}}_\mathbf{r}\tilde{\mathbf{w}}$, let

$$\underset{n_r\times 1}{\mathbf{y}} = \mathbf{J_r} \underset{N_r\times 1}{\tilde{\mathbf{y}}} ,$$

where $n_r = N_r - 1$, and $\mathbf{J_r} = \left[\mathbf{I_{n_r}} \;\vdots\; \mathbf{0}\right]$. Then $\mathbf{y}_t = \mathbf{J_r}\tilde{\mathbf{A}}_\mathbf{r}\tilde{\mathbf{w}}_t$, and, as before, the model for $\mathbf{y}$ would follow quite simply from that assumed for $\tilde{\mathbf{w}}$.

The expression for $\mathbf{y}$ simplifies even further if we assume a system in which at least one category (such as "Food", in the example above) is common to all subperiods, as we can then order the commodities such that

$$\tilde{\mathbf{A}}_\mathbf{r} = \begin{bmatrix} \mathbf{A_r} & \mathbf{0} \\ \mathbf{0}' & 1 \end{bmatrix},$$

where the top-left submatrix $\mathbf{A_r}$ is $n_r \times n$. Consequently, $\mathbf{J_r}\tilde{\mathbf{A}}_\mathbf{r} = \left[\mathbf{A_r} \quad \mathbf{0}\right]$, and

$$\mathbf{y} = \left[\mathbf{A_r} \quad \mathbf{0}\right]\begin{pmatrix} \mathbf{w} \\ w_N \end{pmatrix} = \mathbf{A_r}\mathbf{w} .$$

Exclusion of the last commodity equation to avoid the adding-up problem now corresponds to deletion of the last row and column of $\tilde{\mathbf{A}}$, and we have, for $t \in \mathcal{T}_r$,

$$\mathbf{y}_t' = \mathbf{w}_t'\mathbf{A}_\mathbf{r}' = \mathcal{W}(\tilde{\mathbf{p}}_t', m_t, \theta)\mathbf{A}_\mathbf{r}' + \mathbf{u}_t'\mathbf{A}_\mathbf{r}' \; ; \quad \mathbf{A}_\mathbf{r}\mathbf{u}_t \sim \mathrm{N}(0, \mathbf{A}_\mathbf{r}\Sigma\mathbf{A}_\mathbf{r}') \; .$$

Assuming T_r such observations then yields the log-likelihood for the r^{th} subperiod as

$$\ell_r(\theta, \Sigma) = -\frac{T_r}{2}\ln|\Sigma_\mathbf{r}| - \frac{1}{2}\mathrm{tr}(\Sigma_\mathbf{r}^{-1}\mathbf{U}_\mathbf{r}'\mathbf{U}_\mathbf{r}) \, , \tag{3.1}$$

in which
$$\underset{n_r \times n_r}{\Sigma_\mathbf{r}} = \mathbf{A}_\mathbf{r}\Sigma\mathbf{A}_\mathbf{r}' \; ,$$

and
$$\underset{T_r \times n_r}{\mathbf{U}_\mathbf{r}} = \underset{T_r \times n_r}{\mathbf{Y}_\mathbf{r}} - \mathcal{W}(\underset{T_r \times N}{\tilde{\mathbf{P}}_\mathbf{r}}, \underset{T_r \times 1}{\mathbf{m}_\mathbf{r}}, \theta)\underset{n \times n_r}{\mathbf{A}_\mathbf{r}'} \, . \tag{3.2}$$

Here $\mathbf{U}_\mathbf{r}$ and $\mathbf{Y}_\mathbf{r}$ are the $T_r \times n_r$ matrices of disturbances and observed expenditure shares for the r^{th} subperiod, and $\mathcal{W}(\tilde{\mathbf{P}}_\mathbf{r}, \mathbf{m}_\mathbf{r}, \theta)$, for convenience also denoted $\mathcal{W}_\mathbf{r}(\theta)$, is the $T_r \times n$ matrix of expected expenditure shares, conditional on the $T_r \times N$ matrix[6] of prices pertaining to the r^{th} subperiod $\tilde{\mathbf{P}}_\mathbf{r}$, the $T_r \times 1$ vector of total expenditures $\mathbf{m}_\mathbf{r}$, and the k-vector of mean parameters θ.

Assuming independence of observations across time periods then yields the complete, or "aggregated", log-likelihood

$$\ell(\theta, \Sigma) = \sum_{r=1}^{S} \ell_r = -\frac{1}{2}\sum_{r=1}^{S} T_r \ln\left|\mathbf{A}_\mathbf{r}\Sigma\mathbf{A}_\mathbf{r}'\right| - \frac{1}{2}\sum_{r=1}^{S}\mathrm{tr}\left(\mathbf{U}_\mathbf{r}'\mathbf{U}_\mathbf{r}(\mathbf{A}_\mathbf{r}\Sigma\mathbf{A}_\mathbf{r}')^{-1}\right). \tag{3.3}$$

It is now a straightforward matter, given θ and Σ, to calculate the joint likelihood for the entire sample allowing for the varying levels of aggregation within the sample, provided price data is available on all commodities for the entire period. The only remaining requirement for specification of the aggregated likelihood is a parametric model for the expected expenditure shares; with the comparatively parsimonious Linear Expenditure System serving as a convenient starting point.

[6] Note that it is implicitly assumed that, although expenditure data is not available for all N commodities in all time periods, price data is.

3.1 Application to the Linear Expenditure System

For the LES the t^{th} expenditure on the i^{th} commodity is modelled as

$$x_{it} = p_{it}\gamma_i + \beta_i(m_t - \tilde{\mathbf{p}}_t'\gamma) + v_{it}\ ,\ \ i = 1,\ldots,N,\ \ t = 1,\ldots,T;$$

where β_i, γ_i are parameters, γ is the N-vector $(\gamma_1, \gamma_2, \ldots, \gamma_N)'$, and the adding-up condition implies $\sum_1^N \beta_i = 1$. In expenditure share form this becomes

$$w_{it} = \frac{p_{it}}{m_t}\gamma_i + \beta_i\left(1 - \frac{\tilde{\mathbf{p}}_t'}{m_t}\gamma\right) + u_{it}\ ,\ \ i = 1,\ldots,N,\ \ t = 1,\ldots,T;$$

with total expenditure now subsumed into the price matrix as a divisor. Excluding the N^{th} equation and rewriting this in vector notation as per (2.2) then yields

$$\mathbf{w}_t' = \begin{bmatrix} 1 & \tilde{\mathbf{p}}_t'/m_t \end{bmatrix} \Pi(\beta,\gamma) + \mathbf{u}_t',\quad t = 1,\ldots,T, \tag{3.4}$$

where β is an n-vector excluding β_N and

$$\underset{(N+1)\times n}{\Pi(\theta)} = \Pi(\underset{n\times 1}{\beta}, \underset{N\times 1}{\gamma}) = \begin{bmatrix} \beta_1 & \beta_2 & \cdots & \beta_n \\ \gamma_1(1-\beta_1) & -\gamma_1\beta_n & \cdots & -\gamma_1\beta_n \\ -\gamma_1\beta_1 & \gamma_2(1-\beta_2) & \cdots & -\gamma_1\beta_n \\ \vdots & \vdots & \cdot & \vdots \\ -\gamma_n\beta_1 & -\gamma_n\beta_2 & \cdots & \gamma_n(1-\beta_n) \\ -\gamma_N\beta_1 & -\gamma_N\beta_2 & \cdots & -\gamma_N\beta_n \end{bmatrix}.$$

The expectation of the $T \times n$ matrix of expenditure shares is therefore

$$\mathcal{W}(\tilde{\mathbf{P}}, \mathbf{m}, \theta) = \begin{bmatrix} \iota_{\mathbf{T}} & \mathbf{P} \end{bmatrix} \Pi(\beta,\gamma),$$

where $\iota_{\mathbf{T}}$ is the T-vector of ones, and $\mathbf{P}$ denotes the $T \times N$ matrix of prices scaled by total expenditure (ie, $\mathbf{p}_t = \tilde{\mathbf{p}}_t / m_t$). Accordingly, for the r^{th} subperiod,

$$\underset{T_r\times n}{\mathcal{W}_{\mathbf{r}}(\theta)} \equiv \mathcal{W}(\tilde{\mathbf{P}}_{\mathbf{r}}, \mathbf{m}_{\mathbf{r}}, \theta) = \begin{bmatrix} \underset{T_r\times 1}{\iota_{T_r}} & \underset{T_r\times N}{\mathbf{P}_{\mathbf{r}}} \end{bmatrix} \underset{(N+1)\times n}{\Pi(\beta,\gamma)}\ . \tag{3.5}$$

3.2 AIDS

While the LES has the advantage of deriving directly from a well-defined utility function, and thus automatically satisfying the necessary theoretical restrictions, it can be criticised on the grounds that it simply has too few parameters to adequately model, in particular, the $\frac{1}{2}n(n-1)$ substitution effects involved in a N-commodity demand system. We therefore also consider the Almost Ideal Demand System of Deaton and Muellbauer (1980), viz

$$w_{it} = \alpha_i + \beta_i \ln\frac{m_t}{P_t} + \sum_{j=1}^{N} \gamma_{ij} \ln p_{jt} + u_{it}\ , \ \ i = 1,\ldots,N,\ \ t = 1,\ldots,T; \qquad (3.6)$$

where adding-up implies $\sum_{i=1}^{N} \alpha_i = 1$, $\sum_{i=1}^{N} \beta_i = 0$, and $\sum_{i=1}^{N} \gamma_{ij} = 0$; homogeneity requires $\sum_{j=1}^{N} \gamma_{ij} = 0$; and $\gamma_{ij} = \gamma_{ji}$ ensures Slutsky symmetry.

Strictly speaking, the deflator P_t should enter (3.6) via the translog price index

$$\ln P_t = \alpha_0 + \sum_{j=1}^{N} \alpha_j \ln p_{jt} + \frac{1}{2}\sum_{j=1}^{N}\sum_{\ell=1}^{N} \gamma_{j\ell} \ln p_{jt} \ln p_{\ell t}\ .$$

However, as we shall see, for any more than a few commodity categories the computational burden imposed by the symmetry restriction is already sufficiently onerous without imposing another level of nonlinearity. It is therefore common to either replace $\ln P$ by, for instance, Stone's price index $\sum_{i=1}^{N} w_i \ln p_i$; or, even more simply, to use real expenditure directly if this is already available. If this is done we have

$$\underset{1\times N}{\tilde{\mathbf{w}}'_t} \ = \ \tilde{\alpha}' + \left(\ln\frac{m_t}{P_t}\right)\tilde{\beta}' + \left(\ln \mathbf{p}'_t\right)\underset{N\times N}{\tilde{\Gamma}} + \tilde{\mathbf{u}}'_t\ ,\ \ t = 1,\ldots,T\ .$$

Deleting the N^{th} equation and imposing homogeneity directly then yields

$$\underset{1\times n}{\mathbf{w}'_t} \ = \ \alpha' + \left(\ln\frac{m_t}{P_t}\right)\beta' + \left(\ln\frac{\mathbf{p}'_t}{p_{Nt}}\right)\underset{n\times n}{\Gamma} + \mathbf{u}'_t\ ,\ \ t = 1,\ldots,T\ , \qquad (3.7)$$

implying

$$\underset{T\times n}{\mathbf{W}} = \iota_{\mathbf{T}}\alpha' + \mathbf{h}\beta' + \mathbf{R}\underset{n\times n}{\Gamma} + \mathbf{U},$$

where **h** is the *T*-vector with t^{th} element equal to the logarithm of real income $\ln(m_t / P_t)$, **R** is the $T \times n$ matrix with t^{th} row equal to the logarithm of the vector of normalised prices $\ln(\mathbf{p}_t' / p_{Nt})$, and Γ is $n \times n$ symmetric. Clearly, it is only the cross-equation symmetry restrictions that now make the system nonlinear.

The r^{th} subperiod matrix of expected expenditure shares required by (3.2) is therefore

$$\underset{T_r\times n}{\mathcal{W}_{\mathbf{r}}(\theta)} = \left[\underset{T_r\times 1}{\iota_{\mathrm{T}_r}} \quad \underset{T_r\times 1}{\mathbf{h}_{\mathbf{r}}} \quad \underset{T_r\times n}{\mathbf{R}_{\mathbf{r}}}\right]\left[\alpha \quad \beta \quad \Gamma\right]', \tag{3.8}$$

where $\mathbf{h_r}$ and $\mathbf{R_r}$ denote the T_r -vector of log-real incomes and the $T_r \times n$ matrix of log-normalised prices in the r^{th} subperiod.

4. Maximizing the aggregated likelihood

Specification of an "aggregated" likelihood is thus relatively straightforward. Estimation of the parameters of such a likelihood is, however, another matter. To see this, reconsider the conventional Gaussian likelihood of Section 2. It so happens, in this case, that the first order condition (FOC) for $\mathbf{\Sigma}$ has a simple closed form solution, enabling the construction of a profile, or "concentrated" likelihood for θ of the familiar log-determinant form

$$\ell^*(\theta) = -\frac{T}{2}\ln|\mathbf{U}'\mathbf{U}| \ . \tag{4.1}$$

An optimization problem previously involving *k* mean parameters θ, plus $\frac{1}{2}n(n+1)$ covariance parameters, now depends only the former, and so is far more likely to be feasible. Indeed, it is not, in general, possible to maximize (2.3) with respect to both θ and $\mathbf{\Sigma}$ numerically unless the number of expenditure categories is very small.

Exploitation of the closed form MLE of Σ can therefore be crucial to estimation of the system.

Contrast this with the situation pertaining in the context of "aggregated" likelihood (3.3). The $n \times n$ matrix of scores with respect to the elements of Σ is now

$$\frac{\partial \ell}{\partial \Sigma} = \sum_{r=1}^{S} \frac{\partial \ell_r}{\partial \Sigma} ;$$

where ℓ_r is a scalar-valued function of the quadratic $\mathbf{\Sigma_r} = \mathbf{A_r}\Sigma\mathbf{A_r'}$ as per (3.1), and the $(i, j)^{\text{th}}$ element of $\mathbf{\Sigma_r}$ is just $(\mathbf{\Sigma_r})_{\text{ij}} \equiv \mathbf{a}_{\text{i}}'^{(\text{r})}\Sigma\,\mathbf{a}_{\text{j}}^{(\text{r})}$ with $\mathbf{a}_{\text{i}}^{(\text{r})}$ denoting the $n \times 1$ vector obtained by transposing the i^{th} row of aggregation matrix $\mathbf{A_r}$. The contribution of the r^{th} subperiod to the score with respect to Σ then proceeds by application of Lemmas A.1 – A.3 (Appendix A) as

$$\frac{\partial \ell_r}{\partial \Sigma} = \sum_{i=1}^{n_r} \sum_{j=i}^{n_r} \bar{q}_{\text{ij}}^{(\text{r})} \underset{n\times n}{\overline{\overline{\mathbf{A}}}_{\text{ij}}^{(\text{r})}} , \tag{4.2}$$

in which $\bar{q}_{\text{ij}}^{(\text{r})} \equiv \partial \ell_r / \partial(\mathbf{\Sigma_r})_{\text{ij}}$ is the $(i, j)^{\text{th}}$ element of

$$\underset{n_r\times n_r}{\overline{\mathbf{Q}}_{\mathbf{r}}} \equiv \frac{\partial \ell_r}{\partial \mathbf{\Sigma_r}} = \mathbf{\Sigma_r^{-1}}\left\{\mathbf{U_r'U_r} - T_r\mathbf{\Sigma_r}\right\}\mathbf{\Sigma_r^{-1}} - \frac{1}{2}\mathbf{D}_{\mathbf{\Sigma_r^{-1}}\{\mathbf{U_r'U_r}-T_r\mathbf{\Sigma_r}\}\mathbf{\Sigma_r^{-1}}} ;$$

$\overline{\overline{\mathbf{A}}}_{\text{ij}}^{(\text{r})}$ is the $(i, j)^{\text{th}}$ $n \times n$ submatrix of $\underset{nn_r\times nn_r}{\overline{\overline{\mathbf{A}}}_{\mathbf{r}}} = \frac{\partial \mathbf{\Sigma_r}}{\partial \Sigma} = \frac{\partial \mathbf{A_r}\Sigma\mathbf{A_r'}}{\partial \Sigma}$ defined by

$$\underset{n\times n}{\overline{\overline{\mathbf{A}}}_{\text{ij}}^{(\text{r})}} \equiv \frac{d(\mathbf{\Sigma_r})_{\text{ij}}}{d\Sigma} = \mathbf{a}_{\text{i}}^{(\text{r})}\mathbf{a}_{\text{j}}'^{(\text{r})} + \mathbf{a}_{\text{j}}^{(\text{r})}\mathbf{a}_{\text{i}}'^{(\text{r})} - \mathbf{D}_{\mathbf{a}_{\text{i}}^{(\text{r})}\mathbf{a}_{\text{j}}'^{(\text{r})}} ;$$

$\mathbf{D_A}$ denotes the $n \times n$ diagonal matrix with $(i, i)^{\text{th}}$ element equal to the corresponding diagonal element of the $n \times n$ matrix $\mathbf{A}$[7], and the matrix differential $\partial\mathbf{\Sigma_r}/\partial\Sigma$ is

[7] The same notation is, without ambiguity, employed for the "diagonalisation" of a vector; ie; if **a** is an n-vector then $\mathbf{D_a}$ will denote the $n \times n$ diagonal matrix with $(i, i)^{\text{th}}$ element equal to the corresponding

defined as per MacRae (1974)[8]. The double-sum in (4.2) will be referred to (with notation and terminology borrowed from MacRae) as a modified “star product”, and written

$$\frac{\partial \ell_r}{\partial \boldsymbol{\Sigma}} = \underset{n_r \times n_r}{\overline{\mathbf{Q}}_{\mathbf{r}}} \,\dot{*}\, \underset{nn_r \times nn_r}{\overline{\overline{\mathbf{A}}}_{\mathbf{r}}} \quad ,$$

implying $$\frac{\partial \ell}{\partial \boldsymbol{\Sigma}} = \sum_{r=1}^{S} \big(\underset{n_r \times n_r}{\overline{\mathbf{Q}}_{\mathbf{r}}} \,\dot{*}\, \underset{nn_r \times nn_r}{\overline{\overline{\mathbf{A}}}_{\mathbf{r}}} \big) \,.$$

Thus the FOC for $\boldsymbol{\Sigma}$ consists of the sum, over *r*, of functions in which, even leaving aside the complication posed by the star-product with $\overline{\overline{\mathbf{A}}}_{\mathbf{r}}$, $\boldsymbol{\Sigma}$ only ever appears via the quadratic $\boldsymbol{\Sigma}_{\mathbf{r}} = \mathbf{A}_{\mathbf{r}}\boldsymbol{\Sigma}\mathbf{A}'_{\mathbf{r}}$. It is evident that we can no longer obtain a closed form solution for the MLE $\hat{\boldsymbol{\Sigma}}$, and so cannot derive, after the manner of (4.1), a profile likelihood for θ. The $\frac{1}{2}n(n+1)$ parameters of the covariance matrix must, as a result, be estimated directly, along with the *k* (= 2*n*+1 for the LES, $(n^2+5n)/2$ for symmetry-restricted AIDS) parameters of the mean. This was found, even for the comparatively parsimonious LES, and even leaving aside the matter of the missing expenditure data, to be infeasible[9] for any realistic sample size at the level of commodity disaggregation contemplated here[10].

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element of **a**. The converse operation, in which the diagonal elements of the $n \times n$ matrix **A** are “extracted” into an *n*-vector, will be denoted *dv*(**A**).

[8] That is, $d\mathbf{Y}/d\mathbf{X} \equiv \mathbf{Y} \otimes d/d\mathbf{X}$. *Cf.* “definition 2” in §3 of Magnus and Neudecker (1988, p.171).

[9] In that the optimizing procedure (Gauss module CO) iterates *ad infinitum* without finding a set of parameter values that can with any confidence be said to be “maximizing”.

[10] The data available consisted of 27 annual, or 95 quarterly, observations on up to 18 expenditure categories. It should be noted that even an LES-based version of profile likelihood (4.1) cannot be maximized with only 27 observations – and if we prefer AIDS then 95 observations is similarly inadequate. Estimation of a model with so many commodity categories is thus problematic even without the additional complication of differing degrees of disaggregation.

4.1 De Boer and Harkema's covariance matrix.

An obvious solution to the problem described above is to reduce the dimension of the optimization problem by considering a reduced-order parameterization of the covariance matrix. This has typically been accomplished by setting $\tilde{\Sigma} = \sigma^2 \tilde{\mathbf{C}}$, where $\tilde{\mathbf{C}}$ is a symmetric $N \times N$ matrix of constants (which may be functions of the data) devised such that $\tilde{\Sigma}\iota_N = \mathbf{0}$, where ι_N is the N-vector of ones, and rank($\tilde{\Sigma}$) = n. The $\frac{1}{2}n(n+1)$ unknown covariance parameters are thereby reduced to just one – a degree of parameter reduction which might be thought somewhat extreme. Furthermore, the most common data-independent specification (see §4.2 following) imposes the less than reasonable restriction that all category variances are equal, as are all the cross-category covariances.

Accordingly, consider the less restrictive order-N parameterization devised by De Boer and Harkema (1986), in which the singular $N \times N$ covariance matrix $\tilde{\Sigma}$ is parameterized on an N-vector ξ according to

$$\underset{N\times N}{\tilde{\Sigma}}(\underset{N\times 1}{\xi}) = \mathbf{D}_\xi - \xi\xi'/\iota_N'\xi,$$

$\mathbf{D}_\xi = diag(\xi_1,\ldots,\xi_N)$. Then $\tilde{\Sigma}(\xi)$ clearly satisfies $\tilde{\Sigma}\iota_N = \mathbf{0}$, and the submatrix defined by ommission of the last (or any) category,

$$\underset{n\times n}{\Sigma}(\xi) = \mathbf{J}\tilde{\Sigma}\mathbf{J}' = \mathbf{D}_{\mathbf{J}\xi} - \mathbf{J}\xi\xi'\mathbf{J}'/\iota_N'\xi, \tag{4.3}$$

$\mathbf{J}\xi = (\xi_1,\ldots,\xi_n)'$, is positive definite if either (i) all ξ_i are strictly positive; ie, $\xi_i > 0$ $\forall\ i = 1,\ldots,N$ (in which case all the cross-covariances will be negative); or (ii) a single ξ_i is negative, and of sufficient magnitude that $\iota_N'\xi$ is negative also.

Substituting (4.3) into (3.3) then implies an "aggregated" likelihood parameterized on θ and ξ. Most importantly, the number of covariance parameters to be estimated is now $O(n)$ rather than $O(n^2)$.

4.2 Order-1 parameterization

The most restricted such parameterization of the covariance matrix is obviously just $\xi = \sigma^2 \iota_N$, leading to the well known specification

$$\tilde{\Sigma} = \sigma^2 (\mathbf{I}_N - \iota_N \iota_N' / N) , \tag{4.4}$$

implying $\Sigma = \sigma^2 (\mathbf{I}_n - \iota_n \iota_n' / N)$, $\Sigma^{-1} = (\mathbf{I}_n + \iota_n \iota_n') / \sigma^2$, and $|\Sigma| = \sigma^{2n} / N$. Substituting the last two into (3.3) then yields aggregated log-likelihood $\ell(\theta, \sigma^2) = \sum_{r=1}^{S} \ell_r$ with

$$\ell_r \;=\; -\frac{T_r n_r}{2} \ln \sigma^2 - \frac{T_r}{2} \ln|\mathbf{C_r}| - \frac{1}{2\sigma^2} \operatorname{tr}(\mathbf{C_r^{-1} U_r' U_r}) , \tag{4.5}$$

and $$\underset{n_r \times n_r}{\mathbf{C_r}} = \mathbf{A_r} (\mathbf{I}_n - \iota_n \iota_n' / N) \mathbf{A_r'} .$$

The MLE of our single remaining covariance parameter is then easily obtained as

$$\hat{\sigma}^2 = \sum_{r=1}^{S} \operatorname{tr}(\mathbf{C_r^{-1} U_r' U_r}) \Big/ \sum_{r=1}^{S} T_r n_r . \tag{4.6}$$

Consequently, and in contrast to the situation for more general Σ (including Σ(ξ) of the previous subsection), σ^2 can be concentrated out of $\sum_{r=1}^{S} \ell_r(\theta, \sigma^2)$. The result is an “aggregated” log-profile likelihood for θ of the form (ignoring all constants)

$$\ell^*(\theta) = -\frac{1}{2} \left\{ \sum_{r=1}^{S} T_r n_r \right\} \ln \left\{ \sum_{r=1}^{S} \operatorname{tr}(\mathbf{C_r^{-1} U_r' U_r}) \right\} . \tag{4.7}$$

This expression can naturally be used as a basis for the estimation of θ as an end in itself – provided we are prepared to accept the accompanying, possibly over-restrictive, covariance structure. For our purposes (4.7) and (4.6) are most useful as a means of obtaining starting values for the maximization of the aggregated likelihood with Σ as per (4.3).

4.3 Re-parameterizing AIDS for small datasets

The estimation problem is compounded if we attempt estimation of the Almost Ideal Demand System with a very small dataset (such as the annual dataset used in the Example following). Even after restricting the covariance matrix we find that symmetry-restricted AIDS cannot be estimated unless the dataset is reasonably large[11]. The problem, once again, is simply too many parameters ($(n^2+5n)/2$) to permit non-linear estimation.

Pursuing the same strategy as that employed for the covariance matrix, a feasible, though somewhat *ad hoc*, solution is to reparameterize the Γ matrix in such a way as to considerably reduce the number of free parameters to be estimated, while ensuring symmetry and adding-up. The obvious choice is, once again, De Boer and Harkema's parameterization, with the minor difference that we no longer require positive-definiteness of any $n \times n$ submatrix. Accordingly, let

$$\underset{N\times N}{\tilde{\Gamma}} = \mathbf{D}_{\eta} - \eta\eta'/\iota_N'\eta\,, \tag{4.8}$$

where the N-vector η is unrestricted, implying

$$\underset{n\times n}{\Gamma} = \mathbf{D}_{\mathbf{J}\eta} - \mathbf{J}\eta\eta'\mathbf{J}'/\iota_N'\eta\,. \tag{4.9}$$

The model now involves just $3n+1$ free mean parameters, plus the N parameters of De Boer and Harkema's covariance matrix.

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[11] Such as the 95 observations of the quarterly dataset, for which AIDS with De Boer and Harkema's covariance matrix could be estimated without difficulty.

5. Example

The LES, with De Boer and Harkema's covariance matrix (hereafter designated LES(1)), and AIDS, with De Boer and Harkema's parameterization applied to both Γ and Σ (hereafter AIDS(1)), were estimated for a demand system comprised of up to 18 expenditure categories, over the period 1969/70 – 1995/96. The data[12] used for the main example was collected annually, and included 3 subperiods of differing expenditure aggregation, due, in this case, to successive divisions of the "Other goods and services" category. The three subperiods were defined according to the then published data, with expenditure data disaggregated as follows.

1969/70 – 1980/81. 12 categories: *Food, Cigarettes and Tobacco, Alcohol and spirits, Clothing and footwear, Household appliances, Other household durables, Dwelling rent, Gas, electricity and fuel, Fares, Purchase of motor vehicles, Postal and telecommunications, and Other goods and services.*

1981/82 – 1986/87. Other goods and services (G&S) split into: *Operation of motor vehicles, Health, Entertainment and recreation, Financial services, Other goods and services* $\Rightarrow$ 16 categories.

1986/87 – 1995/96. Other G&S split into: *Other goods, Other services, Net expenditure overseas (LES only)* $\Rightarrow$ 18 categories for LES, 17 for AIDS[13].

The experiment was repeated with quarterly data[14], as this allowed the estimation of AIDS with Γ symmetric but otherwise unrestricted; though, of course, still with De Boer and Harkema's covariance matrix (hereafter designated AIDS(2)). The quarterly dataset extended from 1974(3rd quarter) to 1998(1st quarter), with the 12, 16, and

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[12] Australian Bureau of Statistics National Accounts: Private Final Consumption Expenditure.

[13] Net overseas expenditure (NEO), alone of the categories, can take both negative and positive values. More crucially, the nominal and real data do not always have the same sign, making the actual definition of an IPD rather problematic in any case, and the log-price undefined. Total expenditure for AIDS was thus calculated net of NEO, and the category excluded.

[14] Australian Bureau of Statistics, Private Final Consumption Expenditure (quarterly estimates).

18(17) category subperiods covering, respectively, 1974(3) – 1985(3), 1985(4) – 1989(3), and 1989(4) – 1998(1). The three subperiods were, once again, defined according to the then published data.

In summary, the 1st subperiod consists of 12 annual (45 quarterly) observations on 12 expenditure categories; the 2nd involves 6 annual (16 quarterly) observations on 16 categories; and the 3rd involves 9 annual (34 quarterly) observations on 18 (17) categories. All expenditures are in A$ per capita. Prices are measured by the IPD for each expenditure category, and equal unity in 1989/90.

Each model was estimated by ML in two stages, with *Food* as the "omitted" category in all subperiods. The first stage assumes that Σ is parameterized on the scalar σ^2 as per (4.4); and so consists of maximization of likelihood (4.7) with respect to θ[15]. The MLE of σ^2 then follows via (4.6). The 1st stage thus supplies starting values for θ and ξ (the latter via $\xi = \hat{\sigma}^2 \iota_N$) for the 2nd stage, in which likelihood (3.3), with Σ parameterized on ξ as per (4.3), is maximized with respect to θ and ξ, subject to the restriction that $\xi_i > 0$, $i = 1,\ldots,N$. As remarked above, this is slightly more restrictive than necessary, and has the disadvantage that it forces all the cross covariances to be negative, but is trivial to implement. As it happens, replacing "all $\xi_i > 0$" with the requirement that all eigenvalues of Σ be strictly positive had no effect other than to slow the optimization.

Results for LES(1) and AIDS(1) (annual data) are given in Tables 2 and 4. Tables 3, 5 and 6 give analogous results based on the quarterly dataset and models LES(1) and AIDS(2). Standard errors were computed via the inverse Hessian evaluated at the maximum; the Hessian itself being computed via forward difference approximation of

[15] Starting values for the 1st stage were, for the LES, β = average expenditure share in the final subperiod, and $\gamma = \mathbf{0}$. 1st stage starting values for AIDS were obtained via the unrestricted regression of the matrix of expenditure shares on log-real income, log-normalised prices, and a constant; with the exception of the initial η for AIDS(1), which we started at $0.01\iota_N$.

the derivatives of the analytic gradient[16]. The time required for the estimation was less than 30 seconds for the LES, and about 1½ minutes for AIDS.

We find that our estimated coefficients are, for the most part, statistically significant at the 5% level, and have signs that are usually plausible. Thus, for the LES, it is not unreasonable to suppose that most of those categories attracting a significantly negative γ_i are indeed price elastic; while for AIDS most of the positive β_i are attached to categories that might be regarded as "luxuries". However, as is common when estimating consumer demand based on aggregate data, theoretical restrictions not explicitly imposed during estimation are not in general satisfied. In particular, for the LES, estimates of the income effects parameter β, which should in theory lie between 0 and 1, are occasionally negative; implying both a negative Engel elasticity and positive own-price substitution effect. Similarly, for AIDS most γ_{ii} are positive, suggesting that negativity is again likely to be violated. We emphasise, however, that such criticisms should be regarded separately from the feasibility of the suggested method of estimating a demand system with differently aggregated data.

6. Conclusion

A simple method has been proposed for the ML estimation of a consumer demand system in the situation where not all expenditures are observed for all commodity categories in all time periods. The major difficulty with the estimation of such a system is that, while the likelihood function can be written down simply enough (particularly if we assume serially uncorrelated Gaussian errors), its maximization is problematic because of the $\frac{1}{2}n(n+1)$ covariance parameters that must now also be included in the objective function. In essence, the complete log-likelihood cannot be satisfactorily maximized *unless* Σ can be concentrated out. It is worth noting that this

[16] Analytic expressions for the scores with respect to the components of θ in each model, and with respect to the De Boer and Harkema (1986) covariance vector ξ, are given in Appendix B.

would be the case even if we had a complete set of quarterly data (95 observations) available on all 18 expenditure categories.

The obvious strategy, and the one considered in this paper, is to reduce the number of covariance parameters to be estimated by a suitable re-parameterization, leading to the adoption of De Boer and Harkema's (1986) covariance matrix. We find that the "aggregated" likelihood based on the LES can now be maximized without difficulty, even for the annual (27 observation) dataset. Furthermore, while such estimation cannot easily be carried out in a standard econometric package such as TSP or Shazam, it can be coded and computed quite simply in a programming language such as GAUSS.

Estimation of the aggregated likelihood based on AIDS was (unsurprisingly) more problematic, even after reparameterizing Σ. The method is perfectly feasible if sufficient data is available; however, for practical purposes this means the use of quarterly data. Estimation of "aggregated" AIDS with annual data was, at least for our dataset, possible only if the number of free parameters in the Γ matrix was also greatly reduced. As implemented here this leaves us with only N parameters to estimate the substitution effects. Nonetheless this still represents a distinct advance over the LES, which imposes, among other things, the "hidden" restriction that the Allen-Uzawa substitution elasticities be proportional to the product of the corresponding Engel elasticities.

The need to impose De Boer and Harkema's still fairly restrictive parameterization on the covariance structure of the model might be thought something of a disadvantage. It seems that the price of being able to use differently aggregated data from earlier time periods without sacrificing some commodity subcategories is a somewhat ad hoc covariance structure. We find, however, that not even the *concentrated* log-likelihood, which we would expect to use if there were no missing expenditure data, can be reliably maximized if the annual dataset is preferred. That is, 27 observations are insufficient to allow maximization of the conventional likelihood with symmetric but otherwise unrestricted covariance matrix and more than 7 or 8 categories, even for the extremely parsimonious LES. If we prefer AIDS then the (95 observation) quarterly dataset is similarly insufficient. Indeed, restricting the covariance matrix may well be essential to the estimation of AIDS for a large number of commodities, even without the problem of missing expenditure data. Of course, if there are insufficient data[17] available on all N commodities then an N-commodity model cannot be estimated in any case without resort to additional information – such as that implicit in more highly aggregated data in previous time periods.

Naturally, implementation of our approach requires, fairly obviously, that there be expenditure data available on all commodities in at least one time period[18]. Also note that we must have data on the complete set of explanatory variables for all time periods; that is, only the dependent variable (expenditure) can be "missing". Since (in Australia) price (CPI) data has been collected for a greater degree of disaggregation over longer time periods than almost any other series this may not be too onerous a requirement, at least as regards the estimation of demand systems.

[17] ML requires, at the very least, $T \geq N - 1$ observations to be feasible in an N-commodity system.

[18] Though it is difficult to say how few observations on the full system it would be possible to have before the problem became, in some sense, ill-conditioned, and the maximization infeasible.

Appendix A.

LEMMA A.1. *Given* $\ell = -\frac{T}{2}\ln|\Sigma| - \frac{1}{2}\mathrm{tr}(\mathbf{\Sigma}^{-1}\mathbf{U}'\mathbf{U})$ *where* Σ *is symmetric,*

$$\frac{\partial \ell}{\partial \Sigma} = \mathbf{\Sigma}^{-1}\{\mathbf{U}'\mathbf{U} - T\Sigma\}\Sigma^{-1} - \frac{1}{2}\mathbf{D}_{\Sigma^{-1}\{\mathbf{U}'\mathbf{U}-T\Sigma\}\Sigma^{-1}} . \tag{A.1}$$

LEMMA A.2. *For* $\boldsymbol{X}$ $n \times n$ *symmetric and n-vectors* $\boldsymbol{a}$ *and* $\boldsymbol{b}$,

$$\frac{d\mathbf{a}'\mathbf{X}\mathbf{b}}{d\mathbf{X}} = \mathbf{ab}' + \mathbf{ba}' - \mathbf{D}_{\mathbf{ab}'}$$

LEMMA A.3. *Let w be a scalar-valued function of a* $n \times n$ *symmetric matrix* $\boldsymbol{Y}$ *which is in turn a function of matrix* $\boldsymbol{X}$*. Then the derivative of w with respect to* $\boldsymbol{X}$ *is*

$$\frac{dw}{d\mathbf{X}} = \sum_{i=1}^{n}\sum_{j=i}^{n}\frac{\partial w}{\partial y_{ij}} \cdot \frac{dy_{ij}}{d\mathbf{X}} .$$

LEMMA A.1 is a straightforward application of Graybill (1983, pp.354-358), theorems 10.8.8 and 10.8.11. LEMMA A.2 extends Graybill (1983) Theorem 10.8.4. LEMMA A.3 follows from the ordinary chain rule, bearing in mind that, because **Y** is symmetric the summation is to be taken over only “half” of **Y** to avoid double-counting. The lemma thus modifies Theorem 8 of MacRae (1974) regarding differentiation of a scalar-valued function of a matrix so as to correctly handle symmetric matrices.

Appendix B.

The score with respect to general θ in likelihood (2.3) can readily be shown to be

$$\frac{\partial \ell}{\partial \theta} = \sum_{t=1}^{T}\frac{\partial \mathcal{W}_t'}{\partial \theta}\Sigma^{-1}\mathbf{u}_t \equiv \frac{\partial \underset{\sim}{\mathcal{W}}'}{\partial \theta}\left[\mathbf{I_T} \otimes \Sigma^{-1}\right]\underset{\sim}{\mathbf{u}} ; \tag{B.1}$$

where $\mathcal{W}_t' = \mathcal{W}(\tilde{\mathbf{p}}_t', m_t, \theta)$ is the n-vector of expected expenditure shares at time t as per (2.2), $\underset{\sim}{\mathbf{u}} = vec(\mathbf{U}')$, and $\underset{\sim}{\mathcal{W}} = vec(\mathcal{W}')$; $\mathbf{U}$ and $\mathcal{W}$ being defined as per (2.4).

In the context of "aggregated" likelihood (3.3) this becomes $\frac{\partial \ell}{\partial \theta} = \sum_{r=1}^{S} \frac{\partial \ell_r}{\partial \theta}$, where

$$\frac{\partial \ell_r}{\partial \theta} = \frac{\partial \underset{\sim}{\mathcal{W}}_r'}{\partial \theta}\left[\mathbf{I}_{\mathbf{T}_r} \otimes \mathbf{A}_\mathbf{r}'\boldsymbol{\Sigma}_\mathbf{r}^{-1}\right]\underset{\sim}{\mathbf{u}}_r \equiv \sum_{t \in \mathcal{T}_r} \frac{\partial \mathcal{W}_t'}{\partial \theta} \mathbf{A}_\mathbf{r}'\boldsymbol{\Sigma}_\mathbf{r}^{-1}\mathbf{u}_t^{(r)}, \tag{B.2}$$

$\underset{n_r T_r \times 1}{\underset{\sim}{\mathbf{u}}_\mathbf{r}} = vec(\mathbf{U}_\mathbf{r}')$, $\underset{n T_r \times 1}{\underset{\sim}{\mathcal{W}}_\mathbf{r}} = vec(\mathcal{W}_\mathbf{r}')$, $\mathbf{u}_t^{(r)}$ is the t^{th} row of $\mathbf{U_r}$; and $\mathbf{U_r}$, $\mathcal{W}_\mathbf{r}$ are as per (3.2).

$\frac{\partial \ell_r}{\partial \theta}$ simplifies considerably, as might be expected, for the LES and AIDS. In particular, for the LES, with $\mathcal{W}(\tilde{\mathbf{p}}_t', m_t, \theta)$ as per (3.4), we find that

$$\frac{\partial \ell_r}{\partial \alpha} = \mathbf{A}_\mathbf{r}'\boldsymbol{\Sigma}_\mathbf{r}^{-1}\mathbf{U}_\mathbf{r}'(\iota_{\mathbf{T}_r} - \mathbf{P}_\mathbf{r}\gamma),$$

and
$$\frac{\partial \ell_r}{\partial \gamma} = -\mathbf{P}_\mathbf{r}'\mathbf{U}_\mathbf{r}\boldsymbol{\Sigma}_\mathbf{r}^{-1}\mathbf{A}_\mathbf{r}\alpha + dv\left\{\mathbf{P}_\mathbf{r}'\mathbf{U}_\mathbf{r}\boldsymbol{\Sigma}_\mathbf{r}^{-1}\mathbf{A}_\mathbf{r}\mathbf{J}\right\},$$

where $dv(\underset{n\times n}{\mathbf{A}}) \triangleq (a_{11}, \ldots, a_{nn})'$ and $\mathbf{J} = [\mathbf{I_n} \;\; \mathbf{0}]$.

For AIDS (equation (3.7)), which is in any case linear with respect to α and β, we simply require the sum over r of $\frac{\partial \ell_r}{\partial \alpha} = \mathbf{A}_\mathbf{r}'\boldsymbol{\Sigma}_\mathbf{r}^{-1}\mathbf{U}_\mathbf{r}'\iota_{\mathbf{T}_r}$, $\frac{\partial \ell_r}{\partial \beta} = \mathbf{A}_\mathbf{r}'\boldsymbol{\Sigma}_\mathbf{r}^{-1}\mathbf{U}_\mathbf{r}'\mathbf{h_r}$, and, for symmetric Γ, $\frac{\partial \ell_r}{\partial \Gamma} = \mathbf{G_r} + \mathbf{G_r}' - \mathbf{D}_{\mathbf{G_r}}$, where $\mathbf{G_r} = \mathbf{A}_\mathbf{r}'\boldsymbol{\Sigma}_\mathbf{r}^{-1}\mathbf{U}_\mathbf{r}'\mathbf{R_r}$ and
$\mathbf{D_A} \triangleq diag(a_{11}, \ldots, a_{nn})$ for any $n \times n$ $\mathbf{A}$.

Finally, for De Boer and Harkema's (1986) covariance matrix, in which Σ is parameterized on ξ as per (4.3), it can be shown that the score with respect to ξ, for both conventional likelihood (2.3) and aggregated likelihood (3.3), is given by

$$\frac{\partial \ell}{\partial \xi} = dv\left\{\frac{\partial \ell}{\partial \boldsymbol{\Sigma}}\mathbf{J}\right\} - \mathbf{J}'\left\{\frac{\partial \ell}{\partial \boldsymbol{\Sigma}} + \mathbf{D}_{\partial\ell/\partial\Sigma}\right\}\zeta + \iota_N \zeta'\left\{\frac{\partial \ell}{\partial \boldsymbol{\Sigma}} + \mathbf{D}_{\partial\ell/\partial\Sigma}\right\}\zeta, \tag{B.3}$$

where $\zeta \equiv \mathbf{J}\xi / \iota_N' \xi$, and $\dfrac{\partial \ell}{\partial \Sigma}$ is as per (4.2) in the case of the aggregated likelihood, (A.1) otherwise. Hence, for AIDS with Γ parameterized on η as per (4.8), the score with respect to η is also given by (B.3), with η and Γ replacing ξ and Σ respectively.

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Tables

Table 1. Abbreviations for expenditure categories

Food	FOD
Cigarettes and Tobacco	CGT
Alcohol and spirits	ALC
Clothing and footwear	CFF
Dwelling rent	RNT
Purchase of motor vehicles	MVP
Household appliances	HAP
Other household durables	HDU
Postal and telecommunications	TEL
Gas, electricity and fuel	GEF
Fares	FRS
Operation of motor vehicles	MVO
Health	MED
Entertainment and recreation	REC
Financial services	FIN
Other goods	OGD
Other services	OSV
Net expenditure overseas	NEO

Table 2. ML estimation of the LES. De Boer and Harkema's covariance matrix; 3 subperiods; 18 expenditure categories; annual data

	β		γ		$\xi\ (\times 10^4)^{‡}$	
Expenditure category	final estimate	standard error	final estimate†	standard error	final estimate	standard error
FOD	0.0870	*0.0052*	1.2954	*0.0550*	0.1434	*0.0420*
CGT	0.0244	*0.0132*	0.0949	*0.0948*	0.3015	*0.1026*
ALC	-0.0033	*0.0040*	0.5957	*0.0231*	0.0853	*0.0241*
CFF	0.0051	*0.0036*	0.7215	*0.0198*	0.0954	*0.0272*
RNT	0.2502	*0.0110*	0.4118	*0.1993*	0.4344	*0.1835*
MVP	0.0112	*0.0035*	0.4124	*0.0219*	0.0847	*0.0240*
HAP	0.0454	*0.0044*	0.0539	*0.0084*	0.1441	*0.0424*
HDU	0.0171	*0.0034*	0.3955	*0.0212*	0.0707	*0.0200*
TEL	0.0298	*0.0019*	-0.0076	*0.0093*	0.0084	*0.0024*
GEF	0.0154	*0.0006*	0.1627	*0.0108*	0.0014	*0.0004*
FRS	0.0230	*0.0019*	0.1854	*0.0159*	0.0225	*0.0063*
MVO	0.0526	*0.0063*	0.4167	*0.0422*	0.0390	*0.0146*
MED	0.1143	*0.0076*	0.0580	*0.0728*	0.0267	*0.0104*
REC	0.1138	*0.0074*	-0.2007	*0.1028*	0.0153	*0.0059*
FIN	0.0674	*0.0092*	-0.0577	*0.0585*	0.0678	*0.0264*
OGD	-0.0232	*0.0278*	1.0468	*0.1732*	0.7423	*0.6550*
OSV	0.1878	*0.0120*	-0.2244	*0.1273*	0.0145	*0.0069*
NEO	-0.0179	*0.0036*	-0.0081	*0.0062*	0.2806	*0.1475*

Initial log-likelihood	1651.50	Final log-likelihood	1754.27
Number of iterations	26	Norm of the gradient	2.09×10^{-5}
Time to convergence	38.89 seconds	Number of observations	27 (annual)

† For estimation purposes the matrix of price ratios (that is, the ratio of price (an index, =1 in 1989/90) to total expenditure per capita (in Australian $)) has been scaled up by 10^3. Estimates of γ in Tables 2 and 3 are thus in units of thousands of 1989/90 A$.

‡ The system covariances are recoverable via $\sigma_{ii} = \xi_i - \xi_i^2 / \iota_N' \xi$, $\sigma_{ij} = -\xi_i \xi_j / \iota_N' \xi$, $i \neq j$.

Table 3. ML estimation of the LES. De Boer and Harkema's covariance matrix; 3 subperiods; 18 expenditure categories; quarterly data

	β		γ		$\xi\ (\times 10^4)^{\ddagger}$	
Expenditure category	final estimate	standard error	final estimate	Standard error	final estimate	standard error
FOD	0.0902	*0.0041*	0.2650	*0.0163*	0.2207	*0.0337*
CGT	0.0300	*0.0032*	-0.0083	*0.0035*	0.0527	*0.0104*
ALC	-0.0105	*0.0040*	0.1674	*0.0088*	0.1459	*0.0217*
CFF	0.0179	*0.0050*	0.1479	*0.0108*	0.7158	*0.1188*
RNT	0.2592	*0.0137*	-0.0677	*0.0640*	2.6211	*0.7294*
MVP	0.0243	*0.0035*	0.0663	*0.0095*	0.1590	*0.0238*
HAP	0.0303	*0.0018*	0.0192	*0.0016*	0.1044	*0.0156*
HDU	0.0130	*0.0039*	0.0983	*0.0089*	0.2670	*0.0406*
TEL	0.0262	*0.0013*	-0.0095	*0.0017*	0.0229	*0.0035*
GEF	0.0136	*0.0037*	0.0355	*0.0085*	0.1326	*0.0197*
FRS	0.0269	*0.0011*	0.0228	*0.0041*	0.0228	*0.0034*
MVO	0.0155	*0.0017*	0.1682	*0.0049*	0.0094	*0.0019*
MED	0.0855	*0.0062*	0.0174	*0.0202*	0.1285	*0.0262*
REC	0.0977	*0.0096*	-0.0780	*0.0301*	0.1804	*0.0377*
FIN	0.0338	*0.0030*	0.0310	*0.0081*	0.0728	*0.0147*
OGD	0.1208	*0.0072*	-0.0755	*0.0272*	0.0808	*0.0195*
OSV	0.1383	*0.0117*	-0.0325	*0.0356*	0.3527	*0.0821*
NEO	-0.0126	*0.0023*	-0.0093	*0.0050*	0.6465	*0.1437*

Initial log-likelihood	5159.35	Final log-likelihood	5588.39
Number of iterations	18	Norm of the gradient	2.95×10^{-7}
Time to convergence	28.18 seconds	Number of observations	95 (quarterly)

‡ $\sigma_{ii} = \xi_i - \xi_i^2 / \iota_{\mathbf{N}}' \xi$, $\sigma_{ij} = -\xi_i \xi_j / \iota_{\mathbf{N}}' \xi$, $i \neq j$.

Table 4. ML estimation of AIDS(1). De Boer and Harkema's covariance matrix; 3 subperiods; 18 expenditure categories; annual data

	α		β		$\eta^{\dagger}$		ξ ($\times 10^4$)‡	
Expenditure category	final estimate	standard error	final estimate	standard error	final estimate	standard error	final estimate	standard error
FOD	0.4852	*0.1162*	-0.0355	*0.0123*	0.1781	*0.0373*	0.0937	*0.0307*
CGT	0.3406	*0.0115*	-0.0342	*0.0012*	0.0111	*0.0006*	0.0020	*0.0005*
ALC	0.5316	*0.0239*	-0.0515	*0.0026*	0.0212	*0.0118*	0.0289	*0.0082*
CFF	0.8824	*0.0487*	-0.0870	*0.0052*	-0.0046	*0.0088*	0.0435	*0.0125*
RNT	-0.4252	*0.1496*	0.0639	*0.0158*	0.2214	*0.0385*	0.3386	*0.1605*
MVP	0.3116	*0.0414*	-0.0290	*0.0044*	0.0249	*0.0093*	0.0873	*0.0265*
HAP	-0.0306	*0.2806*	0.0064	*0.0296*	0.0066	*0.0089*	0.1272	*0.0411*
HDU	0.0109	*0.0985*	0.0030	*0.0104*	0.1255	*0.0298*	0.0359	*0.0110*
TEL	-0.2305	*0.0166*	0.0261	*0.0018*	0.0059	*0.0011*	0.0028	*0.0008*
GEF	0.0656	*0.0094*	-0.0047	*0.0010*	0.0161	*0.0011*	0.0015	*0.0004*
FRS	0.1622	*0.0293*	-0.0144	*0.0031*	-0.0099	*0.0051*	0.0125	*0.0035*
MVO	0.4169	*0.0615*	-0.0372	*0.0065*	-0.0036	*0.0075*	0.0198	*0.0082*
MED	-0.4675	*0.1037*	0.0566	*0.0109*	0.0782	*0.0327*	0.0576	*0.0302*
REC	-0.2767	*0.0513*	0.0344	*0.0055*	0.0528	*0.0159*	0.0072	*0.0029*
FIN	0.2227	*0.0620*	-0.0195	*0.0065*	-0.0775	*0.0076*	0.0079	*0.0032*
OGD	-0.2786	*0.0299*	0.0369	*0.0031*	-0.0095	*0.0089*	0.0021	*0.0010*
OSV	-0.7207	*0.0645*	0.0858	*0.0068*	-0.0303	*0.0119*	0.0315	*0.0148*

Initial log-likelihood	1726.74	Final log-likelihood	1908.03
Number of iterations	55	Norm of the gradient	7.21×10^{-4}
Time to convergence	99.97 seconds	Number of observations	27 (annual)

† To recover the matrix of price effects $\tilde{\Gamma}$, recall that $\gamma_{ij} = -\eta_i \eta_j / \iota_{\mathbf{N}}' \eta$, $i \neq j$ and $\gamma_{ii} = \eta_i - \eta_i^2 / \iota_{\mathbf{N}}' \eta$. Note that the estimated $\iota_{\mathbf{N}}' \eta$ = 0.6065, implying $\gamma_{ii} < 0$ iff $\eta_i < 0$ or $\eta_i > \iota_{\mathbf{N}}' \eta$.

Table 5. ML estimation of AIDS(2). De Boer and Harkema's covariance matrix; 3 subperiods; 18 expenditure categories; quarterly data

	α		β		γ_{ii} †		ξ ($\times 10^4$)‡	
Expenditure category	final estimate	standard error	final estimate	standard error	final estimate	standard error	Final estimate	standard error
FOD	0.2340	*0.0706*	-0.0106	*0.0087*	0.0909	*0.0186*	0.0920	*0.0186*
CGT	0.0451	*0.0147*	-0.0034	*0.0018*	0.0074	*0.0011*	0.0033	*0.0005*
ALC	-0.1052	*0.0441*	0.0184	*0.0055*	0.0127	*0.0083*	0.0330	*0.0054*
CFF	-1.0580	*0.0865*	0.1382	*0.0107*	0.1409	*0.0250*	0.1526	*0.0264*
RNT	1.4678	*0.0461*	-0.1587	*0.0057*	0.1190	*0.0170*	0.0297	*0.0060*
MVP	0.0908	*0.0644*	-0.0066	*0.0080*	-0.0084	*0.0082*	0.1045	*0.0180*
HAP	-0.0575	*0.0907*	0.0106	*0.0112*	0.0105	*0.0045*	0.2713	*0.0604*
HDU	-0.7723	*0.0408*	0.1005	*0.0051*	0.0218	*0.0171*	0.0244	*0.0037*
TEL	0.0443	*0.0140*	-0.0037	*0.0017*	0.0066	*0.0016*	0.0028	*0.0005*
GEF	-0.0420	*0.0809*	0.0077	*0.0100*	0.0186	*0.0074*	0.1870	*0.0347*
FRS	0.1288	*0.0298*	-0.0125	*0.0037*	0.0028	*0.0040*	0.0138	*0.0023*
MVO	0.3723	*0.0215*	-0.0382	*0.0027*	0.0532	*0.0041*	0.0034	*0.0008*
MED	0.4543	*0.0531*	-0.0478	*0.0066*	-0.0099	*0.0324*	0.0210	*0.0050*
REC	0.3201	*0.0319*	-0.0338	*0.0040*	0.0963	*0.0245*	0.0059	*0.0014*
FIN	0.2740	*0.0300*	-0.0293	*0.0037*	0.0317	*0.0059*	0.0061	*0.0018*
OGD	-0.8655	*0.0696*	0.1158	*0.0086*	-0.0407	*0.0474*	0.0261	*0.0075*
OSV	0.4691	*0.1042*	-0.0466	*0.0129*	-0.0068	*0.0710*	0.1344	*0.0431*

Initial log-likelihood	6063.93	Final log-likelihood	6431.78
Number of iterations	15	Norm of the gradient	2.73×10^{-4}
Time to convergence	77.77 seconds	Number of observations	95 (quarterly)

† γ_{ii}, $i = 1,\ldots,N$, are the diagonal elements of the $N \times N$ matrix of price effects $\tilde{\Gamma}$.

Table 6. ML estimation of AIDS(2) (*continued*): estimated price effects matrix $\tilde{\Gamma}$ (*standard errors in italics*)

	FOD	CGT	ALC	CFF	RNT	MVP	HAP	HDU	TEL	GEF	FRS	MVO	MED	REC	FIN	OGD	OSV
FOD	0.0909																
	0.0186																
CGT	-0.0054	0.0074															
	0.0032	*0.0011*															
ALC	-0.0102	-0.0042	0.0127														
	0.0093	*0.0022*	*0.0083*														
CFF	-0.0846	0.0016	-0.0094	0.1409													
	0.0160	*0.0040*	*0.0103*	*0.0250*													
RNT	0.0150	-0.0121	0.0070	-0.0479	0.1190												
	0.0128	*0.0031*	*0.0080*	*0.0141*	*0.0170*												
MVP	0.0091	0.0014	-0.0024	-0.0523	0.0444	-0.0084											
	0.0089	*0.0020*	*0.0061*	*0.0111*	*0.0071*	*0.0082*											
HAP	0.0279	0.0041	0.0254	0.0369	-0.0683	0.0124	0.0105										
	0.0051	*0.0011*	*0.0032*	*0.0060*	*0.0043*	*0.0037*	*0.0045*										
HDU	-0.0170	-0.0004	-0.0067	0.0622	-0.0644	-0.0083	0.0278	0.0218									
	0.0115	*0.0030*	*0.0074*	*0.0139*	*0.0125*	*0.0066*	*0.0037*	*0.0170*									
TEL	0.0077	0.0023	0.0060	-0.0137	-0.0147	0.0024	-0.0088	-0.0044	0.0066								
	0.0040	*0.0010*	*0.0025*	*0.0044*	*0.0038*	*0.0022*	*0.0013*	*0.0037*	*0.0016*								
GEF	-0.0069	-0.0023	0.0092	0.0444	-0.0288	-0.0265	0.0000	0.0294	-0.0064	0.0186							
	0.0075	*0.0017*	*0.0050*	*0.0096*	*0.0059*	*0.0056*	*0.0041*	*0.0057*	*0.0019*	*0.0074*							
FRS	-0.0042	0.0024	0.0021	0.0078	-0.0044	-0.0056	-0.0009	0.0034	-0.0024	-0.0091	0.0028						
	0.0063	*0.0015*	*0.0041*	*0.0075*	*0.0058*	*0.0039*	*0.0021*	*0.0056*	*0.0018*	*0.0036*	*0.0040*						
MVO	-0.0276	0.0025	-0.0106	0.0048	-0.0006	0.0086	0.0033	-0.0040	-0.0069	0.0030	0.0000	0.0532					
	0.0068	*0.0018*	*0.0054*	*0.0089*	*0.0061*	*0.0033*	*0.0049*	*0.0059*	*0.0022*	*0.0075*	*0.0032*	*0.0041*					
MED	-0.0346	0.0009	0.0236	0.0388	-0.0141	0.0170	-0.0499	0.0247	0.0049	0.0110	-0.0119	-0.0031	-0.0099				
	0.0177	*0.0044*	*0.0130*	*0.0215*	*0.0155*	*0.0084*	*0.0125*	*0.0146*	*0.0054*	*0.0182*	*0.0071*	*0.0081*	*0.0324*				
REC	-0.0200	0.0044	0.0057	-0.0442	-0.0488	-0.0105	-0.0286	0.0049	-0.0031	0.0109	0.0219	-0.0359	-0.0165	0.0962			
	0.0133	*0.0034*	*0.0104*	*0.0157*	*0.0133*	*0.0059*	*0.0100*	*0.0130*	*0.0049*	*0.0118*	*0.0052*	*0.0056*	*0.0176*	*0.0245*			
FIN	0.0057	0.0075	-0.0135	0.0135	-0.0030	0.0067	-0.0301	0.0288	-0.0003	-0.0119	0.0083	-0.0030	-0.0029	0.0064	0.0317		
	0.0086	*0.0022*	*0.0066*	*0.0123*	*0.0099*	*0.0042*	*0.0060*	*0.0075*	*0.0027*	*0.0085*	*0.0039*	*0.0035*	*0.0097*	*0.0067*	*0.0059*		
OGD	-0.0422	-0.0081	-0.0321	0.0168	0.0178	0.0110	0.0463	-0.0483	0.0095	-0.0183	-0.0158	-0.0015	-0.0032	0.0825	0.0137	-0.0407	
	0.0164	*0.0037*	*0.0115*	*0.0208*	*0.0138*	*0.0108*	*0.0135*	*0.0129*	*0.0042*	*0.0156*	*0.0067*	*0.0095*	*0.0252*	*0.0186*	*0.0112*	*0.0474*	
OSV	0.0964	-0.0020	-0.0027	-0.1156	0.1039	0.0013	-0.0081	-0.0494	0.0212	-0.0162	0.0058	0.0179	0.0249	-0.0254	-0.0577	0.0125	-0.0068
	0.0207	*0.0048*	*0.0148*	*0.0270*	*0.0187*	*0.0141*	*0.0147*	*0.0168*	*0.0057*	*0.0193*	*0.0096*	*0.0120*	*0.0365*	*0.0253*	*0.0154*	*0.0480*	*0.0710*