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Abstract

In this paper we compare two non-stationary time series using non-parametric procedures. Evolutionary spectra are estimated for the two series. Randomization tests are performed on groups of spectral estimates for both related and independent time series. Simulation studies show that in certain cases the tests perform reasonably well. The tests are applied to observed geological and financial time series.

Keywords: Evolutionary Spectra, Lag Window, Time Window, Randomization Tests

1 Introduction

The comparison of two or more time series is useful in many different situations. A geological application would be the detection of differences between the waveforms of an earthquake and a nuclear explosion of similar strength. A medical application would be the comparison of different sections of a particular brain wave recording and a financial application would be the comparison of the performance of different stocks and shares or the comparison of interest rates between various countries.

Identification of similarities or differences in such time series is useful for decision making and forecasting. Suppose that we have a number of time series that we want to forecast. As a result of testing for differences between underlying processes, groups of similar time series can be identified. Then instead of fitting models to all the given series and forecasting each of them, a model can be fitted to a representative of each group and forecasting can then be performed on this representative. This is especially useful if one has to forecast a large number of time series, as can often be the case in inventory control. In terms of reduced time and costs this would certainly be more practical. It is also well known that better estimates are obtained by pooling similar data sets. These similar data sets, which in this case would be non-stationary time series, can be identified on the basis of the techniques for differentiating between them.

Most existing time series comparison techniques are applicable to time series that are stationary, or to non-stationary time series that can be transformed to stationary time series by some simple transformation such as differencing. These comparison techniques have been put forward by authors such as: Jenkins (1961), De Sousa and Thomson (1982), Shumway (1982), Basawa *et al*. (1984), Coates and Diggle (1986), Swanepoel and Van Wyk (1986), Diggle and Fisher (1991), Guo (1999), Timmer *et al*. (1999), and Maharaj (2000). Since many time series in various fields are not easily transformable non-stationary series, the existing time series comparison procedures cannot be used if one is required to test for differences between such series.

In this paper we will consider time series that may or may not be stationary in the mean but are variance non-stationary. For such series no variance reduction transformation will make them variance stationary. Some examples of such time series are waveforms of earthquakes and nuclear explosions and certain financial series.

In Section 2, we briefly describe the estimation of the evolutionary spectrum and in Section 3, we describe the non-parametric tests that will be used to compare the evolutionary spectra of two different series. In Section 4, we describe the simulation study and report the results, while in Section 5, we apply the tests to observed non-stationary time series.

2 Evolutionary Spectra

Priestley (1965) developed the evolutionary spectra approach to the spectral analysis of non-stationary time series. Because the structure of non-stationary series changes over time, estimating a conventional spectrum will not be appropriate. In order to take into account these structural changes over time, evolutionary spectra, that is, successive spectra of overlapping portions of the time series are estimated. This can be likened to viewing the series through a moving time window of fixed length.

Let $\{X_t\ , t = 1, 2, \ldots, T\}$ be a discrete parameter stochastic process. If $\{X_t\}$ is stationary then the spectral density function or spectrum is defined by

$$f_X(\omega) = \frac{1}{\pi}\sum_{k=-\infty}^{\infty} c_k e^{i\omega k} \qquad (2.1)$$

where c_k is the covariance function of X_t and ω, the frequency, is in the range $(0,\pi)$. A smoothed estimate of the spectrum is

$$\hat{f}_X(\omega) = \frac{1}{\pi}\sum_{k=-m}^{m} \lambda_k c_k e^{i\omega k} \qquad (2.2)$$

were λ_k is a suitably chosen lag window and $m < T$ is called the truncation point. The choice of lag window is discussed in Priestley (1966). If X_t is non-stationary, then choosing a weight function u_t of suitable length, the estimate of the evolutionary spectrum is

$$\hat{h}_t(w) = \sum_v u_t \left| \hat{f}_{tX}(w) \right|^2 \quad , \qquad (2.3)$$

where $\hat{f}_{tX}(w)$ is the smoothed estimate of the spectrum in the neighbourhood of t.

3 Testing Procedure

Given two semi-stationary series $\{x_t\}$ and $\{y_t\}$, that is, series that are stationary in the mean but non-stationary in the variance, evolutionary spectra are estimated using the "double window technique" of equations (2.2) and (2.3). In what follows, we will use the Bartlett window for λ_k and the Daniel window for the u_t. The Bartlett window is defined as

$$\lambda_k = \begin{cases} 1/2\sqrt{m\pi} & |k| \le m \\ 0 & |k| > m \end{cases} \qquad (3.1)$$

with the modulus of the Fourier transform being

$$|\Gamma(w)|^2 = \frac{1}{p} \frac{\sin^2 mw}{mw^2}.$$

This is the spectral window corresponding to λ_k and it has bandwidth π/m. The Daniel window is defined as

$$u_t = \begin{cases} 1/T' & |t| \le T'/2 \\ 0 & |t| > T'/2 \end{cases} \qquad (3.2)$$

where T' is the length of the corresponding time window $U_{T'}(w)$ which is the Fourier transform of u_t, with

$$\int_{-\infty}^{\infty} |U_{T'}(w)|^2 dw = O(1/T').$$

This length of the time window $U_{T'}(w)$ must be long enough so that fairly stable estimates are obtainable for a reasonable number of spectral components but not too long so that the occurrence of a fundamental change will be lost in all the averaging.

3.1 Randomisation Tests

Randomisation tests (see Siegel, 1956) are non-parametric tests with which one can obtain the exact probability under the null hypothesis without making any assumptions about normality or homogeneity of variance.

In order to use these tests, the spectral estimates must be uncorrelated. Thus in order to obtain approximately uncorrelated estimates, the frequencies $\{w_j\}$ and the time points $\{t_i\}$ should be chosen so that spacings between $\{w_j\}$ are at least π/m and the spacings between $\{t_i\}$ are at least T' (see Priestley, 1965).

Then within each time window of length T', the approximately uncorrelated spectral estimates of the two series are compared by means of the randomisation tests. In this case within each time window, the null hypothesis is that there is no difference between the evolutionary spectra of the two time series. If there are b time windows each of length T', then it is expected that if the two non-stationary time series under consideration have similar patterns, the test would be non-significant for most of the b time windows. On the other hand if the two time series have markedly different patterns, it is expected that the test would be significant for most of the b time windows.

Within each time window, let D be any measure of distance between the estimated spectra $\hat{h}_{tx}(w_j)$ and $\hat{h}_{ty}(w_j)$, j = 1, 2, . . . , p of the two non-stationary time series x_t and y_t respectively. Then, under the null hypothesis the distribution of D will be invariant under 2^p possible interchanges of $\hat{h}_{tx}(w_j)$ and $\hat{h}_{ty}(w_j)$. In practice it will not be feasible to determine the distribution of D but it can be approximated adequately by calculating D_1, D_2, . . . , D_s for some large number s-1 of interchanges of the spectral estimates at each frequency and by calculating the significance probability of the observed D-value, say D_1, as the proportion of values D_1, D_2, . . . , D_s at least as large as D_1.

Following Siegel (1956), for related time series for each time window we will use

$$D = \sum_{j=1}^{p} d_j$$

where

$$d_j = \hat{h}_{tx}(w_j) - \hat{h}_{ty}(w_j)$$

and for independent time series we will use

$$D = \sum_{j=1}^{p} \hat{h}_{tx}(w_j) - \sum_{j=1}^{p} \hat{h}_{ty}(w_j).$$

4 Simulation Study[1]

4.1 Design

On comparing the theoretical and estimated evolutionary spectra, Priestley (1965) showed that the method for estimating evolutionary spectra described in Section 2,

[1] All programs were written in *Gauss* and are available from the author on request.

using the spectral and time windows in (3.1) and (3.2) respectively, appeared to work quite well for series generated from the semi-stationary process

$$X(t) = C(t)Y_t$$

with

$$C(t) = exp\left(\frac{-(t-500)^2}{2(200)^2}\right)$$

and Y_t being an ARMA process. Hence we have used $X(t)$ which generates series that are non-stationary in variance as the generating process for our study with Y_t being an ARMA process.

To gauge how the independent case randomization test performed for series generated from the same non-stationary process, unrelated series of length $T = 200$ and $T = 500$ were simulated from each of $X(t)$, with Y_t being AR(1): $f = 0, 0.5, 0.9$; MA(1): $q = 0.5$; and ARMA(1,1), $f = -0.6$, $q = 0.3$. These ARMA models were chosen, so that both first and second order processes as well as a range of parameter values would be considered. To gauge how the test performed for series generated from different non-stationary processes, unrelated series were simulated from $X(t)$ with AR(1): $f = 0$ versus $f > 0$.

The performance of the related case randomisation test was assessed using the same scenarios as for the independent case except that it was assumed that ARMA innovations were correlated at 0.5.

The evolutionary spectra were estimated for various values of pair (m, T') for the Bartlett and Daniel windows in Equation (3.1) and (3.2) respectively. Values of m were chosen to be $T^{0.4}$, $T^{0.5,}$ $T^{0.6}$, $T^{0.7}$ and $T^{0.8}$, and values of T' were chosen to be $T/8$ and

$T/4$. That is, spectral estimates were obtained for eight time windows of length $T/8$, and four time windows of length $T/4$.

For each pair of series, the relevant test was performed within each time window and a count was made of the number of times the test was significant at the 5% level over all the time windows. This count was recorded for each of 100 simulations. An average count was then obtained over the 100 simulations. Low averages compared to the number of time windows would indicate a greater similarity between the generating processes whereas high averages compared to the number of time windows would indicate a greater difference between the generating processes.

4.2 Results and Discussion

For $T = 200$, for the combinations ($h = T^{0.8}$, $T' = T/8$) and ($h = T^{0.8}$, $T' = T/4$), the tests for related and independent series performed reasonably well in terms of size and power. For the eight-time-window scenario, where each window was of length $T' = T/8$, the average number of windows in which the null hypothesis was rejected was no more than 3.50, when it was true, while when it was false, the average number of windows in which it was rejected was as high as 7.46 for extremely different series. For the four-time-window scenario, where each window was of length $T' = T/4$, the average number of windows in which the null hypothesis was rejected was no more than 1.80, when it was true, while when it was false, the average number of windows in which it was rejected was as high as 3.76 for extremely different series. Similar averages where obtained in terms of size for the combinations ($h = T^{0.7}$, $T' = T/8$) and ($h = T^{0.7}$, $T' = T/4$). However in terms of power, the tests did not perform as well for these combinations.

For T = 500, for the combinations ($h = T^{0.8}$, T' = T/8) and ($h = T^{0.8}$, T'= T/4), the tests performed much better for both related and independent series, in terms of size and power than for T = 200. The average number of windows in which the null hypothesis was rejected when it was true, was no more than 2.59 for the eight-time-window scenario and no more than 1.12 for the four-time-window scenario. When the null hypothesis was false, the average number of windows in which it was rejected was as high as 7.86 for extremely different series for the eight-time window scenario, and as high as 3.99 for extremely different series for the four-time-window scenario.

For all other combinations of h and T', for both T = 200 and T = 500 and for both independent and related series, the tests' performances were reasonably good in terms of size but very poor in terms of power. Results for the combinations $h = T^{0.8}$ and T' =T/8, and $h = T^{0.8}$ and T' =T/4 are given in Tables 1 to 4.

<Table 1>

<Table 2>

<Table 3>

<Table 4>

A simulation study with the same generating processes described in Section 4.1 was also carried out using a Parzen lag window with the Daniel lag window to estimate the evolutionary spectra. Fairly similar results to those described above were obtained.

5 Applications

5.1 Financial Data: Related Series

Graphs of monthly interest rates[2] from July 1980 to June 2000 of four OECD countries, Australia, the United Kingdom (UK), France and the United States of America (USA) are given in Figure 1. Clearly these time series are non-stationary. While differencing renders them stationary in the mean, it does not render them stationary in variance as seen in Figure 2. Since the same economic and financial factors affect interest rates in the OECD countries, we will apply the test for related series to each pair of differenced series. We use the differenced series since the test (as demonstrated in the simulation study) is applicable to series that are stationary in the mean but are non-stationary in variance.

It can be seen from Figure 1 that interest rates from about July 1980 to January 1982, and from about January 1989 to June 2000 appear to move in tandem for all four countries whereas from about January 1982 to January 1989 Australia‘s interest rate appears to follow a different pattern from the others. From the months from about April 1984 and June 2000 the interest rates for the UK , USA and France appear to move in tandem , whereas from about January 1982 and April 1984, France’s interest rates appear to follow a different pattern from the other countries.

The results of the test for related series for combinations of m and T' for $m = T^{0.7}$ and $T^{0.8}$, and $T' = T/8$ and $T/4$ are given in Table 5.

<Figure 1>

<Table 5>

[2] Source: Australian Bureau of Statistics

The results reveal that over the period from January 1980 to June 2000 there is not much difference between the interest rates patterns of France and the UK, that is, in 1 out of 8 time windows and in 0 out of the 4 time-windows, the null hypothesis of no difference is rejected. There are some differences between the interest rates patterns of the USA and France, and the USA and the UK, that is, between 3 to 5 out of 8 time windows and in 2 out of 4 time-windows, the null hypothesis of no difference is rejected. However for the interest rate patterns between Australia and the other countries, there appear to be fairly large difference, that is, the null hypothesis of no difference is rejected for between 5 to 8 out of 8 time windows and, for between 3 to 4 out of 4 time windows. These results appear to be consistent with some of the observations made from Figure 1.

5.2 Geological Data: Independent Series

It is clear from an examination of earthquake and nuclear explosion waveforms that there are some differences in their patterns. It is therefore expected that there will be differences in their spectra and indeed their evolutionary spectra as well. Figures 3 and 4 show the standardised waveforms[3] of a nuclear explosion detonated in China in August 1995 and an earthquake that occurred in the Solomon Islands in September 1995. The two events, which were of similar strength, were recorded at the same seismological station. Each series consists of 600 observations recorded over a 30 second interval. Clearly it can be seen that the series are variance non-stationary. Furthermore, it can be seen that their patterns differ considerably over some time periods but less so over others.

[3] Source: Australian Geological Survey Organisation.

The results of the test for independent series for combinations of m and T' for $m = T^{0.7}$ and $T^{0.8}$, and $T' = T/8$ and $T/4$ are given in Table 6.

<Figure 3>

<Figure 4>

<Table 6>

The results reveal that there appears to be some differences between the waveforms of the earthquake and the nuclear explosion, in that for between 4 to 5 out of 8 time windows and for 2 out of 4 time windows, the null hypothesis of no difference is rejected. These results appear to be consistent with the observations made from Figures 3 and 4.

6. Concluding Remarks

In summary then, it would appear from the simulation study that for certain combinations of lag and time window lengths, the tests based on the evolutionary spectra for both independent and related series cases perform reasonably well. The applications to real data demonstrate that tests can be quite successfully applied. Hence it seems that these tests can be quite useful for differentiating between time series that are variance non-stationary.

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Table 1 Average number of time windows associated with size: *T* = 200

	Independent Series		Related Series	
Process	$m = T^{0.8}$ $T' = T/8$	$m = T^{0.8}$ $T' = T/4$	$m = T^{0.8}$ $T' = T/8$	$M = T^{0.8}$ $T' = T/4$
AR(1) 0	2.75	1.42	3.13	1.49
0.5	3.33	1.55	2.85	1.62
0.9	3.29	1.69	3.40	1.80
MA(1) 0.5	3.15	1.54	2.70	1.69
ARMA 0.6; 0.3	3.19	1.66	3.50	1.54

Table 2 Average number of time windows associated with power: *T* = 200
AR(1) ϕ = 0 versus AR(1) ϕ > 0

	Independent Series		Related Series	
Process	$m = T^{0.8}$ $T' = T/8$	$m = T^{0.8}$ $T' = T/4$	$M = T^{0.8}$ $T' = T/8$	$M = T^{0.8}$ $T' = T/4$
AR(1) 0	2.75	1.42	3.13	1.49
0.2	2.87	1.47	2.71	1.25
0.4	3.27	1.69	3.26	1.76
0.6	4.86	2.51	4.88	2.68
0.8	7.04	3.76	7.46	3.74

Table 3 Average number of time-windows associated with size: *T* = 500

	Independent Series		Related Series	
Process	$M = T^{0.8}$ $T' = T/8$	$m = T^{0.8}$ $T' = T/4$	$m = T^{0.8}$ $T' = T/8$	$m = T^{0.8}$ $T' = T/4$
AR(1) 0	1.99	0.82	1.80	1.12
0.5	2.21	1.03	2.07	0.98
0.9	2.59	1.12	2.29	0.99
MA(1) 0.5	2.05	0.96	2.10	0.97
ARMA 0.6; 0.3	2.03	1.07	1.97	0.99

Table 4 **Average number of time windows associated with power: T = 500 AR(1) $\phi = 0$ versus AR(1) $\phi > 0$**

	Independent Series		Related Series	
Process	$m = T^{0.8}$ $T' = T/8$	$m = T^{0.8}$ $T' = T/4$	$m = T^{0.8}$ $T' = T/8$	$m = T^{0.8}$ $T' = T/4$
AR(1) 0	1.99	0.82	1.80	1.12
0.2	2.18	0.91	2.08	0.98
0.4	2.81	1.45	3.20	1.59
0.6	5.53	3.15	5.47	3.24
0.8	7.75	3.92	7.86	3.99

Table 5 Number of time windows for which the null hypothesis was rejected

$m = T^{0.8}$ $T' = T/8$ 8 Time Windows				$m = T^{0.7}$ $T' = T/8$ 8 Time Windows			
	AUS	**USA**	**FRA**		**AUS**	**USA**	**FRA**
USA	8			**USA**	7		
FRA	8	4		**FRA**	7	3	
UK	6	5	1	**UK**	5	4	1
$m = T^{0.8}$ $T' = T/4$ 4 Time Windows				$m = T^{0.7}$ $T' = T/4$ 4 Time Windows			
	AUS	**USA**	**FRA**		**AUS**	**USA**	**FRA**
USA	3			**USA**	3		
FRA	4	2		**FRA**	4	2	
UK	4	2	0	**UK**	4	2	0

Table 6 Number of time windows for which the null hypothesis was rejected

Comparison of Earthquake and Nuclear explosion waveforms		
	$T' = T/8$ **8 time windows**	$T' = T/4$ **4 time windows**
$m = T^{0.8}$	4	2
$m = T^{0.7}$	5	2

Figure 1 Interest rates of the four OECD countries: January 1980 - June 2000

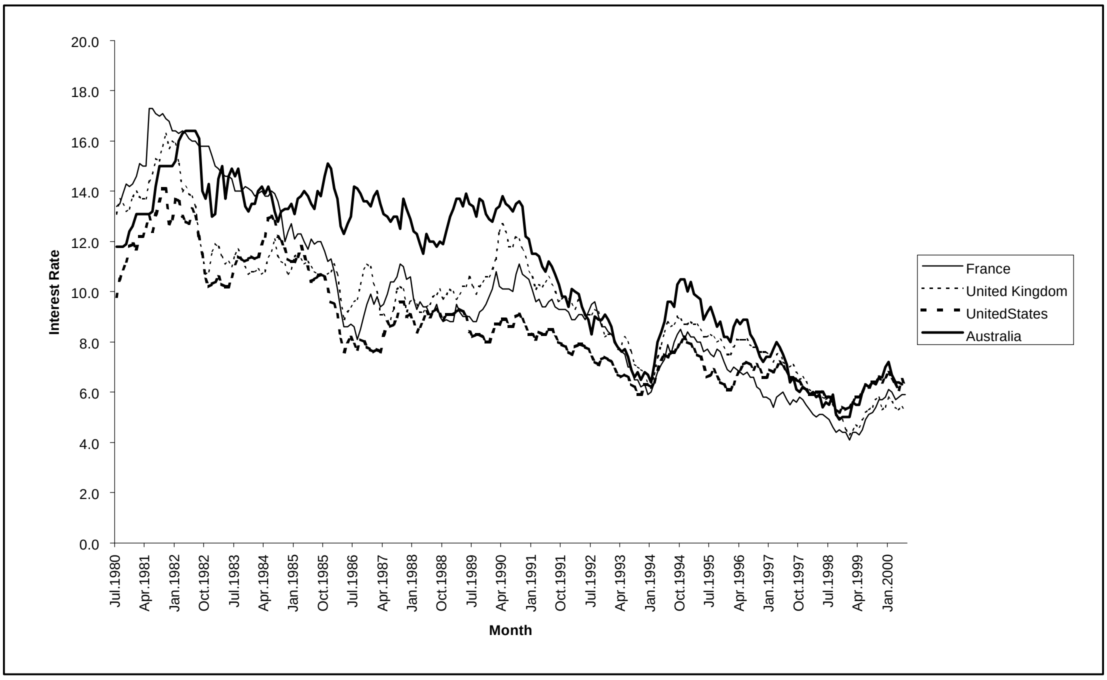

Figure 2 Series of first differences of the interest rates of the four OECD countries: January 1980 - June 2000

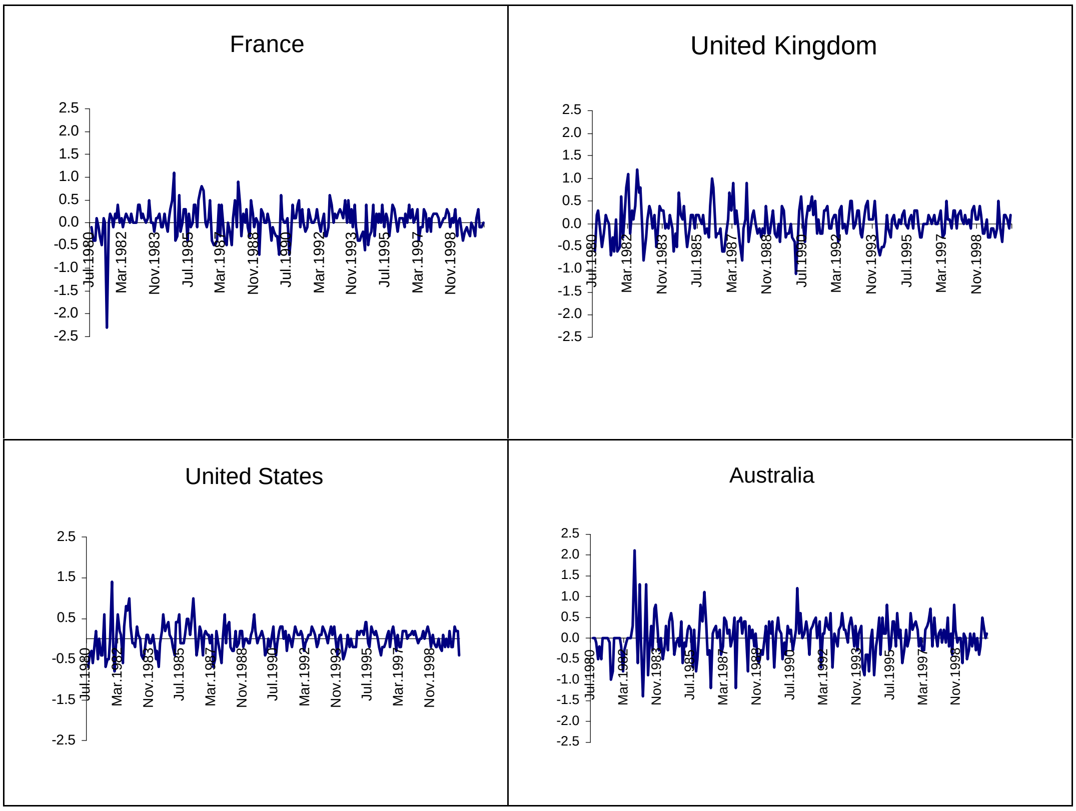

Figure 3 Standardised waveform of a nuclear explosion

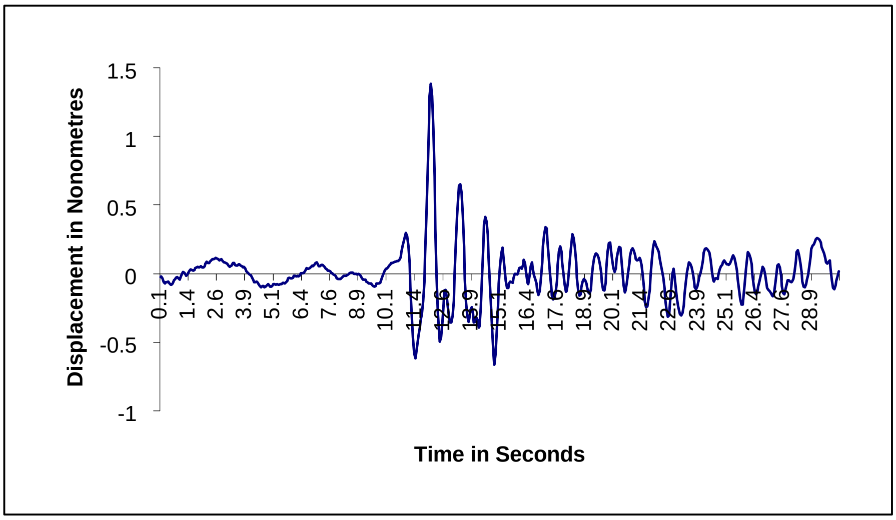

Figure 4 Standardised waveform of an earthquake