Valuing Bonds with Embedded Average Price Options

by

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ABSTRACT

Average price options are based on the average (either arithmetic or geometric) price of the underlying asset during an option's life. Recently, Australia's largest private bank, the National Australia Bank, and the regional Metway Bank, have issued bonds that contain embedded arithmetic average share index options.

The purpose of this paper is to value these options using Monte Carlo simulation, and then to value the bonds themselves. Using a wide range of estimates of the parameters that determine the values of these bonds, it would appear that the fixed-term deposits offered by these same banks represent more profitable investments.

KEYWORDS

Average Price Options; Bonds; Monte Carlo Simulation.

1. Introduction

Average price options are based on the average (either arithmetic or geometric) price of the underlying asset during an option's life. There are many uses of these so-called Asian options, and they are now common in the foreign exchange, interest rate and commodity markets.[1] To illustrate one use of these options, assume that a company buys oil throughout the year and wishes to lock in the price of oil for that period. If the company buys a few barrels of oil each day, its exposure to the price of oil consists of the daily movement in the spot price during the year. The company can hedge this exposure by buying an average call option, where the settlement price is the average of the daily oil prices. If the average daily oil price is above the exercise price of the option, then the company will receive this difference at the end of the year. The company is thereby protected from upward movements in the average oil price.

Hedging is not the only use of average options. By their design, average options reduce the significance of the closing price at the maturity of the option. This reduces the effects of any abnormal price movements at the maturity of the option. Thus, average options provide a way to ameliorate any price distortions that might arise because of a lack of depth in the market for the underlying asset.

Recently, two Australian banks have employed average options in another context. Both the National Australia Bank and Metway Bank have issued bonds that contain embedded arithmetic average share index options. The purpose of this paper is to value the arithmetic average share index options embedded in these bonds, and then to value the bonds themselves. To provide an insight into the nature of the arithmetic average share index

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1 For details of some of the types of average options that exist, see Budd (1983), Boyle and Kirzner (1985), Kemna and Vorst (1990), Turnbull and Wakeman (1991), Hunter and Stowe (1992a and 1992b), and Geman and Yor (1993).

options, their values are also compared with the values of otherwise equivalent European share index options.

The features of the bonds are described in Section 2, and the technique used to value them is discussed in Section 3. The parameter inputs are discussed in Section 4, and the results are presented in Section 5. A summary is provided in Section 6.

2. Bonds with Embedded Average Price Options

The securities to be valued in this paper are the National Sharemarket Bond, a five-year bond issued by the National Australia Bank, and a similar one-year bond issued by Metway Bank. The National Sharemarket Bond was issued in May 1994, and the Metway Bank bond was issued in September 1994.

The information provided in Table 1 may be used to illustrate the calculation of the annual amount payable on the National Sharemarket Bond.[2]

INSERT TABLE 1 HERE

Suppose $10 000 was deposited on 15 April 1993. An initial index value equal to the value of the All Ordinaries Price Index at the close of trading on the next business day is assigned to the deposit. In this case the next business day is 16 April 1993, and the index value is 1703.2.

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2 This illustration was provided in promotional material published by the National Australia Bank (1994, pp. 5-7).

At the end of the 12-month period, on 15 April 1994, an average index value is calculated by taking the arithmetic average of the values of the All Ordinaries Price Index at the close of trading on the fifteenth day of each month from May 1993 to April 1994 inclusive.[3] In this case the average index value is 1992.4. The interest rate for this year would then be (1992.4 - 1703.2)/1703.2 = 16.98%. Had the average index value been less than the initial index value, then no return would have been earned, but the capital would have been maintained. The initial index value used for the calculation of the return in the second year is 2080.6. The annual amount payable on the National Sharemarket Bond may therefore be represented as:

$$Max\ [\ P\ (\ I_a\ -\ I_0\ )\ /\ I_0,\ O\ ] \qquad (1)$$

where P is the initial investment, I_a is the average index value, and I_0 is the initial index value.

The above discussion shows that the annual amount payable component of this bond has all the properties of an arithmetic average share index call option. Specifically, the National Sharemarket Bond comprises five arithmetic average share index call options, one issued at the beginning of each of the five years, together with the repayment of the principal at the end of the five years. This is a risky investment. Therefore, if it is assumed that investors are risk averse, the required rate of return will be greater than the risk-free rate of interest. Cash flows discounted at the risk-free rate of interest will therefore provide a maximum net present value. If a flat yield curve is also assumed, then using the formula for an annuity due, the maximum net present value of an initial investment of P in this bond will be:

3 If the fifteenth day of the month is not a business day, then the index value at the close of trading on the previous trading day is used in the calculation.

$$-P + c_A[1 + (1 - 1/(1 + \gamma)^4)/\gamma] + P/(1 + \gamma)^5 \qquad (2)$$

where c_A is the value of each of the options, and γ is the discrete per annum risk-free rate of interest.[4]

The bond issued by Metway Bank is the same as that issued by the National Australia Bank, except that it is a one-year investment. The maximum net present value of an initial investment of P in this bond will be:

$$-P + c_A + P/(1 + \gamma) \qquad (3)$$

It may be seen from Equations (2) and (3) that the maximum net present value of an investment in the National Australia Bank bond is equal to a five-year annuity due, where the annuity is equal to the maximum net present value of an investment in the Metway Bank bond.[5]

3. Valuation Issues

A difficulty in valuing arithmetic average options is that there is no closed-form solution for such options. Ritchken, Sankarnsubramanian and Vijh (1990), and Turnbull and Wakeman (1991) have derived closed-form solutions for European geometric average options, but Kemna and Vorst (1990) have shown that it is impossible to derive an exact closed-form

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4 If the yield curve is upward-sloping, as it was at the time of the release of the National Sharemarket Bond, this will, *ceteris paribus*, reduce the maximum net present value of the investment.

5 The National Australia Bank and Metway Bank could hedge these options by purchasing a portfolio of Australian shares, together with put options on the All Ordinaries Price Index.

solution for arithmetic average options.[6] Further, many numerical procedures, such as simple lattice approaches, are appropriate only when the option depends on the current value of the underlying asset, not when it depends on the history of those values. A numerical procedure that may be used to value arithmetic average options is Monte Carlo simulation.

In this paper, Monte Carlo simulation is used to value the average options embedded in the National Australia Bank and Metway Bank bonds. To perform the simulation, the term to maturity of the options, namely one year, is divided into 900 discrete time periods.[7] The random sequence of index values $I_1, \ldots, I_n$ is generated as follows:

$$I_t = I_{t-1} + ((r - q)\, I_{t-1}) \,/\, n + \sigma I_{t-1}\, \epsilon\sqrt{1/n} \tag{4}$$

where I_t is the value of the All Ordinaries Price Index at time t, r is the continuous risk-free rate of interest, q is the continuous dividend yield on the All Ordinaries Price Index, σ is the standard deviation of returns on the All Ordinaries Price Index, n is the number of discrete time periods, and ϵ is a random observation from the standard normal distribution. The average index value will be equal to $\Sigma\, I_t/12$, t = 75, 150, . . . , 900. To ensure that the standard errors from the analysis are satisfactorily low, the Monte Carlo simulation in this paper relied on a total of 100 000 series. The antithetic variable technique was used to reduce further the variance of the simulation results.[8]

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6 Various studies have provided analytical approximations, including Carverhill and Clewlow (1990), Kemna and Vorst (1990), Ruttiens (1990), Turnbull and Wakeman (1990), Levy (1992), and Geman and Yor (1993).

7 This equates to a time period of less than half a day. By comparison, Kemna and Vorst (1990) divide the options' lives into discrete periods of approximately one day.

8 For a discussion of the antithetic Monte Carlo method, see Boyle (1977) and Rubinstein (1986).

The estimated value of the average option is:

$$c_A = \sum_{s=1}^{100\,000} e^{-r} Max\ [\ P\ (\ I_{as} - I_{0s}\ )\ /\ I_{0s},\ 0\ ]\ /\ 100\ 000 \tag{5}$$

To enable comparisons to be drawn between average and European share index options, it is necessary to value the corresponding European options. Merton's (1973) formula for the value of the equivalent European share index option is:

$$c_E = I_o e^{-qT} N(d_1) - X e^{-rT} N(d_2) \tag{6}$$

where, in addition to those parameters already defined,

T	=	term to maturity of the option
X	=	exercise price of the option
d_1	=	$(\ \ln\ (\ I_0\ /\ X\ )\ +\ (\ r - q + \sigma^2\ /\ 2\ )T\ )\ /\ \sigma\ \sqrt{T}$
d_2	=	$d_1 - \sigma\sqrt{T}$

and $N(d)$ is the cumulative standard normal density function with upper integral d. For the European share index option that corresponds to the average share index option embedded in the bonds, the exercise price is equal to the initial index value and the term to maturity is one year.

4. Parameter Inputs

To value the options embedded in the bonds, estimates of the discrete γ and continuous r risk-free rates of interest, the dividend yield q on the All Ordinaries Price Index, and the standard deviation σ of returns on the All Ordinaries Price Index are required.

The National Sharemarket Bond was introduced in May 1994. At that time the interest rate on five-year fixed-term bank deposits was 7.50% per annum, and the interest rate on five-year Treasury bonds was 8.15% per annum.[9] The Metway Bank bond was issued in September 1994. At that time the interest rate on one-year fixed-term bank deposits was 7.00% per annum, and the interest rate on 26-week Treasury notes was 6.55% per annum. The dividend yield on the All Ordinaries Price Index was measured as the return on the All Ordinaries Accumulation Index minus the return on the All Ordinaries Price Index. During the period from 16 April 1993 to 15 April 1994, the dividend yield was 4.12%.[10] This equates to a continuous dividend yield of 4.04%. The per annum standard deviation of returns on the All Ordinaries Price Index over the next five years is the most difficult parameter to estimate accurately. In this paper, using daily closing index values from 16 April 1993 to 15 April 1994, the standard deviation of returns was estimated as 12.97%.[11]

Given that all of these estimates may be subject to estimation error, especially the standard deviation of returns on the All Ordinaries Price Index, a range of parameter values was used. The analysis was undertaken by allowing the continuous risk-free rate of interest to vary from 5 to 10% per annum, the continuous dividend yield on the All Ordinaries Price Index to vary from 3 to 5% per annum, and the standard deviation of returns on the All Ordinaries Price Index to vary from 10 to 20% per annum.

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[9] Information on interest rates was obtained from the Reserve Bank of Australia *Bulletin*, October 1994.

[10] The period from 16 April 1993 to 15 April 1994 was used to estimate the dividend yield and the standard deviation of returns on the All Ordinaries Price Index. This period was used by the National Australia Bank (1994) to illustrate the features of the National Sharemarket Bond.

[11] Daily values of the All Ordinaries Price and Accumulation Indices were obtained from issues of the Australian Stock Exchange publication *Personal Investment*.

5. Results

Table 2 provides values of both the average and European options across a range of values of r and σ. Columns 3, 4 and 5 provide values of the average option, the standard error of the average option, and the European option respectively. In all cases, q = 4% per annum. Columns 6, 7 and 8 provide values of the average option as a percentage of the value of the European option, with q = 3, 4, and 5% per annum respectively. In all cases, a $10 000 investment is assumed.

INSERT TABLE 2 HERE

From Table 2 it may be seen that simulation analysis using 100,000 series, together with the antithetic variable technique, is sufficient to produce standard errors for the estimates of the average option that are in all cases less than 0.5% of the respective estimated values.

It may be seen from Table 2 that the value of an average option as a percentage of the value of a European option is a positive function of σ and q and a negative function of r. These relationships may be explained as follows. First, as σ approaches infinity, the values of both the average option and the European option approach the value of the index. Therefore, as σ approaches infinity, the value of the average option as a percentage of the value of the European option approaches unity. Second, as r increases and the effect of σ on the value of the average option becomes relatively less important, the value of the average option approaches the average index value less the exercise price. But as r increases the value of the European option approaches the final index value less the exercise price. An increase in r therefore leads to a smaller increase in the value of the average option than in the value of the European option. The value of the average option as a percentage of the value of the European option therefore falls. Third, an increase in d may be considered as having the same effect as a decrease in *r*. Hence as *d* increases the value of the average option as a percentage of the value of the European option therefore rises.

It should be noted that across the range of parameter values used, the value of the average option lies between 56% and 62% of the value of the corresponding European option.

Table 3 provides estimates of the maximum net present value of a $10 000 investment in the National Sharemarket Bond across a range of values of σ, *r*, and *q*. It may be seen that the value of the bond is a positive function of σ, and a negative function of *r* and *q*. Figure 1 shows the maximum net present value of a $10 000 investment in the bond where $q = 4\%$ per annum. Specifically, with a risk-free rate of interest of 7% per annum, a dividend yield of 4% per annum, and a standard deviation of returns on the All Ordinaries Price Index of 15% per annum, a $10 000 investment in the bond has an estimated maximum net present value of -$1 086. Using the data from Table 2 and subtracting (adding) two standard errors of $1.88 from the average call option value of $427.34 and recalculating the estimated maximum net present value provides a lower (upper) value of -$1 103 (-$1 070). The estimated average option values derived using Monte Carlo simulation are clearly sufficiently reliable to enable one to demonstrate that the estimated maximum net present value of the bond is negative.

As stated earlier, the expected standard deviation of returns on the All Ordinaries Index is very difficult to measure. However, even if it is as high as 20% per annum, then the investment is estimated to be profitable only if the risk-free rate of interest is as low as 5% per annum. At the time of the introduction of the bond, the interest rate on five-year fixed-term bank deposits was 7.50% per annum.

INSERT TABLE 3 HERE

INSERT FIGURE 1 HERE

Recall that the maximum net present value of an investment in the National Australia Bank bond is equal to a five-year annuity due, where the annuity is equal to the maximum net present value of an investment in the Metway Bank bond. Therefore, Table 4, which provides estimates of the maximum net present value of a $10,000 investment in the Metway Bank bond across a range of values of σ, r, and q, is similar to Table 3. With a risk-free rate of interest of 7% per annum, a dividend yield of 4% per annum, and a standard deviation of returns on the All Ordinaries Price Index of 15% per annum, a $10,000 investment in the Metway Bank bond has an estimated maximum net present value of -$249. This investment is also estimated to be profitable only if the risk-free rate of interest is as low as 5% per annum.

INSERT TABLE 4 HERE

6. Summary

During 1994, two Australian banks, the National Australia Bank and Metway Bank, issued bonds that contained embedded arithmetic average share index options. In this paper, Monte Carlo simulation was used to value these options, and thereby to value the bonds themselves. Given reasonable estimates of the parameters that determine the value of the bonds, namely the risk-free rate of interest, the continuous dividend yield on the All Ordinaries Price Index, and the standard deviation of returns on the All Ordinaries Price Index, these bonds do not appear to represent profitable investments. It would appear that fixed-term deposits offered by these same banks represent more profitable investments.

References

Boyle, P.P., 1977, "Options: a Monte Carlo approach", *Journal of Financial Economics*, May, 4, 323-338.

Boyle, P.P. and E.F. Kirzner, 1985, "Pricing complex options: Echo Bay Ltd. Gold Purchase Warrants", Canadian Journal of Administrative Studies, December, 2, 294-306.

Budd, N., 1983, "The future of commodity indexed financing", Harvard Business Review, July-August, 61, 44-50.

Carverhill, A.P. and L.J. Clewlow, 1990, "Valuing average rate (Asian) options", Risk, April, 3, 25-29.

Geman, H. and M. Yor, 1993, "Bessel processes, Asian options, and perpetuities", Mathematical Finance, October, 3, 4, 349-375.

Hunter, W.C. and D.W. Stowe, 1992a, "Path-dependent options", Federal Reserve Bank of Atlanta, Economic Review, March/April, 77, 29-34.

Hunter, W.C. and D.W. Stowe, 1992b, "Path-dependent options", Federal Reserve Bank of Atlanta, Economic Review, July/August, 77, 30-43.

Kemna, A.G.Z. and A.C.F. Vorst, 1990, "A pricing method for options based on average asset values", Journal of Banking and Finance, March, 14, 113-129.

Levy, E., 1992, "Pricing European average rate currency options", Journal of International Money and Finance, 11, 474-491.

Levy, E. and S. Turnbull, 1992, "Average intelligence", Risk, February, 5, 2, 53-59.

Merton, R., 1973, "Theory of rational option pricing", Bell Journal of Economics and Management Science, Spring, 4, 141-183.

National Australia Bank, 1994, National sharemarket bond: a capital guaranteed investment with sharemarket related performance, National Australia Bank Ltd.

Ritchken, P., Sankarnsubramanian, L. and A.M. Vijh, 1990, "The valuation of path dependent contracts on the average", University of Southern California, School of Business Administration, Working Paper, September.

Rubenstein, R. Y., 1986, Monte Carlo optimization, simulation and sensitivity of queuing networks, John Wiley and Sons, New York.

Ruttiens, A., 1990, "Classical replica", Risk, February, 3, 33-36.

Turnbull, S.M. and L.M. Wakeman, 1991, "A quick algorithm for pricing European average options", Journal of Finance and Quantitative Analysis, September, 26, 3, 377 - 389.

Table 1

Illustration of Interest Calculation on the National Sharemarket Bond

Date	**All Ordinaries Price Index Value**
16 April 1993	1703.2
Opening Index Value	= 1703.2
14 May 1993	1698.5
15 June 1993	1724.0
15 July 1993	1805.9
13 August 1993	1855.9
15 September 1993	1903.6
15 October 1993	2074.1
15 November 1993	2082.9
15 December 1993	2069.9
14 January 1994	2206.4
15 February 1994	2234.0
15 March 1994	2172.5
15 April 1994	2080.6
	23908.3/12
Average Index Value	= 1992.4

Table 2

Comparison of the Values of Average Price and European Options

Standard Deviation of Returns[a]	Continuous Risk-Free Rate of Interest	Average Call Option Value q = 4%[b]	Standard Error of Average Call Option Value q = 4%	European Call Option Value q = 4%[c]	Ratio of Average/European Call Option Value		
					q = 3%	q = 4%	q = 5%
(%)	(%)	($)	($)	($)	(%)	(%)	(%)
10	5	259.93[d]	1.18	430.94	59.48	60.32	61.51
	6	286.94	1.23	482.03	58.60	59.53	60.34
	7	314.91	1.28	536.30	58.05	58.72	59.66
	8	344.11	1.32	593.62	57.36	57.97	58.85
	9	375.15	1.37	653.81	56.71	57.38	57.97
	10	407.00	1.41	716.68	56.31	56.79	57.49
15	5	376.06	1.78	620.62	59.98	60.59	61.53
	6	401.55	1.83	668.89	59.26	60.03	60.61
	7	427.34	1.88	719.16	58.90	59.42	60.19
	8	453.66	1.92	771.38	58.31	58.81	59.58
	9	481.65	1.97	825.48	57.72	58.35	58.82
	10	509.67	2.01	881.38	57.41	57.83	58.47
20	5	492.22	2.41	810.26	60.26	60.75	61.56
	6	516.64	2.46	856.64	59.63	60.31	60.76
	7	540.97	2.51	904.43	59.38	59.81	60.49
	8	565.34	2.54	953.59	58.86	59.29	59.98
	9	591.50	2.59	1004.08	58.32	59.91	59.29
	10	617.22	2.63	1055.86	58.09	58.46	59.05

[a] The standard deviation of returns on the All Ordinaries Price Index.

[b] q is the continuous dividend yield on the All Ordinaries Price Index.

[c] European call option values were found using Merton's (1973) formula.

[d] The average call option values were derived using Monte Carlo simulation. For each option value, 100 000 series were used. The antithetic variance technique was used to reduce the variance of the simulation results.

Table 3
Maximum Net Present Value of a $10 000 Investment in the National Sharemarket Bond

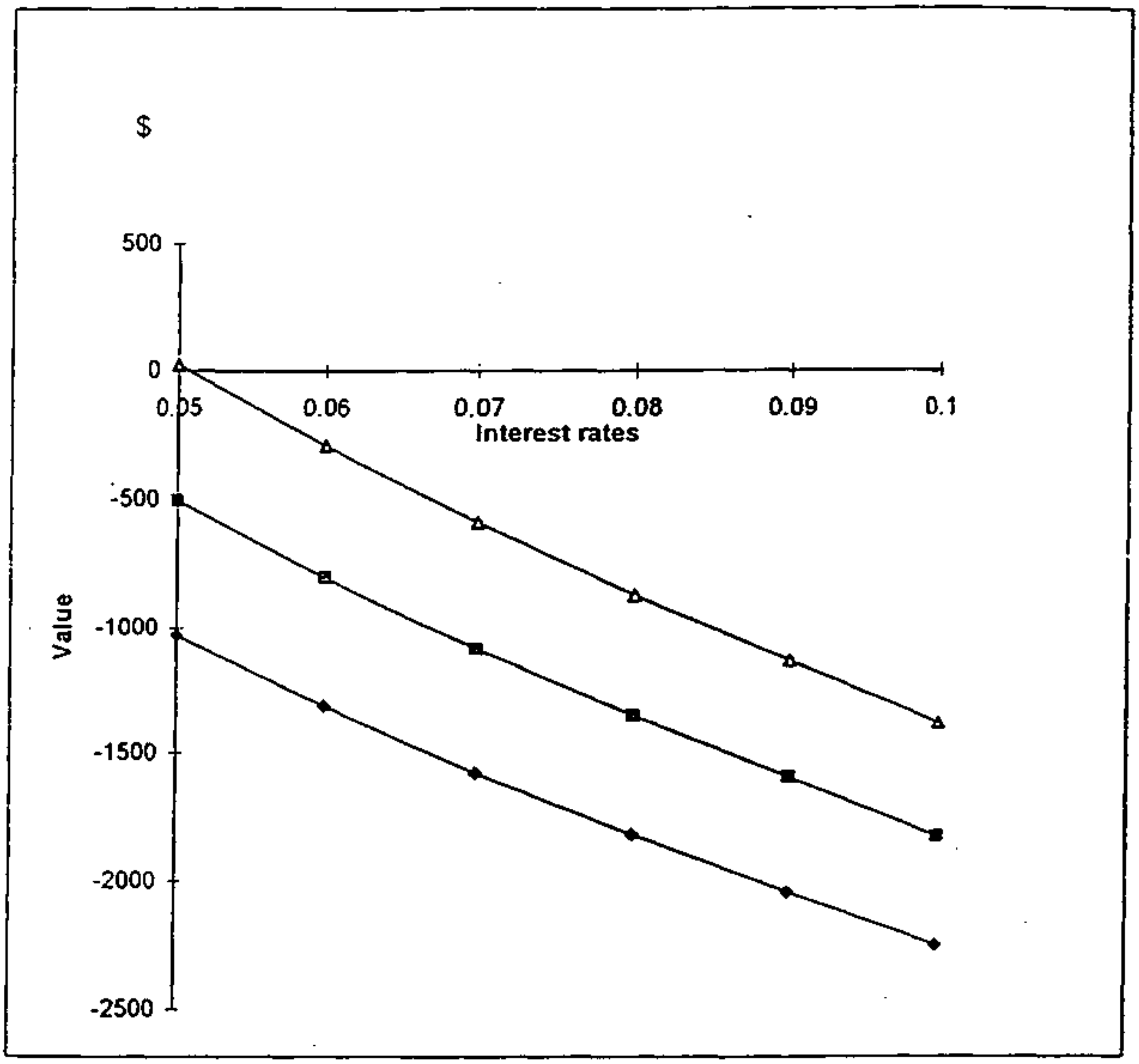
Standard Deviation of Returns[a]	Continuous Risk-Free Rate of Interest	Maximum Net Present Value of a $10 000 Investment in the National Sharemarket Bond		
		q = 3%[b]	q = 4%	q = 5%
(%)	(%)	($)	($)	($)
10	5	-899	-1033	-1154
	6	-1179	-1315	-1446
	7	-1433	-1578	-1709
	8	-1672	-1821	-1957
	9	-1895	-2044	-2189
	10	-2096	-2252	-2396
15	5	-374	-506	-625
	6	-676	-805	-934
	7	-949	-1086	-1212
	8	-1212	-1352	-1478
	9	-1460	-1596	-1733
	10	-1683	-1827	-1959
20	5	153	20	-96
	6	-168	-292	-423
	7	-455	-590	-712
	8	-737	-873	-994
	9	-1005	-1133	-1267
	10	-1244	-1383	-1508

a The standard deviation of returns on the All Ordinaries Price Index.

b q is the continuous dividend yield on the All Ordinaries Price Index.

Figure 1

Maximum Net Present Value of a $10 000
Investment in the National Sharemarket Bond



The standard deviation of returns on the All Ordinaries Price Index takes on three values, namely 10% (represented by a ♦), 15% (represented by a ■), and 20% (represented by a Δ). In all cases, the continuous dividend yield on the All Ordinaries Price Index was set equal to 4%.

Table 4

Maximum Net Present Value of a $10 000 Investment in the Metway Bank Bond

Standard Deviation of Returns[a]	Continuous Risk-Free Rate of Interest	Maximum Net Present Value of a $10 000 Investment in the Metway Bank Bond		
		q = 3%[b]	q = 4%	q = 5%
(%)	(%)	($)	($)	($)
10	5	-198	-228	-254
	6	-265	-295	-325
	7	-328	-361	-391
	8	-390	-425	-456
	9	-450	-486	-520
	10	-507	-545	-580
15	5	-82	-112	-138
	6	-152	-181	-210
	7	-217	-249	-277
	8	-283	-315	-345
	9	-347	-379	-411
	10	-407	-442	-474
20	5	34	5	-21
	6	-38	-66	-95
	7	-104	-135	-163
	8	-172	-203	-232
	9	-239	-269	-301
	10	-301	-334	-365

[a] The standard deviation of returns on the All Ordinaries Price Index.

[b] q is the continuous dividend yield on the All Ordinaries Price Index.